G-CONVERGENCE OF MIXED TYPE EVOLUTION OPERATORS

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Abstract. We give a compactness result with respect to G-convergence for sequences of mixed evolution (elliptic-parabolic) equations $P_h u = \partial_t(\mu_h u) - \mathrm{div}(a_h(x,t,Du)) = f$, μ_h positive, null and negative. We show that the limit operator is of the form $Pu = \partial_t(\mu u) - \mathrm{div}(a(x,t,Du))$ and that μ and a are independent of each other. Under some time regularity we show that this convergence is equivalent to the pointwise (in time) elliptic G-convergence.

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1. Introduction

The asymptotic behaviour as $h \to \infty$, from the point of view of G-convergence, of equations like

$$P_h u = f\,, \quad u \in X_h \tag{1}$$

(X_h suitable space) has been widely studied when P_h is an elliptic or a parabolic operator in divergence form of the type

$$P_h = -\mathrm{div}\,(a_h(x,D)) \qquad \text{or} \qquad P_h = \frac{\partial}{\partial t} - \mathrm{div}\,(a_h(x,t,D))$$

where $x \in \Omega$ and $t \in (0,T)$ with Ω open set of $\mathbf{R}^n$ and $T > 0$. G-convergence was introduced for operator P_h in the linear case by Spagnolo: we recall here only [32], [33] among the first papers about linear and self-adjoint operators, [12] in which for the first time an expression of the limit operator is given (in the case of homogenization), [29] in which operators as $P_h u + b_h \cdot Du + c_h u$ are considered, [35] in which an extension for the non-symmetric and non-linear case is given, [7] in which a comparison between linear elliptic and parabolic G-convergence is made, [39] in which linear operator also of degree greater than two are considered. Besides, as regards the non-linear case, we recall [5] and [34].
G-convergence is designed to express the convergence of the solutions of partial differential equations and, sometimes, of minimum problems when the equations represent the Euler-Lagrange equations of a family of functionals. The interest arises just in this property which is not directly connected with the convergence of the coefficients of the differential operators. A particular case of interest is *homogenization*, the case when one considers fast-oscillating inhomogeneities which in the limit (with respect to G-convergece) often disappear producing an apparent *homogeneous* behaviour.

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Here we consider operators like P_h with a_h satisfying ($x \in \Omega \subset \mathbf{R}^n$, $t \in [0,T]$, $n \geqslant 2$, $p \geqslant 2$)

$$
\begin{aligned}
&\lambda_h(x,t)|\xi-\eta|^p \leqslant (a_h(x,t,\xi)-a_h(x,t,\eta),\xi-\eta) \\
&|a_h(x,t,\xi)-a_h(x,t,\eta)| \leqslant L\,\lambda_h(x,t)\big(1+|\xi|^p+|\eta|^p\big)^{(p-2)/p}|\xi-\eta|
\end{aligned}
\tag{2}
$$

for every $\xi,\eta \in \mathbf{R}^n$, where λ_h are weights, i.e. suitable almost everywhere positive functions, and $L > 1$. The classical case, considered, for instance, in [32], [33], [12], [7] with $p = 2$ and a_h linear in the third variable, corrisponds to $\lambda_h \equiv \lambda_0$ with $\lambda_0 > 0$ (and $X_h = X$ for every $h \in \mathbf{N}$).
In the degenerate setting, i.e. when λ_h are weights, an approach via Γ-convergence (used to study limit behaviour for functionals; we refer to [8] for it) for linear functionals is consider in [18] with $\lambda_h = \lambda$ for every $h \in \mathbf{N}$, λ weight. In [11] a more general situation is considered: in the elliptic case a sequence of non-linear equations is considered, with $\lambda_h = \lambda_h(x)$ satisfying a uniform Muckenhoupt's condition A_p (see Section 2) and a_h verifying a more general condition than (2). For the parabolic case we recall [27] in which the linear case is studied with $\lambda_h = \lambda_h(x)$ satisfying a uniform Muckenhoupt's condition.
About the degenerate situation we want also to recall [37] in which a simple situation is considered in the elliptic case, namely $P_h u = -\mathrm{div}\,(\lambda_h(x)Du)$, λ_h weights, in which it is shown that knowing the limit behaviour of λ_h is not sufficient to characterize the limit problem.

In the present paper we consider operators like

$$
P_h u = \partial_t(\mu_h u) - \mathrm{div}\big(a_h(x,t,Du)\big) \tag{3}
$$

with a_h satisfying (2), λ_h and μ_h suitable functions, and give a compactness result with respect to G-convergence for P_h (see (61) for the precise problems with suitable boundary and initial/final conditions). By this we extend some known result regarding the linear case to the monotone framework, with degeneration depending also on time (see below and Section 8). But the main originality of this paper is that we consider μ_h which may be positive, null and negative, i.e. the equations may be forward parabolic, elliptic, backward parabolic or all these types together. In this situation the greatest problem for our purposes is the lack of a natural compactness result which we need for the sequence of the solutions (for this we refer to [24], Theorem 2.18 and Example 2.19). To bypass this difficulty we first study the problem in the situation

$$
L^1\text{-weak}\lim_{h\to\infty} \mu_h = \mu \neq 0 \text{ a.e. in } \Omega \times (0,T)\,,
$$

i.e. μ invertible, in which we are able to give a compactness result for the solutions (see Theorem 3.7). Then we give the compactness theorem, one of the main results (stated in Theorem 6.9), using an approximation argument (see Theorem 3.8 and Lemma 3.13) and show that the operators P_h converge, up to a subsequence, to an operator P_μ defined as

$$
P_\mu u = \partial_t(\mu u) - \mathrm{div}\,\big(a_\mu(x,t,Du)\big) \tag{4}
$$

for suitable μ and a_μ.
The main thing we want to show is that if we look the operators defined in (3) as evolution operators whatever is μ_h, i.e. also when $\mu_h \equiv 0$ for every $h \in \mathbf{N}$, the limit behaviour is in some sense independent of the coefficients μ_h. The

meaning is stated in Lemma 6.3 which shows that if $\partial_t(\mu_h\cdot) - \text{div }(a_h(x,t,D\cdot)$ G-converges to $\partial_t(\mu\cdot) - \text{div }(a_\mu(x,t,D\cdot)$ and $\partial_t(\nu_h\cdot) - \text{div }(a_h(x,t,D\cdot)$ G-converges to $\partial_t(\nu\cdot) - \text{div }(a_\nu(x,t,D\cdot)$ then $a_\mu = a_\nu$.

We recall that this problem is partially studied in simpler situations: in [1] (see also [2]) in the case of homogenization with where a_h and μ_h are bounded, linear, periodic and independent of time. In this situation the periodicity simplifies the lack of compactess and the general case is not solved neither in the linear case.
We give a partial solution in [26] with $\mu_h \geqslant 0$ and a_h linear. In that case the lack of compactness is bypassed using a regularity result (with respect to time) for the solutions (see Theorem 3.11 in [24]), but we are able to prove it only with $\mu_h \geqslant 0$ and $p = 2$.

We also recall that the problem we treat, as far as the periodic case is concerned, is mentioned in [21], Appendix B.

The case in which we consider time-pointwise G-convergence is different. In Section 7 we show that under some regularity assumption (which could be refined) on time of $(a_h)_h$ G-convergence for the evolution equations (PG-convergence defined in Section 6) is equivalent to time-pointwise G-convergence, but this in general does not hold (we refer to an example in [7] for this).
In the last section we consider some particular cases (Theorem 8.1 extending the result contained in [27] and Theorem 8.2 extending the result contained in [23]). In particular we get that the class $A_{1+p/n}$ is sharp among the Muckenhoup classes to have compactness with respect to G-convergence for operators $P_h = \partial_t - \text{div }(a_h(x,t,D\cdot)$, i.e. $\mu_h \equiv 1$ for every h.

Mixed equations arise in many fields. As regards the case $\mu_h \geqslant 0$ one can think, for instance, to heterogeneous material in which an evolution equation may *degenerate* to an elliptic one in some regions where the diffusion is very quick. The coefficients μ_h may represent, for instance, the electrical conductivity. An interesting example which lead to elliptic-parabolic equations is the study of Maxwell's equations, in dimension two, for quasistationary electromagnetic fields (for this example we refer to [36]). Here we briefly refer to [3] for other examples with $\mu_h \geqslant 0$.
As regards the more general case, i.e. μ_h which may be also negative, equations of the type (3), with $\mu_h = \mu_h(x)$, may arise from kinetic theory (the stationary form of the Fokker Plank equation, see, e.g., [19] and [4]) and stochastic processes (see, e.g., [14], [20] and [15] and the references therein).

The paper is organized as follows. In Section 2 we recall some results about Muckenhoupt weights and show some results with weights depending on time. Section 3 is devoted to give some results for mixed equations we will need later (some of them are generalizations of results contained in [24] needed since we are dealing with weights depending on time). In Section 4 we set the problem we want to study and fix assumptions and in Section 5 we give some compactness results needed later. Finally in Section 6 we give the main results and in Section 7 we make a comparison with the standard elliptic G-convergence. In the last section we give some examples and particular compactness results.

2. Review of Muckenhoupt weights

With the word *weight* we will mean a function η such that

$$\eta \text{ weight if:} \quad \eta \geqslant 0 \ \text{ a. e. in } \mathbf{R}^n \quad \text{and} \quad \eta \in L^1_{\rm loc}(\mathbf{R}^n).$$

From now on ν and λ will be functions such that (p will be a constant greater than 1 and by p' we will denote the number $p/(p-1)$)

$$\nu, \lambda, \lambda^{-1/(p-1)} \qquad \text{weights}. \tag{5}$$

The following definition is usually given for pairs (ν, λ) with $\nu, \lambda > 0$ a. e.: we extend this to pairs of *weights*.

Definition 2.1. *Let $p, q > 1$, $K > 0$ be constants, ν, λ two weights, $\lambda > 0$ a. e., $\alpha \in [0, n)$ and Q_0 a cube in $\mathbf{R}^n$ (even the whole $\mathbf{R}^n$ if $\alpha = 0$). We say that the pair (ν, λ) belongs to the class $A^\alpha_{p,q}(Q_0, K)$ if*

$$|Q|^{\frac{\alpha}{n}} \left(\fint_Q \nu dx \right)^{1/q} \left(\fint_Q \lambda^{-1/(p-1)} dx \right)^{(p-1)/p} \leqslant K \quad \text{for every cube } Q \subset Q_0. \tag{6}$$

We denote by $A^\alpha_{p,q}(Q_0)$ the class $\bigcup_{K>0} A^\alpha_{p,q}(Q_0, K)$. Moreover we denote by $A_{p,q}(Q_0, K)$ the class $A^\alpha_{p,q}(Q_0, K)$ when $\alpha = 0$ and by $A_{p,q}(K)$ the same class when $Q_0 = \mathbf{R}^n$. Finally we denote by $A_p(Q_0, K)$ the Muckenhoupt class of weights for which (6) *holds with $\alpha = 0$, $p = q$ and $\nu \equiv \lambda$, by $A_p(K)$ the class $A_p(Q_0, K)$ with $Q_0 = \mathbf{R}^n$, by $A_p = \bigcup_{K \geqslant 1} A_p(K)$ and $A_\infty = \bigcup_{p>1} A_p$.*

Remark 2.2. - If $\lambda \in A_p(Q_0, K)$ there is $\lambda' \in A_p(K)$ such that $\lambda' \equiv \lambda$ in Q_0.

Remark 2.3. - From the definition it follows that $\lambda \in A_p(K)$ if and only if $\lambda^{1-p'} = \lambda^{-1/(p-1)} \in A_{p'}(K^{p'-1})$.

Remark 2.4. - By definition we have that $A^\alpha_{p,q}(Q_0, K) \subset A^\beta_{p,q}(Q_0, |Q_0|^{(\beta-\alpha)/n} K)$ if $0 \leqslant \alpha < \beta < n$; $A_p(K) \subset A_q(K)$ if $1 < p < q < \infty$.

Remark 2.5. - Consider $0 < s < r < +\infty$, $1 < q \leqslant p < +\infty$, $\alpha \in [0, n/r)$. Then if $(\nu^r, \lambda^r) \in A^{\alpha r}_{p,q}(Q_0, K)$ then $(\nu^s, \lambda^s) \in A^{\alpha s}_{p,q}(Q_0, K')$ ($K' = K^{s/r} |Q_0|^{\frac{r-s}{r}(\frac{1}{q} - \frac{1}{p})}$).

A_p weights verify the following higher sommability property (see [6, 9]): for every $K \geqslant 1$, for every $p > 1$, there exist two positive constants $c = c(n, p, K)$ and $\delta = \delta(n, p, K)$ (depending only on n, p and K) such that

$$\left(\fint_Q \lambda^{1+\delta} dx \right)^{\frac{1}{(1+\delta)}} \leqslant c \left(\fint_Q \lambda dx \right), \quad \left(\fint_Q \lambda^{-\frac{1+\delta}{p-1}} dx \right)^{\frac{1}{(1+\delta)}} \leqslant c \left(\fint_Q \lambda^{-\frac{1}{p-1}} dx \right), \tag{7}$$

for every cube Q and $\lambda \in A_p(K)$. A_p weights also verify the *doubling property*, i. e. if $\lambda \in A_p(K)$ one has that for every $t > 0$ there exists a constant $c = c(t, n, p, K)$ (depending only on t, n, p, K) such that

$$\lambda(tQ) \leqslant c \lambda(Q) \tag{8}$$

for every cube Q of $\mathbf{R}^n$ (see for instance [16]). The condition (8) implies the *reverse doubling* property, which we will call $(\delta_1, \delta_2)-$*reverse doubling*, defined as:

(9)

$$\text{there exist } \delta_1, \delta_2 \in (0, 1) \text{ such that} \quad \lambda(\delta_1 Q) \leqslant \delta_2 \lambda(Q) \qquad \text{for every cube } Q \text{ of } \mathbf{R}^n.$$

The following result, for positive weights, is contained in [28] (for ν non-negative it is sufficient to consider the positive weight

$$\nu_\lambda = \begin{cases} \nu & \text{in } \{x \in \mathbf{R}^n \mid \nu > 0\} \\ \lambda & \text{in } \{x \in \mathbf{R}^n \mid \nu = 0\}\,; \end{cases} \tag{10}$$

see Remark 3.8 in [25]).

Theorem 2.6. *Suppose Q_0 is a proper cube of $\mathbf{R}^n$, $1 < p < +\infty$, and u is a Lipschitz continuous function defined on Q_0, with either support in Q_0 or $\int_{Q_0} f(x)dx = 0$. Consider $\nu \geqslant 0$, $\lambda > 0$ a. e., weights on $\mathbf{R}^n$. Then*

$$\Big[\int_{Q_0} |u(x)|^p \nu(x)dx\Big]^{1/p} \leqslant \mathcal{C}(\nu,\lambda,Q_0)\Big[\int_{Q_0} |Du(x)|^p \lambda(x)dx\Big]^{1/p}$$

where

$$\mathcal{C} = \mathcal{C}(\nu,\lambda,Q_0) = c(p,r) \sup_{Q\subset 8Q_0} |Q|^{1/n}\Big(\fint_Q \nu^r\Big)^{1/pr}\Big(\fint_Q \lambda^{-\frac{r}{p-1}}\Big)^{(p-1)/pr}$$

for any $r > 1$, or

$$\mathcal{C} = \mathcal{C}(\nu,\lambda,Q_0) = c(p) \sup_{Q\subset 8Q_0} |Q|^{1/n-1}\Big(\int_Q \nu\Big)^{1/p}\Big(\int_Q \lambda^{-\frac{1}{p-1}}\Big)^{(p-1)/p}$$

if both ν_λ and $\lambda^{-1/(p-1)}$ satisfy (9).

Remark 2.7. - Observe that $\lambda \in A_p(K)$ satisfies Theorem 2.6 (see Remark 2.5).

Now, considered Ω an open subset of $\mathbf{R}^n$ and $p > 1$, define the space

$$W^{1,p}(\Omega,\nu,\lambda) = \Big\{u \in W^{1,1}_{\text{loc}}(\Omega) \;\Big|\; u \in L^p(\Omega,\nu) \text{ and } D_i u \in L^p(\Omega,\lambda)\,, i = 1, ...n\Big\}.$$

With hypotheses (5) the space $W^{1,p}(\Omega,\nu,\lambda)$ endowed with the topology induced by the norm

$$\|u\|_{W^{1,p}(\Omega,\nu,\lambda)} := \Big(\int_\Omega (|u|^p\nu + |Du|^p\lambda)dx\Big)^{1/p} \tag{11}$$

is a separable Banach space and $C^1_c(\Omega) \subset W^{1,p}(\Omega,\nu,\lambda)$ (see Theorem 2.3 in [25]). We will denote by $W^{1,p}_0(\Omega,\nu,\lambda)$ the closure of $C^1_c(\Omega)$ in the topology of $W^{1,p}(\Omega,\nu,\lambda)$ and by $W^{-1,p'}(\Omega,\nu,\lambda)$ its dual space.
Under assumptions of Theorem 2.6, if Ω is a bounded open set, the following weighted Poincaré type inequality holds

$$\int_\Omega |u|^p\nu dx \leqslant c\int_\Omega |Du|^p\lambda dx \qquad \text{for every } u \in W^{1,p}_0(\Omega,\nu,\lambda) \tag{12}$$

for a suitable positive constant c and then we endow $W^{1,p}_0(\Omega,\nu,\lambda)$ by the norm

$$\|u\|_{W^{1,p}_0(\Omega,\nu,\lambda)} := \Big(\int_\Omega |Du|^p\lambda dx\Big)^{1/p}. \tag{13}$$

Moreover, by Theorem 2.6 (with $\nu = \lambda$) and Theorem 1.4 in [30], if Ω is a bounded open set with Lipschitz boundary and $\lambda \in A_p$, we have that for every ν such that the pair (ν,λ) satisfies hypotheses of Theorem 2.6 we have that

$$W^{1,p}_0(\Omega,\nu,\lambda) = W^{1,p}(\Omega,\lambda,\lambda) \cap W^{1,1}_0(\Omega)\,. \tag{14}$$

Suppose now to consider

$$\lambda \in A_p(K_1) \quad \text{and} \quad (\nu,\lambda) \text{ satisfying Theorem 2.6}\,.$$

By (7), (8) and Remark 2.3 if Ω is a bounded open set there exist $\sigma = \sigma(n,p,K_1) > 0$ (depending only on n,p,K_1) and $c_i = c_i(n,p,K_1,\Omega) > 0$ (depending only on n,p,K_1,Ω), $i = 1,2$, such that

(15)

$$c_1\, \|\lambda^{-1/(p-1)}\|_{L^1(\Omega)}^{-(p-1)/p} \|w\|_{L^{1+\sigma}(\Omega)} \leqslant \|w\|_{L^p(\Omega,\lambda)} \leqslant c_2\, \|\lambda\|_{L^1(\Omega)}^{1/p} \|w\|_{L^{(1+\sigma)'}(\Omega)}$$
$$c_1\, \|\lambda\|_{L^1(\Omega)}^{-1/p} \|w\|_{L^{1+\sigma}(\Omega)} \leqslant \|w\|_{L^{p'}(\Omega,\lambda^{-1/(p-1)})} \leqslant c_2\, \|\lambda^{-1/(p-1)}\|_{L^1(\Omega)}^{(p-1)/p} \|w\|_{L^{(1+\sigma)'}(\Omega)}$$

for every $w \in L^{(1+\sigma)'}(\Omega)$. Applying (15) to $w = |Du|$, using also Theorem 2.6 with $\nu = \lambda$ where Q_0 is a cube containing Ω, there exist $\sigma = \sigma(n,p,K_1) > 0$ (depending only on n,p,K_1) and $c_i = c_i(n,p,K_1,\mathcal{C},\Omega) > 0$ (depending only on $n,p,K_1,\mathcal{C},\Omega$), $\mathcal{C}$ being the constant appearing in Theorem 2.6, $i = 1,2$, such that

(16)

$$c_1\, \|\lambda^{-1/(p-1)}\|_{L^1(\Omega)}^{-(p-1)/p} \||Du|\|_{L^{1+\sigma}(\Omega)} \leqslant \||Du|\|_{L^p(\Omega,\lambda)} \leqslant c_2\, \|\lambda\|_{L^1(\Omega)}^{1/p} \||Du|\|_{L^{(1+\sigma)'}(\Omega)}$$

for every $u \in W^{1,p}_0(\Omega,\nu,\lambda)$, so that

$$W^{1,(1+\sigma)'}_0(\Omega) \subset W^{1,p}_0(\Omega,\nu,\lambda) \subset W^{1,1+\sigma}_0(\Omega), \qquad W^{-1,p'}(\Omega,\nu,\lambda) \subset W^{-1,1+\sigma}(\Omega)\,.$$

Moreover, since $W^{1,p}_0(\Omega,\nu,\lambda)$ continuously embeds in $W^{1,1}(\Omega)$ which continuously embeds in $L^{n/(n-1)}(\Omega)$ if $n \geqslant 2$ (and in $L^2(\Omega)$ if $n = 1$), there exists a positive constant $c_3 = c_3(n,p,K_1,\mathcal{C},\Omega)$ (depending only on $n,p,K_1,\mathcal{C},\Omega$) such that

$$\|f\|_{W^{-1,p'}(\Omega,\nu,\lambda)} \leqslant c_3 \left(\int_\Omega \lambda^{-1/(p-1)} dx\right)^{(p-1)/p} \|f\|_{L^n(\Omega)}. \tag{17}$$

Weights depending on time - Suppose now to consider weights $\nu(x,t), \lambda(x,t)$ depending also on time. Consider the space

$$\mathcal{X} = \{u \in L^p(\Omega\times(0,T);\nu) \mid D_i u \in L^p(\Omega\times(0,T);\lambda)\,, i = 1,...n\}$$

($D_i u$, $i = 1,...n$, denote the first n derivatives, i. e. the spatial derivatives) endowed with the norm

$$\|u\|^p_{\mathcal{X}} = \int_0^T \int_\Omega |u|^p \nu\, dxdt + \int_0^T \int_\Omega |Du|^p \lambda\, dxdt\,.$$

Define, analogously to (10), the positive weight (one can consider ν defined not only in $Q_0 \times (0,T)$ but in $\mathbf{R}^n \times (0,T)$)

$$\nu_\lambda = \begin{cases} \nu & \text{in } \{(x,t) \in \mathbf{R}^n \times (0,T) \,|\, \nu > 0\} \\ \lambda & \text{in } \{(x,t) \in \mathbf{R}^n \times (0,T) \,|\, \nu = 0\}\,. \end{cases} \tag{18}$$

Suppose that the pair (ν,λ) satifies

$$\nu, \lambda, \lambda^{-1/(p-1)} \in L^\infty(0,T;L^1(\Omega))\,. \tag{19}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \sup_{Q\subset\mathbf{R}^n} \left(⨍_Q \lambda(x,t)dx\right)^{1/p} \left(⨍_Q \lambda^{-1/(p-1)}(x,t)dx\right)^{(p-1)/p} = K_1 \tag{20}$$

and one of the following (see assumptions of Theorem 2.6)
(21)

a) $\displaystyle \operatorname*{ess\,sup}_{t\in(0,T)} \sup_{Q\subset Q_0} |Q|^{1/n}\left(\fint_Q \nu^r(x,t)dx\right)^{1/rp} \left(\fint_Q \lambda^{-r/(p-1)}(x,t)dx\right)^{(p-1)/rp} = K_2\,,$

b) $\displaystyle \operatorname*{ess\,sup}_{t\in(0,T)} \sup_{Q\subset Q_0} |Q|^{1/n}\left(\fint_Q \nu(x,t)dx\right)^{1/p} \left(\fint_Q \lambda^{-1/(p-1)}(x,t)dx\right)^{(p-1)/p} = K_2$

and $\nu_\lambda(\cdot,t)$ uniformly (in t) reverse doubling, i. e. satisfying (9).

for any $r > 1$.
Then on $C^1_c(\Omega\times(0,T))$, if (21)-a) or (21)-b) is satisfied, the following Poincaré inequality holds

$$\int_0^T\int_\Omega |u|^p\nu\,dxdt \leqslant c\int_0^T\int_\Omega |Du|^p\lambda\,dxdt\,. \tag{22}$$

We then define (observe that $C^1_c(\Omega\times(0,T))\subset\mathcal{X}$)

$$\mathcal{V}_\lambda := \text{completion of } C^1_c(\Omega\times(0,T)) \text{ w.r.t. the topology induced by } \|\cdot\|_{\mathcal{X}}\,. \tag{23}$$

endowed by the norm

$$\|u\|^p_{\mathcal{V}_\lambda} = \int_0^T\int_\Omega |Du|^p\lambda\,dxdt\,, \tag{24}$$

$\mathcal{V}'_\lambda$ its dual space with the dual norm,

$$\mathcal{H}_{\nu,\lambda} := L^2(\Omega\times(0,T),\nu_\lambda) \qquad \text{(denoted by } \mathcal{H}_\lambda \text{ if } \nu=\lambda)$$

endowed with the norm

$$\|u\|^2_{\mathcal{H}_{\nu,\lambda}} = \int_0^T\int_\Omega |u|^2\,\nu_\lambda\,dxdt\,.$$

Observe that

$$\mathcal{V}_\lambda\subset L^p(\Omega\times(0,T);\lambda) \quad\text{ and }\quad \mathcal{V}_\lambda\subset L^2(\Omega\times(0,T),\nu_\lambda)\,. \tag{25}$$

As a consequence of (19) and (20) (thanks to (7)) for a function $u\in L^p(\Omega\times(0,T);\lambda)$ we derive from (16) the following estimates ($c'_i = c'_i(n,p,K_1,K_2,\Omega)$, $i=1,2$)

$$\begin{aligned} &\|u\|_{L^p(0,T;L^{1+\sigma}(\Omega))} \leqslant c'_1\ \|\lambda^{-1/(p-1)}\|^{(p-1)/p}_{L^\infty(0,T;L^1(\Omega))}\ \|u\|_{L^p(\Omega\times(0,T),\lambda)} \\ &\|u\|_{L^p(\Omega\times(0,T),\lambda)} \leqslant c'_2\ \|\lambda\|^{1/p}_{L^\infty(0,T;L^1(\Omega))}\|u\|_{L^p(0,T;L^{(1+\sigma)'}(\Omega))}\end{aligned} \tag{26}$$

and consequently, assuming (21) and taking $u = |Dv|$ for $v\in\mathcal{V}_\lambda$, we have the following continuous embedding (the constants estimating the embedding operators being the constants appearing in (26))
(27)

$$L^p(0,T;W^{1,(1+\sigma)'}_0(\Omega))\subset\mathcal{V}_\lambda\subset L^p(0,T;W^{1,1+\sigma}_0(\Omega))\,,\quad \mathcal{V}'_\lambda\subset L^{p'}(0,T;W^{-1,1+\sigma}(\Omega))\,.$$

Moreover by (17) we also get

$$\|f\|_{\mathcal{V}'_\lambda} \leqslant c_3\|\lambda^{-1/(p-1)}\|_{L^\infty(0,T;L^1(\Omega))}\|f\|_{L^{p'}(0,T;L^n(\Omega))}\,. \tag{28}$$

Finally we present a slight generalization of a result contained in [22].
Fix ρ a radial non-negative function in $C^\infty(\mathbf{R}^n)$, supported in $B(0,1)$, such that

$\int_{\mathbf{R}^n} \rho = 1$, and and consider $(\rho_\delta)_{\delta>0}$ a family of mollifiers, i. e. $\rho_\delta(x) = \delta^{-n}\rho(x/\delta)$. We define

$$v_\delta(x,t) \overset{def}{=} \int_{\mathbf{R}^n} \rho_\delta(x-y)\overline{v}(y,t)\,dy \quad (x,t) \in \mathbf{R}^n \times [0,T]. \tag{29}$$

(where $\overline{v}(y,t) = v(y,t)$ if $y \in \Omega$, 0 otherwise) if $v \in \mathcal{V}_\lambda$.

Proposition 2.8. *Let Ω be a bounded open set of $\mathbf{R}^n$, Q_0 a cube containing Ω, $T > 0$, $K_1 \geqslant 1$. Then for every (ν, λ) satisfying*

(30)

$$K_2 = \operatorname*{ess\,sup}_{t\in(0,T)} \sup_{Q\subset 3Q_0} |Q|^{\alpha/n} \left(⨍_Q \nu(x,t)dx \right)^{1/p} \left(⨍_Q \lambda^{-1/(p-1)}(x,t)dx \right)^{(p-1)/p} < +\infty$$

with $\alpha \in [0,1)$ there exists a positive constant $c = c(n,\rho,p,\alpha,K_2)$ (depending only on n,ρ,α,K_2) such that:

$$\int_0^T \int_{Q_0} |v_\delta - v|^p \nu\,dx\,dt \leqslant c\ \delta^{\,p(1-\alpha)} \int_0^T \int_{Q_0} |Dv|^p \lambda\,dx\,dt$$

for every $\delta > 0$, for every $v \in \mathcal{V}_\lambda$.

Remark 2.9. - As a consequence of this proposition, by (26), we deduce that, for a fixed $v \in \mathcal{V}_\lambda$, we have that

$$v_\delta \to v \qquad \text{in} \quad L^p(0,T;L^{1+\sigma}(\Omega))$$

since as particular case one has $\nu \equiv \lambda$ (which satisfies (30) with $\alpha = 0$).

Remark 2.10. - Another particular case in the proposition above is $\nu \equiv 1$. In this last case $v_\delta \to v$ in $L^p(\Omega \times (0,T))$.

Remark 2.11. - Notice that if $(\nu^r, \lambda^r) \in A^{\alpha r}_{p,p}(K_2)$, $K_2 \in L^\infty(0,T)$, for some $\alpha \in [0,1)$ then by Remark 2.5 assumption (30) is satisfied.

