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Residually many BV homeomorphisms map a null set in a set of full measure

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Abstract. Let Q be the open unit square in $\mathbb{R}^2$. We prove that in a natural complete metric space of BV homeomorphisms $f : Q \to Q$ with $f_{|\partial Q} = Id$, residually many homeomorphisms (in the sense of Baire categories) map a null set in a set of full measure, and vice versa. Moreover we observe that, for $1 \le p < 2$, the family of $W^{1,p}$ homemomorphisms satisfying the above property is of first category.

Keywords: Sobolev homeomorphism, Baire categories, piecewise affine homeomorphism.

MSC (2010):46B35, 26B35 .

1. Introduction

1.1. **Notation and main result.**
Denote by $|\cdot|_\infty$ the norm on $\mathbb{R}^4$ given by

$$|(a, b, c, d)|_\infty = \max\{|a|, |b|, |c|, |d|\}.$$

Let $Q = (0, 1)^2$ be the open unit square in $\mathbb{R}^2$. Consider a BV map $f : Q \to Q$ and denote by $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ its components, relative to the usual coordinates in $\mathbb{R}^2$.

Denote the *variation* of f in Q by

$$Var(f, Q) := \sup \left\{ \int_Q (f_1 \mathrm{div}\phi_1 + f_2 \mathrm{div}\phi_2) dx : |\phi(x)|_\infty \le 1 \text{ for all } x \in Q \right\},$$

where $\phi = (\phi_1, \phi_2) \in C^1_c(Q, \mathbb{R}^2 \times \mathbb{R}^2)$ and the integration is with respect to the Lebesgue measure. Compare this definition with [1] Definiton 3.4: we do not use the usual notation $V(f, Q)$, because we do not compute the norm of ϕ in the standard way.

Fix a constant $M > 2$ and consider the set

$$X := \{f : Q \to Q : \ f \text{ is a } BV \text{ homeomorphism}, \ f_{|\partial Q} = Id, \ Var(f, Q) < M\},$$

and the distance on X

$$d_X(f, g) := \|f - g\|_\infty + \|f^{-1} - g^{-1}\|_\infty + \left| \frac{1}{M - Var(f, Q)} - \frac{1}{M - Var(g, Q)} \right| .$$

We will prove in §1 that (X, d_X) is a complete metric space. Now let us consider the following subset of X:

$$A := \{f \in X : \exists E \subset Q, E \text{ Borel}, |E| = 0, |f(E)| = 1\}.$$

The main result of the present paper is the following

1.2. Theorem. *The set A is residual in X, i.e. it contains the intersection of countably many open dense subsets of X.*

The Baire theorem (see for instance [7]) implies that the set A is non-empty and, more precisely, that it is dense in X.

1.3. **Strategy for the proof of Theorem 1.2.** We introduce the following family of subsets of X. For every $n \in \mathbb{N}$, we denote

$$A_n := \{f \in X : \exists E \subset Q, |E| < 1/n, |f(E)| > 1 - 1/n\},$$

where the set E is a union of finitely many pairwise disjoint open triangles (the number of such triangles may depend both on n and on the function f).

It is easy to see that the set A contains the intersection of the A_n's (see §4). Hence, to prove Theorem 1.2 it is sufficient to show that the sets A_n are open and dense in X. The openness is an easy issue (see Lemma 4.1), while density is more delicate (see Lemma 4.2), so we will give here a sketch of the proof of the second property. The actual proof of both properties is postponed to §4.

Fix $n \in \mathbb{N}$, $f \in X$ and $\varepsilon > 0$. We want to find $f_\varepsilon \in A_n$ with $d_X(f, f_\varepsilon) < \varepsilon$. Firstly we use the main result of [6] to obtain an orientation preserving, (finitely) piecewise affine homeomorphism $g_\varepsilon \in X$ with

$$d_X(f, g_\varepsilon) < \varepsilon/4.$$

Then we take a finite triangulation of Q such that g_ε is affine on each triangle. If necessary, we can refine such triangulation in order to obtain a new finite triangulation τ such that the diameter of all triangles $T \in \tau$ and of their images through g_ε do not exceed $\varepsilon/8$. This ensures that any perturbation h of g_ε, which agrees with g_ε on ∂T for every $T \in \tau$, satisfies

$$\|h - g_\varepsilon\|_\infty + \|h^{-1} - g_\varepsilon^{-1}\|_\infty \leq \varepsilon/4.$$

