A SHARP STABILITY RESULT
FOR THE GAUSS MEAN VALUE FORMULA

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ABSTRACT. We prove a quantitative stability result for the Gauss mean value formula. We also show by an example that the estimate proved here is sharp.

1. INTRODUCTION

Among the various rigidity properties satisfied by balls one of the best known example is provided by the Gauss mean value formula for harmonic functions. A simple proof of this fact was given by Kuran in [9]. Denoting by D an open set in $\mathbb{R}^n$, $n \geq 2$, and by $\mathcal{H}(D)$ the family of the harmonic functions in D, his result reads as follows:

(Kuran) *Let $D \subset \mathbb{R}^n$ be an open set with finite measure and let $x_0 \in D$ be such that*

$$u(x_0) = \fint_D u(x)\,dx \qquad \forall u \in \mathcal{H}(D) \cap L^1(D). \tag{1.1}$$

Then D is a Euclidean ball centered at x_0.

In view of this harmonic characterization of balls it is natural to raise the question of the stability of the mean value equality (1.1). Roughly speaking, the problem can be stated as follows:

(*) *Let D and x_0 be as above. If $u(x_0)$ is close to $\fint_D u\,dx$ for every $u \in \mathcal{H}(D) \cap L^1(D)$, is it true that D is close to a Euclidean ball centered at x_0?*

To put the previous question in a precise form we have introduced the Gauss mean value gap. Precisely, given an open set D of finite measure and $x_0 \in D$ we define the rescaled *Gauss mean value gap of D relative to x_0* as

$$G(D, x_0) := \sup_{u \in \mathcal{H}(D) \cap L^1(D), u \not\equiv 0} \frac{\left| u(x_0) - \fint_D u(x)\,dx \right|}{\|u\|_{\widetilde{L}^1(D)}}, \tag{1.2}$$

where we have set

$$\|u\|_{\widetilde{L}^1(D)} := \fint_D |u(x)|\,dx.$$

It is easy to verify that $G(D, x_0)$ is translation and scaling invariant. Moreover, using the Gauss mean value property on the ball $B(x_0, r_{x_0})$, where $r_{x_0} = \operatorname{dist}(x_0, \partial D)$, an elementary

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computation shows that

$$G(D,x_0) \leq 1 + \frac{|D|}{|B(x_0,r_{x_0})|} < \infty.$$

A common way to measure the distance of a measurable set $D \subset \mathbb{R}^n$, $|D| < \infty$, from a ball is provided by the so called *Fraenkel asymmetry* which is defined by setting

$$\alpha(D) := \min_{x \in \mathbb{R}^n} \frac{|D \triangle B(x,r_D)|}{|D|},$$

where r_D is the radius of a ball with the same measure of D and

$$|D \triangle B(x,r_D)| := |D \setminus B(x,r_D)| + |B(x,r_D) \setminus D|.$$

By using the Gauss gap G and the Fraenkel asymmetry α, question (*) can be formulated as follows:

$$\textit{is it true that} \quad \lim_{G(D,x_0)\to 0} \alpha(D) = 0\ ? \tag{1.3}$$

In this paper we give a positive answer to this question. Actually we prove a stronger result; i.e, the following stability inequality.

Theorem 1.1. *There exists a constant $C(n)$ such that if $D \subset \mathbb{R}^n$ is an open set of finite measure and $x_0 \in D$, then*

$$\frac{|D \setminus B(x_0,r_{x_0})|}{|D|} \leq C(n)\, G(D,x_0), \tag{1.4}$$

where, as above, $r_{x_0} = \operatorname{dist}(x_0, \partial D)$.

Since $|D \triangle B(x_0,r_D)| = 2|D \setminus B(x_0,r_D)| \leq 2|D \setminus B(x_0,r_{x_0})|$, the stability estimate (1.4) implies that

if D is an open set of finite measure and $x_0 \in D$, then

$$\alpha(D) \leq 2C(n)\, G(D,x_0), \tag{1.5}$$

that trivially implies (1.3). Note also that Theorem 1.1 has Kuran's Theorem as a corollary. In fact, if $G(D,x_0) = 0$ from (1.4) we have that $|D \setminus B(x_0,r_{x_0})| = 0$. Thus, since D contains $B(x_0,r_{x_0})$ and is open, $D = B(x_0,r_{x_0})$.

