THE SURFACE DIFFUSION FLOW WITH ELASTICITY IN THE PLANE

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ABSTRACT. In this paper we prove short-time existence of a smooth solution in the plane to the surface diffusion equation with an elastic term and without an additional curvature regularization. We also prove the asymptotic stability of strictly stable stationary sets.

CONTENTS

1. INTRODUCTION

In the last years, the physical literature has shown a rapidly growing interest toward the study of the morphological instabilities of interfaces between elastic phases generated by the competition between elastic and surface energies, the so called stress driven rearrangement instabilities (SDRI). They occur, for instance, in hetero-epitaxial growth of elastic films when a lattice mismatch between film and substrate is present, or in certain alloys that, under specific temperature conditions, undergo a phase separation in many small connected phases (that we call particles) within a common elastic body. A third interesting situation is represented by the formation and evolution of material voids inside a stressed elastic solid. Mathematically, the common thread is that equilibria are identified with local or global minimizers under a volume constraint of a free energy functional, which is given by the sum of the stored elastic energy and the surface energy (of isotropic or anisotropic perimeter type) accounting for the surface tension along the unknown profile of the film or the interface between the phases. The associated variational problems can be seen as non-local instances of the *isoperimetric principle*, where the non-locality is given by the elastic term. They are very well studied in the physical and numerical literature, but the available rigorous mathematical results are very few. We refer to [6, 8, 10, 21, 25, 27] for some existence, regularity and stability results related to a variational model describing the equilibrium configurations of two-dimensional epitaxially strained elastic films, and to [9, 15] for results in three-dimensions. We also mention that a hierarchy of variational principles to describe the equilibrium shapes in the aforementioned contexts has been introduced in [30]. The simplest prototypical example is perhaps given by the following problem, which can be used to describe the equilibrium shapes of voids in elastically stressed solids (see for instance [36]):

$$\text{minimize } J(F) := \frac{1}{2}\int_{\Omega\setminus F} \mathbb{C}E(u_F) : E(u_F)\, dz + \int_{\partial F} \varphi(\nu_F)\, d\sigma \tag{1.1}$$

where minimization takes place among all sets $F \subset \Omega$ with prescribed measure $|F| = m$. Here, the set F represents the shape of the void that has formed within the elastic body Ω (an open subset of $\mathbb{R}^2$ or $\mathbb{R}^3$), u_F stands for the equilibrium elastic displacement in $\Omega \setminus F$ subject to a prescribed boundary conditions $u_F = w_0$ on $\partial\Omega$ (see (2.10) below), $\mathbb{C}$ is the elasticity tensor of the (linearly) elastic material, $E(u_F) := (\nabla u_F + \nabla^T u_F)/2$ denotes the elastic strain of u_F, and $\varphi(\nu_F)$ is the anisotropic surface energy density evaluated at the outer unit normal ν_F to F. The presence of a nontrivial Dirichlet boundary condition $u_F = w_0$ on $\partial\Omega$ is what causes the solid $\Omega \setminus F$ to be elastically stressed. Indeed, note that when $w_0 = 0$ the elastic term becomes irrelevant and (1.1) reduces to the classical Wulff shape problem (with the confinement constraint $F \subseteq \Omega$). We refer to [14, 24] for related existence, regularity and stability results in two dimensions. See also [11] for a relaxation result valid in all dimensions for a variant of (1.1).

In this paper we address the evolutive counterpart of (1.1) in two-dimensions, namely the morphologic evolution of shapes towards equlbria of the functional J, driven by stress and surface diffusion. Assuming that mass transport in the bulk occurs at a much faster time scale, see [34], we have, according to the Einstein–Nernst relation, that the evolution is governed by the *area preserving* evolution law

$$V_t = \partial_{\sigma\sigma}\mu_t \qquad \text{on } \partial F(t) \tag{1.2}$$

where V_t denotes the (outer) normal velocity of the evolving curve $\partial F(t)$ at time t and $\partial_{\sigma\sigma}\mu_t$ stands for the tangential laplacian of the chemical potential μ_t on $\partial F(t)$. In turn, μ_t is given by the *first variation* of the free-energy functional J at $F(t)$, and thus (see (2.12) below) (1.2) takes the form

$$V_t = \partial_{\sigma\sigma}\big(k_{\varphi,t} - Q(E(u_{F(t)}))\big)\,, \tag{1.3}$$

where $k_{\varphi,t}$ is the anisotropic curvature of $\partial F(t)$, $u_{F(t)}$ denotes as before the elastic equilibrium in $\Omega \setminus F(t)$ subject to $u_{F(t)} = w_0$ on $\partial\Omega$, and Q is the quadratic form defined as $Q(A) := \frac{1}{2}\mathbb{C}A : A$ for all 2×2-symmetric matrices A. Note that when $w_0 = 0$ the elastic term vanishes and thus (1.2) reduces to the *surface diffusion flow* equation

$$V_t = \partial_{\sigma\sigma}k_{\varphi,t} \tag{1.4}$$

for evolving curves, studied in [19] in the isotropic case (see also [20] for the N-dimensional case). Thus, we may also regard (1.3) as a sort of prototypical nonlocal perturbation of (1.4) by an additive elastic contribution.

As observed by Cahn and Taylor in the case without elasticity (see [13]), the evolution equation (1.3) can be seen as the gradient flow of the energy functional J with respect to a suitable Riemannian structure of H^{-1} type, see Remark 3.1.

When the anisotropy φ is *strongly elliptic*, that is when it satisfies

$$D^2\varphi(\nu)\,\tau\cdot\tau > 0 \qquad \text{for all } \nu \in \mathbb{S}^1 \text{ and all } \tau \perp \nu,\ \tau \neq 0 \tag{1.5}$$

the evolution (1.3) yields a *parabolic* 4-th order (geometric) equation, time by time coupled with the *elliptic system* describing the elastic equilibrium in $\Omega \setminus F(t)$.

However, we mention here that for some physically relevant anisotropies the ellipticity condition (1.5) may fail at some directions ν, see for instance [18, 36]. Whenever this happens, (1.3) becomes *backward parabolic* and thus ill-posed. To overcome this ill-posedness, a canonical approach inspired by Herring's work [31] consists in considering a *regularized curvature-dependent* surface energy of the form

$$\int_{\partial F} \Big(\varphi(\nu_F) + \frac{\varepsilon}{2}k^2\Big)\, d\mathcal{H}^1,$$

where $\varepsilon > 0$ and k denotes the standard curvature, see [18, 29]. In this case (1.2) yields the following 6-th order area preserving evolution equation

$$V_t = \partial_{\sigma\sigma}\Big(k_{\varphi,t} - Q(E(u_{F(t)})) - \varepsilon\big(\partial_{\sigma\sigma}k + \frac{1}{2}k^3\big)\Big)\,. \tag{1.6}$$

This equation was studied numerically in [36] (see also [35, 12] and references therein) and analytically in [22], where local-in-time existence of a solution was established in the context of periodic graphs, modelling the evolution of epitaxially strained elastic films. We refer also to [23] for corresponding results in three-dimensions. We remark that the analysis of [22] (and of [23]), which is based on the so-called minimizing movements approach, relies heavily on the presence of the curvature regularization and, in fact, all the estimates provided there are ε-dependent and degenerate as $\varepsilon \to 0^+$, even when φ is strongly elliptic. Thus, the methods developed in [22, 23] do not apply to the case $\varepsilon = 0$ in (1.6).

In this paper we are able to address the case $\varepsilon = 0$ and in one of the main results (see Theorems 3.2 and 3.8 below) we prove short time existence and uniqueness of a smooth solution of (1.3) starting from sufficiently regular initial sets. To the best of our knowledge this is *the first existence result for the surface diffusion flow with elasticity and without curvature regularization.* Note that in general one cannot expect global-in-time existence. Indeed, even when no elasticity is present and φ is isotropic, singularities such as pinching may develop in finite time, see for instance [26].

In the second main result of the paper we establish global-in-time existence and study the long-time behavior for a class of initial data: we show that *strictly stable stationary sets*, that is, sets E that are stationary for the energy functional J and with positive second variation $\partial^2 J(E)$ are *exponentially stable* for the flow (1.3). More precisely, if the initial set F_0 is sufficiently close to the strictly stable set E and has the same area, then the flow (1.3) starting from F_0 exists for all times and converges to E exponentially fast as $t \to +\infty$ (see Theorem 4.1 for the precise statement).

A few comments on the strategy of the proofs are in order. The main technical difficulties in proving short-time existence clearly originate from the presence of the nonlocal elastic term $Q(E(u_{F(t)}))$ in (1.3). When a curvature regularization as in (1.6) is present, the elastic term may be regarded and treated as a lower order perturbation and thus is more easily handled. When $\varepsilon = 0$ this is no longer possible and so one has to find a way to show that the parabolicity of the geometric part of the equation still tames the elastic contribution. Our strategy is based on the natural idea of thinking of Q as a *forcing term* in order to set up a fixed point argument. Roughly speaking, given an initial set F_0 and a forcing term f, we let $t \mapsto F(t)$ be the flow starting from F_0 and solving

$$V_t = \partial_{\sigma\sigma}\big(k_{\varphi,t} - f\big),$$

and we consider the correspondig $t \mapsto Q(E(u_{F(t)}))$, with $u_{F(t)}$ being as usual the elastic equilibrium in $\Omega \setminus F(t)$. The existence proof then amounts to finding a fixed point for the map $f \mapsto Q(E(u_{F(\cdot)}))$. In order to implement this strategy, the crucial idea is to look at the squared L^2-norm of the tangential gradient of the chemical potential $(k_{\varphi,t} - f)$, that is, to study the behavior of the quantity

$$\int_{\partial F(t)} \big(\partial_\sigma (k_{\varphi,t} - f)\big)^2 \, d\mathcal{H}^1 \tag{1.7}$$

with respect to time. More precisely, by computing the time derivative of (1.7) we derive suitable energy inequalities involving (1.7) (see Lemma 3.3) yielding the a priori regularity estimates needed to carry out the fixed point argument. The quantity (1.7), with f now given by the elastic term Q, is also crucial in the aforementioned asymptotic stability analysis. Here, by adapting to the present situation the methods developed in [1] for the surface diffusion flow without elasticity, we are able to show that for properly chosen initial sets, (1.7) becomes monotone decreasing in time and, in fact, exponentially decays to zero, thus giving the desired exponential convergence result.

This paper is organized as follows. In Section 2 we set up the problem, introduce the main notations and collect several auxiliary results concerning the energy functional J in (1.1). Some of these results, which deal with the properties of strictly stable stationary sets, are then crucial for the asymptotic stability analysis carried out in Section 4. The short-time existence, uniqueness and regularity of the flow (1.3) for sufficiently regular initial data is addressed in Section 3. In Section 5 we briefly illustrate how to apply our main existence and asymptotic stability results in the case of evolving periodic graphs, that is in the geometric setting considered in [22]. In particular, in Theorem 5.1 we address and analytically characterize the exponential asymptotic stability of *flat configurations*, thus extending to the evolutionary setting the results of [25, 8]. In the final Appendix, for the reader's convenience we provide the proof of an interpolation result, probably known to the experts, that is used throughout the paper.

We conclude this introduction by mentioning that it would be interesting to investigate whether under the assumption (1.5) the flows (1.6) studied in [22] converge to (1.3) as $\varepsilon \to 0^+$, perhaps using the methods developed in [7]. This issue as well as the extension of the results of this paper to three-dimensions will be addressed in future investigations.

2. Preliminary results

2.1. **Geometric preliminaries and notation.** Let $F \subset \mathbb{R}^2$ be a bounded open set of class C^2. We denote the unit outer normal to F by ν_F and the tangent vector τ_F. Throughout the paper we choose the orientation so that $\tau_F = \mathcal{R}\nu_F$, where $\mathcal{R}$ is the counterclockwise rotation by $\pi/2$.

The differential of a vector field X along ∂F is denoted by $\partial_\sigma X$. We recall that

$$\partial_\sigma \nu_F = k_F \tau_F \qquad \text{and} \qquad \partial_\sigma \tau_F = -k_F \nu_F,$$

where k_F is the curvature of ∂F. When no confusion arises, we will simply write ν, τ, and k in place of ν_F, τ_F and k_F. The tangential divergence of X is $\operatorname{div}_\tau X := \partial_\sigma X \cdot \tau$. The divergence theorem on ∂F states that for every vector field $X \in C^1(\partial F; \mathbb{R}^2)$ it holds

$$\int_{\partial F} \operatorname{div}_\tau X \, d\mathcal{H}^1 = \int_{\partial F} k \, X \cdot \nu \, d\mathcal{H}^1. \tag{2.1}$$

If the boundary of F is of class C^m, with $m \geq 2$, then the signed distance function d_F is of class C^m in a tubular neighborhood of ∂F. We may extend ν, τ and k to such a neighborhood of ∂F by setting $\nu := \nabla d_F$, $\tau := \mathcal{R}\nu$ and $k := \operatorname{div} \nu = \Delta d_F$.

Throughout the paper, we fix a bounded Lipschitz open set $\Omega \subset \mathbb{R}^2$. Moreover, G will always denote a smooth reference set, with the property that $G \subset\subset \Omega$. We will also denote by π_G the orthogonal projection on ∂G and by $\bar\eta$ a positive number such that

$$d_G \text{ and } \pi_G \text{ are smooth in } \mathcal{N}_{\bar\eta}(\partial G), \tag{2.2}$$

where $\mathcal{N}_{\bar\eta}(\partial G)$ denotes the $\bar\eta$-tubular neighborhood of ∂G.

We now introduce a class of sets F sufficiently "close" to G so that the boundary can be written as

$$\partial F = \{x + h_F(x)\nu_G(x) \mid x \in \partial G\}, \tag{2.3}$$

for a suitable function h_F defined on ∂G. More precisely, for $k \in \mathbb{N}$ and $\alpha \in (0,1)$ we set

$$\begin{aligned} \mathfrak{h}^{k,\alpha}_M(G) := \{F \subset\subset \Omega :\ &(2.3) \text{ holds for some } h_F \in C^{k,\alpha}(\partial G), \\ &\text{with } \|h_F\|_{L^\infty} \leq \bar\eta/2 \text{ and } \|h_F\|_{C^{k,\alpha}} \leq M\}. \end{aligned} \tag{2.4}$$

For such sets F we also denote by $\pi_F^{-1} : \partial G \to \partial F$ the map $\pi_F^{-1}(x) = x + h_F(x)\nu_G(x)$ and set

$$J_F := \sqrt{(1 + h_F k_G)^2 + (\partial_\sigma h_F)^2},$$

that is the tangential Jacobian on ∂G of the map π_F^{-1}. We recall now some useful transformation formulas:

$$\tau_F \circ \pi_F^{-1} = \frac{(1 + h_F k_G)\tau_G + \partial_\sigma h_F \nu_G}{J_F} \tag{2.5}$$

and

$$\nu_F \circ \pi_F^{-1} = \frac{-\partial_\sigma h_F \tau_G + (1 + h_F k_G)\nu_G}{J_F}. \tag{2.6}$$

Similarly, the curvature k_F of F at $y = \pi_F^{-1}(x)$ is given by

$$k_F \circ \pi_F^{-1} = \frac{-\partial_{\sigma\sigma} h_F (1 + h_F k_G) + 2(\partial_\sigma h_F)^2 k_G + (1 + h_F k_G)^2 k_G + h_F \partial_\sigma h_F\, \partial_\sigma k_G}{J_F^3}. \tag{2.7}$$

We now fix some notation, which will be used throughout the paper. If $t \mapsto F_t$ is a (smooth) flow of sets, in order to simplify the notation, we will sometimes write h_t, ν_t, τ_t, and k_t instead of h_{F_t}, ν_{F_t}, τ_{F_t}, and k_{F_t}, respectively. Similarly, we will set $k_{\varphi,t} := g(\nu_t)k_t$.

Moreover, whenever we have a one-parameter family $(g_t)_t$ of functions (or vector fields) we shall denote by $\dot g_t$ the partial derivative with respect to t of the function $(x,t) \mapsto g_t(x)$, and by $\nabla^k g_t$ the k-th order differential of the function $(x,t) \mapsto g_t(x)$ with respect to x.

2.2. **The energy functional.** In this section we introduce the energy functional that underlies the flow. We also introduce the proper notions of stationary points and stability that will be needed in the study of the long-time behavior of the flow, see Section 4.

As explained in the introduction, the free energy functional is the sum of an anisotropic perimeter and a bulk elastic term.

We start by introducing the anisotropic surface energy density, which is given by a positively one-homogeneous function $\varphi \in C^\infty(\mathbb{R}^2 \setminus \{0\}; (0, +\infty))$

$$D^2\varphi(\nu)\tau\cdot\tau \geq c_0 > 0 \tag{2.8}$$

for every $\nu \in \mathbb{S}^1$ and every $\tau \in \mathbb{S}^1$ such that $\tau \perp \nu$. Note that the above condition is equivalent to requiring that the level sets of φ have positive curvature.

Concerning the elastic part, for $F \subset\subset \Omega$ and for the elastic displacement $u : \Omega \setminus F \to \mathbb{R}^2$ we denote by $E(u)$ the symmetric part of ∇u, that is, $E(u) := \frac{\nabla u + (\nabla u)^T}{2}$. In what follows, $\mathbb{C}$ stands for a fourth order *elasticity tensor* acting on 2×2 symmetric matrices A, such that $\mathbb{C}A : A > 0$ if $A \neq 0$. Finally we shall denote by $Q(A) := \frac{1}{2}\mathbb{C}A : A$ the *elastic energy density.*

We are now ready to write the energy functional. For a fixed *boundary displacement* $w_0 \in H^{\frac{1}{2}}(\partial\Omega)$, we set

$$J(F) := \int_{\partial F} \varphi(\nu_F)\, d\mathcal{H}^1 + \int_{\Omega\setminus F} Q(E(u_F))\, dx, \tag{2.9}$$

where u_F is the elastic equilibrium satisfying the Dirichlet boundary condition w_0 on a fixed relatively open subset $\partial_D\Omega \subseteq \partial\Omega$. More precisely, u_F is the unique solution in $H^1(\Omega\setminus F;\mathbb{R}^2)$ of the following elliptic system

$$\begin{cases} \operatorname{div}\mathbb{C}E(u_F) = 0 & \text{in } \Omega\setminus F, \\ \mathbb{C}E(u_F)[\nu_F] = 0 & \text{on } \partial F \cup (\partial\Omega\setminus\partial_D\Omega), \\ u_F = w_0 & \text{on } \partial_D\Omega. \end{cases} \tag{2.10}$$

Next, we provide the first and the second variation formulas for (2.9). We start by recalling the well-known first variation formula for the anisotropic perimeter. To this aim, for any vector field $X \in C^1_c(\mathbb{R}^2;\mathbb{R}^2)$, let $(\Phi_t)_{t\in(-1,1)}$ be the associated flow, that is the solution of

$$\begin{cases} \dfrac{\partial\Phi_t}{\partial t} = X(\Phi_t), \\ \Phi_0 = Id. \end{cases} \tag{2.11}$$

Then we have

$$\frac{d}{dt}\Big|_{t=0}\int_{\partial\Phi_t(F)} \varphi(\nu_{\Phi_t(F)})\, d\mathcal{H}^1 = \int_{\partial F} k_\varphi X\cdot\nu\, d\mathcal{H}^1,$$

where the *anisotropic curvature* k_φ of ∂F is given by $k_\varphi := \operatorname{div}_\tau(\nabla\varphi(\nu))$ and can be written also as

$$\begin{aligned} k_\varphi = \operatorname{div}_\tau(\nabla\varphi(\nu)) &= \operatorname{div}(\nabla\varphi(\nu)) = D^2\varphi(\nu) : D\nu \\ &= (D^2\varphi(\nu)\tau\cdot\tau)\, k \\ &=: g(\nu)\, k, \end{aligned}$$

on ∂F.