Finally we modify the homeomorphism g_ε inside each triangle $T \in \tau$ in order to obtain a new orientation preserving homeomorphism $f_\varepsilon \in X$ with the following properties:

(1) f_ε is (finitely) piecewise affine on each triangle $T \in \tau$;
(2) for every $T \in \tau$
$$f_{\varepsilon|\partial T} = g_{\varepsilon|\partial T};$$
(3) $\left|\frac{1}{M\text{-}Var(f_\varepsilon,Q)} - \frac{1}{M\text{-}Var(g_\varepsilon,Q)}\right| \leq \varepsilon/2;$
(4) for every $T \in \tau$ there exists a set $F \subset T$ which is the union of finitely many pairwise disjoint open triangles and satisfies
$$\frac{Area(F)}{Area(T)} < 1/n; \ \frac{Area(f_\varepsilon(F))}{Area(f_\varepsilon(T))} > 1 - 1/n.$$

Clearly the property of the triangulation τ implies that

$$= \|f_\varepsilon - g_\varepsilon\|_\infty + \|f_\varepsilon^{-1} - g_\varepsilon^{-1}\|_\infty < \varepsilon/4$$

and, together with property (3), this implies that $d_X(f, f_\varepsilon) < \varepsilon$. Moreover properties (1),(2) and (4) imply that $f_\varepsilon \in A_n$.

The construction of f_ε starting from g_ε uses a piecewise affine homeomorphism ϕ_n (defined in §3), which maps a square Q onto a parallelogram P and coincides with the affinity between Q and P on ∂Q. This map is similar in spirit to the "basic building block" used in [3] to construct Sobole homeomorphisms with zero distributional Jacobian almost everywhere. The main difference between the two maps is that, although in both cases the aim is to map a small subset F of Q in a (proportionally) much larger set F', with a small cost in the variation, we want in addition that F' is almost a set of full measure in P.

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2. The metric space X

In this section we prove that the pair (X, d_X) defined in the Introduction actually identifies a complete metric space. Since there are no doubts that d_X defines a metric on X, we will focus on the completeness.

2.1. Proposition. *The metric space (X, d_X) is complete.*

Proof. It is a well known result in functional analysis that a sequence $(f_i)_{i\in\mathbb{N}}$ of BV maps which is Cauchy with respect to the supremum norm and such that the variations $Var(f_i, Q) < M$ are equi-bounded converges to a BV map f with $Var(f, Q) \leq M$ (See e.g. propositions 3.6 and 3.13 of [1]: clearly our renorming of $\mathbb{R}^4$ does not affect the validity of such statements). We need to prove that if the sequence is Cauchy with respect to the distance d_X, then the limit f remains a homeomorphism, and moreover $Var(f, Q) < M$. To prove that f is a homeomorphism it is sufficient to observe that the sequence of continuous functions $(f_i^{-1})_{i\in\mathbb{N}}$ is Cauchy with respect to the supremum norm and therefore it converges to a continuous function g, which is the inverse of f (the latter is a trivial fact of general topology).

Assume now by contradiction that $Var(f, Q) = M$. The lower semicontinuity of the variation with respect to the uniform convergence implies that

$$\lim_{i\to\infty} Var(f_i, Q) = M,$$

which implies that, for every fixed $m \in \mathbb{N}$ the quantity

$$\left| \frac{1}{M - Var(f_m, Q)} - \frac{1}{M - Var(f_j, Q)} \right|$$

is unbounded in j, hence $(f_i)_{i\in\mathbb{N}}$ is not Cauchy with respect to d_X, which is a contradiction. □

2.2. Remark. The completeness of the metric space (X, d_X) is necessary in order to apply the Baire theorem. A non-strict inequality on the variation in the definition of X would be probably a more natural choice (in particular this would allow us to drop the last term in the definition of d_X). Nevertheless we introduced such metric space, because in order both to perform the piecewise affine approximation of [6] and to modify such approximation using the homeomorphism ϕ_n, we may need to increase the variation of a small amount.

It is actually possible to circumvent this issue even in the setting mentioned above, i.e. when we just require $Var(f,Q) \leq M$: indeed it is sufficient to approximate preliminarily a BV homeomorphism f by homeomorphisms having smaller variation. This can be always achieved if $Var(f,Q) > 2$, by "interpolating" a contraction of the original homeomorphism f in an inner square concentric to Q and the identity on the outer frame.

