On a differential model for growing sandpiles with non-regular sources *

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February 20, 2008

Abstract

We consider a variational model that describes the growth of a sandpile on a bounded table under the action of a vertical source. The possible equilibria of such a model solve a boundary value problem for a system of nonlinear partial differential equations that we analyse when the source term is merely integrable. In addition, we study the asymptotic behavior of the dynamical problem showing that the solution converges asymptotically to an equilibrium that we characterise explicitly.

Key words: granular matter, asymptotic profile, uniqueness of solutions, distance function

MSC Subject classifications: 35C15, 35F30 (primary), 47J20, 35Q99 (secondary).

1 Introduction

Several differential systems have been proposed for describing the growth of a sandpile on a bounded table under the action of a vertical source. Here we investigate the one proposed by L. Prigozhin in the seminal paper [13]. This problem is strongly related to the fast/slow diffusion model studied by many authors (see, e.g., [1, 9, 10]) as well as to the so–called BCRE model (see, e.g., [3, 12]).

In the model we consider, the table $\Omega \subset \mathbb{R}^n$ is a given bounded connected domain with smooth boundary. The source $f \geq 0$ is an integrable function in $\overline{\Omega}$. The height of the sand, denoted by u, satisfies the following parabolic problem:

*This work has been partially supported by the Italian PRIN 2005 Program "Metodi di viscosità, metrici e di teoria del controllo in equazioni alle derivate parziali nonlineari".

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$$
\begin{array}{lr}
u_t = \mathrm{div}(vDu) + f & \text{in } \mathbb{R}^+ \times \Omega \\
|Du| \le 1 \ , \quad |Du| < 1 \Rightarrow v = 0 & \text{in } \mathbb{R}^+ \times \Omega \\
u = 0 \quad \text{on } \partial\Omega \ , \qquad u(0, \cdot) = u_0 \quad \text{in } \Omega \ , &
\end{array}
\tag{1}
$$

where $v(t,x) \ge 0$ is an auxiliary function, to be determined, and u_0 is the initial configuration, such that $\|Du_0\|_\infty \le 1$, $u_0 = 0$ on $\partial\Omega$.

In [13], it is proved—under very general assumptions—that system (1) has a unique weak solution (u, v). Moreover, the first component u of the solution is characterized by the variational inequality

$$
\begin{cases} f - u_t \in \partial I(u) & \text{in } L^2(\Omega) \\ u(0, \cdot) = u_0 \ . \end{cases}
\tag{2}
$$

Here, ∂I denotes the subdifferential of the convex function $I : L^2(\Omega) \to [0, \infty]$ defined by

$$
I(u) = \begin{cases} 0 & \text{if } u \in \mathbb{K}_0 \\ +\infty & \text{otherwise} \end{cases}
$$

where

$$
\mathbb{K}_0 := \{ v \in W^{1,\infty}(\Omega) : \|Dv\|_\infty \le 1, v|_{\partial\Omega} = 0 \} \, .
$$

Once well-posedness is established, the next natural question is whether the solution $u(t, \cdot)$ converges as $t \to \infty$. At least formally, one would expect the asymptotic limit to be an equilibrium configuration of the dynamical system and, therefore, to satisfy

$$
-\mathrm{div}(vDu) = f \ , \quad (1 - |Du|)v = 0 \quad \text{in } \Omega \ , \quad u|_{\partial\Omega} = 0 \, .
\tag{3}
$$

This system has also been found by Hadeler and Kuttler [12] in order to describe the equilibria of the aforementioned BCRE model. In that same paper, the authors gave the explicit solution for this equilibrium for $n = 1$. Later on, system (3) was analysed in [4] for $n = 2$ and then in [5] for arbitrary space dimension, obtaining the existence, partial uniqueness and representation formula for the solution when the source, f, is *continuous*. In this case, the solution (u, v) turns out to be continuous in Ω, with u equal to the distance from $\partial\Omega$ on the support of v. In particular, the continuity of v is a special—to some extent, surprising—property of the solution that cannot be expected if f is discontinuous.

On the other hand, from both the theoretical and the applied point of view it is interesting to study problem (3) for an integrable source term. This is one of the aims of this paper: we will show that the theory of [4, 5] (existence, partial uniqueness, representation) can be extended to $f \in L^1(\Omega)$. For this, several new ideas will be necessary.

For instance, the representation formula (19) for the solution of (3) involves integrals of f along line segments: when f is just measurable, it should be checked that such a formula remains meaningful, and we do this in section 4.1. As for uniqueness, in the continuous case an important step of the proof was to show that v vanishes on the cut locus $\bar{\Sigma}$ of Ω, which is a set of measure zero. Clearly, the sense of such a property should be made precise when v is just integrable. In fact, it is even false if f is unbounded (see Example 4.7). Therefore, we have to develop a new strategy to prove uniqueness: the new proof we give in section 4.2 turns out to be both simpler and more powerful than the one given in [4, 5], and could possibly be applied to similar problems with different boundary conditions, such as the one considered in [6]. Like in the continuous case, full uniqueness holds just for the v component of the solution. Indeed, one cannot expect uniqueness for u: the structure of (3) only allows to determine u on the support of v (where it coincides with the distance from $\partial\Omega$).

So, returning to the original problem of studying the asymptotic limit of the solution of (2), the above discussion explains why the stationary problem (3) does not suffice to uniquely determine such a function on the whole domain Ω. For this purpose, we will study problem (2) directly, showing that the solution $u(t, \cdot)$ converges, as $t \to \infty$, to a limit that we characterize in section 3.1. Such a limit depends on Ω, the support of f, and the initial condition u_0. Moreover, if f is bounded away from zero in a neighbourhood of the cut locus, then the equilibrium is attained in finite time, as we prove in section 3.2.

Finally, setting up the theory in $L^1(\Omega)$—or, more generally, in $L^p(\Omega)$ for $1 \le p \le \infty$— one can derive Lipschitz estimates in $L^p(\Omega)$ for the v-component of the solution. Such estimates, that are reminiscent of the L^p-regularity results obtained in [7] for the Monge-Kantorovich problem, are derived in section 4.3 of this paper.

2 Notation and preliminaries

Let $n \ge 2$ be an integer. We denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ the Euclidean scalar product and norm in $\mathbb{R}^n$ respectively. For any $x_0 \in \mathbb{R}^n$ and $r > 0$ we set

$$B_r(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\} .$$

For a given function $g \in L^1(A)$, where A is an open subset of $\mathbb{R}^n$, we call the *support* of g the set of all $x \in A$ such that $\int_{B_\varepsilon(x)\cap A} |g(y)|dy > 0$ for all $\varepsilon > 0$ (or, equivalently, such that $\{y \in B_\varepsilon(x) \cap A \;:\; g(y) \neq 0\}$ has positive measure for all $\varepsilon > 0$). It is easy to see that the support of g is a closed set (in the relative topology of A) and that it coincides with the usual notion of support if g is continuous. Clearly, if $\phi \in C(A)$ is nonnegative and $g \in L^1(A)$ is such that $\int_A \phi(x)|g(x)|dx = 0$, then $\phi \equiv 0$ on $\mathrm{spt}(g)$.