Concerning the full functional J, we have the following theorem.

Theorem 2.1. *Let $F \subset\subset \Omega$ be a smooth set, $X \in C^1_c(\Omega;\mathbb{R}^2)$ and let $(\Phi_t)_{t\in(-1,1)}$ be the associated flow as in* (2.11)*. Set $\psi := X\cdot\nu_F$ and $X_\tau := (X\cdot\tau_F)\tau_F$ on ∂F. Then,*

$$\frac{d}{dt}J(\Phi_t(F))\Big|_{t=0} = \int_{\partial F} (g(\nu_F)k_F - Q(E(u_F)))\psi\, d\mathcal{H}^1. \tag{2.12}$$

If in addition $\operatorname{div} X = 0$ *in a neighborhood of* ∂F *we have*

$$\frac{d^2}{dt^2}J(\Phi_t(F))\Big|_{t=0} = \int_{\partial F}\big[g(\nu_F)(\partial_\sigma\psi)^2 - g(\nu_F)k_F^2\psi^2\big]\,d\mathcal{H}^1 - 2\int_{\Omega\setminus F} Q(E(u_\psi))\,dx$$

$$- \int_{\partial F}\partial_{\nu_F}(Q(E(u_F)))\psi^2\,d\mathcal{H}^1 - \int_{\partial F}(g(\nu_F)k_F - Q(E(u_F)))\operatorname{div}_\tau(\psi X_\tau)\,d\mathcal{H}^1, \tag{2.13}$$

where the function u_ψ *is the unique solution in* $H^1(\Omega\setminus F;\mathbb{R}^2)$, *with* $u_\psi = 0$ *on* $\partial_D\Omega$, *of*

$$\int_{\Omega\setminus F}\mathbb{C}E(u_\psi) : E(\varphi)\,dx = -\int_{\partial F}\operatorname{div}_\tau(\psi E(u_F))\cdot\varphi\,d\mathcal{H}^1 \tag{2.14}$$

for all $\varphi\in H^1(\Omega\setminus F;\mathbb{R}^2)$ *such that* $\varphi = 0$ *on* $\partial_D\Omega$.

Formulas (2.12) and (2.13) have been derived in [14] for the case where ϕ is the Euclidean norm, and in a slightly different setting, namely when F is the subgraph of a periodic function, in [25, 9]. The very same calculations apply to the more general situation considered here.

Throughout the paper, given a (sufficiently smooth) set $F\subset\subset\Omega$, we denote by $\Gamma_{F,1},\dots,\Gamma_{F,m}$ the m connected components of ∂F and by F_i the bounded open set enclosed by $\Gamma_{F,i}$. Note that the F_i's are not in general the connected components of F and it may happen that $F_i\subset F_j$ for some $i\neq j$.

We are interested in area preserving variations, in the following sense.

Definition 2.2. Let $F\subset\subset\Omega$ be a smooth set. Given a vector field $X\in C_c^\infty(\Omega;\mathbb{R}^2)$, we say that the associated flow $(\Phi_t)_{t\in(-1,1)}$ is *admissible for* F if there exists $\varepsilon_0\in(0,1)$ such that

$$|\Phi_t(F_i)| = |F_i| \quad \text{for } t\in(-\varepsilon_0,\varepsilon_0) \text{ and } i = 1,\dots,m.$$

Remark 2.3. Note that if the flow associated with X is admissible in the sense of the previous definition, then for $i = 1,\dots,m$ we have

$$\int_{\Gamma_{F,i}} X\cdot\nu_F\,d\mathcal{H}^1 = 0.$$

In view of this remark it is convenient to introduce the space $\tilde H^1(\partial F)$ consisting of all functions $\psi\in H^1(\partial F)$ with zero average on each component of ∂F, i.e.,

$$\int_{\Gamma_{F,i}}\psi\,d\mathcal{H}^1 = 0 \qquad \text{for every } i = 1,\dots,m.$$

We observe that given $\psi\in\tilde H^1(\partial F)\cap C^\infty(\partial F)$ it is possible to construct a sequence of vector fields $X_n\in C_c^\infty(\Omega;\mathbb{R}^2)$, with $\operatorname{div} X_n = 0$ in a neighborhood of ∂F, such that $X_n\cdot\nu_F\to\psi$ in $C^1(\partial F)$, see [2, Proof of Corollary 3.4] for the details. Note that in particular the flows associated with X_n are admissible.

Definition 2.4. Let $F\subset\subset\Omega$ be a set of class C^2. We say that F is *stationary* if

$$\frac{d}{dt}J(\Phi_t(F))\Big|_{t=0} = 0$$

for all admissible flows in the sense of Definition 2.2.

Remark 2.5. By Remark 2.3 and in view of (2.12) it follows that a set $F\subset\subset\Omega$ of class C^2 is stationary if and only if there exist constants $\lambda_1,\dots,\lambda_m$ such that

$$g(\nu_F)k_F - Q(E(u_F)) = \lambda_i \qquad \text{on } \Gamma_{F,i}$$

for every $i = 1, \dots, m$. In turn, note that if F is stationary, then the second variation formula (2.13) reduces to

$$\begin{aligned}\frac{d^2}{dt^2} J(\Phi_t(F))\Big|_{t=0} &= \int_{\partial F} \big[g(\nu_F)(\partial_\sigma \psi)^2 - g(\nu_F) k_F^2 \psi^2\big]\, d\mathcal{H}^1 \\ &\quad - 2\int_{\Omega\setminus F} Q(E(u_\psi))\, dx - \int_{\partial F} \partial_{\nu_F}(Q(E(u_F)))\psi^2\, d\mathcal{H}^1,\end{aligned} \tag{2.15}$$

where we recall that $\psi = X \cdot \nu_F$ and u_ψ is the function satisfying (2.14).

Note that if F is a sufficiently regular (local) minimizer of (2.9) under the constraint $|F| = const.$, then there exists a constant λ such that

$$g(\nu_F)k_F - Q(E(u_F)) = \lambda \qquad \text{on } \partial F.$$

Thus, our notion of stationarity differs from the usual notion of criticality just recalled.

In view of (2.15), for any set $F \subset\subset \Omega$ of class C^2 it is convenient to introduce the quadratic form $\partial^2 J(F)$ defined on $\tilde{H}^1(\partial F)$ as

$$\begin{aligned}\partial^2 J(F)[\psi] &:= \int_{\partial F} \big[g(\nu_F)(\partial_\sigma \psi)^2 - g(\nu_F) k_F^2 \psi^2\big]\, d\mathcal{H}^1 \\ &\quad - 2\int_{\Omega\setminus F} Q(E(u_\psi))\, dx - \int_{\partial F} \partial_{\nu_F}(Q(E(u_F)))\psi^2\, d\mathcal{H}^1,\end{aligned} \tag{2.16}$$

where u_ψ is the unique solution of (2.14) under the Dirichlet condition $u_\psi = 0$ on $\partial_D\Omega$. We conclude this section by defining the notion of stability for a stationary point.

Definition 2.6. Let $F \subset\subset \Omega$ be a stationary set in the sense of Definition 2.4. We say that F is *strictly stable* if

$$\partial^2 J(F)[\psi] > 0 \qquad \text{for all } \psi \in \tilde{H}^1(\partial F) \setminus \{0\}. \tag{2.17}$$

It is not difficult to see that (2.17) is equivalent to the coercivity of $\partial^2 J(F)$ on $\tilde{H}^1(\partial F)$. More precisely, (2.17) holds if and only if there exists $m_0 > 0$ such that

$$\partial^2 J(F)[\psi] \geq m_0 \|\psi\|^2_{\tilde{H}^1(\partial F)} \qquad \text{for all } \psi \in \tilde{H}^1(\partial F), \tag{2.18}$$

see [25]. In turn the latter coercivity property is stable with respect to small $C^{2,\alpha}$-perturbations. More precisely, we have:

Lemma 2.7. *Assume that the reference set $G \subset\subset \Omega$ is a (smooth) strictly stable stationary set in the sense of Definition 2.6 and fix $\alpha \in (0,1)$. Then, there exists $\sigma_0 > 0$ such that for all $F \in \mathfrak{h}^{2,\alpha}_{\sigma_0}(G)$ (see (2.4)) we have*

$$\partial^2 J(F)[\psi] \geq \frac{m_0}{2} \|\psi\|^2_{\tilde{H}^1(\partial F)} \text{ for all } \psi \in \tilde{H}^1(\partial F),$$

where m_0 is the constant in (2.18).

Proof. The proof of the lemma goes as in [23, Proof of Lemma 4.12], where the case of F being the subgraph of a periodic function of two variables is considered. Although the geometric framework here is more general, we can follow exactly the same line of argument up to the obvious changes due to the different setting (and some simplifications due the fact that here we work in two-dimensions). We refer the reader to the aforementioned reference for the details. □

Recall that $G_1, \ldots, G_m$ are the bounded open sets enclosed by the connected components $\Gamma_{G,1}, \ldots, \Gamma_{G,m}$ of the boundary ∂G of the reference set and observe that if $F \in \mathfrak{h}^{2,\alpha}_M(G)$, then ∂F has the same number m of connected components $\Gamma_{F,1}, \ldots, \Gamma_{F,m}$, which can be numbered in such a way that

$$\Gamma_{F,i} = \{x + h_F(x)\nu_G(x) \mid x \in \Gamma_{G,i}\}, \tag{2.19}$$

for suitable $h_F \in C^{k,\alpha}(\partial G)$.

In the next lemma we show that pairs of sets which are sufficiently close in a $C^{2,\alpha}$-sense can always be connected through area preserving flows in the sense of Definition 2.2. More precisely we have:

Lemma 2.8. *Let $M > 0$ and $\alpha \in (0,1)$. There exists $C > 0$ with the following property: If F_1, $F_2 \in \mathfrak{h}^{2,\alpha}_M(G)$ are such that $|F_{1,i}| = |F_{2,i}|$, $i = 1, \ldots, m$, then there exists a flow $(\Phi_t)_{t\in(-1,1)}$ admissible for F_1 in the sense of Definition 2.2, such that $\Phi_0(F_1) = F_1$, $\Phi_1(F_1) = F_2$, $|\Phi_t(F_{1,i})| = |F_{1,i}|$ for all $i = 1, \ldots, m$ and $t \in [0,1]$. Moreover*

$$\sup_{t\in[0,1]} \|\Phi_t - Id\|_{C^{2,\alpha}(\mathcal{N}_{\bar\eta/2}(\partial G))} \le C\|h_{F_1} - h_{F_2}\|_{C^{2,\alpha}(\partial G)}, \tag{2.20}$$

and the velocity field X satisfies $\operatorname{div} X = 0$ *in the $\bar\eta/2$-neighborhood $\mathcal{N}_{\bar\eta/2}(\partial G)$. Here $F_{i,1}, \ldots, F_{i,m}$ denote the bounded open sets enclosed by the connected components $\Gamma_{F_i,1}, \ldots, \Gamma_{F_i,m}$ of ∂F_i, $i = 1, 2$, which are supposed to be numbered as in* (2.19).

Proof. We adapt the construction of [33, Proposition 3.4]. We start by constructing a C^∞ vector-field $\tilde X : \mathcal{N}_{\bar\eta}(\partial G) \to \mathbb{R}^2$ satisfying

$$\operatorname{div} \tilde X = 0 \quad \text{in } \mathcal{N}_{\bar\eta}(\partial G), \qquad \tilde X = \nu_G \quad \text{on } \partial G. \tag{2.21}$$

To this aim, let ζ be the solution of

$$\begin{cases} \nabla\zeta \cdot \nabla d_G + \zeta\Delta d_G = 0 & \text{in } \mathcal{N}_{\bar\eta}(\partial G), \\ \zeta = 1 & \text{on } \partial G. \end{cases}$$

We may solve the above PDE by the method of characteristics, constructing such a ζ amounts to solving for every $x \in \partial G$ the Cauchy problem

$$\begin{cases} (f_x)'(t) + f_x(t)\Delta d_G(x + t\nu_G(x)) = 0 & \text{in } (-\bar\eta, \bar\eta), \\ f_x(0) = 1\,, \end{cases}$$

and setting

$$\zeta(x + t\nu_G(x)) := f_x(t) = \exp\Big(-\int_0^t \Delta d_G(x + s\nu_G(x))\, ds\Big)\,.$$

We can now define $\tilde X := \zeta\nabla d_G$ and check that $\operatorname{div}(\zeta\nabla d_G) = \nabla\zeta \cdot \nabla d_G + \zeta\Delta d_G = 0$.

Let now F_1 and F_2 be as in the statement. We choose $X \in C^\infty_c(\Omega;\mathbb{R}^2)$ in such a way that

$$X(x) := \bigg(\int_{h_{F_1}(\pi_G(x))}^{h_{F_2}(\pi_G(x))} \frac{ds}{\zeta(\pi_G(x) + s\nu_G(\pi_G(x)))}\bigg)\tilde X(x) \qquad \text{for every } x \in \mathcal{N}_{\bar\eta/2}(\partial G) \tag{2.22}$$

and we let Φ be the associated flow. Notice that the integral appearing in (2.22) represents the time needed to go from $\pi_G(x) + h_{F_1}(\pi_G(x))\nu_G(\pi_G(x))$ to $\pi_G(x) + h_{F_2}(\pi_G(x))\nu_G(\pi_G(x))$ along the trajectory of the vector field $\tilde X$. Therefore the above definition ensures that the time needed to go from $\pi_G(x)+h_{F_1}(\pi_G(x))\nu_G(\pi_G(x))$ to $\pi_G(x)+h_{F_2}(\pi_G(x))\nu_G(\pi_G(x))$ along the modified vector field X is one. This is equivalent to saying that for all $x \in \partial G$ we have

$\Phi_1(x + h_{F_1}(x)\nu_G(x)) = x + h_{F_2}(x)\nu_G(x)$ and, in turn, $\Phi_1(F_1) = F_2$. Moreover, recalling the first equation in (2.21) and using the fact that the function

$$x \mapsto \int_{h_{F_1}(\pi_G(x))}^{h_{F_2}(\pi_G(x))} \frac{ds}{\zeta(\pi_G(x) + s\nu_G(\pi_G(x)))}$$

is constant along the trajectories of $\tilde{X}$, we deduce from (2.22) that the modified field X is divergence free in $\mathcal{N}_{\bar{\eta}/2}(\partial G)$. Note that by (2.22) it also follows

$$\|X\|_{C^{2,\alpha}(\mathcal{N}_{\bar{\eta}/2}(\partial G))} \le C\|h_{F_1} - h_{F_2}\|_{C^{2,\alpha}(\partial G)}$$

for a constant $C > 0$ depending on G, and thus (2.20) easily follows.

Observe now that for $i = 1, \dots, m$ and for $\varepsilon_0 > 0$ small enough by [17, equation (2.30)] we have

$$\frac{d^2}{dt^2}|\Phi_t(F_{1,i})| = \int_{\Phi_t(\Gamma_{F_i,1})} (\operatorname{div} X)(X \cdot \nu_{\Phi_t(F_{1,i})}) = 0 \qquad \text{for all } t \in [-\varepsilon_0, 1],$$

where we used the fact that X is divergence free in $\mathcal{N}_{\bar{\eta}/2}(\partial G)$. Hence the function $t \mapsto |\Phi_t(F_{1,i})|$ is affine in $[-\varepsilon_0, 1]$. Since by assumption $|\Phi_0(F_{1,i})| = |F_{1,i}| = |F_{2,i}| = |\Phi_1(F_{1,i})|$, it is in fact constant. This concludes the proof of the lemma. □

We conclude this section by showing that in a sufficiently small $C^{2,\alpha}$-neighborhood of G the stationary sets are isolated, once we fix the areas enclosed by the connected components of the boundary.

Proposition 2.9. *Assume that the reference set $G \subset\subset \Omega$ is a (smooth) strictly stable stationary set in the sense of Definition 2.6, fix $\alpha \in (0,1)$, and let σ_0 be the constant provided by Lemma 2.7. There exists $\sigma_1 \in (0, \sigma_0)$ with the following property: Let F_1, $F_2 \in \mathfrak{h}^{2,\alpha}_{\sigma_1}(G)$ be stationary sets in the sense of Definition 2.4 and (with same notation of Lemma 2.8) assume that $|F_{1,i}| = |F_{2,i}|$ for $i = 1, \dots, m$. Then $F_1 = F_2$.*

Proof. We start by observing that for any $\eta \in (0, \sigma_0)$ we may choose $\sigma_1 > 0$ so small that for any pair F_1, $F_2 \in \mathfrak{h}^{2,\alpha}_{\sigma_1}(G)$ the flow Φ_t provided by Lemma 2.8 satisfies

$$\Phi_t(F_1) \in \mathfrak{h}^{2,\alpha}_{\eta}(G) \subseteq \mathfrak{h}^{2,\alpha}_{\sigma_0}(G) \qquad \text{for all } t \in [0,1], \tag{2.23}$$

Notice that this is possible thanks to (2.20).