3. The homeomorphism ϕ_n

Let (x, y) denote the usual coordinates on the plane and let $\overline{Q}$ be the unit square $[0,1] \times [0,1]$. Consider a linear map

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $\det(A) > 0$. Clearly A identifies the linear, orientation preserving homeomorphism ψ which maps the points $(1,0)$ and $(0,1)$ in (a,c) and (b,d) respectively. Fix $n \in \mathbb{N}$, $n > 2$. We will define a piecewise affine, orientation preserving homeomorphism ϕ_n such that

$$Var(\phi_n, Q) \leq (1 + (2/n^{1/2}))Var(\psi, Q)$$

and $\phi_{n|\partial Q} = \psi_{|\partial Q}$. Moreover we construct ϕ_n in such a way that there exists a set $F \subset Q$ which is a union of finitely many pairwise disjoint open triangles, satisfying

$$Area(F) < \frac{1}{n^{1/2}}; \quad Area(\phi_n(F)) \geq \left(1 - \frac{1}{2n}\right)\left(1 - \frac{1}{n^{1/2}}\right)\det(A). \tag{3.1}$$

For $i = 0, \dots, n^2 - 1$, let R_i be the rectangle

$$R_i := [0,1] \times [i/n^2, (i+1)/n^2].$$

Denote

$$R' := [1/n, 1 - (1/n)] \times [0, 1/n^{5/2}] \subset R_0$$

and

$$R'' := [1/n, 1 - (1/n)] \times [1/n^{5/2}, 1/n^2] \subset R_0.$$

Finally consider $R_0 \setminus (R' \cup R'')$. We define a partition of the left rectangle

$$R''' := [0, 1/n] \times [0, 1/n^2]$$

and on the right rectangle we define the symmetric partition with respect to the axis $x = 1/2$. Let us write

$$R''' = T_1 \cup T_2 \cup T_3 \cup T_4,$$

where (see Figure 1):

- T_1 has vertices in $(0,0)$, $(1/n, 0)$, and $(0, 1/n^{5/2})$,
- T_2 has vertices in $(0, 1/n^{5/2})$, $(1/n, 0)$, and $(1/n, 1/n^{5/2})$,
- T_3 has vertices in $(0, 1/n^{5/2})$, $(1/n, 1/n^{5/2})$, and $(1/n, 1/n^2)$,
- T_4 has vertices in $(0, 1/n^{5/2})$, $(1/n, 1/n^2)$, and $(0, 1/n^2)$.

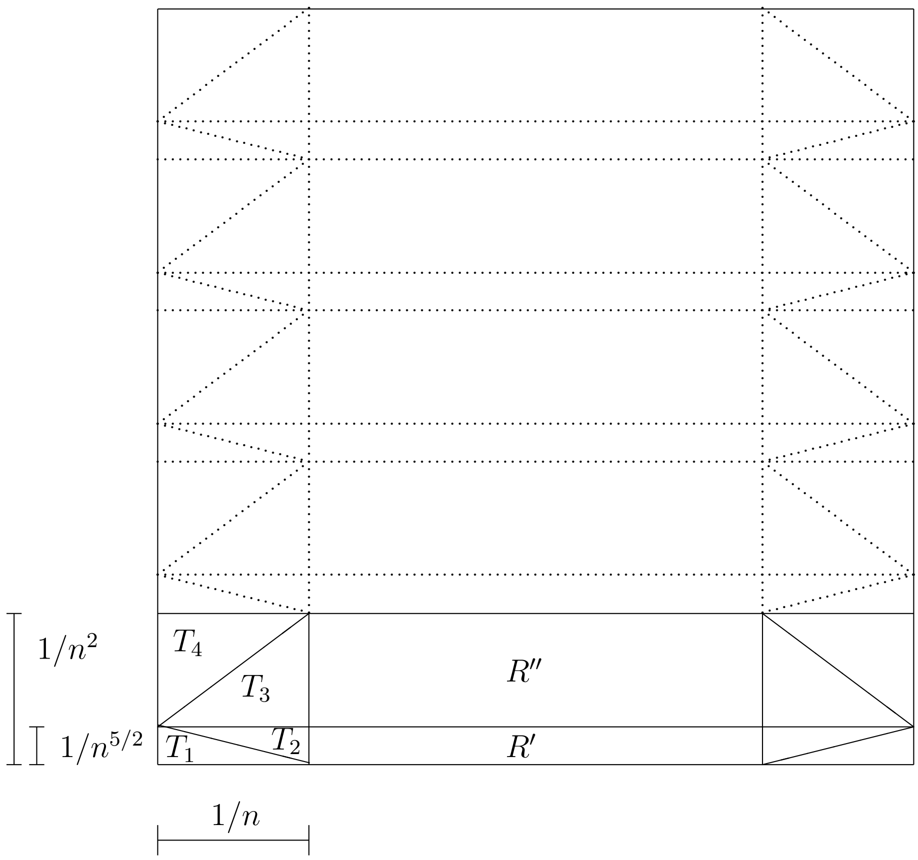

FIGURE 1. Tiling of the square Q.

