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THE ISOPERIMETRIC PROBLEM FOR NONLOCAL PERIMETERS

Annalisa Cesaroni

Department of Statistical Sciences
University of Padova
Via Cesare Battisti 141, 35121 Padova, Italy

Matteo Novaga*

Department of Mathematics
University of Pisa
Largo Bruno Pontecorvo 5, 56127 Pisa, Italy

Abstract. We consider a class of nonlocal generalized perimeters which includes fractional perimeters and Riesz type potentials. We prove a general isoperimetric inequality for such functionals, and we discuss some applications. In particular we prove existence of an isoperimetric profile, under suitable assumptions on the interaction kernel.

1. **Introduction.** In this paper we consider a family of geometric functionals, which in particular contains the fractional isotropic and anisotropic perimeter. More precisely, we define the following energy defined on measurable subsets $E \subset \mathbb{R}^N$:

$$\mathrm{Per}_K(E) := \int_E \int_{\mathbb{R}^N \setminus E} K(x-y)dxdy = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\chi_E(x)-\chi_E(y)|K(x-y)dxdy \quad (1)$$

where the kernel $K : \mathbb{R}^N \to [0, +\infty)$ satisfies the following assumptions:

$$K(x) = K(-x) \quad (2)$$

$$\min(|x|, 1)\, K(x) \in L^1(\mathbb{R}^N). \quad (3)$$

The functional (1) measures the interaction between points in E and in $\mathbb{R}^N \setminus E$, weighted by the kernel K.

Note that it is not restrictive to assume (2) since $\mathrm{Per}_K(E) = \mathrm{Per}_{\widetilde{K}}(E)$ for every E, where $\widetilde{K}(x) := (K(x) + K(-x))/2$. Notice also that, if $K \in L^1(\mathbb{R}^N)$, then for every E with $|E| < \infty$, we have

$$\mathrm{Per}_K(E) = |E|\|K\|_{L^1(\mathbb{R}^N)} - \int_E \int_E K(x-y)dxdy. \quad (4)$$

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* Corresponding author.

In the first part of the paper we deal with isoperimetric inequalities for such functionals. The main result is the following (see Corollary 3.4): if $K(x) \geq \mu\chi_{B_r}(x)$ for some constants $\mu > 0$ and $r > 0$, then for all measurable sets E there holds

$$\mathrm{Per}_K(E) \geq \min(g(|E|), g(|\mathbb{R}^N \setminus E|)),$$

where $g(m) := \mathrm{Per}_{K^\star}(B_m)$, with $K^\star$ the symmetric decreasing rearrangment of K, and B_m the ball with volume m centered at 0. We discuss some property of the function g and we provide a Poincaré type inequality (see Proposition 4.1).

We recall that, in the case of fractional perimeters, sharp quantitative isoperimetric inequality, uniform with respect to the fractional exponent bounded away from 0, have been obtained in [9] (see also [11] for an anisotropic version), whereas Poincaré type inequalities have been discussed in [13].

An interesting related question is understanding which conditions on K imply the compact embedding of the functions with bounded energy J_K into L^p spaces, for some $p \geq 1$.

In the second part of the paper, we consider the isoperimetric problem

$$\min_{E\colon\, |E|=m} \mathrm{Per}_K(E), \tag{5}$$

for a fixed volume $m > 0$.

In the case of the fractional perimeter, the existence of isoperimetric sets solving (5) has been studied in [2, 9] (see also [4] where a bulk term is added to the energy), where it is shown that balls are the unique minimizers of the fractional perimeter among sets with the same volume. In the general case, the same result holds if the kernel K is a radially symmetric decreasing function, as a straightforward consequence of the Riesz rearrangement inequality [14]. So, we focus on the case in which K is not radially symmetric and decreasing. We provide an existence result of minimizers of the relaxed problem associated to (5) under the additional assumption that $K \in L^1(\mathbb{R}^N)$ (see Theorem 5.6). The proof is based on a concentration compactness type argument. Finally, we show that if K has maximum at the origin (in an appropriate sense, see condition (38)), then every minimizer of the relaxed problem is actually the characteristic function of a compact set (see Theorem 5.7).

We are left with the open problem of extending the existence result to more general interaction kernels satisfying only (3).

Another interesting problem is to consider kernels which are just Radon measures on $\mathbb{R}^N$. In this case we don't expect in general compactness of minimizers.

Notation. We denote by $B_m(x)$ the ball centered at x with volume m, that is, the ball with radius $r = m^{\frac{1}{N}}\omega_N^{-\frac{1}{N}}$, and by B_m for the ball centered at 0 with volume m. We also denote by $B(x, r)$ the ball of center x and radius r.

For every measurable set $E \subseteq \mathbb{R}^N$, χ_E denotes the characteristic function of E, that is the function which is 1 on E and 0 outside.

We recall that given a set E with $|E| < \infty$, its symmetric rearrangement $E^\star$ is the ball $B_{|E|}$ that is the ball centered at 0 with volume $|E|$. Moreover the symmetric decreasing rearrangement of a nonnegative measurable function h with level sets of finite measure is defined as

$$h^\star(x) = \int_0^{+\infty} \chi_{\{h>t\}^\star}(x)dt.$$

Note that if h is radially symmetric and decreasing, then $h = h^\star$. Moreover, $h \in L^p(\mathbb{R}^n)$ if and only if $h^\star \in L^p(\mathbb{R}^n)$ with $\|h\|_{L^p(\mathbb{R}^n)} = \|h^\star\|_{L^p(\mathbb{R}^n)}$, for all $p \geq 1$.

2. **Generalized fractional perimeters.** In this section we discuss some properties of the K perimeters.

Remark 2.1. Condition (3) implies that if E is a set with $|E| < \infty$ and $\mathcal{H}^{N-1}(\partial E) < \infty$, then $\mathrm{Per}_K(E) < \infty$ (see [7, Remark 1.4]). Indeed

$$\begin{aligned}\mathrm{Per}_K(E) &= \int_E \int_{\mathbb{R}^N \setminus E} K(x-y)dxdy \\ &= \int_{\mathbb{R}^N} |(E+x) \cap (\mathbb{R}^N \setminus E)| K(x)dx \le C \int_{\mathbb{R}^N} (|x| \wedge 1) K(x)dx\end{aligned}$$

where C is a constant which depends on E.

