SADDLE-SHAPED SOLUTIONS FOR THE FRACTIONAL ALLEN-CAHN EQUATION

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Abstract. We establish existence and qualitative properties of solutions to the fractional Allen-Cahn equation, which vanish on the Simons cone and are even with respect to the coordinate axes. These solutions are called saddle-shaped solutions.

More precisely, we prove monotonicity properties, asymptotic behaviour, and instability in dimensions $2m = 4, 6$. We extend to any fractional power s of the Laplacian, some results obtained for the case $s = 1/2$ in [19].

The interest in the study of saddle-shaped solutions comes in connection with a celebrated De Giorgi conjecture on the one-dimensional symmetry of monotone solutions and of minimizers for the Allen-Cahn equation. Saddle-shaped solutions are candidates to be (not one-dimensional) minimizers in high dimension, a property which is not known to hold yet.

1. Introduction

In this paper we study existence and qualitative properties of saddle-shaped solutions (see definition 1.1) to the equation

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $s \in (0, 1)$, $n = 2m$ is an even integer and f is of bistable type. In particular, we extend the results contained in [19] for the case $s = 1/2$ to any fractional power of the Laplacian $0 < s < 1$.

The saddle-shaped solutions that we consider are even with respect to the coordinate axes and odd with respect to the Simons cone, which is defined as follows. For $n = 2m$ the Simons cone $\mathcal{C}$ is given by:

$$\mathcal{C} = \{x \in \mathbb{R}^{2m} : x_1^2 + ... + x_m^2 = x_{m+1}^2 + ... + x_{2m}^2\}.$$

We recall that the Simons cone has zero mean curvature at every point $x \in \mathcal{C} \setminus \{0\}$, in every dimension $2m \geq 2$. Moreover in dimensions $2m \geq 8$ it is a minimizer of the area functional, as established by Bombieri, De Giorgi, and Giusti in [4].

We define two new variables

$$s = \sqrt{x_1^2 + \cdots + x_m^2} \quad \text{and} \quad t = \sqrt{x_{m+1}^2 + \cdots + x_{2m}^2}, \tag{1.2}$$

for which the Simons cone becomes $\mathcal{C} = \{s = t\}$.

Definition 1.1. We say that a solution u of (1.1) is a *saddle-shaped solution*, if u depends only on s and t, and it satisfies $u > 0$ for $s > t$ and $u(s, t) = -u(t, s)$.

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Observe that, obviously, the Simons cone $\mathcal{C}$ is the zero-level set of u.

We are interested in the study of this type of solutions since they are relevant in connection to the following well known De Giorgi conjecture for the (classical) Allen-Cahn equation: let u be a bounded monotone (in some direction) solution to

$$-\Delta u = u - u^3 \quad \text{in } \mathbb{R}^n, \tag{1.3}$$

then, if $n \leq 8$, u depends on only one Euclidean variable, that is all its level sets are hyperplanes.

The motivation for this conjecture relies on the classical Modica-Mortola result which states that the energy functional associated to equation (1.3), after a suitable rescaling, Γ-converges to the area functional. This result establish a very deep connection between the classification of area-minimizing surfaces and the classification of certain solutions to (1.3). It is well known that when $n \leq 7$, any area minimzing surface in the all $\mathbb{R}^n$ is necessarely an hyperplane. The Simons cone is the first example of a singular area minimizing surface in $\mathbb{R}^8$.

The De Giorgi conjecture has been proven to be true in dimension $n = 2$ by Ghoussoub and Gui [27] and in dimension $n = 3$ by Ambrosio and Cabré [2]. For $4 \leq n \leq 8$, if $\partial_{x_n} u > 0$, and assuming the additional condition

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \quad \text{for all } x' \in \mathbb{R}^{n-1},$$

it has been established by Savin [31]. Moreover Savin proved the conjecture for solutions that are minimizers of the associated energy functional (without any monotonicity assumption) in any dimension $4 \leq n \leq 7$. A counterexample to the conjecture for $n \geq 9$ in the setting of monotone solutions has been found by del Pino, Kowalczyk and Wei [24], while for $n \geq 8$ in the setting of minimizers by Liu, Wang, Wei in the recent contribution [30]. In these last two references, the minimality property of the Simons cone plays a crucial role. Besides the solutions constructed in [30], saddle-shaped solutions are candidates to be minimizers in dimensions $2m \geq 8$, a property which is not established yet and turns out to be a difficult problem.

For the fractional equation $(-\Delta)^s u = f(u)$ in $\mathbb{R}^n$ with $0 < s < 1$, the De Giorgi conjecture has been proven to be true when $n = 2$ and $s = 1/2$ by Cabré and Solà-Morales [14], and when $n = 2$ and for every $0 < s < 1$ by Cabré and Sire [12], and by Sire and Valdinoci [36]. In all these references the proof of the conjecture makes use of the so-called Caffarelli-Silvestre extension. We emphasize that in the recent contribution [6], Bucur and Valdinoci give an alternative proof of the conjecture in dimension $n = 2$, which does not use the extension. When $n = 3$ and for every power $1/2 \leq s < 1$ the conjecture has been established in [8, 9] by Cabré and the author. Very recently Savin [32] extended his result for the local case (in dimensions $4 \leq n \leq 8$ for monotone solutions with limits at ∞ and in dimensions $4 \leq n \leq 7$ for minimizers) to any fractional power of the Laplacian $1/2 < s < 1$; in [25] under the same assumption on the limits at ∞ the conjecture has been established in dimension $n = 3$ and for $0 < 1/2 < 1$. Finally in [28] and in the forthcoming paper [11], the conjecture has been established in dimension $n = 3$ and for any $0 < s < 1/2$ for any monotone solutions (without any additional assumption). Counterexample to the conjecture in high dimension for the fractional problem are not known yet.

We remind also the analogue of the Modica-Mortola result in this fractional setting: in [34], Savin and Valdinoci proved that, after a suitable rescaling, the energy functional associated to the fractional Allen-Cahn equation Γ-converges to the (classical) perimeter functional when $1/2 \le s < 1$ and to the fractional perimeter when $0 < s < 1/2$. Hence when $1/2 \le s < 1$ the De Giorgi conjecture is expected to be true up to dimension $n = 8$ (as already mentioned, Savin proved this fact for $1/2 < s < 1$), while when $0 < s < 1/2$ to guess which is the critical dimension for the validity of the conjecture, one should know the classification for minimizers of the fractional perimeter. The notion of fractional perimeter was introduced in [17] by Caffarelli, Roquejoffre, and Savin, but the only available classification results are just in dimension $n = 2$ (see [33] and [21]) and there are some partial results in dimension $n = 3$ (more precisely, for minimal graphs see [26] and for stable cones but only for s sufficiently close to $1/2$ see [10]).

Saddle solutions for the classical equation $-\Delta u = f(u)$ when $n = 2$ were studied in [22, 35]. In higher dimension they were considerd in [7, 15, 16], where existence and qualitative properties were established.

In [19], the author started the study of these solutions for the fractional Allen-Cahn equation, for $s = 1/2$. The main result of [19] are: existence of saddle-shaped solutions in all $\mathbb{R}^{2m}$, monotonicity properties, asymptotic behaviour, and instability in dimensions $2m = 4, 6$. In this contribution we extend the results in [19] to any fractional power of the Laplacian $0 < s < 1$.

The instability properties in low dimensions are interesting in connection with the problem, described above, of understanding the minimality property for saddle-shaped solutions (which is still an open problem even in the classical case). Regarding stability properties, we mention the recent contribution [23] where the authors study stability for the, so called, nonlocal minimal Lawson cones (that is, Lowson cones which are stationary for the fractional perimeter functional). The class of Lowson cones includes, in particular, the Simons cone. In [23], the authors prove that for s sufficiently close to 0, nonlocal minimal Lawson cones are unstable in $\mathbb{R}^n$ for $n \le 6$ and stable for $n \ge 7$. This result shows an interesting difference between the nonlocal setting and the local one, indeed Lawson cones are unstable for the classical perimeter also in dimension $n = 7$. On the other side, the instability result of [23] in the nonlocal setting up to dimension 6 is consistent with the instability result for saddle-shaped solution to (1.1) in dimension $n = 4, 6$ that we prove in this paper (see Theorem 2.2 below). We stress however that, while the result in [23] holds only for $s \sim 0$, our instability result holds for any $0 < s < 1$.

2. Some preliminaries and main results

To study the nonlocal problem (1.1) we will use the so-called Caffarelli-Silvestre extension [18], which allow to realize it as a local problem in $\mathbb{R}^{n+1}_+$ with a nonlinear Neumann condition on $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n$. More precisely, given $u = u(x)$ defined on $\mathbb{R}^n$, we consider its s-harmonic extension $v = v(x, \lambda)$ in $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, +\infty)$. It is well known (see [18]) that u is a solution of (1.1) if and only if v satisfies

$$\begin{cases} \operatorname{div}(\lambda^{1-2s}\nabla v) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\dfrac{1}{c_s}\lim\limits_{\lambda\to 0}\lambda^{1-2s}\partial_\lambda v = f(v) & \text{on } \mathbb{R}^n = \partial\mathbb{R}^{n+1}_+, \end{cases} \tag{2.1}$$

where c_s is a constant whose precise value is given by

$$c_s = \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}. \tag{2.2}$$

Since the constant c_s will not play any role in this paper, we will omit it in the sequel.

With this local formulation at hand, we can now give the notion of *stability* and *minimality* in a standard way.

First, we consider the energy functional associated to problem (2.1). Let $\widetilde{\Omega}$ be a bounded Lipschitz domain in $\mathbb{R}^{n+1}_+$. We denote by $\partial^0\widetilde{\Omega}$ the subset of $\partial\widetilde{\Omega}$ which lies on the boundary of $\mathbb{R}^{n+1}$, that is $\partial^0\widetilde{\Omega} = \partial\widetilde{\Omega} \cap \{\lambda = 0\}$. For $f : \mathbb{R} \to \mathbb{R}$, we call G the potential such that $G' = -f$.

The energy functional associated to (2.1) is given by

$$\mathcal{E}_{\widetilde{\Omega}}(v) = \int_{\widetilde{\Omega}} \frac{1}{2}\lambda^{1-2s}|\nabla v|^2 \, dx\, d\lambda + \int_{\partial^0\widetilde{\Omega}} G(v)\, dx. \tag{2.3}$$

We say that v is a minimizer for problem (2.1) in $\widetilde{\Omega}$ if

$$\mathcal{E}_{\widetilde{\Omega}}(v) \le \mathcal{E}_{\widetilde{\Omega}}(w),$$

for any w which coincides with v on $\partial\widetilde{\Omega} \cap \{\lambda > 0\}$. Observe that an admissible competitor w is "free" on the bottom boundary $\Omega \times \{0\}$, due to the Neumann condition in (2.1). We say that v is a global minimizer if it is a minimizer in $\widetilde{\Omega}$, for any bounded $\widetilde{\Omega} \subset \mathbb{R}^{n+1}_+$.