Now we define the homeomorphism ϕ_n in the rectangle R_0, in such a way that

$$\phi_{n|\partial R_0} = \psi_{|\partial R_0}. \tag{3.2}$$

Then we will be able to extend ϕ_n to the square Q, requiring its continuity and defining, for every $(x, y) \in \text{int}(R_i)$,

$$\phi_n(x, y) = \phi_n((x, y) - (0, i/n^2)) + i/n^2(b, d) \tag{3.3}$$

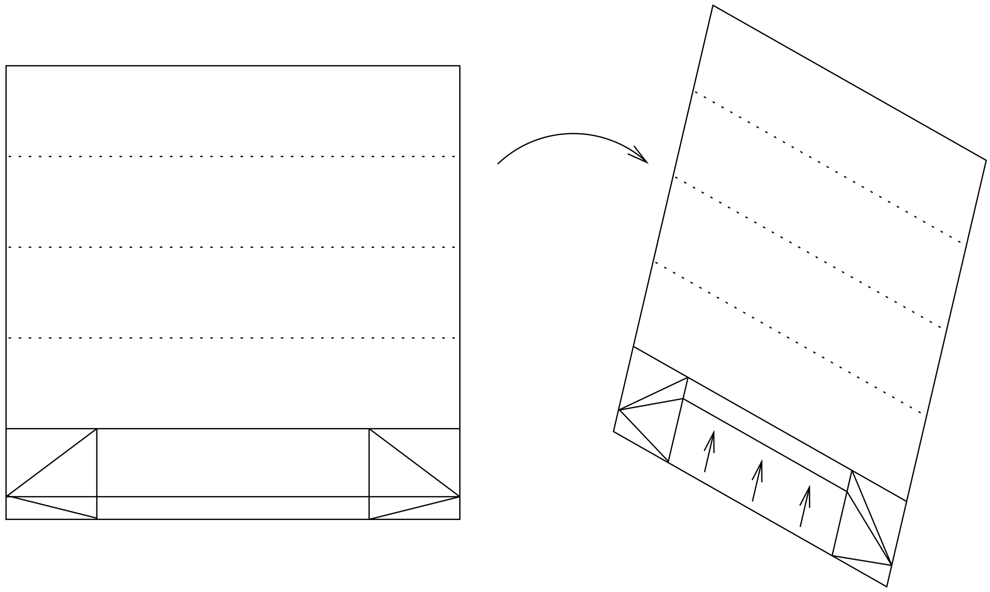

FIGURE 2. Representation of the map ϕ_n.

(notice that the point $((x,y)-(0,i/n^2))$ belongs to $\mathrm{int}(R_0)$). We define ϕ_n on R' as the linear map $\phi_n(x,y) = A'(x,y)^t$, where

$$A' := \begin{pmatrix} a & (n^{1/2}-1)b \\ c & (n^{1/2}-1)d \end{pmatrix}.$$

The map ϕ_n is now uniquely defined on Q by the conditions (3.2), (3.3) and by the requirement that ϕ_n is continuous on Q and affine on R', R'' and on the triangles $T_1, \ldots, T_4$ (and on their symmetric copies).

In particular on R'' there holds

$$\nabla\phi_n = \begin{pmatrix} a & (1/(n^{1/2}-1))b \\ c & (1/(n^{1/2}-1))d \end{pmatrix}.$$

Denoting F the set

$$F := \bigcup_{i=0}^{n^2-1} (\mathrm{int}(R') + (0, i/n^2)),$$

it is easy to see that (3.1) holds. More precisely, since we want that F is the union of pairwise disjoint open triangles, we should replace the set $\mathrm{int}(R')$ with the union of two disjoint open triangles such that the closure of their union is R'. Notice also that $\phi_n = \psi$ on T_1 and T_4.