Recall that by Remark 2.5 there exist constants λ_i such that $g(\nu_G)k_G - Q(E(u_G)) = \lambda_i$ on $\Gamma_{G,i}$ for $i = 1, \dots, m$. In what follows, the subscript t stands for the subscript $\Phi_t(F_1)$, where Φ_t is the flow of Lemma 2.8. Fix $\varepsilon > 0$ and observe that by taking η in (2.23) and, in turn, σ_1 smaller if needed, we may ensure that

$$\sup_{t \in [0,1]} \|\nu_t - \nu_G \circ \pi_G\|_{C^1(\Phi_t(\partial F))} \le \varepsilon \tag{2.24}$$

and

$$\sup_{i=1,\dots,m} \sup_{t \in [0,1]} \|g(\nu_t)k_t - Q(E(u_t)) - \lambda_i\|_{C^0(\Phi_t(\Gamma_{F,i}))} \le \varepsilon. \tag{2.25}$$

The latter condition can be guaranteed thanks also to the elliptic estimates proved later in Lemma 3.6. Let X be the velocity field of Φ_t and recall that by the explicit construction given in the proof of Lemma 2.8 we have $X = [X \cdot (\nu_G \circ \pi_G)]\nu_G \circ \pi_G$ in $\mathcal{N}_{\bar{\eta}/2}(\partial G)$. Thus, writing

$X = [X \cdot (\nu_G \circ \pi_G - \nu_t)]\nu_G \circ \pi_G + (X \cdot \nu_t)\nu_G \circ \pi_G$ on $\Phi_t(\partial F)$ and using (2.24) with ε (and in turn σ_1) sufficiently small, we find that for all $t \in [0, 1]$

$$|X| \leq 2|X \cdot \nu_t| \quad \text{and} \quad |\partial_\sigma X| \leq C\big(|X \cdot \nu_t| + |\partial_\sigma(X \cdot \nu_t)|\big) \quad \text{on } \Phi_t(\partial F), \tag{2.26}$$

for some constant $C > 0$ depending only on G.

Let now F_1 and F_2 be as in the statement of the proposition and Φ_t as above. Recalling (2.13) and (2.16), for every $s \in [0, 1]$ we may write

$$\begin{aligned} \frac{d^2}{dt^2} J(\Phi_t(F))\Big|_{t=s} &= \partial^2 J(\Phi_s(F_1))[X \cdot \nu_s] \\ &\quad - \int_{\Phi_s(\partial F_1)} \big[g(\nu_s)k_s - Q(E(u_s))\big] \operatorname{div}_\tau\big(X_\tau(X \cdot \nu_s)\big)\, d\mathcal{H}^1 \\ &= \partial^2 J(\Phi_s(F_1))[X \cdot \nu_s] \\ &\quad - \sum_{i=1}^m \int_{\Phi_s(\Gamma_{F_1,i})} \big[g(\nu_s)k_s - Q(E(u_s)) - \lambda_i\big] \operatorname{div}_\tau\big(X_\tau(X \cdot \nu_s)\big)\, d\mathcal{H}^1. \end{aligned} \tag{2.27}$$

Recall that (Φ_t) is an admissible flow and thus $X \cdot \nu_s \in \tilde{H}^1(\Phi_s(\partial F_1))$ for every $s \in [0, 1]$ due to Remark 2.3. In turn, by (2.23) and Lemma 2.7 we deduce that

$$\partial^2 J(\Phi_s(F_1))[X \cdot \nu_s] \geq \frac{m_0}{2} \|X \cdot \nu_s\|^2_{\tilde{H}^1(\Phi_s(\partial F_1))}.$$

Note also that by (2.26) it is easily checked that

$$\|\operatorname{div}_\tau\big(X_\tau(X \cdot \nu_s)\big)\|_{L^1(\Phi_s(F_1))} \leq C\|X \cdot \nu_s\|^2_{\tilde{H}^1(\Phi_s(\partial F_1))}.$$

Thus, collecting all the above observations and recalling also (2.25), we deduce from (2.27) that for every $s \in [0, 1]$

$$\frac{d^2}{dt^2} J(\Phi_t(F))\Big|_{t=s} \geq \Big(\frac{m_0}{2} - Cm\varepsilon\Big)\|X \cdot \nu_s\|^2_{\tilde{H}^1(\Phi_s(\partial F_1))} \geq \frac{m_0}{4}\|X \cdot \nu_s\|^2_{\tilde{H}^1(\Phi_s(\partial F_1))},$$

where the last inequality holds true provided we choose in (2.25) a sufficiently small ε (and σ_1). Since on the other hand by the stationarity of F_1 and F_2 we have

$$\frac{d}{dt} J(\Phi_t(F))\Big|_{t=0} = \frac{d}{dt} J(\Phi_t(F))\Big|_{t=1} = 0,$$

we infer that $\frac{d^2}{dt^2} J(\Phi_t(F))\Big|_{t=s} = 0$ and in turn $X \cdot \nu_s = 0$ on $\Phi_s(\partial F_1)$ for all $s \in [0, 1]$. This means that $s \mapsto \Phi_s(F_1)$ is constant in $[0, 1]$ and, in particular, $F_1 = F_2$. □

3. Short-time existence and regularity

We are interested in the evolution law

$$V_t = \partial_{\sigma\sigma}\big(g(\nu_t)k_t - Q(E(u_t))\big) \quad \text{on } \partial F_t, \tag{3.1}$$

where V_t stands for the outer normal velocity of ∂F_t, and $u_t \in H^1(\Omega \setminus F_t; \mathbb{R}^2)$ is the unique solution of

$$\begin{cases} \operatorname{div} \mathbb{C}E(u_t) = 0 & \text{in } \Omega \setminus F_t, \\ \mathbb{C}E(u_t)[\nu_t] = 0 & \text{on } \partial F_t \cup (\partial\Omega \setminus \partial_D\Omega), \\ u_t = w_0 & \text{on } \partial_D\Omega. \end{cases} \tag{3.2}$$

Remark 3.1. We remark that (3.1) can be regarded as the gradient flow of (2.9) with respect to a suitable Riemannian structure of H^{-1}-type. To illustrate this fact, consider the dual H_t^{-1} of $H_t^1 := H^1(\partial F(t))$ endowed with the scalar product

$$\langle \psi_1, \psi_2 \rangle_{H_t^{-1}} := \int_{\partial F(t)} \partial_\sigma v_{\psi_1} \partial_\sigma v_{\psi_2} \, d\mathcal{H}^1 = -\langle \partial_{\sigma\sigma} v_{\psi_2}, v_{\psi_1} \rangle_{H_t^{-1} \times H_t^1} = \langle \psi_2, v_{\psi_1} \rangle_{H_t^{-1} \times H_t^1} = \langle \psi_1, v_{\psi_2} \rangle_{H_t^{-1} \times H_t^1}, \tag{3.3}$$

where ∂_σ denotes the tangential derivative on $\partial F(t)$ and for any $\psi \in H_t^{-1}$ the function v_ψ is the unique function in H_t^1 satisfying

$$\begin{cases} -\partial_{\sigma\sigma} v_\psi = \psi & \text{on } \partial F(t)\,, \\ \displaystyle\int_{\partial F(t)} v_\psi \, d\mathcal{H}^1 = 0\,. \end{cases} \tag{3.4}$$

As already recalled, the first variation $\partial J(F(t))$ satisfies

$$\partial J(F(t))[\psi] = \int_{\partial F(t)} \big(k_{\varphi,t} - Q(E(u_{F(t)}))\big) \psi \, d\mathcal{H}^1$$

for all $\psi \in C^\infty(\partial F(t))$. Thus, recalling also (3.1), (3.3) and (3.4), we have

$$\begin{aligned} \langle V_t, \psi \rangle_{H^{-1}(\partial F(t))} &= \int_{\partial F(t)} V_t \, v_\psi \, d\mathcal{H}^1 = \int_{\partial F(t)} \partial_{\sigma\sigma} \big(k_{\varphi,t} - Q(E(u_{F(t)}))\big) v_\psi \, d\mathcal{H}^1 \\ &= \int_{\partial F(t)} \big(k_{\varphi,t} - Q(E(u_{F(t)}))\big) \partial_{\sigma\sigma} v_\psi \, d\mathcal{H}^1 = -\partial J(F(t))[\psi]. \end{aligned}$$

Hence, time by time the normal velocity V_t is the element of H_t^{-1} that represents the action of $-\partial J(F(t))$ with respect to the scalar product defined in (3.3). This formally establishes the H^{-1}-gradient flow structure of (3.1).

The following theorem establishes the short-time existence of a unique weak solution of (3.1). In Theorem 3.8 below we will show that the weak solution is in fact smooth and therefore classical.

Theorem 3.2. *Let $F_0 \subset\subset \Omega$ be such that*

$$\partial F_0 = \{x + h_0(x)\nu_G(x) \mid h_0 \in H^3(\partial G)\}. \tag{3.5}$$

There exist δ and $T > 0$, which depend on the H^3-norm of h_0, such that if $\|h_0\|_{L^2(\partial G)} \le \delta$ then the flow (3.1) admits a unique local-in-time weak solution $(F_t)_{t \in (0,T)}$ with an initial set F_0. More precisely, we have $\partial F_t = \{x + h_t(x)\nu_G(x)\}$, where $(h_t)_t \in H^1(0,T; H^1(\partial G)) \cap L^2(0,T; H^3(\partial G))$. Moreover $(h_t)_t \in C^0([0,T); C^{2,\alpha}(\partial G))$ for all $\alpha \in (0, \frac{1}{2})$ and $\big([g(\nu_t)k_t - Q(E(u_t))] \circ \pi_{F_t}^{-1}\big)_t \in L^2(0,T; H^3(\partial G))$.

Note that when the initial set F_0 is smooth we may take $G = F_0$. We give the proof of Theorem 3.2 at the end of the section. We will first prove a sequence of lemmas needed for the proof of the short-time existence result.

We will need some preliminary results. Our proof of Theorem 3.2 is based on a fixed point argument. To this aim, for a given smooth function $f : \partial G \times (0,T) \to \mathbb{R}$, we consider the forced surface diffusion flow given by

$$V_t = \partial_{\sigma\sigma}\big(g(\nu_t)k_t + f_t \circ \pi_G\big) \quad \text{on } \partial F_t \tag{3.6}$$

with initial datum F_0 of class H^3, where we denoted $f_t := f(\cdot, t)$. To simplify the notation we will denote

$$R_t := g(\nu_t)k_t + f_t \circ \pi_G \quad \text{on } \partial F_t. \tag{3.7}$$

The following monotone quantities are the starting point of our analysis.

Lemma 3.3. *Let F_0 be a set with smooth boundary, $f \in C^\infty(\partial G \times [0,\infty))$, and let $(F_t)_{t\in(0,T)}$ be a smooth flow satisfying (3.6), with initial datum F_0. Then we have*

$$\frac{d}{dt}\int_{F_t \Delta G} dist(x, \partial G)\, dx = \int_{\partial F_t} d_G\, \partial_{\sigma\sigma} R_t \, d\mathcal{H}^1 \le P(F_t)^{\frac{1}{2}} \left(\int_{\partial F_t} (\partial_\sigma R_t)^2 \, d\mathcal{H}^1 \right)^{\frac{1}{2}}, \tag{3.8}$$

whenever the flow (3.6) exists. Moreover there exists C_1, which depends on $\sup_{(0,T)} ||h_t||_{C^{2,\alpha}}$ and $\sup_{(0,T)} ||f_t||_{C^{1,\alpha}}$, such that

$$\begin{aligned} &\frac{d}{dt}\int_{\partial F_t} (\partial_\sigma R_t)^2 \, d\mathcal{H}^1 + c_0 \int_{\partial F_t} (\partial_{\sigma\sigma\sigma} R_t)^2 \, d\mathcal{H}^1 \\ &\qquad\qquad \le C_1 \|\dot{f}_t\|^2_{H^{-\frac{1}{2}}(\partial G)} + C_1 \left(1 + \int_{\partial F_t} (\partial_\sigma R_t)^2 \, d\mathcal{H}^1 \right)^q, \end{aligned} \tag{3.9}$$

for some $q > 1$.

Proof. Let X_t be the velocity field associated with the flow. In particular we have that

$$X_t \cdot \nu_t = \partial_{\sigma\sigma} R_t.$$

For $t \in (0,T)$ and $s > 0$ $\Phi_s : \partial F_t \to \partial F_{t+s}, \Phi_s = \pi^{-1}_{F_{t+s}} \circ \pi_{F_t}$ are admissible diffeomorphisms and by the above equality it holds $\frac{\partial}{\partial s}\Phi_s\big|_{s=0} = X_t$.

As mentioned in the previous section we can extend ν_t, τ_t and k_t by means of the signed distance function d_{F_t} in a tubular neighborhood of ∂F_t. This extension yields the following identities (see for instance [9, Lemma 4.2]):

$$\partial_{\nu_t} k_{\varphi,t} = -k_t^2 g(\nu_t), \tag{3.10}$$

$$\dot{\nu}_t = -\partial_\sigma(X_t \cdot \nu_t)\, \tau_t = -\partial_{\sigma\sigma\sigma} R_t \, \tau_t \tag{3.11}$$

and

$$\dot{k}_{\varphi,t} = \mathrm{div}(D^2\varphi(\nu_t)\, \dot{\nu}_t) = -\partial_\sigma(g(\nu_t)\partial_{\sigma\sigma\sigma} R_t). \tag{3.12}$$

Note that

$$\int_{F_t \Delta G} \mathrm{dist}(x, \partial G)\, dx = \int_{F_t} d_G \, dx - \int_G d_G \, dx.$$

Thus,

$$\begin{aligned} \frac{d}{dt}\int_{F_t \Delta G} \mathrm{dist}(x, \partial G)\, dx &= \frac{d}{dt}\int_{F_t} d_G\, dx = \int_{F_t} \mathrm{div}(d_G X_t)\, dx \\ &= \int_{\partial F_t} d_G (X_t \cdot \nu_t)\, d\mathcal{H}^1 = \int_{\partial F_t} d_G\, \partial_{\sigma\sigma} R_t \, d\mathcal{H}^1 \\ &= -\int_{\partial F_t} \partial_\sigma d_G \, \partial_\sigma R_t \, d\mathcal{H}^1 \le P(F_t)^{\frac{1}{2}} \left(\int_{\partial F_t} (\partial_\sigma R_t)^2 \, d\mathcal{H}^1 \right)^{\frac{1}{2}}. \end{aligned}$$

This proves (3.8). Concerning (3.9) we begin by calculating

(3.13)
$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int_{\partial F_t}(\partial_\sigma R_t)^2\,d\mathcal{H}^1\right) &= \frac{\partial}{\partial s}\left(\frac{1}{2}\int_{\partial F_t}\big((\nabla R_{t+s})(\Phi_s(x))\cdot\tau_{t+s}(\Phi_s(x))\big)^2\,J_\tau\Phi_s\,d\mathcal{H}^1\right)\Big|_{s=0}\\ &= \frac{1}{2}\int_{\partial F_t}(\partial_\sigma R_t)^2\,\mathrm{div}_\tau X_t\,d\mathcal{H}^1 + \int_{\partial F_t}\partial_\sigma R_t\,\frac{\partial}{\partial s}\Big(\nabla R_{t+s}(\Phi_s(x))\cdot\tau_{t+s}(\Phi_s(x))\Big)\Big|_{s=0}\,d\mathcal{H}^1.\end{aligned}$$

Using our notation we write the last term as

$$\begin{aligned}&\frac{\partial}{\partial s}\Big(\nabla R_{t+s}(\Phi_s(x))\cdot\tau_{t+s}(\Phi_s(x))\Big)\Big|_{s=0}\\ &\quad= \partial_\sigma\dot R_t + (\nabla^2 R_t\,X_t)\cdot\tau_t + \nabla R_t\cdot\dot\tau_t + \nabla R_t\cdot(\nabla\tau_t X_t).\end{aligned}$$

We write $X_{t,\tau} := X_t\cdot\tau_t$. Note that by (3.11) we have that $\dot\tau_t = \mathcal{R}\dot\nu_t = \partial_{\sigma\sigma\sigma}R_t\,\nu_t$. Moreover it holds $D\tau_t\,\nu_t = 0$. Therefore we get

$$\begin{aligned}&\frac{\partial}{\partial s}\Big(\nabla R_{t+s}(\Phi_s(x))\cdot\tau_{t+s}(\Phi_s(x))\Big)\Big|_{s=0}\\ &\quad= \partial_\sigma\dot R_t + \partial_{\sigma\sigma\sigma}R_t\,\partial_{\nu_t}R_t + \partial_{\sigma\sigma}R_t(\nabla^2R_t\,\nu_t)\cdot\tau_t + (\nabla^2R_t\,\tau_t)\cdot\tau_t\,X_{t,\tau} + \nabla R_t\cdot(\nabla\tau_t\,\tau_t)\,X_{t,\tau}.\end{aligned}$$

Therefore, using the fact that $\partial_\sigma(\partial_{\nu_t}R_t) = k_t\,\partial_\sigma R_t + (\nabla^2R_t\,\nu_t)\cdot\tau_t$ and integrating by parts, (3.13) can be written as

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int_{\partial F_t}(\partial_\sigma R_t)^2\,d\mathcal{H}^1\right) &= \int_{\partial F_t}\frac{1}{2}(\partial_\sigma R_t)^2\,\mathrm{div}_\tau X_t + \partial_\sigma R_t\,\partial_\sigma\dot R_t + \partial_\sigma R_t\,\partial_{\sigma\sigma\sigma}R_t\,\partial_{\nu_t}R_t\,d\mathcal{H}^1\\ &\quad+ \int_{\partial F_t}\partial_\sigma R_t\,\partial_{\sigma\sigma}R_t(\nabla^2R_t\,\nu_t)\cdot\tau_t + \partial_\sigma R_t\,X_{t,\tau}\,(\nabla^2R_t\,\tau_t)\cdot\tau_t + \partial_\sigma R_t\,\nabla R_t\cdot(\nabla\tau_t\,\tau_t)\,X_{t,\tau}\,d\mathcal{H}^1\\ &= \int_{\partial F_t}\frac{1}{2}(\partial_\sigma R_t)^2\,\mathrm{div}_\tau X_t - \partial_{\sigma\sigma}R_t\dot R_t - (\partial_{\sigma\sigma}R_t)^2\partial_{\nu_t}R_t - k_t\,(\partial_\sigma R_t)^2\partial_{\sigma\sigma}R_t\,d\mathcal{H}^1\\ &\quad+ \int_{\partial F_t}\partial_\sigma R_t\,(\nabla^2R_t\,\tau_t)\cdot\tau_t\,X_{t,\tau} + \partial_\sigma R_t\nabla R_t\cdot(\nabla\tau_t\,\tau_t)\,X_{t,\tau}\,d\mathcal{H}^1.\end{aligned}$$