Moroever, we say that a bounded solution v of (2.1) is *stable* if the second variation of the energy $\delta^2\mathcal{E}/\delta^2\xi$, with respect to perturbations ξ compactly supported in $\overline{\mathbb{R}^{n+1}_+}$, is nonnegative. That is, if

$$Q_v(\xi) := \int_{\mathbb{R}^{n+1}_+} \lambda^{1-2s}|\nabla \xi|^2 \, dx d\lambda - \int_{\partial\mathbb{R}^{n+1}_+} f'(v)\xi^2 \, dx \ge 0 \tag{2.4}$$

for every $\xi \in C_0^\infty(\overline{\mathbb{R}^{n+1}_+})$.

In what follows we will assume some or all of the following properties on f:

$$f \ \text{ is odd;} \tag{2.5}$$

$$G \ge 0 = G(\pm 1) \text{ in } \mathbb{R}, \text{ and } G > 0 \text{ in } (-1,1); \tag{2.6}$$

$$f' \ \text{ is decreasing in } (0,1). \tag{2.7}$$

We observe that, if (2.5) and (2.6) hold, then $f(0) = f(\pm 1) = 0$. Conversely, if f is odd in $\mathbb{R}$, positive with f' decreasing in $(0,1)$ and negative in $(1,\infty)$ then f satisfies (2.5), (2.6) and (2.7). A typical example of such stable nonlinearity is $f(u) = u - u^3$, which appears in the well studied Allen-Cahn equation.

Our first result is an existence result for a saddle-shaped solution for problem (1.1) in every even dimension $n = 2m$.

For R, $L > 0$, we define the cylinder

$$C_{R,L} = B_R \times (0, L),$$

where B_R is the open ball in $\mathbb{R}^{2m}$ centered at the origin and of radius R.

Theorem 2.1. *Let $2m \ge 2$ and $0 < s < 1$. Assume that f satisfy (2.5) and (2.6). Then, there exists a saddle solution u, with $|u| < 1$, of the problem*

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^{2m}.$$

Let v be the s-harmonic extension of the saddle solution u in $\mathbb{R}^{2m+1}_+$. If in addition f satisfies (2.7), then

$$Q_v(\xi) \ge 0$$

for any test function $\xi \in C^1(\overline{\mathbb{R}^{2m+1}_+})$ *with compact support in* $\overline{\mathbb{R}^{2m+1}_+}$ *and vanishing on* $\mathcal{C} \times [0, +\infty)$.

The proof of this existence result will be given in Section 3.

For solutions of problem (2.1) depending only on the coordinates s, t and λ, problem (2.1) becomes

$$\begin{cases} -(v_{ss} + v_{tt} + v_{\lambda\lambda}) - (m-1)\left(\dfrac{v_s}{s} + \dfrac{v_t}{t}\right) - \dfrac{1-2s}{\lambda} v_\lambda = 0, & \text{in } \mathbb{R}^{2m+1}_+ \\ -\lim\limits_{\lambda\to 0} \lambda^{1-2s}\partial_\lambda v = f(v) & \text{on } \partial\mathbb{R}^{2m+1}_+, \end{cases} \tag{2.8}$$

while the energy functional becomes

$$\mathcal{E}(v, \widetilde{\Omega}) = c_m \left\{ \int_{\widetilde{\Omega}} \lambda^{1-2s}\, s^{m-1} t^{m-1} \frac{1}{2}(v_s^2 + v_t^2 + v_\lambda^2)\, ds\, dt\, d\lambda + \int_{\partial^0\widetilde{\Omega}} s^{m-1} t^{m-1} G(v)\, ds\, dt \right\}, \tag{2.9}$$

where c_m is a positive constant depending only on m—here we have assumed that $\widetilde{\Omega} \subset \mathbb{R}^{2m+1}_+$ is radially symmetric in the first m variables and also in the last m variables.

As said in the Introduction, saddle-shaped solutions are candidates to be minimizers of the energy functional (2.3), in high dimension (at least for $s \geq 1/2$ one would expect for dimensions $n \geq 8$, due to the minimality of the Simons cone for the classical perimeter functional).

Proving minimality of saddle-shaped solution is a difficult task, which is still open even for the classical Allen-Cahn equation. In this direction, it is interesting to have informations at least on the stability properties of these solutions (we recall that minimality implies stability). In our second main result, we show that saddle-shaped solution are *unstable* in dimensions $2m = 4$, 6 for any $0 < s < 1$.

Theorem 2.2. *Let f satisfy conditions* (2.5), (2.6), (2.7).

Then, every bounded solution u of $(-\Delta)^s u = f(u)$ in $\mathbb{R}^{2m}$ such that $u = 0$ on the Simons cone $\mathcal{C} = \{s = t\}$ and u has the same sign as $s - t$, is unstable in dimensions $2m = 4$ and $2m = 6$.

We stress that in Theorem 2.1 we have established the stability of u under perturbations that vanish on $\mathcal{C} \times [0, +\infty)$, hence our instability result Theorem 2.2 above (where perturbations do not vanish in general on $\mathcal{C} \times [0, +\infty)$) relies crucially on the instability property of the Simons cone in low dimensions.

The proof of this result uses several ingredients, that we will develop in Sections 4-6. More precisely, we will need to establish existence and monotonicity properties for a maximal saddle-solution (see Proposition 5.1 in Section 5). The proof of this result will rely on maximum principles for a fractional Laplacian in bounded domains, that we will provide in Section 4. A last fundamental ingredient in the proof of the instability property is given by the asymptotic behaviour of our saddle-solution, which will be established in Proposition 5.2 in Section 5. Finally, in Section 6, we prove our main Theorem 2.2.

3. The existence result

In this section we prove the existence of a saddle solution u for problem (1.1), by proving the existence of a solution v for the extended problem (2.1) satisfying the following properties:

(1) v depends only on the variables s, t and λ. We write $v = v(s, t, \lambda)$;

(2) $v > 0$ for $s > t$;
(3) $v(s,t,\lambda) = -v(t,s,\lambda)$.

The proof follows the analogue proof of the existence result (Theorem 1.6) in [19]. For the convenience of the reader, we present here the main steps of the proof and we explicitely emphasize when computations differ from the case $s = 1/2$ and when some difficulties arise, due to the presence of the weight λ^{1-2s} in the equations satisfied by v.

We introduce the following sets:

$$\mathcal{O} := \{x \in \mathbb{R}^{2m} : s > t\} \subset \mathbb{R}^{2m},$$
$$\widetilde{\mathcal{O}} := \{(x,\lambda) \in \mathbb{R}^{2m+1}_+ : x \in \mathcal{O}\} \subset \mathbb{R}^{2m+1}_+.$$

Note that

$$\partial\mathcal{O} = \mathcal{C}.$$

Let B_R be the open ball in $\mathbb{R}^{2m}$ centered at the origin and of radius R. We will consider the open bounded sets

$$\mathcal{O}_R := \mathcal{O} \cap B_R = \{s > t, |x|^2 = s^2 + t^2 < R^2\} \subset \mathbb{R}^{2m}.$$
$$\widetilde{\mathcal{O}}_{R,L} := \mathcal{O}_R \times (0,L) = \{(x,\lambda) \in \mathbb{R}^{2m+1}_+ : s > t,\ |x|^2 = s^2 + t^2 < R^2, \lambda < L\}.$$

Note that

$$\partial\mathcal{O}_R = (\mathcal{C} \cap \overline{B}_R) \cup (\partial B_R \cap \mathcal{O}).$$

We define now the sets

$$H^1(\widetilde{\mathcal{O}}_{R,L}, \lambda^{1-2s}) = \{v : \widetilde{\mathcal{O}}_{R,L} \to \mathbb{R} \ :\ \lambda^{1-2s}(v^2 + |\nabla v|^2) \in L^1(\widetilde{\mathcal{O}}_{R,L})\},$$
$$\widetilde{L}^2(\widetilde{\mathcal{O}}_{R,L}) = \{v \in L^2(\widetilde{\mathcal{O}}_{R,L}) : v = v(s,t,\lambda) \text{ a.e.}\}$$

and

$$\widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L}, \lambda^{1-2s}) = \{v \in H^1(\widetilde{\mathcal{O}}_{R,L}, \lambda^{1-2s}) : v \equiv 0 \text{ on } \partial^+\widetilde{\mathcal{O}}_{R,L},\ v = v(s,t,\lambda) \text{ a.e.}\}.$$

We recall that the inclusion

$$\widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L}, \lambda^{1-2s}) \subset\subset L^2(\partial^0\widetilde{\mathcal{O}}_{R,L}) \tag{3.1}$$

is compact (see the proof of Lemma 4.1 in [13] and Section 2 in [19]).

We can now give the proof of Theorem 2.1.

Proof of Theorem 2.1. As already mentioned, we prove the existence of a solution v for the problem (2.1) such that $v = v(s,t,\lambda)$ and $v(s,t,\lambda) = -v(-t,s,\lambda)$.

Consider the energy functional in $\widetilde{\mathcal{O}}_{R,L}$,

$$\mathcal{E}_{\widetilde{\mathcal{O}}_{R,L}}(v) = \frac{1}{2}\int_{\widetilde{\mathcal{O}}_{R,L}} \lambda^{1-2s}|\nabla v|^2 + \int_{\partial^0\widetilde{\mathcal{O}}_{R,L}} G(v) \qquad \text{for every } v \in \widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L}, \lambda^{1-2s}).$$

Next, we prove the existence of a minimizer of the functional among functions in $\widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L}, \lambda^{1-2s})$. Recall that we assume condition (2.6) on G, that is,

$$G(\pm 1) = 0 \quad \text{and } G > 0 \text{ in } (-1,1).$$

We define a continuous extension $\widetilde{G}$ of G in $\mathbb{R}$ such that

- $\widetilde{G} = G$ in $[-1,1]$,
- $\widetilde{G} > 0$ in $\mathbb{R} \setminus [-1,1]$,
- $\widetilde{G}$ is even,
- $\widetilde{G}$ has linear growth at infinity.

We consider the new energy functional

$$\widetilde{\mathcal{E}}_{\widetilde{\mathcal{O}}_{R,L}}(v) = \frac{1}{2}\int_{\widetilde{\mathcal{O}}_{R,L}} \lambda^{1-2s}|\nabla v|^2 + \int_{\partial^0\widetilde{\mathcal{O}}_{R,L}} \widetilde{G}(v) \qquad \text{for every} \quad v \in \widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L}, \lambda^{1-2s}).$$

We observe that every minimizer w of $\widetilde{\mathcal{E}}_{\widetilde{\mathcal{O}}_{R,L}}(\cdot)$ in $\widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L}, \lambda^{1-2s})$ such that $-1 \leq w \leq 1$ is also a minimizer of $\mathcal{E}_{\widetilde{\mathcal{O}}_{R,L}}(\cdot)$ in the set

$$\{v \in \widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L}, \lambda^{1-2s}) : -1 \leq v \leq 1\}.$$

To show that $\widetilde{\mathcal{E}}_{\widetilde{\mathcal{O}}_{R,L}}(\cdot)$ admits a minimizer we use a standard variational argument and the compactness of the inclusion (3.1) above. Hence, taking a minimizing sequence $\{v^k_{R,L}\} \in \widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L}, \lambda^{1-2s})$ and a subsequence convergent in $\widetilde{L}^2(\partial^0\widetilde{\mathcal{O}}_{R,L})$, we conclude that $\widetilde{\mathcal{E}}_{\widetilde{\mathcal{O}}_{R,L}}(\cdot)$ admits an absolute minimizer $v_{R,L}$ in $\widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L}, \lambda^{1-2s})$.