Proposition 2.2. *The following properties hold:*

1. *$Per_K(E) = Per_K(\mathbb{R}^N \setminus E)$ and*
$$Per_K(E \cap F) + Per_K(E \cup F) \le Per_K(E) + Per_K(F). \tag{6}$$
2. *$E \to Per_K(E)$ is lower semicontinuous with respect to the L^1_{loc}-convergence.*

Proof. We start by proving 1. The first equality is a direct consequence of the definition of Per_K. In order to prove (6), we observe that

$$\begin{aligned}&\int_{E\cup F} \int_{\mathbb{R}^N \setminus (E\cup F)} \\ &\quad = \int_E \int_{\mathbb{R}^N \setminus E} + \int_F \int_{\mathbb{R}^N \setminus F} - \int_{E\cap F} \int_{\mathbb{R}^N \setminus (E\cup F)} - \int_E \int_{F\setminus (E\cap F)} - \int_F \int_{E \setminus (E\cap F)} .\end{aligned}$$

and

$$\begin{aligned}&\int_{E\cap F} \int_{\mathbb{R}^N \setminus (E\cap F)} \\ &\quad = \int_{E\cap F} \int_{\mathbb{R}^N \setminus (E\cup F)} + \int_{E\cap F} \int_{E\setminus (F\cap E)} + \int_{E\cap F} \int_{F\setminus (F\cap E)} .\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{Per}_K(E\cap F) + \mathrm{Per}_K(E\cup F) &= \mathrm{Per}_K(E) + \mathrm{Per}_K(F) \\ &\quad - 2\int_{E\setminus(E\cap F)} \int_{F\setminus (E\cap F)} K(x-y)dxdy,\end{aligned}$$

which gives (6).

The proof of 2. is a consequence of Fatou lemma, observing that $\mathrm{Per}_K(E) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\chi_E(x) - \chi_E(y)| K(x-y)dxdy$. □

2.1. **Examples.** A first class of examples is given by the kernels $K(x)$ which satisfies

$$\lambda \frac{1}{|x|^{N+s}} \le K(x) \le \Lambda \frac{1}{|x|^{N+s}},$$

for some $s \in (0,1)$ and $0 < \lambda \le \Lambda$. This class includes the fractional perimeters, and its inhomogeneous and anisotropic versions.

The fractional perimeter, which has been introduced in [13] and further developed in [2], is defined as

$$P_s(E) := \int_E \int_{\mathbb{R}^N \setminus E} \frac{1}{|x-y|^{N+s}} \, dx \, dy, \tag{7}$$

for $s \in (0,1)$. It is also possible to substitute the kernel $\frac{1}{|x-y|^{N+s}}$ with more general heterogeneous, isotropic kernels of the type

$$K(x) = \frac{a(x)}{|x|^{N+s}}$$

where $a : \mathbb{R}^N \to (0, +\infty)$ is a measurable function such that $0 < \lambda \leq a(x) \leq \Lambda$. The anisoptropic fractional perimeters have been defined in [11] as follows: let $B \subseteq \mathbb{R}^N$ be a convex set which is symmetric with respect to the origin and let $|\cdot|_B$ the norm in $\mathbb{R}^N$ with unitary ball B, then we define

$$P_{s,B}(E) := \int_E \int_{\mathbb{R}^N \setminus E} \frac{1}{|x-y|_B^{N+s}}\, dx\, dy. \tag{8}$$

Another class of examples, relevant for this paper, is given by the kernels $K(x) \in L^1(\mathbb{R}^N)$, for which the representation formula (4) holds.

2.2. **Coarea formula.** We introduce the following functional on functions $u \in L^1_{loc}(\mathbb{R}^N)$:

$$J_K(u) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)| K(x-y) dx dy. \tag{9}$$

Note that $J_K(\chi_E) = \mathrm{Per}_K(E)$ for all measurable $E \subset \mathbb{R}^N$.

We provide a coarea formula, linking the functional Per_K to J_K.

Proposition 2.3 (Coarea formula)**.** *The following formula holds*

$$J_K(u) = \int_{-\infty}^{+\infty} Per_K(\{u > s\}) ds. \tag{10}$$

Proof. First of all we observe that, for every measurable function u,

$$|u(x) - u(y)| = \int_{-\infty}^{+\infty} |\chi_{\{u>s\}}(x) - \chi_{\{u>s\}}(y)| ds.$$

Moreover for every $s \in \mathbb{R}$

$$|\chi_{\{u>s\}}(x) - \chi_{\{u>s\}}(y)| = \chi_{\{u>s\}}(x)\chi_{\mathbb{R}^N \setminus \{u>s\}}(y) + \chi_{\{u>s\}}(y)\chi_{\mathbb{R}^N \setminus \{u>s\}}(x).$$

Therefore we get, recalling (2), and using Tonelli theorem,

$$\begin{aligned} 2J_K(u) &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)| K(x-y) dx dy \\ &= 2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{-\infty}^{+\infty} \chi_{\{u>s\}}(x) \chi_{\mathbb{R}^N \setminus \{u>s\}}(y) K(x-y) ds dx dy \\ &= 2 \int_{-\infty}^{+\infty} \int_{\{u>s\}} \int_{\mathbb{R}^N \setminus \{u>s\}} K(x-y) dx dy ds \\ &= 2 \int_{-\infty}^{+\infty} \mathrm{Per}_K(\{u > s\}) ds. \end{aligned}$$

□

3. **Isoperimetric inequality.** In this section we prove an isoperimetric inequality for generalized nonlocal perimeters.

Proposition 3.1. *For every measurable set $E \subseteq \mathbb{R}^N$ such that $|E| < \infty$, there holds*

$$Per_K(E) \geq Per_{K^*}(B_{|E|}),$$

where K^* is the symmetric decreasing rearrangement of K.

In particular, if K is radially symmetric and decreasing, then

$$Per_K(E) \geq Per_K(B_{|E|}).$$

Moreover, equality holds if and only if E is a translated of $B_{|E|}$.

Proof. First of all we consider the case in which $K \in L^1(\mathbb{R}^N)$. Note that $(\chi_E)^\star = \chi_{B_{|E|}}$. By Riesz rearrangement inequality [14], we get that

$$\int_E \int_E K(x-y)dxdy = \int_{\mathbb{R}^N} \chi_E(x)(\chi_E * K)(x)dx$$
$$\leq \int_{\mathbb{R}^N} \chi_{B_{|E|}}(x)(\chi_{B_{|E|}} * K^\star)(x)dx = \int_{B_{|E|}} \int_{B_{|E|}} K^\star(x-y).$$

So, recalling (4) we get the conclusion.

Finally, if $K = K^\star$, we have that equality in the Riesz rearrangement inequality holds if and only if χ_E is equal, up to translation, to its symmetric-decreasing rearrangement, therefore if and only if E is equal, up to translation, to $B_{|E|}$.

Now if $K \notin L^1(\mathbb{R}^N)$, we define $K_\varepsilon(x) = K(x) \wedge \frac{1}{\varepsilon}$. Then $K_\varepsilon \in L^1(\mathbb{R}^N)$ and K_ε converges to K monotonically increasing. Note that

$$K_\varepsilon^\star(x) = \int_0^{\frac{1}{\varepsilon}} \chi_{\{K>t\}^\star}(x)dt.$$

So as $\varepsilon \to 0$, also $K_\varepsilon^\star \to K^\star$ monotonically increasing. Therefore by the monotone convergence theorem if E is a measurable set we get that

$$\lim_{\varepsilon\to 0} \mathrm{Per}_{K_\varepsilon}(E) = \mathrm{Per}_K(E) \qquad \lim_{\varepsilon\to 0} \mathrm{Per}_{K_\varepsilon^\star}(E) = \mathrm{Per}_{K^\star}(E).$$

By the previous argument we get $\mathrm{Per}_{K_\varepsilon}(E) \geq \mathrm{Per}_{K_\varepsilon^\star}(B_{|E|})$. So, we conclude sending $\varepsilon \to 0$. □

For every $m \geq 0$, we define

$$g(m) := \mathrm{Per}_{K^\star}(B_m)$$

where we recall that B_m is the ball centered at 0 with volume m. We also set $g(+\infty) = +\infty$.