Remark 2.12. - The following weak compactness result holds: if $(\lambda_h)_h$ is a sequence of weights satisfying

(31)

$$\begin{aligned} &\lambda_h, \lambda_h^{-1/(p-1)} \in C^0([0,T];L^1(\Omega))\,, \quad \text{equi-continuous in } L^1(\Omega)\,, \\ &\|\lambda_h\|_{L^\infty(0,T;L^1(\Omega))} \leqslant c\,, \|\lambda_h^{-1/(p-1)}\|_{L^\infty(0,T;L^1(\Omega))} \leqslant c \quad \text{for some constant } c\,, \\ &\lambda_h(t) \in A_p(K_1) \quad \text{for every } t \in [0,T]\,, \end{aligned}$$

then there are a subsequence (still indexed by h) and two comparable weights (we say ν_1, ν_2 *comparable* weights if there are two positive constants c_1, c_2 such that $c_1\nu_1 \leqslant \nu_2 \leqslant c_2\nu_1$) $\lambda, \tilde{\lambda} \in C^0([0,T];L^1(\Omega))$, $\lambda(t), \tilde{\lambda}(t) \in A_p(K_1)$ for every $t \in [0,T]$ such that, up to a subsequence,

$$\lambda_h(t) \to \lambda(t), \quad \lambda_h^{-\frac{1}{p-1}}(t) \to \tilde{\lambda}^{-\frac{1}{p-1}}(t) \quad \text{in } L^1(\Omega)\text{-weak} \quad \text{for every } t \in [0,T]$$

and moreover

$$\tilde{\lambda} \leqslant \lambda \leqslant K_1\tilde{\lambda}\,. \tag{32}$$

Indeed by (7) and (31) we have that $t \mapsto \int_\Omega g(x)\lambda_h(x,t)dx$ are equibounded and equicontinuous for every $g \in L^{(1+\delta)'}(\Omega)$ (the same holds for $\lambda_h^{-\frac{1}{p-1}}$). Since $L^{(1+\delta)'}(\Omega)$ is separable we conclude. Finally (32) can be obtained (using (8)) as in [9] or [27].

REMARK 2.13. - If two weights ν_1, ν_2 defined in a domain $D \subset \mathbf{R}^k$ are comparable weights (see the previous remark), then $L^p(D,\nu_1) = L^p(D,\nu_2)$.

3. EQUATIONS OF MIXED TYPE

In this section we want to present some generalizations to spaces depending on time of results contained in [24] (see also the references therein for similar results).

Consider the following family of evolution triplets

$$V(t) \subset H(t) \subset V(t)' \qquad t \in [0,T]$$

where $H(t)$ is a Hilbert space, $V(t)$ a reflexive Banach space which continuously and densely embeds in $H(t)$ and $V(t)'$ the dual space of $V(t)$.
Moreover we will suppose the existence of a set U such that

$$U \subset V(t) \quad \text{dense in } V(t) \text{ for a. e. } t \in [0,T] \tag{33}$$

In [24] a triplet $V \subset H \subset V'$ of spaces not depending on time is considered, so we do not give proofs since they are very similar to those contained in [24].
We define $\mathcal{U}$ the set of polynomials $v(t) = \sum_{k=0}^N u_k t^k$ with $u_k \in U$, $N \in \mathbf{N}$ and suppose that the functions

$$t \mapsto \|v(t)\|_{V(t)}\,, \quad t \mapsto \|v(t)\|_{H(t)}\,, \quad t \mapsto \|v(t)\|_{V(t)'}\,, \quad t \in [0,T]\,,$$

are measurable for every $v \in \mathcal{U}$. Consider $p \in [2,+\infty)$; we denote (improperly) the spaces

$$\mathcal{V} = L^p(0,T;V(t))\,, \qquad \mathcal{H} = L^2(0,T;H(t))\,, \qquad \mathcal{V}' = L^{p'}(0,T;V(t)')$$

($p' = p/(p-1)$) defining respectively by $\mathcal{V}$ and $\mathcal{H}$ the closure of $\mathcal{U}$ with respect to the following norms

$$\|v\|^p_{\mathcal{V}} := \int_0^T \|v(t)\|^p_{V(t)} dt \qquad \text{and} \qquad \|v\|^2_{\mathcal{H}} := \int_0^T \|v(t)\|^2_{H(t)} dt$$

and by $\mathcal{V}'$ the dual space of $\mathcal{V}$.

Suppose to have a family $R(t)$ of linear operators from $H(t)$ to $H(t)$ satisfying

$$\text{(34)} \qquad \begin{array}{ll} \text{(i)} & R : [0,T] \longrightarrow \mathcal{L}\big(H(t)\big)\,, \quad \text{for every } t \in [0,T]\,, \\ \text{(ii)} & R(t) \quad \text{self-adjoint} \\ \text{(iii)} & \sup_{t\in(0,T)} \|R(t)\|_{\mathcal{L}(H(t))} \leqslant C_1\,, \\ \text{(iv)} & t \mapsto \big(R(t)u,v\big)_{H(t)} \text{ absolutely continuos on } [0,T]\,, \\ \text{(v)} & \big|\dfrac{d}{dt}\big(R(t)u,v\big)_{H(t)}\big| \leqslant C_2\,\|u\|_{V(t)}\|v\|_{V(t)} \quad \text{for a. e. } t \in [0,T] \\ \text{(vi)} & \dfrac{d}{dt}\big(R(t)u,u\big)_{H(t)} \geqslant 0 \quad \text{for a. e. } t \in [0,T] \end{array}$$

for every $u, v \in U$. Assumptions (33) are needed to define the derivative of the operator $\mathcal{R}$. If U is a Banach space one can consider $C^1([0,T];U)$ instead of $\mathcal{U}$ and

simply $\mathcal{V}$ if $V(t) = V$ for every $t \in [0,T]$.
By (34) we can define an operator $\mathcal{R}$ by

$$\mathcal{R} : \mathcal{H} \longrightarrow \mathcal{H} \qquad \text{by} \qquad \mathcal{R}u(t) := R(t)u(t) \tag{35}$$

which turns out to be linear and bounded by the constant C_1.
Now define the Banach space

$$\mathcal{W}_{\mathcal{R}} = \big\{ u \in \mathcal{V} \; \big| \; (\mathcal{R}u)' \in \mathcal{V}' \big\} , \qquad \|u\|_{\mathcal{W}_{\mathcal{R}}} := \|u\|_{\mathcal{V}} + \|(\mathcal{R}u)'\|_{\mathcal{V}'} . \tag{36}$$

where $(\mathcal{R}u)'$ denotes the derivative of $\mathcal{R}u$ with respect to the variable t in the following distributional sense

$$\big\langle (\mathcal{R}u)', \phi \big\rangle_{\mathcal{V}' \times \mathcal{V}} = - \int_0^T \big(\mathcal{R}u(t), \phi'(t) \big)_{H(t)} dt \qquad \text{for every } \phi \in \mathcal{U} \text{ such that } R(0)\phi(0) = R(T)\phi(T) = 0 .$$

Observe that in (34)-(v) we require a control on the temporal derivative of $(R(t)u, v)_{H(t)}$ by the norm of u and v in the space $V(t)$ and not $H(t)$.
By (34) we can also define a family of equibounded operators

$$R' : [0,T] \to \mathcal{L}(V(t), V(t)') \qquad \text{by} \qquad \langle R'(t)u, v \rangle_{V(t)' \times V(t)} := \frac{d}{dt} \big(R(t)u, v \big)_{H(t)}$$

and, by density of $\mathcal{U}$ in $\mathcal{V}$, an operator

$$\mathcal{R}' : \mathcal{V} \to \mathcal{V}' \qquad \text{by} \qquad \big\langle \mathcal{R}'u, v \big\rangle_{\mathcal{V}' \times \mathcal{V}} := \int_0^T \langle R'(t)u(t), v(t) \rangle_{V(t)' \times V(t)} dt$$

which turns out to be linear and bounded by C_2.

Remark 3.1. - Observe that, if $H(t) = H$ for every t, $R'(t)$ is the derivative of $R(t)$, while it could be something else in a more general situation (see Section 4 for the situation in which we are interested on).

Some other notations: for every $t \in [0,T]$ we will write $H(t)$ as sum of three subspaces

$$H(t) = H_+(t) \oplus H_0(t) \oplus H_-(t) , \qquad R(t) = R_+(t) + R_0(t) + R_-(t)$$

where $H_0(t)$ is the kernel of $R(t)$ and $H_+(t)$ and $H_-(t)$ are the invariant subspaces of $H(t)$ associated to respectively the positive and negative part of the spectrum of $R(t)$. By $R_+(t)$ we denote the restriction of $R(t)$ to $H_+(t)$, i.e. $R_+(t)u = R(t)u$ for $u \in H_+(t)$, and by $R_-(t)$ we denote the operator defined by $R_-(t)u = -R(t)u$ for $u \in H_-(t)$, in such a way that both $R_+(t)$ and $R_-(t)$ are positive (analogously we define $R_0(t)$ which turns out to be simply such that $R_0(t)u = 0$ for every $u \in H_0(t)$).
We moreover define
(37)

$$\begin{aligned} &\tilde{H}_+(t) := \text{completion of } H_+(t) \text{ w.r.t. the norm } \|w\|_{\tilde{H}_+(t)} = \|R_+(t)^{1/2} w\|_{H(t)} , \\ &\tilde{H}_-(t) := \text{completion of } H_-(t) \text{ w.r.t. the norm } \|w\|_{\tilde{H}_-(t)} = \|R_-(t)^{1/2} w\|_{H(t)} , \\ &\tilde{H}(t) := \tilde{H}_+(t) \oplus H_0(t) \oplus \tilde{H}_-(t) . \end{aligned}$$

For the following density result one can simply adact Proposition 2.4 in [24].

Proposition 3.2. *The space $\mathcal{U}$ is dense in $\mathcal{W}_{\mathcal{R}}$.*

Also for the following result one can simply adact Proposition 2.6 in [24].

Theorem 3.3. *For every $u, v \in \mathcal{W}_{\mathcal{R}}$ the following holds:*
(38)
$$\frac{d}{dt}(\mathcal{R}u(t), v(t))_{H(t)} = \\ = \langle \mathcal{R}'u(t), v(t)\rangle_{V(t)'\times V(t)} + \langle \mathcal{R}u'(t), v(t)\rangle_{V(t)'\times V(t)} + \langle \mathcal{R}v'(t), u(t)\rangle_{V(t)'\times V(t)}\,,$$
so in particular we have
(39)
$$2\int_s^t \big\langle (\mathcal{R}u)'(\tau), u(\tau)\big\rangle_{V(t)'\times V(t)} d\tau = \\ = (R(t)u(t), u(t))_{H(t)} - (R(s)u(s), u(s))_{H(s)} + \int_s^t \langle \mathcal{R}'u(\tau), u(\tau)\rangle_{V(\tau)'\times V(\tau)} d\tau\,.$$
Moreover the function
$$t \mapsto (R(t)u(t), u(t))_{H(t)} \qquad \text{is continuous}$$
and there is constant $c = c(T, \|\mathcal{R}\|)$ (depending only on $T, \|\mathcal{R}\| \leqslant C_1$) such that
$$\max_{[0,T]} |(R(t)u(t), u(t))_{H(t)}| \leqslant c\ \|u\|^2_{\mathcal{W}_{\mathcal{R}}}\,. \tag{40}$$

Thanks to the previous proposition it makes sense to evaluate $u(t)$ in $\tilde{H}(t)$ for every $t \in [0, T]$. Then for every $t \in [0, T]$ we can consider the orthogonal projections
(41)
$$\pi_+(t) : \tilde{H}(t) \longrightarrow \tilde{H}_+(t)\,, \qquad \pi_-(t) : \tilde{H}(t) \longrightarrow \tilde{H}_-(t)\,, \qquad \pi_0(t) : \tilde{H}(t) \longrightarrow \tilde{H}_0(t)$$
and, once defined $\mathcal{H}_+$, $\mathcal{H}_-$ as the subspaces of $\mathcal{H}$ associated to respectively the positive and negative part of the spectrum of $\mathcal{R}$, $\mathcal{H}_0 = \mathrm{Ker}\ \mathcal{R}$, we defined $\mathcal{R}_+$, $\mathcal{R}_0$, $\mathcal{R}_-$ analogously as R_+, R_0 and R_- we define
$$\begin{aligned} &\tilde{\mathcal{H}}_+ := \text{completion of } \mathcal{H}_+ \text{ w.r.t. the norm } \|w\|_{\tilde{\mathcal{H}}_+} = \|\mathcal{R}_+^{1/2} w\|_{\mathcal{H}}\,, \\ &\tilde{\mathcal{H}}_- := \text{completion of } \mathcal{H}_- \text{ w.r.t. the norm } \|w\|_{\tilde{\mathcal{H}}_-} = \|\mathcal{R}_-^{1/2} w\|_{\mathcal{H}}\,, \\ &\tilde{\mathcal{H}} := \tilde{\mathcal{H}}_+ \oplus \mathcal{H}_0 \oplus \tilde{\mathcal{H}}_-\,. \end{aligned} \tag{42}$$
For given $\varphi \in \tilde{H}_+(0)$ and $\psi \in \tilde{H}_-(T)$ we can also define the space
$$\mathcal{W}_{\mathcal{R}}^{\varphi,\psi} = \big\{u \in \mathcal{W}_{\mathcal{R}}\ \big|\ \pi_+(0)u(0) = \varphi\,,\ \pi_-(T)u(T) = \psi\big\} \tag{43}$$
Now we give an existence result (see Theorem 3.4 and Theorem 3.5 in [24]) for the following problem: suppose you are given $f \in \mathcal{V}'$, $\varphi \in \tilde{H}_+(0)$ and $\psi \in \tilde{H}_-(T)$. Consider the problem
$$\begin{cases} (\mathcal{R}u)' + \mathcal{A}u = f \\ \pi_+(0)u(0) = \varphi \\ \pi_-(T)u(T) = \psi \end{cases} \tag{44}$$
where $\mathcal{A}$ is defined by a family of operators
(45)
$$A(t) : V(t) \to V(t)'\,,\ t \in [0, T]\,, \qquad \text{with} \quad t \mapsto \big\langle A(t)u, v\big\rangle_{V(t)'\times V(t)} \qquad \text{measurable}$$
as follows
$$\mathcal{A} : \mathcal{V} \longrightarrow \mathcal{V}'\,, \qquad \mathcal{A}u(t) = A(t)u(t) \qquad 0 \leqslant t \leqslant T\,. \tag{46}$$

Definition 3.4. *A function $u \in \mathcal{W}_{\mathcal{R}}$ is a solution of* (44) *if*

$$\langle (\mathcal{R}u)'(t), v\rangle_{V(t)'\times V(t)} + \langle \mathcal{A}u(t), v\rangle_{V(t)'\times V(t)} = \langle f(t), v\rangle_{V(t)'\times V(t)}$$
$$\pi(0)u(0) = \varphi \quad \text{in } \tilde{H}_+(0) , \qquad \pi_-(T)u(T) = \psi \quad \text{in } \tilde{H}_-(T)$$

for every $v \in U$ and for a.e. $t \in [0, T]$ (the initial data make sense thanks to Theorem 3.3*) or equivalently (see, e. g.* [31]*, Proposition* 2.1 *in Chap.* 3 *or* [36]*, Theorem* 43.3*) if*

$$- \int_0^T \Big((\mathcal{R}u)(t), \frac{\partial\phi}{\partial t}\Big)_{H(t)} dt + \int_0^T \big\langle \mathcal{A}u(t), \phi(t)\big\rangle_{V(t)'\times V(t)} dt =$$
$$= \int_0^T \big\langle f(t), \phi(t)\big\rangle_{V(t)'\times V(t)} dt + \big(R_+(0)\varphi, \phi(0)\big)_{H(0)} + \big(R_-(T)\psi, \phi(T)\big)_{H(T)}$$

for every $\phi \in \mathcal{U}$ such that $\pi_+(T)\phi(T) = 0$, $\pi_-(0)\phi(0) = 0$. If $\mathcal{R} \equiv 0$ the initial condition has no meaning and in this case a solution is a function $u \in \mathcal{V}$ such that

$$\langle \mathcal{A}u(t), v\rangle_{V(t)'\times V(t)} = \langle f(t), v\rangle_{V(t)'\times V(t)} \qquad \text{for every } v \in U, \text{ for a. e. } t \in [0, T] .$$

Theorem 3.5. *Define the operator $\mathcal{P} : \mathcal{W}_{\mathcal{R}} \to \mathcal{V}'$ by $\mathcal{P}u = (\mathcal{R}u)' + \mathcal{A}u$ where $\mathcal{R}$ satisfies* (34) *and $\mathcal{A} : \mathcal{V} \to \mathcal{V}'$ is continuous. Suppose that there exist two constants $\alpha, \beta > 0$ such that either*

$$\begin{aligned} &\mathcal{A}u = 0 \qquad \text{if } u = 0 \\ &\big\langle \mathcal{A}u - \mathcal{A}v + \tfrac{1}{2}(\mathcal{R}'u - \mathcal{R}'v), u - v\big\rangle_{\mathcal{V}'\times\mathcal{V}} \geqslant \alpha\|u - v\|^2_{\mathcal{V}} , \\ &\|\mathcal{A}u + \tfrac{1}{2}\mathcal{R}'u - \mathcal{A}v - \tfrac{1}{2}\mathcal{R}'u\|_{\mathcal{V}'} \leqslant \beta\|u - v\|_{\mathcal{V}} \end{aligned} \tag{47}$$

or for some $p > 2$,

$$\begin{aligned} &\mathcal{A}u = 0 \qquad \text{if } u = 0 \\ &\langle \mathcal{A}u - \mathcal{A}v, u - v\rangle_{\mathcal{V}'\times\mathcal{V}} \geqslant \alpha\|u - v\|^p_{\mathcal{V}} , \\ &\|\mathcal{A}u - \mathcal{A}v\|_{\mathcal{V}'} \leqslant \beta\big(1 + \|u\|^p_{\mathcal{V}} + \|v\|^p_{\mathcal{V}}\big)^{\frac{p-2}{p-1}} \|u - v\|_{\mathcal{V}}^{\frac{1}{p-1}} \\ &\langle \mathcal{R}'u, u\rangle_{\mathcal{V}'\times\mathcal{V}} \geqslant 0 \end{aligned} \tag{48}$$

for every $u, v \in \mathcal{V}$. Then for every $f \in \mathcal{V}'$, $\varphi \in \tilde{H}_+(0)$, $\psi \in \tilde{H}_-(T)$, problem (44) *has a unique solution and moreover there is a constant $c_1 = c_1(\alpha, p)$ (depending only on α and p) and a constant $c_2 = c_2(\alpha, \beta, p, \|\mathcal{R}'\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')})$ (depending only on α, β, p and, if $p = 2$, possibly also on $\|\mathcal{R}'\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}$) such that for every $u \in \mathcal{W}_{\mathcal{R}}$*

$$\begin{aligned} \|u - v\|_{\mathcal{W}_{\mathcal{R}}} \leqslant\ & \|\mathcal{P}u - \mathcal{P}v\|_{\mathcal{V}'} + c_1\Big[\|\mathcal{P}u - \mathcal{P}v\|_{\mathcal{V}'}^{\frac{1}{p-1}} \\ & + \|R_+^{1/2}(0)(u(0) - v(0))\|^{2/p}_{H_+(0)} + \|R_-^{1/2}(T)(u(T) - v(T))\|^{2/p}_{H_-(T)}\Big] + \\ & + c_2(1 + \|u\|^p_{\mathcal{V}} + \|v\|^p_{\mathcal{V}})^{\frac{p-2}{p-1}} \cdot \Big[\|\mathcal{P}u - \mathcal{P}v\|_{\mathcal{V}'}^{\frac{1}{p-1}} + \\ & + \|R_+^{1/2}(0)(u(0) - v(0))\|^{2/p}_{H_+(0)} + \|R_-^{1/2}(T)(u(T) - v(T))\|^{2/p}_{H_-(T)}\Big]^{\frac{1}{p-1}} . \end{aligned} \tag{49}$$

Proof - We confine to prove the estimate (49). For the proof of the existence (and uniqueness) we refer to Theorem 3.8 in [24] (but see also [31], Section III.3).

Observe that for $w_1, w_2 \in \mathcal{W}_{\mathcal{R}}$ we have that, both if $p = 2$ and if $p > 2$, $\alpha\|w_1 - w_2\|^p_{\mathcal{V}} \leqslant \langle \mathcal{A}w_1 - \mathcal{A}w_2, w_1 - w_2\rangle + \frac{1}{2}\langle \mathcal{R}'(w_1 - w_2), w_1 - w_2\rangle$. Then by (39) we derive

$$
\begin{aligned}
\alpha\|w_1 - w_2\|^p_{\mathcal{V}} \ &\leqslant\ \langle \mathcal{P}w_1 - \mathcal{P}w_2, w_1 - w_2\rangle + \\
&\quad + \frac{1}{2}(R(0)(w_1(0) - w_2(0)), (w_1(0) - w_2(0)))_{H(0)} + \\
&\quad - \frac{1}{2}(R(T)(w_1(T) - w_2(T)), (w_1(T) - w_2(T)))_{H(T)} \leqslant \\
&\leqslant\ \langle \mathcal{P}w_1 - \mathcal{P}w_2, w_1 - w_2\rangle + \\
&\quad + \frac{1}{2}(R_+(0)(w_1(0) - w_2(0)), (w_1(0) - w_2(0)))_{H_+(0)} + \\
&\quad + \frac{1}{2}(R_-(T)(w_1(T) - w_2(T)), (w_1(T) - w_2(T)))_{H_-(T)}\,.
\end{aligned} \tag{50}
$$

By this we derive first, using Young inequality,

$$
\|w_1 - w_2\|_{\mathcal{V}} \leqslant c_1 \left[\|\mathcal{P}w_1 - \mathcal{P}w_2\|^{1/(p-1)}_{\mathcal{V}'} + \|w_1(0) - w_2(0)\|^{2/p}_{\tilde{H}_+(0)} + \|w_1(T) - w_2(T)\|^{2/p}_{\tilde{H}_-(T)} \right], \tag{51}
$$

where $c_1 = c_1(\alpha, p)$. Moreover, since $(\mathcal{R}w_1)' - (\mathcal{R}w_2)' = \mathcal{P}w_1 - \mathcal{P}w_2 - \mathcal{A}w_1 + \mathcal{A}w_2$, using (48) we also obtain, for $p > 2$,

$$
\begin{aligned}
&\|(\mathcal{R}w_1)' - (\mathcal{R}w_2)'\|_{\mathcal{V}'} \leqslant \\
&\qquad \leqslant \|\mathcal{P}w_1 - \mathcal{P}w_2\|_{\mathcal{V}'} + \beta\big(1 + \|w_1\|^p_{\mathcal{V}} + \|w_2\|^p_{\mathcal{V}}\big)^{\frac{p-2}{p-1}} \|w_1 - w_2\|^{\frac{1}{p-1}}_{\mathcal{V}}
\end{aligned}
$$

and conclude using (51). For $p = 2$ we can proceed analogously using (47) and (34)-(v). □

In the classical situation (i. e. $\mathcal{R}$ =Id) the following compactness result holds (see [17]).

Theorem 3.6. *Consider three Banach spaces B_0, B, B_1 such that the embeddings $B_0 \subset B \subset B_1$ are continuous, with B_0 and B_1 reflexive. Fix two numbers $p_0, p_1 \in (1, \infty)$ and $T > 0$. The space $\{v \in L^{p_0}(0, T; B_0) \mid v' \in L^{p_1}(0, T; B_1)\}$ compactly embeds in $L^{p_0}(0, T; B)$.*

In this framework the natural compactness result reads as follows (see Theorem 2.12 and Theorem 2.16 in [24]).

Theorem 3.7. *Consider a family $R : [0, T] \to \mathcal{L}(H)$ of operators satisfying* (34)-(i)*,* (34)-(ii)*,* (34)-(iii)*. Consider the operator $\mathcal{R}$ and suppose $\mathcal{R} \not\equiv 0$. Suppose the embedding $V(t) \subset H(t)$ is compact for every $t \in [0, T]$. Then the space $\mathcal{W}_{\mathcal{R}}$ compactly embeds in $\tilde{\mathcal{H}}_+ \oplus \tilde{\mathcal{H}}_-$.*

In general, i.e. if $\mathcal{R}$ is not invertible, a compactness result (in $\mathcal{H}$) is un-natural (see Example 2.19 in [24]) and clearly also if we have a sequence of operators $\mathcal{R}_h \to \mathcal{R}$, $\mathcal{R}$ not invertible, as we will consider in the next sections.
For this reason we will consider a family of parabolic operators approximating the elliptic-parabolic ones, in the sense stated in the theorem below.

Therefore we introduce the following approximation argument. Consider a family of operators $\mathcal{R}_\epsilon$ defined analogously to $\mathcal{R}$ and satisfying (34), a family $\mathcal{A}_\epsilon : \mathcal{V} \to \mathcal{V}'$, all satisfying assumptions of Theorem 3.5, $f \in \mathcal{V}'$, $\varphi, \psi \in U$. If we denote by $\tilde{H}^\epsilon_+(t)$,

$\tilde{H}^\epsilon_-(t)$, $H^\epsilon_0(t)$ the analogous of the spaces defined in (37) defined via $\mathcal{R}_\epsilon$ and $\pi_{\epsilon,+}(t)$ and $\pi_{\epsilon,-}(t)$ the corresponding projections, we consider the following problems

$$(E_0)\begin{cases} (\mathcal{R}u)' + \mathcal{A}u = f \\ \pi_+(0)u(0) = \pi_+(0)\varphi \\ \pi_-(T)u(T) = \pi_-(T)\psi \end{cases} \qquad (E_\epsilon)\begin{cases} (\mathcal{R}_\epsilon u)' + \mathcal{A}_\epsilon u = f \\ \pi_{\epsilon,+}(0)u(0) = \pi_{\epsilon,+}(0)\varphi \\ \pi_{\epsilon,-}(T)u(T) = \pi_{\epsilon,-}(T)\psi \end{cases}$$

and define

$$\mathcal{W}_\epsilon = \big\{u \in \mathcal{V} \;\big|\; (\mathcal{R}_\epsilon u)' \in \mathcal{V}'\big\}\,.$$

Theorem 3.8. *Denote by $u \in \mathcal{W}_\mathcal{R}$ the solution of (E_0), by $u_\epsilon \in \mathcal{W}_\epsilon$ the solution of (E_ϵ). Suppose*

$$(\mathcal{R}_\epsilon w)' \to (\mathcal{R}w)' \quad \text{and} \quad \mathcal{A}_\epsilon w \to \mathcal{A}w \qquad \text{in } \mathcal{V}' \qquad \text{for every } w \in \mathcal{V}\,.$$

Then

$$\begin{array}{lll} u_\epsilon \to u & \text{in } \mathcal{V} & \\ (\mathcal{R}_\epsilon u_\epsilon)' \to (\mathcal{R}u)' & \text{in } \mathcal{V}' & \qquad \text{when } \epsilon \to 0^+\,. \\ \mathcal{A}_\epsilon u_\epsilon \to \mathcal{A}u & \text{in } \mathcal{V}' & \end{array}$$

Remark 3.9. - We will use this result with $\mathcal{R}_\epsilon = \mathcal{R} + \epsilon\,\mathcal{S}$ for a suitable $\mathcal{S}$. Notice that $((\mathcal{R}+\epsilon\,\mathcal{S})z)' \to (\mathcal{R}z)'$ in $\mathcal{V}'$ for every $z \in \mathcal{V}$ since, for regular z, $((\mathcal{R}+\epsilon\,\mathcal{S})z)' = (\mathcal{R}z)' + \epsilon(\mathcal{S}z)'$.

Proof - By Proposition 3.2 we can find $w \in C^1([0,T];U)$ such that

$$w(0) = \varphi\,, \quad w(T) = \psi\,, \qquad \|u - w\|_{\mathcal{W}_\mathcal{R}} < \delta\,. \tag{52}$$

Then by (50) we can write

$$\alpha\|w - u_\epsilon\|^p_\mathcal{V} \leqslant \big\langle \mathcal{P}_\epsilon w - \mathcal{P}_\epsilon u_\epsilon, w - u_\epsilon\big\rangle$$

where $\mathcal{P}_\epsilon z = (\mathcal{R}_\epsilon z)' + \mathcal{A}_\epsilon z$ and $\mathcal{P}z = (\mathcal{R}z)' + \mathcal{A}z$. Notice that $\mathcal{P}_\epsilon u_\epsilon = \mathcal{P}u$. Then

$$\begin{aligned} \alpha\|w - u_\epsilon\|^p_\mathcal{V} &\leqslant \big\langle \mathcal{P}_\epsilon w - \mathcal{P}_\epsilon u_\epsilon, w - u_\epsilon\big\rangle = \\ &= \big\langle \mathcal{P}_\epsilon w - \mathcal{P}u, w - u_\epsilon\big\rangle = \\ &= \big\langle (\mathcal{R}_\epsilon w)' - (\mathcal{R}u)' + \mathcal{A}_\epsilon w - \mathcal{A}u, w - u_\epsilon\big\rangle\,. \end{aligned}$$

Now since $(\mathcal{R}_\epsilon w)' - (\mathcal{R}u)' + \mathcal{A}_\epsilon w - \mathcal{A}u = (\mathcal{R}_\epsilon w)' - (\mathcal{R}w)' + (\mathcal{R}w)' - (\mathcal{R}u)' + \mathcal{A}_\epsilon w - \mathcal{A}w + \mathcal{A}w - \mathcal{A}u$, by (52) we conclude ($c = \beta(1 + \|w\|^p_\mathcal{V} + \|u\|^p_\mathcal{V})^{\frac{p-2}{p-1}}$)

$$\alpha\|w - u_\epsilon\|^{p-1}_\mathcal{V} \leqslant \delta + c\,\delta^{1/(p-1)} + \|(\mathcal{R}_\epsilon w)' - (\mathcal{R}w)'\|_{\mathcal{V}'} + \|\mathcal{A}_\epsilon w - \mathcal{A}w\|_{\mathcal{V}'}\,. \tag{53}$$

To prove that $u_\epsilon \to u$ in $\mathcal{V}$ it is sufficient to estimate

$$\|u_\epsilon - u\|_\mathcal{V} \leqslant \|u_\epsilon - w\|_\mathcal{V} + \|w - u\|_\mathcal{V}\,,$$

and let ϵ go to zero. Observing that $\mathcal{A}_\epsilon$ are equibounded we derive that $\mathcal{A}_\epsilon u_\epsilon \to \mathcal{A}u$ and, since $\mathcal{P}_\epsilon u_\epsilon = \mathcal{P}u$, we also conclude that $(\mathcal{R}_\epsilon u_\epsilon)' - (\mathcal{R}u)'$. □

Lemma 3.10. *Consider a sequence $(w_h)_{h\in\mathbf{N}} \subset L^p(A)$ (A open set and $p \geqslant 1$), $w_h \to w$ in $L^p(A)$ and a sequence of mollifiers $(\rho_n)_{n\in\mathbf{N}}$ defined as usual by a function ρ. Then, for every $\delta > 0$, it is possible to choose $n_h \in \mathbf{N}$ such that $\|w_h * \rho_{n_h} - w_h\|_{L^p} < \delta$ and $\sup_{h\in\mathbf{N}} n_h < +\infty$.*

Proof - Fix $\delta > 0$ and consider $n_h \in \mathbf{N}$ the minimum integer for which

$$\|w_h * \rho_{n_h} - w_h\|_{L^p} < \delta\,.$$

By this choice it follows that $\sup_h n_h$ is finite. Indeed one has

$$\|w_h * \rho_{n_h - 1} - w_h\|_{L^p} \geqslant \delta$$

and consequently

$$\liminf_h \|w_h * \rho_{n_h - 1} - w\|_{L^p} \geqslant \delta$$

which implies that $\limsup n_h < +\infty$ since $\lim_h \|w_h * \rho_{j_h} - w\|_{L^p} = 0$ for every sequence $j_h \to_h +\infty$. □

Remark 3.11. - Clearly, in this last lemma, the strong convergence $w_h \to w$ is fundamental. If $(w_h)_h$ is only bounded the result is not true. Take for instance $w_h(x) := w(hx)$ in $L^1(0,1)$ where $w(x) = \chi_{[0,1]}(x)$ for $x \in [0,2]$ extended by periodicity to the whole $\mathbf{R}$.