Observe that, without loss of generality, we may assume $0 \leq v^k_{R,L} \leq 1$ in $\widetilde{\mathcal{O}}_{R,L}$ because, if not, we can replace the minimizing sequence $v^k_{R,L}$ with the sequence $\min\{|v^k_{R,L}|, 1\}$. Indeed, it is also minimizing because $\widetilde{G}$ is even and $\widetilde{G} \geq \widetilde{G}(1)$. Then the absolute minimizer $v_{R,L}$ is such that $0 \leq v_{R,L} \leq 1$ in $\widetilde{\mathcal{O}}_{R,L}$.

To prove that $v_{R,L}$ is indeed a solution of (2.8) in $\widetilde{\mathcal{O}}_{R,L}$ some care is needed, we refer to [19] for details.

To get a solution in all of $B_R \times (0, L)$, we consider now the odd reflection of $v_{R,L}$ with respect to $\mathcal{C} \times \mathbb{R}^+$,

$$v_{R,L}(s, t, \lambda) = -v_{R,L}(t, s, \lambda).$$

This is a solution in $B_R \setminus \{0\} \times (0, L)$. Using a cutoff argument, precisely as in [19], we conclude that $v_{R,L}$ is also a solution around 0, and hence in all of $B_R \times (0, L)$.

We now wish to pass to the limit in R and L, and obtain a solution in all of $\mathbb{R}^{2m+1}_+$. Let $S > 0$, $L' > 0$ and consider the family $\{v_{R,L}\}$ of solutions in $B_{S+2} \times [0, L' + 2]$, with $R > S + 2$ and $L > L' + 2$. Since $|v_{R,L}| \leq 1$, regularity results proved in [12], Proposition 4.5, give a uniform $C^\alpha(\overline{B}_S \times [0, L'])$ bound for $v_{R,L}$ (uniform with respect to R and L).

Moreover $v_{R,L}$ satisfies (see Proposition 4.6 in [12])

$$|\nabla v_{R,L}(x, \lambda)| \leq \frac{C}{\lambda} \quad \text{in } B_R \times (1, L), \tag{3.2}$$

$$\|\lambda^{1-2s}\partial_\lambda u\|_\infty \leq C \quad \text{in } B_R \times (1, L). \tag{3.3}$$

Choose now $L = R^\gamma$, with $1/2 < \gamma < \frac{1}{2(1-s)}$ (this choice will be used later to prove that the solution that we construct is not identically zero). By the Arzelà-Ascoli Theorem, using the uniform C^α estimates and the bound (3.3), we deduce that a subsequence of $\{v_{R,R^\gamma}\}$ converges in $C^\alpha(\overline{B}_S \times [0, S^\gamma])$ to a solution in $B_S \times (0, S^\gamma)$. Taking $S = 1, 2, 3, \ldots$ and making a Cantor diagonal argument, we obtain a sequence v_{R_j, R_j^γ} converging in $C^\alpha_{loc}(\mathbb{R}^{2m+1}_+)$ to a solution $v \in C^\alpha(\mathbb{R}^{2m+1}_+)$. By construction we have found a solution v in $\mathbb{R}^{2m+1}_+$ depending only on s, t and λ, such that $v(s, t, \lambda) = -v(t, s, \lambda)$, $|v| \leq 1$ and $v \geq 0$ in $\{s > t\}$.

The fact that $|v| \leq 1$ follows easily using that $f(1) = 0$ and applying the maximum principle (see Remark 4.2 in [12]) as in [19].

We prove now that $v \not\equiv 0$ in $\mathbb{R}^{2m+1}_+$. Then, the strong maximum principle and Hopf's Lemma (see again Remark 4.2 in [12]) lead to $v > 0$ in $\{s > t\} \times \mathbb{R}^+$ since $f(0) = 0$ and $v \geq 0$ in $\{s > t\} \times \mathbb{R}^+$.

To prove that v does not identically vanish, we establish an energy estimate using a comparison argument (based on the minimality property of $v_{R,L}$ in the set $\widetilde{\mathcal{O}}_{R,L}$). We give all the detailed computations for this part of the proof because of the precence of the weight λ^{1-2s} in the Dirichlet energy.

Let $1/2 < \gamma < \frac{1}{2(1-s)}$ as above and β be a positive real number depending only on γ and such that $1/2 \leq \beta < \gamma < \frac{1}{2(1-s)}$. Let $S < R-2$, since we have chosen before $L = R^\gamma$, then $S^\gamma < L$. We consider a C^1 function $g : \widetilde{\mathcal{O}}_{S,S^\gamma} \to \mathbb{R}$ defined as follows:

$$g(x,\lambda) = g(s,t,\lambda) = \eta(s,t)\min\left\{1, \frac{s-t}{\sqrt{2}}\right\} + (1-\eta(s,t))v_{R,L}(s,t,\lambda),$$

where η is a smooth function depending only on $r^2 = s^2+t^2$ such that $\eta \equiv 1$ in B_{S-1} and $\eta \equiv 0$ outside B_S. Observe that g agrees with $v_{R,L}$ on the lateral boundary of $\widetilde{\mathcal{O}}_{S,S^\gamma}$ and g is identically 1 inside $(\mathcal{O}_{S-1} \cap \{(s-t)/\sqrt{2} > 1\}) \times (0, S^\gamma)$.

Now, we introduce the following C^1 function $\xi : (0, S^\gamma) \to (0, +\infty)$:

$$\xi(\lambda) = \begin{cases} 1 & \text{if } 0 < \lambda \leq S^\gamma - S^\beta \\ \dfrac{\log S^\gamma - \log \lambda}{\log S^\gamma - \log(S^\gamma - S^\beta)} & \text{if } S^\gamma - S^\beta < \lambda \leq S^\gamma. \end{cases}$$

Finally, we define $w : \widetilde{\mathcal{O}}_{S,S^\gamma} \to (-1,1)$ as follows

$$w(x,\lambda) = \xi(\lambda)g(x,\lambda) + [1-\xi(\lambda)]v_{R,L}(x,\lambda). \tag{3.4}$$

We set $\hat{\mathcal{O}} = \mathcal{O}_{S-1} \cap \{(s-t)/\sqrt{2} > 1\} \times (0, S^\gamma - S^\beta)$. Observe that w agree with $v_{R,L}$ on $\partial^+\widetilde{\mathcal{O}}_{S,S^\gamma}$ and $w \equiv 1$ in $\hat{\mathcal{O}}$. We extend w to be identically equal to $v_{R,L}$ in $\widetilde{\mathcal{O}}_{R,L} \setminus \widetilde{\mathcal{O}}_{S,S^\gamma}$. Using the minimality of $v_{R,L}$ in $\widetilde{\mathcal{O}}_{R,L}$, we deduce

$$\mathcal{E}_{\widetilde{\mathcal{O}}_{R,L}}(v_{R,L}) \leq \mathcal{E}_{\widetilde{\mathcal{O}}_{R,L}}(w).$$

Thus, since $w = v_{R,L}$ in $\widetilde{\mathcal{O}}_{R,L} \setminus \widetilde{\mathcal{O}}_{S,S^\gamma}$, we get

$$\mathcal{E}_{\widetilde{\mathcal{O}}_{S,S^\gamma}}(v_{R,L}) \leq \mathcal{E}_{\widetilde{\mathcal{O}}_{S,S^\gamma}}(w).$$

We want now to estimate $\mathcal{E}_{\widetilde{\mathcal{O}}_{S,S^\gamma}}(w)$. We start by observing that, since $w \equiv 1$ on $\mathcal{O}_{S-1} \cap \{(s-t)/\sqrt{2} > 1\}$, then

$$\int_{\mathcal{O}_S} G(w) = \int_{\mathcal{O}_S \setminus (\mathcal{O}_{S-1} \cap \{(s-t)/\sqrt{2} > 1\})} G(w) \leq C|\mathcal{O}_S \setminus (\mathcal{O}_{S-1} \cap \{(s-t)/\sqrt{2} > 1\})| \leq CS^{2m-1}. \tag{3.5}$$

It remains to bound the Dirichlet energy of w. We have

$$\begin{aligned}\int_{\widetilde{\mathcal{O}}_{S,S^\gamma}} \lambda^{1-2s}|\nabla w(x,\lambda)|^2 dx d\lambda &= \int_{\widetilde{\mathcal{O}}_{S,S^\gamma - S^\beta}} \lambda^{1-2s}|\nabla w(x,\lambda)|^2 dx d\lambda \\ &\quad + \int_{\widetilde{\mathcal{O}}_{S,S^\gamma} \setminus \widetilde{\mathcal{O}}_{S,S^\gamma - S^\beta}} \lambda^{1-2s}|\nabla w(x,\lambda)|^2 dx d\lambda. \end{aligned} \tag{3.6}$$

Using again that $w \equiv 1$ in $\hat{O}$ and that $|\mathcal{O}_S \setminus \{(s-t)/\sqrt{2} > 1\}| \le CS^{2m-1}$, we deduce that

$$\begin{aligned}
&\int_{\widetilde{\mathcal{O}}_{S,S^\gamma}} \lambda^{1-2s} |\nabla w(x,\lambda)|^2 dx d\lambda \\
&\quad \le CS^{2m-1} \int_0^{S^\gamma - S^\beta} \lambda^{1-2s}\, d\lambda + \int_{\widetilde{\mathcal{O}}_{S,S^\gamma} \setminus \widetilde{\mathcal{O}}_{S,S^\gamma - S^\beta}} \lambda^{1-2s} |\nabla w(x,\lambda)|^2 dx d\lambda \qquad (3.7)\\
&\quad \le CS^{2m-1+2\gamma(1-s)} + \int_{\widetilde{\mathcal{O}}_{S,S^\gamma} \setminus \widetilde{\mathcal{O}}_{S,S^\gamma - S^\beta}} \lambda^{1-2s} |\nabla w(x,\lambda)|^2 dx d\lambda.
\end{aligned}$$

We give now an estimate for the last term on the right-hand side of (3.7). By the definition of w in (3.4), we have that

$$|\nabla w(x,\lambda)|^2 \le |\xi'(\lambda)|^2 [g(x,\lambda) + v_{R,L}(x,\lambda)]^2 + \{|\nabla g|^2 + |\nabla v_{R,L}(x,\lambda)|^2\}[1+\xi(\lambda)]^2.$$