We provide some estimates on the function g.

Lemma 3.2.

1. *If $K \notin L^1(\mathbb{R}^N)$, then*
$$\lim_{m\to 0^+} \frac{g(m)}{m} = +\infty. \tag{11}$$
2. *If $K \in L^1(\mathbb{R}^N)$, then*
$$g(m) \leq \|K\|_{L^1} m, \qquad \lim_{m\to 0^+} \frac{g(m)}{m} = \|K\|_{L^1}. \tag{12}$$

Proof. For $x \in \mathbb{R}^N$ and $m > 0$ we define

$$K^\star_m(x) = \frac{1}{m}\int_{B_m(x)} K^\star(y)dy.$$

We have

$$\begin{aligned}\int_{\mathbb{R}^N} K^\star_m(x)dx &= \frac{1}{m}\int_{\mathbb{R}^N}\int_{B_m} K^\star(x+y)dydx \\ &= \frac{1}{m}\int_{B_m}\int_{\mathbb{R}^N} K^\star(x)dxdy = \int_{\mathbb{R}^N} K^\star(x)dx.\end{aligned}$$

In particular $K^\star \in L^1(\mathbb{R}^N)$ if and only if $K^\star_m \in L^1(\mathbb{R}^N)$ and $\|K^\star\|_{L^1} = \|K^\star_m\|_{L^1}$. We recall that $\|K^\star\|_{L^1} = \|K\|_{L^1}$.

We then compute

$$\mathrm{Per}_{K^\star}(B_m) = \int_{B_m}\int_{\mathbb{R}^N\setminus B_m} K^\star(x-y)dxdy = m\int_{\mathbb{R}^N\setminus B_m} K^\star_m(x)dx,$$

which gives (11) and (12), sending $m \to 0$. □

Proposition 3.3. *Assume that the kernel K satisfies the following condition:*

$$\text{there exist } \mu, r > 0 \text{ such that } K(x) \geq \mu \text{ for all } x \in B(0,r). \tag{13}$$

Let E be a measurable set such that $Per_K(E) < \infty$, then $|E| < \infty$ or $|\mathbb{R}^N \setminus E| < \infty$.

Proof. Let $\{Q_i\}_{i\in\mathbb{N}}$ be a partition of $\mathbb{R}^N$ made of cubes of sidelength $r/\sqrt{n}$, where r is as in (13). Note that for all $x, y \in Q_i$ we have $K(x-y) \geq \mu$.

Assume by contradiction that $|E| = |\mathbb{R}^N \setminus E| = \infty$. Then three possible cases may verify: either there exists $\delta > 0$ such that $\limsup_i |E \cap Q_i| \leq (1-\delta)|Q_i|$ or $\limsup_i |Q_i \setminus E| \leq (1-\delta)|Q_i|$ or there exist two subsequences Q_{i_n}, Q_{j_n} such that $\lim_n |Q_{i_n} \cap E| = \lim_n |Q_{j_n} \setminus E| = |Q_i|$.

Case 1: assume there exists $\delta > 0$ such that $\limsup_i |E \cap Q_i| \leq (1-\delta)|Q_i|$. So there exists $i_0 > 0$ such that for all $i \geq i_0$, $|Q_i \setminus E| \geq \delta/2|Q_i|$. Then we get

$$\mathrm{Per}_K(E) \geq \sum_i \int_{Q_i\cap E}\int_{Q_i\setminus E} K(x-y)dxdy \geq \mu \sum_{i\geq i_0} |Q_i\cap E||Q_i\setminus E| \geq \mu\frac{\delta}{2}\sum_{i\geq i_0}|E\cap Q_i|.$$

This implies, recalling that $\mathrm{Per}_K(E) < \infty$, that $\sum_i |E \cap Q_i| < \infty$, which is in contradiction with $|E| = \infty$.

Case 2: assume there exists $\delta > 0$ such that $\limsup_i |E\cap Q_i| \leq (1-\delta)|Q_i|$. Then the argument is the same as in *Case 1*, substituting E with $\mathbb{R}^N \setminus E$.

Case 3: assume that here exist two subsequences Q_{i_n}, Q_{j_n} such that $\lim_n |Q_{i_n} \cap E| = \lim_n |Q_{j_n} \setminus E| = |Q_i|$. Therefore for $\delta > 0$ there exists i_0 such that for all $j_n, i_n \geq i_0$, we get $|Q_{i_n} \cap E|, |Q_{j_n} \setminus E| > (1-\delta)|Q_i|$. By continuity we get that there exists a subsequence $\bar{Q}_i$ such that $|\bar{Q}_i \cap E|, |\bar{Q}_i \setminus E| \geq \delta|\bar{Q}_i|$, So,

$$\begin{aligned}\mathrm{Per}_K(E) &\geq \sum_i \int_{\bar{Q}_i\cap E}\int_{\bar{Q}_i\setminus E} K(x-y)dxdy \\ &\geq \mu\sum_i |\bar{Q}_i \cap E||\bar{Q}_i \setminus E| \geq \mu\delta^2\sum_i |\bar{Q}_i|^2 = \infty\end{aligned}$$

giving a contradiction to $\mathrm{Per}_K(E) < \infty$.

As a consequence, neither of the three cases can arise, which implies that either $|E|$ or $|R^N \setminus E|$ are finite.

□

From Propositions 3.1 and 3.3 we immediately get the following result:

Corollary 3.4. *Assume that K satisfies condition* (13). *Then, for all measurable sets E there holds*

$$Per_K(E) \geq \min(g(|E|), g(|\mathbb{R}^N \setminus E|)).$$

4. **Poincaré inequality.**

Proposition 4.1. *Assume that the kernel K satisfies* (13) *and that there exist $k \geq 1$ and a constant C depending on k, N such that*

$$g(m) \geq \frac{m^k}{C} \qquad \forall m > 0. \tag{14}$$

Then, for all $u \in L^1_{loc}$ with $J_K(u) < \infty$, there holds

$$\|u - m(u)\|_{L^k(\mathbb{R}^N)} \leq C J_K(u), \tag{15}$$

where

$$m(u) = \inf\{s \mid |\{u(x) > s\}| < \infty\} \in \mathbb{R}. \tag{16}$$

Moreover if $u \in L^1(\mathbb{R}^N)$, then $m(u) = 0$.