Let Ω be a bounded domain in $\mathbb{R}^n$ with $\mathcal{C}^2$ boundary $\partial\Omega$. We briefly recall some properties of the distance function in Ω; some more details can be found e.g. in [4]. In what follows we denote by $d : \overline{\Omega} \to \mathbb{R}$ the distance function from the boundary of Ω and by Σ the singular set of d, that is, the set of points $x \in \Omega$ at which d is not differentiable. Since d is Lipschitz continuous, Σ has Lebesgue measure zero. Introducing the projection $\Pi(x)$ of x onto $\partial\Omega$ in the usual way, Σ is also the set of all points x at which $\Pi(x)$ is not a singleton.

For any $x \in \partial\Omega$ and $i = 1, \dots, n-1$, the number $\kappa_i(x)$ denotes the $i - th$ principal curvature of $\partial\Omega$ at the point x, corresponding to a principal direction $e_i(x)$ orthogonal to $Dd(x)$, with the sign convention $\kappa_i \ge 0$ if the normal section of Ω along the direction e_i is convex. Also, we will label in the same way the extension of κ_i to $\overline{\Omega} \setminus \Sigma$ given by

$$\kappa_i(x) = \kappa_i(\Pi(x)) \qquad \forall x \in \overline{\Omega} \setminus \Sigma . \tag{4}$$

Notice that the regularity of Ω guarantees that the principal curvatures κ_i are continuous functions on $\partial\Omega$. For any $x \in \overline{\Omega}$ and any $y \in \Pi(x)$ we recall that

$$\kappa_i(y)d(x) \le 1 \qquad \forall i = 1, \dots, n-1\,. \tag{5}$$

If, in addition, $x \in \overline{\Omega}\backslash\overline{\Sigma}$, then

$$\kappa_i(x)d(x) < 1 \qquad \text{and} \qquad D^2d(x) = -\sum_{i=1}^{n-1} \frac{\kappa_i(x)}{1-\kappa_i(x)d(x)}\, e_i(x) \otimes e_i(x)$$

where $e_i(x)$ is the unit eigenvector corresponding to $\frac{\kappa_i(x)}{1-\kappa_i(x)d(x)}$ and $\otimes$ stands for tensor product (see, e.g., [11]).

Remark 2.1 The set Γ of points $x \in \Omega \setminus \Sigma$ such that the equality sign holds in (5) for some index i is called the set of regular *focal* (or *conjugate*) points. It represents the "boundary" of the singular set Σ in the sense that $\overline{\Sigma} \subset \Omega$ and $\overline{\Sigma} = \Sigma \cup \Gamma$. The set $\overline{\Sigma}$ is called the *cut locus* (or the *ridge*) of Ω. We recall that under our assumptions, $\overline{\Sigma}$ is a set of zero Lebesgue measure.

Let us introduce the function

$$\tau(x) = \begin{cases} \min\left\{t \ge 0 \ : \ x + tDd(x) \in \overline{\Sigma}\right\} & \forall x \in \overline{\Omega}\backslash\overline{\Sigma} \\ 0 & \forall x \in \overline{\Sigma}\,. \end{cases} \tag{6}$$

Since the map $x \mapsto x + \tau(x)Dd(x)$ is a natural retraction of $\overline{\Omega}$ onto $\overline{\Sigma}$, we will refer to $\tau(\cdot)$ as the *maximal retraction length* of Ω onto $\overline{\Sigma}$ or *normal distance to* $\overline{\Sigma}$. It is easy to see that

$$d(x + tDd(x)) = d(x) + t, \qquad \forall\, t \in [-d(x), \tau(x)] \tag{7}$$

It can be proved that τ is continuous in $\overline{\Omega}$ (see [4, Lemma 2.14]). The function τ actually enjoys finer regularity properties, which will not be needed in this paper.

To a closed set $C \subset \Omega$, let us associate the map u_C defined as follows:

$$u_C(x) = \max_{y \in C} [d(y) - |y - x|]_+, \qquad x \in \Omega\,. \tag{8}$$

If C is empty then we set $u_C \equiv 0$.

Proposition 2.2 *The function u_C satisfies the following properties:*

(i) *u_C is the smallest nonnegative function on Ω such that $u_C \equiv d$ on C and $Lip(u_C) \le 1$.*

(ii) *$u_C \le d$ in Ω; in addition $u_C \equiv d$ in Ω if and only if $\overline{\Sigma} \subset C$.*

Proof — Property (i) is a straightforward consequence of the definition. Since d is nonnegative with Lipschitz constant one, we also deduce from (i) that $u_C \leq d$. To prove the equivalence in (ii), suppose first that $\overline{\Sigma} \subset C$. Then (i) implies that $u_C = d$ on $\overline{\Sigma}$. If we take any $x \in \Omega$, we have that $x + \tau(x)Dd(x) \in \overline{\Sigma}$, and therefore

$$u_C(x) \geq u_C(x + \tau(x)Dd(x)) - |\tau(x)Dd(x)| = d(x + \tau(x)Dd(x)) - \tau(x) = d(x),$$

where we have also used (7). Thus we have that $u_C \equiv d$ everywhere in Ω. To prove the converse implication, we argue by contradiction and suppose that there exists $x_0 \in \overline{\Sigma}$ such that $x_0 \notin C$ and $d(x_0) = u_C(x_0)$. By definition of u_C, there exists $y_0 \in C$ such that $u_C(x_0) = d(y_0) - |y_0 - x_0|$. Let us take any $z_0 \in \Pi(x_0)$. Then $d(y_0) \leq |y_0 - z_0|$ and

$$d(x_0) = u_C(x_0) = d(y_0) - |y_0 - x_0| \leq |y_0 - z_0| - |y_0 - x_0| \leq |x_0 - z_0| = d(x_0) \ .$$

So equality holds everywhere in the above inequalities. In particular, $d(y_0) = |y_0 - z_0|$ and x_0 belongs to the interior of the segment $[z_0, y_0]$. This is impossible since it is well known that all points of the segment joining a point of Ω to one of its projections on $\partial\Omega$ do not belong to $\overline{\Sigma}$, except possibly for the initial endpoint. □

3 Asymptotic behavior

In this section we investigate the variational inequality

$$\begin{cases} f - u_t \in \partial I(u) & \text{in } L^2(\Omega) \\ u(\cdot, 0) = u_0 \end{cases} \tag{9}$$

where $f \in L^2(\Omega)$, $I(u)$ is defined by

$$I(u) = \begin{cases} 0 & \text{if } u \in K_0 \\ +\infty & \text{otherwise,} \end{cases}$$

and where

$$\mathbb{K}_0 := \{v \in W^{1,\infty}(\Omega) : \|Dv\|_\infty \leq 1, v|_{\partial\Omega} = 0\}$$

The initial position u_0 is also assumed to belong to $\mathbb{K}_0$.