The estimate from below of the Gauss mean value gap in Theorem 1.1 is sharp in the following sense: if we consider the family of ellipsoids

$$D_\varepsilon := \left\{ x \in \mathbb{R}^n \,:\, (\varepsilon x_1)^2 + x_2^2 + \cdots + x_n^2 < 1 \right\}, \qquad \frac{1}{2} < \varepsilon < 1$$

and denote x_0 the origin, then $r_{x_0} = 1$,

$$\lim_{\varepsilon \to 1} |D_\varepsilon \setminus B(0,1)| = 0$$

and there exists a constant $c > 0$, independent of ε, such that

$$\frac{1}{C(n)} \frac{|D_\varepsilon \setminus B(0,1)|}{|D_\varepsilon|} \leq G(D_\varepsilon, 0) \leq c \frac{|D_\varepsilon \setminus B(0,1)|}{|D_\varepsilon|}. \tag{1.6}$$

The first inequality comes from Theorem 1.1 and the second one is a straightforward consequence of the following continuity-type result for the Gauss mean value gap: the $C^{1,\alpha}$-convergence of domains to a Euclidean ball forces the Gauss gap to go to zero. Precisely,

Theorem 1.2. *Let* $d \in C^{1,\alpha}(B(0,2))$, $\alpha \in]0,1[$, *and let*

$$D := \{x \in B(0,2) \, : \, d(x) < 1\}$$

be such that

$$\partial D = \{x \in B(0,2) \, : \, d(x) = 1\}$$

and

$$B(0,1/2) \subseteq D \subseteq \overline{D} \subseteq B(0,2).$$

We let

$$d_e : \mathbb{R}^n \to \mathbb{R}, \quad d_e(x) := |x|^2.$$

Then there exists a positive constant c, *only depending on* n *and the* $C^{1,\alpha}$*-norm of* d *in* $B(0,2)$, *such that*

$$G(D,0) \leq c \, \|d - d_e\|_{C^{1,\alpha}(B(0,2))} \, . \tag{1.7}$$

Roughly, the above results says that the Gauss mean value gap of domains converging to the unit ball in $C^{1,\alpha}$-norm goes to 0, and it provides a quantitative estimate, see (1.7).

The inequality (1.7) is almost sharp. Indeed, we cannot replace the $C^{1,\alpha}$-norm at the right hand side with a $C^{0,\beta}$-norm, for any $\beta < 1$, or even with a $W^{1,p}$-norm, for any fixed $p < \infty$, as the following result shows.

Theorem 1.3. *For every* $\varepsilon \in]0,1[$ *there exists a family of Lipschitz continuous functions* $d_\varepsilon : \mathbb{R}^n \to [0,\infty[$, *such that, letting*

$$D_\varepsilon = \{x \in \mathbb{R}^n \, : \, d_\varepsilon(x) < 1\}$$

we have

(i) $B(0,1/2) \subseteq D_\varepsilon \subseteq B(0,2)$
(ii) ∂D_ε *piecewise smooth and* $\partial D_\varepsilon = \{x \in \mathbb{R}^n \, : \, d_\varepsilon(x) = 1\}$
(iii) $\|d_\varepsilon - d_e\|_{W^{1,p}(D_\varepsilon)} \to 0$ *as* $\varepsilon \to 0$ *for every* $p > 1$
(iv) $\liminf_{\varepsilon \to 0} G(D_\varepsilon, 0) > 0$.

Now, a few remarks are in order. The stability inequality (1.5) is reminiscent of other stability estimates, such as the quantitative isoperimetric inequality, see [7], see also [6] for the anisotropic case, which states that there exists a constant $c(n)$ such that if $D \subset \mathbb{R}^n$ is a measurable set of finite measure, then

$$\alpha(D)^2 \leq c(n) \bigg(\frac{P(D) - P(B(0,r_D))}{P(B(0,r_D))} \bigg), \tag{1.8}$$

where $P(\cdot)$ stands for the perimeter. Here, the right hand side of (1.8) represents the gap between the perimeter of D and the (minimal) perimeter of the ball with the same volume.