By these two last lemmas we can now state the following corollary of Theorem 3.8. Before suppose to have three sequences of operators $\mathcal{A}_h : \mathcal{V} \to \mathcal{V}'$, satisfying (45), $\mathcal{R}_h, \mathcal{S}_h : \mathcal{H} \to \mathcal{H}$ defined by R_h and S_h satisfying (34), $h \in \mathbf{N}$. Consider $\tilde{H}_{h,+}(t)$, $\tilde{H}_{h,-}(t)$ the analogous of the spaces defined in (37) defined via $\mathcal{R}_h$, $\tilde{H}^{\epsilon}_{h,+}(t)$, $\tilde{H}^{\epsilon}_{h,-}(t)$, the analogous of the spaces defined in (37) defined via $\mathcal{R}_h + \epsilon\,\mathcal{S}_h$ and $\pi_{h,+}(t)$ and $\pi_{h,-}(t)$, $\pi_{h,\epsilon,+}(t)$ and $\pi_{h,\epsilon,-}(t)$ the corresponding orthogonal projections. Consider the following sequences of problems for $f \in \mathcal{V}'$, $\varphi, \psi \in U$,

$$(E_0)_h \begin{cases} (\mathcal{R}_h u)' + \mathcal{A}_h u = f \\ \pi_{h,+}(0)u(0) = \pi_{h,+}(0)\,\varphi \\ \pi_{h,-}(T)u(T) = \pi_{h,-}(T)\,\psi \end{cases} \qquad (E_\epsilon)_h \begin{cases} ((\mathcal{R}_h + \epsilon\,\mathcal{S}_h)u)' + \mathcal{A}_h u = f \\ \pi_{h,\epsilon,+}(0)u(0) = \pi_{h,\epsilon,+}(0)\,\varphi \\ \pi_{h,\epsilon,-}(T)u(T) = \pi_{h,\epsilon,-}(T)\,\psi \end{cases}$$

where $\mathcal{P}_h u = (\mathcal{R}_h u)' + \mathcal{A}_h u$ and $\mathcal{P}_{h,\epsilon} u = ((\mathcal{R}_h + \epsilon\,\mathcal{S}_h)u)' + \mathcal{A}_h u$ satisfy the assumptions of Theorem 3.5.

Corollary 3.12. *Denote by u_h and $u_{h,\epsilon}$ the solutions respectively of $(E_0)_h$ and $(E_\epsilon)_h$. Suppose that there are two operators $\mathcal{R}, \mathcal{S} : \mathcal{H} \to \mathcal{H}$ such that*

$$\mathcal{R}_h w \to_h \mathcal{R} w\,,\ \ \mathcal{S}_h w \to_h \mathcal{S} w \quad \text{in } \mathcal{V}'\,, \qquad \text{for every } w \in \mathcal{H}$$

Then for every $\delta \in (0,1)$ there is $h_\delta \in \mathbf{N}$ such that for every $\epsilon \in (0,1)$

$$\|u_h - u_{h,\epsilon}\|_{\mathcal{V}}^{p-1} \leqslant c\left[\epsilon + \delta^{1/(p-1)}\right] \qquad \text{for every } h \in \mathbf{N}\,, h \geqslant h_\delta\,.$$

with $c > 0$ depends only on $\alpha, \beta, p, \|f\|_{\mathcal{V}'}, \|R_h(0)^{1/2}\varphi\|_{H(0)}, \|R_h(T)^{1/2}\psi\|_{H(T)}, \|S_h(0)^{1/2}\varphi\|_{H(0)}, \|S_h(T)^{1/2}\psi\|_{H(T)}$.

Proof - As in the proof of Theorem 3.8 we fix a datum $\eta, \theta \in V$ and denote by v_h and $v_{h,\epsilon}$ the solutions respectively of $(E_0)_h$ and $(E_\epsilon)_h$ with initial/final data η and θ in such a way, for a fixed $\delta > 0$, $\|v_h - u_h\|_{\mathcal{V}} < \delta/3$, $\|v_{h,\epsilon} - u_{h,\epsilon}\|_{\mathcal{V}} < \delta/3$.
Now we approximate v_h. First, for a fixed $\delta > 0$, we consider a sequence $(\tilde{w}_h)_h \subset L^p(0,T;U)$ such that $\|v_h - \tilde{w}_h\|_{\mathcal{V}} < \delta/3$ and then regularise the sequence via a family of mollifiers $(\rho_n)_n$. In particular for each $h \in \mathbf{N}$ we can find n_h such that $\|\tilde{w}_h - \tilde{w}_h * \rho_{n_h}\|_{\mathcal{V}} < \delta/3$. By Lemma 3.10 we can choose n_h in such a way

$$\sup_{h \in \mathbf{N}} n_h < +\infty\,.$$

Then, defined $w_h := \tilde{w}_h * \rho_{n_h}$, we have

$$\|u_h - w_h\|_{\mathcal{V}} < \delta\,, \qquad \|v_h - w_h\|_{\mathcal{V}} < \delta\,.$$

Arguing as in the proof of Theorem 3.8 we get, as in (53), that

$$\|w_h - v_{h,\epsilon}\|_{\mathcal{V}}^{p-1} \leqslant \epsilon\,\|(\mathcal{S}_h w_h)'\|_{\mathcal{V}'} + \delta + c\,\delta^{1/(p-1)}$$

where $c = c(\alpha, \beta, p, \|f\|_{\mathcal{V}'}, \|R_h(0)^{1/2}\varphi\|_{H(0)}, \|(R_h(T)^{1/2}\psi\|_{H(T)}, \|S_h(0)^{1/2})\varphi\|_{H(0)}, \|S_h(T)^{1/2})\psi\|_{H(T)}$.
We are done if $\|(\mathcal{S}_h w_h)'\|_{\mathcal{V}'}$ is equibounded. But notice that

$$\begin{aligned} \|(\mathcal{S}_h w_h)'\|_{\mathcal{V}'} &\leqslant \|\mathcal{S}_h' w_h\|_{\mathcal{V}'} + \|\mathcal{S}_h w_h'\|_{\mathcal{V}'} \leqslant \|\mathcal{S}_h'\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}\|w_h\|_{\mathcal{V}} + \|\mathcal{S}_h\|_{\mathcal{L}(\mathcal{H})}\|w_h'\|_{\mathcal{V}'} \leqslant \\ &\leqslant \|\mathcal{S}_h'\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}(\|v_h\|_{\mathcal{V}} + \delta) + c\,\|\mathcal{S}_h\|_{\mathcal{L}(\mathcal{H})}\Big[\int_{\mathbf{R}} |\rho_{n_h}'(\tau)|^{p'} d\tau\Big]^{1/p'} \|v_h\|_{\mathcal{V}} \end{aligned}$$

and since $\mathcal{S}_h, \mathcal{S}_h'$ and v_h are equibounded, using Lemma 3.10 to estimate $\int_{\mathbf{R}} |\rho_{n_h}'(\tau)|^{p'} d\tau$, we conclude as in Theorem 3.8. □

We conclude this section with an abstract result which will be useful to prove the G-compactness result.

Lemma 3.13. *Consider an operator $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ defined by a family $S(t)$ satisfying* (34)*, $\mathcal{S} \not\equiv 0$. Suppose $Ker\mathcal{R} \cap Ker\mathcal{S} = \{0\}$ and $\mathcal{H}$ is separable.*
Then for every $\epsilon \in (0, 1]$, except at most a countable subset of $(0, 1]$, the operator $\mathcal{R} + \epsilon\,\mathcal{S}$ is invertible.

Proof - We denote by K_ϵ the subspace $\mathrm{Ker}\,(\mathcal{R}_\epsilon)$ where $\mathcal{R}_\epsilon := \mathcal{R} + \epsilon\,\mathcal{S}$, possibly $K_\epsilon = \{0\}$. Clearly if $\delta, \epsilon > 0$, $\delta \neq \epsilon$, $K_\epsilon \cap K_\delta = \{0\}$. Notice that $\oplus_{\delta\in(0,1]}K_\delta$ is a subspace of $\mathcal{H}$. Now we affirm that $K_\delta = \{0\}$ for every $\delta > 0$, except at most a countable set of positve values of δ. Indeed it is impossible that $K_\delta \neq \{0\}$ for $\delta \in D \subset (0, 1]$ and D more than countable. If this were true, we could find for each $\delta \in D$ $u_\delta \in K_\delta$, $u_\delta \neq 0$, with the elements of the family $\{u_\delta\}_{\delta\in D}$ which turn out to be linearly independent. But since $\mathcal{H}$ is separable this is not possible. □

4. Position of the problem

In this section we are going to define the problem we want to study. We will consider a sequence of elliptic-parabolic problem, so first we are going to fix assumptions we will make about the coefficients of these equations.
From now on we will consider **fixed** an open bounded set Ω with Lipschitz boundary, a cube Q_Ω containing Ω, and some positive constants $T, p, L, K_1, C_1, r, \delta_1, \delta_2, C_2$ with $p \geqslant 2$, $L, K_1 \geqslant 1$, $r > 1$, $0 < \delta_1, \delta_2 < 1$, a constant $\alpha \in [0, 1)$, two non-negative constants C_3, K_2 and a constant $\sigma = \sigma(n, p, K_1) > 0$ in such a way (16) hold. We moreover denote by F a continuous function

$$F : [0, T] \to [0, +\infty) \qquad \text{s.t. } F(0) = 0 \text{ and } F(t) > 0 \text{ for } t \in (0, T]\,. \tag{54}$$

Finally we denote and consider fixed also the spaces

$$\begin{gathered} U_1 := W_0^{1,1+\sigma}(\Omega)\,, \qquad U_2 := W_0^{1,(1+\sigma)'}(\Omega)\,, \\ \mathcal{U}_1 := L^p(0, T; U_1),\ \mathcal{U}_2 := L^p(0, T; U_2), \quad \mathcal{U} = \{u \in \mathcal{U}_1 \mid u' \in \mathcal{U}_2'\}\,. \end{gathered} \tag{55}$$

Definition 4.1. *We denote by* $\mathcal{M}_{\Omega\times(0,T)}(p, L, K_1, C_1, F)$ *(sometimes we will omit the constants if not necessary) the class of Carathéodory functions*

$$a : \Omega \times (0,T) \times \mathbf{R}^n \to \mathbf{R}^n\,, \qquad a = a(x,t,\xi)$$

for which there exists a function $\lambda_a = \lambda_a(x,t)$ *such that the following hold*

$$\text{(S.1)} \quad a(x,t,0) = 0\,,$$

$$\text{(S.2)} \quad |a(x,t,\xi) - a(x,t,\eta)| \leqslant L\,\lambda_a(x,t)\big(1 + |\xi|^p + |\eta|^p\big)^{(p-2)/p}|\xi - \eta|\,,$$

$$\text{(S.3)} \quad (a(x,t,\xi) - a(x,t,\eta), \xi - \eta) \geqslant \lambda_a(x,t)|\xi - \eta|^p\,,$$

$$\text{(S.4)} \quad \lambda_a(t) \in A_p(K_1)\,, \ \textit{i. e. } (20) \textit{ is satisfied,}$$

$$\text{(S.5)} \quad \|\lambda_a(t)\|_{L^1(\Omega)} + \|\lambda_a^{-1/(p-1)}(t)\|_{L^1(\Omega)} \leqslant C_1\,,$$

$$\text{(S.6)} \quad \|\lambda_a(t) - \lambda_a(s)\|_{L^1(\Omega)} \leqslant F(|t-s|)\,,$$

$$\text{(S.7)} \quad \|\lambda_a^{-1/(p-1)}(t) - \lambda_a^{-1/(p-1)}(s)\|_{L^1(\Omega)} \leqslant F(|t-s|)\,,$$

where (S.1), (S.2), (S.3) *hold for a. e.* $(x,t) \in \Omega \times (0,T)$ *and for every* $\xi, \eta \in \mathbf{R}^n$, *and* (S.4), (S.5), (S.6), (S.7) *hold for every* $t, s \in [0,T]$ *and* F *is the function defined in* (54).
We denote by $\mathcal{N}_{\Omega\times(0,T)}(p, L, K_1, C_1, F)$ *the class of Carathéodory functions* $a : \Omega \times (0,T) \times \mathbf{R}^n \to \mathbf{R}^n$ *for which* (S.1), (S.3)-(S.6) *and the following hold*

$$\text{(S.2)}' \quad |a(x,t,\xi) - a(x,t,\eta)| \leqslant L\,\lambda_a(x,t)(1 + |\xi|^p + |\eta|^p)^{\frac{p-2}{p-1}}|\xi - \eta|^{\frac{1}{p-1}}\,.$$

Finally we denote by $\Lambda(a)$ *the classes of weights* λ_a *for which* (S.1)-(S.7) *hold if* $a \in \mathcal{M}_{\Omega\times(0,T)}$ *or for which* (S.1), (S.2)′, (S.3)-(S.6) *hold if* $a \in \mathcal{N}_{\Omega\times(0,T)}$.

Remark 4.2. - Observe that $\mathcal{M}_{\Omega\times(0,T)}(p, L, K_1, C_1, F) \subset \mathcal{N}_{\Omega\times(0,T)}(p, L', K_1, C_1, F)$ for $L' = L'(p, L)$ depending only on p, L (see the proof of Lemma 4.9).

Remark 4.3. - It follows from the definition that two weights $\lambda_1, \lambda_2 \in \Lambda(a)$ are comparable.

Definition 4.4. *For a given* $a \in \mathcal{M}_{\Omega\times(0,T)}$ *and* $\lambda \in \Lambda(a)$ *we denote by* $\mathcal{F}(\lambda, K_2, r, \alpha, \delta_1, \delta_2, C_2, C_3, F)$*, or simply by* $\mathcal{F}(\lambda)$ *since the constants will be fixed, the class of functions* μ

such that (T.2)-(T.6) *and one amongs* (T.1)-a) *and* (T.1)-b) *hold*

$$\text{(T.1)}\qquad \begin{array}{ll} \text{a)} & (|\mu|^r(t), \lambda^r(t)) \in A^{\alpha r}_{p,p}(K_2, 8Q_\Omega)\,, \\ \text{b)} & \left| \begin{array}{l} (|\mu|(t), \lambda(t)) \in A^{\alpha}_{p,p}(K_2, 8Q_\Omega)) \\ |\mu|_\lambda(\cdot, t) \quad (\delta_1, \delta_2) - \textit{reverse doubling uniformly in } t,\ t \in [0,T] \end{array} \right. \end{array}$$

$$\text{(T.2)}\qquad \|\mu\|_{L^\infty(0,T;L^r(\Omega))} \leqslant C_2$$

$$\text{(T.3)}\qquad \|\mu(t) - \mu(s)\|_{L^r(\Omega)} \leqslant F(|t-s|) \quad \textit{for every } s,t \in [0,T]\,,$$

$$\text{(T.4)}\qquad t \mapsto \int_\Omega u(x)v(x)\mu(x,t)dx\,, \quad t \mapsto \int_\Omega u(x)v(x)\lambda(x,t)dx \quad \in AC([0,T])\,,$$

$$\text{(T.5)}\qquad \Big| \frac{d}{dt} \int_\Omega u(x)v(x)\mu(x,t)dx \Big| \leqslant C_3\ \|u\|_{W^{1,p}_0(\Omega,|\mu|(t),\lambda(t))} \|v\|_{W^{1,p}_0(\Omega,|\mu|(t),\lambda(t))}\,,$$

$$\Big| \frac{d}{dt} \int_\Omega u(x)v(x)\lambda(x,t)dx \Big| \leqslant C_3\ \|u\|_{W^{1,p}_0(\Omega,|\mu|(t),\lambda(t))} \|v\|_{W^{1,p}_0(\Omega,|\mu|(t),\lambda(t))}\,,$$

$$\text{(T.6)}\qquad \frac{d}{dt} \int_\Omega u^2(x)\mu(x,t)dx \geqslant 0\,, \quad \frac{d}{dt} \int_\Omega u^2(x)\lambda(x,t)dx \geqslant 0\,,$$

for a. e. $t \in [0,T]$ *and for every* $u, v \in U_2$ *and where the function* $|\mu|_\lambda$ *is defined in (18),* F *is the function defined in* (54) *and the space* $W^{1,p}_0(\Omega, |\mu|(t), \lambda(t))$ *is defined in Section* 2*.*

Remark 4.5. - Assumption (T.1) is needed for Theorem 2.6 and Proposition 2.8 to hold (using Remark 2.5). Assumption (T.6) could be weakened, at least if $p = 2$. Indeed in this case one could consider the derivative of $t \mapsto \int_\Omega u^2(x)\mu(x,t)$ also negative, but suitable bounded by below (see Theorem 3.5 and [26] where linear operators are considered).

Remark 4.6. - The class just defined is compact: consider a sequence $(\lambda_h)_h$ ($\lambda_h \in \Lambda(a_h)$ for $(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(p, L, K_1, C_1, F)$), and a sequence $(\mu_h)_h$ such that $\mu_h \in \mathcal{F}(\lambda_h, K_2, r, \alpha, \delta_1, \delta_2,\ C_2, C_3, F)$. First of all (see Remark 2.12) there is a weight $\tilde\lambda$ satisfying (S.4), (S.5), (S.7) such that $\lambda_{h_j}^{-\frac{1}{p-1}}(t) \to \tilde\lambda^{-\frac{1}{p-1}}(t)$ in $L^1(\Omega)$-weak for every $t \in [0,T]$ and a weight λ satisfying (S.4), (S.5), (S.6) such that $\lambda_{h_j}(t) \to \lambda(t)$ in $L^1(\Omega)$-weak for every $t \in [0,T]$ with λ and $\tilde\lambda$ comparable. Moreover, by (T.2) and (T.3) there is a subsequence, still indexed for simplicity by h_j, $(\mu_{h_j})_j$ and a function μ such that $\mu_{h_j} \to \mu$ in $C^0([0,T]; L^r(\Omega)$-weak) (this can be proved as done for $(\lambda_h)_h$ in Remark 2.12). Therefore $(|\mu|, \tilde\lambda)$, and since $\tilde\lambda \leqslant \lambda$ also $(|\mu|, \lambda)$, satisfies (T.1) and μ (T.2) and (T.3). Now one can consider a countable set U, dense in U_1, and for every $u, v \in U$ define the functions $F^{u,v}_{h_j}(t) = \int_\Omega u(x)v(x)\mu_{h_j}(x,t)dx$, which turn out to be equicontinuous and equibounded, and conclude (following the proof in Remark 4.3 in [26]) that $\mu \in \mathcal{F}(\lambda, K_2, r, \alpha, \delta_1, \delta_2, C_2, C_3, F)$.

Now we consider a sequence of functions $(a_h)_h$ and two sequences of weights $(\lambda_h)_h$, $(\mu_h)_h$ such that

$$\text{(56)}\qquad \begin{array}{ll} (a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(p, L, K_1, C_1, F)\,, \quad \lambda_h \in \Lambda(a_h) & \\ \mu_h \in \mathcal{F}(\lambda_h, K_2, r, \alpha, \delta_1, \delta_2, C_2, C_3, F) & \text{for every } h \in \mathbf{N}\,. \end{array}$$

Denote by $|\mu|_{\lambda_h}$ the extension $|\mu_h|_{\lambda_h}$ defined in (18). For simplicity we denote by

$$\mathcal{H}_h \qquad \text{and} \qquad \mathcal{V}_h$$

respectively the spaces $\mathcal{H}_{|\mu|_{\lambda_h}}$ and $\mathcal{V}_{\lambda_h}$ defined in Section 2. Then, once considered (it is possible to define $\omega_h(t)$ for every t thanks to assumption (T.4))

(57)
$$\begin{aligned}
&\omega_h^+(t) =: \big\{x \in \Omega \;\big|\; \mu_h(x,t) > 0\big\}\,, && \Omega^+_{T,h} =: \big\{(x,t) \in \Omega \times (0,T) \;\big|\; \mu_h(x,t) > 0\big\}\,,\\
&\omega_h^-(t) =: \big\{x \in \Omega \;\big|\; \mu_h(x,t) < 0\big\}\,, && \Omega^-_{T,h} =: \big\{(x,t) \in \Omega \times (0,T) \;\big|\; \mu_h(x,t) < 0\big\}\\
&\omega_h(t) =: \omega_h^+(t) \cup \omega_h^-(t)\,, && \Omega_{T,h} =: \Omega^+_{T,h} \cup \Omega^-_{T,h}\,.
\end{aligned}$$

Consider the positive and negative part of μ_h denoted by $\mu_{h,+}$, $\mu_{h,-}$ respectively. We define the operators ($H_h(t) := L^2(\Omega, |\mu|_{\lambda_h}(\cdot,t))$, $H_{h,+}(t) := L^2(\omega_h^+(t), \mu_{h,+}(\cdot,t))$, $H_{h,-}(t) := L^2(\omega_h^-(t), \mu_{h,-}(\cdot,t))$, $\mathcal{H}_{h,+} := L^2(\Omega^+_{T,h}; \mu_h)$, $\mathcal{H}_{h,-} := L^2(\Omega^-_{T,h}; -\mu_h)$)

$$(58)\qquad \begin{aligned}
&R_h^+(t) : H_h(t) \to H_{h,+}(t)\,, && \mathcal{R}_h^+ : \mathcal{H}_h \to \mathcal{H}_{h,+}\,,\\
&R_h^-(t) : H_h(t) \to H_{h,-}(t)\,, && \mathcal{R}_h^- : \mathcal{H}_h \to \mathcal{H}_{h,-}\,,
\end{aligned}$$

the orthogonal projections, and finally

$$(59)\qquad R_h(t) := R_h^+(t) - R_h^+(t)\,, \qquad \mathcal{R}_h := \mathcal{R}_h^+ - \mathcal{R}_h^-\,,$$

in such a way that

$$\|R_h(t)\|_{\mathcal{L}(L^2(\Omega,|\mu|_{\lambda_h}(\cdot,t)),L^2(\omega_h(t),|\mu|_h(\cdot,t)))} \leqslant 1\,, \qquad \|\mathcal{R}_h\|_{\mathcal{L}(\mathcal{H}_h,\mathcal{H}_{h,+}\oplus\mathcal{H}_{h,-})} \leqslant 1\,.$$

Remark 4.7. - Notice that, once defined $r_h(x,t) = \chi_{\Omega^+_{T,h}} - \chi_{\Omega^-_{T,h}}$, one can represent $\mathcal{R}_h$ by r_h, i.e. $(\mathcal{R}_h u)(x,t) = r_h(x,t)u(x,t)$.

Finally denote

$$\mathcal{A}_h u = -\mathrm{div}(a_h(x,t,Du))\,, \quad \mathcal{A}_h : \mathcal{V}_h \to \mathcal{V}_h'\,.$$

For fixed

$$f \in L^{p'}(0,T;L^n(\Omega))\,, \qquad \varphi \in L^\infty(\Omega)\,, \qquad \psi \in L^\infty(\Omega)$$

we consider the sequence of abstract problems (see Theorem 3.5 for existence of the solutions)

$$(60)\qquad \begin{cases}
(\mathcal{R}_h u)' + \mathcal{A}_h u = f\\
R_h^+(0)u(0) = R_h^+(0)\varphi\\
R_h^-(T)u(T) = R_h^-(T)\psi
\end{cases}$$

which correspond to

$$(61)\qquad (P_h)\ \begin{cases}
\dfrac{\partial}{\partial t}(\mu_h u) - \mathrm{div}\big(a_h(x,t,Du)\big) = f & \text{on } \Omega \times (0,T)\\
u = 0 & \text{on } \partial\Omega \times (0,T)\\
u(x,0) = \varphi(x) & \text{on } \omega_h^+(0)\\
u(x,T) = \psi(x) & \text{on } \omega_h^-(T)\,.
\end{cases}$$

Remark 4.8. - The choice of $f \in L^{p'}(0,T;L^n(\Omega))$ garantees (see (28)) that $f \in \mathcal{V}_h'$ for every $h \in \mathbf{N}$. Analogously $\varphi \in L^\infty(\Omega)$ (and the same holds for ψ) garantees that $\varphi \in L^2(\Omega, \lambda_h(\cdot,0))$ and $\varphi \in L^2(\Omega, \mu_h(\cdot,0))$ for every $h \in \mathbf{N}$. In fact, in order that $\varphi \in L^p(\Omega, \lambda_h(\cdot,0))$, and consequently $\varphi \in L^2(\Omega, \lambda_h(\cdot,0))$, and $\varphi \in L^2(\Omega, |\mu_h|(\cdot,0))$, it would be sufficient to consider $\varphi \in L^q(\Omega)$ where $q = \max\{(1+\sigma)', 2r'\}$ (see (15) and use Hölder's inequality since $\mu_h(\cdot, t) \in L^r(\Omega)$ for every $t \in [0, T]$.)

Notice that in this case, since we identify the dual space of $\mathcal{H}_h$ with itself, and then have the triplet

$$\mathcal{V}_h \subset \mathcal{H}_h \subset \mathcal{V}_h' , \qquad \langle \cdot, \cdot \rangle_h := \langle \cdot, \cdot \rangle_{\mathcal{V}_h' \times \mathcal{V}_h} ,$$

(and also $\mathcal{V}_h \subset \mathcal{K}_h \subset \mathcal{V}_h'$) we have

$$\big\langle (\mathcal{R}_h u)', \phi \big\rangle_h = - \int_0^T \int_\Omega u \frac{\partial \phi}{\partial t} \mu_h \, dxdt \qquad \text{for every } \phi \in C^1_c(\Omega \times (0,T)) .$$

and the solution u_h will belong to the space $\mathcal{W}_h^{\varphi,\psi}$ defined as follows:

$$\begin{aligned} & \mathcal{W}_h := \{ u \in \mathcal{V}_h \mid (\mathcal{R}_h u)' \in \mathcal{V}_h' \} , \\ & \mathcal{W}_h^{\varphi,\psi} := \{ u \in \mathcal{W}_h \mid R_h^+(0) u(0) = R_h^+(0)\varphi, \, R_h^-(T) u(T) = R_h^-(T)\psi \} . \end{aligned} \tag{62}$$

Lemma 4.9. *Given* $a \in \mathcal{M}_{\Omega \times (0,T)}(p, L, K_1, C_1, F)$ *and* $\lambda \in \Lambda(a)$ *the operator* $\mathcal{A}_a u = -\mathrm{div}(a(x,t,Du))$ *satisfies, for* $p \geqslant 2$, $(L' = L'(L,p))$

$$\| \mathcal{A}u - \mathcal{A}v \|_{\mathcal{V}'} \leqslant L' \big(\|\lambda\|_{L^1(\Omega \times (0,T))} + \|u\|^p_{\mathcal{V}} + \|v\|^p_{\mathcal{V}} \big)^{\frac{p-2}{p-1}} \|u - v\|_{\mathcal{V}}^{\frac{1}{p-1}} .$$

Proof - By (S.1), (S.2), (S.3) and using suitably Young's inequality we derive

$$\begin{aligned} & |a(x,t,\xi) - a(x,t,\eta)| \leqslant \\ & \quad \leqslant L \, \lambda(x,t)^{1/p} \big(\lambda(x,t) + (a(x,t,\xi), \xi) + (a(x,t,\eta), \eta) \big)^{\frac{p-2}{p}} (a(x,t,\xi) - a(x,t,\eta), \xi - \eta)^{\frac{1}{p}} \\ & \quad \leqslant L' \, \lambda(x,t) (1 + |\xi|^p + |\eta|^p)^{\frac{p-2}{p-1}} |\xi - \eta|^{\frac{1}{p-1}} . \end{aligned}$$

Then

$$\begin{aligned} \| \mathcal{A}_a u - \mathcal{A}_a v \|_{\mathcal{V}_\lambda'} & \leqslant \Big(\int_0^T \int_\Omega |a(x,t,Du) - a(x,t,Dv)|^{p'} \lambda^{1-p'} dxdt \Big)^{1/p'} \leqslant \\ & \leqslant \Big(\int_0^T \int_\Omega \Big[L' \lambda (1 + |Du|^p + |Dv|^p)^{\frac{p-2}{p-1}} |Du - Dv|^{\frac{1}{p-1}} \Big]^{p'} \lambda^{1-p'} dxdt \Big)^{1/p'} \leqslant \\ & \leqslant L' \Big(\int_0^T \int_\Omega \lambda dx + \int_0^T \int_\Omega |Du|^p \lambda dx + \int_0^T \int_\Omega |Dv|^p \lambda dx \Big)^{\frac{p-2}{p-1}} \|u - v\|_{\mathcal{V}_\lambda}^{\frac{1}{p-1}} . \end{aligned}$$ □

Theorem 4.10. *Problem* (61) *admits a unique solution. Moreover there is a positive constant c, independent of h, such that, denoted by* u_h *the solution of* (P_h)*, we have*

$$\|u_h\|_{\mathcal{W}_h} \leqslant c$$

where c depends (*only*) *on* $f, \varphi, p, L, C_1, C_2, C_3$.