Equation (9) has been interpreted by several authors [1, 9, 10, 13] as a natural model for growing sandpiles. We are interested in the behavior as $t \to +\infty$ of the solution of (9). We recall that u is a solution of (9) if, for any $T > 0$, $u \in H^1((0,T), L^2(\Omega))$, $u(\cdot, t) \in \mathbb{K}_0$ for any $t \geq 0$ and $f - u_t(\cdot, t) \in \partial I(u(\cdot, t))$ a.e., where $\partial I(u(\cdot, t))$ denotes the subdifferential (in the sense of convex analysis) of the convex map I at $u(\cdot, t)$. Note that this is equivalent to say that

$$\langle u_t(\cdot, t) - f, \phi - u(\cdot, t)\rangle_{L^2(\Omega)} \geq 0 \qquad \forall \phi \in \mathbb{K}_0, \qquad \text{for almost all } t \geq 0$$

where $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ stands for the scalar product in $L^2(\Omega)$. It is well-known that (9) has a unique solution, see, e.g., [2].

The following comparison principle for solutions can be found in [13].

Lemma 3.1 *Suppose that $f^1 \geq f^2$ and that $u_0^1 \geq u_0^2$. Then $u^1 \geq u^2$, where u^i, $i = 1, 2$, is the solution of the variational problem* (9) *with $f = f^i$ and $u(0, \cdot) = u_0^i$.*

In particular, a solution u of (9) is non–decreasing in time (compare u with the constant solution $u^2 \equiv u_0$ given for $f^2 \equiv 0$). We give the proof of the above lemma for the reader's convenience.
Proof — Let us set

$$u^+(t, x) = \max\{u^1(t, x), u^2(t, x)\} \quad \text{and} \quad u^-(t, x) = \min\{u^1(t, x), u^2(t, x)\}.$$

Then $u^\pm$ are continuous functions with $u_t^\pm \in L^2((0, T) \times \Omega)$ and $u^\pm \in \mathbb{K}$. Using u^+ as a test function in the variational problem for u^1, we obtain

$$\langle f^1 - u_t^1, u^+ - u^1 \rangle_{L^2(\Omega)} \leq 0 \qquad \text{a.e. } t \in [0, T] \,.$$

Since $f^1 \geq f^2$ and $u^+ \geq u^1$, then also

$$\langle f^2 - u_t^1, u^+ - u^1 \rangle_{L^2(\Omega)} \leq 0 \qquad \text{a.e. } t \in [0, T].$$

Analogously, by looking at the variational problem for u^2, we get

$$\langle f^2 - u_t^2, u^- - u^2 \rangle_{L^2(\Omega)} \leq 0 \,.$$

Now, let us denote by $\mathbf{1}_A$ the characteristic function of a set A, that is $\mathbf{1}_A(x) = 1$ for $x \in A$ and $\mathbf{1}_A(x) = 0$ otherwise. Thus, since $\{(t, x) : u^1(t, x) < u^2(t, x)\}$ is an open set (u^i are continuous functions), we have

$$u^- - u^2 = (u^1 - u^2)\mathbf{1}_{\{u^1 < u^2\}} = (u^1 - u^+)\mathbf{1}_{\{u^1 < u^2\}} = u^1 - u^+,$$

while $u_t^2 \mathbf{1}_{\{u^1 < u^2\}} = u_t^+ \mathbf{1}_{\{u^1 < u^2\}}$. Therefore,

$$\langle f^2 - u_t^2, u^- - u^2 \rangle_{L^2(\Omega)} = \langle f^2 - u_t^+, u^1 - u^+ \rangle_{L^2(\Omega)} \leq 0 \qquad \text{a.e. } t \in [0, T] \,,$$

and then

$$\begin{aligned} \tfrac{d}{dt}\tfrac{1}{2}|u^+ - u^1|_{L^2(\Omega)}^2 &= \langle u_t^+ - u_t^1, u^+ - u^1 \rangle_{L^2(\Omega)} \\ &= \langle u_t^+ - f^2, u^+ - u^1 \rangle_{L^2(\Omega)} + \langle f^2 - u_t^1, u^+ - u^1 \rangle_{L^2(\Omega)} \leq 0 \,. \end{aligned}$$

Since $u^+ = u^1$ at time $t = 0$ and the functions u^i, $u^\pm$ are continuous, we conclude that $u^+ \equiv u^1$. $\square$

3.1 Identification of the asymptotic limit

Let u be a solution of (9) defined on $[0, +\infty)$. Since u is nondecreasing and bounded from above by d—as are all elements of $\mathbb{K}_0$—the limit

$$u_\infty(x) = \lim_{t \to +\infty} u(t, x)$$

exists and satisfies

$$u_0(x) \leq u_\infty(x) \leq d(x) \qquad \forall x \in \Omega \,.$$

Moreover $u_\infty \in \mathbb{K}_0$ because $u(\cdot, t) \in \mathbb{K}_0$ for any t.

Theorem 3.2 *We have*

$$u_\infty(x) = \max\{u_0(x)\ ,\ u_f(x)\} \qquad \forall x \in \Omega\,, \tag{10}$$

where u_f is the map defined by

$$u_f(x) = \max_{y \in \mathrm{spt}(f)} [d(y) - |y - x|]_+ \qquad x \in \Omega\,. \tag{11}$$

Proof — Let us introduce the function

$$\psi(t) = \int_\Omega u(t,x)dx \qquad \forall t \geq 0\ .$$

Since $u_t \in L^2([0,T] \times \Omega)$ for any $T > 0$, ψ is absolutely continuous. The map $t \mapsto u(t,x)$ being nondecreasing for any x, we have that $u_t \geq 0$ a.e. and

$$\psi'(t) = \int_\Omega u_t(t,x)dx \geq 0 \qquad \text{for almost all } t \geq 0\,.$$

Since $\psi(t) \to \int_\Omega u_\infty$ as $t \to +\infty$, there is a sequence $t_k \to +\infty$ such that $\psi'(t_k) \to 0$ and for which $u_t(t_k, \cdot)$ exists and satisfies

$$\langle f - u_t(t_k,\cdot), \phi - u(t_k,\cdot)\rangle \leq 0 \qquad \forall \phi \in \mathbb{K}_0\,.$$

Note that $\psi'(t_k) \to 0$ implies that $u_t(t_k,\cdot) \to 0$ in $L^1(\Omega)$. Passing to the limit in the above equation gives

$$\langle f, \phi - u_\infty\rangle_{L^2(\Omega)} \leq 0 \qquad \forall \phi \in \mathbb{K}_0\,. \tag{12}$$

In particular, plugging $\phi = d$ in the above inequality entails

$$\int_\Omega f(d - u_\infty) \leq 0\,.$$

Since $f \geq 0$ and $u_\infty \leq d$, we conclude that $u_\infty = d$ on $\mathrm{spt}(f)$.

To complete the proof of the theorem, we first observe that $\bar{u} := \max\{u_0, u_f\}$ is a stationary solution of (9) because $\bar{u} = d \geq \phi$ on $\mathrm{spt}(f)$ for any $\phi \in \mathbb{K}_0$ and $f \geq 0$. Since $u_0 \leq \bar{u}$, we get, by comparison, that $u(t,x) \leq \bar{u}$ for any $t \geq 0$. Hence $u_\infty \leq \bar{u}$.