We integrate now in $\widetilde{\mathcal{O}}_{S,S^\gamma} \setminus \widetilde{\mathcal{O}}_{S,S^\gamma - S^\beta}$, use that g, $|\nabla g|$, $v_{R,L}$, and ξ are bounded, the definition of ξ, and the gradient bound (3.2) for $v_{R,L}$, and we recall that $\nabla g \equiv 0$ in $\mathcal{O} \cap \{(s-t)/\sqrt{2} > 1\} \times (0, S^\gamma)$, to obtain

$$\begin{aligned}
\int_{\widetilde{\mathcal{O}}_{S,S^\gamma} \setminus \widetilde{\mathcal{O}}_{S,S^\gamma - S^\beta}} \lambda^{1-2s} |\nabla w(x,\lambda)|^2 &\le C \int_{\mathcal{O}_S} \int_{S^\gamma - S^\beta}^{S^\gamma} \lambda^{1-2s} |\xi'(\lambda)|^2 d\lambda dx \\
&\quad + C \int_{\mathcal{O}_S} \int_{S^\gamma - S^\beta}^{S^\gamma} \frac{\lambda^{1-2s}}{\lambda^2} d\lambda dx + CS^{2m-1+2\gamma(1-s)} \\
&\le C \left[\frac{1}{\left(\log \frac{S^\gamma}{S^\gamma - S^\beta}\right)^2} + 1 \right] \int_{\mathcal{O}_S} \int_{S^\gamma - S^\beta}^{S^\gamma} \lambda^{-1-2s} d\lambda dx + CS^{2m-1+2\gamma(1-s)} \\
&\le CS^{2m} \left[\frac{1}{(-\log(1 - S^{\beta-\gamma}))^2} + 1 \right] \left[\frac{1}{S^{2s\gamma} - S^{2s\beta}} - \frac{1}{S^{2s\gamma}} \right] + S^{2m-1+2\gamma(1-s)} \\
&\le CS^{2m} \cdot S^{2(\gamma-\beta)} \cdot S^{-2s\gamma} + CS^{2m-1+2\gamma(1-s)} \\
&\le CS^{2m+2\gamma(1-s)-2\beta} + CS^{2m-1+2\gamma(1-s)}. \qquad (3.8)
\end{aligned}$$

Combining (3.5), (3.7) and (3.8), we get

$$\mathcal{E}_{\widetilde{\mathcal{O}}_{S,S^\gamma}}(w) \le C(S^{2m-1} + CS^{2m+2\gamma(1-s)-2\beta} + CS^{2m-1+2\gamma(1-s)}). \qquad (3.9)$$

Since, by hypothesis, γ and $\beta = \beta(\gamma)$ satisfy $1/2 \le \beta < \gamma < \frac{1}{2(1-s)}$, then there exists $\varepsilon = \varepsilon(\gamma) > 0$ such that

$$\mathcal{E}_{\widetilde{\mathcal{O}}_{S,S^\gamma}}(w) \le CS^{2m-\varepsilon}.$$

Thus by minimality of $v_{R,L}$, we get

$$\mathcal{E}_{\widetilde{\mathcal{O}}_{S,S^\gamma}}(v_{R,L}) \le CS^{2m-\varepsilon}.$$

We now let R and $L = R^\gamma$ tend to infinity to obtain

$$\mathcal{E}_{\widetilde{\mathcal{O}}_{S,S^\gamma}}(v) \le CS^{2m-\varepsilon}.$$

Making an odd reflection with respect to $\mathcal{C}$, the previous bound leads to the energy estimate

$$\mathcal{E}_{C_{S,S^\gamma}}(v) \le CS^{2m-\varepsilon}.$$

This last estimate implies that v cannot be identically 0. Indeed $v \equiv 0$ would imply

$$c_m G(0) S^{2m} = \mathcal{E}_{C_{S,S^\gamma}}(v) \leq C S^{2m-\varepsilon},$$

which is a contradiction for S large.

We prove now the last part of the statement, on the stability of saddle-shaped solutions under perturbations vanishing on $\mathcal{C} \times (0, +\infty)$.

Since $f(0) = 0$, concavity leads to $f'(w) \leq f(w)/w$ for all real numbers $w \in (0, 1)$. Hence we have

$$\begin{cases} -\mathrm{div}(\lambda^{1-2s} \nabla v) = 0 & \text{in } \widetilde{\mathcal{O}} \\ -\lim\limits_{\lambda \to 0} \lambda^{1-2s} \partial_\lambda v \geq f'(v) v & \text{on } \mathcal{O} \times \{0\}. \end{cases}$$

Following a simple argument (see the proof of Proposition 4.2 of [1] and the proof of Theorem 1.6 in [19]), we multiply the equation $-\mathrm{div}(\lambda^{1-2s} \nabla v) = 0$ by ξ^2/v, where $\xi \in C^1(\mathbb{R}^{2m+1}_+)$ has compact support in $\widetilde{\mathcal{O}} \cup \partial^0 \widetilde{\mathcal{O}}$, integrate by parts in $\widetilde{\mathcal{O}}$, and use Cauchy-Schwarz inequality to get:

$$\begin{aligned} 0 &= \int_0^{+\infty} \int_{\mathcal{O}} -\mathrm{div}(\lambda^{1-2s} \nabla v) \frac{\xi^2}{v} = \int_0^{+\infty} \int_{\mathcal{O}} \lambda^{1-2s} \nabla v \cdot \nabla \xi \frac{2\xi}{v} \\ &\quad - \int_0^{+\infty} \int_{\mathcal{O}} \lambda^{1-2s} |\nabla v|^2 \frac{\xi^2}{v^2} + \int_{\mathcal{O}} \lambda^{1-2s} \frac{\xi^2}{v} \frac{\partial v}{\partial \lambda} \\ &\leq \int_0^{+\infty} \int_{\mathcal{O}} \lambda^{1-2s} |\nabla \xi|^2 - \int_{\mathcal{O}} f'(v) \xi^2 = Q_v(\xi). \end{aligned}$$

By an approximation argument, the same holds for all $\xi \in C^1$ with compact support in $\overline{\widetilde{\mathcal{O}}}$ and vanishing on $\mathcal{C} \times \mathbb{R}^+$. Finally, by odd symmetry with respect to $\mathcal{C} \times \mathbb{R}^+$, we deduce that $Q_v(\xi) \geq 0$ for any C^1 functions ξ with compact support in $\overline{\mathbb{R}^{2m+1}_+}$ and vanishing on $\mathcal{C} \times \mathbb{R}^+$. □

4. Fractional Laplacians on domains and maximum principles

In this section we introduce the operator A^s, which is a fractional power of the Laplacian on a domain $\Omega \subset \mathbb{R}^n$ with 0-Dirichlet boundary condition.

Let u be a function defined on Ω, where Ω is a sufficiently regular (say Lipschitz) domain of $\mathbb{R}^n$. Following [5, 37], we can consider the s-harmonic extension v of u in the cylinder $\Omega = \Omega \times (0, \infty)$ which vanishes on the all lateral boundary $\partial\Omega \times (0, \infty)$, i.e the solution of the problem

$$\begin{cases} \mathrm{div}(\lambda^{1-2s} \nabla v) = 0 & \text{in } \Omega \times (0, \infty) \\ v = 0 & \text{on } \partial\Omega \times (0, \infty) \\ v = u & \text{on } \Omega \times \{y = 0\}. \end{cases} \tag{4.1}$$

We can define

$$A^s u(x) = -\frac{1}{c_\alpha} \lim_{\lambda \to 0} \lambda^{1-2s} \partial_\lambda v(x, \lambda),$$

where

$$c_s = \frac{2^{1-2s} \Gamma(1-s)}{\Gamma(s)}. \tag{4.2}$$

Since, as before, the constant c_s will not be important for our purposes, we will omit it in the sequel.

In [5, 37] bounded domain Ω were considered. For our porpuse Ω is not necessarily bounded (later we will consider $\Omega = \mathcal{O}$) and we observe that the definition of A^s as the (weighted) Dirichlet to Neumann operator can be given also for unbounded domains.

With this definition in mind, we will study the problem

$$\begin{cases} A^s u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega \end{cases} \tag{4.3}$$

by studing the corresponding local problem

$$\begin{cases} \operatorname{div}(\lambda^{1-2s}\nabla v) = 0 & \text{in } \Omega \times (0,\infty) \\ v = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u > 0 & \text{in } \Omega \times (0,\infty) \\ -\lim_{\lambda\to 0} \lambda^{1-2s}\partial_\lambda v = f(v) & \text{on } \partial\Omega \times \{0\}. \end{cases} \tag{4.4}$$

We recall now an existence result for layer solutions in the all $\mathbb{R}$ for the one-dimensional problem

$$(-\partial_{xx})^s u = f(u) \quad \text{in } \mathbb{R}. \tag{4.5}$$

In Theorem 2.4 in [13], Cabré and Sire established that under our assumption on the nonlinearity f (see (2.5), (2.6), (2.7)), there exists a monotone solution of (4.5) going from -1 to 1. This kind of solutions are usually called *layers*. Moreover, they are unique up to translations.

In the sequel we will call u_0 the layer solution of (4.5) which vanish at 0, and v_0 its s-harmonic extension in $\mathbb{R}^2_+$.

The following Proposition is the analogue of Proposition 1.8 in [19] and it provides a supersolution for the extension problem (4.4) in $\mathcal{O} \times (0,\infty)$. We remind that $|s-t|/\sqrt{2}$ is the distance to the Simons cone.

Proposition 4.1. *Let f satisfy hypothesis* (2.5), (2.6), (2.7)*. Let u_0 be the layer solution, vanishing at the origin, of problem* (1.1) *in $\mathbb{R}$ and let v_0 be its s-harmonic extension in $\mathbb{R}^2_+$.*

Then, the function $v_0\left(\frac{s-t}{\sqrt{2}},\lambda\right)$ satisfies

$$\begin{cases} \mathit{div}(\lambda^{1-2s}\nabla v_0) \geq 0 & \text{in } \widetilde{\mathcal{O}} \\ -\lim_{\lambda\to 0} \lambda^{1-2s}\partial_\lambda v_0 \geq f(v_0) & \text{on } \mathcal{O} \times \{0\}. \end{cases} \tag{4.6}$$

Proof. The proof is the same as the proof of Proposition 3.3 in [19]. It is enough to write problem (4.6) in the (s,t,λ) variables:

$$\begin{cases} v_{ss} + v_{tt} + v_{\lambda\lambda} + (m-1)\left(\frac{v_s}{s} + \frac{v_t}{t}\right) + \frac{1-2s}{\lambda} v_\lambda = 0 & \text{in } \widetilde{\mathcal{O}} \\ v = 0 & \text{on } \mathcal{C} \times [0,\infty) \\ v > 0 & \text{in } \widetilde{\mathcal{O}} \\ -\lim_{\lambda\to 0} \lambda^{1-2s}\partial_\lambda v = f(v) & \text{on } \mathcal{O} \times \{0\}. \end{cases}$$

A direct computations shows that $v_0((s-t)/\sqrt{2},\lambda)$ is a supersolution in the set $\{(s,t,\lambda) : s > t > 0\}$. As in [19], a cut-off argument implies that $v_0((s-t)/\sqrt{2},\lambda)$ is a supersolution in all of $\widetilde{\mathcal{O}}$ in dimensions $2m+1 \geq 5$. In dimension $2m+1 = 3$, the same holds true, since the outer flux $-\partial_t v_0((s-t)/\sqrt{2},\lambda)$ is positive. □

We have the following

Corollary 4.2. *Let f satisfy hypothesis* (2.5), (2.6), (2.7)*. Let v_0 be as above and assume* $K \geq 1$.