Proof. The argument is similar to the one in [1, Theorem 3.47]. First of all we observe that, since $J_K(u) < \infty$, by the coarea formula (10) the set S of $s \in \mathbb{R}$ such that $\mathrm{Per}_K(\{x \,|u(x) > s\}) < \infty$ is dense in $\mathbb{R}$. So by Proposition 3.3 for every $s \in S$, either $|\{u(x) > s\}| < \infty$ or $|\{u(x) \leq s\}| < \infty$. Note that if $s > m(u)$, then there exists $t \in (m(u), s)$ such that $|\{u(x) > t\}| < \infty$ and then $|\{u(x) > s\}| < \infty$. Analogously, if $s < m(u)$, then $|\{u(x) \leq t\}| < \infty$. Moreover $m(u) \in \mathbb{R}$. Indeed, if by contradiction this were not true, and e.g. $m(u) = -\infty$ (the other case being similar), we would get that $|\{u(x) > t\}| < \infty$ for every $t \in \mathbb{R}$. By the coarea formula

$$\int_{-n-1}^{-n} \mathrm{Per}_K(\{u(x) > t\})dt \leq \frac{1}{2} J_K(u) \qquad \forall n \in \mathbb{N},$$

so there exist $r > 0$ and $t_n \in [-n-1, -n]$ such that $\mathrm{Per}_K(\{u(x) > t_n\} \leq r$. By the isoperimetric inequality $|\{u(x) > t_n\}| \leq (Cr)^k$, but this is contradiction with the fact that $\{u(x) > t_n\} \to \mathbb{R}^N$ as $n \to +\infty$.

Finally, if $u \in L^1(\mathbb{R}^N)$, $m(u) = 0$. Indeed, by Chebychev inequality for every $s > 0$, we have that

$$|\{x \,|u(x) \leq -s\}| + |\{x \,|u(x) > s\}| \leq |\{x \,||u(x)| \geq s\}| \leq \frac{1}{s} \int_{\mathbb{R}^N} |u| dx < +\infty.$$

We denote by $u^+ = \max(u - m(u), 0)$ the positive part of $u - m(u)$. Then, by definition of $m(u)$, we get that $|\{x \,|u^+(x) > s\}| < \infty$ for all $s > 0$.

After a change of variable, we get

$$\begin{aligned}\int_{\mathbb{R}^N} (u^+)^k dx = \int_0^{+\infty} |\{x \,|u^+(x) > t^{\frac{1}{k}}\}| dt \\ = k \int_0^{+\infty} |\{x \,|u^+(x) > s\}| s^{k-1} ds.\end{aligned} \tag{17}$$

In [1, Lemma 3.48]) it is shown that, if $f : (0, +\infty) \to [0, +\infty)$ is decreasing and $k \geq 1$, then

$$k \int_0^T f(s) s^{k-1} ds \leq \left(\int_0^T f(s)^{\frac{1}{k}} ds \right)^k \qquad \forall T > 0.$$

So, we apply this inequality to $f(s) = |\{x \,|u^+(x) > s\}|$. This gives, by recalling the definition of $m(u)$ and using isoperimetric inequality (14),

$$k \int_0^{+\infty} |\{x \,|u^+(x) > s\}| s^{k-1} ds \leq \left(\int_0^{+\infty} |\{x \,|u^+(x) > s\}|^{\frac{1}{k}} ds \right)^k \leq \left(C \int_0^{+\infty} J_K(\{x \,|u(x) > s\}) ds \right)^k . \quad (18)$$

Therefore, putting together (17) and (18) and recalling the coarea formula (10), we get

$$\left(\int_{\mathbb{R}^N} (u^+)^k dx \right)^{\frac{1}{k}} \leq C J_K(u).$$

Repeating the same argument for the negative part of $u - m(u)$, i.e. $u^- = -\min(u - m(u), 0)$, we conclude. Indeed, again by definition of $m(u)$, we get that for all $s > 0$, $|\{x \,|u^-(x) > s\}| < \infty$. □

Remark 4.2. If $u \in L^1(\mathbb{R}^N)$ assumption (13) is not needed, indeed for all $s \in \mathbb{R}$ with $s \neq 0$ either $|\{u(x) > s\}| < \infty$ or $|\{u(x) \leq s\}| < +\infty$, so that it is not necessary to use Proposition 3.3.

5. **Existence of an isoperimetric profile.** In this section we show the existence of an isoperimetric profile, that is, a solution to Problem (5), under suitable assumptions on the kernel K. First of all we will assume throughout this section that $K \not\equiv 0$ and

$$K \in L^1(\mathbb{R}^N). \quad (19)$$

We observe that, if $K = K^\star$, then by Proposition 3.1 we know that the ball of volume m is the unique minimizer of (5), up to translations.

In order to get existence of minimizers, we first consider a relaxed version of the perimeter functional, obtained by extending it to general densities functions. More precisely, we define the new energy as follows: given $f : \mathbb{R}^N \to [0, 1]$, with $f \in L^1(\mathbb{R}^N)$, we let

$$\mathcal{P}_K(f) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x)[1 - f(y)] K(x - y) dx dy. \quad (20)$$

Note that $\mathcal{P}_K(\chi_E) = \mathrm{Per}_K(E)$ and the constraint $0 \leq f \leq 1$ is inherited by the original problem, naturally arising from the relaxation procedure.

Note that the previous energy can be written as

$$\mathcal{P}_K(f) = \|f\|_{L^1} \|K\|_{L^1} - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x) f(y) K(x - y) dx dy. \quad (21)$$

The relaxed version of the isoperimetric problem (5) can be restated as follows. Given $m \geq 0$, we consider

$$\inf_{f \in \mathcal{A}_m} \mathcal{P}_K(f), \quad (22)$$

where the set of admissible functions is defined as

$$\mathcal{A}_m = \left\{ f \in L^1(\mathbb{R}^N, [0, 1]),\ \int_{\mathbb{R}^N} f(x) dx = m \right\}.$$

Notice also that

$$\liminf_n \mathrm{Per}_K(E_n) = \mathcal{P}_K(f)$$

where the liminf is taken over all sequences E_n with $|E_n| = m$, such that $\chi_{E_n} \xrightharpoonup{*} f$ weakly* in L^∞.

Due to (21), the minimization problem (22) is equivalent to the maximization problem

$$\sup_{f\in\mathcal{A}_m} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)f(y)K(x-y)dxdy. \tag{23}$$

We now show monotonicity and subadditivity of the energy in (23) with respect to m.

Lemma 5.1.

i) If $m_1 > m_2 > 0$ then

$$\sup_{f\in\mathcal{A}_{m_1}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)f(y)K(x-y)dxdy \geq \sup_{f\in\mathcal{A}_{m_2}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)f(y)K(x-y)dxdy.$$

ii) If $\sum_{i=1}^l m_i = m$ then

$$\sum_{i=1}^{l} \sup_{f_i\in\mathcal{A}_{m_i}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_i(x)f_i(y)K(x-y)dxdy \leq \sup_{f\in\mathcal{A}_m} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)f(y)K(x-y)dxdy.$$

Moreover, if the equality holds in the above inequality and if the supremum in (23) is attained for all volumes m_i's, then $m_i = 0$ for all i's except one.