Another important property of balls, the Faber-Krahn inequality, states that they minimize the first Dirichlet eigenvalue of the Laplacian $\lambda(D)$ among all open sets D with the same volume. The quantitative version of this inequality has been recently established in [3], where it is proved that there exists a constant $\kappa(n)$ such that if D is an open set of finite measure then

$$\alpha(D)^2 \leq \kappa(n) \bigg(\frac{\lambda(D) - \lambda(B(0,r_D))}{\lambda(B(0,r_D))} \bigg). \tag{1.9}$$

Note that in (1.8) and (1.9), as well as in many other stability estimates of this kind, the Fraenkel asymmetry always appears with a power 2. In our case the sharp exponent of the Fraenkel asymmetry is 1. Indeed

$$\alpha(D)^\gamma \leq C(n,\gamma)\, G(D,x_0) \qquad \forall \gamma \geq 1 \tag{1.10}$$

with $C(n,\gamma) = 2^\gamma C(n)$, as it immediately follows by (1.5), since $\alpha(D) \leq 2$. On the other hand, our estimate (1.6) straightforwardly implies that inequality (1.10) does not hold for $\gamma < 1$.

We remark that our technique to prove the stability result Theorem 1.1 does not seem suitable to obtain a similar result for the Gauss gap related to the surface average. An interesting stability result in this direction has been obtained in dimension $n = 2$ by Agostiniani and Magnanini in [1], see also [10].

The plan of the paper is the following. In Section 2 we prove the stability result Theorem 1.1. In Section 3 we provide the proofs of the estimates from above of the Gauss mean value gap, precisely, the proofs of Theorem 1.2 and of (1.6). In Section 4 we give the proof of Theorem 1.3.

2. Proof of Theorem 1.1

Let $D \subseteq \mathbb{R}^n$, $n \geq 2$, be an open set with finite Lebesgue measure. Fixed $x_0 \in D$, denote r_{x_0} :=dist$(x_0, \partial D)$. Then there exists $x_1 \in \partial B(x_0, r_{x_0}) \cap \partial D$.

As in Kuran [9], we define $h : D \to \mathbb{R}$,

$$h(x) := 1 + r_{x_0}^{n-2} \frac{|x-x_0|^2 - r_{x_0}^2}{|x-x_1|^n}.$$

Lemma 2.1. *$h \in \mathcal{H}(D) \cap L^1(D)$ and*

$$\|h\|_{L^1(D)} \leq c(n, |D|),$$

where $c(n,|D|)$ denotes a constant only depending on n and $|D|$.

Proof. Let P be the Poisson kernel for the ball $B(0, r_{x_0})$. Then

$$P(x,y) = c(r_{x_0}, n) \frac{r_{x_0}^2 - |x|^2}{|x-y|^n} \quad x \in B(0, r_{x_0}), y \in \partial B(0, r_{x_0}),$$

for some positive constant $c(r_{x_0}, n)$. It is well known that $x \mapsto P(x,y)$ is harmonic in $\mathbb{R}^n \setminus \{y\}$. By a translation argument and taking into account that $x_1 \notin D$, we conclude that $h \in \mathcal{H}(D)$.

Let us prove that $h \in L^1(D)$ and let us estimate its L^1-norm.
Using the inequalities

$$||x-x_0| - r_{x_0}| = ||x-x_0| - |x_1 - x_0|| \leq |x - x_1|,$$

$$|x - x_0| + r_{x_0} \leq |x - x_1| + 2r_{x_0},$$

and taking into account that

$$r_{x_0} \leq \left(\frac{|D|}{\omega_n}\right)^{\frac{1}{n}}, \tag{2.1}$$

where ω_n is the Lebesgue measure of the unit ball of $\mathbb{R}^n$, we get

$$\begin{aligned}\frac{\left||x-x_0|^2-r_{x_0}^2\right|}{|x-x_1|^n} &\le \frac{|x-x_1|(|x-x_1|+2r_{x_0})}{|x-x_1|^n}\\ &\le \frac{1}{|x-x_1|^{n-2}}+\left(\frac{|D|}{\omega_n}\right)^{\frac{1}{n}}\frac{2}{|x-x_1|^{n-1}}.\end{aligned} \tag{2.2}$$

Thus, recalling (2.1), h is summable in $D\cap B(x_1,1)$ and

$$\|h\|_{L^1(D\cap B(x_1,1))}\le c(|D|,n).$$

If $x\notin B(x_1,1)$ then $|x-x_1|\ge 1$, so (2.2) implies

$$\frac{\left||x-x_0|^2-r_{x_0}^2\right|}{|x-x_1|^n}\le 1+2\left(\frac{|D|}{\omega_n}\right)^{\frac{1}{n}} \qquad \forall x\in D\setminus B(x_1,1).$$

Therefore, using (2.1),

$$\int_{D\setminus B(x_1,1)}|h(x)|\,dx\le\left(1+\left(\frac{|D|}{\omega_n}\right)^{\frac{n-2}{n}}+2\left(\frac{|D|}{\omega_n}\right)^{\frac{n-1}{n}}\right)|D|<\infty.$$

We conclude that the L^1-norm of h in D can be estimated by a constant only depending on n and $|D|$. □

We are now ready to prove our main stability result.