We now want to compute the variation $Var(\phi_n, Q)$. Since ϕ_n is piecewise affine, this is equivalent to compute the *energy*

$$\mathbb{E}(\phi_n) := \int_Q |\nabla\phi_n|_1 dx,$$

where we denoted by $|\cdot|_1$ the norm on Mat(2×2) given by

$$\left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|_1 = |a|+|b|+|c|+|d|.$$

By construction, we have

$$\mathbb{E}(\phi_n) = n^2\mathbb{E}(\phi_{n|R_0}).$$

Moreover, by the 1-homogeneity of the energy, we can compute

$$\begin{aligned} n^2\mathbb{E}(\phi_{n|R'}) &= n^2\left(1-\frac{2}{n}\right)\left(\frac{1}{n^{5/2}}\right)\left|\begin{pmatrix} a & (n^{1/2}-1)b \\ c & (n^{1/2}-1)d \end{pmatrix}\right|_1 \\ &= \left(1-\frac{2}{n}\right)\left|\begin{pmatrix} (1/n^{1/2})a & (1-(1/n^{1/2}))b \\ (1/n^{1/2})c & (1-(1/n^{1/2}))d \end{pmatrix}\right|_1 \end{aligned} \tag{3.4}$$

and analogously

$$\begin{aligned} n^2\mathbb{E}(\phi_{n|R''}) &= n^2\left(1-\frac{2}{n}\right)\left(\frac{1}{n^2}-\frac{1}{n^{5/2}}\right)\left|\begin{pmatrix} a & (1/(n^{1/2}-1))b \\ c & (1/(n^{1/2}-1))d \end{pmatrix}\right|_1 \\ &= \left(1-\frac{2}{n}\right)\left|\begin{pmatrix} (1-(1/n^{1/2}))a & (1/n^{1/2})b \\ (1-(1/n^{1/2}))c & (1/n^{1/2})d \end{pmatrix}\right|_1 . \end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5) we have that

$$\mathbb{E}(\phi_{n|(R'\cup R'')}) = \mathbb{E}(\psi_{|(R'\cup R'')}). \tag{3.6}$$

Regarding the energy of ϕ_n in the triangles $T_1,\dots,T_4$, we have, trivially

$$\mathbb{E}(\phi_{n|T_1}) = \mathbb{E}(\psi_{|T_1})$$

and

$$\mathbb{E}(\phi_{n|T_4}) = \mathbb{E}(\psi_{|T_4}).$$

Moreover on T_2, we have

$$\nabla\phi_n = \begin{pmatrix} a+((1/n)-2/n^{3/2})b & (n^{1/2}-1)b \\ c+((1/n)-2/n^{3/2})d & (n^{1/2}-1)d \end{pmatrix}$$

and therefore

$$|\nabla\phi_n|_1 \le n^{1/2}|\nabla\psi|_1. \tag{3.7}$$

Finally on T_3, we have

$$\nabla\phi_n = \begin{pmatrix} a+((1/n)-2/n^{3/2})b & (1/(n^{1/2}-1))b \\ c+((1/n)-2/n^{3/2})d & (1/(n^{1/2}-1))d \end{pmatrix}$$

and therefore

$$|\nabla\phi_n|_1 \le 2|\nabla\psi|_1. \tag{3.8}$$

With similar computations, one can verify that the equations (3.7) and (3.8) are satisfied also on the symmetric copies of T_2 and T_3, respectively. Hence, combining (3.6) with (3.7) and (3.8), we have

$$\mathbb{E}(\phi_n) \le (1+(2/n^{1/2}))\mathbb{E}(\psi).$$

Let us summarize the conclusions of the computations above in the following proposition. In order to obtain a clean statement, we denote by ϕ_n the map we constructed above relative to a parameter which is actually larger than n.

3.1. Proposition. *For every affine homeomeorphism ϕ defined on a square Q with edges parallel to the coordinate lines and for every $n \in \mathbb{N}$ there exists piecewise affine map ϕ_n on Q and a set F_n which is a finite union of pairwise disjoint open triangles such that*

(1) $|Var(\phi_n, Q) - Var(\phi, Q)| \leq (1/n)Var(\phi, Q)$;
(2) $\phi_{n|\partial Q} = \phi_{|\partial Q}$;
(3)
$$\frac{Area(F_n)}{Area(Q)} < 1/n; \ \frac{Area(\phi_n(F_n))}{Area(\phi(Q))} > 1 - 1/n.$$

4. Proof of Theorem 1.2

We have to prove that the sets A_n defined in the Introduction are open and dense in (X, d_X).