Conversely, we already know that $u_0 \leq u_\infty$. Since $u_\infty \in \mathbb{K}_0$ and $u_\infty = d$ on $\mathrm{spt}(f)$, we obtain $u_\infty \geq u_f$ because u_f is the smallest function in $\mathbb{K}_0$ which coincides with d on $\mathrm{spt}(f)$ (Proposition 2.2). Thus, $u_\infty \geq \bar{u}$. □

3.2 Convergence in finite time

In this subsection we assume that f is positive in a neighborhood of the ridge, that is,

$$\exists r > 0 \text{ such that } f \geq r \text{ a.e. in } B_r(x) \text{ for any } x \in \Sigma\,. \tag{13}$$

Such an assumption implies, in particular, that $\overline{\Sigma} \subset \mathrm{spt} f$. Therefore, by Proposition 2.2 and Theorem 3.2, the asymptotic limit u_∞ of the solution to (9) is given by the distance function d. Our next result shows that, in this case, convergence takes place in finite time.

Theorem 3.3 *Under assumption* (13) *there is a time* T *such that, for any initial position* $u_0 \in \mathbb{K}_0$*, the solution* $u(\cdot, t)$ *of* (9) *becomes stationary after* T*, that is,*

$$u(t, \cdot) = d \qquad \forall t \geq T\ . \tag{14}$$

Proof — Let $R = \max_\Omega d$ and let $r > 0$ be given by assumption (13). Let us set $T = R^{n+1}/((n+1)r^{n+1} + 1$. We will show that (14) holds for such a choice of T. Fix $\bar{x} \in \Sigma$ and define, for all $x \in \Omega$ and $t \geq 0$,

$$f^1(t, x) = \begin{cases} (r - |x - \bar{x}|)_+ & \text{if } t \in [0, 1) \\ r & \text{if } t \geq 1 \text{ and } |x - \bar{x}| \leq r \\ 0 & \text{otherwise.} \end{cases}$$

Let u^1 be the solution of (9) with initial condition $u^1_0 :\equiv 0$ and source f^1. One readily checks that

$$u^1(t, x) = t\,(r - |x - \bar{x}|)_+ \qquad \forall\, t \in [0, 1]\,,\ \forall\, x \in \Omega\,.$$

Let α be given by

$$\alpha(t) = \big(r^{n+1} + (n+1)r^{n+1}(t-1)\big)^{\frac{1}{n+1}} \qquad \text{for } t \geq 1.$$

Observe that $\alpha'(t) = r^{n+1}\alpha(t)^{-n}$. We claim that

$$u^1(t, x) = (\alpha(t) - |x - \bar{x}|)_+ \qquad \text{if } t \in [1, \bar{t}]\,,$$

where $\bar{t} = (d^{n+1}(\bar{x}) - r^{n+1})/((n+1)r^{n+1}) + 1$. To prove this let us denote by u^2 the right-hand side of the equality. Then $u^2(\cdot, t) \in \mathbb{K}_0$ for any $t \in [1, \bar{t}]$, $u^2(\cdot, 1) = u^1(\cdot, 1)$ and $u^2_t \in L^2((1, \bar{t}) \times \Omega)$. Let us now check that u^2 satisfies the variational inequality

$$\langle u^2_t - f^1, \phi - u^2 \rangle_{L^2(\Omega)} \geq 0 \qquad \forall \phi \in \mathbb{K}_0\ .$$

For any $\phi \in \mathbb{K}_0$ we have (in polar coordinates)

$$\begin{aligned} &\langle u^2_t - f^1, \phi - u^2 \rangle_{L^2(\Omega)} \\ &= \int_{S^{n-1}} d\mathcal{H}^{n-1}(\omega) \int_0^r (\alpha'(t) - r)(\phi(t, \bar{x} + \rho\omega) - u^2(t, \bar{x} + \rho\omega))\rho^{n-1} d\rho \\ &\quad + \int_{S^{n-1}} d\mathcal{H}^{n-1}(\omega) \int_r^{\alpha(t)} \alpha'(t)(\phi(t, \bar{x} + \rho\omega) - u^2(t, \bar{x} + \rho\omega))\rho^{n-1} d\rho\,. \end{aligned}$$

From the definition of u^2 and the fact that $\mathrm{Lip}(\phi) \leq 1$ we deduce that the map

$$\rho \mapsto \phi(t, \bar{x} + \rho\omega) - u^2(t, \bar{x} + \rho\omega)$$

is nondecreasing on $[0, \alpha(t)]$. Therefore, since $0 < \alpha'(t) \leq r$, we have

$$(\alpha'(t) - r)(\phi(t, \bar{x} + \rho\omega) - u^2(t, \bar{x} + \rho\omega)) \geq (\alpha'(t) - r)(\phi(t, \bar{x} + r\omega) - u^2(t, \bar{x} + r\omega))$$

for all $\rho \in [0, r]$. Similarly we have, for all $\rho \in [r, \alpha(t)]$,

$$\alpha'(t)(\phi(t, \bar{x} + \rho\omega) - u^2(t, \bar{x} + \rho\omega)) \geq \alpha'(t)(\phi(t, \bar{x} + r\omega) - u^2(t, \bar{x} + r\omega)).$$

From these two inequalities we obtain that

$$\begin{aligned}
&\langle u_t^2 - f, \phi - u^2 \rangle_{L^2(\Omega)} \\
&\quad \geq \int_{S^{n-1}} d\mathcal{H}^{n-1}(\omega)(\phi(t, \bar{x} + r\omega) - u^2(t, \bar{x} + r\omega)) \int_0^{\alpha(t)} \rho^{n-1}(\alpha'(t) - r\mathbf{1}_{[0,r]})d\rho = 0
\end{aligned}$$

since, by the definition of α, $\int_0^{\alpha(t)} \rho^{n-1}(\alpha'(t) - r\mathbf{1}_{[0,r]}(\rho))d\rho = 0$. This shows that u^2 is a solution and therefore $u^1 = u^2$ on $[0, \bar{t}]$.

By assumption (13), we have that $f^1 \leq f$. Therefore, since $u^1(\cdot, 0) = 0 \leq u_0$, a comparison argument shows that $u^1(\cdot, t) \leq u(\cdot, t)$ for any $t \in [0, \bar{t}]$. Thus, since $\bar{t} \leq T$,

$$u(T, x) \geq u(\bar{t}, x) \geq u^1(\bar{t}, x) = (d(\bar{x}) - |x - \bar{x}|)_+ \qquad \forall \bar{x} \in \Sigma\,. \tag{15}$$

This implies that $u(T, x) \geq u_f(x)$, where $u_f(x)$ is defined in (11). So, in view of Proposition 2.2-(ii) and assumption (13), to obtain the conclusion it suffices to note that $u_f(x)$ coincides with d. □

4 Analysis of the stationary problem

In this section we analyse the system of partial differential equations

$$\begin{cases} -\mathrm{div}\,(vDu) = f & \text{in } \Omega \\ v \geq 0,\ |Du| \leq 1 & \text{in } \Omega \\ |Du| - 1 = 0 & \text{in } \{v > 0\}, \end{cases} \tag{16}$$

complemented with the conditions

$$\begin{cases} u \geq 0, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{17}$$

Such a system describes the stationary states of problem (1). The solution of system (17) is intended in the following sense:

Definition 4.1 A pair of functions $(u, v) \in W_0^{1,\infty}(\Omega) \times L^1(\Omega)$ is a solution of (16)–(17) if

1. $u, v \geq 0$ and $|Du(x)| \leq 1$ almost everywhere in Ω;