Proof of Theorem 1.1. Since $G(D,x_0)=G(D-x_0,0)$, we may assume without loss of generality that $x_0=0$. Moreover, since both $G(D,0)$ and the left hand side of (1.4) are scaling invariant, we may also assume that $|D|=1$.
By Lemma 2.1, $h\in\mathcal{H}(D)\cap L^1(D)$, moreover $h(0)=0$. Therefore

$$G(D,0)\ge\frac{\left|h(0)-⨍_D h(x)\,dx\right|}{\|h\|_{\tilde{L}^1(D)}}=\frac{\left|⨍_D h(x)\,dx\right|}{\|h\|_{\tilde{L}^1(D)}}=\frac{\left|\int_D h(x)\,dx\right|}{\|h\|_{L^1(D)}}.$$

Using $B(0,r_0)\subseteq D$ and the Gauss mean value Theorem, we have

$$\int_D h(x)\,dx=\int_{D\setminus B(0,r_0)}h(x)\,dx+|B(0,r_0)|h(0)=\int_{D\setminus B(0,r_0)}h(x)\,dx.$$

So we have proved that

$$G(D,0)\ge\frac{\left|\int_{D\setminus B(0,r_0)}h(x)\,dx\right|}{\|h\|_{L^1(D)}}.$$

Taking into account that $h(x)\ge 1$ in $D\setminus B(0,r_0)$, we obtain

$$|D\setminus B(0,r_0)|\le G(D,0)\int_D|h(x)|\,dx.$$

We now observe that $r_0\le|B(0,1)|^{-1/n}$ and $|D|=1$. Then Lemma 2.1 immediately implies

$$\int_D|h(x)|\,dx\le C(n)$$

for some positive constant $C(n)$ depending only on the dimension. Hence, (1.4) follows. □

3. Proof of Theorem 1.2 and of inequality (1.6)

Proof of Theorem 1.2. We give a proof in $\mathbb{R}^n$, $n \geq 3$. The case $n = 2$ can be handled exactly in the same way.

We split the proof into steps.

Step I.

We let $\varphi : [0, \infty[\to \mathbb{R}$,

$$\varphi(t) := \frac{n}{n-2} \frac{(\Gamma(1))^{\frac{n}{n-2}}}{(\Gamma(1)+t)^{1+\frac{n}{n-2}}}.$$

Hereafter Γ denotes the fundamental solution of Δ:

$$\Gamma(x) = \Gamma(|x|) := \frac{1}{n(n-2)\omega_n} |x|^{2-n}.$$

A trivial computation shows that

$$\int_0^\infty \varphi(t)\, dt = 1.$$

The function

$$w_D := \varphi(G_D)|\nabla G_D|^2,$$

where G_D stands for $G_D(\cdot, 0)$, the Green function of D with pole at 0, is a density with the mean value property for D at 0; i.e.,

$$u(0) = \int_D u(x) w_D(x)\, dx \qquad \forall u \in \mathcal{H}(D) \cap L^1(D),$$

see Aikawa [2].

Analogously

$$w_B := \varphi(G_B)|\nabla G_B|^2$$

is a density with the mean value property for the Euclidean unit ball B centered at 0.

Since $G_B(x, 0) = \Gamma(x) - \Gamma(1)$, another trivial computation shows that

$$w_B = \frac{1}{\omega_n} = \frac{1}{|B|}.$$

Step II.

Let $\mathcal{U} := \{u \in \mathcal{H}(D) \, : \, ⨍_D |u(x)|\, dx = 1\}$. Then

$$G(D, 0) = \sup_{u \in \mathcal{U}} \left| u(0) - ⨍_D u(x)\, dx \right| = \sup_{u \in \mathcal{U}} \left| \int_D u(x) \left(w_D - \frac{1}{|D|} \right) dx \right|.$$

Hence

$$G(D, 0) \leq |D| \sup_D \left| w_D - \frac{1}{|D|} \right| \leq \omega_n 2^n \sup_D \left| w_D - \frac{1}{|D|} \right|. \tag{3.1}$$

Step III.