Note that

$$\frac{1}{2}\,\mathrm{div}_\tau\big((\partial_\sigma R_t)^2X_t\big) = \frac{1}{2}(\partial_\sigma R_t)^2\,\mathrm{div}_\tau X_t + \partial_\sigma R_t\,(\nabla^2R_t\,\tau_t)\cdot\tau_t\,X_{t,\tau} + \partial_\sigma R_t\nabla R_t\cdot(\nabla\tau_t\,\tau_t)\,X_{t,\tau}$$

Hence, using also (2.1), we get

(3.14)
$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\int_{\partial F_t}(\partial_\sigma R_t)^2\,d\mathcal{H}^1\right)\\ &\quad= \int_{\partial F_t}\frac{1}{2}\,\mathrm{div}_\tau\big((\partial_\sigma R_t)^2X_t\big)\,d\mathcal{H}^1 - \partial_{\sigma\sigma}R_t\dot R_t - (\partial_{\sigma\sigma}R_t)^2\partial_{\nu_t}R_t - k_t\,(\partial_\sigma R_t)^2\partial_{\sigma\sigma}R_t\,d\mathcal{H}^1\\ &\quad= -\int_{\partial F_t}\partial_{\sigma\sigma}R_t\dot R_t + (\partial_{\sigma\sigma}R_t)^2\partial_{\nu_t}R_t + \frac{1}{2}k_t\,(\partial_\sigma R_t)^2\partial_{\sigma\sigma}R_t\,d\mathcal{H}^1.\end{aligned}$$

Therefore, recalling (3.7), by (3.10) and (3.12) we get from (3.14) that

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\partial F_t}(\partial_\sigma R_t)^2\,d\mathcal{H}^1\right) = -\int_{\partial F_t} g(\nu_t)(\partial_{\sigma\sigma\sigma}R_t)^2\,d\mathcal{H}^1 - \int_{\partial F_t}\partial_{\sigma\sigma}R_t(\dot f_t\circ\pi_G)\,d\mathcal{H}^1 - \int_{\partial F_t}\left(\partial_{\nu_t}(f_t\circ\pi_G)(\partial_{\sigma\sigma}R_t)^2 - g(\nu_t)k_t^2(\partial_{\sigma\sigma}R_t)^2 + \frac{1}{2}k_t\,(\partial_\sigma R_t)^2\partial_{\sigma\sigma}R_t\right)d\mathcal{H}^1. \tag{3.15}$$

By the ellipticity assumptions (2.8) we have that $c_0 \le g(\nu_t) \le C_0$ and $|k_t| \le \frac{1}{c_0}|k_{\varphi,t}| \le C$, where C depends also on the $C^{2,\alpha}$-norm of h_t. For $\varepsilon > 0$ to be chosen, using also Young's inequality, we may estimate (3.15) as

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\int_{\partial F_t}(\partial_\sigma R_t)^2\,d\mathcal{H}^1\right) + c_0\int_{\partial F_t}(\partial_{\sigma\sigma\sigma}R_t)^2\,d\mathcal{H}^1\\ &\quad\le C_\varepsilon\|\dot f_t\|^2_{H^{-\frac12}(\partial G)} + \varepsilon\|\partial_{\sigma\sigma}R_t\|^2_{H^{\frac12}(\partial F_t)} + C\int_{\partial F_t}\left(1+(\partial_{\sigma\sigma}R_t)^2 + (\partial_\sigma R_t)^2|\partial_{\sigma\sigma}R_t|\right)d\mathcal{H}^1,\end{aligned}$$

where the constant C depends on the $C^{2,\alpha}$-norm of h_t and the $C^{1,\alpha}$-norm of f_t. Since $\|\partial_{\sigma\sigma}R_t\|_{H^{\frac12}(\partial F_t)} \le C\|\partial_{\sigma\sigma\sigma}R_t\|_{L^2(\partial F_t)}$, by choosing ε small enough we get

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\int_{\partial F_t}(\partial_\sigma R_t)^2\,d\mathcal{H}^1\right) + \frac{2}{3}c_0\int_{\partial F_t}(\partial_{\sigma\sigma\sigma}R_t)^2\,d\mathcal{H}^1\\ &\quad\le C\|\dot f_t\|^2_{H^{-\frac12}(\partial G)} + C\int_{\partial F_t}\left(1+(\partial_{\sigma\sigma}R_t)^2 + (\partial_\sigma R_t)^4\right)d\mathcal{H}^1.\end{aligned}$$

Note now that by Theorem 6.1,

$$\|\partial_{\sigma\sigma}R_t\|^2_{L^2} \le C\|\partial_{\sigma\sigma\sigma}R_t\|_{L^2}\,\|\partial_\sigma R_t\|_{L^2} \le \varepsilon\|\partial_{\sigma\sigma\sigma}R_t\|^2_{L^2} + \frac{C}{\varepsilon}\|\partial_\sigma R_t\|^2_{L^2}$$

and

$$\|\partial_\sigma R_t\|^4_{L^4} \le C\|\partial_{\sigma\sigma\sigma}R_t\|^{\frac12}_{L^2}\|\partial_\sigma R_t\|^{\frac72}_{L^2} \le \varepsilon\|\partial_{\sigma\sigma\sigma}R_t\|^2_{L^2} + \frac{C}{\varepsilon}\|\partial_\sigma R_t\|^{\frac{14}{3}}_{L^2}.$$

Hence the estimate (3.9) follows for $q = \frac{7}{3}$ by choosing ε small enough. □

Next theorem establishes the local in time existence for (3.6). The same result for $f = 0$ is proved in [16]. The case considered here follows essentially from the same argument.

Theorem 3.4. *Let $h_0 \in H^4(\partial G)$ and let $f \in C^\infty(\partial G\times[0,+\infty))$. Then, there exist $\delta > 0$ and $T > 0$ such that if $\|h_0\|_{C^1(\partial G))} \le \delta$, then (3.6) admits a smooth solution $(F_t)_t$ defined for all $t \in (0,T)$. Moreover, setting $\partial F_t = \{x + h_t(x)\nu_G(x)\}$, we have that $(h_t)_t \in H^1(0,T;L^2(\partial G))\cap L^2(0,T;H^4(\partial G))$. Finally, there exists $\bar\delta \in (0,\bar\eta)$, where $\bar\eta$ is as in (2.2), depending only on G and Ω, such that if $\sup_{(0,T)}\|h_t - h_0\|_{C^1(\partial G)} < \bar\delta$, then the solution can be extended beyond the time T.*

Proof. The proof goes exactly as the one of Theorem 2.5 of [16], taking into account the presence of the forcing term f. Note that as in [16] in the first part of the proof we can only conclude that the time T depends on $\|h_0\|_{H^4}$ and on $\|f\|_{L^2(0,T;H^2)}$. However, one can then argue as in the second part of proof of Theorem 2.5 of [16] to conclude that the $\bar\delta$ for which the extension property holds is independent of $\|h_0\|_{H^4}$ and $\|f\|_{L^2(0,T;H^2)}$, as long as $f \in L^2(0,T;H^2)$ (a property which is implied by our assumption on f). Finally, the C^∞-regularity of the solution for $t > 0$ follows by standard arguments (or arguing as in the proof of Theorem 3.8 below where in fact the more complicated equation (3.1) is dealt with). □

In the next lemma we use the monotone quantity (3.9) to deduce regularity estimates for the flow (3.6).

Lemma 3.5. *Let F_0 be a smooth initial set and $h_0 \in C^\infty(\partial G)$ the function representing ∂F_0 as in* (3.5)*. Fix $M_0 > 0$, $\alpha \in (0, \frac{1}{2})$, $\delta_1 \in (0, \bar\delta)$, with $\bar\delta$ as in Theorem 3.4. There exist $\delta_0 > 0$ and T_0, depending on M_0, α, and δ_1, such that if $f \in C^\infty(\partial G \times [0, +\infty))$ satisfies*

$$\sup_{(0,T_0)} \|f_t\|_{C^{1,\alpha}(\partial G)} \le M_0 \qquad \text{and} \qquad \int_0^{T_0} \|\dot f_t\|^2_{H^{-\frac{1}{2}}(\partial G)} \le M_0 \tag{3.16}$$

and if $\|h_0\|_{H^3(\partial G)} \le M_0$, $\|h_0\|_{L^2(\partial G)} < \delta_0$, then the flow (3.6) *exists on $(0, T_0)$ and*

$$\sup_{(0,T_0)} \|h_t\|_{C^{2,\alpha}(\partial G)} \le \delta_1 \qquad \text{and} \qquad \sup_{(0,T_0)} \|\partial_\sigma R_t\|^2_{L^2(\partial F_t)} \le 2C_1 M_0 + \|\partial_\sigma R_0\|^2_{L^2(\partial F_0)}, \tag{3.17}$$

where C_1 is the constant appearing in Lemma 3.3.

Proof. We fix $\delta_1 < \bar\delta$ and observe if $\delta_0 > 0$ is sufficiently small, $\|h_0\|_{H^3(\partial G)} \le M_0$ and $\|h_0\|_{L^2(\partial G)} < \delta_0$, then from (6.3) we get $\|h_0\|_{C^{2,\alpha}(\partial G)} < \bar\delta - \delta_1$. In particular, by Theorem 3.4 the flow exists for a short time and as long as $\|h_t\|_{C^{2,\alpha}(\partial G)} < \delta_1$, since this implies that $\|h_t - h_0\|_{C^1(\partial G)} < \bar\delta$.

Let us denote by T_0 the maximal time such that

$$\|h_t\|_{C^{2,\alpha}(\partial G)} < \delta_1 \text{ and } \|\partial_\sigma R_t\|^2_{L^2(\partial F_t)} < 2C_1 M_0 + \|\partial_\sigma R_0\|^2_{L^2(\partial F_0)} \quad \text{for all } t \in (0, T_0). \tag{3.18}$$

We want to show that if (3.16) is satisfied, then T_0 is bounded away from 0 by a constant depending only M_0, α, and δ_1. Thus, without loss of generality we may assume that $T_0 \le 1$, otherwise there is nothing to prove.

Assume first that $\|h_{T_0}\|_{C^{2,\alpha}(\partial G)} = \delta_1$. From (3.18), from the first inequality in (3.16) and from the assumption $\|h_0\|_{H^3(\partial G)} \le M_0$ we conclude that

$$\|\partial_\sigma R_t\|_{L^2(\partial F_t)} \le C(M_0) \qquad \text{for all } t \in (0, T_0). \tag{3.19}$$

In turn, using again the first inequality in (3.16) we get

$$\|\partial_\sigma k_{\varphi,t}\|^2_{L^2(\partial F_t)} \le C(M_0) \qquad \text{for all } t \in (0, T_0).$$

Now, by the first inequality in (3.18), recalling also formula (2.7), we deduce that

$$\|h_t\|_{H^3(\partial G)} \le C(M_0) \qquad \text{for all } t \in (0, T_0). \tag{3.20}$$

Moreover, we conclude by integrating (3.8) over $(0, t)$ and by (3.19) that

$$\|h_t\|^2_{L^2(\partial G)} \le C(M_0)T_0 + C\|h_0\|^2_{L^2(\partial G)} \le C(M_0)T_0 + C\delta_0^2 \qquad \text{for all } t \in (0, T_0).$$

In turn, by (6.3) and by (3.20) we get

$$\delta_1 = \|h_{T_0}\|_{C^{2,\alpha}(\partial G)} \le C\left(\|h_{T_0}\|^\theta_{H^3(\partial G)} \|h_{T_0}\|^{1-\theta}_{L^2(\partial G)} + \|h_{T_0}\|_{L^2(\partial G)}\right) \le C(M_0)\left(\sqrt{T_0} + \delta_0\right)^{1-\theta},$$

where θ depends only on α. It is clear from the above inequality that if δ_0 is sufficiently small we get that T_0 is bounded away from 0.

Assume now that $y(T_0) = 2C_1 M_0 + y(0)$, where we have set $y(t) := \|R_\sigma\|^2_{L^2(\partial F_t)}$. By integrating (3.9) over the time interval $(0, T_0)$ and using the second inequality in (3.16) we get

$$y(T_0) \le y(0) + C_1 M_0 + C_1 \int_0^{T_0} (1 + y)^q \, dt.$$

Now, using the second inequality in (3.18) we conclude that

$$2C_1M_0 + y(0) = y(T_0) \le y(0) + C_1M_0 + C_1T_0(1 + 2C_1M_0 + y(0))^q.$$

From this estimate we get again that T_0 is bounded away from 0. This concludes the proof of the lemma. ☐

We will need the following result for the elastic equilibrium, which states that if F, $\widetilde{F} \in \mathfrak{h}^{2,\alpha}_M(G)$ are $C^{2,\alpha}$-close, then the corresponding elastic equilibria are $C^{2,\alpha}$-close to each other. More precisely, we have the following lemma.

Lemma 3.6. *Let $0 < \alpha < \beta \le 1$, $M > 0$ and $k \in \mathbb{N}$. Then there exists $C > 0$ such that if $F, \widetilde{F} \in \mathfrak{h}^{k,\beta}_M(G)$ we have*

$$\|u_F \circ \pi_F^{-1} - u_{\widetilde{F}} \circ \pi_{\widetilde{F}}^{-1}\|_{C^{k,\alpha}(\partial G)} \le C\|h_F - h_{\widetilde{F}}\|_{C^{k,\alpha}(\partial G)}. \tag{3.21}$$

Here, we recall that u_F and $u_{\widetilde{F}}$ denote the elastic equilibria corresponding to F and $\tilde{F}$, respectively, as defined in (2.10).

Proof. The case $k = 1$ can be proved as in [22, Lemma 6.10]. We now assume $k \ge 2$.

Denote by $\mathcal{U}$ the open set in $C^{k,\alpha}(\partial G)$ defined as

$$\mathcal{U} := \big\{h \in C^{k,\alpha}(\partial G) : \ \|h\|_{L^\infty(\partial G)} < 2\bar{\eta}/3, \ \|h\|_{C^{k,\alpha}(\partial G)} < M'\big\},$$

where $M' > 0$ is chosen so large that $\mathfrak{h}^{k,\beta}_M(G) \subset \mathcal{U}$.

Given $h \in \mathcal{U}$, we denote by u_h the solution to (2.10), with F replaced by the bounded set F_h whose boundary is given by the (normal) graph of h over ∂G.

Fix $\psi \in C^\infty_c(\Omega)$, $0 \le \psi \le 1$, $\psi \equiv 1$ in $\{d_G \le 2\bar{\eta}/3\}$, and $\operatorname{supp}\psi \subset\subset \{d_G < \bar{\eta}\}$ and notice that if $\|h\|_{C^{k,\alpha}(\partial G)} \le \delta'$, for δ' sufficiently small, then the map $\Phi_h : \Omega \setminus F_h \to \Omega \setminus G$ of the form

$$\Phi_h(x) = x - h(\pi_G(x))\psi(x)\nu_G(\pi_G(x))$$

is a $C^{k,\alpha}$-diffeomorphism.

Then setting $v_h := u_h \circ (\Phi_h)^{-1}$ one can see that v_h is the solution to

$$\begin{cases} \operatorname{div}(\mathbb{A}(y, h(\pi_G(y)), \partial_\sigma h(\pi_G(y)))\nabla v) = 0 & \text{in } \Omega \setminus G \\ \mathbb{A}(y, h(y), \partial_\sigma h(y))\nabla v[\nu_G] = 0 & \text{on } \partial G, \\ v = w_0 & \text{on } \partial_D\Omega, \end{cases}$$

where the entries of the tensor valued function $\mathbb{A}$ are 4-th order polynomials in $h \circ \pi$ and $\partial_\sigma h \circ \pi$ with $C^{k-1,\alpha}$-coefficients. It is easily checked that the map $\mathcal{F} : \mathcal{U} \times C^{k,\alpha}(\Omega \setminus G) \to C^{k-2,\alpha}(\Omega \setminus G) \times C^{k-1,\alpha}(\partial G)$ given by

$$\mathcal{F}(h, v) := (\operatorname{div}(\mathbb{A}(y, h(\pi_G(y)), \partial_\sigma h(\pi_G(y)))\nabla v), \mathbb{A}(y, h(y), \partial_\sigma h(y))\nabla v[\nu_G])$$

is of class C^1. We now check the invertibility (with continuity of the inverse) of $\partial_v\mathcal{F}(h, v_h)$, which is a linear operator from $C^{k,\alpha}(\Omega \setminus G)$ to $C^{k-2,\alpha}(\Omega \setminus G) \times C^{k-1,\alpha}(\partial G)$. This amounts to showing that for every $f \in C^{k-2,\alpha}(\Omega \setminus G)$ and $g \in C^{k-1,\alpha}(\partial G)$ the system

$$\begin{cases} \operatorname{div}(\mathbb{A}(y, h(\pi_G(y)), \partial_\sigma h(\pi_G(y)))\nabla w) = f & \text{in } \Omega \setminus G \\ \mathbb{A}(y, h(y), \partial_\sigma h(y))\nabla w[\nu_G] = g & \text{on } \partial G, \\ w = 0 & \text{on } \partial_D\Omega, \end{cases}$$

admits a unique solution $w \in C^{k,\alpha}(\Omega \setminus G)$ such that $\|w\|_{C^{k,\alpha}(\Omega\setminus G)} \leq C(\|f\|_{C^{k-2,\alpha}(\Omega\setminus G)} + \|g\|_{C^{k-1,\alpha}(\partial G)})$, with C depending only on k, α, G and Ω. This follows from the classical Schauder estimates for linear elliptic systems (see for instance [3]). Thus, since $\mathcal{F}(h, v_h) = 0$, we may apply the Implicit Function Theorem (see [4, Theorem 2.3]) to deduce that there exists $\delta > 0$ such that the map $h \mapsto \mathcal{S}(h) := v_h$ is of class C^1 from a δ-neighborhood of h in the $C^{k,\alpha}(\partial G)$-norm to $C^{k,\alpha}(\Omega \setminus G)$.

Since $\mathcal{S}$ is C^1 in $\mathcal{U}$ and $\mathfrak{h}^{k,\beta}_M(G) \subset \mathcal{U}$ is compact in $C^{k\,\alpha}(\partial G)$, the Fréchet derivative $D\mathcal{S}(h)$ is equibounded in $\mathcal{L}(C^{k,\alpha}(\partial G); C^{k,\alpha}(\Omega \setminus G))$ for $h \in \mathfrak{h}^{k,\beta}_M(G)$. Hence (3.21) easily follows. $\square$

Let us consider the flow (3.6), where f satisfies the assumptions of Lemma 3.5. For every given time t we consider the elastic equilibrium u_t defined in (3.2). We start by observing that arguing as in [25, Theorem 3.2] one can show that $\dot{u}_t$ satisfies

$$\int_{\Omega\setminus F_t} \mathbb{C}E(\dot{u}_t) : E(\varphi)\, dx = -\int_{\partial F_t} \mathrm{div}_\tau(\partial_{\sigma\sigma} R_t\, \mathbb{C}E(u_t)) \cdot \varphi \, d\mathcal{H}^1 \tag{3.22}$$

for all $\varphi \in H^1(\Omega \setminus F_t; \mathbb{R}^2)$ such that $\varphi = 0$ on $\partial_D\Omega$. Note also that $\dot{u}_t = 0$ on $\partial_D\Omega$.