4.1. Lemma. *For every $n \in \mathbb{N}$ the set A_n is open.*

Proof. Take $f \in A_n$. Let $1/n > \varepsilon > 0$ and let $T_1, \dots, T_m$ be pairwise disjoint open triangles in Q such that, denoting $E = \bigcup_i T_i$, there holds

(1) $|E| < 1/n - \varepsilon$;
(2) $|f(E)| > 1 - 1/n + \varepsilon$.

Since the image of each triangle T_i is open, then there exists $\eta > 0$ such that, denoting for every open set B

$$B^\eta := \{x \in B : dist(x, B^C) > \eta\},$$

there holds

$$|f(T_i)^\eta| \geq (1 - \varepsilon)|f(T_i)|,$$

for every $i = 1, \dots, m$.

Consider now $g \in X$ with $d_X(f, g) < \eta$, In particular $\|f - g\|_\infty < \eta$, hence $g(T_i) \supset f(T_i)^\eta$, for every $i = 1, \dots, m$. Therefore

$$|g(E)| = \sum_{i=1}^{m} |g(T_i)| \geq \sum_{i=1}^{m} |f(T_i)^\eta| \geq (1 - \varepsilon)|f(E)| > 1 - 1/n.$$

Hence $g \in A_n$. □

4.2. Lemma. *For every $n \in \mathbb{N}$ the set A_n is dense in X.*

Proof. Fix $f \in X$ and $\varepsilon > 0$. We want to find $f_\varepsilon \in A_n$ with $d_X(f, f_\varepsilon) < \varepsilon$. By [6, Theorem A], we can find a sequence of (finitely) piecewise affine homeomorphisms $(g_i)_{i \in \mathbb{N}} : Q \to Q$ such that

(1) $g_{i|\partial Q} = Id$;
(2) $\|g_i - f\|_\infty$ tends to 0 for $i \to \infty$;

(3) $\|g_i^{-1} - f^{-1}\|_\infty$ tends to 0 for $i \to \infty$;
(4) $\lim_{i\to\infty} Var(g_i, Q) \le Var(f, Q)$.

Given (2), the validity of (3) is a simple consequence of the uniform continuity of f^{-1}. Indeed such property implies that if $\|g_i - f\|_\infty$ is small, then $\|g_i^{-1} - f^{-1}\|_\infty$ is also small. To prove it, fix $\varepsilon > 0$ and let $\delta > 0$ be such that if $|x - y| < \delta$ then $|f^{-1}(x) - f^{-1}(y)| < \varepsilon$. Now take $i \in \mathbb{N}$ such that $\|g_i - f\|_\infty < \delta$. We want to prove that $\|g_i^{-1} - f^{-1}\|_\infty < \varepsilon$. Assume by contradiction there exists x_0 such that $|g_i^{-1}(x_0) - f^{-1}(x_0)| > \varepsilon$. Denoting $x_1 := g_i^{-1}(x_0)$ and $x_2 := f^{-1}(x_0)$, we have $|g_i(x_1) - f(x_1)| < \delta$. Hence, denoting $x_3 := f(x_1)$, we have $|x_3 - x_0| < \delta$, but $|f^{-1}(x_3) - f^{-1}(x_0)| > \varepsilon$, which is a contradiction.

Using the lower semicontinuity of the variation w.r.t. the uniform convergence, from (1)-(4) we deduce that there exists a piecewise affine homeomorphism $g_\varepsilon \in X$ with

$$d_X(f, g_\varepsilon) < \varepsilon/4. \tag{4.1}$$

We can also assume

$$\left| \frac{1}{M - Var(f,Q)} - \frac{1}{M - Var(g_\varepsilon, Q)} \right| < \varepsilon/8. \tag{4.2}$$

Now we take a finite triangulation of Q such that g_ε is affine on each triangle. If necessary, we can refine such triangulation in order to obtain a new finite triangulation τ such that the diameter of all triangles $T_i \in \tau$ and of their images through g_ε are less than $\varepsilon/8$.

By (4.2) we can take $m \in \mathbb{N}$, $m > 2n$ such that

$$\left| \frac{1}{M - Var(f,Q)} - \frac{1}{M - CVar(g_\varepsilon, Q)} \right| < \varepsilon/2, \tag{4.3}$$

for every $C \in [1 - (1/m), 1 + 1/m]$. We define the homeomorphism f_ε as follows. For every $T_i \in \tau$ take finitely many closed squares Q_i^j with pairwise disjoint interiors and with edges parallel to the coordinate lines such that $Q_i^j \subset T_i$ and

$$\left| \bigcup_j Q_i^j \right| \ge \left(1 - \frac{1}{m}\right) |T_i|. \tag{4.4}$$

For every i, j, define f_ε on Q_i^j as the map obtained by replacing the map g_ε with the map $(g_\varepsilon)_m$ obtained applying Proposition 3.1 with $n = m$ and $\phi = g_{\varepsilon|Q_i^j}$. For every i, define $f_\varepsilon := g_\varepsilon$ on $T_i \setminus (\bigcup_j Q_i^j)$.