Then, the function $\min\{Kv_0((s-t)/\sqrt{2}, \lambda), 1\}$ *is a supersolution of problem (4.4) with* $\Omega = \mathcal{O} = \{s > t\}$.

In particular $\min\{Ku_0((s-t)/\sqrt{2}), 1\}$ is a supersolution of problem (4.3) with $\Omega = \mathcal{O} = \{s > t\}$.

Proof. To prove the assertion it is enough to prove that $Kv_0((s-t)/\sqrt{2}, \lambda)$ is a supersolution in the set $\{(x, \lambda) \in \widetilde{\mathcal{O}} : Kv_0(x, \lambda) < 1\}$. This follows by the assumption on the nolinearity f. Indeed (see Remark 3.4 in [19]) we have that $f(\rho)/\rho$ is non-increasing in $(0, 1)$ and hence

$$-\partial_\lambda(Kv_0(z, 0)) = Kf(v_0(z, 0)) \geq f(Kv_0(z, 0)) \quad \text{if } Kv_0(z, 0) < 1.$$

This implies that

$$-\lim_{\lambda\to 0} \lambda^{1-2s}\partial_\lambda(Kv_0(z, 0)) = Kf(v_0(z, 0)) \geq f(Kv_0(z, 0)) \quad \text{on } \{(x, 0) \in \widetilde{\mathcal{O}} : Kv_0(z, 0) < 1\},$$

which concludes the proof of the Corollary. □

We want now to introduce an operator, that we will call $A^s_{\Omega,\varphi}$ acting as A^s on functions u defined in a domain Ω, but satisfying the Dirichlet boundary condition $u = \varphi$ on $\partial\Omega$. Following [19], Section 4, we consider the s-harmonic extension v of u in the cylinder $\Omega \times (0, +\infty)$ which agrees with φ on all the lateral boundary $\partial\Omega \times (0, \infty)$ and we define $A^s_{\Omega,\varphi}$ to be

$$A^s_{\Omega,\varphi} = -\lim_{\lambda\to 0} \lambda^{1-2s}\partial_\lambda v_{|\Omega\times\{0\}}.$$

When $\varphi = 0$, $A^s_{\Omega,\varphi}$ coincides with the operator A^s introduced in the beginning of this section.

Hence, the nonlocal problem

$$\begin{cases} A^s_{\Omega,\varphi} u = f(u) & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

is reformulated in the local problem,

$$\begin{cases} -\mathrm{div}(\lambda^{1-2s}\nabla v) = 0 & \text{in } \Omega \times (0, \infty) \\ v(x, \lambda) = \varphi(x) & \text{on } \partial\Omega \times (0, \infty) \\ -\lim_{\lambda\to 0} \lambda^{1-2s}\partial_\lambda v = f(v) & \text{on } \Omega \times \{0\}. \end{cases} \tag{4.7}$$

In the sequel, we will show that the operator $A^s_{\Omega,\varphi}$ satisfies weak and strong maximum principles. In order to do that, we recall the following Lemma which gives an explicit expression for a solution to the ordinary differential equation

$$\varphi'' + \frac{1-2s}{\lambda}\varphi' - \varphi = 0,$$

in terms of Bessel's functions (see e.g. [3], Lemma 2.2 and [20], Sect. 4).

Lemma 4.3 (Lemma 2.2 in [3])**.** *The solution of the ordinary differential equation*

$$\varphi'' + \frac{1-2s}{\lambda}\varphi' - \varphi = 0 \tag{4.8}$$

may be written as $\varphi(\lambda) = \lambda^s \psi(\lambda)$*, where* ψ *solves the well known Bessel equation*

$$\lambda^2\psi'' + \lambda\psi' - (\lambda^2 + s^2)\psi = 0. \tag{4.9}$$

In addition (4.9) *has two linearly independent solutions,* I_s*,* Z_s*, which are the modified Bessel functions; their asymptotic behaviour is given precisely by*

$$\begin{aligned} I_s(\lambda) &\sim \frac{1}{\Gamma(s+1)}\left(\frac{\lambda}{2}\right)^s\left(1 + \frac{\lambda^2}{4(s+1)} + \frac{\lambda^4}{32(s+1)(s+2)} + \ldots\right), \\ Z_s(\lambda) &\sim \frac{\Gamma(s)}{2}\left(\frac{2}{\lambda}\right)^s\left(1 + \frac{\lambda^2}{4(1-s)} + \frac{\lambda^4}{32(1-s)(2-s)} + \ldots\right) + \\ &\quad + \frac{\Gamma(-s)}{2^s}\left(\frac{\lambda}{2}\right)^s\left(1 + \frac{\lambda^2}{4(s+1)} + \frac{\lambda^4}{32(s+1)(s+2)} + \ldots\right), \end{aligned} \tag{4.10}$$

for $y \to 0^+$*,* $s \notin \mathbb{Z}$*. And when* $\lambda \to +\infty$*,*

$$\begin{aligned} I_s(\lambda) &\sim \frac{1}{\sqrt{2\pi\lambda}}e^{\lambda}\left(1 - \frac{4s^2-1}{8\lambda} + \frac{(4\lambda^2-1)(4\lambda^2-9)}{2!(8\lambda)^2} + \ldots\right), \\ Z_s(\lambda) &\sim \sqrt{\frac{\pi}{2\lambda}}e^{-\lambda}\left(1 - \frac{4s^2-1}{8\lambda} + \frac{(4\lambda^2-1)(4\lambda^2-9)}{2!(8\lambda)^2} + \ldots\right). \end{aligned} \tag{4.11}$$

In the sequel, we will use the solution given in Lemma 4.3 above which grows exponentially as $\lambda \to \infty$. Up to a normalization constant chosen in such a way that $\varphi(0) = 1$, we set

$$\varphi(\lambda) := \lambda^s I_s(\lambda). \tag{4.12}$$

We observe that if φ satisfies (4.8), then the function $\varphi_\mu(\lambda) := \varphi(\sqrt{\mu}\lambda)$ satisfies

$$\partial_\lambda(\lambda^{1-2s}\partial_\lambda\varphi_\mu) = \lambda^{1-2s}\mu\varphi_\mu.$$

Finally, we stress that

$$\varphi(\lambda) \sim \lambda^{s-1/2}e^{\lambda} \quad \text{as } \lambda \to \infty. \tag{4.13}$$

We can now give the following:

Lemma 4.4. *Let* $\widetilde{\Omega} = \Omega \times \mathbb{R}^+$ *be a cylinder in* $\mathbb{R}^{n+1}_+$*, where* $\Omega \subset \mathbb{R}^n$ *is a bounded domain. Suppose that* $v \in C^2(\widetilde{\Omega}) \cap C(\overline{\widetilde{\Omega}})$ *is bounded and satisfies*

$$div(\lambda^{1-2s}\nabla v) = 0 \quad in\ \widetilde{\Omega}. \tag{4.14}$$

Then,

$$\inf_{\widetilde{\Omega}} v = \inf_{\partial\widetilde{\Omega}} v.$$

Proof. Substracting a constant from v, we may assume that v is nonnegative on $\partial\widetilde{\Omega}$ and we need to show $v \geq 0$ in $\widetilde{\Omega}$.

We want to construct a strictly positive function ψ, which is still a solution to (4.14) and which goes to infinity as $|(x,\lambda)| \to \infty$.

First, let B_R be a ball of radius R in $\mathbb{R}^n$ which containes Ω (we recall that Ω is bounded). Let μ_R and ϕ_R be, respectively, the first eigenvalue and the corresponding eigenfunction of the Laplacian $-\Delta$ in B_R with $0-$Dirichlet boundary condition on ∂B_R. Let $\varphi_{\mu_R}(\lambda) = \varphi(\sqrt{\mu_R}\lambda)$, where φ is the solution of (4.8) given in Lemma 4.3, which tends to ∞ as $\lambda \to \infty$.

We define the function $\psi : B_R \times \mathbb{R}^+ \to \mathbb{R}$ as follows

$$\psi(x,\lambda) = \phi_R(x)\varphi_{\mu_R}(\lambda).$$

Then the restriction of ψ in $\widetilde{\Omega}$ is a strictly positive s-harmonic function.
Moreover, since ϕ_R is bounded, we have that

$$\lim_{|(x,\lambda)|\to+\infty} \psi(x,\lambda) = \lim_{\lambda\to+\infty} \psi(x,\lambda) = +\infty. \tag{4.15}$$

We define now the function $w = v/\psi$. Clearly $w \geq 0$ in $\widetilde{\Omega}$.
By an easy computation, we have that w satisfies $w \geq 0$ on $\partial\widetilde{\Omega}$ and

$$\begin{aligned} \operatorname{div}(\lambda^{1-2s}\nabla w) &= \frac{1}{\psi}\operatorname{div}(\lambda^{1-2s}\nabla v) - \frac{v}{\psi^2}\operatorname{div}(\lambda^{1-2s}\nabla\psi) - 2\lambda^{1-2s}\frac{\nabla\psi}{\psi}\cdot\left(\frac{\nabla v}{\psi} - \frac{\nabla\psi}{\psi^2}v\right) \\ &= -2\lambda^{1-2s}\frac{\nabla\psi}{\psi}\cdot\nabla w. \end{aligned}$$

In addition, by (4.15), $w(x,\lambda) \to 0$ as $|(x,\lambda)| \to +\infty$ and thus, by the strong maximum principle (applied, by a contradiction argument, to a possible negative minimum) $w \geq 0$ in $\widetilde{\Omega}$, which implies $v \geq 0$ in $\widetilde{\Omega}$, since w has the same sign of v. □

As a consequence of the previous result, we can deduce the following lemmas (which are the analogous, respectively, of Lemma 4.2, Corollary 4.3, and Lemma 4.4 in [19]).

Lemma 4.5. *Assume that* $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *satisfies*

$$\begin{cases} A^s_{\Omega,\varphi}u + c(x)u \geq 0 & in\ \Omega, \\ u = \varphi & on\ \partial\Omega, \end{cases}$$

where Ω *is a bounded domain in* $\mathbb{R}^n$ *and* $c(x) \geq 0$ *in* Ω*. Suppose that* $\varphi \geq 0$ *on* $\partial\Omega$*. Then* $u \geq 0$ *in* Ω*.*

Proof. The proof follows exactly the proof of Lemma 4.2 in [19]. For the sake of completeness, we recall it here.