Proof. *i*) Let $f \in \mathcal{A}_{m_2}$ and let $\varepsilon > 0$. Let $E \subseteq \mathbb{R}^N$ such that $f(x) < 1-\varepsilon$ in E and $|E| = (m_1 - m_2)/\varepsilon$. We define

$$\tilde{f}(x) := f(x) + \varepsilon\chi_E(x) \qquad x \in \mathbb{R}^N.$$

We have $\tilde{f} \in \mathcal{A}_{m_1}$. Moreover, since $K \geq 0$,

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \tilde{f}(x)\tilde{f}(y)K(x-y)dxdy &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)f(y)K(x-y)dxdy \\ &+ 4\varepsilon \int_{\mathbb{R}^N}\int_E f(x)K(x-y)dxdy + \varepsilon^2 \int_E\int_E K(x-y)dxdy \\ &\geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)f(y)K(x-y)dxdy.\end{aligned}$$

ii) We consider the case $l = 2$, as the case $l > 2$ can be treated analogously. Let $f_i \in \mathcal{A}_{m_i}$, and let $\varepsilon > 0$ and $R > 0$ such that $\int_{\mathbb{R}^N\setminus B(0,R)} f_i(x)dx \leq \varepsilon$ for $i = 1, 2$. Note that

$$\left| \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_i(x)f_i(y)K(x-y)dxdy - \int_{B(0,R)}\int_{B(0,R)} f_i(x)f_i(y)K(x-y)dxdy \right| \leq 2\,\varepsilon\,\|K\|_{L^1}.$$

We fix $x_R \in \mathbb{R}^N$ such that $(f_1(x)\chi_{B(0,R)}(x))(f_2(x-x_R)\chi_{B(x_R,R)}) = 0$ for a.e. x, and we let

$$f(x) := f_1(x)\chi_{B(0,R)}(x) + f_2(x-x_R)\chi_{B(x_R,R)}.$$

Then, $f \in \mathcal{A}_{m'}$ for some $m' \in [m-2\varepsilon, m]$. Hence, by item i), we get

$$\sup_{g\in\mathcal{A}_m} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g(x)g(y)K(x-y)dxdy \geq \sup_{g\in\mathcal{A}_{m'}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g(x)g(y)K(x-y)dxdy$$

$$\geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)f(y)K(x-y)dxdy = \sum_{i=1}^{2} \int_{B(0,R)}\int_{B(0,R)} f_i(x)f_i(y)K(x-y)dxdy$$

$$+ \int_{B(0,R)}\int_{B(0,R)} f_1(x)f_2(y)K(x-y+x_R)dxdy$$

$$\geq \sum_{i=1}^{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_i(x)f_i(y)K(x-y)dxdy - 4\,\varepsilon\,\|K\|_{L^1}, \quad (24)$$

from which we conclude by the arbitrariness of ε.

Note that, since $K \not\equiv 0$, we can always choose x_R such that

$$\int_{B(0,R)}\int_{B(0,R)} f_1(x)f_2(y)K(x-y+x_R)dxdy > 0$$

so that the last inequality in (24) is in fact a strict inequality. □

5.1. **The potential function.** Given a function $f \in \mathcal{A}_m$, we can define the potential of f as

$$V(x) := \int_{\mathbb{R}^N} f(y)K(x-y)dy. \quad (25)$$

In the following we give some properties of the potential V.

Proposition 5.2.

i) $V \in C(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, *with* $0 \leq V \leq \|K\|_{L^1}$ *and* $\|V\|_{L^1} = m\|K\|_{L^1}$.

ii) $\lim_{|x|\to+\infty} V(x) = 0$.

iii) There exists $x \in \mathbb{R}^N$, *density point of* f, *such that* $f(x) < 1$ *and* $V(x) > 0$.

Proof. i) By definition $V \in L^1 \cap L^\infty$, with $0 \leq V \leq \|K\|_{L^1}$ and $\|V\|_{L^1} = m\|K\|_{L^1}$. Observe that since $f \in L^1$ and $K \in L^1$,

$$\lim_{h\to 0}\int_{\mathbb{R}^N} |f(y-x-h) - f(y-x)|K(y)dy = 0 \qquad \text{for every } x.$$

This implies that V is continuous.

ii) Let $\varepsilon > 0$ and let $A_\varepsilon := \{f > \varepsilon\}$. We have

$$V(x) = \int_{\mathbb{R}^N} K(y)f(x-y)dy \leq \varepsilon\|K\|_{L^1} + \int_{x-A_\varepsilon} K(y)dy. \quad (26)$$

For $R > 0$ we have

$$\int_{x-A_\varepsilon} K(y)dy = \int_{x-(A_\varepsilon\cap B(0,R))} K(y)dy + \int_{x-(A_\varepsilon\setminus B(0,R))} K(y)dy. \quad (27)$$

Since $K \in L^1(\mathbb{R}^N)$ and $\lim_{R\to\infty} |A_\varepsilon \setminus B(0,R)| = 0$, for R sufficiently large we have

$$\int_{x-(A_\varepsilon\setminus B(0,R))} K(y)dy \leq \varepsilon\|K\|_{L^1},$$

which gives, recalling (27),

$$\begin{aligned}\int_{x-A_\varepsilon} K(y)dy &\le \int_{x-(A_\varepsilon\cap B(0,R))} K(y)dy + \varepsilon\|K\|_{L^1} \\ &\le \int_{|y|\ge|x|-R} K(y)dy + \varepsilon\|K\|_{L^1} \le 2\,\varepsilon\,\|K\|_{L^1},\end{aligned} \tag{28}$$

for x large enough (depending on R). The thesis now follows from (26), (28) and the arbitrariness of ε.

iii) Since $K \not\equiv 0$, there exist $\delta > 0$ and a bounded set A with $|A| = k > 0$ such that $0 \notin A$ and $K(z) \ge \delta$ for a.e. $z \in A$. Let $x_0 \in \mathbb{R}^N$ be a Lebesgue point of f such that $f(x_0) > 0$. Let $y_0 \in A$ be a point of density 1 in A, such that $x_0 + y_0$ is a density point of f and we compute

$$V(x_0 + y_0) \ge \int_A f(x_0 + y_0 - z)K(z)dz \ge \delta \int_A f(x_0 + y_0 - z)dz > 0.$$

If $f(x_0 + y_0) < 1$, we are done, $c \ge V(x_0 + y_0) > 0$. If, on the other hand, $f(x_0 + y_0) = 1$ for all points y_0 which are points of density 1 of A and density points for $f(x_0 + \cdot)$, then we consider $y_1 \in A$ be a point of density 1 in A, such that $x_0 + y_0 + y_1$ is a density point of f and we compute

$$V(x_0 + y_0 + y_1) \ge \int_A f(x_0 + y_0 + y_1 - z)K(z)dz \ge \delta \int_A f(x_0 + y_0 + y_1 - z)dz > 0.$$

Repeating this argument, we construct a sequence $y_n \in A$ of points of density 1 such that $x_0+y_0+\cdots+y_n$ is a density point of f, and such that $V(x_0+y_0+\cdots+y_n) > 0$. Note that for some $n \ge 0$, we get that $f(x_0 + y_0 + \cdots + y_n) < 1$. If it were not the case, we would get

$$m = \int_{\mathbb{R}^N} f(x)dx \ge \sum_n \int_{x_0+nA} f(x)dx = \sum_n |A| = +\infty$$

which is impossible. □

5.2. **First and second variation.** We now compute the first and second variation of the energy in (22).