Proof - By Theorem 3.5 the existence of a unique solution u_h and the boundedness of the solution follow immediatly. The bound is independent of h since, by Lemma 4.9, the operators $\mathcal{A}_h$ satisfy assumptions of Theorem 3.5. □

To study the limit behaviour, as $h \to +\infty$, of problems (P_h), since a compactness result does not hold in general (see Theorem 3.7) we will approximate problems (P_h) by the following problems

$$\begin{cases} (\mathcal{R}_h u + \epsilon\, \mathcal{S}_h\, u)' + \mathcal{A}_h u = f \\ \big[R_h(0) + \epsilon\, S_h(0)\big]_- u(0) = \big[R_h(0) + \epsilon\, S_h(0)\big]_+ \varphi \\ \big[R_h(T) + \epsilon\, S_h(T)\big]_+ u(T) = \big[R_h(T) + \epsilon\, S_h(0)\big]_- \psi \end{cases} \qquad \epsilon \in (0,1]\,.$$

with the solution u_h^ϵ belonging to the space

$$\mathcal{W}_{h,\epsilon} := \{u \in \mathcal{V}_h \mid (\mathcal{R}_h u + \epsilon \mathcal{S}_h u)' \in \mathcal{V}_h'\} \tag{63}$$

where $S_h(t)$ and $\mathcal{S}_h$ denote the identities in $L^2(\Omega, \lambda_h(\cdot,t)$ and $\mathcal{K}_h := L^2(\Omega \times (0,T); \lambda_h)$ respectively, i.e.

$$\begin{array}{cccc} S_h(t): & L^2(\Omega, \lambda_h(\cdot,t)) & \longrightarrow & L^2(\Omega, \lambda_h(\cdot,t))\,, \\ & u & \mapsto & u \end{array} \qquad \begin{array}{cccc} \mathcal{S}_h: & \mathcal{K}_h & \longrightarrow & \mathcal{K}_h\,. \\ & u & \mapsto & u \end{array} \tag{64}$$

Notice that for $u, v \in \mathcal{K}_h$, $\big(u, v\big)_{\mathcal{K}_h} = \int_0^T \int_\Omega u\, v\, \lambda_h\, dxdt$ and

$$\langle S_h'(t) u, v\rangle_{W^{-1,p'}(\Omega,\lambda_h(\cdot,t),\lambda_h(\cdot,t)) \times W_0^{1,p}(\Omega,\lambda_h(\cdot,t),\lambda_h(\cdot,t))} = \frac{d}{dt}\int_\Omega u(x)v(x)\lambda_h(x,t)dx\,.$$

In this way the abstract problems above correspond to

$$(P_h^\epsilon) \begin{cases} \dfrac{\partial}{\partial t}((\mu_h + \epsilon\lambda_h)u) - \mathrm{div}\big(a_h(x,t,Du)\big) = f & \text{on } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = \varphi(x) & \text{on } \omega_{h,\epsilon}^+(0) \\ u(x,T) = \psi(x) & \text{on } \omega_{h,\epsilon}^-(T) \end{cases} \tag{65}$$

where $\omega_{h,\epsilon}^+(0) = \{x \in \Omega \,|\, (\mu_h + \epsilon\lambda_h)(x,0) > 0\}$, $\omega_{h,\epsilon}^-(T) = \{x \in \Omega \,|\, (\mu_h + \epsilon\lambda_h)(x,T) < 0\}$.

Bt Lemma 3.13 we can find $\epsilon > 0$ such that

$$\mu_h + \epsilon\lambda_h \neq 0 \qquad \text{a.e. in } \Omega \times (0,T)\,.$$

The idea is to study before the limit behaviour of problems (P_h^ϵ) and then use Theorem 3.8 to study the limit behaviour of (P_h).

5. Preliminary compactness results

In this section we will give some compactness results for a sequence of functions $(v_h)_h$, $v_h \in \mathcal{W}_h$. We recall that $T, p, L, K_1, K_2, C_1, C_2, C_3, r, \delta_1, \delta_2, \sigma$ are the constants and F the function fixed at the beginning of Section 4. Then we consider the three sequences $(\mu_h)_h$, $(a_h)_h$, $(\lambda_h)_h$ (considered in (56)) where $\lambda_h(x,t) \in \Lambda(a_h)$. Then we will consider the constants, the Carathéodory functions a_h, the weights λ_h and μ_h fixed in this and in the following sections.
If assumptions for a particular result can be weakened we will stress it (as done,

for instance, in Remark 5.3). We can suppose λ_h, $\lambda_h^{-1/(p-1)}$ and μ_h to converge (otherwise we extract a subsequence); we denote by μ, λ, $\tilde{\lambda}$ the following limits (66)

$$\mu_h(t) \to \mu(t)\,, \quad \lambda_h(t) \to \lambda(t)\,, \quad \lambda_h^{-1/(p-1)}(t) \to \tilde{\lambda}^{-1/(p-1)}(t) \qquad \text{in } L^1(\Omega)\text{-weak}$$

for every $t \in [0, T]$, which we recall to satisfy (see Remark 2.12 and Remark 4.6)

$$\begin{aligned}
&\tilde{\lambda} \leqslant \lambda \leqslant K_1 \tilde{\lambda} \\
&\lambda, \tilde{\lambda} \in C^0([0,T]; L^1(\Omega)) \\
&\lambda(t), \tilde{\lambda}(t) \in A_p(K_1) \quad \text{for every } t \\
&\mu \in L^\infty(0,T; L^1(\Omega)) \quad \text{and} \quad (|\mu|, \lambda) \quad \text{satisfies (T.1) -a) or (T.1) -b)}
\end{aligned}$$

Analogously as done in (59) we associate to μ the operator $\mathcal{R}$. In the same way we denote $\omega^+(t)$, $\omega^-(t)$, μ_+, μ_-, $R^+(t)$, $R^-(t)$ the sets, functions and operators analogous to those defined in (57) corresponding to μ.

Remark 5.1. - Notice that $\mu_h \to \mu$ in $C^0([0,T]; L^r(\Omega)\text{-w})$ if and only if $R_h(t)w \to R(t)w$ in $L^{1+\sigma}(\Omega)$ for every $t \in [0,T]$ and $w \in C^0(\bar{\Omega})$.

By $\mathcal{V}_\lambda, \mathcal{V}_{\tilde{\lambda}}$ we will denote the spaces defined in (23) associated to λ and $\tilde{\lambda}$, by $\mathcal{W}_{\mu,\lambda}$ and $\mathcal{W}_{\mu,\lambda}^{\varphi,\psi}$ the spaces associated to μ and λ defined analogously as done in (36) and (43) (the spaces $\mathcal{V}_\lambda$ and $\mathcal{V}_{\tilde{\lambda}}$ do not depend on μ, while $\mathcal{W}_{\mu,\lambda}$ and $\mathcal{W}_{\mu,\lambda}^{\varphi}$ do).

The compactness result which follows, and consequently some results we state in this section, holds with the **additional assumption** that

A) $\qquad \mu \neq 0 \quad \text{a.e. in } \Omega \times (0,T)\,.$

This result could not be obtained without **A)** (see, e.g., Theorem 2.18 and Example 2.19 in [24] for more details). In any case this assumption is temporary, in the sense that for the G-compactness results we want to state it will be removed.

Theorem 5.2. *Consider a sequence $(v_h)_h$ with $v_h \in \mathcal{W}_h$. Suppose there is a positive constant c for which $\|v_h\|_{\mathcal{W}_h} \leqslant c$ for every $h \in \mathbf{N}$ and moreover suppose* **A)** *holds. Then, up to a subsequence,*

$$v_h \to v \quad \text{in } L^p(0,T; L^1(\Omega))\,.$$

Remark 5.3. - If $\sup_h \inf_t \|\lambda_h(t)\|_{L^1(\Omega)} = +\infty$ (and then (S.5) does not hold) Theorem 5.2 still remains true. Indeed in this case, taking a cube Q containing Ω, one can find a subsequence such that $\lim_{j\to+\infty} \sup_{t\in(0,T)} \int_Q \lambda_{h_j}^{-1/p-1}(x,t)dx = 0$. By (26) one obtain that $v_{h_j} \to 0$ in $L^p(0,T; L^1(\Omega))$.

Proof - We consider, for every positive δ, the sequence $(v_{h,\delta})_h$ obtained as in (29). Now consider $\widetilde{\Omega}$ such that $\bar{\Omega} \subset \widetilde{\Omega}$ and define $\delta_0 = \text{dist}(\Omega, \partial\widetilde{\Omega})$. Consider

$$\mathcal{Z} = \{u \in \mathcal{U}_2 \mid u' \in \mathcal{U}_2'\}$$

where Ω is replaced by $\widetilde{\Omega}$). Then for every $\delta \in (0, \delta_0]$ one can prove that the sequence $(v_{h,\delta})_h$ satisfies, for a.e. t,

$$\|Dv_{h,\delta}(\cdot,t)\|_{L^\infty(\Omega)} \leqslant \|\nabla\rho_\delta\|_\infty \|v_h(\cdot,t)\|_{L^1(\Omega)}.$$

by which we get

$$\|v_{h,\delta}\|_{\mathcal{U}_2} \leqslant c_1 \, \|\nabla \rho_\delta\|_\infty \|v_h\|_{L^p(0,T;L^1(\Omega))}$$

with c_1 depending on $|\widetilde{\Omega}|, \sigma$ (σ is the value fixed at the beginning of Section 4 by which $\mathcal{U}_2$ is defined). As regards $(\mathcal{R}_h v_{h,\delta})'$ first observe that, given $\phi \in C^1_c(\Omega \times (0,T))$ and defined $(J\phi)(t) := \int_0^t \phi(\cdot, s) ds$, one has

$$\big(J(\mathcal{R}_h(\mathcal{R}_h \phi_t)_\delta)\big)(t) = \big(J[\mathcal{R}'_h(\mathcal{R}_h \phi)_\delta - \mathcal{R}_h(\mathcal{R}'_h \phi)_\delta]\big)(t) \, . \tag{67}$$

Then we have (recall that ρ is radial and that $\mathcal{R}_h^2 = \mathrm{Id}$, by **A)**; $\langle \cdot, \cdot \rangle$ denotes the duality betweeen $\mathcal{U}'_2$ and $\mathcal{U}_2$)

$$\begin{aligned}\big\langle (\mathcal{R}_h v_{h,\delta})', \phi \big\rangle = -\big\langle \mathcal{R}_h v_{h,\delta}, \phi' \big\rangle = -\big\langle v_h, (\mathcal{R}_h \phi')_\delta \big\rangle = -\big\langle \mathcal{R}_h v_h, \mathcal{R}_h(\mathcal{R}_h \phi')_\delta \big\rangle = \\ = \big\langle (\mathcal{R}_h v_h)', J\mathcal{R}_h(\mathcal{R}_h \phi')_\delta \big\rangle - \big(\mathcal{R}_h v_h(T), (J\mathcal{R}_h(\mathcal{R}_h \phi')_\delta)(T)\big)_{L^2} \, .\end{aligned}$$

Then, by (67) and the density of $C^1_c(\widetilde{\Omega} \times (0,T))$ in $L^p(0,T; W_0^{1,(1+\sigma)'}(\widetilde{\Omega}))$ we get that $(\mathcal{R}_h v_{h,\delta})'$ is bounded by a constant c_2 which depends on c and δ. Then we conclude that there is $c_3 > 0$ such that

$$\|v_{h,\delta}\|_{L^p(0,T;W_0^{1,(1+\sigma)'}(\widetilde{\Omega}))} + \|(\mathcal{R}_h v_{h,\delta})'\|_{L^{p'}(0,T;W^{-1,1+\sigma}(\widetilde{\Omega}))} \leqslant c_3 \, .$$

Applying Theorem 2.18-ii) in [24] we get that (for a fixed δ)

$$(v_{h,\delta})_h \qquad \text{is precompact in } L^p(0,T; L^{(1+\sigma)'}(\widetilde{\Omega})) \, .$$

Moreover the estimate, thanks to (26) and Proposition 2.8,

$$\|v_{h,\delta} - v_h\|_{L^p(0,T;L^{1+\sigma}(\Omega))} \leqslant c_4 \, \|v_{h,\delta} - v_h\|_{L^p(\Omega \times (0,T);\lambda_h)} \leqslant c_5 \, \delta^{1-\alpha} \|v_h\|_{\mathcal{V}_h}$$

holds (here c_5 depends (only) on $n, p, K_1, K_2, \Omega, C_1, \alpha, \rho$) by which we obtain the compactness of $(v_h)_h$ in $L^p(0,T; L^1(\Omega))$. □

Corollary 5.4. *If there is $\gamma > 0$ such that $|\mu_h| \geqslant \gamma$ (or $\lambda_h \geqslant \gamma$) for every h and $\|v_h\|_{\mathcal{W}_h} \leqslant c$ for every h, the sequence $(v_h)_h$ is compact in $L^p(\Omega \times (0,T))$.*

Proof - From the proof of Theorem 5.2 we know that $(v_{h,\delta})_h$ is compact in $L^p(\Omega \times (0,T))$. Then we conclude thanks to the last estimate of the proof of Theorem 5.2, since we get (σ may be possibly changed and chosen such that $(1+\sigma)' \geqslant p$)

$$\gamma \|v_{h,\delta} - v_h\|_{L^p(\Omega \times (0,T))} \leqslant \|v_{h,\delta} - v_h\|_{L^p(\Omega \times (0,T);\mu_h)} \leqslant C \, \delta^{1-\alpha} \|v_h\|_{\mathcal{V}_h} \, . \qquad \square$$

Before stating the theorem which follows we recall the following lemma (the proof is analogous to that of Lemma 3.7 in [27]).

Lemma 5.5. *Let $k, m \in \mathbf{N}^*$, D be a bounded open set of $\mathbf{R}^k$, $p > 1$, ν, ν_h ($h = 1, 2, ...$) be positive weights on $\mathbf{R}^k$ such that*

$$\nu_h^{-1/(p-1)} \to \nu^{-1/(p-1)} \qquad \text{in } L^1(D)\text{-weak} \, .$$

Let $w_h \in \big(L^p(D;\nu_h)\big)^m$ ($h = 1, 2, ...$) be a sequence of functions such that

$$w_h \to w \qquad \text{in } \big(L^1(D)\big)^m\text{-weak} \, .$$

Then $w \in \big(L^p(D;\nu)\big)^m$ and

$$\|w\|_{L^p(D;\nu)} \leqslant \liminf_{h \to \infty} \|w_h\|_{L^p(D;\nu_h)} .$$

Theorem 5.6. *Consider a sequence $(v_h)_h$ with $v_h \in \mathcal{V}_h$. If*

$$\|v_h\|_{\mathcal{V}_h} \leqslant c \qquad \text{for every } h \in \mathbf{N}$$

then there is a function $v \in \mathcal{U}_1$ ($\mathcal{U}_1$ and $\mathcal{U}_2$ being the spaces defined in (55)) such that, up to subsequences,

(i) *$v_h \to v$ in $\mathcal{U}_1$-weak, $v \in \mathcal{V}_{\tilde\lambda}$ and $\|v\|_{\mathcal{V}_{\tilde\lambda}} \leqslant \liminf_{h\to\infty} \|v_h\|_{\mathcal{V}_h}$.*

If moreover $v_h \in \mathcal{W}_h$ and

$$\|v_h\|_{\mathcal{W}_h} \leqslant c \qquad \text{for every } h \in \mathbf{N}\,.$$

then, up to subsequences,

(ii) *$(\mathcal{R}_h v_h)' \to (\mathcal{R}v)'$ in $\mathcal{U}_2'$-weak, $(\mathcal{R}v)' \in \mathcal{V}_\lambda'$ and $\|(\mathcal{R}v)'\|_{\mathcal{V}_\lambda'} \leqslant \liminf_{h\to\infty} \|(\mathcal{R}_h v_h)'\|_{\mathcal{V}_h'}$.*

Proof - The proof of point i) can be obtained using Lemma 5.5 and following the proof of Proposition 3.8 in [27].
Applying Theorem 3.6 to the sequence $(\mathcal{R}_h v_h)_h$, one has that there is $\xi \in \mathcal{U}_2'$ such that, up to subsequences, $\mathcal{R}_h v_h \to \xi$ and $(\mathcal{R}_h v_h)' \to \xi'$ in $\mathcal{U}_2'$-weak. By Corollary 5.7 we get that for every $\phi \in C^1_c(\Omega \times (0,T))$

$$\langle \xi', \phi \rangle = \lim_h \langle (\mathcal{R}_h v_h)', \phi \rangle = -\lim_h \langle \mathcal{R}_h v_h, \phi' \rangle = \langle \mathcal{R}v, \phi' \rangle = \langle (\mathcal{R}v)', \phi \rangle$$

and then $\xi = \mathcal{R}v$. Then one concludes as for point i). $\square$

Corollary 5.7. *Consider two sequences $(v_h)_h$, $(w_h)_h$ such that $v_h, w_h \in \mathcal{V}_h$ and a constant c and two functions $v, w \in \mathcal{V}_\lambda$ such that*

$$\|v_h\|_{\mathcal{V}_h} \leqslant c\,, \quad \|w_h\|_{\mathcal{V}_h} \leqslant c\,, \quad v_h \to v \quad \text{and} \quad w_h \to w \quad \text{in } L^p(0,T;L^1(\Omega))\,.$$

Then, for every $\varphi \in L^\infty(\Omega \times (0,T))$, we have

$$\int_0^T \int_\Omega v_h w_h \varphi \mu_h dxdt \to \int_0^T \int_\Omega v w \varphi \mu dxdt\,.$$

Remark 5.8. - As particular cases we have:

i) if $w_h = v_h$ and $\varphi \equiv 1$ we have $\int_0^T \int_\Omega v_h^2 \mu_h dxdt \to \int_0^T \int_\Omega v^2 \mu dxdt$;

ii) indeed with a very similar proof one can also obtain
$$\int_0^T \int_\Omega v_h \varphi \mu_h dxdt \to \int_0^T \int_\Omega v \varphi \mu dxdt \qquad \text{for every } \varphi \in L^\infty.$$

Proof - First of all notice that, by Theorem 5.6 - (i), $v, w \in \mathcal{V}_{\tilde\lambda}$ and then in particular $v, w \in L^2(\Omega \times (0,T); \nu_+ + \nu_-)$, where $\nu_+ = \lim_h \mu_{h,+}$ and $\nu_- = \lim_h \mu_{h,-}$, respectively the positive and negative part of μ_h, and then $v, w \in L^2(\Omega \times (0,T); |\mu|)$ since $|\mu| \leqslant \nu_+ + \nu_-$. One can easily see that ($\overline{v}$ and v_δ defined in (29))

$$|v_{h,\delta}(x,t) - v_\delta(x,t)| = \Big| \int_{\mathbf{R}^n} \big(\overline{v}_h(\xi,t) - \overline{v}(\xi,t)\big) \rho_\delta(x-\xi) d\xi \Big| \leqslant$$
$$\leqslant \|\rho_\delta\|_\infty \int_\Omega |v_h(\xi,t) - v(\xi,t)| d\xi$$

from which we obtain that

$$v_{h,\delta} \to_h v_\delta \quad \text{in } L^p(0,T;L^\infty(\Omega))\,. \tag{68}$$

Using this estimate and following the proof of Corollary 4.7 in [22] one can conclude. □

Remark 5.9. - By point (i) of Theorem 5.6 and since λ and $\tilde{\lambda}$ are comparable (see Remark 2.13), if $v_h \to v$ satisfy asssumption of Theorem 5.6, we deduce that $v \in L^2(\Omega \times (0,T); \nu)$ for every weight $\nu \geqslant 0$ satisfying the Poincaré's inequality (22).

Now we show a series of result regarding the pointwise convergence of some functions involving the solutions of the problems

$$(P_h)_{f,\varphi,\psi} \begin{cases} (\mathcal{R}_h u)' + \mathcal{A}_h u = f \\ R_h(0)u(0) = \varphi \\ R_h(T)u(T) = \psi \end{cases} \qquad (P_h)_{g,\eta,\vartheta} \begin{cases} (\mathcal{R}_h u)' + \mathcal{A}_h u = g \\ R_h(0)u(0) = \eta \\ R_h(T)u(T) = \vartheta \end{cases}$$

for $f, g \in L^{p'}(0,T; L^n(\Omega))$ and $\varphi, \psi, \eta, \vartheta \in L^\infty(\Omega)$. As regards the following theorem we adapt the proof of the analogous result in [38]; to do this we need the two lemmas which follows.

Theorem 5.10. *Let $u_h = u_h(f,\varphi,\psi)$ be the solutions of the problems (P_h) above, $h \in \mathbf{N}$. Suppose* **A)** *holds. Then, called u the limit, up to subsequences, of u_h in $L^p(0,T; L^1(\Omega))$, we have that*

$$U_h(t) = \int_\Omega u_h^2(x,t)\mu_h(x,t)dx \ \to_h \ U(t) = \int_\Omega u^2(x,t)\mu(x,t)dx \qquad \text{in } C^0\big([0,T]\big).$$

Before we need a result, stated in Lemma 5.12, for which we need the following lemma.

Lemma 5.11. *Consider a sequence $(w_h)_h$ such that $w_h \in \mathcal{W}_h$ and $\|w_h\|_{\mathcal{W}_h} \leqslant c$ for every $h \in \mathbf{N}$ and for some positive constant c and such that $w_h \to w$ in $L^p(0,T; L^1(\Omega))$ for some $w \in L^p(0,T; L^1(\Omega))$. Then for every $\phi \in C^1_c(\Omega)$ the sequence*

$$\Big(t \mapsto \int_\Omega w_h(x,t)\phi(x)\mu_h(x,t)dx\Big)_h \qquad \text{is equibounded and equicontinuous}$$

and if w is the limit, up to subsequences, of $(w_h)_h$ in $L^p(0,T; L^1(\Omega))$, we get

$$\int_\Omega w_h(x,t)\phi(x)\mu_h(x,t)dx \to_h \int_\Omega w(x,t)\phi(x)\mu(x,t) \qquad \text{in } C^0([0,T])\,.$$

Proof - By Theorem 3.3 the functions we are considering are continuous and equibounded. It is then sufficient to prove the equicontinuity. By (38) we have

$$\begin{aligned} \Big| \int_\Omega w_h(x,t)\phi(x)\mu_h(x,t)dx - \int_\Omega w_h(x,s)\phi(x)\mu_h(x,s)dx \Big| = \\ = \Big| \int_s^t \langle (\mathcal{R}_h w_h)'(\tau), \phi \rangle_{V_h'(\tau)\times V_h'(\tau)} d\tau - \Big| \leqslant \\ \leqslant \int_s^t \|(\mathcal{R}_h w_h)'(\tau)\|_{V_h'(\tau)} \|\phi\|_{V_h(\tau)} d\tau \leqslant \\ \leqslant \|\nabla\phi\|_\infty \int_s^t \|(\mathcal{R}_h w_h)'(\tau)\|_{V_h'(\tau)} \Big(\int_\Omega \lambda_h(x,\tau)dx \Big)^{1/p} d\tau \leqslant \\ \leqslant c(p, C_1, C_2, C_3, \phi) \|(\mathcal{R}_h w_h)'\|_{\mathcal{V}_h'} |t-s|^{1/p}\,. \end{aligned}$$

Therefore the first part is proved. Now for every $\phi \in C^1_c(\Omega)$ define V^ϕ_h the function $\int_\Omega w_h(x,t)\phi(x)\mu_h(x,t)dx$. There is a continuous function V^ϕ such that

$$\int_\Omega w_h(x,t)\phi(x)\mu_h(x,t)dx = V^\phi_h(t) \to V^\phi(t) \qquad \text{in } C^0([0,T])\,.$$

Then we have that for every $\eta \in L^\infty(0,T)$

$$\int_0^T V^\phi_h(t)\eta(t)dt = \int_0^T \Big(\int_\Omega w_h(x,t)\phi(x)\mu_h(x,t)dx\Big)\eta(t)dt \to \int_0^T V^\phi(t)\eta(t)dt\,.$$

Since $w_h \to w$ in $L^p(0,T;L^1(\Omega))$ (see Remark 5.8 - ii)) we derive that for every $\eta \in L^\infty(0,T)$

$$\int_0^T\int_\Omega w_h(x,t)\phi(x)\mu_h(x,t)\eta(t)dxdt \to \int_0^T\int_\Omega w(x,t)\phi(x)\mu(x,t)\eta(t)dxdt$$

and then

$$V^\phi(t) = \int_\Omega w(x,t)\phi(x)\mu(x,t)dx \qquad \text{for every } t \in [0,T]\,.$$

□

Lemma 5.12. *Consider a sequence $(w_h)_h$, $w_h \in \mathcal{W}^{\rho,\varsigma}_h$ for some $\rho,\varsigma \in L^\infty(\Omega)$, $h \in$ **N**; suppose $w_h \to w$ in $L^p(0,T;L^1(\Omega))$. Then, defined $W_h(t) = \int_\Omega w_h^2(x,t)\mu_h(x,t)dx$ and $W(t) = \int_\Omega w^2(x,t)\mu(x,t)dx$,*

$$\limsup_h W_h(0) \leqslant W(0)\,, \qquad \liminf_h W_h(T) \geqslant W(T)\,.$$

Proof - There are two non-negative functions ν_+ and ν_- such that, up to subsequences,

$$\lim_h \mu_{h,+}(\cdot,0) = \nu_+\,, \qquad \lim_h \mu_{h,-}(\cdot,0) = \nu_- \qquad \text{in } L^r(\Omega)\text{-weak}$$

where $\mu_{h,+}(\cdot,0)$ and $\mu_{h,-}(\cdot,0)$ are respectively the positive and negative parts of $\mu_h(\cdot,0)$ and $\nu_+(x) - \nu_-(x) = \mu(x,0)$.
First of all notice that, since by the previous result and the fact that $w_h(x,0)\mu_{h,+}(x,0) = \rho(x)\mu_{h,+}(x,0)$ we have that

$$\int_\Omega w_h(x,0)\phi(x)\mu_h(x,0)dx \to \int_\Omega w(x,0)\phi(x)\mu(x,0)dx$$

and

$$\int_\Omega w_h(x,0)\phi(x)\mu_{h,+}(x,0)dx \to \int_\Omega \rho(x)\phi(x)\nu_+(x)dx\,, \tag{69}$$

we also get that there is a function ζ such that

$$\int_\Omega w_h(x,0)\phi(x)\mu_{h,-}(x,0)dx \to \int_\Omega \zeta(x)\phi(x)dx\,. \tag{70}$$

By density we can reach (69) and (70) for every $\phi \in L^\infty(\Omega)$. Since

$$\Big|\int_\Omega w_h(x,0)\phi(x)\mu_{h,-}(x,0)dx\Big| \leqslant \Big(\int_\Omega \phi^2(x)\mu_{h,-}(x,0)dx \int_\Omega w^2(x,0)\mu_{h,-}(x,0)dx\Big)^{1/2},$$

taking the lim inf we get

$$\Big|\int_\Omega \zeta(x)\phi(x)dx\Big| \leqslant \Big(\int_\Omega \phi^2(x)\nu_-(x)dx\Big)^{1/2}\Big(\liminf_h \int_\Omega w_h^2(x,0)\mu_{h,-}(x,0)dx\Big)^{1/2}.$$

First observe that, since this holds for every $\phi \in L^\infty(\Omega)$ and $w_h^2(x,0)\mu_{h,+}(x,0) \to \rho^2(x)\nu_+(x)$ implies that the sequence $(\int_\Omega w_h^2(x,0)\mu_{h,-}(x,0)dx)_h$ is bounded, we get that

$$\zeta = z\,\nu_-$$

for some function z. Moreover taking the supremum over all $\phi \in C_c^1(\Omega)$ such that $\int_\Omega \phi^2(x)\nu_-(x)dx \leqslant 1$ we conclude

$$\int_\Omega z^2(x)\nu_-(x)dx \leqslant \liminf_h \int_\Omega w_h^2(x,0)\mu_{h,-}(x,0)dx\,. \tag{71}$$

Now

$$W_h(0) = \int_\Omega w_h^2(x,0)\mu_h(x,0)dx = \int_\Omega \rho^2(x)\mu_{h,+}(x,0)dx - \int_\Omega w_h^2(x,0)\mu_{h,-}(x,0)dx\,.$$

Then

$$\begin{aligned}\limsup_h W_h(0) &\leqslant \limsup_h \int_\Omega \rho^2(x)\mu_{h,+}(x,0)dx + \limsup_h \Big(-\int_\Omega w_h^2(x,0)\mu_{h,-}(x,0)dx\Big)\\ &\leqslant \int_\Omega \rho^2(x)\nu_+(x)dx - \liminf_h \int_\Omega w_h^2(x,0)\mu_{h,-}(x,0)dx \qquad (72)\\ &\leqslant \int_\Omega \rho^2(x)\nu_+(x)dx - \int_\Omega z^2(x)\nu_-(x)dx\,.\end{aligned}$$

By Lemma 5.11, (69) and (70) we get that

$$w(x,0)\mu(x,0) = \rho(x)\nu_+(x) - z(x)\nu_-(x)\,. \tag{73}$$

Suppose first $\rho \equiv 0$. Then

$$z(x) = -\frac{1}{\nu_-(x)}w(x,0)\mu(x,0) \qquad \text{where } \nu_-(x) > 0$$

and then

$$-z^2(x)\nu_-(x) = \begin{cases} -w^2(x,0)\mu^2(x,0)\dfrac{1}{\nu_-(x)} & \text{if } \nu_-(x) > 0\,,\\ 0 & \text{if } \nu_-(x) = 0\,.\end{cases}$$

Since $\mu_h(x,0) \geqslant -\mu_{h,-}(x,0)$ taking the (weak) limit one has that $\mu(x,0) \geqslant -\nu_-(x)$ and then

$$-\frac{\mu(x,0)}{\nu_-(x)} \leqslant 1 \qquad \text{where } \nu_-(x) > 0\,.$$

Summing up we get that

$$-z^2(x)\nu_-(x) \begin{cases} \leqslant w^2(x,0)\mu(x,0) & \text{if } \nu_-(x) > 0\,,\\ = 0 & \text{if } \nu_-(x) = 0\,.\end{cases} \tag{74}$$