Let $\delta := \|d - d_e\|_{C(B(0,2))}$. If $\delta < \frac{1}{2}$, then

$$B(0, \sqrt{1-\delta}) \subseteq D \subseteq B(0, \sqrt{1+\delta}),$$

From these inclusions we easily get

$$\left|\frac{1}{|D|}-\frac{1}{|B|}\right|\le c_n\delta,$$

where $c_n>0$ only depends on n.

On the other hand, if $\delta\ge\frac{1}{2}$, one immediately verifies that

$$\left|\frac{1}{|D|}-\frac{1}{|B|}\right|\le c(n)\delta$$

where $c(n)>0$ only depends on n.

Summing up:

$$\left|\frac{1}{|D|}-\frac{1}{|B|}\right|\le C(n)\|d-d_e\|_{C(B(0,2))}.$$

As a consequence, from inequality (3.1) we get

$$G(D,0)\le\omega_n 2^n\sup_D\left|w_D-\frac{1}{|B|}\right|+c(n)\|d-d_e\|_{C(B(0,2))}. \tag{3.2}$$

Step IV.

In this step we estimate the first term at the right hand side of (3.2). We start by recalling that

$$w_D-\frac{1}{|B|}=\varphi(G_D)|\nabla G_D|^2-\varphi(G_B)|\nabla G_B|^2 \quad [1]$$

We now remark that, in D,

$$G_D=(G_D-G_B)+G_B=(\Gamma(1)-h)+G_B,$$

where h solves the Dirichlet problem

$$\begin{cases}\Delta h=0 & \text{in } D\\ h=\Gamma & \text{on } \partial D.\end{cases}$$

Elementary computations and the Widman's estimates of G_D ([11, Theorem 2.3]) allow to show that

$$\sup_D\left|w_D-\frac{1}{|B|}\right|\le c\left(\sup_D|h-\Gamma(1)|+\sup_D|\nabla(h-\Gamma(1))|\right), \tag{3.3}$$

where c is a positive constant only depending on the dimension n and the $C^{1,\alpha}$-norm of d in $B(0,2)$.

Step V.

In this last step we show that

$$\sup_D|h-\Gamma(1)|+\sup_D|\nabla(h-\Gamma(1))|\le c\|d-d_e\|_{C^{1,\alpha}(B(0,2))}, \tag{3.4}$$

where c only depends on n and the $C^{1,\alpha}$-norm of d in $\overline{B(0,2)}$. This inequality, together with (3.3) and (3.2), will prove (1.7).

To prove (3.4) we first observe that $h-\Gamma(1)$ solves

$$\begin{cases}\Delta(h-\Gamma(1))=0 & \text{in } D\\ h-\Gamma(1)=\Psi(\Gamma-\Gamma(1)) & \text{on } \partial D,\end{cases}$$

[1] Since $G_B=\Gamma-\Gamma(1)$ in B, we agree to extend G_B out of B by letting $G_B=\Gamma-\Gamma(1)$ in $\mathbb{R}^n\setminus B$.

where $\Psi \in C^\infty(\overline{B(0,2)})$ is such that

$$\Psi = 1 \text{ on } \partial D \quad \text{and} \quad \Psi = 0 \text{ in } B(0,1/2).$$

Now, if $x \in \partial D$ (i.e. $d(x) = 1$) we have

$$\begin{aligned}\Psi(x)(\Gamma(x) - \Gamma(1)) &= \Psi(x)\Gamma(x)(1 - |x|^{n-2}) = \frac{\Psi(x)\Gamma(x)}{1 + |x|^{n-2}}(1 - |x|^{2(n-2)}) \\ &= \frac{\Psi(x)\Gamma(x)}{1 + |x|^{n-2}}(d^{n-2}(x) - d_e^{n-2}(x)) = \Phi(x)(d(x) - d_e(x)),\end{aligned}$$

where, for every $x \in B(0,2)$,

$$\Phi(x) = \frac{\Psi(x)\Gamma(x)}{1 + |x|^{n-2}}(d^{n-3}(x) + \cdots + d_e^{n-3}(x)).$$

Obviously $\Phi \in C^{1,\alpha}(B(0,2))$ and $h - \Gamma(1)$ satisfies

$$\begin{cases} \Delta(h - \Gamma(1)) = 0 & \text{in } D \\ h - \Gamma(1) = \Phi(d - d_e) & \text{on } \partial D, \end{cases}$$

Thus, inequality (3.4) follows by Theorem 8.33 in [8].