Lemma 5.3. *Let $f \in \mathcal{A}_m$ be a minimizer of (22). Let $S := \{x \,|f(x) = 1\}$ and $N := \{x \,|f(x) = 0\}$.*

i) For every $\psi, \phi \in L^1(\mathbb{R}^N, [0,1])$ with $\int_{\mathbb{R}^N} \phi(x)dx = \int_{\mathbb{R}^N} \psi(x)dx$, and such that $\psi \equiv 0$ a.e. in S and $\phi \equiv 0$ a.e. in N, the following holds

$$\int_{\mathbb{R}^N} (\psi(x) - \phi(x))V(x)dx \le 0. \tag{29}$$

ii) There exists a constant $c > 0$ such that

$$\begin{cases} V(x) \equiv c & \text{for every } x \in \mathbb{R}^N \setminus (N \cup S) \\ V(x) \ge c & \text{for every } x \in S \\ V(x) \le c & \text{for every } x \in N. \end{cases} \tag{30}$$

Proof. i) We argue as in [6, Lemma 1.2]. First observe that for every λ, $\int_{\mathbb{R}^N} f + \lambda(\psi - \phi)dx = \int_{\mathbb{R}^N} f dx = m$. Moreover, for all $\lambda \in [0,1]$ and a.e. $x \in S \cup N$, we get that $f(x) + \lambda(\psi(x) - \phi(x)) \in [0,1]$. We consider two sequences $\psi_\varepsilon \to \psi$, $\phi_\varepsilon \to \phi$ in L^1 such that $\int_{\mathbb{R}^N} \psi_\varepsilon dx = \int_{\mathbb{R}^N} \phi_\varepsilon dx = \int_{\mathbb{R}^N} \phi dx$ and such that $\psi_\varepsilon(x) \equiv 0$ on the set $\{x|f(x) > 1 - \varepsilon\}$ and $\phi_\varepsilon \equiv 0$ on the set $\{x \,|f(x) \le \varepsilon\}$. So, choosing $\lambda > 0$

sufficiently small (depending on ε) we can show that $f + \lambda(\psi_\varepsilon - \phi_\varepsilon) \in \mathcal{A}_m$. By minimality of f we get

$$\frac{1}{\lambda}(\mathcal{P}_K(f + \lambda(\psi_\varepsilon - \phi_\varepsilon)) - \mathcal{P}_K(f)) \geq 0.$$

So, sending $\lambda \to 0$ and recalling that K is symmetric (2), we get

$$\int_{\mathbb{R}^N} (\psi_\varepsilon(x) - \phi_\varepsilon(x))(1 - 2f(y))K(x-y)dx \geq 0.$$

So, sending $\varepsilon \to 0$ and recalling that $\int_{\mathbb{R}^N}(\phi(x) - \psi(x))dx = 0$, we conclude

$$\begin{aligned}
0 &\leq \int_{\mathbb{R}^N} (\psi(x) - \phi(x))(1 - 2f(y))K(x-y)dx \\
&= \|K\|_{L^1} \int_{\mathbb{R}^N} (\psi(x) - \phi(x))dx - 2\int_{\mathbb{R}^N} (\psi(x) - \phi(x))V(x)dx \\
&= -2\int_{\mathbb{R}^N} (\psi(x) - \phi(x))V(x)dx.
\end{aligned}$$

ii) Choosing ψ, ϕ in (29) such that $\psi = 0 = \phi$ on $S \cup N$, we can exchange the role of ψ and ϕ, and obtain that in $\mathbb{R}^N \setminus (N \cup S)$, V has to be constant. So, there exists $c > 0$ (by Proposition 5.2 *iii)*) such that $V(x) \equiv c$ in $\mathbb{R}^N \setminus (N \cup S)$.

Choosing ϕ in (29) such that $\phi = 0$ a.e. in $S \cup N$, we get, since $\int_{\mathbb{R}^N}(\psi(x) - \phi(x))dx = 0$, $\int_{\mathbb{R}^N \setminus (N \cup S)}(\psi(x) - \phi(x))dx = -\int_S \psi(x)dx$. We compute

$$\begin{aligned}
0 &\geq \int_S (\psi(x) - \phi(x))V(x)dx + \int_N (\psi(x) - \phi(x))V(x)dx + c\int_{\mathbb{R}^N \setminus (S \cup N)} (\psi(x) - \phi(x))dx \\
&= \int_N \psi(x)V(x)dx + c\int_{\mathbb{R}^N \setminus (S \cup N)} (\psi(x) - \phi(x))dx = \int_N \psi(x)(V(x) - c)dx
\end{aligned}$$

for all $\psi \in L^1(\mathbb{R}^N, [0,1])$ such that $\psi = 0$ a.e. in S and $\int_{\mathbb{R}^N}(\psi(x) - \phi(x))dx = 0$. This implies that $V \leq c$ in N. With an analogous argument, exchanging the role of ψ and ϕ, we get $V(x) \geq c$ in S.

□

As immediate consequence of the first variation (30) and of the properties of the potential V we obtain that every minimizer of (22) has compact support.

Proposition 5.4. *Every minimizer $f \in \mathcal{A}_m$ of (22) has compact support.*

Proof. By Proposition 5.2 $\lim_{|x| \to +\infty} V(x) = 0$. Hence we can find $R > 0$ such that $0 \leq V(x) < c$ for $|x| > R$, where $c > 0$ is the constant appearing in (30). By (30) this implies immediately that the support of f is contained in $B_R(0)$. □

We now consider the second variation of the functional.