2. for every test function $\phi \in \mathcal{C}_c^\infty(\Omega)$,

$$\int_\Omega v(x)\langle Du(x), D\phi(x)\rangle dx = \int_\Omega f(x)\phi(x)dx\,; \tag{18}$$

3. $\int_\Omega v(x)(|Du(x)|^2 - 1)dx = 0.$

4.1 Existence

In this section we prove that the pair (d, v_f), where d is the distance function from $\partial\Omega$ and

$$v_f(x) = \int_0^{\tau(x)} f(x + tDd(x)) \prod_{i=1}^{n-1} \frac{1 - (d(x)+t)\kappa_i(x)}{1 - d(x)\kappa_i(x)}\, dt \qquad \text{almost every } x \in \Omega \tag{19}$$

is a solution of system (16)–(17). In spite of the terms of form $(1 - d(x)\kappa_i(x))^{-1}$, the product appearing inside the integral is a uniformly bounded function; in fact, it is easy to check (see [5, Proposition 3.2]) that

$$0 < \frac{1 - (d(x)+t)\kappa_i(x)}{1 - d(x)\kappa_i(x)} \leq 1 + \|[\kappa_i]_-\|_\infty \, \|\tau\|_\infty \,, \qquad 0 < t < \tau(x). \tag{20}$$

However, when $f \in L^1(\Omega)$ it is not obvious, at first sight, that the integral in (19) is finite for a.e. x; thus, our first step will be to show that v_f is a well-defined function in $L^1(\Omega)$.

Given $y \in \partial\Omega$, we denote by $\nu(y)$ the interior normal to Ω at y. Then $Dd(y + t\nu(y)) = \nu(y)$ for all $t \in [0, \tau(y))$. Let $\mathcal{O}$ be the subset of $\partial\Omega \times \mathbb{R}_+$ defined by

$$\mathcal{O} = \{(y,t) \in \partial\Omega \times \mathbb{R}_+ \mid 0 < t < \tau(x)\} \,.$$

Then the mapping $X : \mathcal{O} \to \Omega \backslash \overline{\Sigma}$ defined by

$$\forall (y,t) \in \mathcal{O},\ X(y,t) = y + t\nu(y)$$

is one-to-one and $\mathcal{C}^1$ on its domain. Moreover, the volume element changes according to

$$dx = \prod_{i=1}^{n-1} (1 - t\kappa_i(y))\, dt\, d\mathcal{H}^{n-1}(y).$$

Since $|\overline{\Sigma}| = 0$, we deduce the following formula, valid for any $h \in L^1(\Omega)$,

$$\int_\Omega h(x)dx = \int_{\partial\Omega} \int_0^{\tau(y)} h(y + sDd(y)) \prod_{i=1}^{n-1} (1 - s\kappa_i(y))\, ds\, d\mathcal{H}^{n-1}(y) \,. \tag{21}$$

Lemma 4.2 *For $\mathcal{H}^{n-1}$-a.e. $y \in \partial\Omega$ the function $s \to f(y + s\nu(y)) \prod_{i=1}^{n-1}(1 - s\kappa_i(y))$ is in $L^1([0, \tau(y)])$.*

Proof — It is an immediate consequence of formula (21) above. □

In particular, we deduce that for a.e. $y \in \partial\Omega$ one of the two following properties hold: either (i) the map $t \to f(y + t\nu(y))$ is in $L^1([0, \tau(y)])$, or (ii) $\kappa_i(y)\tau(y) = 1$ for some i, i.e. the normal ray starting at y ends at a focal point. Simple examples show that the set of the points $y \in \partial\Omega$ which satisfy (ii) but not (i) can have positive $\mathcal{H}^{n-1}$-measure (see Example 4.7 later).

Lemma 4.3 *Let $g : \Omega \to \mathbb{R}$ be such that the map $x \to d(x)g(x)$ belongs to $L^1(\Omega)$. Then*

$$\int_\Omega \int_0^{\tau(x)} g(x + tDd(x)) \prod_{i=1}^{n-1} \frac{1 - (d(x)+t)\kappa_i(x)}{1 - d(x)\kappa_i(x)} dt dx = \int_\Omega d(x)g(x)dx \, .$$

Proof — It suffices to consider the case where $g \in L^\infty(\Omega)$, since the general case follows by approximation. Let us consider the function

$$h(x) = \int_0^{\tau(x)} g(x + tDd(x)) \prod_{i=1}^{n-1} \frac{1 - (d(x)+t)\kappa_i(x)}{1 - d(x)\kappa_i(x)} dt \, ,$$

which is in $L^\infty(\Omega)$ since we are assuming that $g \in L^\infty(\Omega)$. We first observe that, given any $y \in \partial\Omega$ and $s \in [0, \tau(y))$, we have

$$h(y + s\nu(y)) = \int_0^{\tau(y)-s} g(y + (t+s)\nu(y)) \prod_{i=1}^{n-1} \frac{1 - (s+t)\kappa_i(y)}{1 - s\kappa_i(y)} dt$$

because $d(y+s\nu(y)) = s$, $\tau(y+s\nu(y)) = \tau(y)-s$, $Dd(y+s\nu(y)) = \nu(y)$ and $\kappa_i(y+s\nu(y)) = \kappa_i(y)$, for $s \in [0, \tau(y))$. Thus (21) implies

$$\begin{aligned}
\int_\Omega h(x)dx &= \int_{\partial\Omega} \int_0^{\tau(y)} h(y + s\nu(y)) \prod_{i=1}^{n-1} (1 - s\kappa_i(y)) ds \, d\mathcal{H}^{n-1}(y) \\
&= \int_{\partial\Omega} \int_0^{\tau(y)} \int_0^{\tau(y)-s} g(y + (t+s)\nu(y)) \prod_{i=1}^{n-1} (1 - (s+t)\kappa_i(y)) \, dt \, ds \, d\mathcal{H}^{n-1}(y) \\
&= \int_{\partial\Omega} \int_0^{\tau(y)} \int_s^{\tau(y)} g(y + t\nu(y)) \prod_{i=1}^{n-1} (1 - t\kappa_i(y)) \, dt \, ds \, d\mathcal{H}^{n-1}(y) \\
&= \int_{\partial\Omega} \int_0^{\tau(y)} \int_0^{t} g(y + t\nu(y)) \prod_{i=1}^{n-1} (1 - t\kappa_i(y)) \, ds \, dt \, d\mathcal{H}^{n-1}(y) \\
&= \int_{\partial\Omega} \int_0^{\tau(y)} t \, g(y + t\nu(y)) \prod_{i=1}^{n-1} (1 - t\kappa_i(y)) dt d\mathcal{H}^{n-1}(y) \\
&= \int_\Omega d(x)g(x)dx
\end{aligned}$$

where we have again used (21) in the last equality. This proves our lemma. □

Corollary 4.4 *The function*

$$v_f(x) := \int_0^{\tau(x)} f(x + tDd(x)) \prod_{i=1}^{n-1} \frac{1 - (d(x)+t)\kappa_i(x)}{1 - d(x)\kappa_i(x)} dt$$

is well–defined for almost every $x \in \Omega$ and is in $L^1(\Omega)$.

Now we can prove that the pair (d, v_f) is a solution of our system.