Since we are considering $\rho \equiv 0$ we also get that $w(x,0)\mu(x,0) = -z(x)\nu_-(x)$ and then $w(x,0)\mu(x,0) = 0$ where $\nu_- = 0$ we conclude that

$$-\int_\Omega z^2(x)\nu_-(x)dx \leqslant \int_\Omega w^2(x,0)\mu(x,0)dx$$

and by (72) we get the thesis for $\rho \equiv 0$. In the general case call $\tilde w_h(x,t)$ the functions $w_h(x,t) - \rho(x)$. Then

$$\limsup_h \int_\Omega \tilde w_h^2(x,0)\mu_h(x,0)dx \leqslant \int_\Omega \tilde w^2(x,0)\mu(x,0)dx\,,$$

where $\tilde{w}(x,0) = w(x,0) - \rho(x)$, that is

$$\limsup_h \Big[\int_\Omega w_h^2(x,0)\mu_h(x,0)dx + \int_\Omega \rho^2(x)\mu_h(x,0)dx - 2\int_\Omega w_h(x,0)\rho(x)\mu_h(x,0)dx \Big] \leqslant$$
$$\leqslant \int_\Omega w^2(x,0)\mu(x,0)dx + \int_\Omega \rho^2(x)\mu(x,0)dx - 2\int_\Omega w(x,0)\rho(x)\mu(x,0)dx\,.$$

By Lemma 5.11 and since $\mu_h(x,0) \to \mu(x,0)$ we conclude that

$$\limsup_h \int_\Omega w_h^2(x,0)\mu_h(x,0)dx \leqslant \int_\Omega w^2(x,0)\mu(x,0)dx\,.$$

In an analogous way one can prove the other inequality. □

Proof of Theorem 5.10 - Notice that, by Theorem 3.3, we get that the functions U_h are continuous. Moreover the sequence $(u_h)_h$ admits a subsequence, still denoted by $(u_h)_h$, compact in $L^p(0,T;L^1(\Omega))$. Say u the limit. By Theorem 5.6 we deduce that $u \in \mathcal{V}_\lambda$ and $(\mu u)' \in \mathcal{V}'_{\tilde{\lambda}}$. Finally, thanks to Remark 2.12, we get that $\mathcal{V}_\lambda = \mathcal{V}_{\tilde{\lambda}}$, and (see Remark 4.6) the pair $(\mu, \tilde{\lambda})$ satisfies (T.1); then, by Theorem 3.3, U is continuous. So now we have to prove that $U_h \to U$. By Remark 5.8 we get that $U_h \to U$ a.e. in $[0,T]$. Now the idea is to prove that $(U_h)_h$ are relatively compact in $C^0([0,T])$, from which the thesis will follow.
Now suppose, by contradiction, that U_h does not converge to U uniformly. Then there is $\epsilon > 0$ and a sequence $(t_k)_{k\in\mathbf{N}}$ with $t_k \to t_0$, such that

$$|U_k(t_k) - U(t_0)| > \epsilon \qquad \text{for every } k\,. \tag{75}$$

Notice that, since u_k are the solutions of (61), by Theorem 4.10 and (40) we get that $|U_k(t_k)| \leqslant c$ for some positive constant c. Then, up to subsequence, $U_k(t_k) \to a$ for some constant a. Since $|U_k(t_k) - U(t_0)| > \epsilon$ we have that there is a positive constant σ such that

$$|U(t_0) - a| \geqslant \sigma\,.$$

Then we can suppose that

$$\text{i)} \quad U_k(t_k) > U(t_0) \quad \text{for every } k \in \mathbf{N} \qquad \text{or}$$
$$\text{ii)} \quad U_k(t_k) < U(t_0) \quad \text{for every } k \in \mathbf{N}$$

otherwise we can extract a further subsequence such that i) or ii) holds.
Notice that

$$\text{i)} \quad \Longrightarrow \quad t_0 \neq 0 \qquad \text{and}$$
$$\text{ii)} \quad \Longrightarrow \quad t_0 \neq T\,.$$

Indeed multiply the equation $(\mathcal{R}_k u_k)'(s) + \mathcal{A}_k u_k(s) = f(s)$ in (60) by $u_k(s)$ and integrate in $[0,t_k]$ (φ is the initial condition) and obtain, by (39) and (47)-(48)

$$\frac{1}{2}\big(U_k(t_k) - U_k(0)\big) \leqslant \int_0^{t_k} \langle f(s), u_k(s)\rangle ds$$

If we suppose $f \in L^\infty(\Omega \times (0,T))$

$$\begin{aligned} U_k(t_k) &\leqslant U_k(0) + 2\,\|f\|_\infty \Big[\int_0^{t_k}\!\!\int_\Omega |u_k|^p(x,\tau)\lambda_k(x,\tau)dxd\tau\Big]^{\frac{1}{p}} \Big[\int_0^{t_k}\!\!\int_\Omega \lambda_k^{-1/(p-1)}(x,\tau)dxd\tau\Big]^{\frac{1}{p'}} \\ &\leqslant U_k(0) + c\max_{\tau\in[0,T]}\Big[\int_\Omega \lambda_k^{-1/(p-1)}(x,\tau)d\tau\Big]^{1/p'} {t_k}^{1/p'} \end{aligned}$$

where, thanks to Theorem 2.6 and Theorem 3.5, $c = c(p, L, K_2, r, C_2, C_3, T, \varphi, f)$. Then, using assumption (S.5) about λ_k, if t_0 were 0, using Lemma 5.12 we would have

$$\lim_k U_k(t_k) \leqslant \limsup_k U_k(0) \leqslant U(0) = U(t_0)$$

and this is impossible if i) and (75) hold. In an analogous way, integrating in $[t_k, T]$, and using Lemma 5.12, one can prove that if ii) holds then $t_0 \neq T$.

Suppose now that i) holds. Then $t_0 > 0$ and we can fix $\eta > 0$ in such a way $t_k - \eta > 0$, at least definitively. Multiply the equation $(\mathcal{R}_k u_k)'(s) + \mathcal{A}_k u_k(s) = f(s)$ by $u_k(s)$ and integrate in $[t_k - \eta, t_k]$ and obtain, as above,

$$U_k(t_k) - U(t_k - \eta) \leqslant c\,\eta^{1/p'}$$

We have, for sufficiently big k,

$$U_k(t_k - \eta) \geqslant U_k(t_k) - c\,\eta^{1/p'} \geqslant a - \frac{\sigma}{4} - c\,\eta^{1/p'}\,.$$

By i) we deduce that the inequality $|U(t_0) - a| \geqslant \sigma$ is indeed $a \geqslant U(t_0) + \sigma$. Then, choosing η_0 sufficiently small and $\eta \in (0, \eta_0)$, we have

$$U_k(t_k - \eta) \geqslant U(t_0) + \sigma - \frac{\sigma}{4} - \frac{\sigma}{4} = U(t_0) + \frac{\sigma}{2}$$

By continuity of U we also have $U(t_0) \geqslant U(t_k - \eta) - \sigma/4$ and then finally

$$U_k(t_k - \eta) \geqslant U(t_k - \eta) + \frac{\sigma}{4}$$

for every η sufficiently small. But $U_k(s) \to U(s)$ for almost every $s \in [0, T]$, while for every $\eta \in (0, \eta_0)$

$$U_k(t_k - \eta) - U(t_k - \eta) \geqslant \frac{\sigma}{4}\,.$$

This concludes the proof if i) holds. If ii) holds, and then $t_0 \neq T$, integrating in $[t_k, t_k + \eta]$ we get a positive constant c such that, analogously as before, for $\eta \in (0, \eta_0)$,

$$-U_k(t_k + \eta) \geqslant -U_k(T) - c\eta^{1/p'} \geqslant -U(t_0) + \frac{\sigma}{2}$$

and again, similarly as before,

$$-U_k(t_k + \eta) + U(t_k + \eta) \geqslant \frac{\sigma}{4}$$

for every $\eta \in (0, \eta_0)$, which is impossible. □

Lemma 5.13. *Let u_h and v_h be the solutions of the problems $(P_h)_{f,\varphi,\psi}$ and $(P_h)_{g,\eta,\vartheta}$ respectively, $h \in \mathbf{N}$. Then, called u and v the limits, up to subsequences, of u_h and v_h respectively in $L^p(0, T; L^1(\Omega))$, we have that*

$$\lim_h \int_\Omega u_h(x,0) v_h(x,0) \mu_h(x,0) dx \to \int_\Omega u(x,0) v(x,0) \mu(x,0) dx\,,$$
$$\lim_h \int_\Omega u_h(x,T) v_h(x,T) \mu_h(x,T) dx \to \int_\Omega u(x,T) v(x,T) \mu(x,T) dx\,.$$

Proof - Since

$$\limsup_h \int_\Omega 2u_h(x,0)v_h(x,0)\mu_h(x,0)dx \leqslant$$

$$\leqslant \limsup_h \int_\Omega (u_h(x,0)+v_h(x,0))^2\mu_h(x,0)dx+$$

$$-\lim_h \int_\Omega u_h(x,0)^2\mu_h(x,0)dx - \int_\Omega v_h(x,0)^2\mu_h(x,0)dx$$

by which, applying Lemma 5.12 to the sequence $w_h = u_h + v_h$ and then using the first step we get

$$\limsup_h \int_\Omega u_h(x,0)v_h(x,0)\mu_h(x,0)dx \leqslant \int_\Omega u(x,0)v(x,0)\mu(x,0)dx\,.$$

The same arguement applied to the sequence $w_h = u_h - v_h$ gives

$$\int_\Omega u(x,0)v(x,0)\mu(x,0)dx \leqslant \liminf_h \int_\Omega u_h(x,0)v_h(x,0)\mu_h(x,0)dx$$

and consequently

$$\lim_h \int_\Omega u_h(x,0)v_h(x,0)\mu_h(x,0)dx = \int_\Omega u(x,0)v(x,0)\mu(x,0)dx\,.$$

In an analogous way one can prove

$$\lim_h \int_\Omega u_h(x,T)v_h(x,T)\mu_h(x,T)dx = \int_\Omega u(x,T)v(x,T)\mu(x,T)dx\,. \qquad \square$$

We now give an important corollary of Lemma 5.11 and Theorem 5.10, which will give us information about the initial/final conditions of the limit problem.

Corollary 5.14. *Let $u_h = u_h(f,\varphi,\psi)$ be the solutions of the problems (P_h) above, $h \in \mathbf{N}$. Suppose* **A)** *holds. Then, called $u(f,\varphi,\psi)$ the limit, up to subsequences, of u_h in $L^p(0,T;L^1(\Omega))$, we have that*

$$u(f,\varphi,\psi)(0) = \varphi \quad \text{in } \{x\in\Omega \,|\, \mu(x,0)>0\}$$
$$u(f,\varphi,\psi)(T) = \psi \quad \text{in } \{x\in\Omega \,|\, \mu(x,T)<0\}\,.$$

Proof - We show the proof for $t=0$, being the proof for $t=T$ similar. First of all notice that if we substitute φ with a whatever function $\tilde\varphi$ such that

$$\tilde\varphi \equiv \varphi \qquad \text{in } \omega_+(0) := \{x\in\Omega \,|\, \mu(x,0)>0\}$$

the solutions $u_h(\varphi,\psi,f)$ and $u_h(\tilde\varphi,\psi,f)$ have the same limit. Indeed, by (49), we get

$$\|u_h(\varphi,\psi,f) - u_h(\tilde\varphi,\psi,f)\|_{\mathcal{W}_h} \leqslant c\int_\Omega (\varphi(x)-\tilde\varphi(x))^2\mu_{h,+}(x,0)dx$$

$$= c\int_{\omega_h^+(0)\cap(\omega_+(0))^c} (\varphi(x)-\tilde\varphi(x))^2\mu_{h,+}(x,0)dx$$

which goes to zero. Then if the limit in $L^p(0,T;L^1(\Omega))$ of $(u_h(\varphi,\psi,f))_h$ is u, the same holds for the sequence $(u_h(\tilde\varphi,\psi,f))_h$.

The first step is to show that for every $\phi \in L^\infty(\Omega)$

$$\lim_h \int_\Omega \phi(x) u_h^2(x,0)\mu_h(x,0)dx = \int_\Omega \phi(x) u^2(x,0)\mu_+(x,0)dx\,. \tag{76}$$

Consider, thanks to Proposition 3.2, a family depending on $\delta > 0$ of sequences of functions $u_{h,\delta}$ and a family $(u_\delta)_{\delta>0}$ such that

$$\begin{aligned} &\|u_h - u_{h,\delta}\|_{\mathcal{W}_h} < \delta\,, \qquad \|u - u_\delta\|_{\mathcal{W}_\mathcal{R}} < \delta\,, \\ &\lim_h \|u_{h,\delta} - u_\delta\|_{L^\infty(\Omega\times(0,T))} = 0 \qquad \text{for every fixed } \delta > 0\,. \end{aligned} \tag{77}$$

Since the functions $t \mapsto \int_\Omega u_h^2(x,t)\mu_h(x,t)dx$ and $t \mapsto \int_\Omega u_h(x,t)\phi(x)\mu_h(x,t)dx$ are continuous (for every $\phi \in L^\infty(\Omega)$) the same holds for the analogous functions when u_h is replaced by $u_{h,\delta}$; analogously for u and u_δ. Then we can also choose $u_{h,\delta}$ and u_δ in such a way that for $t = 0$ (the same holds for $t = T$, but only for 0 and T)

$$\begin{aligned} &\int_\Omega \big(u_h(x,0) - u_{h,\delta}(x,0)\big)^2 |\mu_h|(x,0)dx < \delta \\ &\int_\Omega \big(u(x,0) - u_\delta(x,0)\big)^2 |\mu|(x,0)dx < \delta \end{aligned} \tag{78}$$

since $u_h \in \mathcal{W}_h^{\varphi,\psi}$. Then we have

$$\begin{aligned} \int_\Omega \phi(x) u_h^2(x,0)\mu_h(x,0)dx - \int_\Omega \phi(x) u^2(x,0)\mu_+(x,0)dx = \\ = \int_\Omega \phi(x) u_h^2(x,0)\mu_h(x,0)dx - \int_\Omega \phi(x) u_{h,\delta}^2(x,0)\mu_h(x,0)dx + \quad (1^\circ) \\ + \int_\Omega \phi(x) u_{h,\delta}^2(x,0)\mu_h(x,0)dx - \int_\Omega \phi(x) u_\delta^2(x)\mu_h(x,0)dx + \quad (2^\circ) \\ + \int_\Omega \phi(x) u_\delta^2(x,0)\mu_h(x,0)dx - \int_\Omega \phi(x) u_\delta^2(x,0)\mu(x,0)dx + \quad (3^\circ) \\ + \int_\Omega \phi(x) u_\delta^2(x,0)\mu(x,0)dx - \int_\Omega \phi(x) u^2(x,0)\mu(x,0)dx \qquad (4^\circ) \end{aligned}$$

It is easy to check that the first and the fourth terms are $O(\delta)$ by (78) and the third is going to zero since $\mu_h(\cdot,0) \to \mu(\cdot,0)$ weakly in $L^1(\Omega)$. To estimate the second term it is sufficient to use (77) and (78) and then taking the limit for $h \to +\infty$ and since what said holds for every $\delta > 0$ (76) is proved.

Now we want to show that $\int_{\omega_+(0)} (u(x,0) - \varphi(x))^2 dx = 0$. Using (76) with $\phi = \chi_{\{\mu(\cdot,0)>0\}}$ we get

$$\lim_h \int_{\omega_+(0)} u_h^2(x,0)\mu_h(x,0)dx = \int_{\omega_+(0)} u^2(x,0)\mu_+(x,0)dx\,.$$

Using that, the previous computations and (78) we have

$$\begin{aligned}\int_{\omega_+(0)}\big(u(x,0)-\varphi(x)\big)^2\mu_+(x,0)dx &= \lim_h\int_{\omega_+(0)}\big(u_h(x,0)-\varphi(x)\big)^2\mu_h(x,0)dx\leqslant\\ &=\lim_h\int_{\omega_+(0)\cap(\omega_h^+(0))^c}\big(u_h(x,0)-\varphi(x)\big)^2\mu_h(x,0)dx=\\ &=\lim_h\int_{\omega_+(0)\cap(\omega_h^+(0))^c}\big(u_{h,\delta}(x,0)-\varphi(x)\big)^2\mu_h(x,0)dx+\delta\leqslant\\ &\leqslant\limsup_h\|u_{h,\delta}(x,0)-\varphi(x)\|^2_{L^\infty(\Omega)}\int_{\omega_+(0)\cap(\omega_h^+(0))^c}\mu_h(x,0)dx+\delta\leqslant\\ &\leqslant c\,\limsup_h\int_{\omega_+(0)\cap(\omega_h^+(0))^c}\mu_h(x,0)dx+\delta\leqslant\delta\end{aligned}$$

since $\mu_h(x,0)\leqslant 0$ in $\omega_+(0)\cap(\omega_h^+(0))^c$ and where c depends only on $\|\varphi\|_{L^\infty(\Omega)}$ and $\|u_\delta(\cdot,0)\|_{L^\infty(\Omega)}$. Since δ is arbitrary we conclude that $u(\cdot,0)=\varphi$ in $\omega_+(0)$. □

Finally we give an important result, useful later for many of the results of the following sections.

Theorem 5.15. *Consider two sequences $\rho_h,\nu_h\in\mathcal{F}(\lambda_h)$, $h\in\mathbf{N}$, two sequences $(u_h)_h,(v_h)_h$ such that*

$$\|u_h\|_{\mathcal{V}_h}\leqslant c\,,\ \ \|(\rho_h u_h)'\|_{\mathcal{V}_h'}\leqslant c\,,\ \ \|v_h\|_{\mathcal{V}_h}\leqslant c\,,\ \ \|(\nu_h v_h)'\|_{\mathcal{V}_h'}\leqslant c\,,$$
$$u_h\to u\quad\textit{and}\quad v_h\to v\qquad\textit{in}\ \ \mathcal{U}\textit{-weak}.$$

Consider two sequences $(A_h)_h,(B_h)_h\subset L^{p'}(\Omega\times(0,T),\lambda_h^{-1/(p-1)})$ such that

$$\|A_h\|_{L^{p'}(\Omega\times(0,T),\lambda_h^{-1/(p-1)})}\leqslant c\,,\|B_h\|_{L^{p'}(\Omega\times(0,T),\lambda_h^{-1/(p-1)})}\leqslant c$$
$$A_h\to M\quad\textit{and}\quad B_h\to N\qquad\textit{in}\quad L^{p'}(0,T;(L^1(\Omega))^n)\textit{-weak}.$$

Finally suppose that

$$(\rho_h u_h)'-\mathrm{div}A_h\quad\textit{and}\quad(\nu_h v_h)'-\mathrm{div}B_h\qquad\textit{compact in}\quad L^{p'}(0,T;L^\infty(\Omega))\,.$$

Then $(A_h-B_h,Du_h-Dv_h)\to(M-N,Du-Dv)$ in $\mathcal{D}'(\Omega\times(0,T))$.

Remark 5.16. - Indeed it is sufficient to consider $(\rho_h u_h)'-\mathrm{div}A_h$ and $(\nu_h v_h)'-\mathrm{div}B_h$ compact in $L^{p'}(0,T;L^{(1+\sigma)'}(\Omega))$ where σ is the number fixed at the beginning of Section 4 (the quantity u_h-v_h is compact in $L^p(0,T;L^{1+\sigma}(\Omega))$, see the proof of Theorem 5.2).

Proof - Up to a choice of a subsequence we can suppose that $\rho_h\to\rho$, $\nu_h\to\nu$ in $C^0([0,T];L^1(\Omega)\text{-weak})$. Multiply the quantity $((\rho_h u_h)'-\mathrm{div}A_h)-((\nu_h v_h)'-\mathrm{div}B_h)$ by $(u_h-v_h)\varphi$ for a fixed $\varphi\in C_c^\infty(\Omega\times(0,T))$, by which we derive

$$\begin{aligned}&\int_0^T\int_\Omega(A_h-B_h,Du_h-Dv_h)\varphi dxdt=\\ (79)\quad&=\big\langle((\rho_h u_h)'-\mathrm{div}A_h)-((\nu_h v_h)'-\mathrm{div}B_h),(u_h-v_h)\varphi\big\rangle-\\ &\quad-\big\langle(\rho_h u_h)'-(\nu_h v_h)',(u_h-v_h)\varphi\big\rangle-\int_0^T\int_\Omega(A_h-B_h,D\varphi)(u_h-v_h)dxdt\end{aligned}$$

Consider the right hand side terms. Since $(u_h - v_h)\varphi \in L^p(0,T;L^{1+\sigma}(\Omega))$ (see (16)), by assumption we have that there are two functions $f, g \in L^{p'}(0,T;L^\infty(\Omega))$ such that

$$\big\langle ((\rho_h u_h)' - \mathrm{div} A_h) - ((\nu_h v_h)' - \mathrm{div} B_h), (u_h - v_h)\varphi \big\rangle \to \big\langle f - g, (u - v)\varphi \big\rangle\,.$$

As regards the second term: denoting by $\mathcal{R}_{\rho_h}$ and $\mathcal{R}_{\nu_h}$ the operators associated to ρ_h and ν_h as defined in (35), one has that

$$\begin{aligned} 2\langle (\mathcal{R}_{\rho_h} u_h)', u_h\varphi\rangle &= \langle \mathcal{R}'_{\rho_h} u_h, u_h\varphi\rangle - (\mathcal{R}_{\rho_h} u_h, u_h\varphi) + \\ &\quad + \int_\Omega \varphi(x,T) u_h^2(x,T)\rho_h(x,T)dx - \int_\Omega \varphi(x,0) u_h^2(x,0)\rho_h(x,0)dx\,. \end{aligned}$$

To prove that the left hand side term converges we look at the right hand side. The second term converges thanks to Corollary 5.7. The third and fourth terms converge thanks to (76). As regards the first term, adapting Lemma 2.2 in [26] we can relate again to Corollary 5.7 and finally prove that the second term on the right hand side of (79) converges to $\langle (\rho u)' - (\nu v)', (u-v)\varphi\rangle$. For the third term we consider the approximating functions $u_{h,\delta}$ and $v_{h,\delta}$ defined in (29) which satisfy (68) and write

$$\begin{aligned} \int_0^T\!\!\int_\Omega (A_h - B_h, D\varphi)(u_h - v_h)dxdt &= \int_0^T\!\!\int_\Omega (A_h - B_h, D\varphi)(u_{h,\delta} - v_{h,\delta})dxdt + \\ &\quad + \int_0^T\!\!\int_\Omega (A_h - B_h, D\varphi)[(u_h - u_{h,\delta}) - (v_h - v_{h,\delta})]dxdt\,. \end{aligned}$$

Then by Proposition 2.8 and (68) if one lets $h \to \infty$ and then $\delta \to 0$ obtain

$$\int_0^T\!\!\int_\Omega (M - N, D\varphi)(u - v)dxdt\,.$$

Thus we get that

$$\begin{aligned} &\int_0^T\!\!\int_\Omega (A_h - B_h, Du_h - Dv_h)\varphi dxdt \to \\ &\quad \to \big\langle (f - (\rho u)') - (g - (\nu v)'), (u-v)\varphi \big\rangle - \int_0^T\!\!\int_\Omega (M - N, D\varphi)(u - v)dxdt\,. \end{aligned}$$

Since

$$-\mathrm{div} A_h \to f - (\rho u)' \quad \text{and} \quad -\mathrm{div} B_h \to g - (\nu v)' \quad \text{in } \mathcal{D}'$$

we conclude that $-\mathrm{div} A_h \to -\mathrm{div} M$ and $-\mathrm{div} B_h \to -\mathrm{div} N$ and then the thesis. □

Proposition 5.17. *For every* $f \in L^{p'}((0,T);L^n(\Omega))$

$$\|f\|_{\mathcal{V}'_\lambda} \leqslant \liminf_{h\to\infty} \|f\|_{\mathcal{V}'_{\lambda_h}} \leqslant \limsup_{h\to\infty} \|f\|_{\mathcal{V}'_{\lambda_h}} \leqslant \|f\|_{\mathcal{V}'_\lambda} \leqslant K_1^{1/p}\|f\|_{\mathcal{V}'_\lambda}\,.$$

Proof - The proof follows by arguing as in Lemma 1.4 (ii) in [11]. □

6. The PG-compactness result

In this section we will give two definition of convergence (Definition 6.1 and Definition 6.12) for sequences of operators $\mathcal{P}_h u = (\mu_h u)_t - \mathrm{div}(a_h(Du))$ and some results of compactness with respect to these. First (see Theorem 6.8) we give a result under the assumptions **A)** made at the beginning of the Section 5

$$\mu \neq 0 \qquad \text{a.e. in } \Omega \times (0,T)$$

so that we can use all the results of the previous section. Later we give the result for a generic sequence $(\mu_h)_h$ dropping assumption **A)** (see Theorem 6.9). Finally we give a result which shows that the convergence we are going to define below is independent of the coefficients μ_h (Theorem 6.13).

Definition 6.1. *Consider a sequence $(a_h)_h$ and a Carathéodory function a contained in $\mathcal{N}_{\Omega\times(0,T)}(p, L, K_1, C_1, F)$, a sequence $(\mu_h)_h$ with, for $\lambda_h \in \Lambda(a_h)$, $\mu_h \in \mathcal{F}(\lambda_h, K_2, r, \alpha, \delta_1, \delta_2, C_2, C_3, F)$ and a function μ. We say that the sequence $(\mu_h, a_h)_h$ G-converges to (μ, a) in $\Omega \times (0,T)$, and we write*

$$G\text{-}\lim_{h\to+\infty} (\mu_h, a_h) = (\mu, a) \quad \text{or} \quad (\mu_h, a_h) \xrightarrow{G} (\mu, a) \quad \text{in } \Omega\times(0,T),$$

if for every $f \in L^{p'}(0,T; L^n(\Omega))$ and $\varphi, \psi \in L^\infty(\Omega)$ what follows holds:

$$\begin{aligned} & u_h \to u && \text{in } L^p(0,T;L^1(\Omega)) \\ & a_h(\cdot,\cdot,Du_h) \to a(\cdot,\cdot,Du) && \text{in } L^{p'}(0,T;(L^1(\Omega))^n)\text{-weak,} \end{aligned}$$

where u_h and u denote respectively the solutions

$$(80) \quad \begin{aligned} (P_h)_{f,\varphi,\psi} & \begin{cases} \dfrac{\partial}{\partial t}\big(\mu_h(x,t)v\big) - \mathrm{div}\big(a_h(x,t,Dv)\big) = f & \text{in } \Omega\times(0,T) \\ v = 0 & \text{in } \partial\Omega\times(0,T) \\ v(x,0) = \varphi(x) & \text{in } \omega_h^+(0) \\ v(x,T) = \psi(x) & \text{in } \omega_h^-(T)\,. \end{cases} \\ (P)_{f,\varphi,\psi} & \begin{cases} \dfrac{\partial}{\partial t}\big(\mu(x,t)v\big) - \mathrm{div}\big(a(x,t,Dv)\big) = f & \text{in } \Omega\times(0,T) \\ v = 0 & \text{in } \partial\Omega\times(0,T) \\ v(x,0) = \varphi(x) & \text{in } \omega^+(0) \\ v(x,T) = \psi(x) & \text{in } \omega^-(T)\,. \end{cases} \end{aligned}$$

where $\omega^+(0) := \big\{x \in \Omega \;\big|\; \mu(x,0) > 0\big\}$ and $\omega^-(T) := \big\{x \in \Omega \;\big|\; \mu(x,T) < 0\big\}$.

Lemma 6.2. *Given a sequence $(\mu_h, a_h)_h$ as in Definition 6.1, if $(\mu_h, a_h)_h$ G-converges to (μ, a) in $\Omega\times(0,T)$ then $\mu_h \to \mu$ in $L^1(\Omega\times(0,T))$-weak.*

Proof - Consider the operator $\mathcal{Q} : L^{p'}(0,T;L^n(\Omega)) \times L^\infty(\Omega) \to \mathcal{V}_\lambda$ where $\lambda \in \Lambda(a)$ defined as $\mathcal{Q}(f,\varphi)$ is the solution of $(P)_{f,\varphi}$ in (80). Denote by u_h the solution of $(P_h)_{f,\varphi}$ in (80). We get

$$\int_0^T \int_\Omega u_h \frac{\partial \phi}{\partial t} \mu_h \, dxdt \to \int_0^T \int_\Omega u \frac{\partial \phi}{\partial t} \mu \, dxdt$$

for every $\phi \in C^1_c(\Omega \times (0,T))$. On the other hand for every subsequence $(\mu_{h_j})_j$, by (T.2), there is a subsequence $(\mu_{h_{j_k}})_k$ and a weight ν such that $(\mu_{h_{j_k}})_k \to \nu$ in $L^1(\Omega \times (0,T))$-weak and, by Corollary 5.7,

$$\int_0^T \int_\Omega u_{h_{j_k}} \frac{\partial \phi}{\partial t} \mu_{h_{j_k}} \, dxdt \to \int_0^T \int_\Omega u \frac{\partial \phi}{\partial t} \nu \, dxdt \, .$$

Since $\mathcal{Q}(L^{p'}(0,T; L^n(\Omega)) \times L^\infty(\Omega))$ is dense in $\mathcal{V}_\lambda$ in particular one concludes that

$$\int_0^T \int_\Omega u \frac{\partial \phi}{\partial t} (\nu - \mu) \, dxdt = 0 \qquad \text{for every } u \in C^1_c(\Omega \times (0,T)) \, .$$

Then $\nu = \mu$ and since this holds for every subsequence $(\mu_{h_j})_j$ of $(\mu_h)_h$ we conclude that $\mu_h \to \mu$. □

Lemma 6.3. *Suppose to have a sequence $(a_h)_h \subset \mathcal{N}_{\Omega \times (0,T)}$ and, for $\lambda_h \in \Lambda(a_h)$, two sequences $\rho_h, \nu_h \in \mathcal{F}(\lambda_h)$, $h \in \mathbf{N}$, satisfying* **A)**. *Suppose $(\rho_h, a_h)_h$ G-converges to (ρ, a_μ) and $(\nu_h, a_h)_h$ G-converges to (ν, a_ν) in $\Omega \times (0,T)$. Then $a_\rho(x,t,\xi) = a_\nu(x,t,\xi)$ for a.e. $(x,t) \in \Omega \times (0,T)$ and for every $\xi \in \mathbf{R}^n$.*

Proof - Consider $f, g \in L^{p'}(0,T; L^n(\Omega))$, $\varphi, \psi, \eta, \vartheta \in L^\infty(\Omega)$ and denote by $u_h = u_h(f, \varphi, \psi)$ and $v_h = v_h(g, \eta, \vartheta)$ the solutions of problems $(P_h)_{f,\varphi,\psi}$ and $(P_h)_{g,\eta,\vartheta}$ in (80) respectively with coefficients ρ_h and ν_h in the place of μ_h. Denote by u and v the limits, respectively, of u_h and v_h. Now consider a $\phi \in C^\infty_c(\Omega \times (0,T))$, $\phi \geqslant 0$: then

$$\big(a_h(x,t,Du_h) - a_h(x,t,Dv_h), Du_h - Dv_h\big)\phi \geqslant 0 \qquad \text{a.e. in } \Omega \times (0,T) \, .$$

Integrating this quantity and taking the limit, by Theorem 5.6 and Theorem 5.15 we deduce that

$$\int_0^T \int_\Omega \big(a_\rho(x,t,Du) - a_\nu(x,t,Dv), Du - Dv\big)\phi \, dxdt \geqslant 0 \qquad \text{for every } \phi \geqslant 0$$

and then

$$\big(a_\rho(x,t,Du) - a_\nu(x,t,Dv), Du - Dv\big) \geqslant 0 \qquad \text{a.e. in } \Omega \times (0,T) \, .$$

Since $\mathcal{Q}_{\rho,a_\rho}(L^{p'}(0,T; L^n(\Omega)) \times L^\infty(\Omega))$ and $\mathcal{Q}_{\nu,a_\nu}(L^{p'}(0,T; L^n(\Omega)) \times L^\infty(\Omega))$ are dense in $\mathcal{V}_\lambda$ ($\mathcal{Q}_{\rho,a_\rho}$ and $\mathcal{Q}_{\nu,a_\nu}$ defined analogously to the $\mathcal{Q}$ of the proof of Lemma 6.2) the inequality above holds for every $u, v \in \mathcal{V}_\lambda$.
Consider then $v, w \in \mathcal{V}_\lambda$ and define $u := v + \tau w$ with $\tau > 0$: we get $(a_\rho(x,t,Dv + \tau Dw) - a_\nu(x,t,Dv), Dw) \geqslant 0$ a.e. in $\Omega \times (0,T)$. If we let $\tau \to 0$ we obtain

$$\big(a_\rho(x,t,Dv) - a_\nu(x,t,Dv), Dw\big) \geqslant 0 \qquad \text{for every } v, w \in \mathcal{V}_\lambda$$

and for a.e. $(x,t) \in \Omega \times (0,T)$. Then taking $\phi \in C^1_c(\Omega \times (0,T))$ with $\phi \equiv 1$ in $\Omega' \times I \subset\subset \Omega \times (0,T)$, $\xi, \eta \in \mathbf{R}^n$ and inserting $v(x) = (\xi, x)\phi(x)$, $w(x) = (\eta, x)\phi(x)$ in the above inequality we get

$$\big(a_\rho(x,t,\xi) - a_\nu(x,t,\xi), \eta\big) \geqslant 0 \qquad \text{a.e. in } \Omega' \times I$$

and for every $\xi, \eta \in \mathbf{R}^n$. Since Ω', I, ξ, η are arbitrary we derive that

$$a_\rho(x,t,\xi) = a_\nu(x,t,\xi) \qquad \text{for every } \xi \in \mathbf{R}^n \text{ and a.e. in } \Omega \times (0,T) \, .$$ □

From the following proposition the uniqueness of the G-limit follows immediatly.