By (4.1),(4.3), point (1) of Proposition 3.1 and the property of the triangulation, we have

$$d(f, f_\varepsilon) \le d(f, g_\varepsilon) + d(f_\varepsilon, g_\varepsilon) < ((\varepsilon/4) + (\varepsilon/2)) + (\varepsilon/4) = \varepsilon.$$

By point (3) of Proposition 3.1 and (4.4) we have that, denoting by $(F_i^j)_m$ the set given by Proposition 3.1 applied with $n = m$ and $\phi = g_{\varepsilon|Q_i^j}$, and by F the set

$$F := \bigcup_{i,j} (F_i^j)_m,$$

there holds $|F| < 1/m$ and

$$|f_\varepsilon(F)| > (1 - (1/m))(1 - (1/m)) > 1 - (2/m) > 1 - (1/n),$$

hence $f_\varepsilon \in A_n$. □

Proof of Theorem 1.2. The only thing left to show is that $A \supset \bigcap_{n\in\mathbb{N}} A_n$. Fix $f \in \bigcap_{n\in\mathbb{N}} A_n$. In particular, for every $j \in \mathbb{N}$, we have $f \in \bigcap_{i>j} A_{2^i}$, hence for every $i \in \mathbb{N}$ with $i > j$ there exists a Borel set E_i with $|E_i| < 2^{-i}$ such that $|f(E_i)| > 1 - 2^{-i}$. Therefore denoting $E^j := \bigcup_{i>j} E_i$, we have $|E^j| < 2^{-j}$ and $|f(E^j)| = 1$. Since the countable intersection of sets of full measure is a set of full measure we deduce that, denoting $E := \bigcap_{j\in\mathbb{N}} E^j$, we have that $|E| = 0$ and $|f(E)| = 1$, hence $f \in A$. □

5. Final remarks and open questions

5.1. **$W^{1,p}$ homeomorphisms.** In [3], Hencl proves that for $1 \leq p < 2$ there exists a homeomorphism $f : Q \to Q$ in $W^{1,p}$ with $f_{|\partial Q} = Id$ satisfying $Jf = 0$ a.e. It is easy to see that the set of such homemomorphisms is dense in $W^{1,p}$ with respect to the C^0-distance. Therefore a natural question is whether in a suitable complete metric space of $W^{1,p}$ homeomorphisms, these maps are residually many. For $p > 1$ a natural setting to answer this question (i.e. a reasonable complete metric space of $W^{1,p}$ homeomorphisms) is the set

$$Y := \left\{ f : Q \to Q : \ f \text{ is a } W^{1,p} \text{ homeomorphism}, \ f_{|\partial Q} = Id, \ \int_Q |Df|^p \leq M \right\}$$

for an arbitrary constant $M > 1$, with the distance

$$d_Y(f,g) := \|f - g\|_\infty + \|f^{-1} - g^{-1}\|_\infty.$$

Observe that one cannot consider as distance the natural norm of $W^{1,p}$, because the convergence in such norm would also imply the convergence of the Jacobians, almost everywhere.

Trying to imitate our strategy, one could rely on the fact that in [5] the authors prove that it is possible to approximate a $W^{1,p}$ homeomorphism ($p > 1$) uniformly and in the $W^{1,p}$ norm by piecewise affine homeomorphisms. Nevertheless, there is no hope that homeomorphisms with zero Jacobian almost everywhere are residual in the metric space (Y, d_Y), since they are not even dense. Indeed, take a homeomorphism $f \in Y$ with $Jf > 0$ on a set of positive measure and satisfying $\int_Q |Df|^p = M$. Assume that there exist homeomorphisms f_n in Y with

$d_Y(f_n, f) \to 0$ as $n \to \infty$. Since the quantity $\int_Q |Df|^p$ is lower semicontinuous with respect to the uniform convergence, this would force

$$\int_Q |Df_n|^p \to M = \int_Q |Df|^p, \quad \text{as } n \to \infty. \tag{5.1}$$