Theorem 4.5 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $\mathcal{C}^2$ and f be in $L^1(\Omega)$ and nonnegative. Then, the pair (d, v_f) defined above satisfies* (16)–(17) *in the sense of Definition 4.1.*

Proof— Let $\{f_k\}$ be a sequence of continuous functions such that $f_k \to f$ in $L^1(\Omega)$ as $k \to \infty$ and set

$$v_{f_k}(x) = \begin{cases} \int_0^{\tau(x)} f_k(x + tDd(x)) \prod_{i=1}^{n-1} \frac{1-(d(x)+t)\kappa_i(x)}{1-d(x)\kappa_i(x)}\, dt & \forall x \in \Omega \setminus \overline{\Sigma} \\ 0 & \forall x \in \overline{\Sigma}\,. \end{cases}$$

By [5, Theorem 3.1] the pair (d, v_{f_k}) satisfies, for every test function $\phi \in \mathcal{C}_c^\infty(\Omega)$,

$$\int_\Omega v_{f_k}(x)\langle Dd(x), D\phi(x)\rangle dx = \int_\Omega f_k(x)\phi(x) dx\,. \tag{22}$$

In addition, we have, setting $g_k = |f - f_k|$ and applying Lemma 4.3,

$$\begin{aligned} ||v_f - v_{f_k}||_1 &\leq \int_\Omega \int_0^{\tau(x)} g_k(x + tDd(x)) \prod_{i=1}^{n-1} \frac{1-(d(x)+t)\kappa_i(x)}{1-d(x)\kappa_i(x)}\, dx \\ &= \int_\Omega d(x) g_k(x) dx \leq \mathrm{diam}\,(\Omega)||f - f_k||_1. \end{aligned}$$

This shows that $v_{f_k} \to v_f$ in $L^1(\Omega)$. Passing to the limit in (22), we obtain that v_f satisfies point 2 of Definition 4.1. Points 1 and 3 follow immediately from well known properties of the distance function. $\square$

Proposition 4.6 *For $\mathcal{H}^{n-1}$-a.e. $y \in \partial\Omega$ we have*

$$\lim_{t \uparrow \tau(y)} v_f(y + tDd(y)) \prod_{i=1}^{n-1}(1 - t\kappa_i(y)) = 0\,.$$

Proof — We have

$$v_f(y + tDd(y)) \prod_{i=1}^{n-1}(1 - t\kappa_i(y)) = \int_t^{\tau(y)} f(y + s\nu(y)) \prod_{i=1}^{n-1}(1 - s\kappa_i(y))\, ds\,,$$

and we know from Lemma 4.2 that the function inside the integral is in $L^1([0, \tau(y)])$ for a.e. $y \in \partial\Omega$. Therefore the integral tends to zero as the interval of integration shrinks to a point. $\square$

If $f \in L^\infty$ then it is easy to see, directly from the definition, that $v_f(y + tDd(y)) \to 0$ if $t \to \tau(y)$ for a.e. $y \in \partial\Omega$. If f is unbounded, this is no longer true in general, as the following example shows.

Example 4.7 Let $\Omega = B_1 \subset \mathbb{R}^2$ and let $f(x) = 1/|x|$. Then it is easily checked that $d(x) = 1 - |x|$, $\Sigma = \{0\}$, $k(x) \equiv 1$, $\tau(x) = |x|$ and $v_f(x) \equiv 1$.

We prove one last property of v_f which will be needed in the following.

Lemma 4.8 *Let $x \in \mathrm{spt}(v_f)$. Then there exists $t \in [0, \tau(x)]$ such that $x + tDd(x) \in \mathrm{spt} f$.*

Proof — Let us first consider the case where $x \notin \overline{\Sigma}$. We argue by contradiction and suppose that $x + tDd(x) \notin \mathrm{spt} f$ for all $t \in [0, \tau(x)]$. Then there exists a neighbourhood of the segment joining x to $x + \tau(x)Dd(x)$ where $f \equiv 0$ a.e.. Using the definition of v_f and the continuity of τ and of Dd, this easily implies that $v_f \equiv 0$ a.e. in a neighbourhood of x. Thus, x cannot belong to $\mathrm{spt}(v_f)$. If $x \in \overline{\Sigma}$, we prove that $x \in \mathrm{spt} f$ by a similar argument. □

4.2 Uniqueness

In this section we will prove the following uniqueness result.

Theorem 4.9 *If (u, v) is a solution of system (16)–(17) in the sense of Definition 4.1, then $v = v_f$ almost everywhere, where v_f is given by (19), and $u \equiv d$ in $\mathrm{spt}(v_f)$.*

Thus, the v–component of the solution must coincide with v_f, while the u–component is uniquely determined only on the support of v_f. We shall see also that $\mathrm{spt}(v_f)$ coincides with Ω if and only if $\overline{\Sigma} \subset \mathrm{spt}(f)$; in this case the solution is unique. If this condition is not satisfied, there are indeed solutions to the system with different u–components. The possible solutions can be completely described, see Corollary 4.14.

As a first step, we prove the following result.

Lemma 4.10 *Let (u, v) be a solution of system (16)–(17). Then $u \equiv d$ in $\mathrm{spt}(f)$ and $v(x)Du(x) = v(x)Dd(x)$ almost everywhere in Ω.*

Proof — It is well known that $d \geq \phi$ for all $\phi \in \mathbb{K}_0$; in particular, we have that $d \geq u$. By Definition 4.1 we have, for every $\phi \in C_c^\infty(\Omega)$,

$$\int_\Omega v(x)\langle Du(x), D\phi(x)\rangle dx = \int_\Omega f(x)\phi(x)\, dx\,.$$

By approximation, the same property holds for $\phi \in W_0^{1,\infty}(\Omega)$, including the case $\phi = u - d$. Hence,

$$\int_\Omega v(x)\langle Du(x), Du(x) - Dd(x)\rangle dx = \int_\Omega f(x)(u(x) - d(x))\, dx\, \leq 0\,.$$

On the other hand,

$$\begin{aligned}
&\int_\Omega v(x)\langle Du(x), Du(x) - Dd(x)\rangle dx \\
= &\int_\Omega \frac{v(x)}{2}\left(|Du(x) - Dd(x)|^2 + |Du(x)|^2 - |Dd(x)^2|\right) dx \\
= &\int_\Omega \frac{v(x)}{2}\left(|Du(x) - Dd(x)|^2\right) dx + \int_\Omega \frac{v(x)}{2}\left(|Du(x)|^2 - 1\right) dx \\
= &\int_\Omega \frac{v(x)}{2}\left(|Du(x) - Dd(x)|^2\right) dx \geq 0\,,
\end{aligned}$$

where we have used property 3 of Definition 4.1. We conclude that

$$\int_\Omega \frac{v(x)}{2} \left(|Du(x) - Dd(x)|^2\right)\, dx = \int_\Omega f(x)(u(x) - d(x))\, dx\; = 0\,.$$

It follows that both integrands are zero almost everywhere. Since u, d are continuous and $d - u \geq 0$, the vanishing of the first integral implies that $vDu = vDv$ almost everywhere while the vanishing of the second one is equivalent to $u \equiv d$ in $\mathrm{spt}(f)$. □

Corollary 4.11 *If (u, v) is a solution of system* (16)–(17)*, then (d, v) is a solution of the same system.*

Proof — By the previous lemma, the pair (d, v) satisfies point 2 of Definition 4.1. Points 1 and 3 are immediate consequences of the properties of d. □

Observe that, if d, v were smooth functions, then we could integrate equation (16) and apply the divergence theorem to obtain

$$\begin{aligned}\int_\Omega f(x)dx \;&=\; -\int_\Omega \mathrm{div}\,(v(x)Dd(x))dx = \int_{\partial\Omega} v(y)\langle Dd(y), Dd(y)\rangle d\mathcal{H}^{n-1}(y)\\ &=\; \int_{\partial\Omega} v(y)d\mathcal{H}^{n-1}(y),\end{aligned}$$

since Dd coincides with the inner normal on $\partial\Omega$. The next proposition contains a weak formulation of the above equality.