Proposition 6.4. *Consider two sequences* $(a_h)_{h\in\mathbf{N}} \subset \mathcal{N}_{\Omega\times(0,T)}$, $\mu_h \in \mathcal{F}(\lambda_h)$ *for* $\lambda_h \in \Lambda(a_h)$. *Consider* $\Omega_1, \Omega_2 \subset \Omega$ *open sets with Lipschitz boundary and* $I_1, I_2 \subset [0,T]$ *intervals. Suppose*

$$(\mu_h, a_h) \xrightarrow{G} (\nu_i, b_i) \qquad \text{on } \Omega_i \times I_i,\ (i = 1,2)$$

for suitable $b_i \in \mathcal{N}_{\Omega\times(0,T)}$ *and* ν_i *weights* $(i = 1,2)$. *Then* $\nu_1 = \nu_2$ *and* $b_1 = b_2$ *a.e. in* $\Omega_1 \times I_1 \cap \Omega_2 \times I_2$.

Proof - The proof that $b_1 = b_2$ in $\Omega_1 \times I_1 \cap \Omega_2 \times I_2$ can be easily obtained following the proof of Proposition 2.9 in [11]. By Lemma 6.2 we also conclude that $\nu_1 = \nu_2$ in $\Omega_1 \times I_1 \cap \Omega_2 \times I_2$. □

Before proving the compactness result we need several preliminary steps. In all these steps we will consider $a_h, \lambda_h, \mu_h, \lambda, \tilde{\lambda}, \mu$ and all the constants considered in Definition 6.1 fixed as done at the beginning of Section 5. For simplicicty we define the following operators

$$\begin{aligned} &\mathcal{P}_h : \mathcal{W}_h^{\varphi,\psi} \to \mathcal{V}_h', \qquad \mathcal{P}_h v := \frac{\partial}{\partial t}\big(\mu_h(x,t)v\big) - \mathrm{div}\big(a_h(x,t,Dv)\big) \\ &\mathcal{A}_h : \mathcal{V}_h \to \mathcal{V}_h', \qquad \mathcal{A}_h v := -\mathrm{div}\big(a_h(x,t,Dv)\big)\,. \end{aligned} \tag{81}$$

Lemma 6.5. *Denote by* $u_h(f,\varphi,\psi)$ *the solution belonging to* $\mathcal{W}_h^{\varphi,\psi}$ *of* (61). *There exist three continuous operators* $\mathcal{B} : \mathcal{V}_\lambda' \times L^2(\omega^+(0), \mu_+(\cdot,0)) \times L^2(\omega^-(T), \mu_-(\cdot,T)) \to \mathcal{W}_{\mu,\lambda}$, $\mathcal{K} : \mathcal{V}_\lambda' \times L^2(\omega^+(0), \mu_+(\cdot,0)) \times L^2(\omega^-(T), \mu_-(\cdot,T)) \to \mathcal{V}_\lambda'$ *and* $M : \mathcal{V}_\lambda' \times L^2(\omega^+(0), \mu_+(\cdot,0)) \times L^2(\omega^-(T), \mu_-(\cdot,T)) \to L^{p'}(\Omega \times (0,T), \lambda^{-1/(p-1)})$, *such that, up to subsequence, for every* $f \in L^{p'}(0,T; L^n(\Omega))$, $\varphi, \psi \in L^{(1+\sigma)'}(\Omega)$,

$$\begin{aligned} &u_h(f,\varphi,\psi) \to \mathcal{B}(f,\varphi,\psi) \qquad \text{in } L^p(0,T;L^1(\Omega))\,, \\ &a_h(\cdot,\cdot,Du_h(f,\varphi,\psi)) \to M(f,\varphi,\psi) \qquad \text{in } L^2(0,T;(L^1(\Omega))^n)\text{-weak}. \end{aligned}$$

Moreover $\mathcal{B}$ *satisfies*

$$\begin{aligned} &\mathcal{B}(f,\varphi,\psi)(0) = \varphi \qquad \text{in } L^2(\omega^+(0), \mu_+(0)) \\ &\mathcal{B}(f,\varphi,\psi)(T) = \psi \qquad \text{in } L^2(\omega^-(T), \mu_-(T)) \end{aligned}$$

and $\mathcal{K}$ *satisfies*

$$\begin{aligned} &\frac{d}{dt}(\mathcal{B}(f,\varphi,\psi)) + \mathcal{K}(f,\varphi,\psi) = f\,, \quad \text{in } \mathcal{V}_\lambda' \\ &\mathcal{K}(f,\varphi,\psi) = -\mathrm{div}\ M(f,\varphi,\psi) \quad \text{in } \mathcal{V}_\lambda' \end{aligned}$$

for every $f \in \mathcal{V}_\lambda'$, $\varphi \in L^2(\omega^+(0), \mu_+(0))$, $\psi \in L^2(\omega^-(T), \mu_-(T))$.

Proof - Fix X countable and dense in $L^{p'}(0,T;L^n(\Omega)) \times L^{(1+\sigma)'}(\Omega) \times L^{(1+\sigma)'}(\Omega)$. We can consider $L^{(1+\sigma)'}(\Omega)$ instead of $L^\infty(\Omega)$ (see Remark 4.8). First we define $\mathcal{B}$. By Theorem 4.10 we derive, for $(f,\varphi,\psi) \in X$, that $\|u_h(f,\varphi,\psi)\|_{\mathcal{W}_h} \leqslant c$. By Theorem 5.2 we get that a subsequence, still denoted by u_h, and $\mathcal{B}(f,\varphi,\psi) \in L^p(0,T;L^1(\Omega))$ such that $u_h(f,\varphi,\psi) \to_h \mathcal{B}(f,\varphi,\psi)$ in $L^p(0,T;L^1(\Omega))$ and by Theorem 5.6 $u_h'(f,\varphi,\psi) \to_h \mathcal{B}'(f,\varphi,\psi)$ in $L^{p'}(0,T;U_2)$-weak. By Corollary 5.14 we get that $\mathcal{B}(f,\varphi,\psi)(0) = \varphi$ in $L^2(\omega^+(0), \mu_+(0))$, $\mathcal{B}(f,\varphi,\psi)(T) = \psi$ in $L^2(\omega^-(T), \mu_-(T))$. By Theorem 5.6 we get $\mathcal{B}(f,\varphi,\psi) \in \mathcal{V}_{\tilde{\lambda}}$ and $\mathcal{B}(f,\varphi,\psi)' \in \mathcal{V}_\lambda'$. Since $\tilde{\lambda}$ is comparable to λ (see assumptions at the beginning of Section 5 and Remark 2.12) we have that $\mathcal{V}_{\tilde\lambda} = \mathcal{V}_\lambda$ and then $\mathcal{B}(f,\varphi,\psi) \in \mathcal{W}_{\mu,\lambda}$. In fact

$$\big\langle (\mathcal{R}_h u_h)', \phi \big\rangle_{\mathcal{U}_2'\times\mathcal{U}_1} = \big\langle (\mathcal{R}_h u_h)', \phi \big\rangle_{\mathcal{V}_h'\times\mathcal{V}_h} = -\int_0^T \int_\Omega u_h \frac{\partial\phi}{\partial t}\mu_h dxdt$$

and by Corollary 5.7

$$\big\langle (\mathcal{R}_h u_h)', \phi \big\rangle_{\mathcal{U}_2'\times\mathcal{U}_1} \to_h \big\langle (\mathcal{R}\mathcal{B}(f,\varphi,\psi))', \phi \big\rangle_{\mathcal{U}_2'\times\mathcal{U}_1} = \big\langle (\mathcal{R}\mathcal{B}(f,\varphi,\psi))', \phi \big\rangle_{\mathcal{V}_\lambda'\times\mathcal{V}_\lambda},$$

for every $\phi \in C^1_c(\Omega\times(0,T))$. Using a diagonal process, we can extract a subsequence which converges for every tern $(f,\varphi,\psi) \in X$. By (50)

$$\begin{aligned}\alpha\|u_h(f,\varphi,\psi) - u_h(g,\eta,\vartheta)\|^p_{\mathcal{V}_h} &\leqslant \langle f-g, u_h(f,\varphi,\psi) - u_h(g,\eta,\vartheta)\rangle + \\ &+ \frac12\int_\Omega(\varphi(x)-\eta(x))^2\mu_h(x,0)dx - \frac12\int_\Omega(\eta(x)-\vartheta(x))^2\mu_h(x,T)dx\,.\end{aligned}$$

Taking the liminf, using Theorem 5.6, (32) and Theorem 5.10, we get

$$\begin{aligned}\alpha K_1^{-1}\|\mathcal{B}(f,\varphi,\psi) - \mathcal{B}(g,\eta,\vartheta)\|^p_{\mathcal{V}_\lambda} &\leqslant \langle f-g, \mathcal{B}(f,\varphi,\psi) - \mathcal{B}(g,\eta,\vartheta)\rangle + \\ &+ \frac12\int_\Omega(\varphi(x)-\eta(x))^2\mu(x,0)dx - \frac12\int_\Omega(\eta(x)-\vartheta(x))^2\mu(x,T)dx\,.\end{aligned}$$

Then, arguing as in the proof of Theorem 3.5 and using Proposition 5.17, there is a constant c depending only on α, p, K_1, such that

(82)

$$\begin{aligned}&\|\mathcal{B}(f,\varphi,\psi) - \mathcal{B}(g,\eta,\vartheta)\|_{\mathcal{V}_\lambda} \leqslant c\|f-g\|^{1/p}_{\mathcal{V}_\lambda'} + \\ &+c\Big(\int_\Omega(\varphi(x)-\psi(x))^2\mu_+(x,0)dx\Big)^{1/p} + c\Big(\int_\Omega(\eta(x)-\vartheta(x))^2\mu_-(x,T)dx\Big)^{1/p}.\end{aligned}$$

Similarly we get that there is c', independent of h and depending on $\|f\|_{\mathcal{V}_\lambda'}$, $\|g\|_{\mathcal{V}_\lambda'}$, $\|\varphi\|_{L^2(\omega^+(0),\mu_+(\cdot,0))}$, $\|\psi\|_{L^2(\omega^+(T),\mu_+(\cdot,T))}$, $\|\eta\|_{L^2(\omega^+(0),\mu_+(\cdot,0))}$, $\|\vartheta\|_{L^2(\omega^+(T),\mu_+(\cdot,T))}$, L, p, K_1, C_3, such that

$\|\mathcal{B}(f,\varphi,\psi) - \mathcal{B}(g,\eta,\vartheta)\|_{\mathcal{W}_\lambda} \leqslant$

(83)

$$\begin{aligned}&\leqslant c'\Big[\|f-g\|_{\mathcal{V}_\lambda'} + \|f-g\|^{1/p}_{\mathcal{V}_\lambda'} + \|\varphi-\eta\|^{2/p}_{L^2(\omega^+(0),\mu_+(\cdot,0))} + \|\psi-\vartheta\|^{2/p}_{L^2(\omega^-(T),\mu_-(\cdot,T))}\Big] + \\ &+ c'\Big[\|f-g\|^{1/p}_{\mathcal{V}_\lambda'} + \|\varphi-\eta\|^{2/p}_{L^2(\omega^+(0),\mu_+(\cdot,0))} + \|\psi-\vartheta\|^{2/p}_{L^2(\omega^-(T),\mu_-(\cdot,T))}\Big]^{1/(p-1)}.\end{aligned}$$

Then we conclude by density of X in $\mathcal{V}_\lambda' \times L^2(\omega^+(0),\mu_+(\cdot,0)) \times L^2(\omega^-(T),\mu_-(\cdot,T))$ that $\mathcal{B}$ can be extended to $\mathcal{V}_\lambda' \times L^2(\omega^+(0),\mu_+(\cdot,0)) \times L^2(\omega^-(T),\mu_-(\cdot,T))$ to a (locally Hölder) continuous operator.

Now we define M. We estimate, with $(f,\varphi,\psi) \in X$, the quantity $a_h(x,t,Du_h(f,\varphi,\psi))$: by (S.2) we derive

$$|a_h(x,t,Du_h)| \leqslant L\lambda_h\big(1+|Du_h|^p\big)^{\frac{p-2}{p}}|Du_h| \leqslant L\lambda_h\big(1+|Du_h|^p\big)^{\frac{p-1}{p}}$$

by which

$$\begin{aligned}&\int_0^T\int_\Omega |a_h(x,t,Du_h)|^{\frac{p}{p-1}}\lambda_h^{-\frac{1}{p-1}}dxdt \leqslant \\ &\qquad\leqslant L\int_0^T\int_\Omega\big(1+|Du_h|^p\big)\lambda_h dxdt \leqslant LTC_1 + L\|u_h\|^p_{\mathcal{V}_h} \leqslant c\end{aligned}$$

where c is a constant depending (only) on $f, \varphi, \psi, p, L, C_1, T$.
Then $\|a_h(x,t,Du_h(f,\varphi,\psi))\|_{L^{p'}(\Omega\times(0,T),\lambda_h^{-1/(p-1)})}$ is bounded (for every $(f,\varphi,\psi)\in X$ fixed). By (15) we also deduce that $a_h(x,t,Du_h(f,\varphi,\psi))$ is bounded in $L^{p'}(0,T;(L^{1+\sigma}(\Omega))^n)$ and then there exists $M(f,\varphi,\psi)\in L^{p'}(0,T;(L^{1+\sigma}(\Omega))^n)$ such that

$$a_h(x,t,Du_h(f,\varphi,\psi)) \to M(f,\varphi,\psi) \qquad \text{in } L^{p'}(0,T;(L^{1+\sigma}(\Omega))^n)\text{-weak}\,.$$

Moreover by Lemma 5.5 we have that

$$|||M(f,\varphi,\psi)|||_{L^{p'}(\Omega\times(0,T),\lambda^{-1/(p-1)})} \leqslant \liminf_h |||a_h(x,t,Du_h(f,\varphi,\psi))|||_{L^{p'}(\Omega\times(0,T),\lambda_h^{-1/(p-1)})}$$

and then $M(f,\varphi,\psi)\in \big(L^{p'}(\Omega\times(0,T),\lambda^{-1/(p-1)})\big)^n$. Now, for $(f,\varphi,\psi),(g,\eta,\vartheta)\in X$, again by Lemma 5.5 and by Lemma 4.9 we have

$$\begin{aligned}
&|||M(f,\varphi,\psi)-M(g,\eta,\vartheta)|||_{L^{p'}(\Omega\times(0,T),\lambda^{-1/(p-1)})} \leqslant\\
&\quad \leqslant \liminf_h |||a_h(x,t,Du_h(f,\varphi,\psi))-a_h(x,t,Du_h(g,\eta,\vartheta))|||_{L^{p'}(\Omega\times(0,T),\lambda_h^{-1/(p-1)})}\\
&\quad \leqslant \Big(\int_0^T\int_\Omega \lambda_h dx+\int_0^T\int_\Omega |Du_h(f,\varphi,\psi)|^p\lambda_h dx+\int_0^T\int_\Omega |Du_h(g,\eta,\vartheta)|^p\lambda_h dx\Big)^{\frac{p-2}{p-1}}\cdot\\
&\qquad\qquad \cdot\|u_h(f,\varphi,\psi)-u_h(g,\eta,\vartheta)\|_{\mathcal{V}_{\lambda_h}}^{\frac{1}{p-1}}\,.
\end{aligned}$$

Using (82) and (S.5) we finally obtain

(84)
$$\begin{aligned}
&|||M(f,\varphi,\psi)-M(g,\eta,\vartheta)|||_{L^{p'}(\Omega\times(0,T),\lambda^{-1/(p-1)})} \leqslant\\
&\quad \leqslant c'\left[\|f-g\|_{\mathcal{V}'_\lambda}^{1/p}+\|\varphi-\eta\|^{2/p}_{L^2(\omega_+(0),\mu_+(0))}+\|\psi-\vartheta\|^{2/p}_{L^2(\omega_-(T),\mu_-(T))}\right]^{\frac{1}{p-1}}
\end{aligned}$$

with $c'=c'(L,p,K_1,C_1,T,\|f\|_{\mathcal{V}_\lambda},\|g\|_{\mathcal{V}_\lambda},\|\varphi\|_{L^2(\omega_+(0),\mu_+(0))},\|\eta\|_{L^2(\omega_+(0),\mu_+(0))},\|\psi\|_{L^2(\omega_-(T),\mu_-(T))}, \|\vartheta\|_{L^2(\omega_-(T),\mu_-(T))}))$. Then we can extend M to $\mathcal{V}'_\lambda\times L^2(\Omega,\mu(\cdot,0))$ to a continuous operator.
Finally we define $\mathcal{K}$ on $\mathcal{V}'_\lambda\times L^2(\omega_+(0),\mu_+(0))\times L^2(\omega_-(T),\mu_-(T))$ simply as follows

$$\mathcal{K}(f,\varphi,\psi) := f-\frac{d}{dt}\mathcal{B}(f,\varphi,\psi)\,.$$

By Theorem 5.6 we have that

$$\mathcal{A}_h(u_h(f,\varphi,\psi)) \to \mathcal{K}(f,\varphi,\psi) \qquad \text{in } L^{p'}(0,T;U_2)\text{-weak}.$$

To conclude, we multiply $(\mathcal{R}_h u_h)'+\mathcal{A}_h u_h=f$ by u_h and $(\mathcal{R}\mathcal{B}(f,\varphi,\psi))'+\mathcal{K}(f,\varphi,\psi)=f$ by $\mathcal{B}(f,\varphi,\psi)$ and obtain

$$\frac12\Big[\int_\Omega u_h^2(x,T)\mu_h(x,T)dx-\int_\Omega u_h^2(x,0)\mu_h(x,0)dx\Big]+\big\langle \mathcal{A}_h u_h,u_h\big\rangle_{\mathcal{V}'_h\times\mathcal{V}_h}=\big\langle f,u_h\big\rangle_{\mathcal{V}'_h\times\mathcal{V}_h}$$

(85)
$$\begin{aligned}
&\frac12\Big[\int_\Omega(\mathcal{B}(f,\varphi,\psi))^2(x,T)\mu(x,T)dx-\int_\Omega(\mathcal{B}(f,\varphi,\psi))^2(x)\mu(x,0)dx\Big]+\\
&\qquad +\big\langle\mathcal{K}(f,\varphi,\psi),\mathcal{B}(f,\varphi,\psi)\big\rangle_{\mathcal{V}'_\lambda\times\mathcal{V}_\lambda}=\big\langle f,\mathcal{B}(f,\varphi,\psi)\big\rangle_{\mathcal{V}'_\lambda\times\mathcal{V}_\lambda}
\end{aligned}$$

Using Theorem 5.10 we obtain that

(86) $$\big\langle\mathcal{A}_h(u_h(f,\varphi,\psi)),u_h(f,\varphi,\psi)\big\rangle_{\mathcal{V}'_h\times\mathcal{V}_h}\to\big\langle\mathcal{K}(f,\varphi,\psi),\mathcal{B}(f,\varphi,\psi)\big\rangle_{\mathcal{V}'_\lambda\times\mathcal{V}_\lambda}$$

Moreover since by Theorem 5.6-(ii) $(\mathcal{R}_h u_h(f,\varphi,\psi))' \to (\mathcal{R}\mathcal{B}(f,\varphi,\psi))'$ in $L^{p'}(0,T;U_2')$-weak, for every $\phi \in C^1_c(\Omega \times (0,T))$

$$\big\langle \mathcal{A}_h u_h(f,\varphi,\psi), \phi \big\rangle_{\mathcal{V}_h' \times \mathcal{V}_h} = \int_0^T \int_\Omega (a_h(x,t,Du_h(f,\varphi,\psi)), D\phi) dxdt$$

and taking the limit

$$\big\langle \mathcal{K}(f,\varphi,\psi), \phi \big\rangle_{\mathcal{V}_\lambda' \times \mathcal{V}_\lambda} = \int_0^T \int_\Omega (M(f,\varphi,\psi), D\phi) dxdt \, .$$

By density we conclude that

$$\mathcal{K}(f,\varphi,\psi) = -\text{div}\; M(f,\varphi,\psi) \quad \text{on } \mathcal{V}_\lambda' \qquad \text{for every } (f,\varphi,\psi) \in X \, . \tag{87}$$

Now we estimate $\|\mathcal{K}(f,\varphi,\psi) - \mathcal{K}(g,\eta,\vartheta)\|_{\mathcal{V}_\lambda'}$. By the last equality we have that

$$\begin{aligned} \|\mathcal{K}(f,\varphi,\psi) - \mathcal{K}(g,\eta,\vartheta)\|_{\mathcal{V}_\lambda'} &= \sup_{\|v\|_{\mathcal{V}_\lambda} \leqslant 1} \int_0^T \int_\Omega \big(M(f,\varphi,\psi) - M(g,\eta,\vartheta), Dv\big) dxdt \leqslant \\ &\leqslant \Big[\int_0^T \int_\Omega |M(f,\varphi,\psi) - M(g,\eta,\vartheta)|^{p'} \lambda^{-1/(p-1)} dxdt \Big]^{1/p'} \Big[\int_0^T \int_\Omega |Dv|^p \lambda dxdt \Big]^{1/p} \end{aligned}$$

and by (84) we derive that, for every $(f,\varphi,\psi), (g,\eta,\vartheta) \in X$,

$$\begin{aligned} &\|\mathcal{K}(f,\varphi,\psi) - \mathcal{K}(g,\eta,\vartheta)\|_{\mathcal{V}_\lambda'} \leqslant \\ &\qquad \leqslant c \Big[\|f-g\|_{\mathcal{V}_\lambda'}^{1/p} + \|\varphi - \eta\|^{2/p}_{L^2(\omega_+(0),\mu_+(0))} + \|\psi - \vartheta\|^{2/p}_{L^2(\omega_-(T),\mu_-(T))} \Big]^{\frac{1}{p-1}} . \end{aligned}$$

Then $\mathcal{K}$ can be estended to a continuous operator defined in $\mathcal{V}_\lambda' \times L^2(\omega^+(0), \mu_+(\cdot,0)) \times L^2(\omega^-(T), \mu_-(\cdot,T))$ and (87) turns out to hold for every $(f,\varphi,\psi) \in \mathcal{V}_\lambda' \times L^2(\omega^+(0), \mu_+(\cdot,0)) \times L^2(\omega^-(T), \mu_-(\cdot,T))$. □

We recall the following result (see [5], Lemma 7.8).

Lemma 6.6. *Let D be a bounded open set in $\mathbf{R}^k$. Let $\vartheta_1, \dots \vartheta_m$ be non-negative numbers such that $\vartheta_1 + \dots + \vartheta_m \leqslant 1$. Let us assume that $(r_{1,h})_h, \dots, (r_{m,h})_h$ and $(s_h)_h$ are sequences in $L^1(D)$ such that*

$$r_{i,h} \geqslant 0 \textit{ for } i = 1, \dots m, \quad |s_h| \leqslant r_{1,h}^{\vartheta_1} \cdot \ldots \cdot r_{m,h}^{\vartheta_m} \quad \textit{a.e. in } D, \textit{ for every } h$$

and that there exist $r_1, \dots, r_m, s \in L^1(D)$ such that

$$r_{i,h} \to_h r_i \quad \textit{for } i = 1, \dots m, \quad s_h \to s \qquad \textit{in } \mathcal{D}'(D).$$

Then

$$|s| \leqslant r_1^{\vartheta_1} \cdot \ldots \cdot r_m^{\vartheta_m} \qquad \textit{a.e. in } D.$$

Lemma 6.7. *The operator $\mathcal{B} : \mathcal{V}_\lambda' \times L^2(\omega^+(0), \mu_+(\cdot,0)) \times L^2(\omega^-(T), \mu_-(\cdot,T)) \to \mathcal{W}_{\mu,\lambda}$ introduced in Lemma 6.5 is invertible.*

Proof - First of all we prove that $\mathcal{B}$ is injective. Consider $(f,\varphi,\psi), (g,\eta,\vartheta) \in L^{p'}(0,T;L^n(\Omega)) \times L^{(1+\sigma)'}(\Omega) \times L^{(1+\sigma)'}(\Omega)$. We have that, by (S.2) and (S.3),

$$\begin{aligned} &|a_h(x,t,Du_h(f,\varphi,\psi)) - a_h(x,t,Du_h(g,\eta,\vartheta))| \leqslant \\ &\quad \leqslant L\lambda_h (1 + |Du_h(f,\varphi,\psi)|^p + |Du_h(g,\eta,\vartheta)|^p)^{\frac{p-2}{p}} |Du_h(f,\varphi,\psi) - Du_h(g,\eta,\vartheta)| \leqslant \\ &\quad \leqslant L\lambda_h^{1/p} \big(\lambda_h + (a_h(x,t,Du_h(f,\varphi,\psi)), Du_h(f,\varphi,\psi)) + (a_h(x,t,Du_h(g,\eta,\vartheta)), Du_h(g,\eta,\vartheta))\big)^{\frac{p-2}{p}} \cdot \\ &\qquad\qquad \cdot (a_h(x,t,Du_h(f,\varphi,\psi)) - a_h(x,t,Du_h(g,\eta,\vartheta)), Du_h(f,\varphi,\psi) - Du_h(g,\eta,\vartheta))^{1/p} \, . \end{aligned}$$

By Lemma 6.5, Theorem 5.15 and Lemma 6.6 we derive

$$
\begin{aligned}
&|M(f,\varphi,\psi) - M(g,\eta,\vartheta)| \leqslant \\
&\qquad \leqslant L\lambda^{1/p}\big(\lambda + (M(f,\varphi,\psi), D\mathcal{B}(f,\varphi,\psi)) + (M(g,\eta,\vartheta), D\mathcal{B}(g,\eta,\vartheta))\big)^{\frac{p-2}{p}} \cdot \\
&\qquad\qquad \cdot\big(M(f,\varphi,\psi) - M(g,\eta,\vartheta), D\mathcal{B}(f,\varphi,\psi) - D\mathcal{B}(g,\eta,\vartheta)\big)^{1/p}
\end{aligned}
$$

by which (as done in the proof of Lemma 4.9) there is L' depending (only) on p, L such that
(88)

$$
\begin{aligned}
&|M(f,\varphi,\psi) - M(g,\eta,\vartheta)| \leqslant \\
&\qquad \leqslant L'\lambda^{1/(p-1)}\big(\lambda + (M(f,\varphi,\psi), D\mathcal{B}(f,\varphi,\psi)) + (M(g,\eta,\vartheta), D\mathcal{B}(g,\eta,\vartheta))\big)^{\frac{p-2}{p-1}} \cdot \\
&\qquad\qquad \cdot|D\mathcal{B}(f,\varphi,\psi) - D\mathcal{B}(g,\eta,\vartheta)|^{1/(p-1)}.
\end{aligned}
$$