In turn, since the norm on $W^{1,p}$ is uniformly convex, the uniform convergence and (5.1) imply the convergence in norm, which forces the convergence of the Jacobians, too. Indeed, in every uniformly convex space, if $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ then $\|x_n - x\| \to 0$ (see Proposition 3.32 of [2]). In particular we can deduce that it is not possible to extend Proposition 3.1 to the setting of $W^{1,p}$ homeomorphisms: roughly speaking, in this class there is a positive minimal cost in the energy to approximate an affine homeomorphism with homeomorphisms that map a small set in a large one. Notice that, since the subset of homeomorphisms $f \in Y$ satisfying $\int_Q |Df|^p = M$ is residual in Y, then we have actually proved that the set of $W^{1,p}$ homeomorphisms with zero Jacobian almost everywhere is of first category in Y. Indeed we have proved that the set of homeomorphisms $f \in Y$ satisfying $\int_Q |Df|^p = M$ and $Jf > 0$ on a set of positive measure is relatively open. To prove that it is dense, we can use the same construction described at the end of §2. Moreover the result is independent on the dimension of the ambient space: let us summarize all these observations in the following

5.2. Theorem. *Let $d \geq 2$ and $Q^d := (0,1)^d$. Fix $1 < p < d$, $M > 1$. Define*

$$Y := \left\{ f : Q^n \to Q^d : f \text{ is a } W^{1,p} \text{ homeomorphism},\ f_{|\partial Q^d} = Id, \int_{Q^d} |Df|^p \leq M \right\}$$

and the distance on Y

$$d_Y(f, g) := \|f - g\|_\infty + \|f^{-1} - g^{-1}\|_\infty.$$

Then the set A of all homeomorphisms $f \in Y$ with $Jf = 0$ a.e. is of first category in Y, i.e. $Y \setminus A$ is residual in Y.

5.3. $W^{1,1}$ **homeomorphisms.** In [4] the authors prove that it is possible to approximate a $W^{1,1}$ homeomorphism uniformly and in the $W^{1,1}$ norm by piecewise affine homeomorphisms. Moreover in Proposition 3.1 (1), it is equivalent to consider the variation $Var(\phi, Q)$ or the energy $\mathbb{E}(\phi)$, hence if one considers the metric space (Y, d_Y) as defined in the previous subsection, for $p = 1$, it is not difficult to adapt the arguments presented in Section 4, to prove that the set of $W^{1,1}$ homeomorphisms of Q onto itself mapping a set of measure smaller than $1/n$ in a set of measure larger than $1 - 1/n$ are open and dense. The issue here is that (Y, d_Y) is not complete, because a sequence of $W^{1,1}$ maps converging uniformly and with equi-bounded energies may converge to a map which is in BV but not in $W^{1,1}$. Hence the countable intersection of open and dense sets might in principle be empty. The completion of such space is a space of BV homeomorphisms, with a uniform bound on the variation. However, such complete metric space is too large for $W^{1,1}$ homeomorphisms with zero Jacobian almost everywhere to be residual. Indeed in Proposition 5.4, we show that in such metric space the subset of $W^{1,1}$ homeomorphisms is of first category. For the sake of brevity, and since the result is not surprising, we prove such statement only in the metric space defined in the Introduction. The same can be done, with minor changes, in the completion of the metric space defined in the previous subsection, for $p = 1$.

5.4. Proposition. *Let (X, d_X) be the metric space defined in the Introduction. Then the set A of all $W^{1,1}$ homeomorphisms in X is of first category.*

Proof. Define

$$A_n := \{f \in X : \exists E \subset Q, |E| < 1/n, Var(f, E) > 1/2 - 1/n\},$$

where E is the union of finitely many pairwise disjoint open triangles. Clearly the intersection of the A_n's does not contain any $W^{1,1}$ homeomorphism, therefore, to prove the proposition it is sufficient to show that the A_n's are open and dense. The openness is just a consequence of the lower semicontinuity of the variation with respect to the uniform convergence. The density can be achieved as in Lemma 4.2: it is sufficient to observe, from (3.4) that the maps ϕ_n, ψ and the set F constructed in §3 satisfy

$$Var(\phi_n, F) > (1/2 - 1/\sqrt{n})Var(\psi, Q). \tag{5.2}$$

More precisely, such inequality is satisfied tout court if $|b| + |d| \geq |a| + |c|$. In case $|b|+|d| < |a|+|c|$, actually one should slightly modify the map ϕ_n to obtain (5.2): roughly speaking, it is sufficient to "switch" the coordinates (x, y).

□

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