Consider the s-harmonic extension v of u in $\widetilde{\Omega} = \Omega\times(0,+\infty)$ with Dirichlet data $v(x,\lambda) = \varphi(x)$ on the lateral boundary $\partial\Omega\times(0,+\infty)$ (as in the definition of the operator $A^s_{\Omega,\varphi}$). We prove that $v \geq 0$ in $\widetilde{\Omega}$, then in particular $u \geq 0$ in Ω.

Suppose by contradiction that v is negative somewhere in $\Omega\times\mathbb{R}^+$. Since v is s-harmonic, Lemma 4.4 implies that the $\inf_{\widetilde{\Omega}} v < 0$ will be achieved at some point $(x_0, 0) \in \Omega\times\{0\}$. Thus, we have

$$\inf_{\widetilde{\Omega}} v = v(x_0,0) < 0.$$

By Hopf's lemma,

$$v_\lambda(x_0,0) > 0.$$

It follows

$$-\lambda^{1-2s}v_\lambda(x_0,0) = A^s_{\Omega,\varphi}v(x_0,0) < 0.$$

Therefore, since $c \geq 0$,

$$A^s_{\Omega,\varphi}v(x_0,0) + c(x_0)v(x_0,0) < 0.$$

This is a contradiction with the hypothesis $A^s_{\Omega,\varphi}u + c(x)u \geq 0$. □

As a corollary, we deduce

Corollary 4.6. *Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose that u_1 and u_2 are two bounded functions, $u_1,\ u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$, which satisfy*

$$\begin{cases} A^s_{\Omega,\varphi} u_1 \leq A^s_{\Omega,\varphi} u_2 & \text{in } \Omega \\ u_1 = u_2 = \varphi & \text{on } \partial\Omega. \end{cases}$$

Then, $u_1 \leq u_2$ in Ω.

Finally we have the following strong maximum principle, whose proof is similar to the one of Lemma (4.5) above and we omit the details here (see [19], proof of Lemma 4.4).

Lemma 4.7. *Assume that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies*

$$\begin{cases} A^s_{\Omega,\varphi} u + c(x) u \geq 0 & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ and $c \in L^\infty(\Omega)$. Suppose $\varphi \geq 0$ on $\partial\Omega$.

Then, either $u > 0$ in Ω, or $u \equiv 0$ in Ω.

5. Monotonicity properties and asymptotic behaviour

We start by introducing the new variables $(y, z) \in \mathbb{R}^m \times \mathbb{R}^m$ as follows:

$$\begin{cases} y = \dfrac{s+t}{\sqrt{2}} \\ z = \dfrac{s-t}{\sqrt{2}}. \end{cases} \tag{5.1}$$

Note that $|z| \leq y$ and that we may write the Simons cone as $\mathcal{C} = \{z = 0\}$. We observe that problem (2.8), written in these new variables, becomes

$$\begin{cases} v_{yy} + v_{zz} + v_{\lambda\lambda} + \dfrac{2(m-1)}{y^2 - z^2}(y v_y - z v_z) + \dfrac{1-2s}{\lambda} v_\lambda = 0 & \text{in } \mathbb{R}^{2m+1}_+ \\ -\lambda^{1-2s} \partial_\lambda v = f(v) & \text{on } \partial\mathbb{R}^{2m+1}_+. \end{cases}$$

The first result of this section concerns existence and monotonicity properties of a maximal saddle solution.

Proposition 5.1. *Let f satisfy conditions (2.5), (2.6), and (2.7).*

Then, there exists a saddle solution $\overline{u}$ of $(-\Delta)^s \overline{u} = f(\overline{u})$ in $\mathbb{R}^{2m}$, with $|u| < 1$, which is maximal in the following sense. For every solution u of $(-\Delta)^s u = f(u)$ in $\mathbb{R}^{2m}$, vanishing on the Simons cone and such that u has the same sign as $s - t$, we have

$$0 < u \leq \overline{u} \quad \text{in } \mathcal{O}.$$

As a consequence, we also have

$$0 \leq |u| \leq |\overline{u}| \quad \text{in } \mathbb{R}^{2m}.$$

In addition, if $\overline{v}$ is the harmonic extension of $\overline{u}$ in $\mathbb{R}^{2m+1}_+$, then $\overline{v}$ satisfies:

(a) $\partial_s \overline{v} \geq 0$ *in* $\overline{\mathbb{R}^{2m+1}_+}$.
(b) $\partial_t \overline{v} \leq 0$ *in* $\overline{\mathbb{R}^{2m+1}_+}$.
(c) $\partial_z \overline{v} > 0$ *in* $\overline{\mathbb{R}^{2m+1}_+} \setminus \{0\}$;
(d) $\partial_y \overline{v} > 0$ *in* $\{s > t\} \times [0, +\infty)$.

As a consequence, for every direction $\partial_\eta = \alpha\partial_y - \beta\partial_t$, with α and β nonnegative constants, $\partial_\eta \overline{v} > 0$ in $\{s > t > 0\} \times [0, +\infty)$.

This result is the analogue of Theorem 1.7 in [19]. The proof, which follows exactly the one in [19], uses two main ingredients: the maximum principles for the fractional Laplacian in bounded domains with Dirichlet boundary condition established in the previous section, and an upper barrier for our saddle solution, that we construct here below.

Let, as before, v be the s-harmonic extension of a saddle solution u in the half-space $\mathbb{R}^{2m+1}_+$. The regularity results established in Proposition 4.6 of [12] give a uniform upper bound for $|\nabla_x v|$ (where ∇_x denotes the derivatives in the horizontal variable $x \in \mathbb{R}^{2m}$). Then, since $v = 0$ on $\mathcal{C} \times \mathbb{R}^+ = \{z = 0\} \times \mathbb{R}^+$, there exists a constant C, depending only on n, $||u||_\infty$, and $||f||_{C^1}$, such that

$$|v(x, \lambda)| = |v(y, z, \lambda)| \leq C|z|, \quad \text{for every } (x, \lambda) \in \overline{\mathbb{R}^{2m+1}_+}.$$

In particular, we have that $|u(x)| = |v(x, 0)| \leq C|z|$ for every $x \in \mathbb{R}^{2m}$.

Using the properties of the one-dimensional layer solution u_0, we can see that there exists a real number $K \geq 1$ such that

$$\min\{1, C|z|\} \leq \min\{1, K|u_0(z)|\} \quad \text{for every } z.$$

Indeed it is enough to choose

$$K \geq \max\{C/u_0'(0), 1/u_0(C^{-1})\}, \tag{5.2}$$

which is possible since the quantities $u_0'(0)$ and $u_0(C^{-1})$ are strictly positive.

Choosing K as in (5.2), then the s-harmonic extension v in $\mathbb{R}^{2m+1}_+$ of every saddle solution u of (1.1) satisfies

$$|v(x, \lambda)| \leq \min\{1, K|u_0(z)|\} \quad \text{for every } (x, \lambda) \in \overline{\mathbb{R}^{2m+1}_+}. \tag{5.3}$$

We recall that in Corollary 4.2, we have seen that $\min\{1, K|u_0(z)|\}$ is a supersolution for problem 4.3 in $\mathcal{O}$.

We set

$$u_b(z) := \min\{1, K|u_0(z)|\}, \tag{5.4}$$

where K satisfies (5.2). Note that $u_b = 0$ on $\mathcal{C}$.

Sketch of the Proof of Proposition 5.1. Since the proof of Proposition 5.1 is exactly the same as the one of Theorem 1.7 in [19] (once we have maximum principles for the operator $A^s_{\Omega,\varphi}$ and the upper barrier $u_b(z)$ for u), we refer to [19], Sect. 5 for the details. We give here just the main idea of the proof, which can be sketched as follows:

- First, we establish existence of a maximal positive solution $\overline{u}_R$, depending only on s and t, of $A^s_{T_R,u_b} = f(u)$ in the bounded set T_R defined as $T_R = \{x \in \mathbb{R}^{2m} : 0 < t < s < R\}$ (note that $T_R \supset \mathcal{O}_R = \mathcal{O} \cap B_R$) with Dirichlet boundary condition u_b, as defined in (5.4). This is exactly the analogue of Lemma 5.1 in [19]. The proof follows a quite standard argument: we construct a non inceasing sequence of solutions to linear problems involving $A^s_{T_R,u_b}$; this is done by an iterative use of the maximum principle (Lemma 4.5 and Corollary 4.6) and using that the function $\min\{1, K|u_0(z)|\}$ is a supersolution.
- In a second step, we prove that the function $\overline{u}_R$, constructed in Step 1, satisfies the monotonicity property of Proposition 5.1 (in particular $\partial_t \overline{u}_R \leq 0$ and $\partial_y \overline{u}_R \geq 0$). These results are the analogue of Lemmas 5.2 and 5.3 in [19]. The proof consists in

differentiating (in t and y, respectively) the equations satisfied by the s-harmonic extension of $\overline{u}_R$ in $T_R \times (0,\infty)$ and applying, again, maximum principles.
- In the last step, by standard elliptic estimates and a compactness argument (as in the proof of the existence result Theorem 2.1) we let $R \to \infty$ and we obtain a maximal positive solution $\overline{u}$ in all $\mathcal{O} = \{s > t\}$, satisfying the above mentioned monotonicity properties. Finally, we extend $\overline{u}$ to all $\mathbb{R}^{2m}$ by odd reflection with respect to the Simons cone $\mathcal{C}$.

In the second result of this Section we establish the asymptotic behaviour of a saddle solution as $|x| \to \infty$ (the corresping result for $s = 1/2$ is contanied in Theorem 1.9 in [19]).

Proposition 5.2. *Let f satisfy conditions* (2.5), (2.6)*, and* (2.7)*, and let u be a bounded solution of $(-\Delta)^s u = f(u)$ in $\mathbb{R}^{2m}$ such that $u = 0$ on $\mathcal{C}$, $u > 0$ in $\mathcal{O} = \{s > t\}$ and u is odd with respect to $\mathcal{C}$.*

Then, denoting $U(x) := u_0((s-t)/\sqrt{2}) = u_0(z)$ we have,

$$u(x) - U(x) \to 0 \quad \text{and} \quad \nabla u(x) - \nabla U(x) \to 0, \tag{5.5}$$

uniformly as $|x| \to \infty$. That is,

$$||u - U||_{L^\infty(\mathbb{R}^{2m}\setminus B_R)} + ||\nabla u - \nabla U||_{L^\infty(\mathbb{R}^{2m}\setminus B_R)} \to 0 \ \text{ as } \ R \to \infty. \tag{5.6}$$

The proof of this result follows the one in [19] and it is based on the two following Liouville type results for the extended problem in the half-space and in a quarter of space, which have been proven via the method of moving planes in [29] and [37], respectively.

Theorem 5.3 ([29])**.** *Let $\mathbb{R}^{n+1}_+ = \{(x_1, x_2, \cdots, x_n, \lambda) \in \mathbb{R}^{n+1} \mid \lambda > 0\}$ and let f be such that $f(u)/u^{\frac{n+2s}{n-2s}}$ is non-increasing. Assume that v is a solution of problem*

$$\begin{cases} div(\lambda^{1-2s}\nabla v) = 0 & in\, \mathbb{R}^{n+1}_+, \\ -\lambda^{1-2s}\partial_\lambda v = f(v) & on\, \{\lambda = 0\}, \\ v > 0 & in\, \mathbb{R}^{n+1}_+. \end{cases} \tag{5.7}$$

Then v depends only on λ.