We can now prove the regularity for the elastic equilibria u_t.

Lemma 3.7. *Let F_0 and α be as in Lemma 3.5. Fix*

$$M_0 > 2\|Q(E(u_G))\|_{C^{1,\alpha}(\partial G)}. \tag{3.23}$$

There exist $\delta_1 \in (0, \bar\delta)$, $T_0 > 0$ and $\delta_0 > 0$ such that if f satisfies (3.16), $\|h_0\|_{H^3(\partial G)} \leq M_0$ *and* $\|h_0\|_{L^2(\partial G)} < \delta_0$, *then the flow* (3.6) *exists on* $(0, T_0)$, (3.17) *holds true and for every* $t \in (0, T_0)$

$$\|Q(E(u_t)) \circ \pi_{F_t}^{-1}\|_{C^{1,\alpha}(\partial G)} < M_0 \quad \text{and} \quad \int_0^{T_0} \|\partial_t\big(Q(E(u_t)) \circ \pi_{F_t}^{-1}\big)\|^2_{H^{-\frac12}(\partial G)}\, dt < M_0. \tag{3.24}$$

Proof. Let u_G be the elastic equilibrium in G. We first recall that if δ_0 is as in Lemma 3.5, then by the first inequality in (3.17) and (3.21) we have

$$\|u_t \circ \pi_{F_t}^{-1} - u_G\|_{C^{2,\alpha}(\partial G)} \leq C\|h_t\|_{C^{2,\alpha}(\partial G)} \leq C\delta_1. \tag{3.25}$$

Therefore, choosing $\delta_1 \in (0, \bar\delta)$ sufficiently small (depending on M_0) and recalling (3.23), the first estimate in (3.24) follows.

For the second estimate we calculate

$$\begin{aligned} &\partial_t\big(Q(E(u_t))(x + h_t(x)\nu_G(x))\big) \\ &\quad = \mathbb{C}E(u_t) \circ \pi_{F_t}^{-1} : \big((\nabla E(u_t) \circ \pi_{F_t}^{-1})[\dot{h}_t \nu_G]\big) + (\mathbb{C}E(u_t) : E(\dot{u}_t)) \circ \pi_{F_t}^{-1}. \end{aligned}$$

Therefore by the $C^{2,\alpha}$-bound (3.17) on h_t and by (3.25) we have that

$$\|\partial_t\big(Q(E(u_t))\circ\pi_{F_t}^{-1}\big)\|_{H^{-\frac12}(\partial G)} \leq C(M_0)\|\dot{h}_t\|_{L^2(\partial G)} + C(M_0)\|\mathbb{C}E(u_t) : E(\dot{u}_t)\|_{H^{-\frac12}(\partial F_t)}. \tag{3.26}$$

Observe first that the normal velocity of ∂F_t can be written on ∂G as

$$V \circ \pi_{F_t}^{-1} = \dot{h}_t \frac{1 + h_t k_G}{J_t}.$$

Therefore by the definition of the flow, recalling the interpolation inequality (6.2), we get

$$\|\dot{h}_t\|_{L^2(\partial G)} \leq C\|V\|_{L^2(\partial F_t)} = C\|\partial_{\sigma\sigma} R_t\|_{L^2(\partial F_t)} \leq C\|\partial_{\sigma\sigma\sigma} R_t\|^{\frac12}_{L^2(\partial F_t)} \|\partial_\sigma R_t\|^{\frac12}_{L^2(\partial F_t)}. \tag{3.27}$$

In order to estimate second term in (3.26) we recall that $\mathbb{C}E(u_t)[\nu] = 0$. Therefore, using (3.25) again, we get

$$
\begin{aligned}
\|\mathbb{C}E(u_t) : E(\dot u_t)\|_{H^{-\frac12}(\partial F_t)} &= \|\mathbb{C}E(u_t) : \nabla \dot u_t\|_{H^{-\frac12}(\partial F_t)} \\
&= \|\mathbb{C}E(u_t) : \nabla_\tau \dot u_t\|_{H^{-\frac12}(\partial F_t)} \leq C\|\nabla_\tau \dot u_t\|_{H^{-\frac12}(\partial F_t)}.
\end{aligned} \tag{3.28}
$$

Choosing as a test function $\varphi = \dot u_t$ in the equation (3.22) we get arguing as above that

$$
\begin{aligned}
2\int_{F_t} Q(E(\dot u_t))\,dx &= -\int_{\partial F_t} \mathrm{div}_\tau(\partial_{\sigma\sigma} R_t\, \mathbb{C}E(u_t))\cdot \dot u_t\, d\mathcal{H}^1 \\
&= \int_{\partial F_t} \partial_{\sigma\sigma} R_t\, \mathbb{C}E(u_t) : \nabla_\tau \dot u_t\, d\mathcal{H}^1 \\
&\leq C(M_0)\|\partial_{\sigma\sigma} R_t\|_{H^{\frac12}(\partial F_t)} \|\nabla_\tau \dot u_t\|_{H^{-\frac12}(\partial F_t)}.
\end{aligned} \tag{3.29}
$$

Recall that $\dot u_t = 0$ on $\partial_D \Omega$. Therefore by Korn's inequality, by (3.29) and by the interpolation inequality (6.1) we get

$$
\begin{aligned}
\|\nabla_\tau \dot u_t\|^2_{H^{-\frac12}(\partial F_t)} &\leq C\|\dot u_t\|^2_{H^{\frac12}(\partial F_t)} \leq C\int_{\Omega\setminus F_t} |\nabla \dot u_t|^2\,dx \\
&\leq C\int_{F_t} Q(E(\dot u_t))\,dx \leq C(M_0)\|\partial_{\sigma\sigma} R_t\|_{H^{\frac12}(\partial F_t)} \|\nabla_\tau \dot u_t\|_{H^{-\frac12}(\partial F_t)} \\
&\leq C(M_0)\big(\|\partial_{\sigma\sigma\sigma} R_t\|^{\frac34}_{L^2}\|\partial_\sigma R_t\|^{\frac14}_{L^2} + \|\partial_\sigma R_t\|_{L^2}\big)\|\nabla_\tau \dot u_t\|_{H^{-\frac12}(\partial F_t)}.
\end{aligned}
$$

Note that the first inequality above uses the fact that the tangential derivative is a continuous operator from $H^{1/2}$ to $H^{-1/2}$. This is a well-known fact, see for instance [25, Theorem 8.6]. This inequality together with (3.28) yields

$$
\|\mathbb{C}E(u_t) : E(\dot u_t)\|_{H^{-\frac12}(\partial F_t)} \leq C(M_0)\big(\|\partial_{\sigma\sigma\sigma} R_t\|^{\frac34}_{L^2(\partial F_t)}\|\partial_\sigma R_t\|^{\frac14}_{L^2(\partial F_t)} + \|\partial_\sigma R_t\|_{L^2(\partial F_t)}\big). \tag{3.30}
$$

By combining the estimates (3.26), (3.27) and (3.30) we deduce that for every $\varepsilon > 0$

$$
\int_0^{T_0} \|\big(\partial_t Q(E(u_t))\big)\circ \pi_{F_t}^{-1}\|^2_{H^{-\frac12}(\partial G)}\,dt \leq \int_0^{T_0} \big(\varepsilon\|\partial_{\sigma\sigma\sigma} R_t\|^2_{L^2(\partial F_t)} + C_\varepsilon \|\partial_\sigma R_t\|^2_{L^2(\partial F_t)}\big)\,dt.
$$

Hence, taking ε sufficiently small we obtain by (3.9) and by Lemma 3.5

$$
\begin{aligned}
&\int_0^{T_0} \|\big(\partial_t Q(E(u_t))\big)\circ \pi_{F_t}^{-1}\|^2_{H^{-\frac12}(\partial G)}\,dt \leq \\
&\qquad\leq \frac12\int_0^{T_0} \|\dot f_t\|^2_{H^{-\frac12}(\partial G)}\,dt + C(M_0)\sup_{(0,T_0)}\big(1+\|\partial_\sigma R_t\|^2_{L^2(\partial F_t)}\big)^q T_0 \leq \frac12 M_0 + CT_0.
\end{aligned}
$$

Thus the second estimate in (3.24) follows by choosing T_0 sufficiently small. □

Proof of Theorem 3.2. We divide the proof into several steps.
Step 1. Fix $\mu \in (0,1)$ and let M_0, T_0, α, δ_1 and δ_0 be as in Lemma 3.7. Let $f_1, f_2 \in C^\infty(\partial G\times [0,+\infty)$ satisfy the assumptions of Lemma 3.7, let $h_0 \in C^\infty(\partial G)$ satisfy $\|h_0\|_{H^3(\partial G)} \leq M_0$, $\|h_0\|_{L^2(\partial G)} < \delta_0$, and let $F_{t,i}$ be a solution of (3.6) with f replaced by f_i. Denote by $h_{t,i}$ the

function such that $\partial F_{t,i} = \{x + h_{t,i}(x)\nu_G(x) : x \in \partial G\}$. We start by showing that there exists $T \in (0, T_0)$ such that

$$\int_0^T \int_{\partial G} (h_{t,2} - h_{t,1})^2 \, d\mathcal{H}^1 dt \le \mu \int_0^T \int_{\partial G} (f_2 - f_1)^2 \, d\mathcal{H}^1 dt. \tag{3.31}$$

Note in particular that the above inequality implies the uniqueness of the solution of (3.6) when all the data are smooth.

Recall that by Lemma 3.5 we have that $\|h_{t,i}\|_{C^{2,\alpha}} \le \delta_1$ for all $t \in (0, T_0)$, for $i = 1, 2$. Note that we may write the equation (3.6) as

$$\frac{(1 + h_{t,i}k_G)}{J_{t,i}} \, \dot{h}_{t,i} = \frac{1}{J_{t,i}} \partial_\sigma \left(\frac{1}{J_{t,i}} \partial_\sigma \left((g(\nu_{F_{t,i}}) k_{F_{t,i}}) \circ \pi_{F_{t,i}}^{-1} + f_i \right) \right) \qquad \text{on } \partial G \tag{3.32}$$

for $i = 1, 2$ respectively, where $J_{t,i} = \sqrt{(1 + h_{t,i}k_G)^2 + (\partial_\sigma h_{t,i})^2}$. To simplify the notation we write $g_{t,i}$ and $k_{t,i}$ in place of $g(\nu_{F_{t,i}})$ and $k_{F_{t,i}}$, respectively. Note that by the $C^{2,\alpha}$ bounds on $h_{t,2}$ and $h_{t,1}$, and by the expressions (2.5) and (2.6) we may estimate

$$|g_{t,2}(\pi_{F_{t,2}}^{-1}(x)) - g_{t,1}(\pi_{F_{t,1}}^{-1}(x))| \le C \left(|\partial_\sigma (h_{t,2} - h_{t,1})(x)| + |(h_{t,2} - h_{t,1})(x)| \right) \tag{3.33}$$

for every $x \in \partial G$. Moreover from the expression (2.7), from the $C^{2,\alpha}$-bounds on h_2 and h_1, and from the ellipticity assumptions on φ we deduce that there are positive constants c and C such that

$$\begin{aligned} &[k_{t,2}(\pi_{F_{t,2}}^{-1}) - k_{t,1}(\pi_{F_{t,1}}^{-1})] \, \partial_{\sigma\sigma}(h_{t,2} - h_{t,1}) \\ &\qquad \le -c|\partial_{\sigma\sigma}(h_{t,2} - h_{t,1})|^2 + C \left(|\partial_\sigma (h_{t,2} - h_{t,1})|^2 + |(h_{t,2} - h_{t,1})|^2 \right) \end{aligned} \tag{3.34}$$

on ∂G.

Multiply the equation (3.32) by $h_{t,2} - h_{t,1}$ for $i = 1, 2$, integrate over ∂G and integrate by parts twice to get

$$\begin{aligned} &\int_{\partial G} \dot{h}_{t,i} \, (h_{t,2} - h_{t,1}) \, d\mathcal{H}^1 \\ &\quad = \int_{\partial G} \partial_\sigma \left(\frac{1}{J_{t,i}} \partial_\sigma \left((g_{t,i} \, k_{t,i}) \circ \pi_{F_{t,i}}^{-1} + f_i \right) \right) \left(\frac{1}{1 + h_{t,i}k_G} (h_{t,2} - h_{t,1}) \right) d\mathcal{H}^1 \\ &\quad = \int_{\partial G} \left((g_{t,i} \, k_{t,i}) \circ \pi_{F_{t,i}}^{-1} + f_i \right) \partial_\sigma \left(\frac{1}{J_{t,i}} \partial_\sigma \left(\frac{1}{1 + h_{t,i}k_G} (h_{t,2} - h_{t,1}) \right) \right) d\mathcal{H}^1. \end{aligned}$$

Substract the equation for $i = 1$ from the equation for $i = 2$ to get

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \int_{\partial G} (h_{t,2} - h_{t,1})^2 \, d\mathcal{H}^1 \right) &= \int_{\partial G} (h_{t,2} - h_{t,1}) \, (\dot{h}_{t,2} - \dot{h}_{t,1}) \, d\mathcal{H}^1 \\ &= \int_{\partial G} ((g_{t,2} \, k_{t,2}) \circ \pi_{F_{t,2}}^{-1} + f_2) \, \partial_\sigma \left(\frac{1}{J_{t,2}} \partial_\sigma \left(\frac{1}{1 + h_{t,2}k_G} (h_{t,2} - h_{t,1}) \right) \right) d\mathcal{H}^1 \\ &\quad - \int_{\partial G} ((g_{t,1} \, k_{t,1}) \circ \pi_{F_{t,1}}^{-1} + f_1) \, \partial_\sigma \left(\frac{1}{J_{t,1}} \partial_\sigma \left(\frac{1}{1 + h_{t,1}k_G} (h_{t,2} - h_{t,1}) \right) \right) d\mathcal{H}^1. \end{aligned}$$

By the $C^{2,\alpha}$-bounds on $h_{t,2}, h_{t,1}$, by $C^{0,\alpha}$-bounds on f_2, f_1, by (2.7) and by (3.33) and (3.34) we conclude that there are positive constants c and C such that

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int_{\partial G}(h_{t,2}-h_{t,1})^2\,d\mathcal{H}^1\right)+c\int_{\partial G}|\partial_{\sigma\sigma}(h_{t,2}-h_{t,1})|^2\,d\mathcal{H}^1\\
&\qquad\leq C\int_{\partial G}(f_2-f_1)^2\,d\mathcal{H}^1+C\int_{\partial G}|\partial_\sigma(h_{t,2}-h_{t,1})|^2+(h_{t,2}-h_{t,1})^2\,d\mathcal{H}^1.
\end{aligned}
\tag{3.35}
$$

Denote $w_t(x) := h_{t,2}(x) - h_{t,1}(x)$. By interpolation we have

$$
\|\partial_\sigma w_t\|_{L^2}^2 \leq C\,\|\partial_{\sigma\sigma} w_t\|_{L^2}\|w_t\|_{L^2} + C\|w_t\|_{L^2}^2 \leq \varepsilon\|\partial_{\sigma\sigma} w_t\|_{L^2}^2 + C_\varepsilon\|w_t\|_{L^2}^2.
$$

Hence, for ε small enough, we obtain by (3.35) that

$$
\frac{d}{dt}\left(\int_{\partial G} w_t^2\,\mathcal{H}^1\right) \leq C\int_{\partial G} w_t^2\,d\mathcal{H}^1 + C\int_{\partial G}(f_2-f_1)^2\,d\mathcal{H}^1. \tag{3.36}
$$

Take $T \in (0, T_0)$. Recall that $w_0 \equiv 0$. Therefore integrating (3.36) over $(0,t)$, with $t \in (0,T)$, implies

$$
\int_{\partial G} w_t^2\,d\mathcal{H}^1 \leq C\int_0^T\int_{\partial G} w_s^2\,d\mathcal{H}^1 ds + C\int_0^T\int_{\partial G}(f_2-f_1)^2\,d\mathcal{H}^1 ds. \tag{3.37}
$$

Integrating (3.37) over $(0,T)$ yields

$$
\int_0^T\int_{\partial G} w_t^2\,d\mathcal{H}^1 dt \leq CT\int_0^T\int_{\partial G} w_t^2\,d\mathcal{H}^1 dt + CT\int_0^T\int_{\partial G}(f_2-f_1)^2\,d\mathcal{H}^1 dt.
$$

Therefore (3.31) follows by taking $T \in (0, T_0)$ sufficiently small.

Step 2. Fix $M_0 > 2\|Q(E(u_G))\|_{C^{1,\alpha}(\partial G)}$, $\mu \in (0,1)$, and let T, δ_0 be as in Step 1. Define the function space

$$
\mathcal{C} := \Big\{ f \in L^2(0,T;L^2(\partial G)) \,:\, \sup_{(0,T)} \|f\|_{C^{1,\alpha}} \leq M_0,\ \int_0^T \|\dot{f}_t\|_{H^{-\frac{1}{2}}(\partial G)}^2 \leq M_0 \Big\}.
$$

We want to show that for every $f \in \mathcal{C}$ equation (3.6) has a solution in the interval $(0,T)$, and that (3.31) holds for f_1, $f_2 \in \mathcal{C}$.

To this end, fix $h_0 \in H^3(\partial G)$ satisfying $\|h_0\|_{H^3(\partial G)} \leq M_0$, $\|h_0\|_{L^2(\partial G)} < \delta_0$, and let $f \in \mathcal{C}$. Consider a sequence $f_n \in \mathcal{C} \cap C^\infty(\partial G \times [0,+\infty))$ such that $f_n \to f$ in $L^2(0,T;L^2(\partial G))$ and a sequence of smooth functions h_n such that $\|h_n\|_{H^3(\partial G)} \leq M_0$ and $h_n \to h_0$ in $L^2(\partial G)$. Denote by $F_{t,n}$ the solution of (3.6) with forcing term f_n and initial datum h_n, and let $h_{t,n}$ be the function on ∂G such that $\partial F_{t,n} = \{x + h_{t,n}(x)\nu_G(x) : x \in \partial G\}$.