Proposition 4.12 *Let us set $\Omega_\varepsilon = \{x \in \Omega \;:\; d(x) \leq \varepsilon\}$. Then,*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} v(x)dx = \int_\Omega f(x)dx\,.$$

Proof — For any $\varepsilon > 0$, let us set $\phi_\varepsilon(x) = \min\{1, \varepsilon^{-1}d(x)\}$. Then

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} v(x)dx \;&=\; \lim_{\varepsilon\to 0^+}\int_\Omega v(x)\langle D\phi_\varepsilon(x), Dd(x)\rangle dx\\ &=\; \lim_{\varepsilon\to 0^+}\int_\Omega f(x)\phi_\varepsilon(x)dx = \int_\Omega f(x)\end{aligned}$$

as required. □

Theorem 4.13 *If (u, v) is a solution of system* (16)–(17)*, then $v = v_f$ a.e..*

Proof — It is convenient to change coordinates. Let us consider a parametrization of a portion of boundary of Ω, given by $\Phi : A \to \partial\Omega$, with $A \subset \mathbb{R}^{n-1}$. Then the map $(z, t) \longrightarrow \Phi(z) + t\nu(z)$ (where $\nu(z)$ is the inner normal) is a diffeomorphism for $(z, t) \in \tilde{A}$, where

$$\tilde{A} = \{(z, t) \;:\; z \in A, t \in (0, \tau(z))\}.$$

Given a function h defined on Ω, let us denote by $\tilde{h}$ the corresponding function on $\tilde{A}$ defined by $\tilde{h}(z,t) = h(\Phi(z) + t\nu(z))$. If h is differentiable, then we have that

$$\frac{\partial \tilde{h}}{\partial t}(t,z) = \langle Dh(x), Dd(x)\rangle|_{x=\Phi(z)+t\nu(z)} \,.$$

In addition, the volume element changes according to $dx = \prod_{i=1}^{n-1}(1-\kappa_i(z)t)m(z)\,dz\,dt$, where $\kappa_i(z) = \kappa_i(\Phi(z))$ and $m(z) = J\Phi(z)$ is the jacobian of Φ defined as in [8, Section 3.2.2]. Since, by Corollary 4.11, the pair (v,d) solves our system, $\tilde{v}$ satisfies

$$\int_{\tilde{A}} \tilde{v}(z,t)\frac{\partial \psi}{\partial t}(z,t)\prod_{i=1}^{n-1}(1-\kappa_i(z)t)m(z)\,dz\,dt = \int_{\tilde{A}} \tilde{f}(z,t)\psi(z,t)\prod_{i=1}^{n-1}(1-\kappa_i(z)t)m(z)\,dz\,dt.$$

for any $\psi \in W_0^{1,\infty}(\tilde{A})$; indeed, any such ψ can be seen as $\psi = \tilde{\phi}$ for some $\phi \in W_0^{1,\infty}(\Omega)$. Since (v_f, d) is also a solution, the same relation is satisfied by the function $\tilde{v}_f$. Therefore, taking $w(z,t) = \tilde{v}(z,t) - \tilde{v}_f(z,t)$, we have

$$\int_{\tilde{A}} w(z,t)\frac{\partial \psi}{\partial t}(z,t)\prod_{i=1}^{n-1}(1-t\kappa_i(z))m(z)\,dz\,dt = 0$$

for any $\psi \in W_0^{1,\infty}(\tilde{A})$. From this it is easy to deduce that $w(z,t)\prod_{i=1}^{n-1}(1-t\kappa_i(z)) = \bar{w}(z)$ a.e. in $\tilde{A}$ for a suitable function $\bar{w}$ of z only. Since the argument can be repeated on any part of $\partial\Omega$, we conclude that there exists a function $W \in L^1(\partial\Omega)$ such that

$$v(y+t\nu(y)) = v_f(y+t\nu(y)) + W(y)\prod_{i=1}^{n-1}(1-t\kappa_i(y))^{-1}, \qquad y\in\partial\Omega,\ t\in[0,\tau(y)) \text{ a.e. } .$$

We need to show that $W = 0$ a.e.. First we show it is nonnegative. In fact, we have

$$W(y) = [v(y+t\nu(y)) - v_f(y+t\nu(y))]\prod_{i=1}^{n-1}(1-t\kappa_i(y)), \qquad y\in\partial\Omega,\ t\in[0,\tau(y)) \text{ a.e. } .$$

Thus, letting $t \to \tau(y)$ and using Proposition 4.6, we obtain that

$$W(y) = \lim_{t\to\tau(y)} v(y+t\nu(y))\prod_{i=1}^{n-1}(1-t\kappa_i(y))\,,$$

which is nonnegative a.e. since both factors are nonnegative. Next we observe that, by Proposition 4.12,

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}[v(x)-v_f(x)]dx = 0\,.$$

On the other hand

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}[v(x)-v_f(x)]dx \\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\partial\Omega}\int_0^\varepsilon [v(y+tDd(y)) - v_f(y+tDd(y))]\prod_{i=1}^{n-1}(1-t\kappa_i(y))\,dt d\mathcal{H}^{n-1}(y) \\
&= \int_{\partial\Omega} W(y)\,d\mathcal{H}^{n-1}(y).
\end{aligned}$$

Since the integrand is nonnegative, we have a contradiction unless $W(y) = 0$ a.e. $\square$

Corollary 4.14 *A pair of functions* $(u, v) \in W_0^{1,\infty} \times L^1(\Omega)$ *is a solution of* (16)–(17) *if and only if*

(i) $v = v_f$ *a.e. in* Ω*;*

(ii) $\|Du\|_\infty \leq 1$ *and* $u_f \leq u \leq d$ *in* Ω*, where* u_f *is given by* (11)*.*

In addition, the solution of system (16)–(17) *is unique if and only if* $\overline{\Sigma} \subset \mathrm{spt} f$*.*

Proof — Suppose that the pair (u, v) is a solution. Then $v = v_f$ a.e. in Ω by Theorem 4.13. In addition, $u = d$ on $\mathrm{spt}(f)$ by Lemma 4.10. By definition, u is nonnegative, vanishes on $\partial\Omega$ and has Lipschitz constant at most one. Then $u \geq u_f$ by Proposition 2.2-(i) and $u \leq d$ by the maximality of d. This proves (ii).

Conversely, suppose that (u, v) satisfy (i)–(ii). Then, since $u = d$ on $\mathrm{spt}(v) = \mathrm{spt}(v_f)$, and since (d, v_f) is a solution, we easily verify that (u, v) is also a solution.