Lemma 5.5. *Let $f \in \mathcal{A}_m$ be a minimizer of (22) and let S, N be as in Lemma 5.3. Then, for every $\xi \in L^1(\mathbb{R}^N, [-1,1])$ with $\int_{\mathbb{R}^N} \xi(x)dx = 0$, and such that $\xi \equiv 0$ a.e. in $N \cup S$, it holds*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \xi(x)\xi(y)K(x-y)dxdy \leq 0. \tag{31}$$

Proof. We argue as in [6, Lemma 1.5]. Reasoning as in the proof of Lemma 5.3, item i), we can assume that there exists a sequence $\xi_\varepsilon \to \xi$ in L^1 such that $\int_{\mathbb{R}^N} \xi_\varepsilon(x)dx = 0$ and $\xi_\varepsilon \equiv 0$ a.e. in $\{x \mid f(x) \le \varepsilon \text{ or } f(x) \ge 1-\varepsilon\}$. So for λ sufficiently small $f + \lambda\xi_\varepsilon \in \mathcal{A}_m$ and by minimality we get

$$
\begin{aligned}
&0 \le \mathcal{P}_K(f+\lambda\xi_\varepsilon) - \mathcal{P}_K(f)\\
&= \lambda \int_{\mathbb{R}^N\setminus(N\cup S)} (\|K\|_{L^1} - 2V(x))\xi_\varepsilon(x)dx - \lambda^2 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \xi_\varepsilon(x)\xi_\varepsilon(y)K(x-y)dxdy.
\end{aligned}
$$

Recalling (30) and the fact that $\int_{\mathbb{R}^N} \xi_\varepsilon(x)dx = 0$, we conclude the desired inequality by letting $\varepsilon \to 0$. □

5.3. **Existence of minimizers.**

Theorem 5.6. *For every $m > 0$ there exists at least one $f \in \mathcal{A}_m$ which solves the minimization problem* (22).

Proof. The proof is similar to that in [6, Theorem 1.9] (see also [4]), and is based on a concentration compactness argument.

Let $f_n \in \mathcal{A}_m$ be a minimizing sequence, and recall that the energy can be written as

$$
\mathcal{P}_K(f_n) = m\|K\|_{L^1} - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_n(x)f_n(y)K(x-y)dxdy.
$$

We consider a partition of $\mathbb{R}^N$ in disjoint cubes: let $Q = [0,1]^n$ and let $Q^z := z+Q$ for $z \in \mathbb{Z}^N$. We fix $\varepsilon > 0$ small and divide the cubes in two subsets:

$$
I_{\varepsilon,n} := \{z \in \mathbb{Z}^N \mid \int_{Q^z} f_n(x)dx =: m_{n,z} \le \varepsilon\}, \qquad A_{\varepsilon,n} := \cup_{z\in I_{\varepsilon,n}} Q^z
$$

and

$$
J_{\varepsilon,n} := \{z \in \mathbb{Z}^N \mid \int_{Q^z} f_n(x)dx = m_{n,z} > \varepsilon\}, \qquad E_{\varepsilon,n} := \cup_{z\in J_{\varepsilon,n}} Q^z.
$$

For $z \in I_{\varepsilon,n}$ and $w \in \mathbb{Z}^N$, recalling Riesz rearrangement inequality, we have

$$
\begin{aligned}
&\int_{Q^z}\int_{Q^w} f_n(x)f_n(y)K(x-y)dxdy \le \int_{B_{m_{n,w}}}\int_{B_{m_{n,z}}} K^\star(x-y)dxdy\\
&= \int_{B_{m_{n,w}}}\int_{B_{m_{n,z}}(x)} K^\star(y)dydx \le m_{n,w}\int_{B_{m_{n,z}}} K^\star(y)dy \le m_{n,w}\int_{B_\varepsilon} K^\star(y)dy,
\end{aligned} \tag{32}
$$

where the last inequality follows from the fact that $z \in I_{\varepsilon,n}$, and the previous inequality from the fact that $K^\star$ is symmetrically decreasing.

For $R > 0$, we compute

$$
\begin{aligned}
&\int_{A_{\varepsilon,n}}\int_{\mathbb{R}^N} f_n(x)f_n(y)K(x-y)dxdy\\
&\qquad = \sum_{w\in\mathbb{Z}^N, z\in I_{\varepsilon,n}, |z-w|>R} \int_{Q^z}\int_{Q^w} f_n(x)f_n(y)K(x-y)dxdy\\
&\qquad\quad + \sum_{w\in\mathbb{Z}^N, z\in I_{\varepsilon,n}, |z-w|\le R} \int_{Q^z}\int_{Q^w} f_n(x)f_n(y)K(x-y)dxdy.
\end{aligned}
$$

By (32) the second addendum can be bounded as follows

$$\sum_{w\in\mathbb{Z}^N, z\in I_{\varepsilon,n}, |z-w|\le R} \int_{Q^z}\int_{Q^w} f_n(x)f_n(y)K(x-y)dxdy$$
$$\le \sum_{w\in\mathbb{Z}^N} m_{n,w}(2R)^N\int_{B_\varepsilon} K^\star(y)dy \le m(2R)^N \int_{B_\varepsilon} K^\star(y)dy.$$

On the other hand the first addendum can be bounded as

$$\sum_{w\in\mathbb{Z}^N, z\in I_{\varepsilon,n}, |z-w|> R} \int_{Q^z}\int_{Q^w} f_n(x)f_n(y)K(x-y)dxdy$$
$$\le \sum_{z\in I_{\varepsilon,n}} m_{z,n}\int_{|y|>R-\sqrt{N}} K(y)dy \le m \int_{|y|>R-\sqrt{N}} K(y)dy.$$

Collecting the two estimates, we get

$$\int_{A_{\varepsilon,n}}\int_{\mathbb{R}^N} f_n(x)f_n(y)K(x-y)dxdy \le m(2R)^N\int_{B_\varepsilon} K^\star(y)dy + m\int_{|y|>R-\sqrt{N}} K(y)dy. \tag{33}$$

Since $K\in L^1$, hence also $K^\star\in L^1$, we can choose $R=R(\varepsilon)$ in such a way that

$$\lim_{\varepsilon\to 0} R(\varepsilon) = +\infty \quad \text{and} \quad \lim_{\varepsilon\to 0} R(\varepsilon)^N\int_{B_\varepsilon} K^\star(y)dy = 0.$$

With this choice of R, from (33) we get

$$\int_{A_{\varepsilon,n}}\int_{\mathbb{R}^N} f_n(x)f_n(y)K(x-y)dxdy \le r(\varepsilon) \tag{34}$$

where $r(\varepsilon)\to 0$ as $\varepsilon\to 0$, uniformly in n. As a consequence, we obtain

$$\mathcal{P}_K(f_n) \ge m\|K\|_{L^1} - \int_{E_{\varepsilon,n}}\int_{E_{\varepsilon,n}} f_n(x)f_n(y)K(x,y)dxdy - 2r(\varepsilon). \tag{35}$$

Observe that, due to the fact that $\int_{\mathbb{R}^N} f(x)dx = m$, we have $\#J_{\varepsilon,n}\le m/\varepsilon$. Given $w_i, w_j\in J_{\varepsilon,n}$, up to subsequence we get that $|w_i-w_j|\to c_{i,j}\in\mathbb{N}\cup\{+\infty\}$ as $n\to+\infty$. We consider the following sets, for $l=1,\dots,H_\varepsilon$, with $H_\varepsilon\le\frac{m}{\varepsilon}$

$$\mathcal{Q}^l_{\varepsilon,n} = \bigcup_{w_i\in J_{\varepsilon,n},\ c_{il}<+\infty} Q^{w_i}. \tag{36}$$

Note that by construction $dist(\mathcal{Q}^l_{\varepsilon,n}, \mathcal{Q}^k_{\varepsilon,n})\to+\infty$ if $k\ne l$ as $n\to+\infty$. Moreover, always by construction, we get that

$$diam(\mathcal{Q}^l_{\varepsilon,n}) \le \sum_{i\in\{1,\dots H_\varepsilon\}, c_{il}<\infty} (2c_{il}+2\sqrt{N}) \le M_\varepsilon,$$

where M_ε does not depend on n.