Observe that from (3.17) we have that

$$
\sup_n \sup_{(0,T)} \big(\|h_{t,n}\|_{C^{2,\alpha}(\partial G)} + \|\partial_\sigma R_{t,n}\|_{L^2(\partial F_t)}\big) < +\infty,
$$

where $R_{t,n}$ is defined as in (3.7) with f replaced by f_n. In turn (3.9) yields that $R_{t,n}$ is uniformly bounded in $L^2(0,T;H^3(\partial G))$ and thus $h_{t,n}$ is uniformly bounded in $H^1(0,T;H^1(\partial G))$. Therefore, up to a (not relabelled) subsequence, we may assume that $h_{t,n} \rightharpoonup h_t$ weakly in $H^1(0,T;H^1(\partial G))$ and, recalling the uniform $C^{2,\alpha}$ bounds on $h_{t,n}$ we may conclude that in fact $h_{t,n} \to h_t$ in $C^{2,\beta}(\partial G)$ for all $\beta \in (0,\alpha)$ and for all $t \in (0,T)$ and thus $R_{t,n} \circ \pi_{F_{t,n}}^{-1} \to R_t \circ \pi_{F_t}^{-1}$ in $C^{0,\beta}(\partial G)$, where F_t is the set corresponding to h_t. It is now easy to see that the equation passes to the limit and F_t is a solution of (3.6) with initial datum h_0 and forcing term f.

Note also that the same approximation argument yields that (3.31) holds true also in the case where $f_1,\ f_2 \in \mathcal{C}$, so that in particular the solution is unique also in this case. Moreover, again by approximation, the conclusions of Lemmas 3.5 and 3.7 remain true.

Step 3. Fix h_0 as in Step 2 and consider the map $\mathcal{T}:\mathcal{C}\to\mathcal{C}$ defined by $Tf(\cdot,t) = -Q(E(u_t))\circ \pi_{F_t}^{-1}$ for all $t\in[0,T)$, where F_t is the solution of (3.6) with initial datum h_0 and forcing term f (and u_t is the elastic equilibrium in $\Omega\setminus F_t$). From (3.24), which holds also in our case thanks to the previous step, it follows that the map is well defined. In order to conclude the proof it is enough to show that $\mathcal{T}$ is a contraction and thus it admits a fixed point. To this aim, with the same notation of Step 1, for any $f_1,\ f_2\in\mathcal{C}$ and for any $\varepsilon>0$ we have

$$\begin{aligned}\int_0^T &\|Q(E(u_{t,1}))\circ\pi_{F_{t,1}}^{-1} - Q(E(u_{t,2}))\circ\pi_{F_{t,2}}^{-1}\|^2_{L^2(\partial G)}\,dt\\
&\le C\int_0^T \|Q(E(u_{t,1}))\circ\pi_{F_{t,1}}^{-1} - Q(E(u_{t,2}))\circ\pi_{F_{t,2}}^{-1}\|^2_{C^{0,\alpha}(\partial G)}\,dt\\
&\le C\int_0^T \|h_{t,1}-h_{t,2}\|^2_{C^{1,\alpha}(\partial G)}\,dt\\
&\le C\int_0^T \big[\|\partial_{\sigma\sigma}(h_{t,1}-h_{t,2})\|^{2\theta}_{L^2}\|h_{t,1}-h_{t,2}\|^{2(1-\theta)}_{L^2} + \|h_{t,1}-h_{t,2}\|^2_{L^2}\big]\,dt\\
&\le \int_0^T \big[\varepsilon\|\partial_{\sigma\sigma}(h_{t,1}-h_{t,2})\|^2_{L^2} + C_\varepsilon\|h_{t,1}-h_{t,2}\|^2_{L^2}\big]\,dt,\end{aligned}$$

where we used (3.21) and (6.3). We use (3.35) and (6.2), argue as in Step 1, to control the last integral in the above chain of inequalities and deduce that there exists $C_1>0$ independent of ε such that

$$\begin{aligned}\int_0^T &\|Q(E(u_{t,1}))\circ\pi_{F_{t,1}}^{-1} - Q(E(u_{t,2}))\circ\pi_{F_{t,2}}^{-1}\|^2_{L^2(\partial G)}\,dt\\
&\le C_1\varepsilon\int_0^T \|f_1-f_2\|^2_{L^2}\,dt + C_\varepsilon\int_0^T\|h_{t,1}-h_{t,2}\|^2_{L^2}\,dt\\
&\le C_1\varepsilon\int_0^T \|f_1-f_2\|^2_{L^2}\,dt + C_\varepsilon\mu\int_0^T\|f_1-f_2\|^2_{L^2}\,dt,\end{aligned}$$

where the last inequality follows from (3.31). The conclusion follows by taking ε and then μ sufficiently small. □

We conclude this section by showing that the solution provided by Theorem 3.2 is in fact classical, namely of class C^∞.

Theorem 3.8. *Under the assumptions of Theorem 3.2 we have* $(h_t)_{t\in(0,T)}\in C^\infty(0,T;C^\infty(\partial G))$.

Proof. As in the proof of Theorem 3.2 we rewrite the equation on ∂G, see (3.32), thus getting

$$\frac{(1+h_tk_G)}{J_t}\,\dot h_t = \frac{1}{J_t}\partial_\sigma\left(\frac{1}{J_t}\partial_\sigma\left((g(\nu_{F_t})k_{F_t})\circ\pi_{F_t}^{-1} - Q_t\right)\right)\qquad\text{on } \partial G, \tag{3.38}$$

where we have set $Q_t := Q(E(u_t))\circ\pi_{F_t}^{-1}$. We divide the proof in four steps.

Step 1. Given $\Delta t\neq 0$, let us subtract the equation above at time t from the same equation at time $t+\Delta t$ and multiply both sides by $h_{t+\Delta t}-h_t$. Then, integrating by part and arguing as in the proof of (3.35) we get, using also Proposition 6.1 to estimate $\|\partial_\sigma(h_{t+\Delta t}-h_t)\|_{L^2(\partial G)}$,

(3.39)
$$\frac{d}{dt}\Big(\frac12\int_{\partial G}(h_{t+\Delta t}-h_t)^2\,d\mathcal{H}^1\Big)+c\int_{\partial G}|\partial_{\sigma\sigma}(h_{t+\Delta t}-h_t)|^2\,d\mathcal{H}^1 \le C\int_{\partial G}(Q_{t+\Delta t}-Q_t)^2\,d\mathcal{H}^1+C\int_{\partial G}(h_{t+\Delta t}-h_t)^2\,d\mathcal{H}^1.$$

Fix now $\alpha\in(0,\frac12)$. Using Proposition 6.1 and the estimate (3.21) with F and G replaced respectively by $F_{t+\Delta t}$ and F_t we have

$$\begin{aligned}\|Q_{t+\Delta t}-Q_t\|_{L^2(\partial G)}&\le C\|Q_{t+\Delta t}-Q_t\|_{C^{0,\alpha}(\partial G)}\le C\|h_{t+\Delta t}-h_t\|_{C^{1,\alpha}(\partial G)}\\&\le C\big(\|\partial_{\sigma\sigma}(h_{t+\Delta t}-h_t)\|^{\vartheta}_{L^2(\partial G)}\|h_{t+\Delta t}-h_t\|^{1-\vartheta}_{L^2(\partial G)}+\|h_{t+\Delta t}-h_t\|_{L^2(\partial G)}\big).\end{aligned}$$

Inserting this inequality in (3.39) we get

$$\frac{d}{dt}\Big(\frac12\int_{\partial G}(h_{t+\Delta t}-h_t)^2\,d\mathcal{H}^1\Big)+c\int_{\partial G}|\partial_{\sigma\sigma}(h_{t+\Delta t}-h_t)|^2\,d\mathcal{H}^1\le C\|h_{t+\Delta t}-h_t\|^2_{L^2(\partial G)}.$$

Then for $\mathcal{L}^1$-a.e. t_0,t_1 with $0<t_0<t_1<T$, integrating the above inequality in (t_0,t_1), we have

$$\begin{aligned}&\|h_{t_1+\Delta t}-h_{t_1}\|^2_{L^2(\partial G)}+\int_{t_0}^{t_1}\|\partial_{\sigma\sigma}(h_{t+\Delta t}-h_t)\|^2_{L^2(\partial G)}\,dt\\&\qquad\le\|h_{t_0+\Delta t}-h_{t_0}\|^2_{L^2(\partial G)}+C\int_{t_0}^{t_1}\|h_{t+\Delta t}-h_t\|^2_{L^2(\partial G)}\,dt.\end{aligned}$$

Finally, dividing both sides of this inequality by $(\Delta t)^2$, letting $\Delta t\to0$ and recalling that $h\in H^1(0,T;H^1(\partial G))$, we conclude that for any time interval $J\subset\subset(0,T)$

$$\sup_{t\in J}\|\dot h_t\|^2_{L^2(\partial G)}+\int_J\|\partial_t(\partial_{\sigma\sigma}h_t)\|^2_{L^2(\partial G)}\,dt<\infty. \tag{3.40}$$

Step 2. We start again by subtracting equation (3.38) at time t from the same equation at time $t+\Delta t$. We now multiply both sides by $\partial_{\sigma\sigma}(h_{t+\Delta t}-h_t)$. Then, arguing as in the proof of (3.39) we get the following estimate

$$\begin{aligned}&\frac{d}{dt}\Big(\frac12\int_{\partial G}(\partial_\sigma(h_{t+\Delta t}-h_t))^2\,d\mathcal{H}^1\Big)+c\int_{\partial G}|\partial_{\sigma\sigma\sigma}(h_{t+\Delta t}-h_t)|^2\,d\mathcal{H}^1\\&\qquad\le C\int_{\partial G}(\partial_\sigma(Q_{t+\Delta t}-Q_t))^2\,d\mathcal{H}^1+C\int_{\partial G}(h_{t+\Delta t}-h_t)^2\,d\mathcal{H}^1.\end{aligned}\tag{3.41}$$

As in the previous step we may estimate, using (3.21) and Proposition 6.1,

$$\begin{aligned}\|\partial_\sigma(Q_{t+\Delta t}-Q_t)\|_{L^2(\partial G)}&\le C\|\partial_\sigma(Q_{t+\Delta t}-Q_t)\|_{C^{0,\alpha}(\partial G)}\le C\|h_{t+\Delta t}-h_t\|_{C^{2,\alpha}(\partial G)}\\&\le C\|\partial_{\sigma\sigma\sigma}(h_{t+\Delta t}-h_t)\|^{\vartheta}_{L^2}\|h_{t+\Delta t}-h_t\|^{1-\vartheta}_{L^2}+C\|h_{t+\Delta t}-h_t\|_{L^2}.\end{aligned}$$

Using this estimate and integrating (3.41) in (t_0, t_1) for $\mathcal{L}^1$-a.e. t_0, t_1 with $0 < t_0 < t_1 < T$, we have

$$\|\partial_\sigma(h_{t_1+\Delta t} - h_{t_1})\|^2_{L^2(\partial G)} + c\int_{t_0}^{t_1} \|\partial_{\sigma\sigma\sigma}(h_{t+\Delta t} - h_t)\|^2_{L^2(\partial G)}\, dt \leq \|\partial_\sigma(h_{t_0+\Delta t} - h_{t_0})\|^2_{L^2(\partial G)} + C\int_{t_0}^{t_1} \|h_{t+\Delta t} - h_t\|^2_{L^2(\partial G)}\, dt$$

Divide both sides of this inequality by $(\Delta t)^2$ and recall that $\partial_{\sigma\sigma\sigma} h_t \in L^2(0,T; L^2(\partial G))$ and that by (3.40) $\partial_t(\partial_\sigma h_t) \in L^2_{loc}(0,T; H^1(\partial G))$. Using this information and letting $\Delta t \to 0$ we conclude that for every interval $J \subset\subset (0,T)$

$$\sup_{t\in J} \|\partial_t(\partial_\sigma h_t)\|^2_{L^2(\partial G)} + \int_J \|\partial_t(\partial_{\sigma\sigma\sigma} h_t)\|^2_{L^2(\partial G)}\, dt < \infty. \tag{3.42}$$

Note that from the previous inequality and by the equation (3.32) we have that for every interval $J \subset\subset (0,T)$

$$\sup_{t\in J} \big(\|\partial_{\sigma\sigma\sigma} R_t\|_{L^2(\partial G)} + \|\partial_{\sigma\sigma\sigma} h_t\|_{L^2(\partial G)}\big) < \infty.$$

In particular, from this inequality we deduce that

$$\sup_{t\in J} \big(\|\partial_\sigma R_t\|_{C^{0,\alpha}(\partial G)} + \|h_t\|_{C^{2,\alpha}(\partial G)}\big) < \infty.$$

In turn, since $\|\partial_\sigma Q_t\|_{C^{0,\alpha}(\partial G)} \leq C(G, \|h_t\|_{C^{2,\alpha}(\partial G)})$, the above inequality implies immediately that

$$\sup_{t\in J} \|h_t\|_{C^{3,\alpha}(\partial G)} < \infty \tag{3.43}$$

Step 3. At this point we would like to continue as before, subtracting the equations (3.38) at times $t + \Delta t$ an t and multiplying the resulting difference by $\partial_{\sigma\sigma\sigma\sigma}(h_{t+\Delta t} - h_t)$. However this argument only works provided we know that $h \in L^2_{loc}(0,T; H^4(\partial G))$.

To prove this property of h we go back to equation (3.38) and, denoting by s the arclength on ∂G, we subtract the equation for h from the same equation for $h(\cdot + \Delta s)$, where Δs is a non zero increment of the arclength. Then we multiply both sides by $\partial_{\sigma\sigma} h(\cdot + \Delta s) - \partial_{\sigma\sigma} h$ to deduce with the usual calculations that

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\partial G} (\partial_\sigma(h_t(\cdot{+}\Delta s) - h_t))^2\, d\mathcal{H}^1\right) + c\int_{\partial G} |\partial_{\sigma\sigma\sigma}(h_t(\cdot + \Delta s) - h_t)|^2\, d\mathcal{H}^1 \leq C\int_{\partial G} (\partial_\sigma(Q_t(\cdot + \Delta s) - Q_t))^2\, d\mathcal{H}^1 + C\int_{\partial G} (h_t(\cdot + \Delta s) - h_t)^2\, d\mathcal{H}^1.$$

As before we estimate

$$\|\partial_\sigma(Q_t(\cdot + \Delta s) - Q_t)\|_{L^2(\partial G)} \leq C\|h_t(\cdot + \Delta s) - h_t)\|_{C^{2,\alpha}(\partial G)} \leq C\|\partial_{\sigma\sigma\sigma}(h_t(\cdot + \Delta s) - h_t)\|^{\vartheta}_{L^2} \|h_t(\cdot + \Delta s) - h_t\|^{1-\vartheta}_{L^2} + C\|h_t(\cdot + \Delta s) - h_t\|_{L^2}$$

so to obtain that for $\mathcal{L}^1$-a.e. t_0, t_1 with $0 < t_0 < t_1 < T$

$$\|\partial_\sigma(h_{t_1}(\cdot + \Delta s) - h_{t_1})\|^2_{L^2(\partial G)} + \int_{t_0}^{t_1} \|\partial_{\sigma\sigma\sigma}(h_t(\cdot + \Delta s) - h_t)\|^2_{L^2(\partial G)}\, dt \leq \|\partial_\sigma(h_{t_0}(\cdot + \Delta s) - h_{t_0})\|^2_{L^2(\partial G)} + C\int_{t_0}^{t_1} \|h_t(\cdot + \Delta s) - h_t\|^2_{L^2(\partial G)}\, dt.$$

Thus, we may conclude that for every interval $J \subset\subset (0,T)$

$$\sup_{t\in J} \|\partial_{\sigma\sigma} h_t\|^2_{L^2(\partial G)} + \int_J \|\partial_{\sigma\sigma\sigma\sigma} h_t\|^2_{L^2(\partial G)}\, dt < \infty.$$

We now use this estimate, together with the estimate (3.43) obtained in the previous step, in order to show (3.44) below.

To this end, we subtract equation (3.38) at time t from the same equation at time $t+\Delta t$ and multiply both sides by $\partial_{\sigma\sigma\sigma\sigma}(h_{t+\Delta t} - h_t)$ and we obtain

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int_{\partial G} (\partial_{\sigma\sigma}(h_{t+\Delta t}-h_t))^2\, d\mathcal{H}^1\right) + c\int_{\partial G} |\partial_{\sigma\sigma\sigma\sigma}(h_{t+\Delta t} - h_t)|^2\, d\mathcal{H}^1 \\ \leq C\int_{\partial G} (\partial_{\sigma\sigma}(Q_{t+\Delta t} - Q_t))^2\, d\mathcal{H}^1 + C\int_{\partial G} (h_{t+\Delta t} - h_t)^2\, d\mathcal{H}^1.\end{aligned}$$

Then using, (3.43), (3.21) and Proposition 6.1, we have

$$\begin{aligned}\|\partial_{\sigma\sigma}(Q_{t+\Delta t} - Q_t)\|_{L^2(\partial G)} &\leq C\|\partial_{\sigma\sigma}(Q_{t+\Delta t} - Q_t)\|_{C^{0,\alpha}(\partial G)} \leq C\|h_{t+\Delta t} - h_t\|_{C^{3,\alpha}(\partial G)} \\ &\leq C\|\partial_{\sigma\sigma\sigma}(h_{t+\Delta t} - h_t)\|^{\vartheta}_{L^2}\|h_{t+\Delta t} - h_t\|^{1-\vartheta}_{L^2} + C\|h_{t+\Delta t} - h_t\|_{L^2}.\end{aligned}$$

Then, arguing as in the proof of (3.42), we get

$$\sup_{t\in J} \|\partial_t(\partial_{\sigma\sigma} h_t)\|^2_{L^2(\partial G)} + \int_J \|\partial_t(\partial_{\sigma\sigma\sigma\sigma} h_t)\|^2_{L^2(\partial G)}\, dt < \infty. \tag{3.44}$$

Then, arguing as in the proof of (3.43) we have that

$$\sup_{t\in J} \|h_t\|_{C^{4,\alpha}(\partial G)} < \infty$$

At this point we proceed by induction, obtaining at each step first an increment in the space regularity and then the corresponding estimate with respect to time. More precisely, for every interval $J \subset\subset (0,T)$ and every integer $k \geq 2$ we first have that

$$\sup_{t\in J} \|\partial^k_\sigma h_t\|^2_{L^2(\partial G)} + \int_J \|\partial^{k+2}_\sigma h_t\|^2_{L^2(\partial G)}\, dt < \infty$$

Then from this we deduce that again for every interval $J \subset\subset (0,T)$

$$\sup_{t\in J} \|\partial_t(\partial^k_\sigma h_t)\|^2_{L^2(\partial G)} + \int_J \|\partial_t(\partial^{k+2}_\sigma h_t)\|^2_{L^2(\partial G)}\, dt < \infty$$

and in turn that

$$\sup_{t\in J} \|h_t\|_{C^{k+2,\alpha}(\partial G)} < \infty$$

In conclusion this proves that $h \in W^{1,\infty}_{loc}(0,T; C^\infty(\partial G))$.