More precisely, there exist $a \geq 0$ and $b > 0$ such that

$$v(x,\lambda) = v(\lambda) = a\lambda + b \ \text{ and } \ f(b) = a.$$

Remark 5.4. As already mentioned in the proof of Corollary 4.2, under our assumptions on f, we have that $f(u)/u$ is non-increasing in $(0,1)$ and hence so it is $f(u)/u^{\frac{n+2s}{n-2s}}$, since

$$\frac{f(u)}{u^{\frac{n+2s}{n-2s}}} = \frac{f(u)}{u} \cdot u^{1-\frac{n+2s}{n-2s}}.$$

Hence, applying Theorem 5.3 above we deduce that if f satisfy (2.5), (2.6), (2.7), then any bounded solution v of problem (5.7) is necessarely $v \equiv 0$ or $v \equiv 1$. Indeed, since f is bistable, we have that f is odd, $f(0) = f(\pm 1) = 0$, $f > 0$ in $(0,1)$ and $f < 0$ in $(1, +\infty)$. Thus, the Liouville theorem above and the boundedness of v, imply that $v(x,\lambda) = b$ with $f(b) = 0$, that is $v \equiv 0$ or $v \equiv 1$. This fact will be used in the proof of Proposition 5.2.

The following theorem, proven in [37], gives an analog symmetry property but for solutions in a quarter of space.

Theorem 5.5 ([37]). *Let* $\mathbb{R}^{n+1}_{++} = \{(x_1, x_2, \cdots, x_n, \lambda) \in \mathbb{R}^{n+1} \mid x_n > 0, \lambda > 0\}$ *and let* f *be such that* $f(u)/u^{\frac{n+2s}{n-2s}}$ *is non-increasing. Assume that* v *is a solution of problem*

$$\begin{cases} -div(\lambda^{1-2s}\nabla v)v = 0 & \text{in } \mathbb{R}^{n+1}_{++}, \\ -\partial_\lambda v = f(v) & \text{on } \{x_n > 0, \lambda = 0\}, \\ v = 0 & \text{on } \{x_n = 0, \lambda \geq 0\}, \\ v > 0 & \text{in } \mathbb{R}^{n+1}_{++}, \end{cases}$$

Then v *depends only on* x_n *and* λ*.*

We give now a
Sketch of the Proof of Proposition 5.2. The proof of Proposition 5.2 follows the one of the analogue result contained in [19]. The proof uses a compactness argument as follows: we assume by contradiction that there exists $\epsilon > 0$ and a sequence $\{x_k\}$ with

$$|x_k| \to \infty \quad \text{and} \quad |v(x_k, \lambda) - V(x_k, \lambda)| + |\nabla v(x_k, \lambda) - \nabla V(x_k, \lambda)| \geq \epsilon. \tag{5.8}$$

By continuity we may move slightly x_k and assume $x_k \notin \mathcal{C}$ for all k. Moreover, up to a subsequence (which we still denote by $\{x_k\}$), either $\{x_k\} \subset \{s > t\}$ or $\{x_k\} \subset \{s < t\}$. By the symmetries of the problem we may assume $\{x_k\} \subset \{s > t\} = \mathcal{O}$.

We distinguish the two cases: $\{ \text{dist}(x_k, \mathcal{C}) = d_k\}$ unbounded or bounded.

In the first case, we show that, up to a subsequence, a suitable translation of the solutions $v(x_k)$ converges to a solution of the semilienar Neumann problem in the half-space given in the statement of Theorem 5.3. Using Theorem 5.3 and the stability of v_k we get a contradiction. Similarly, in the second case ($\{ \text{dist}(x_k, \mathcal{C}) = d_k\}$ bounded) we reach a contradiction using the second Liouville type result in a quarter of space.

6. Instability in low dimensions

In this Section we give the proof of our instability result Theorem 2.2, which follows the proof of the analogue result Theorem 1.10 in [19].

We start by observing the following easy fact. Assume that the nonlinearity f satisfies conditions (2.5), (2.6), (2.7) and that v is a bounded solution of (2.1) in $\mathbb{R}^{n+1}_+$. If w is a function such that $|v| \leq |w| \leq 1$ in $\mathbb{R}^{n+1}_+$, Then,

$$Q_v(\xi) \leq Q_w(\xi) \quad \text{for all } \xi \in C_0^\infty(\overline{\mathbb{R}^{n+1}_+}),$$

where the second variation of the energy functional Q_w is given by

$$Q_w(\xi) = \int_{\mathbb{R}^{n+1}_+} \lambda^{1-2s}|\nabla \xi|^2 dx d\lambda - \int_{\partial \mathbb{R}^{n+1}_+} f'(w)\xi^2 dx. \tag{6.1}$$

In particular, if there exists a function $\xi \in C_0^\infty(\overline{\mathbb{R}^{n+1}_+})$ such that $Q_w(\xi) < 0$, then v is unstable. This follows by the easy observation that, since f' is decreasing in $(0, 1)$, we have that $f'(|v|) \geq f'(|w|)$ in $\mathbb{R}^{n+1}_+$. Moreover, using that f' is even, we deduce that $f'(v) \geq f'(w)$ in $\mathbb{R}^{n+1}_+$, which yelds (6.1).

The main ingredients in the proof of Theorem 2.2 are the following. First, by the observation above, in order to prove that a saddle solution v is unstable, it is enough to prove that the maximal solution $\overline{v}$ (constructed in Section 5) is unstable. To do that we provide an explicit test function ξ for which $Q_{\overline{v}}(\xi) \leq 0$. The idea is to choose ξ to be a truncation (in the variables y and λ) of the function $\overline{v}_z$. To conclude the proof of instability, we will use crucially: the monotonicity properties established in Proposition 5.1, the asymptotic behaviour given in Proposition 5.2, and the optimal constant for the Hardy inequality (6.7).

We can give now the

Proof of Theorem 2.2. As said above, it is enough to show that the maximal solution $\overline{v}$ is unstable in dimension $2m = 4$ and $2m = 6$.

We recall that the second variation of the energy is given by the expression

$$Q_{\overline{v}}(\xi) = \int_{\mathbb{R}^{2m+1}_+} \lambda^{1-2s} |\nabla \xi|^2 dx d\lambda - \int_{\partial \mathbb{R}^{2m+1}_+} f'(\overline{v}) \xi^2 dx,$$

for every test function ξ.

We consider now a test function ξ of the form $\xi = \xi(y, z, \lambda) = \eta(y, z, \lambda)\psi(y, z, \lambda)$, with η and ψ Lipschitz functions with compact support in $y \in [0, +\infty)$ and $\lambda \in [0, +\infty)$. The expression for $Q_{\overline{v}}$ becomes,

$$\begin{aligned} Q_{\overline{v}}(\xi) = \int_0^{+\infty} \lambda^{1-2s} \int_{\mathbb{R}^{2m}} \left(|\nabla \eta|^2 \psi^2 + \eta^2 |\nabla \psi|^2 + 2\eta\psi \nabla \eta \cdot \nabla \psi \right) dx d\lambda \\ - \int_{\mathbb{R}^{2m}} f'(\overline{v}) \eta^2 \psi^2 dx. \end{aligned}$$

Observing that $2\eta\psi\nabla\eta \cdot \nabla\psi = \psi\nabla(\eta^2) \cdot \nabla\psi$ and integrating by parts, we obtain

$$\begin{aligned} Q_{\overline{v}}(\xi) = \int_0^{+\infty} \int_{\mathbb{R}^{2m}} \left(\lambda^{1-2s} |\nabla \eta|^2 \psi^2 - \eta^2 \psi \operatorname{div}(\lambda^{1-2s} \nabla \psi) \right) dx d\lambda \\ - \int_{\mathbb{R}^{2m}} \eta^2 \psi (\lambda^{1-2s} \partial_\lambda \psi + f'(\overline{v}) \psi) dx. \end{aligned}$$

We recall that problem (2.1) (satisfied by $\overline{v}$) written in the (y, z, λ) variables, reads

$$\begin{cases} \overline{v}_{yy} + \overline{v}_{zz} + \overline{v}_{\lambda\lambda} + \dfrac{2(m-1)}{y^2 - z^2} (y \overline{v}_y - z \overline{v}_z) + \dfrac{1-2s}{\lambda} \overline{v}_\lambda = 0 & \text{in } \mathbb{R}^{2m+1}_+ \\ -\lambda^{1-2s} \partial_\lambda \overline{v} = f(\overline{v}) & \text{on } \partial \mathbb{R}^{2m+1}_+. \end{cases} \tag{6.2}$$

Differentiating the above problem with respect to z, we have

$$\begin{cases} \Delta \overline{v}_z - \dfrac{2(m-1)}{y^2 - z^2} \overline{v}_z + \dfrac{4(m-1)z}{(y^2 - z^2)^2} (y \overline{v}_y - z \overline{v}_z) + \dfrac{1-2s}{\lambda} \overline{v}_{\lambda z} = 0 & \text{in } \mathbb{R}^{2m+1}_+ \\ -\lambda^{1-2s} \partial_\lambda \overline{v}_z = f'(\overline{v}) \overline{v}_z & \text{on } \partial \mathbb{R}^{2m+1}_+. \end{cases} \tag{6.3}$$

We choose now $\psi(y, z, \lambda) = \overline{v}_z(y, z, \lambda)$ and we use the equations satisfied by $\overline{v}_z$ to get

$$\begin{aligned} Q_{\overline{v}}(\xi) = \int_0^{+\infty} \lambda^{1-2s} \int_{\mathbb{R}^{2m}} \Big(|\nabla \eta|^2 \overline{v}_z^2 - \\ -\eta^2 \Big\{ \frac{2(m-1)(y^2 + z^2)}{(y^2 - z^2)^2} \overline{v}_z^2 - \frac{4(m-1)zy}{(y^2 - z^2)^2} \overline{v}_y \overline{v}_z \Big\} \Big) dx d\lambda. \end{aligned}$$

Next we change coordinates to (y, z, λ) and we have, for some positive constant c_m,

$$\begin{aligned} c_m Q_{\overline{v}}(\xi) = \int_0^{+\infty} \lambda^{1-2s} \int_{\{-y<z<y\}} (y^2 - z^2)^{m-1} \Big(|\nabla \eta|^2 \overline{v}_z^2 - \\ -\eta^2 \Big\{ \frac{2(m-1)(y^2 + z^2)}{(y^2 - z^2)^2} \overline{v}_z^2 - \frac{4(m-1)zy}{(y^2 - z^2)^2} \overline{v}_y \overline{v}_z \Big\} \Big) dy dz d\lambda. \end{aligned}$$