Then, if we suppose $\mathcal{B}(f,\varphi,\psi) = \mathcal{B}(g,\eta,\vartheta)$, we deduce that $M(f,\varphi,\psi) = M(g,\eta,\vartheta)$. By (87) we deduce that $\mathcal{K}(f,\varphi,\psi) = \mathcal{K}(g,\eta,\vartheta)$ and finally, since by definition of $\mathcal{K}$ we have $\mathcal{K}(f,\varphi,\psi) = f - \frac{d}{dt}(\mathcal{B}(f,\varphi,\psi))$ and $\mathcal{K}(g,\eta,\vartheta) = g - \frac{d}{dt}(\mathcal{B}(g,\eta,\vartheta))$, we conclude that $f = g$. Since, by Lemma 6.5, $\varphi = \eta$ in $L^2(\omega^+(0), \mu_+(\cdot,0))$ and $\psi = \vartheta$ in $L^2(\omega^-(T), \mu_-(\cdot,T))$ and $\mathcal{B}$ is injective.
We denote by $\mathcal{D}$ the set of possible data $\mathcal{V}'_\lambda \times L^2(\omega^+(0), \mu_+(\cdot,0)) \times L^2(\omega^-(T), \mu_-(\cdot,T))$.
Now we show that $\mathcal{B}(\mathcal{D})$ is dense in $\mathcal{V}_\lambda$. Consider $f_0 \in \mathcal{V}'_\lambda$ such that $\langle f_0, \mathcal{B}(f,\varphi,\psi)\rangle_{\mathcal{V}'_\lambda\times\mathcal{V}_\lambda} = 0$ for every $(f,\varphi,\psi) \in \mathcal{D}$. In particular $\langle f_0, \mathcal{B}(f_0,\varphi,\psi)\rangle_{\mathcal{V}'_\lambda\times\mathcal{V}_\lambda} = 0$ for every $\varphi \in L^2(\omega^+(0), \mu_+(\cdot,0))$, $\psi \in L^2(\omega^-(T), \mu_-(\cdot,T))$. From (85) and Lemma 6.5 , taking $\varphi \equiv 0$ and $\psi \equiv 0$, we derive that $\langle \mathcal{K}(f_0,0,0), \mathcal{B}(f_0,0,0)\rangle_{\mathcal{V}'_\lambda\times\mathcal{V}_\lambda} \leqslant 0$. On the other side, by (S.3) and (86), we get that $\langle \mathcal{K}(f_0,0,0), \mathcal{B}(f_0,0,0)\rangle_{\mathcal{V}'_\lambda\times\mathcal{V}_\lambda} \geqslant 0$ and than $\langle \mathcal{K}(f_0,0,0), \mathcal{B}(f_0,0,0)\rangle_{\mathcal{V}'_\lambda\times\mathcal{V}_\lambda} = 0$. Again by (S.3) and (86), using Theorem 5.6-(i) and the fact that $\lambda \leqslant K_1\tilde{\lambda}$ (see Remark 2.12) we have that

$$
\|\mathcal{B}(f_0,0,0)\|^p_{\mathcal{V}_\lambda} \leqslant K_1\big\langle \mathcal{K}(f_0,0,0), \mathcal{B}(f_0,0,0)\big\rangle_{\mathcal{V}'_\lambda\times\mathcal{V}_\lambda}
$$

and then $\mathcal{B}(f_0,0,0) = 0$. Since $\mathcal{B}(0,0,0) = 0$ and $\mathcal{B}$ is injective we get $f_0 = 0$ and then the density of $\mathcal{B}(\mathcal{D})$ in $\mathcal{V}_\lambda$.
Then $\mathcal{B}$ is invertible and the inverse is defined on a dense subset of $\mathcal{V}_\lambda$.
Now we construct the inverse. First we define $\mathcal{A} : \mathcal{B}(\mathcal{D}) \to \mathcal{V}'_\lambda$ as

$$
\mathcal{A}(\mathcal{B}(f,\varphi,\psi)) := \mathcal{K}(f,\varphi,\psi).
$$

Take $(f,\varphi,\psi), (g,\eta,\vartheta) \in \mathcal{D}$ and consider the equations (in the corresponging spaces)
(89)

$$
\begin{aligned}
&(\mathcal{R}\mathcal{B}(f,\varphi,\psi))' + \mathcal{A}(\mathcal{B}(f,\varphi,\psi)) = f\,, \quad \mathcal{B}(f,\varphi,\psi)(0) = \varphi \quad \mathcal{B}(f,\varphi,\psi)(T) = \psi \\
&(\mathcal{R}\mathcal{B}(g,\eta,\vartheta))' + \mathcal{A}(\mathcal{B}(g,\eta,\vartheta)) = g\,, \quad \mathcal{B}(g,\eta,\vartheta)(0) = \eta\,, \quad \mathcal{B}(g,\eta,\vartheta)(T) = \vartheta\,.
\end{aligned}
$$

Subtracting one to the other, multiplying by $\mathcal{B}(f,\varphi,\psi) - \mathcal{B}(g,\eta,\vartheta)$ and integrating one obtains

(90)

$$\begin{aligned}&\frac{1}{2}\Big[\int_\Omega (\mathcal{B}(f,\varphi,\psi)(x,T) - \mathcal{B}(g,\eta,\vartheta)(x,T))^2\mu(x,T)dx - \\ &\quad - \int_\Omega (\mathcal{B}(f,\varphi,\psi)(x,0) - \mathcal{B}(g,\eta,\vartheta)(x,0))^2\mu(x,0)dx\Big] + \\ &\quad + \big\langle \mathcal{A}(\mathcal{B}(f,\varphi,\psi)) - \mathcal{A}(g,\eta,\vartheta)\,, \mathcal{B}(f,\varphi,\psi) - \mathcal{B}(g,\eta,\vartheta)\big\rangle_{\mathcal{V}'_\lambda\times\mathcal{V}_\lambda} = \\ &\qquad\qquad = \big\langle f-g, \mathcal{B}(f,\varphi,\psi) - \mathcal{B}(g,\eta,\vartheta)\big\rangle_{\mathcal{V}'_\lambda\times\mathcal{V}_\lambda}.\end{aligned}$$

Now consider the equations (in the corresponging spaces)

$$\begin{aligned}&(\mathcal{R}_h(u_h(f,\varphi,\psi)))' + \mathcal{A}_h u_h(f,\varphi,\psi) = f\,, \quad u_h(f,\varphi,\psi)(0) = \varphi\,, \quad u_h(f,\varphi,\psi)(T) = \psi\,, \\ &(\mathcal{R}_h u_h(g,\eta,\vartheta))' + \mathcal{A}_h u_h(g,\eta,\vartheta) = g\,, \qquad u_h(g,\eta,\vartheta)(0) = \eta\,, \quad u_h(g,\eta,\vartheta)(T) = \vartheta\,,\end{aligned}$$

subtract one to the other and multiply by $u_h(f,\varphi,\psi) - u_h(g,\eta,\vartheta)$. One obtains

$$\begin{aligned}&\frac{1}{2}\Big[\int_\Omega (u_h(f,\varphi,\psi)(x,T) - u_h(g,\eta,\vartheta)(x,T))^2\mu_h(x,T)dx - \\ &\quad - \int_\Omega (u_h(f,\varphi,\psi)(x,0) - u_h(g,\eta,\vartheta)(x,0))^2\mu_h(x,0)dx\Big] + \\ &\quad + \big\langle \mathcal{A}_h(u_h(f,\varphi,\psi)) - \mathcal{A}_h(u_h(g,\eta,\vartheta))\,, u_h(f,\varphi,\psi) - u_h(g,\eta,\vartheta)\big\rangle_{\mathcal{V}'_h\times\mathcal{V}_h} = \\ &\qquad\qquad = \big\langle f-g, u_h(f,\varphi,\psi) - u_h(g,\eta,\vartheta)\big\rangle_{\mathcal{V}'_h\times\mathcal{V}_h}.\end{aligned}$$

Since the right hand side converge to the right hand side of (90) and, by Theorem 5.10, also the first term of the left hand side converge to the first term of the left hand side of (90), we deduce that

$$\begin{aligned}&\big\langle \mathcal{A}_h(u_h(f,\varphi,\psi)) - \mathcal{A}_h(u_h(g,\eta,\vartheta))\,, u_h(f,\varphi,\psi) - u_h(g,\eta,\vartheta)\big\rangle_{\mathcal{V}'_h\times\mathcal{V}_h} \to_h \\ &\qquad \to_h \big\langle \mathcal{A}(\mathcal{B}(f,\varphi,\psi)) - \mathcal{A}(\mathcal{B}(g,\eta,\vartheta))\,, \mathcal{B}(f,\varphi,\psi) - \mathcal{B}(g,\eta,\vartheta)\big\rangle_{\mathcal{V}'_\lambda\times\mathcal{V}_\lambda}.\end{aligned}$$

By this convergence, (S.3), Theorem 5.6-(i) and (32) we have that

$$\|\mathcal{B}(f,\varphi,\psi) - \mathcal{B}(g,\eta,\vartheta)\|^p_{\mathcal{V}_\lambda} \leqslant K_1\big\langle \mathcal{A}(\mathcal{B}(f,\varphi,\psi)) - \mathcal{A}(\mathcal{B}(g,\eta,\vartheta))\,, \mathcal{B}(f,\varphi,\psi) - \mathcal{B}(g,\eta,\vartheta)\big\rangle_{\mathcal{V}'_\lambda\times\mathcal{V}_\lambda}.$$

Moreover, by Lemma 6.5, we have that for every $\phi \in C^1_c(\Omega\times(0,T))$

$$\begin{aligned}\big\langle \mathcal{A}(\mathcal{B}(f,\varphi,\psi)) - \mathcal{A}(\mathcal{B}(g,\eta,\vartheta)), \phi\big\rangle_{\mathcal{V}'_\lambda\times\mathcal{V}_\lambda} &= \lim_h \big\langle \mathcal{A}_h(u_h(f,\varphi,\psi)) - \mathcal{A}_h(u_h(g,\eta,\vartheta)), \phi\big\rangle_{\mathcal{V}'_h\times\mathcal{V}_h} = \\ &= \int_0^T\int_\Omega \big(M(f,\varphi,\psi) - M(g,\eta,\vartheta), D\phi\big)dxdt\,.\end{aligned}$$

Since $|(M(f,\varphi,\psi), D\mathcal{B}(f,\varphi,\psi))| \leqslant (a_h(x,t,Du_h(f,\varphi,\psi)), Du_h(f,\varphi,\psi)) \leqslant \lambda_h|Du_h(f,\varphi,\psi)|^p$ (idem for $u_h(g,\eta,\vartheta)$), using (88), estimating $\int_0^T\int_\Omega \lambda dxdt$ by C_1T, taking the supremum over all ϕ with $\|\phi\|_{\mathcal{V}_\lambda} = 1$ and arguing as in the proof of Lemma 4.9 we obtain there exists $L'' = L''(p, L, C_1, T)$ (depending only on p, L, C_1, T) such that

$$\|\mathcal{A}(\mathcal{B}(f,\varphi,\psi)) - \mathcal{A}(\mathcal{B}(g,\eta,\vartheta))\|_{\mathcal{V}'_\lambda} \leqslant$$
$$\leqslant L''\big(1 + \|\mathcal{B}(f,\varphi,\psi)\|^p_{\mathcal{V}_\lambda} + \|\mathcal{B}(g,\eta,\vartheta)\|^p_{\mathcal{V}_\lambda}\big)^{\frac{p-2}{p-1}} \|\mathcal{B}(f,\varphi,\psi) - \mathcal{B}(g,\eta,\vartheta)\|^{\frac{1}{p-1}}_{\mathcal{V}_\lambda} .$$

By the density of $\mathcal{B}(\mathcal{D})$ in $\mathcal{V}_\lambda$ we can estend the operator $\mathcal{A}$ to another operator, still denoted by $\mathcal{A}$, $\mathcal{A} : \mathcal{V}_\lambda \to \mathcal{V}'_\lambda$ such that

$$\begin{aligned} &\|u - v\|^p_{\mathcal{V}_\lambda} \leqslant K_1 \langle \mathcal{A}u - \mathcal{A}v \,, u - v\rangle_{\mathcal{V}'_\lambda \times \mathcal{V}_\lambda} , \\ &\|\mathcal{A}u - \mathcal{A}v\|_{\mathcal{V}'_\lambda} \leqslant L''\big(1 + \|u\|^p_{\mathcal{V}_\lambda} + \|v\|^p_{\mathcal{V}_\lambda}\big)^{\frac{p-2}{p-1}} \|u - v\|^{\frac{1}{p-1}}_{\mathcal{V}_\lambda} . \end{aligned} \tag{91}$$

Finally, by definition, we have that

$$\begin{aligned} &\frac{d}{dt}(\mathcal{R}\mathcal{B}(f,\varphi,\psi)) + \mathcal{A}(\mathcal{B}(f,\varphi,\psi)) = f && \text{in } \mathcal{V}'_\lambda , \\ &\mathcal{B}(f,\varphi,\psi)(0) = \varphi && \text{in } L^2(\omega^+(0), \mu_+(\cdot,0)) , \\ &\mathcal{B}(f,\varphi,\psi)(T) = \psi && \text{in } L^2(\omega^-(T), \mu_-(\cdot,T)) \end{aligned}$$

for every $f \in \mathcal{V}'_\lambda$ and $\varphi \in L^2(\omega^+(0), \mu_+(\cdot,0))$, $\psi \in L^2(\omega^-(T), \mu_-(\cdot,T))$. □

Theorem 6.8. *Consider $(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(p, L, K_1, C_1, F)$, $(\lambda_h)_h, (\mu_h)_h$ with $\lambda_h \in \Lambda(a_h)$ $\mu_h \in \mathcal{F}(\lambda_h,\ K_2, r, \alpha, \delta_1, \delta_2, C_2,\ C_3, F)$ satisfying* **A)**. *There is a subsequence $(a_{h_j})_j$, $a \in \mathcal{N}_{\Omega\times(0,T)}(p, L^*,\ K_1, C_1, K_1^{-1}F)$, $L^* = L'K_1$ with $L' = L'(L,p)$, such that*

$$(\mu_{h_j}, a_{h_j}) \ \stackrel{G}{\longrightarrow}\ (\mu, a) \quad \text{in } \Omega \times (0,T) .$$

Proof - As done in the proof of the previous lemma, denote by $\mathcal{D}$ the set of possible data $\mathcal{V}'_\lambda \times L^2(\omega^+(0), \mu_+(\cdot,0)) \times L^2(\omega^-(T), \mu_-(\cdot,T))$. Consider the space $\mathcal{Y} = \mathcal{B}(\mathcal{D})$ and call $\mathcal{P} : \mathcal{W}_{\mu,\lambda} \to \mathcal{D}$ the inverse of $\mathcal{B}$ ($\mathcal{B}$ is the operator of Lemma 6.7). Define an operator $\mathcal{M} : \mathcal{Y} \to \big(L^{p'}(\Omega\times(0,T); \lambda^{-1/(p-1)})\big)^n$ as $\mathcal{M}u := M \circ \mathcal{B}^{-1}u = M \circ \mathcal{P}u$ (M defined in Lemma 6.5) which can be extended to $\mathcal{V}_\lambda$ (by the proof of Lemma 6.7 the space $\mathcal{Y}$ is dense in $\mathcal{V}_\lambda$). Since $Du_{h_j} := D(\mathcal{P}^{-1}_{h_j}\mathcal{P}u) \to_j Du$ and $Dv_{h_j} := D(\mathcal{P}^{-1}_{h_j}\mathcal{P}v) \to_j Dv$ in $L^p(0,T; (L^1(\Omega))^n)$-weak (see Theorem 5.6-(i)) and

$$|D(\mathcal{P}^{-1}_{h_j}\mathcal{P}u) - D(\mathcal{P}^{-1}_{h_j}\mathcal{P}v)|^p \leqslant$$
$$\leqslant \lambda^{-1}_{h_j}\big(a_{h_j}(x,t,D(\mathcal{P}^{-1}_{h_j}\mathcal{P}u)) - a_{h_j}(x,t,D(\mathcal{P}^{-1}_{h_j}\mathcal{P}v)), D(\mathcal{P}^{-1}_{h_j}\mathcal{P}u) - D(\mathcal{P}^{-1}_{h_j}\mathcal{P}v)\big)$$

by Lemma 6.6 (and also (32) and Theorem 5.15) we get

$$\frac{1}{K_1}|Du - Dv|^p\lambda \leqslant \big(\mathcal{M}u - \mathcal{M}v, Du - Dv\big) \qquad \text{a.e. in } \Omega\times(0,T) . \tag{92}$$

By (88) we also get, for $u, v \in \mathcal{V}_\lambda$,

$$|\mathcal{M}u - \mathcal{M}v| \leqslant L'\lambda^{1/(p-1)}\big(\lambda + (\mathcal{M}u, Du) + (\mathcal{M}v, Dv)\big)^{\frac{p-2}{p-1}} \cdot |Du - Dv|^{1/(p-1)} . \tag{93}$$

Now we consider a sequence $\omega_k \times I_k$, $k \in \mathbf{N}$, ω_k open sets such that $\overline{\omega}_k \subset \Omega$, I_k open intervals such that $\overline{I}_k \subset (0,T)$ in such a way $\cup_{k=1}^\infty \omega_k \times I_k = \Omega \times (0,T)$. Now choose $\phi_k \in C^1_0(\Omega\times(0,T))$ such that $\phi_k \equiv 1$ on $\omega_k \times I_k$ and define, for a fixed $\xi \in \mathbf{R}^n$, $\phi^{(\xi)}_k(x,t) = (\xi, x)\phi_k(x,t)$. Finally define

$$a(x,t,\xi) = \mathcal{M}(\phi^{(\xi)}_k)(x,t) \qquad (x,t) \in \Omega\times(0,T) .$$

Taking $v = 0$ in (93) and then $u = \phi_k^{(\xi)}(x,t)$ with $\xi = 0$ we deduce that $a(x,t,0) = 0$ almost everywhere. Moreover for $\xi, \eta \in \mathbf{R}^n$ consider $u = \phi_k^{(\xi)}(x,t)$ and $v = \phi_k^{(\eta)}(x,t)$. Now for almost every $(x,t) \in \omega_k \times (0,T)$ we derive from (92) and (93)

$$\text{(S.1)} \quad a(x,t,0) = 0\,,$$

$$\text{(S.2)}' \quad |a(x,t,\xi) - a(x,t,\eta)| \leqslant L'\lambda(x,t)(1 + |\xi|^p + |\eta|^p)^{\frac{p-2}{p-1}}|\xi - \eta|^{\frac{1}{p-1}}$$

$$\text{(S.3)} \quad (a(x,t,\xi) - a(x,t,\eta), \xi - \eta) \geqslant \frac{1}{K_1}\lambda(x,t)|\xi - \eta|^p\,,$$

with λ satisfying (S.4), (S.5) in Definition 4.1. Defining $\lambda_a := \frac{1}{K_1}\lambda$ we get that

$$\|\lambda_a(t) - \lambda_a(s)\|_{L^1(\Omega)} \leqslant K_1^{-1}F(|t-s|)$$

$$\|\tilde{\lambda}^{-1/(p-1)}(t) - \tilde{\lambda}^{-1/(p-1)}(s)\|_{L^1(\Omega)} \leqslant F(|t-s|)\,,$$

(with $\tilde{\lambda}$ comparable to λ_a) and consequently $a \in \mathcal{N}_{\Omega\times(0,T)}(p, L^*, K_1, C_1, K_1^{-1}F)$ where $L^* = L'K_1$ and L' depends only on p, L.
The proof is complete observing that since $\mu_h \to \mu$ we also have $\mu_{h_j} \to \mu$, taking the solutions of the problems (P_{h_j}) in (61) and applying the preceeding lemmas. □

Now, in the following theorem, we drop assumption **A)** required at the beginning of the section and in the previous theorem.

Theorem 6.9. *Consider three sequences $(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(p, L, K_1, C_1, F)$, $(\lambda_h)_h, (\mu_h)_h$ with $\lambda_h \in \Lambda(a_h)$ $\mu_h \in \mathcal{F}(\lambda_h, K_2, r, \alpha, \delta_1, \delta_2, C_2, C_3, F)$. There are two subsequences $(a_{h_j})_j$, $(\mu_{h_j})_j$, $a \in \mathcal{N}_{\Omega\times(0,T)}(p, L^*, K_1, C_1, K_1^{-1}F)$, $L^* = L'K_1$ with L' depending only on p and L, $\lambda \in \Lambda(a)$ and $\mu \in \mathcal{F}(\lambda, K_2, r, \alpha, \delta_1, \delta_2, C_2, C_3, F)$ such that*

$$(\mu_{h_j}, a_{h_j}) \xrightarrow{G} (\mu, a) \quad \text{in } \Omega\times(0,T)\,.$$

Proof - Consider $\epsilon > 0$ and the problems (P_h^ϵ) defined in (65) and call, for simplicity,

$$\mu_h^\epsilon := \mu_h + \epsilon\lambda_h\,.$$

Consider μ, λ the weights defined in (66). We have (up to subsequences)

$$\mu_h^\epsilon \to \mu^\epsilon := \mu + \epsilon\lambda\,,$$

Thanks to Lemma 3.13 we can find $\mathcal{E} \subset (0,1]$, $\mathcal{E}$ countable and dense in $(0,1]$, in such a way that for every $\epsilon \in \mathcal{E}$

$$\mu + \epsilon\lambda \neq 0 \quad \text{a.e. in } \Omega\times(0,T)\,.$$

By Theorem 6.8 we have that $(\mu_h^\epsilon, a_h)_h$ is relatively compact with respect to G-convergence. Thus one can find a subsequence $(\mu_{h_j}^\epsilon, a_{h_j})_j$ and a Carathéodory function $a^\epsilon \in \mathcal{N}_{\Omega\times(0,T)}(p,\ L'K_1,\ K_1, C_1, K_1^{-1}F)$ $(L' = L'(p,L))$ such that $(\mu_{h_j}^\epsilon, a_{h_j})_j$ G-converges to $(\mu + \epsilon\,\lambda, a^\epsilon)$. By a diagonal process one can find a further subsequence, still denoted by h_j, such that $(\mu_{h_j}^\epsilon, a_{h_j}) \xrightarrow{G} (\mu + \epsilon\lambda, a^\epsilon)$ for every $\epsilon \in \mathcal{E}$. By Lemma 6.3 a^ϵ is in fact independent of ϵ so we denote it by a and

$$(94) \qquad (\mu_{h_j}^\epsilon, a_{h_j}) \xrightarrow{G} (\mu + \epsilon\lambda, a) \quad \text{in } \Omega\times(0,T) \qquad \text{for every } \epsilon \in \mathcal{E}\,.$$

In particular for every $\epsilon \in \mathcal{E}$

$$\begin{aligned} & u^\epsilon_{h_j} \to_j u^\epsilon && \text{in } L^p(0,T;L^1(\Omega)) \\ & a_{h_j}(\cdot,\cdot,Du^\epsilon_{h_j}) \to_j a(\cdot,\cdot,Du^\epsilon) && \text{in } L^{p'}(0,T;(L^1(\Omega))^n)\text{-weak} \end{aligned}$$

where $u^\epsilon_{h_j}$ and u^ϵ solve respectively the following problems

$$\begin{cases} \dfrac{\partial}{\partial t}\big((\mu_{h_j} + \epsilon\,\lambda_{h_j})v\big) - \operatorname{div}\big(a_{h_j}(x,t,Dv)\big) = f & \text{in } \Omega \times (0,T) \\ v = 0 & \text{in } \partial\Omega \times (0,T) \\ v(x,0) = \varphi(x) & \text{in } \ \omega^+_{h,\epsilon}(0) \\ v(x,T) = \psi(x) & \text{in } \ \omega^-_{h,\epsilon}(T)\,, \end{cases}$$

$$\begin{cases} \dfrac{\partial}{\partial t}\big((\mu + \epsilon\lambda)v\big) - \operatorname{div}\big(a(x,t,Dv)\big) = f & \text{in } \Omega \times (0,T) \\ v = 0 & \text{in } \partial\Omega \times (0,T) \\ v(x,0) = \varphi(x) & \text{in } \ \omega^+_{\epsilon}(0) \\ v(x,T) = \psi(x) & \text{in } \ \omega^-_{\epsilon}(T)\,, \end{cases}$$

where $\omega_{h,\epsilon}$ and ω_ϵ are the subset analogous to those defined in (57) and corresponding to μ^ϵ_h and μ^ϵ. By Theorem 3.8 we get that

$$\lim_{\substack{\epsilon \to 0^+ \\ \epsilon \in \mathcal{E}}} u^\epsilon = u \qquad \text{in } \mathcal{V}_\lambda$$

and by Corollary 3.12 for every $\sigma > 0$ there are $\epsilon_\sigma > 0$ and j_σ such that

$$\|u^\epsilon_{h_j} - u_{h_j}\|_{\mathcal{V}_{h_j}} < \sigma \qquad \text{for every } \epsilon \in \mathcal{E} \cap (0,\epsilon_\sigma) \text{ and } j \geqslant j_\sigma\,,$$

where u_{h_j} and u are respectively the solutions of the problems $(P_{h_j})_{f,\varphi,\psi}$ and $(P)_{f,\varphi,\psi}$ in (80). Moreover, since a_{h_j} and a are Carathéodory functions, by the convergence above we also get that, for every $\tau > 0$ there is $\sigma > 0$ such that

$$\begin{aligned} & \|a_{h_j}(\cdot,\cdot,Du^\epsilon_{h_j}) - a_{h_j}(\cdot,\cdot,Du_{h_j})\|_{L^{p'}(0,T;(L^1(\Omega))^n)} < \tau && \text{for every } \epsilon \in \mathcal{E} \cap (0,\epsilon_\sigma), j \geqslant j_\sigma\,, \\ & \lim_{\epsilon \to 0^+} a(\cdot,\cdot,Du^\epsilon) = a(\cdot,\cdot,Du) && \text{in } L^{p'}(0,T;(L^1(\Omega))^n)\,. \end{aligned}$$

Then in particular we conclude that

$$\begin{aligned} & u_{h_j} \to_j u && \text{in } L^p(0,T;L^1(\Omega)) \\ & a_{h_j}(\cdot,\cdot,Du_{h_j}) \to_j a(\cdot,\cdot,Du) && \text{in } L^{p'}(0,T;(L^1(\Omega))^n)\text{-weak} \end{aligned}$$

which concludes the proof. □

Remark 6.10. - By Theorem 6.9 and Theorem 3.8 we get that (94) holds in fact for every $\epsilon \in (0,1]$.
Moreover from the proof of Theorem 6.9 one gets that

$$\operatorname*{G\text{-}lim}_{h \to +\infty} \operatorname*{G\text{-}lim}_{\epsilon \to 0^+} (\mu_h + \epsilon\lambda_h, a_h) = \operatorname*{G\text{-}lim}_{\epsilon \to 0^+} \operatorname*{G\text{-}lim}_{h \to +\infty} (\mu_h + \epsilon\lambda_h, a_h)\,.$$

Lemma 6.11. *Suppose to have a sequence* $(a_h)_h \subset \mathcal{N}_{\Omega\times(0,T)}$ *and, for* $\lambda_h \in \Lambda(a_h)$, *two sequences* $\mu_h, \nu_h \in \mathcal{F}(\lambda_h)$, $h \in \mathbf{N}$. *Suppose* $(\mu_h, a_h)_h$ *G-converges to* (μ, a_μ) *and* $(\nu_h, a_h)_h$ *G-converges to* (ν, a_ν) *in* $\Omega \times (0,T)$. *Then* $a_\mu(x,t,\xi) = a_\nu(x,t,\xi)$ *for a.e.* $(x,t) \in \Omega \times (0,T)$ *and for every* $\xi \in \mathbf{R}^n$.

Proof - Fix $\epsilon > 0$ and consider the sequences $(\mu_h + \epsilon\lambda_h, a_h)_h$, $(\nu_h + \epsilon\lambda_h, a_h)_h$. By assumptions and Theorem 3.8 we have that

$$\underset{h\to+\infty}{G\text{-lim}}\ \underset{\epsilon\to0^+}{G\text{-lim}}\ (\mu_h + \epsilon\lambda_h, a_h) = (\mu, a_\mu)\,,$$
$$\underset{h\to+\infty}{G\text{-lim}}\ \underset{\epsilon\to0^+}{G\text{-lim}}\ (\nu_h + \epsilon\lambda_h, a_h) = (\nu, a_\nu)\,.$$

By Remark 6.10 and Lemma 6.3 (if necessary taking a further subsequence $(\lambda_{h_j})_j$) we conclude. □

Thanks to this result we can give the following definition and conlude the section stating the theorem below.

Definition 6.12. *Consider a sequence* $(a_h)_h$ *and a Carathéodory function a with* $a_1, a_2, \ldots,\ a \in \mathcal{N}_{\Omega\times(0,T)}(p, L, K_1, C_1, F)$ *and* $(\lambda_h)_h$ *with* $\lambda_h \in \Lambda(a_h)$ *for every* $h \in \mathbf{N}$. *We say that the sequence* $(a_h)_h$ *PG-converges to a in* $\Omega \times (0,T)$, *and we write*

$$a_h \xrightarrow{PG} a \quad in\ \Omega \times (0,T),$$

if for every sequence $(\mu_h)_h$ *with* $\mu_h \in \mathcal{F}(\lambda_h, K_2, r,\ \alpha, \delta_1, \delta_2, C_2, C_3, F)$ *we have that*

$$(\mu_h, a_h) \xrightarrow{G} (\mu, a) \quad in\ \Omega \times (0,T)\,.$$

Theorem 6.13. *There is* $L' = L'(p, L)$ *such that the class* $\mathcal{M}_{\Omega\times(0,T)}(p, L, K_1, C_1, F)$ *is relatively compact in* $\mathcal{N}_{\Omega\times(0,T)}(p,\ L'K_1, K_1, C_1, K_1^{-1}F)$ *with respect to PG-convergence.*

Proof - The proof follows immediatly by Theorem 6.9 and Lemma 6.11. □

7. Comparison with the elliptic case

The goal of this section is to compare the results of the previous section with the known results about the elliptic case. First we introduce some classes of functions we need in this section and recall the definition of convergence in the elliptic case.

Definition 7.1. *By* $\mathcal{M}_\Omega(p, L, K_1, C_1)$ *and* $\mathcal{N}_\Omega(p, L, K_1, C_1)$ *we denote the subclasses respectively of* $\mathcal{M}_{\Omega\times(0,T)}(p, L, K_1, C_1, F)$ *and of* $\mathcal{N}_{\Omega\times(0,T)}(p, L, K_1, C_1, F)$ *of Carathéodory functions independent of time.*
By $\mathcal{M}_{\Omega\times(0,T)}(p, L, K_1, C_1, F, G)$ *and* $\mathcal{N}_{\Omega\times(0,T)}(p, L, K_1, C_1, F, G)$ *we denote the subclasses respectively of* $\mathcal{M}_{\Omega\times(0,T)}(p, L, K_1, C_1, F)$ *and* $\mathcal{N}_{\Omega\times(0,T)}(p, L, K_1, C_1, F)$ *for which the following holds:*

$$\begin{aligned} &\text{there is } \lambda \in \Lambda(a) \text{ with } \lambda = \lambda(x) \text{ such that} \\ &\big|a(x,t,\xi) - a(x,s,\xi)\big| \leqslant \lambda(x) G(|t-s|)\big(1 + |\xi|^{p-1}\big) \end{aligned} \tag{95}$$

for almost every $x \in \Omega$, *for every* $t, s \in [0,T]$, *for every* $\xi \in \mathbf{R}^n$, *where* $G : [0,T] \to [0,+\infty)$ *is a bounded function, countinuos in zero and such that* $G(0) = 0$.