Step 4. Let us now show the full regularity of h with respect to time. As in Step 1 we fix $\Delta t \neq 0$ and subtract equation (3.38) from the same equation at time $t+\Delta t$. However, differently from before, we multiply both sides of this difference by $\dot h_{t+\Delta t} - \dot h_t$. Then, a simple use of Young's inequality and Proposition 6.1 yields

$$\begin{aligned}\int_{\partial G} (\dot h_{t+\Delta t} - \dot h_t)^2\, d\mathcal{H}^1 &\leq C\int_{\partial G} |\partial_{\sigma\sigma\sigma\sigma}(h_{t+\Delta t} - h_t)|^2\, d\mathcal{H}^1 \\ &\quad + C\int_{\partial G} (\partial_{\sigma\sigma}(Q_{t+\Delta t} - Q_t))^2\, d\mathcal{H}^1 + C\int_{\partial G} (h_{t+\Delta t} - h_t)^2\, d\mathcal{H}^1.\end{aligned} \tag{3.45}$$

Then we estimate as usual

$$\begin{aligned}\|\partial_{\sigma\sigma}(Q_{t+\Delta t}-Q_t)\|_{L^2} &\le C\|\partial_{\sigma\sigma}(Q_{t+\Delta t}-Q_t)\|_{C^{0,\alpha}} \le C\|h_{t+\Delta t}-h_t\|_{C^{3,\alpha}}\\ &\le C\|\partial_{\sigma\sigma\sigma\sigma}(h_{t+\Delta t}-h_t)\|^{\vartheta}_{L^2}\|h_{t+\Delta t}-h_t\|^{1-\vartheta}_{L^2} + C\|h_{t+\Delta t}-h_t\|_{L^2}.\end{aligned}$$

Thus, from (3.45) one gets

$$\int_{\partial G}(\dot h_{t+\Delta t}-\dot h_t)^2\,d\mathcal{H}^1 \le C\int_{\partial G}|\partial_{\sigma\sigma\sigma\sigma}(h_{t+\Delta t}-h_t)|^2\,d\mathcal{H}^1 + C\int_{\partial G}(h_{t+\Delta t}-h_t)^2\,d\mathcal{H}^1.$$

Dividing this inequality by $(\Delta t)^2$ and recalling what was proved in Step 3 we conclude that for every interval $J \subset\subset (0,T)$

$$\sup_{t\in J}\|\partial_{tt}h_t\|^2_{L^2(\partial G)} < \infty.$$

Similarly, differentiating k times the equation (3.38) and arguing as before we conclude that indeed for every integer k and for every interval $J \subset\subset (0,T)$

$$\sup_{t\in J}\|\partial_{tt}(\partial^k_\sigma h_t)\|^2_{L^2(\partial G)} < \infty.$$

Then we have that $h \in W^{2,\infty}_{loc}(0,T;C^\infty(\partial G))$. Finally, differentiating (3.38) with respect to t and repeating the same argument as before we end up by proving that $h \in W^{k,\infty}_{loc}(0,T;C^\infty(\partial G))$ for every integer $k \ge 2$. This concludes the proof. □

4. Asymptotic Stability

In this section we address the long-time behavior of the flow for a special class of initial data.

To this aim, we start by noticing that if G is stationary, then a standard bootstrap argument shows that in fact G is of class C^∞. Moreover, by the results in [32] G turns out to be analytic. Recall also that the definition of stationary set is weaker than the notion of criticality, where one requires the first variation to be constant on the whole ∂G (see Remark 2.5).

However, the above definition fits better in our framework, since during the evolution there is no mass transfer from one Jordan component to the other. More precisely, denoting as before by $F_{t,i}$ the bounded open set enclosed by the i-th connected component $\Gamma_{F_t,i}$ of ∂F_t, one has that the area $|F_{t,i}|$ is preserved during the flow. Indeed, one has

$$\frac{d}{ds}|F_{t+s,i}|_{|s=0} = \int_{\Gamma_{F_t,i}} V_t\,d\mathcal{H}^1 = \int_{\Gamma_{F_t,i}}\partial_{\sigma\sigma}R_t\,d\mathcal{H}^1 = 0. \tag{4.1}$$

We are now ready to state the main result of this section.

Theorem 4.1. *Let $G \subset\subset \Omega$ be a regular strictly stable stationary set in the sense of Definition 2.6 and fix $M > 0$, $\alpha \in (0,1)$. There exists $\delta_0 > 0$ with the following property: Let $F_0 \in \mathfrak{h}^{2,\alpha}_M(\partial G)$ be such that*

$$|F_0 \Delta G| < \delta_0, \qquad \text{and} \qquad \int_{\partial F_0}(\partial_\sigma R_0)^2\,d\mathcal{H}^1 < \delta_0,$$

where $R_0 := g(\nu_{F_0})k_{F_0} - Q(E(u_{F_0}))$ on ∂F_0. Then the unique solution $(F_t)_{t>0}$ of the flow (3.1) with intial datum F_0 is defined for all times $t > 0$.

Moreover $F_t \to F_\infty$ H^3-exponentially fast, where F_∞ is the unique stationary set in $\mathfrak{h}^{2,\alpha}_{\sigma_1}(\partial G)$ (see Proposition 2.9) such that $|F_{\infty,i}| = |F_{0,i}|$ for $i = 1,\dots,m$. In particular, if $|F_{0,i}| = |G_i|$ for $i = 1,\dots,m$, then $F_t \to G$ H^3-exponentially fast.

Here $(F_{\infty,i})_{i=1,\dots,m}$ *and* $(F_{0,i})_{i=1,\dots,m}$ *denote the open sets enclosed by the connected components* $(\Gamma_{F_\infty,i})_{i=1,\dots,m}$ *of* ∂F_∞ *and* $(\Gamma_{F_0,i})_{i=1,\dots,m}$ *of* ∂F_0*, respectively, numbered according to* (2.19)*.*

Remark 4.2. In the previous statement by H^3 exponential convergence of F_t to F_∞ we mean precisely the following: writing $\partial F_t := \{x + \tilde h_t(x)\nu_{F_\infty}(x) : x \in \partial F_\infty\}$, we have

$$\|\tilde h_t\|_{H^3(\partial F_\infty)} \le Ce^{-ct}.$$

for suitable constants $C, c > 0$.

For an example of strictly stable set G to which Theorem 4.1 applies we refer to [14].

In order to proof the theorem, we need the following preliminary energy identities.

Proposition 4.3. *Let* $(F_t)_{t\in(0,T)}$ *solve* (3.1)*. Then we have:*

$$\frac{d}{dt}J(F_t) = -\int_{\partial F_t} (\partial_\sigma R_t)^2 \, d\mathcal{H}^1$$

and

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\partial F_t} (\partial_\sigma R_t)^2 \, d\mathcal{H}^1\right) = -\partial^2 J[\partial_{\sigma\sigma}R_t] - \frac{1}{2}\int_{\partial F_t} k_t\,(\partial_\sigma R_t)^2\,\partial_{\sigma\sigma}R_t\,d\mathcal{H}^1.$$

Proof. The first identity follows immediately recalling that $R_t = g(\nu_t)k_t - Q(E(u_t))$ is the first variation of the energy at $J(F_t)$, and thus

$$\frac{d}{dt}J(F_t) = \int_{\partial F_t} R_t(X_t\cdot\nu_t)\,d\mathcal{H}^1 = \int_{\partial F_t} R_t\,\partial_{\sigma\sigma}R_t\,d\mathcal{H}^1 = -\int_{\partial F_t}(\partial_\sigma R_t)^2\,d\mathcal{H}^1.$$

For the second identity we note that the calculations leading to (3.15) still apply with $f_t\circ\pi$ replaced by $-Q(E(u_t))$ on ∂F_t. Hence we have

(4.2)
$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int_{\partial F_t}(\partial_\sigma R_t)^2\,d\mathcal{H}^1\right) &= -\int_{\partial F_t} g(\nu_t)(\partial_{\sigma\sigma\sigma}R_t)^2\,d\mathcal{H}^1 + \int_{\partial F_t}\partial_{\sigma\sigma}R_t\frac{\partial}{\partial t}\left(Q(E(u(\cdot,t)))\right)d\mathcal{H}^1\\ &\quad + \int_{\partial F_t} g(\nu_t)k_t^2(\partial_{\sigma\sigma}R_t)^2\,d\mathcal{H}^1 + \int_{\partial F_t}\partial_{\nu_t}(Q(E(u_t)))(\partial_{\sigma\sigma}R_t)^2\,d\mathcal{H}^1\\ &\quad - \frac{1}{2}\int_{\partial F_t} k_t\,(\partial_\sigma R_t)^2\partial_{\sigma\sigma}R_t\,d\mathcal{H}^1.\end{aligned}$$

In order to conclude, we need to show that

$$\int_{\partial F_t}\partial_{\sigma\sigma}R_t\frac{\partial}{\partial t}\left(Q(E(u(\cdot,t)))\right)d\mathcal{H}^1 = 2\int_{\Omega\setminus F_t} Q(E(\dot u_t))\,dx$$

to recognize the quadratic form $-\partial^2 J[\partial_{\sigma\sigma}R_t]$ in the four first terms of (4.2). To this aim, observe that since $\mathbb{C}E(u_t)[\nu_t] = 0$ on ∂F_t, we have

$$\begin{aligned}\int_{\partial F_t}\partial_{\sigma\sigma}R_t\frac{\partial}{\partial t}\left(Q(E(u(\cdot,t)))\right)d\mathcal{H}^1 &= \int_{\partial F_t} R_{\sigma\sigma}\mathbb{C}E(u_t) : E(\dot u_t)\,d\mathcal{H}^1 = \int_{\partial F_t} R_{\sigma\sigma}\mathbb{C}E(u_t) : D\dot u_t\,d\mathcal{H}^1\\ &= \int_{\partial F_t}\partial_{\sigma\sigma}R_t\mathbb{C}E(u_t) : D_\tau\dot u_t\,d\mathcal{H}^1 = -\int_{\partial F_t}\operatorname{div}_\tau(\partial_{\sigma\sigma}R_t\mathbb{C}E(u_t))\cdot\dot u_t\,d\mathcal{H}^1\\ &= 2\int_{\Omega\setminus F_t} Q(E(\dot u_t))\,dx,\end{aligned}$$

where the last equality follows by choosing $\varphi = \dot{u}_t$ as a test function in the equation (3.22). □

Proof of Theorem 4.1. Throughout the proof, C will denote a constant depending only on the $C^{2,\alpha}$-bounds on the boundary of the set G. Here we always assume that $\alpha < 1/2$ and the value of C may change from line to line. For any set $F \in \mathfrak{h}^{2,\alpha}_M(G)$ consider

$$D(F) := \int_{F\Delta G} \operatorname{dist}(x, \partial G)\, dx \tag{4.3}$$

and note that

$$|F\Delta G| \leq C\|h_F\|_{L^2(\partial G)} \leq C\sqrt{D(F)}$$

for constants depending only on G. Recall that h_F is the function such that

$$\partial F = \{x + h_F(x)\nu_G(x) : x \in \partial G\}.$$

For every $\varepsilon_1 > 0$ sufficiently small, there exists $\delta_1 \in (0,1)$ so small that for any set $F \in \mathfrak{h}^{2,\alpha}_M(G)$ the following implications hold true:

$$F \in \mathfrak{h}^{2,\alpha}_M(G) \text{ and } D(F) \leq \delta_1 \Longrightarrow \|h_F\|_{C^{1,\alpha}(\partial G)} \leq \frac{\varepsilon_1}{2}\,, \tag{4.4}$$

and

$$\|h_F\|_{C^{1,\alpha}(\partial G)} \leq \varepsilon_1 \text{ and } \int_{\partial F} (\partial_\sigma R_F)^2\, d\mathcal{H}^1 \leq 1 \Longrightarrow \|h_F\|_{C^{2,\alpha}(\partial G)} \leq \omega(\varepsilon_1) \leq \sigma_1 \wedge M, \tag{4.5}$$

where ω is a positive non-decreasing function such that $\omega(\varepsilon_1) \to 0$ as $\varepsilon_1 \to 0^+$, and σ_1 is the constant provided by Proposition 2.9. Here R_F stands, as usual, for $g(\nu_F)k_F - Q(E(u_F))$ on ∂F.

Fix ε_1, $\delta_1 \in (0,1)$ satisfying (4.4) and (4.5) and choose an initial set $F_0 \in \mathfrak{h}^{2,\alpha}_M(G)$ such that

$$D(F_0) \leq \delta_0 \qquad \text{and} \qquad \int_{\partial F_0} (\partial_\sigma R_0)^2\, d\mathcal{H}^1\, dx \leq \delta_0\,, \tag{4.6}$$

where the choice of $\delta_0 < \delta_1$ will be made later. Here, we denote R_0 instead of R_{F_0}.

Let $(F_t)_{t\in(0,T(F_0))}$ the unique classical solution of the flow (3.1) provided by Theorem 3.2. Here $T(F) \in (0,+\infty]$ stands for the maximal time of existence of the classical solution starting from F. By the same theorem, there exists $\delta > 0$ and $T_0 > 0$ such that

$$T(F) \geq T_0 \quad \text{for all } F \subset\subset \Omega \text{ s.t. } \|h_F\|_{L^2(\partial G)} \leq \delta \text{ and } \|h_F\|_{H^3(\partial G)} \leq 1. \tag{4.7}$$

Without loss of generality, in what follows we may also assume δ_1 to be so small that $D(F) \leq \delta_1$ implies $\|h_F\|_{L^2(\partial G)} \leq \delta$, with δ as in (4.7).

We now split the rest of the proof into two steps.

Step 1. *(Stopping-time)* Let $\bar{t} \leq T(F_0)$ be the maximal time such that

$$\|h_t\|_{C^{1,\alpha}(\partial G)} < \varepsilon_1 \quad \text{and} \quad \int_{\partial F_t} (\partial_\sigma R_t)^2\, d\mathcal{H}^1 < 2\delta_0. \quad \text{for all } t \in (0,\bar{t}), \tag{4.8}$$

Note that such a maximal time is well defined in view of (4.4) and (4.6). We claim that by taking ε_1 and δ_0 smaller if needed, we have $\bar{t} = T(F_0)$. To this aim, assume by contradiction that $\bar{t} < T(F_0)$. Then,

$$\|h_{\bar{t}}\|_{C^{1,\alpha}(\partial G)} = \varepsilon_1 \quad \text{or} \quad \int_{\partial F_{\bar{t}}} (\partial_\sigma R_{\bar{t}})^2\, d\mathcal{H}^1 = 2\delta_0$$

We split the proof into steps, according to the two alternatives above.

Step 1-(a). Assume that

$$\int_{\partial F_{\bar{t}}} (\partial_\sigma R_{\bar{t}})^2 \, d\mathcal{H}^1 = 2\delta_0. \tag{4.9}$$

Since (4.5) holds for h_t for every $t \in (0, \bar{t})$ then by Lemma 2.7 (and the fact that $\sigma_1 < \sigma_0$) we get

$$J(F_t)[\partial_{\sigma\sigma} R_t] \geq \frac{m_0}{2} \|\partial_{\sigma\sigma} R_t\|^2_{H^1(\partial F_t)}.$$

Therefore by Proposition 4.3 we get

$$\begin{aligned}
\frac{d}{dt}\Big(\frac{1}{2}\int_{\partial F_t} (\partial_\sigma R_t)^2 \, d\mathcal{H}^1\Big) &= -J(F_t)[\partial_{\sigma\sigma} R_t] - \frac{1}{2}\int_{\partial F_t} k_t \, (\partial_\sigma R_t)^2 \, \partial_{\sigma\sigma} R_t \, d\mathcal{H}^1 \\
&\leq -\frac{m_0}{2} \|\partial_{\sigma\sigma} R_t\|^2_{H^1(\partial F_t)} + C \int_{\partial F_t} (\partial_\sigma R_t)^2 \, |\partial_{\sigma\sigma} R_t| \, d\mathcal{H}^1 \\
&\leq -\frac{m_0}{2} \int_{\partial F_t} (\partial_{\sigma\sigma\sigma} R_t)^2 d\mathcal{H}^1 + C \int_{\partial F_t} |\partial_\sigma R_t|^3 + |\partial_{\sigma\sigma} R_t|^3 \, d\mathcal{H}^1.
\end{aligned} \tag{4.10}$$

In turn, by Proposition 6.1 and the Poincaré Inequality we get

$$\begin{aligned}
\|\partial_\sigma R_t\|^3_{L^3(\partial F_t)} &\leq C \, \|\partial_{\sigma\sigma\sigma} R_t\|^{\frac{1}{4}}_{L^2(\partial F_t)} \|\partial_\sigma R_t\|^{\frac{11}{4}}_{L^2(\partial F_t)} \\
&\leq C \|\partial_{\sigma\sigma\sigma} R_t\|^2_{L^2(\partial F_t)} \|\partial_\sigma R_t\|_{L^2(\partial F_t)} \\
&\leq C\sqrt{\delta_0} \|\partial_{\sigma\sigma\sigma} R_t\|^2_{L^2(\partial F_t)},
\end{aligned}$$

where the last inequality follows from (4.8). Similarly we get

$$\begin{aligned}
\|\partial_{\sigma\sigma} R_t\|^3_{L^3(\partial F_t)} &\leq C \, \|\partial_{\sigma\sigma\sigma} R_t\|^{\frac{7}{4}}_{L^2(\partial F_t)} \|\partial_\sigma R_t\|^{\frac{5}{4}}_{L^2(\partial F_t)} \\
&\leq C\sqrt{\delta_0} \|\partial_{\sigma\sigma\sigma} R_t\|^2_{L^2(\partial F_t)}.
\end{aligned}$$

Therefore, choosing δ_0 small enough, we deduce from (4.10) that

$$\begin{aligned}
\frac{d}{dt}\Big(\frac{1}{2}\int_{\partial F_t} (\partial_\sigma R_t)^2 \, d\mathcal{H}^1\Big) &\leq \big(-\frac{m_0}{2} + C\sqrt{\delta_0}\big) \int_{\partial F_t} (\partial_{\sigma\sigma\sigma} R_t)^2 \, d\mathcal{H}^1 \\
&\leq -\frac{m_0}{4} \int_{\partial F_t} (\partial_{\sigma\sigma\sigma} R_t)^2 \, d\mathcal{H}^1 \\
&\leq -m_1 \int_{\partial F_t} (\partial_\sigma R_t)^2 \, d\mathcal{H}^1
\end{aligned}$$

for all $t \leq \bar{t}$ and for some $m_1 > 0$. Note that the last inequality above follows from the Poincaré Inequality. Integrating the above differential inequality implies

$$\int_{\partial F_t} (\partial_\sigma R_t)^2 \, d\mathcal{H}^1 \leq e^{-m_1 t} \int_{\partial F_0} (\partial_\sigma R_0)^2 \, d\mathcal{H}^1 \leq \delta_0 \, e^{-m_1 t} \tag{4.11}$$

for every $t \leq \bar{t}$. This contradicts (4.9).