□

Proof of (1.6). The left inequality comes from Theorem 1.1. We now prove the right inequality, by using Theorem 1.2.
Define $d_\varepsilon : \mathbb{R}^n \to \mathbb{R}$,

$$d_\varepsilon(x) := (\varepsilon x_1)^2 + x_2^2 + x_3^2 + \cdots + x_n^2 \qquad \varepsilon \in \Big]\frac{1}{2}, 1\Big[.$$

Then

$$D_\varepsilon = \{x \in \mathbb{R}^n \ : \ d_\varepsilon < 1\},$$

so that

$$B(0,1) \subseteq D_\varepsilon \subseteq B\big(0, \frac{1}{\varepsilon}\big) \subseteq B(0,2).$$

Of course $d_\varepsilon \in C^\infty(\mathbb{R}^n)$ and

$$d_\varepsilon(x) - d_e(x) = (\varepsilon^2 - 1)x_1^2.$$

Then

$$\|d_\varepsilon - d_e\|_{C^{1,\alpha}(B(0,2))} = C_\alpha(1 - \varepsilon^2), \tag{3.5}$$

where C_α is the $C^{1,\alpha}(B(0,2))$-norm of $x \mapsto x_1^2$. On the other hand,

$$\frac{|D_\varepsilon \setminus B(0,1)|}{|D_\varepsilon|} = 1 - \frac{|B(0,1)|}{|D_\varepsilon|} = 1 - \varepsilon \to_{\varepsilon \to 1^-} 0. \tag{3.6}$$

By Theorem 1.2, (3.5) and (3.6) it immediately follows the right inequality in (1.6). □

4. PROOF OF THEOREM 1.3

Proof of Theorem 1.3. In $\mathbb{R}^n$, $n \geq 2$, let us denote a vector $x \in \mathbb{R}^n$ as $x = (x_1, \hat{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Fixed $\varepsilon \in]0, \frac{1}{n}[$, let us denote $B(\varepsilon)$ the ball centered at $x_\varepsilon := (1+\varepsilon, \hat{0})$ and radius 1; i.e.,

$$B(\varepsilon) := B(x_\varepsilon, 1).$$

Let us consider the truncated cone

$$K := \left\{ x \in \mathbb{R}^n \ : \ \frac{|x|}{\sqrt{n}} < x_1 < \frac{2}{n} \right\} = \left\{ (x_1, \hat{x}) \in \mathbb{R}^n \ : \ \frac{|\hat{x}|}{\sqrt{n-1}} < x_1 < \frac{2}{n} \right\}.$$

Consider the open connected set $D_\varepsilon \subseteq \mathbb{R}^n$ with Lipschitz boundary

$$D_\varepsilon := B(\varepsilon) \cup K \setminus \overline{B(0, \varepsilon^2)} \quad \text{(see fig. 1)}$$

that converges, with respect to the Hausdorff metric, to the Euclidean unit ball centered at $(1, \hat{0})$.

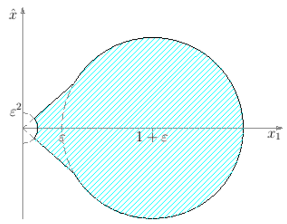

Figure 1. The set D_ε

Let us consider the Gauss gap of D_ε with respect to x_ε; i.e,

$$G(D_\varepsilon, x_\varepsilon) = \sup_{v \in \mathcal{H}(D_\varepsilon), v \not\equiv 0} \frac{\left| \fint_{D_\varepsilon} v(x)\, dx - v(x_\varepsilon) \right|}{\fint_{D_\varepsilon} |v(x)|\, dx}.$$

We claim that

(a) $\liminf_{\varepsilon \to 0} G(D_\varepsilon, x_\varepsilon) > 0$

(b) there exists a Lipschitz function $d_\varepsilon : \mathbb{R}^n \to [0, \infty[$, such that

$$D_\varepsilon = \{x \in \mathbb{R}^n \ : \ d_\varepsilon < 1\}, \quad \partial D_\varepsilon = \{x \in \mathbb{R}^n \ : \ d_\varepsilon = 1\}$$

and, for any $p \in [1, \infty[$,

$$\|d_\varepsilon - d_{\varepsilon,e}\|_{W^{1,p}(D_\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0, \tag{4.1}$$

where

$$d_{\varepsilon,e}(x) := |x - x_\varepsilon|^2.$$

Let us prove (a).