Definition 7.2. *A sequence $(a_h)_h \subset \mathcal{N}_\Omega(p, L, K_1, C_1)$ is said to EG-converge to $a \in \mathcal{N}_\Omega(p, L,\ K_1, C_1)$ in Ω, and we will write*

$$a_h \stackrel{EG}{\longrightarrow} a \quad in\, \Omega\,,$$

if for every $f \in L^n(\Omega)$ we have that

$$\begin{aligned} & u_h \to u && in\ L^1(\Omega) \\ & a_h(\cdot, Du_h) \to a(\cdot, Du) && in\ L^1(\Omega)^n\text{-}weak, \end{aligned}$$

where u_h and u are respectively the solutions of

$$\begin{cases} -\mathrm{div}\big(a_h(x, Dw)\big) = f & \text{on } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} -\mathrm{div}\big(a(x, Dw)\big) = f & \text{on } \Omega \\ w = 0 & \text{on } \partial\Omega\,. \end{cases}$$

The following result is a particular case of Theorem 3.5 in [11].

Theorem 7.3. *There is $L' = L'(p, L)$ such that the class $\mathcal{M}_\Omega(p, L, K_1, C_1)$ is relatively compact in $\mathcal{N}_\Omega(p, L'K_1, K_1, C_1)$ with respect to EG-convergence.*

Lemma 7.4. *Consider $(a_h)_h, (b_h)_h \subset \mathcal{M}_\Omega(p, L, K_1, C_1)$ such that $a_h \stackrel{EG}{\longrightarrow} a$ and $b_h \stackrel{EG}{\longrightarrow} b$ in Ω. Suppose moreover there is a constant $c > 0$ such that*

$$\big|a_h(x, \xi) - b_h(x, \xi)\big| \leqslant c\, \lambda_h(x)\, \big(1 + |\xi|^{p-1}\big)$$

where $\lambda_h(x) := \lambda_{a_h}(x)$ is such that $\lambda_h \to \lambda$ in $L^1(\Omega)$-weak. Then there is a constant $c' \geqslant c$, c' depending (only) on $c, p, L, |\Omega|$, such that

$$\big|a(x, \xi) - b(x, \xi)\big| \leqslant c'\, \lambda(x)\, \big(1 + |\xi|^{p-1}\big)\,.$$

Proof - Consider a sequence $(\lambda_h)_h \subset A_p(K)$, $\lambda_h = \lambda_h(x)$ and suppose that $\lambda_h \to \lambda$, $\lambda_h^{-1/(p-1)} \to \tilde{\lambda}^{-1/(p-1)}$ in $L^1(\Omega)$-weak. Consider $f, g \in L^n(\Omega)$ and let $-\mathrm{div}\,(a_h(x, Du_h)) = f$, $-\mathrm{div}\,(b_h(x, Dv_h)) = g$, $u, v \in W_0^{1,p}(\Omega, \lambda_h, \lambda_h)$. Let λ be the weak limit (up to subsequences) of λ_h and let $u, v \in W_0^{1,p}(\Omega, \lambda, \lambda)$ such that $-\mathrm{div}\,(a(x, Du)) = f$, $-\mathrm{div}\,(b(x, Dv)) = g$. First of all using the assumption and the Young's inequality (in what follows $\epsilon > 0$, c_ϵ a constant depending only on ϵ, c_1 depending on c and p) we get

$$\begin{aligned} |Du_h - Dv_h|^p & \leqslant \lambda_h^{-1}\big(b_h(x, Du_h) - b_h(x, Dv_h), Du_h - Dv_h\big) \\ & \leqslant \lambda_h^{-1}\big[\big(b_h(x, Du_h) - a_h(x, Du_h), Du_h - Dv_h\big) + \\ & \qquad + \big(a_h(x, Du_h) - b_h(x, Dv_h), Du_h - Dv_h\big)\big] \\ & \leqslant c_1\, (1 + |Du_h|^{p-1})|Du_h - Dv_h| + \\ & \qquad + \lambda_h^{-1}\big(a_h(x, Du_h) - b_h(x, Dv_h), Du_h - Dv_h\big) \\ & \leqslant c_1\, \frac{c_\epsilon}{p'}(1 + |Du_h|^{p-1})^{p'} + c\, \frac{\epsilon}{p}|Du_h - Dv_h|^p + \\ & \qquad + \lambda_h^{-1}\big(a_h(x, Du_h) - b_h(x, Dv_h), Du_h - Dv_h\big) \end{aligned}$$

by which, choosing a suitable $\epsilon > 0$, we find a constant $c_2 > c$ depending on c, ϵ and p such that

$$
\begin{aligned}
&|Du_h - Dv_h|^p \leqslant \\
&\leqslant c_2\big[(1+|Du_h|^{p-1})^{p'} + \lambda_h^{-1}\big(a_h(x,Du_h) - b_h(x,Dv_h), Du_h - Dv_h\big)\big] \\
&\leqslant c_3\big[1 + \lambda_h^{-1}(a_h(x,Du_h), Du_h) + \\
&\qquad + \lambda_h^{-1}\big(a_h(x,Du_h) - b_h(x,Dv_h), Du_h - Dv_h\big)\big]
\end{aligned} \tag{96}
$$

with $c_3 = c_3(c,\epsilon,p) \geqslant c_2$. Consider $\varphi \geqslant 0$, $\varphi \in C_c^\infty(\Omega)$. Since by assumption we have $a_h(x,Du_h) - b_h(x,Dv_h) \to a(x,Du) - b(x,Dv)$ in $(L^1(\Omega))^n$-weak, by Lemma 5.5 we derive that (c_4 a constant depending only on p)

$$
\begin{aligned}
&\int_\Omega \big|a(x,Du) - b(x,Dv)\big|^{p'} \lambda^{1-p'} \varphi\, dx \leqslant \\
&\leqslant \liminf_h \int_\Omega \big|a_h(x,Du_h) - b_h(x,Dv_h)\big|^{p'} \lambda_h^{1-p'} \varphi\, dx \leqslant \\
&\leqslant c_4\Big[\liminf_h \int_\Omega \big|a_h(x,Du_h) - b_h(x,Du_h)\big|^{p'} \lambda_h^{1-p'} \varphi\, dx + \\
&\qquad + \liminf_h \int_\Omega \big|b_h(x,Du_h) - b_h(x,Dv_h)\big|^{p'} \lambda_h^{1-p'} \varphi\, dx\Big].
\end{aligned} \tag{97}
$$

As regard the first term of the right hand side from the assumptions we get

$$
\begin{aligned}
&\int_\Omega \big|a_h(x,Du_h) - b_h(x,Du_h)\big|^{p'} \lambda_h^{1-p'} \varphi\, dx \leqslant c \int_\Omega \lambda_h^{p'} \big(1+|Du_h|^{p-1}\big)^{p'} \lambda_h^{1-p'} \varphi\, dx \leqslant \\
&\leqslant c\, c_5 \int_\Omega \lambda_h \big(1+|Du_h|^p\big)\varphi\, dx \leqslant \\
&\leqslant c\, c_5 \int_\Omega \lambda_h \big(1+(a_h(x,Du_h),Du_h)\lambda_h^{-1}\big)\varphi\, dx = \\
&= c\, c_5 \int_\Omega \big(\lambda_h + (a_h(x,Du_h),Du_h)\big)\varphi\, dx
\end{aligned} \tag{98}
$$

with $c_5 = c_5(p)$. By Theorem 2.3 in [10] and by assumption we derive that

$$
\lim_{h\to\infty} \int_\Omega \big(\lambda_h + (a_h(x,Du_h),Du_h)\big)\varphi\, dx = \int_\Omega \big(\lambda + (a(x,Du),Du)\big)\varphi\, dx.
$$

As regard the second term of the right hand side in (97) we estimate

$$
\begin{aligned}
&\int_\Omega \big|b_h(x,Du_h) - b_h(x,Dv_h)\big|^{p'} \lambda_h^{1-p'} \varphi\, dx \leqslant \\
&\leqslant \int_\Omega L^{p'} \lambda_h^{p'} \big(1+|Du_h|^p + |Dv_h|^p\big)^{\frac{p-2}{p}p'} |Du_h - Dv_h|^{p'} \lambda_h^{1-p'} \varphi\, dx = \\
&= \int_\Omega L^{p'} \lambda_h \big(1+|Du_h|^p + |Dv_h|^p\big)^{\frac{p-2}{p-1}} |Du_h - Dv_h|^{\frac{p}{p-1}} \varphi\, dx \leqslant \\
&\leqslant \int_\Omega L^{p'} \lambda_h \big(1+\lambda_h^{-1}(a_h(x,Du_h),Du_h) + \lambda_h^{-1}(b_h(x,Dv_h),Dv_h)\big)^{\frac{p-2}{p-1}} |Du_h - Dv_h|^{\frac{p}{p-1}} \varphi\, dx \\
&= \int_\Omega L^{p'} \lambda_h^{\frac{1}{p-1}} \big(\lambda_h + (a_h(x,Du_h),Du_h) + (b_h(x,Dv_h),Dv_h)\big)^{\frac{p-2}{p-1}} |Du_h - Dv_h|^{\frac{p}{p-1}} \varphi\, dx.
\end{aligned}
$$

To estimate this we put, for simplicity, $A_h := (a_h(x, Du_h), Du_h)$, $B_h := (b_h(x, Dv_h), Dv_h)$ and $D_h := (a_h(x, Du_h) - b_h(x, Dv_h), Du_h - Dv_h)$ and use inequality (96) to obtain

$$\int_\Omega \big|b_h(x, Du_h) - b_h(x, Dv_h)\big|^{p'} \lambda_h^{1-p'} \varphi\, dx \leqslant$$
$$\leqslant L^{p'} \int_\Omega \lambda_h^{\frac{1}{p-1}} \big(\lambda_h + A_h + B_h\big)^{\frac{p-2}{p-1}} \big[c_3(1 + \lambda_h^{-1} A_h + \lambda_h^{-1} D_h)\big]^{\frac{1}{p-1}} \varphi\, dx\,.$$

Then using Hölder's inequality we find a constant, denoted by c_6, depending on L, c_3 and $|\Omega|$, such that

$$\int_\Omega \big|b_h(x, Du_h) - b_h(x, Dv_h)\big|^{p'} \lambda_h^{1-p'} \varphi\, dx \leqslant$$
$$\leqslant c_6 \Big[\int_\Omega \big(\lambda_h + A_h + B_h\big)\varphi\, dx\Big]^{\frac{p-2}{p-1}} \Big[\int_\Omega \big(\lambda_h + A_h + D_h\big)\varphi\, dx\Big]^{\frac{1}{p-1}}\,.$$

Taking the limit and using Theorem 2.3 in [10] we get

$$\liminf_{h\to\infty} \int_\Omega \big|b_h(x, Du_h) - b_h(x, Dv_h)\big|^{p'} \lambda_h^{1-p'} \varphi\, dx \leqslant$$
$$\leqslant c_6 \Big[\int_\Omega \big(\lambda + A + B\big)\varphi\, dx\Big]^{\frac{p-2}{p-1}} \Big[\int_\Omega \big(\lambda + A + D\big)\varphi\, dx\Big]^{\frac{1}{p-1}}$$

where $A := (a(x, Du), Du)$, $B := (b(x, Dv), Dv)$ and $D := (a(x, Du) - b(x, Dv), Du - Dv)$. Taking into account this last inequality and (98) in (97) we get

$$\int_\Omega \big|a(x, Du) - b(x, Dv)\big|^{p'} \lambda^{1-p'} \varphi\, dx \leqslant c_7 \Big[\int_\Omega (\lambda + (a(x, Du), Du))\varphi\, dx +$$
$$+ \Big(\int_\Omega (\lambda + (a(x, Du), Du) + (b(x, Dv), Dv))\varphi\, dx\Big)^{\frac{p-2}{p-1}} \cdot$$
$$\cdot \Big(\int_\Omega (\lambda + (a(x, Du), Du) + (a(x, Du) - b(x, Dv), Du - Dv))\varphi\, dx\Big)^{\frac{1}{p-1}}\Big]$$

where $c_7 = \max\{c\, c_5, c_6\}$. This last inequlity holds for every $u \in E_a^{-1}(L^n(\Omega))$ and $v \in E_b^{-1}(L^n(\Omega))$, where $E_a u = -\mathrm{div}(a(x, Du))$, $E_b v = -\mathrm{div}(b(x, Dv))$. By density this holds for every $u, v \in W_0^{1,p}(\Omega, \lambda, \lambda)$. In particular choosing $u = v = \psi(x)\big(x, \xi\big)$ where $\xi \in \mathbf{R}^n$, $\psi \in C_c^1(\Omega)$, $\psi \equiv 1$ in $\omega \subset\subset \Omega$, we obtain, since the above inequality holds for every $\varphi \in C_c^\infty(\Omega)$, $\varphi \geqslant 0$,

$$\big|a(x, \xi) - b(x, \xi)\big|^{p'} \lambda^{1-p'}(x) \leqslant$$
$$\leqslant c_7 \Big[(\lambda(x) + (a(x, \xi), \xi)) + \Big(\lambda(x) + (a(x, \xi), \xi) + (b(x, \xi), \xi)\Big)^{\frac{p-2}{p-1}} \cdot$$
$$\cdot \Big((\lambda(x) + (a(x, \xi), \xi)\Big)^{\frac{1}{p-1}}\Big]$$
$$\leqslant c_7 \Big[(\lambda(x) + 3\lambda(x)(1 + |\xi|^{p-2})|\xi|^2\Big]$$

almost everywhere in ω (and then in Ω). Finally by this we conclude deriving the existence of a constant $c' > c$ for which

$$\big|a(x, \xi) - b(x, \xi)\big| \leqslant c'\, \lambda(x)\big(1 + |\xi|^{p-1}\big)\,. \qquad \square$$

Corollary 7.5. *Consider* $(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(p, L, K_1, C_1, F, G)$*, i.e. such that*

$$\big|a_h(x,t,\xi) - a_h(x,s,\xi)\big| \leqslant \lambda_h(x)\, G(|t-s|)\big(1+|\xi|^{p-1}\big)\,.$$

Then there is a constant $c > 1$*, a constant* L' *depending (only) on* L, p *and a map* $a \in \mathcal{N}_\Omega(p, L'K_1, K_1, C_1, F, c\,G)$ *such that, up to subsequences,* $a_h(\cdot,t,\cdot) \xrightarrow{EG} a(\cdot,t,\cdot)$ *for every* $t \in [0,T]$*.*

Proof - For every fixed $t \in [0,T]$ the sequence $(a_h(\cdot,t,\cdot))_h \subset \mathcal{M}_\Omega(p,L,K_1,C_1)$. Then by Theorem 7.3, by lower semicontinuity and by a diagonalization argument we can find a subsequence, denoted by $(a_{h_j})_j$, such that

$$a_{h_j}(\cdot,t,\cdot) \xrightarrow{EG} a(\cdot,t,\cdot) \qquad \text{for every } t \in \mathbf{Q}\cap[0,T]\,.$$

By Lemma 7.4 we also get that there is $\lambda \in A_p$ and $c > 1$ such that

$$\big|a(x,t,\xi) - a(x,s,\xi)\big| \leqslant c\,\lambda(x)\, G(|t-s|)\big(1+|\xi|^{p-1}\big) \qquad \text{for every } t,s \in \mathbf{Q}\cap[0,T]$$

which can be estended by continuity to a map a defined for every $t \in [0,T]$.
Now fix $\bar t \in [0,T]\setminus\mathbf{Q}$: for every subsequence of $(a_{h_j})_j$ we can find a further subsequence, denoted by $(a_{h_{j_k}})_k$, and a map b such that

$$a_{h_{j_k}}(\cdot,\bar t,\cdot) \xrightarrow{EG} b(\cdot,\bar t,\cdot)\,.$$

By Lemma 7.4 we have that

$$\big|b(x,\bar t,\xi) - a(x,t,\xi)\big| \leqslant c\,\lambda(x)\, G(|\bar t-t|)\big(1+|\xi|^{p-1}\big) \qquad \text{for every } t \in \mathbf{Q}\cap[0,T]\,.$$

By the continuity of G we derive that $b(x,\bar t,\cdot) = a(\cdot,\bar t,\cdot)$. □

Proposition 7.6. *Consider a sequence* $(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(p,L,K_1,C_1,F,G)$*. Then*

$$(0,a_h) \xrightarrow{G} (0,a) \quad \text{in } \Omega\times(0,T)$$

if and only if

$$a_h(\cdot,t,\cdot) \xrightarrow{EG} a(\cdot,t,\cdot) \quad \text{in } \Omega \qquad \text{for every } t\in[0,T]\,,$$

$a \in \mathcal{M}_{\Omega\times(0,T)}(p,L,K_1,C_1,F,c\,G)$ *for any* $c > 0$*.*

Proof - We can suppose that $(0,a_h) \xrightarrow{G} (0,a)$ in $\Omega\times(0,T)$ and $a_h(\cdot,t,\cdot) \xrightarrow{EG} b(\cdot,t,\cdot)$ in Ω for every $t \in [0,T]$, otherwise we can extract a common subsequence, thanks to Theorem 6.9 and Corollary 7.5, and prove the result for every further subsequence. Consider a function $f(x,t) = g(x)h(t)$, $g \in L^n(\Omega)$ and $h \in C([0,T])$, and denote by u_h the solution of

$$\begin{cases} -\mathrm{div}\big(a_h(x,t,Dw)\big) = f & \text{in } \Omega\times(0,T) \\ w = 0 & \text{in } \partial\Omega\times(0,T) \end{cases}$$

which coincides, for a.e. $t \in [0,T]$, with the solution

$$\begin{cases} -\mathrm{div}\big(a_h(x,t,Dw)\big) = f(\cdot,t) & \text{in } \Omega \\ w = 0 & \text{in } \partial\Omega\,. \end{cases}$$

and the sequence $(u_h)_h$ is such that $u_h \to u$ in $L^p(0,T;L^1(\Omega))$. By Theorem 5.15 we get that for every $\phi \in C_c^\infty(\Omega \times (0,T))$

$$\int_0^T \int_\Omega \big(a_h(x,t,Du_h), Du_h\big)\phi(x,t)dxdt \to \int_0^T \int_\Omega \big(a(x,t,Du), Du\big)\phi(x,t)dxdt$$

and by Theorem 2.3 in [10] for every $\varphi \in C_c^\infty(\Omega)$ and $t \in [0,T]$

$$\int_\Omega \big(a_h(x,t,Du_h), Du_{h_j}\big)\varphi(x)dx \to \int_\Omega \big(b(x,t,Du), Du\big)\varphi(x)dx\,.$$

Define the functions

$$\zeta_h^u(t) = \int_\Omega \big(a_h(x,t,Du_h), Du_h\big)\varphi(x)dx\,.$$

Since by Proposition 2.7 in [11] and (17) we get that there is a constant c (depending only on $n, p, K_1, K_2, \Omega, C_1, L$) such that

$$\big|\zeta_h^u(t)\big| \leqslant c\,\big|h(t)\big|\|\varphi\|_\infty \|g\|_{L^n(\Omega)}$$

with $h \in C([0,T])$ and $\zeta_h^u(t) \to \zeta_b^u(t) := \displaystyle\int_\Omega \big(b(x,t,Du), Du\big)\varphi(x)dx$ for every $t \in [0,T]$ we get that

$$\int_0^T \tau(t)\zeta_h^u(t)dt \to \int_0^T \tau(t)\zeta_b^u(t)dt$$

for every $\tau \in C_c^\infty(0,T)$. On the other side, taking $\phi(x,t) = \varphi(x)\tau(t)$ in the limit above, we get

$$\int_0^T \tau(t)\zeta_h^u(t)dt \to \int_0^T \tau(t)\zeta_a^u(t)dt$$

for every $\tau \in C_c^\infty(0,T)$, where $\zeta_a^u(t) := \displaystyle\int_\Omega \big(a(x,t,Du), Du\big)\varphi(x)dx$. We conclude that

$$\int_\Omega \big(a(x,t,Du), Du\big)\varphi(x)dx = \int_\Omega \big(b(x,t,Du), Du\big)\varphi(x)dx$$

for every $\varphi \in C_c^\infty(\Omega)$ and for almost every $t \in [0,T]$. Then $(a(x,t,Du), Du) = (b(x,t,Du), Du)$. Since this holds for every $u \in \mathcal{A}^{-1}(L^n(\Omega) \times C([0,T]))$ where $\mathcal{A} = -\mathrm{div}(a(\cdot,\cdot,D))$, by the density in $L^{p'}(0,T;L^n(\Omega))$ of the finite sums of functions in $L^n(\Omega) \times C([0,T])$ and arguing as in the last part of the proof of Lemma 7.4 we conclude that $b(x,t,\xi) = a(x,t,\xi)$ (for a.e. $x \in \Omega$, every $t \in [0,T]$ and every $\xi \in \mathbf{R}^n$). □

Theorem 7.7. *Consider a sequence* $(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(p, L, K_1, C_1, F, G)$. *The following are equivalent:*

i) $a_h(\cdot,t,\cdot) \xrightarrow{EG} a(\cdot,t,\cdot)$ *in* Ω *for every* $t \in [0,T]$,

ii) $a_h \xrightarrow{PG} a$ *in* $\Omega \times (0,T)$

Proof - The proof follows immediatly from Theorem 6.13 and Proposition 7.6. □

Remark 7.8. - If we drop assumption of continuity in time, i.e. we consider a sequence $(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(p, L, K_1, C_1, F) \setminus \mathcal{M}_{\Omega\times(0,T)}(p, L, K_1, C_1, F, G)$, the last theorem could not hold (see [7]).

8. Some particular cases

Consider the equations $\partial_t(\mu_h u) - \mathrm{div}\big(a_h(x,t,Du)\big) = f$ for some functions $a_h \in \mathcal{M}_{\Omega\times(0,T)}$ and $\mu_h \in \mathcal{F}(\lambda_h)$ for $\lambda_h \in \Lambda(a_h)$. Here we analyse briefly some particular examples, first the "classical" case, i.e. $\lambda_0 \leqslant \mu_h, \lambda_h \leqslant \Lambda_0$ with $\lambda_0 > 0$, then we give some compactness result in two particular cases, one with $\mu_h \equiv 1$ for every $h \in \mathbf{N}$ and finally a situation in which assumptions (T.1) are easily satisfied.

Example 1 - Consider $\lambda_h \equiv \lambda_0$, λ_0 positive constant. In this case every sequence $(\mu_h)_h$ equibounded in $L^\infty(\Omega\times(0,T))$ satisfying (T.5) and (T.6) belongs to $\mathcal{F}(\lambda_0)$, but also unbounded μ_h can be considered, provided that $\sup_t \sup_Q \sup_h |Q|^{\frac{\alpha}{n}} (|Q|^{-1}\int_\Omega \mu_h(x,t)dx)^{1/p}$ is bounded, i.e. (T.1)-b) holds. Also μ_h valued in $\{-1,0,1\}$, i.e. discontinuous, can be considered. For this we refer to examples in [24] (see also below).
In particular Theorem 6.8 with $\mu_h \equiv 1$ for every $h \in \mathbf{N}$ and Theorem 7.7 give the results contained in [34]. Observe that in particular the convergence

$$\begin{cases} -\mathrm{div}\big(a_h(x,t,Du)\big) = f & \text{in } \Omega\times(0,T) \\ u = 0 & \text{in } \partial\Omega\times(0,T) \end{cases} \to \begin{cases} -\mathrm{div}\big(b_1(x,t,Du)\big) = f & \text{in } \Omega\times(0,T) \\ u = 0 & \text{in } \partial\Omega\times(0,T) \end{cases}$$

in the sense of Definition 6.1 and the convergence

$$\begin{cases} -\mathrm{div}\big(a_h(x,t,Du)\big) = f(t) & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases} \to \begin{cases} -\mathrm{div}\big(b_2(x,t,Du)\big) = f(t) & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

for almost every $t \in (0,T)$ in the sense of Definition 7.2 do not garantee $b_1 = b_2$ (see [7]). A sufficient condition is (95).

Example 2 - $\mu_h \equiv 1$ - In this case Theorem 6.8 gives a compactness result for a sequence of classical evolution equations of the type $\partial_t u - \mathrm{div}\big(a_h(x,t,Du)\big) = f$. Notice that (what we are going to say can be derived following the analogous results in [27], precisely Theorem 2.4, Corollary 2.5, Remark 2.6 and Remark 2.7) assumption (T.1)-b), i.e. $(1,\lambda_h(t)) \in A^\alpha_{p,p}(K_2)$ for every t is satisfied with $\alpha < 1$ if $\lambda_h(t) \in A_q(K_1) \subseteq A_p(K_1)$ for any K_1 (for every $h \in \mathbf{N}$ and every $t \in [0,T]$) with $q = \min\{p, 1+p/n\} = 1+p/n$ (since we are taking $n, p \geqslant 2$). Moreover the Poincaré inequality (22) holds with $\mu \equiv 1$ (in general) again if $(\lambda_h(t))_h \subset A_{1+p/n}(K_1)$ and if moreover $\int_\Omega \lambda_h^{-n/p}(x,t)dx \leqslant C_1$ for every $t \in [0,T]$. Therefore taking $\lambda_h = \lambda_h(x)$ so that (T.5) and (T.6) are easily satisfied and modifying (S.4) and (S.5) as follows

$$\lambda \in A_{1+p/n}(K_1)\,, \qquad \int_\Omega \lambda^{-n/p}(x)dx \leqslant C_1\,, \tag{99}$$

Theorem 6.8 reads as follows (compare with [27] in which for the linear case the class $A_{1+2/n}$ is considered).

Theorem 8.1. *$(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(p,L,K_1,C_1,F)$, and suppose $\lambda_h \in \Lambda(a_h)$ is independent of time and satisfies (99) for every $h \in \mathbf{N}$. Then there is $a \in \mathcal{N}_{\Omega\times(0,T)}(p,L^*, K_1, C_1, K_1^{-1}F)$, $L^* = L'K_1$ with $L' = L'(L,p)$, a weight $\lambda \in \Lambda(a)$ independent of time such that $\lambda \in A_{1+p/n}(K_1)$ and for which, up to a subsequence,*

$$(1,a_h) \xrightarrow{G} (1,a) \quad \text{in } \Omega\times(0,T)\,.$$

Notice that in this case the convergence of the solutions are in the topology of $L^p(\Omega \times (0,T))$ (see Corollary 5.4).

Example 3 - A general situation in which (T.1) is easily satisfied is the following: given $(\lambda_h)_h$ consider a sequence $(\mu_h)_h$ satisfying

$$0 \leqslant |\mu_h| \leqslant c_h \, \lambda_h$$

for some positive constants c_h, $c_h \leqslant c < +\infty$. A simple situation could be the following:

$$\lambda_h = \lambda_h(x)\,, \qquad \mu_h = r_h(x,t)\lambda_h(x)$$

with r_h equibounded in L^∞. As in Example 1 r_h may be not regular, also in time; for instance $r_h = \chi_{A_h}$ the characteristic function of $A_h \subset \Omega \times (0,T)$, denoting by $A_h(t) = \{x \in \Omega \,|\, (x,t) \in A\}$, assumptions (34) become requirements on the sets $A_h(t)$ since in this case the functions

$$t \mapsto \int_{A_h(t)} u(x)v(x)\lambda(x)dx$$

are to be regular and satisfy (34)(iv)-(vi) for every $u, v \in C^1_c(\Omega)$ (see, e.g, Proposition 3, section 3.4.4, in [13] for more detalis on differentiability of these functions). For more examples of admissible μ_h we refer to [24] and [26].
A particular case of this situation is that in which $\mu_h \equiv \lambda_h = \lambda_h(x)$. In this case we obtain that (T.1) is free and then we obtain the following result (compare with [23]).

Theorem 8.2. *Consider $(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(p, L, K_1, C_1, F)$ and $\lambda_h \in \Lambda(a_h)$ independent of time. Then there is $a \in \mathcal{N}_{\Omega\times(0,T)}(p, L^*, K_1, C_1, K_1^{-1}F)$, $L^* = L'K_1$ with $L' = L'(L,p)$, a weight $\lambda \in \Lambda(a)$ independent of time for which, up to a subsequence,*

$$(\lambda_h, a_h) \overset{G}{\longrightarrow} (\lambda, a) \quad \text{in } \Omega \times (0,T)\,.$$

As corollary of this theorem and Theorem 3.8 (as done in the proof of Theorem 6.9 one can consider $(\epsilon\lambda_h, a_h)$ and then let $\epsilon \to 0$) we obtain what follows.

Corollary 8.3. *Consider $(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(p, L, K_1, C_1, F)$. Then there is $a \in \mathcal{N}_{\Omega\times(0,T)}(p, L^*, K_1, C_1, K_1^{-1}F)$, $L^* = L'K_1$ with $L' = L'(L,p)$, a weight $\lambda \in \Lambda(a)$ for which, up to a subsequence,*

$$(0, a_h) \overset{G}{\longrightarrow} (0, a) \quad \text{in } \Omega \times (0,T)\,.$$

Thanks to Proposition 7.6, this result applied to the class $\mathcal{M}_{\Omega\times(0,T)}(p, L, K_1, C_1, F, G)$ (defined at the beginning of Section 7) allow us to reobtain, as a particular case, Theorem 7.3, i.e. the result contained in [11] for elliptic equations.

Corollary 8.4. *The class $\mathcal{M}_{\Omega\times(0,T)}(p, L, K_1, C_1, F, G)$ is relatively compact in $\mathcal{N}_{\Omega\times(0,T)}(p, L'K_1, K_1, C_1, K_1^{-1}F, c\,G)$ with respect to the pointwise (with respect to time) EG-convergence in Ω ($L' = L'(p, L)$, $c = c(p, L, |\Omega|)$).*

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