Step 1-(b). Assume now that

$$\|h_{\bar{t}}\|_{C^{1,\alpha}(\partial F)} = \varepsilon_1 \, . \tag{4.12}$$

By (3.8) we have that

$$\frac{d}{dt}D(F_t) \le P(F_t)^{\frac{1}{2}} \left(\int_{\partial F_t} (\partial_\sigma R_t)^2 \, d\mathcal{H}^1 \right)^{\frac{1}{2}}, \tag{4.13}$$

where $D(F_t)$ is defined in (4.3). Therefore we may use (4.11) to estimate

$$\frac{d}{dt}D(F_t) \le C\sqrt{\delta_0 e^{-m_1 t}} \tag{4.14}$$

for every $t \le \bar{t}$. This implies

$$D(F_t) \le D(F_0) + C\sqrt{\delta_0} \le C\sqrt{\delta_0}$$

for every $t \le \bar{t}$. We may choose δ_0 so small enough the above estimate implies $D(F_t) \le \delta_1$ and, in turn, by (4.4), $\|h_t\|_{C^{1,\alpha}(\partial F)} \le \frac{\varepsilon_1}{2}$ for every $t \le \bar{t}$. This contradicts (4.12).

Step 2. *(Global-in-time existence and convergence)* By the previous step we have that as long as the flow is defined, i.e., over $(0, T(F_0))$, the estimates (4.8) hold. In turn, by taking ε_1 (and δ_1, δ_0) smaller if needed, we have $\|h_t\|_{L^2(\partial G)} \le \delta$ and $\|h_t\|_{H^3(\partial G)} \le 1$ for all $t \in (0, T(F_0))$. By (4.7) and a standard continuation argument, we deduce that $(F_t)_t$ is defined for all times, i.e., $T(F_0) = \infty$.

From (4.11) we also deduce that

$$\int_{\partial F_t} (\partial_\sigma R_t)^2 \, d\mathcal{H}^1 \le e^{-m_1 t} \int_{\partial F_0} (\partial_\sigma R_0)^2 \, d\mathcal{H}^1 \le \delta_0 \, e^{-m_1 t} \tag{4.15}$$

for all $t > 0$, and in turn, by (4.5) we have

$$\|h_t\|_{C^{2,\alpha}(\partial G)} \le M \tag{4.16}$$

for all $t > 0$. Therefore, we deduce that there exists $h_\infty \in C^{2,\alpha}(\partial G)$ and a sequence $t_n \to +\infty$ such that

$$h_{t_n} \to h_\infty \qquad \text{in } C^{2,\beta}(\partial G) \text{ for all } \beta < \alpha. \tag{4.17}$$

Moreover, by (4.15) we have $\partial_\sigma R_\infty = 0$, and thus, the set $F_\infty \in \mathfrak{h}^{2,\alpha}_M(G)$ such that

$$\partial F_\infty = \{x + h_\infty(x)\nu_G(x) : x \in \partial G\}$$

is stationary. Recall that for every $t \in [0, +\infty]$, $(\Gamma_{F_t,i})_{i=1,\dots,m}$ denote the connected components of ∂F_t, numbered according to (2.19). Denote also as usual by $F_{t,i}$ the bounded open set enclosed by $\Gamma_{F_t,i}$. Since $|F_{t,i}| = |F_{i,0}|$ for every $t > 0$ by (4.1), taking also into account (4.5) and Proposition 2.9, we deduce that in fact F_∞ is the unique stationary set in $\mathfrak{h}^{2,\alpha}_{\sigma_1}(\partial G)$ such that $|F_{\infty,i}| = |F_{0,i}|$ for $i = 1, \dots, m$.

It remains to show that the whole flow exponentially converges to F_∞. To this aim, define

$$D_\infty(E) := \int_{E \Delta F_\infty} \operatorname{dist}(x, \partial F_\infty) \, dx \, .$$

The same calculations and arguments leading to (4.13) and (4.14) show that

$$\frac{d}{dt}D_\infty(F_t) \le P(F_t)^{\frac{1}{2}} \left(\int_{\partial F_t} (\partial_\sigma R_t)^2 \, d\mathcal{H}^1 \right)^{\frac{1}{2}} \le C\sqrt{\delta_0 e^{-m_1 t}} \tag{4.18}$$

for all $t > 0$. From this inequality it is easy to deduce that $\lim_{t\to+\infty} D_\infty(F_t)$ exists. Thus, by (4.17), $D_\infty(F_t) \to 0$ as $t \to +\infty$. In turn, integrating (4.18) and writing $\partial F_t = \{x + \tilde{h}_t(x)\nu_{F_\infty}(x) : x \in \partial F_\infty\}$ we get

$$\|\tilde{h}_t\|^2_{L^2(\partial F_\infty)} \le CD_\infty(F_t) \le \int_t^{+\infty} C\sqrt{\delta_0 e^{-m_1 s}}, ds \le C\sqrt{\delta_0 e^{-m_1 t}}\,.$$

Since $(\tilde{h}_t)_{t>0}$ are bounded in $C^{2,\alpha}(\partial G)$ by (4.16), we obtain by the above estimate together with standard interpolation that also $\|\tilde{h}_t\|_{C^{2,\beta}(\partial G)} \to 0$ exponentially fast to zero for $\beta < \alpha$. Finally, using also (4.15) and Lemma 3.6 (with $G = F_\infty$), we deduce that $\|\tilde{h}_t\|_{H^3(\partial G)} \to 0$ exponentially fast. □

5. Periodic graphs

In this section we briefly describe how our main results read in the context of evolving periodic graphs.

In this framework, given a (sufficiently regular) non-negative ℓ-periodic function $h : [0,\ell] \to [0,+\infty)$, the free energy associated with it reads

$$J(h) := \int_{\Omega_h} Q(E(u_h))\,dx + \int_{\Gamma_h} \varphi(\nu_{\Omega_h})\,d\mathcal{H}^1, \tag{5.1}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, Γ_h denotes the graph of h, Ω_h is the subgraph of h, i.e.,

$$\Omega_h := \{(x_1, x_2) \in (0,\ell) \times \mathbb{R} : 0 < x_2 < h(x_1)\},$$

and u_h is the elastic equilibrium in Ω_h, namely the solution of the elliptic system

$$\begin{cases} \operatorname{div} \mathbb{C}E(u_h) = 0 & \text{in } \Omega_h, \\ \mathbb{C}E(u_h)[\nu_{\Omega_h}] = 0 & \text{on } \Gamma_h, \\ \nabla u_h(\cdot, x_2) & \text{is } \ell\text{-periodic}, \\ u(x_1, 0) = e_0(x_1, 0)\,, \end{cases} \tag{5.2}$$

for a suitable fixed constant $e_0 \neq 0$. As mentioned already in the introduction, the above energy relates to a variational model for epitaxial growth: The graph Γ_h describes the (free) profile of the elastic films, which occupies the region Ω_h and is grown on a (rigid) and much thicker substrate, while the *mismatch strain* constant e_0 appearing in the Dirichlet condition for u_h at the interface $\{x_1 = 0\}$ between film and substrate measures the mismatch between the characteristic atomic distances in the lattices of the two materials. In this framework, the (local) minimizers of (5.1) under an area constraint on Ω_h describe the equilibrium configurations of epitaxially strained elastic films, see [21, 22, 23, 25] and the reference therein.

In the context of periodic graphs, given an initial ℓ-periodic profile $\bar{h} \in H^3(0,\ell)$ (in short $\bar{h} \in H^3_{\mathrm{per}}(0,\ell)$), we look for a solution $(h_t)_{t\in[0,T)}$ of the following problem:

$$\begin{cases} \frac{1}{J_t}\dot{h}_t = (g(\nu_t)k_t + Q(E(u_t)))_{\sigma\sigma} & \text{on } \Gamma_{h_t} \text{ and for all } t \in (0,T), \\ h_t \text{ is } \ell\text{-periodic} & \text{for all } t \in (0,T), \\ h_0 = \bar{h}\,, \end{cases} \tag{5.3}$$

where we set $J_t := \sqrt{1 + \left|\frac{\partial h_t}{\partial x_1}\right|^2}$, u_t stands for the solution of (5.2), with Ω_{h_t} in place of Ω_h, and we wrote ν_t, k_t instead of $\nu_{\Omega_{h_t}}$ and $k_{\Omega_{h_t}}$, respectively. Note that in the first equation of (5.3) we have $+Q(E(u_t))$ instead of $-Q(E(u_t))$. This is due to the fact that in (5.1) the vector ν_{Ω_h} now point outwards with respect to the elastic body.

Although the setting is slightly different from that of the previous sections, the short-time existence and regularity theory of Section 3 clearly extends also to the present situation, with the same arguments. In this way we improve upon the results of [22], showing that there is no need of a curvature regularization in the case where the anisotropy φ is convex and satisfies the ellipticity condition (2.8). Also the stabilty analysis of Section 4 goes through without changes, thus showing that strictly stable stationary ℓ-periodic configurations are H^3-exponentially stable (in the sense made precise by Remark 4.2).

In the case of flat configurations, that is, of constant profiles $h \equiv a$ for some $a > 0$, and when Q is of the form

$$Q(E) := \mu|E|^2 + \frac{\lambda}{2}(\operatorname{trace} E)^2$$

for some constants $\mu > 0$ and $\lambda > -\mu$ (the so called *Lamé coefficients*), the relation between the a, μ, λ, ℓ, and e_0 (see (5.2)) that guarantees the strict stability of flat configuration $h \equiv a$ with respect to ℓ-periodic perturbations is analytically determined. For the reader's convenience, we recall the results. Consider the *Grinfeld function* K defined by

$$K(s) := \max_{n\in\mathbb{N}} \frac{1}{n} H(ns)\,, \quad s \geq 0\,,$$

where

$$H(s) := \frac{y + (3-4\nu_p)\sinh s \cosh s}{4(1-\nu_p)^2 + s^2 + (3-4\nu_p)\sinh^2 s}\,,$$

and ν_p is the *Poisson modulus* of the elastic material, i.e., $\nu_p := \frac{\lambda}{2(\lambda+\mu)}$.

It turns out that K is strictly increasing and continuous, $K(s) \leq Cs$, and $\lim_{s\to+\infty} K(s) = 1$, for some positive constant C, see [25, Corollary 5.3]. Set $\mathbf{e_1} := (1,0)$ and $\mathbf{e_2} := (0,1)$. Combining [25, Theorem 2.9] and [8, Theorem 2.8] with the results of the previous section, we obtain the following theorem.

Theorem 5.1. *Let $a_{\text{stable}} : (0,+\infty) \to (0,+\infty]$ be defined as $a_{\text{stable}}(\ell) := +\infty$, if $0 < \ell \leq \frac{\pi}{4}\frac{(2\mu+\lambda)\partial_{\mathbf{e_1}\mathbf{e_1}}\varphi(\mathbf{e_2})}{e_0^2\mu(\mu+\lambda)}$, and as the solution of*

$$K\Big(\frac{2\pi a_{\text{stable}}(\ell)}{\ell}\Big) = \frac{\pi}{4}\frac{(2\mu+\lambda)\partial_{\mathbf{e_1}\mathbf{e_1}}\varphi(\mathbf{e_2})}{e_0^2\mu(\mu+\lambda)}\frac{1}{\ell}\,,$$

otherwise. Then $h \equiv a$ is an ℓ-periodic strictly stable stationary configuration for (5.1) if and only if $0 < a < a_{\text{stable}}(\ell)$. In particular, for all $a \in (0, a_{\text{stable}}(\ell))$ there exists $\delta > 0$ such that if $\|\bar{h} - a\|_{H^3_{\text{per}}(0,\ell)} \leq \delta$ and $|\Omega_{\bar{h}}| = a\ell$, then the unique solution $(h_t)_t$ of (5.3) is defined for all times and satisfies

$$\|h_t - a\|_{H^3_{\text{per}}(0,\ell)} \leq Ce^{-ct} \qquad \text{for all } t > 0,$$

for suitable constants $C, c > 0$.

6. Appendix

Let $s \in (0,1)$ and $p \geq 1$. We recall that for a function $f : \mathbb{S}^1 \to \mathbb{R}$ the Gagliardo seminorm $[f]_{s,p}$ is defined as

$$[f]^p_{s,p} := \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|f(x)-f(y)|^p}{|x-y|^{1+sp}}\,dxdy\,.$$

If $s > 0$ and ℓ is the integer part of s, the Sobolev space $W^{s,p}(\mathbb{S}^1)$ is the space of all functions f in $W^{\ell,p}(\mathbb{S}^1)$ such that $[\partial^\ell f]_{s-\ell,p}$ is finite, endowed with the norm $\|f\|^p_{W^{s,p}(\mathbb{S}^1)} := \|f\|^p_{W^{\ell,p}(\mathbb{S}^1)} + [\partial^\ell_\sigma f]^p_{s-\ell,p}$. Here we used the convention $W^{0,p} = L^p$ and $[\partial^\ell_\sigma f]^p_{s-\ell,p} = \|\partial^\ell_\sigma f\|_{L^p(\Gamma)}$. We recall also that for $p = 2$ the seminorm $[\partial^\ell_\sigma f]^p_{s-\ell,p}$ is equivalent to

$$\Big(\sum_{k\in\mathbb{Z}} k^{2s} a_k(f)^2\Big)^{\frac{1}{2}},$$

where $\{a_k(f)\}$ is the sequence of the Fourier coefficients of f with respect to the orthornormal basis $\{(2\pi)^{\frac{1}{2}}\mathrm{e}^{-ikz}\}_{k\in\mathbb{Z}}$. These definitions extend in the obvious way to the case where $\mathbb{S}^1$ is replaced by any regular Jordan curve Γ.

We prove the following interpolation inequality for curves. Note that in the statement we are using and $W^{t,2} = H^t$ for all $t > 0$.

Proposition 6.1. *Let Γ be a regular Jordan curve. Let $m \geq 1$ be an integer, $0 \leq s < m$ and $p \in [2, +\infty)$ such that $s + 1/2 - 1/p < m$. There exists a constant $C > 0$, depending only on m, s, p and on the length of Γ such that for every $f \in H^m(\Gamma)$*

$$\|f\|_{W^{s,p}(\Gamma)} \leq C\big(\|\partial^m_\sigma f\|^\theta_{L^2(\Gamma)}\|f\|^{1-\theta}_{L^2(\Gamma)} + \|f\|_{L^2(\Gamma)}\big), \tag{6.1}$$

where

$$\theta = \frac{s + 1/2 - 1/p}{m}.$$

If s is a positive integer, then

$$||\partial^s_\sigma f||_{L^p(\Gamma)} \leq C\, ||\partial^m_\sigma f||^\theta_{L^2(\Gamma)}||f||^{1-\theta}_{L^2(\Gamma)}, \tag{6.2}$$

with θ as before. The same inequality also holds if $s = 0$, provided that f has zero average.

Finally, if $0 < \alpha < \frac{1}{2}$, there exists θ', depending only on m and α, such that for every $f \in H^m(\Gamma)$

$$\|f\|_{C^{m-1,\alpha}(\Gamma)} \leq C(\|\partial^m_\sigma f\|^{\theta'}_{L^2(\Gamma)}\|f\|^{1-\theta'}_{L^2(\Gamma)} + \|f\|_{L^2(\Gamma)}). \tag{6.3}$$

Proof. It is enough to prove the statement for $\Gamma = \mathbb{S}^1$. The general case will follow by parametrizing Γ by the arclength and by rescaling. Let $t = s + \frac{1}{2} - \frac{1}{p}$. Observe that

$$\Big(\sum^\infty_{k\in\mathbb{Z}} k^{2t} a_k(f)^2\Big)^{\frac{1}{2}} \leq \Big(\sum_{k\in\mathbb{Z}} k^{2m} a_k(f)^2\Big)^{\frac{\theta}{2}}\Big(\sum_{k\in\mathbb{Z}} a_k(f)^2\Big)^{\frac{1-\theta}{2}}, \tag{6.4}$$

and thus (6.1) follows with $\|f\|_{W^{s,p}(\mathbb{S}^1)}$ replaced by $\|f\|_{W^{t,2}(\mathbb{S}^1)}$. The general case follows recalling that $W^{t,2}(\mathbb{S}^1)$ is continuously embedded in $W^{s,p}(\mathbb{S}^1)$, since $t = s + \frac{1}{2} - \frac{1}{p}$ (see [28, Th. 1.4.4.1]). Observe that it is enough to prove (6.2) for functions f with zero average also when s is a positive integer. On the other hand if f has zero average, (6.2) follows from (6.4) and the aforementioned Sobolev Embedding after observing that the $W^{s,p}$-norm of f is equivalent to the L^p-norm of $\partial^s_\sigma f$.

Finally, to prove (6.3) it is enough to assume $m = 1$ and then to argue by induction with respect to m. To this aim, we observe that for every z, $w \in \mathbb{S}^1$

$$|f(z) - f(w)| \leq c|z - w|^{\frac{1}{2}}\|\partial_\sigma f\|_{L^2},$$

for some universal constant c. Then if $0 < \alpha < \frac{1}{2}$

$$|f(z) - f(w)| \leq |f(z) - f(w)|^{2\alpha} |f(z) - f(w)|^{1-2\alpha} \leq c|z - w|^{\alpha} \|\partial_\sigma f\|_{L^2}^{2\alpha} \|f\|_{L^\infty}^{1-2\alpha} .$$

The conclusion follows by estimating $\|f\|_{L^\infty}$ by (6.1), with $m = 1$. □

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