We conclude that the solution to the system is unique if and only if $u_f \equiv d$ everywhere in Ω, and this is equivalent to $\overline{\Sigma} \subset \mathrm{spt}(f)$ by Proposition 2.2(ii). $\square$

Proof of Theorem 4.9 — The only part of the statement which is not included in the previous corollary is the property that $u_f = d$ on $\mathrm{spt}(v_f)$. To prove this, let us take any $x \in \mathrm{spt}(v_f)$. Then Lemma 4.8 implies that $x + tDd(x) \in \mathrm{spt}(f)$ for some $t \in [0, \tau(x)]$. But then

$$u_f(x) \geq u_f(x + tDd(x)) - |tDd(x)| = d(x + tDd(x)) - t = d(x) + t - t = d(x).$$

Since the reverse inequality holds for any x, we obtain that $u_f(x) = d(x)$ on $\mathrm{spt}(v_f)$. By party (ii) of the previous corollary, we conclude that $u = d$ on $\mathrm{spt}(v_f)$. $\square$

Remark 4.15 A result related to Corollary 4.14 has been recently obtained by Crasta and Finzi Vita in [6]. The authors consider a stationary problem with an integrable source in the presence of walls on some parts of the boundary, obtaining existence of solutions in agreement with our corollary. However, the uniqueness of v is left as an open problem in [6]. It is likely that the ideas of our paper can be applied to prove the uniqueness of v for the problem with walls as well.

4.3 Regularity

In this last part of our paper, we shall investigate the regularity properties of the mapping which associates to a function $f \in L^\infty(\Omega)$ the solution (u, v_f) of (16). Since we can always choose $u = d$, we only consider the second component $f \mapsto v_f$ of this mapping.

Proposition 4.16 *We have, for any* $p \in [1, +\infty]$*,*

$$\|v_{f_1} - v_{f_2}\|_p \leq C_p(\Omega) \|f_1 - f_2\|_p \qquad \forall f_1, f_2 \in L^\infty(\Omega)\,.$$

where

$$C_p(\Omega) = \mathrm{diam}\,(\Omega)\,(1 + \|[\kappa]_-\|_\infty \mathrm{diam}\,(\Omega))^{(n-1)(1-\frac{1}{p})}$$

with $[\kappa]_- = \max_{1 \leq i \leq n-1} \max\{0, -\kappa_i\}$*.*

Remark 4.17 1. If we choose $f_2 = 0$, then $v_{f_2} = 0$ and we have the following bounds on v_f:

$$\|v_f\|_p \le C_p(\Omega)\|f\|_p \qquad \forall f \in L^\infty(\Omega)\,.$$

2. If $p = 1$ or if Ω is convex, then the constant $C_p(\Omega)$ only depends on p, n and the diameter of Ω.

3. The above estimates still hold if $p > 1$ and $\partial\Omega$ is of class $\mathcal{C}^{1,1}$.

Proof — Let us compute $\|v_{f_1} - v_{f_2}\|_p^p$. We have

$$\begin{aligned}
&\|v_{f_1} - v_{f_2}\|_p^p \\
&= \int_\Omega \Big| \int_0^{\tau(x)} (f_1 - f_2)(x + tDd(x)) \prod_{i=1}^{n-1} \frac{1 - (d(x)+t)\kappa_i(x)}{1 - d(x)\kappa_i(x)} dt \Big|^p dx \\
&\le \int_\Omega (\tau(x))^{p-1} \int_0^{\tau(x)} |(f_1 - f_2)(x + tDd(x))|^p \left(\prod_{i=1}^{n-1} \frac{1 - (d(x)+t)\kappa_i(x)}{1 - d(x)\kappa_i(x)} \right)^p dtdx
\end{aligned}$$

thanks to Jensen's inequality. Taking $C = 1 + \|[\kappa]_-\|_\infty \|\tau\|_\infty$, we obtain, by (20),

$$\begin{aligned}
&\|v_{f_1} - v_{f_2}\|_p^p \\
&\le \int_\Omega (\tau(x))^{p-1} C^{(n-1)(p-1)} \int_0^{\tau(x)} |(f_1 - f_2)(x + tDd(x))|^p \prod_{i=1}^{n-1} \frac{1 - (d(x)+t)\kappa_i(x)}{1 - d(x)\kappa_i(x)} dtdx \\
&\le C^{(n-1)(p-1)} (\|\tau\|_\infty)^{p-1} \int_\Omega \int_0^{\tau(x)} |(f_1 - f_2)(x + tDd(x))|^p \prod_{i=1}^{n-1} \frac{1 - (d(x)+t)\kappa_i(x)}{1 - d(x)\kappa_i(x)} dtdx\,.
\end{aligned}$$

Hence, by Lemma 4.3, we have

$$\begin{aligned}
&\|v_{f_1} - v_{f_2}\|_p^p \\
&\le C^{(n-1)(p-1)} (\|\tau\|_\infty)^{p-1} \int_\Omega d(x) |(f_1 - f_2)(x)|^p dx \\
&\le C^{(n-1)(p-1)} (\|\tau\|_\infty)^{p-1} \operatorname{diam}(\Omega) \int_\Omega |(f_1 - f_2)(x)|^p dx\,.
\end{aligned}$$

We can then complete the proof noting that $\tau(x) \le \operatorname{diam}(\Omega)$ for any x. □

Let us underline that, in Proposition 4.16, $C_p(\Omega)$ strongly depends on the curvature of the set Ω. However, we can get rid of this dependence introducing a weight in the L^p norm.

Proposition 4.18 *For any $p \in [1, +\infty]$, we have*

$$\|d^{(n-1)(1-\frac{1}{p})}(v_{f_1} - v_{f_2})\|_p \le C'_p(\Omega)\|f_1 - f_2\|_p \qquad \forall f_1, f_2 \in L^\infty(\Omega)\,,$$

where

$$C'_p(\Omega) = (\operatorname{diam}(\Omega))^{(np-n+1)/p}\,.$$

Proof — One can argue as in the proof of Proposition 4.16, replacing estimate (20) by the following one: for all $x \in \Omega \backslash \overline{\Sigma}$ and $t \in [0, \tau(x))$ we have

$$\frac{1-(d(x)+t)\kappa_i(x)}{1-d(x)\kappa_i(x)} \leq 1 + \frac{\tau(x)}{d(x)} \qquad \forall 1 \leq i \leq n-1\,.$$

We omit the easy details. □

Remark 4.19 1. In particular, the above proposition implies that the map $f \mapsto v_f$ can be defined on any bounded domain Ω and that, for any $p \in (1, +\infty]$, v_f belongs to $L^p_{loc}(\Omega)$ if f belongs to $L^p(\Omega)$.

2. In general, if Ω is not smooth, one cannot expect v_f to be in $L^p(\Omega)$ for f in $L^p(\Omega)$, unless Ω is convex or $p = 1$. For instance, in the case of $n = 2$, $\Omega = B_2(0) \backslash \{0\}$ and $f = 1$, we have that $v_f(x) = (1 - |x|^2)/(2|x|)$ in $B_1(0) \backslash \{0\}$, which is unbounded although $f \in L^\infty(\Omega)$. Notice, however that the map $x \mapsto d(x)v_f(x)$ is bounded.

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