Define the function $u : D_\varepsilon \to \mathbb{R}$,

$$u(x) := \frac{1}{|x|^n} \left(\frac{x_1^2}{|x|^2} - \frac{1}{n} \right).$$

Notice that $u \in \mathcal{H}(D_\varepsilon)$, since

$$u = c_n \frac{\partial^2 \Gamma}{\partial x_1^2},$$

where Γ is the fundamental solution of the Laplace operator with pole at 0 and c_n is a dimensional constant.

By the mean value theorem and taking into account that $D_\varepsilon \supseteq B(\varepsilon)$ and $u > 0$ in $C_\varepsilon := D_\varepsilon \setminus B(\varepsilon)$ (see fig. 2)

$$G(D_\varepsilon, x_\varepsilon) \geq \frac{\left| ⨍_{D_\varepsilon} u(x)\, dx - u(x_\varepsilon) \right|}{⨍_{D_\varepsilon} |u(x)|\, dx} = \frac{\left| \int_{D_\varepsilon} u(x)\, dx - \frac{|D_\varepsilon|}{|B(\varepsilon)|} \int_{B(\varepsilon)} u(x)\, dx \right|}{\int_{D_\varepsilon} |u(x)|\, dx}$$

$$\geq \frac{\int_{D_\varepsilon \setminus B(\varepsilon)} u(x)\, dx - \frac{|D_\varepsilon \setminus B(\varepsilon)|}{|B(\varepsilon)|} \int_{B(\varepsilon)} |u(x)|\, dx}{\int_{D_\varepsilon} |u(x)|\, dx}.$$

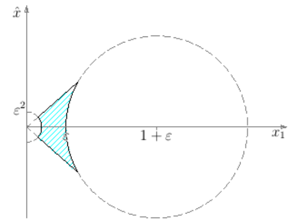

Figure 2. The set $C_\varepsilon := D_\varepsilon \setminus B(\varepsilon)$

Therefore, recalling that $B(\varepsilon)$ has radius 1, we get

$$G(D_\varepsilon, x_\varepsilon) \geq \frac{\int_{C_\varepsilon} u(x)\, dx}{\int_{D_\varepsilon} |u|\, dx} - \frac{|C_\varepsilon|}{\omega_n} =: I_\varepsilon + J_\varepsilon. \tag{4.2}$$

Trivially

$$J_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.3}$$

Let us prove that

$$\liminf_{\varepsilon \to 0} I_\varepsilon > 0.$$

Since $D_\varepsilon \subseteq B(0, 2+\varepsilon) \setminus B(0, \varepsilon^2)$ and $|u(x)| \leq \frac{1}{|x|^n}$ we get

$$\int_{D_\varepsilon} |u|\, dx \leq n\omega_n \int_{\varepsilon^2}^{2+\varepsilon} \frac{1}{\rho}\, d\rho = n\omega_n \log \frac{2+\varepsilon}{\varepsilon^2}. \tag{4.4}$$

Since $u > 0$ in C_ε and

$$C_\varepsilon \supseteq \{x \in K \,:\, \varepsilon^2 < |x| < \varepsilon\},$$

then

$$\int_{C_\varepsilon} u\,dx \geq \int_{\varepsilon^2}^{\varepsilon} \frac{1}{\rho}\left(\int_\Sigma \left(\frac{x_1^2}{|x|^2} - \frac{1}{n}\right) d\sigma(x)\right) d\rho = c\log\frac{1}{\varepsilon}, \tag{4.5}$$

where

$$\Sigma := \left\{\xi = (\xi_1, \hat{\xi}) \in \mathbb{R}^n : |\xi| = 1, |\xi| < \sqrt{n}\xi_1\right\}$$

and $c := \int_\Sigma \left(\frac{x_1^2}{|x|^2} - \frac{1}{n}\right) d\sigma(x) > 0$.