Let $f_n^{l,\varepsilon} := f_n\chi_{\mathcal{Q}^l_{\varepsilon,n}}$ and let $x_{l,n}$ such that $f_n^{l,\varepsilon}(x_{l,n})>0$. Up to subsequences we can assume that $f_n^{l,\varepsilon}(\cdot + x_{n,l}) \overset{*}{\rightharpoonup} f^{l,\varepsilon}$ weakly* in L^∞, as $n\to+\infty$. Observe that the support of $f_n^{l,\varepsilon}(\cdot+x_{n,l})$ is contained in $B(0,M_\varepsilon)$ for every n. Moreover, since the functional $\mathcal{P}_K$ is continuous with respect to the tight convergence (see for instance [6]), we get that

$$\lim_n \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_n^{l,\varepsilon}(x+x_{n,l})f_n^{l,\varepsilon}(y+x_{n,l})K(x-y)dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^{l,\varepsilon}(x)f^{l,\varepsilon}(y)K(x-y)dxdy.$$

Therefore

$$\begin{aligned}\inf_{\mathcal{A}_m} \mathcal{P}_K &= \lim_n \mathcal{P}_K(f_n) \geq m\|K\|_{L^1} - \lim_n \int_{E_{\varepsilon,n}}\int_{E_{\varepsilon,n}} f_n(x)f_n(y)K(x-y)dxdy - 2r(\varepsilon)\\ &= m\|K\|_{L^1} - \sum_{l=1}^{H_\varepsilon} \lim_n \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_n^{l,\varepsilon}(x+x_{n,l})f_n^{l,\varepsilon}(y+x_{n,l})K(x-y)dxdy - 2r(\varepsilon)\\ &= m\|K\|_{L^1} - \sum_{l=1}^{H_\varepsilon} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^{l,\varepsilon}(x)f^{l,\varepsilon}(y)K(x-y)dxdy - 2r(\varepsilon). \end{aligned} \tag{37}$$

We pass to a subsequence $\varepsilon_k \to 0$ such that ε_k is decreasing. So $H_{\varepsilon_k} \to H \in (0, +\infty]$. Moreover, we can relabel the sequence in such a way that f_n^{l,ε_k} and then also their limit f^{l,ε_k} are monotone in ε_k. By monotone convergence $f^{l,\varepsilon_k} \to f^l$ strongly in L^1. Moreover if $m_l = \int_{\mathbb{R}^N} f^l(x)dx$, then $\sum_{l=1}^H m_l = \tilde{m} \leq m$. Again by continuity of the functional with respect to the L^1-convergence, from (37) and from Lemma 5.1 we get that

$$\begin{aligned}&\sup_{f\in\mathcal{A}_m} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)f(y)K(x-y)dxdy\\ &= \lim_n \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_n(x)f_n(y)K(x-y)dxdy) \leq \sum_{l=1}^{H} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^l(x)f^l(y)K(x-y)dxdy\\ &\leq \sum_{l=1}^{H} \sup_{\mathcal{A}_{m_l}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)f(y)K(x-y)dxdy \leq \sup_{\mathcal{A}_{\tilde{m}}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)f(y)K(x-y)dxdy\\ &\leq \sup_{\mathcal{A}_m} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)f(y)K(x-y)dxdy.\end{aligned}$$

Therefore the previous are all equalities, and f^l is a minimizer of $\mathcal{P}_K$ in $\mathcal{A}_{m_l}$ for all l's. In particular, recalling again Lemma 5.1, we get that $H = 1$, and f^1 is a minimizer of $\mathcal{P}_K$ in $\mathcal{A}_m$. $\square$

We finally show that, under a further condition on K, the isoperimetric problem (5) admits a solution.

Theorem 5.7. *Assume that for a.e. $x \in \mathbb{R}^N$ there exists $\varepsilon_x > 0$ such that for all $\varepsilon < \varepsilon_x$*

$$\int_{B(0,2\varepsilon)} |B(0,\varepsilon) \cap B(z,\varepsilon)|(K(z) - K(x+z))dz > 0. \tag{38}$$

Then, for every $m > 0$ there exists a compact set $E \subseteq \mathbb{R}^N$ such that $|E| = m$ and E solves the isoperimetric problem (5).

Proof. By Theorem 5.6 there exists at least one $f \in \mathcal{A}_m$ which solves the minimization problem (22). Moreover the support of f is compact due to Proposition 5.4.

Assume by contradiction that f is not a characteristic function. Then there exist $\bar{x} \neq \bar{y}$ Lebesgue points of f such that $0 < f(\bar{x}), f(\bar{y}) < 1$. Let $\varepsilon < \frac{1}{2}|\bar{x} - \bar{y}|$ and define the function $\xi(x) := \chi_{B(\bar{x},\varepsilon)} - \chi_{B(\bar{y},\varepsilon)}$. Therefore, by the second variation formula (31), for every $\varepsilon < \frac{1}{2}|\bar{x} - \bar{y}|$ we get

$$
\begin{aligned}
0 \geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \xi(y)\xi(x)K(x-y)dxdy \\
&= 2\int_{B(0,\varepsilon)}\int_{B(0,\varepsilon)} (K(x-y) - K(\bar{x} - \bar{y} - (x-y))dxdy \\
&= 2\int_{B(0,2\varepsilon)} |B(0,\varepsilon) \cap B(z,\varepsilon)|(K(z) - K(\bar{x} - \bar{y} + z))dz,
\end{aligned}
$$

which contradicts (38), and concludes the proof. $\square$

Remark 5.8. A sufficient condition for (38) to hold is that

$$
\liminf_{z \to 0} \left(K(z) - K(z+x)\right) > 0 \qquad \text{for a.e. } x \in \mathbb{R}^N. \tag{39}
$$

In particular this condition is always verified if K is positive definite, that is,

$$
\begin{cases}
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi(x)\phi(y)K(x-y)dxdy \geq 0 & \forall \phi \in L^1(\mathbb{R}^N) \\
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi(x)\phi(y)K(x-y)dxdy = 0 & \text{iff } \phi \equiv 0.
\end{cases}
$$

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E-mail address: annalisa.cesaroni@unipd.it

E-mail address: matteo.novaga@unipi.it