Now we take $\eta(y,z,\lambda)=\eta_1(y)\eta_2(\lambda)$, where η_1 and η_2 are smooth functions with compact support in $[0,+\infty)$. Moreover we require that $\eta_2(\lambda)\equiv 1$ for $\lambda<N$ and $\eta_2(\lambda)\equiv 0$ for $\lambda>N+1$, where N is a large positive number that we will choose later. For $a>1$, a constant that we will make tend to infinity, let $\phi=\phi(\rho)$ be a Lipschitz function of $\rho:=y/a$ with compact support $[\rho_1,\rho_2]\subset[0,+\infty)$. Let us denote by

$$\eta_1^a(y):=\phi(y/a) \quad \text{and}$$
$$\xi_a(y,z,\lambda)=\eta_1^a(y)\eta_2(\lambda)\overline{v}_z(y,z,\lambda)=\phi(y/a)\eta_2(\lambda)\overline{v}_z(y,z,\lambda).$$

The change $y=a\rho, dy=ad\rho$ yields,

$$c_mQ_{\overline{v}}(\xi_a)=a^{2m-3}\int_0^{N+1}\lambda^{1-2s}\int_{\rho_1}^{\rho_2}\int_{-a\rho}^{a\rho}\rho^{2(m-1)}\left(1-\frac{z^2}{a^2\rho^2}\right)^{m-1}\Big(\phi_\rho^2\eta_2^2(\lambda)\overline{v}_z^2$$
$$+a^2\phi^2(\rho)(\eta_2')^2\overline{v}_z^2-\phi^2\eta_2^2\Big\{\frac{2(m-1)(1+\frac{z^2}{a^2\rho^2})}{\rho^2(1-\frac{z^2}{a^2\rho^2})^2}\overline{v}_z^2-\frac{4(m-1)z}{a\rho^3(1-\frac{z^2}{a^2\rho^2})^2}\overline{v}_y\overline{v}_z\Big\}\Big)dzd\rho. \tag{6.4}$$

We divide now by $a^{2m-3}N^{2-2s}$ and use that $\left(1-\frac{z^2}{a^2\rho^2}\right)^2\le 1$ and $1+\frac{z^2}{a^2\rho^2}\ge 1$, to get

$$\frac{c_mQ_{\overline{u}}(\xi_a)}{a^{2m-3}N^{2-2s}}\le$$
$$\le\frac{1}{N^{2-2s}}\int_0^{N+1}\lambda^{1-2s}\int_{\rho_1}^{\rho_2}\int_{-a\rho}^{a\rho}\rho^{2(m-1)}\eta_2^2\overline{v}_z^2(a\rho,z,\lambda)\left(\phi_\rho^2-\frac{2(m-1)}{\rho^2}\phi^2\right)dzd\rho d\lambda$$
$$+\frac{a^2}{N^{2-2s}}\int_N^{N+1}\lambda^{1-2s}\int_{\rho_1}^{\rho_2}\int_{-a\rho}^{a\rho}\rho^{2(m-1)}\phi^2(\eta_2')^2\overline{v}_z^2dzd\rho d\lambda$$
$$+\frac{1}{N^{2-2s}}\int_0^{N+1}\lambda^{1-2s}\int_{\rho_1}^{\rho_2}\int_{-a\rho}^{a\rho}\frac{4(m-1)z\rho^{2m-5}\eta_2^2\phi^2(\rho)}{a}\overline{v}_y(a\rho,z,\lambda)\overline{v}_z(a\rho,z,\lambda)dzd\rho d\lambda.$$
$$=I_1+I_2+I_3.$$

We estimate the three terms above separately.

We start by I_3. We use the asymptotic behaviour established in Proposition 5.2, and in particular that $\overline{v}_y(a\rho,z,\lambda)\to 0$ uniformly, for all $\rho\in[\rho_1,\rho_2]=\text{supp}\phi$, as a tends to infinity. Given $\epsilon>0$, for a sufficiently large, $|\overline{v}_y(a\rho,z)|\le\epsilon$. Moreover, by Theorem 5.1 we have that $\overline{v}_z\ge 0$. All these facts, together with the fact that ϕ is bounded, implies that for a large we have

$$I_3\le\left|\frac{1}{N^{2-2s}}\int_0^{N+1}\lambda^{1-2s}\eta_2^2\int\frac{4(m-1)z\rho^{2m-5}\phi^2(\rho)}{a}\overline{v}_y\overline{v}_zd\rho dzd\lambda\right|$$
$$\le\frac{1}{N^{2-2s}}\int_0^{N+1}\lambda^{1-2s}\eta_2^2\int\left|\frac{4(m-1)z\rho^{2m-5}\phi^2(\rho)}{a}\right||\overline{v}_y|\overline{v}_zd\rho dzd\lambda$$
$$\le\frac{1}{N^{2-2s}}\int_0^{N+1}\lambda^{1-2s}\eta_2^2\int 4(m-1)\rho^{2m-4}\phi^2(\rho)|\overline{v}_y|\overline{v}_zd\rho dzd\lambda$$
$$\le\frac{C\epsilon}{N^{2-2s}}\int_0^{N+1}\lambda^{1-2s}\eta_2^2d\lambda\int_{\rho_1}^{\rho_2}\rho^{2m-4}d\rho\int_{-a\rho}^{a\rho}\overline{v}_zdz$$
$$=\frac{C\epsilon}{N^{2-2s}}\int_0^{N+1}\lambda^{1-2s}\eta_2^2\int_{\rho_1}^{\rho_2}\left(\overline{v}(a\rho,a\rho,\lambda)-\overline{v}(a\rho,-a\rho,\lambda)\right)d\rho d\lambda$$
$$\le C\epsilon,$$

where C are different constants depending on ρ_1 and ρ_2. Hence, as a tends to infinity, this integral tends to zero.

We consider now I_2. Let N be such that $N > a^4$. With this choice of N, we have

$$
\begin{aligned}
I_2 = \frac{a^2}{N^{2-2s}} \int_N^{N+1} \lambda^{1-2s} \int_{\rho_1}^{\rho_2} \int_{-a\rho}^{a\rho} \rho^{2(m-1)} \phi^2 (\eta_2')^2 \overline{v}_z^2 \leq \\
\leq C \frac{a^3}{N^{2-2s}} \sup \overline{v}_z^2 \int_N^{N+1} \lambda^{1-2s} \leq C \frac{a^3}{N^{2-2s}} \sup \overline{v}_z^2 [(N+1)^{2-2s} - N^{2-2s}] \\
\leq C \sup \overline{v}_z^2 \frac{a^3}{N^{2-2s}} N^{1-2s} \leq C \sup \overline{v}_z^2 \frac{1}{a}.
\end{aligned}
$$

Thus, also I_2 tends to 0 as $a \to \infty$.

Finally, we consider I_1. We have that, again by Proposition 5.2, $\overline{v}_z(a\rho, z, \lambda)$ converges to $\partial_z v_0(z, \lambda)$ which is a bounded positive integrable function. We write

$$
\begin{aligned}
I_1 &= \frac{1}{N^{2-2s}} \int_0^{N+1} \lambda^{1-2s} \eta_2^2 \int_{\rho_1}^{\rho_2} \int_{-a\rho}^{a\rho} \rho^{2(m-1)} \overline{v}_z^2(a\rho, z, \lambda) \left(\phi_\rho^2 - \frac{2(m-1)}{\rho^2} \phi^2 \right) d\rho dz d\lambda = \\
&= \frac{1}{N^{2-2s}} \int_0^{N+1} \lambda^{1-2s} \eta_2^2 \int_{\rho_1}^{\rho_2} \int_{-a\rho}^{a\rho} (\partial_z v_0)^2 \rho^{2(m-1)} \left(\phi_\rho^2 - \frac{2(m-1)}{\rho^2} \phi^2 \right) d\rho dz d\lambda \\
&\quad + \frac{1}{N^{2-2s}} \int_0^{N+1} \lambda^{1-2s} \eta_2^2 \int_{\rho_1}^{\rho_2} \int_{-a\rho}^{a\rho} \rho^{2(m-1)} (\overline{v}_z(a\rho, z, \lambda) - \partial_z v_0(z, \lambda))(\overline{v}_z(a\rho, z, \lambda) \\
&\quad + \partial_z v_0(z, \lambda)) \left(\phi_\rho^2 - \frac{2(m-1)}{\rho^2} \phi^2 \right) d\rho dz d\lambda.
\end{aligned}
$$

For a large, $|\overline{v}_z(a\rho, z, \lambda) - \partial_z v_0(z, \lambda)| \leq \epsilon$ in $[\rho_1, \rho_2]$. Moreover, $\overline{v}_z(a\rho, z, \lambda) + \partial_z v_0(z, \lambda)$ is positive and it is integrable in z since it is the derivative with respect to z of a bounded function. Thus, since $\phi = \phi(\rho)$ is smooth with compact support, the second integral converges to zero as a tends to infinity. Therefore, letting a tend to infinity, we obtain

$$
\begin{aligned}
&\limsup_{a \to \infty} \frac{c_m Q_{\overline{v}}(\xi_a)}{a^{2m-3} N^{2-2s}} \leq \qquad (6.5) \\
&\leq \limsup_{a \to \infty} \frac{1}{N^{2-2s}} \left(\int_0^{N+1} \lambda^{1-2s} \, d\lambda \eta_2^2 \int_0^{+\infty} (\partial_z v_0)^2(z) \, dz \right) \int \rho^{2(m-1)} \left(\phi_\rho^2 - \frac{2(m-1)}{\rho^2} \phi^2 \right) d\rho \\
&\leq C \int_0^{+\infty} (\partial_z v_0)^2(z) dz \int \rho^{2(m-1)} \left(\phi_\rho^2 - \frac{2(m-1)}{\rho^2} \phi^2 \right) d\rho.
\end{aligned}
$$

The conclusion of the proof is as in [19]. We show that when $2m = 4$ and $2m = 6$, there exists a test function ϕ for which

$$
\int \rho^{2(m-1)} \left(\phi_\rho^2 - \frac{2(m-1)}{\rho^2} \phi^2 \right) d\rho < 0, \qquad (6.6)
$$

where we observe that the integral in ρ can be seen as an integral in $\mathbb{R}^{2m-1}$ of radial functions $\phi = \phi(|\zeta|) = \phi(\rho)$

Comparing the constant $2(m-1)$ with the optimal constant of the well known Hardy inequality in $\mathbb{R}^{2m-1}$

$$
\frac{(2m-1-2)^2}{4} \int_{\mathbb{R}^{2m-1}} \frac{\varphi^2}{|\zeta|^2} dx \leq \int_{\mathbb{R}^{2m-1}} |\nabla \varphi|^2 dx, \qquad (6.7)
$$

we deduce that the integral in (6.6) is positive for all Lipschitz compactly supported functions ϕ, if and only if

$$2(m-1) \leq \frac{(2m-1-2)^2}{4}.$$

Writing $n = 2m$, the above inequality holds if and only if

$$n^2 - 10n + 17 \geq 0,$$

that is, $n \geq 8$. This shows the instability of $\overline{v}$ in dimensions $2m = 4, 6$ and concludes the proof of Theorem 2.2.

□

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