By (4.2) and the previous estimates (4.3), (4.4) and (4.5) we get

$$\liminf_{\varepsilon\to 0} G(D_\varepsilon, x_\varepsilon) \geq \liminf_{\varepsilon\to 0} \frac{c\log\frac{1}{\varepsilon}}{n\omega_n \log\frac{2+\varepsilon}{\varepsilon^2}} = \frac{c}{2n\omega_n} > 0,$$

and claim (a) is proved.

Let us now prove (b).

Let us consider the function $d_\varepsilon : \mathbb{R}^n \to [0,\infty[$

$$d_\varepsilon(x) := \begin{cases} (1 - \operatorname{dist}(x, \partial D_\varepsilon))^2 & \text{if } x \in D_\varepsilon \\ \\ 1 + \operatorname{dist}(x, \partial D_\varepsilon) & \text{if } x \notin D_\varepsilon \end{cases}$$

Notice that

$$\max_{x\in D_\varepsilon} \operatorname{dist}(x, \partial D_\varepsilon) = 1$$

and that d_ε is globally Lipschitz with Lipschitz constant uniformly bounded with respect to ε.

We have that

$$D_\varepsilon = \{x \in \mathbb{R}^n \,:\, d_\varepsilon < 1\} \quad \text{and} \quad \partial D_\varepsilon = \{x \in \mathbb{R}^n \,:\, d_\varepsilon = 1\}.$$

Notice that

$$\operatorname{dist}(x, \partial D_\varepsilon) = 1 - |x - x_\varepsilon| \qquad x \in D_\varepsilon \setminus \hat{C}_\varepsilon,$$

where $\hat{C}_\varepsilon$ is as in fig. 3.

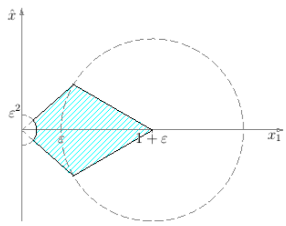

Figure 3. The set $\hat{C}_\varepsilon$

Therefore,

$$d_\varepsilon(x) = (1 - \operatorname{dist}(x, \partial D_\varepsilon))^2 = |x - x_\varepsilon|^2 = d_{\varepsilon,e}(x) \qquad x \in D_\varepsilon \setminus \hat{C}_\varepsilon.$$

This equality, together with

$$|\hat{C}_\varepsilon| \to 0 \quad \text{as } \varepsilon \to 0,$$

easily implies (4.1). □

REFERENCES

[1] V. AGOSTINIANI, R. MAGNANINI: *Stability in an overdetermined problem for the Green's function,* Ann. Mat. Pura Appl. **190** (2011), 21-31.

[2] H. AIKAWA: *Integrability of superharmonic functions and subharmonic functions,* Proc. Amer. Math. Soc. **120** (1994), 109-117.

[3] L. BRASCO, G. DE PHILIPPIS, B. VELICHKOV: *Faber-Krahn inequalities in sharp quantitative form,* Duke Math. J. **164** (2015), 1777-1831.

[4] G. CUPINI, E. LANCONELLI: *On an inverse problem in potential theory,* Atti Acc. Sci. Mat. Ren. Lincei **27** (2016), 431-442.

[5] G. CUPINI, E. LANCONELLI: *Densities with the Mean Value Property for Sub-Laplacians: An Inverse Problem.* Harmonic Analysis, Partial Differential Equations and Applications. S. Chanillo, B. Franchi, G. Lu, C. Perez E.T. Sawyer (eds.). Applied and Numerical Harmonic Analysis, Birkhäuser, 2017, 109-124.

[6] A. FIGALLI, F. MAGGI, A. PRATELLI, *A mass transportation approach to quantitative isoperimetric inequalities,* Invent. Math. **182** (2010), 167-211.

[7] N. FUSCO, F. MAGGI, A. PRATELLI: *The sharp quantitative isoperimetric inequality,* Ann. of Math. **168** (2008), 941-980.

[8] D. GILBARG, N. S. TRUDINGER: Elliptic partial differential equations of second order. Second edition. *Grundlehren der Mathematischen Wissenschaften* **224**. Springer-Verlag, Berlin, 1983.

[9] Ü. KURAN: *On the mean-value property of harmonic functions,* Bull. London Math. Soc. **4** (1972), 311-312.

[10] R. MAGNANINI: *Alexandrov, Serrin, Weinberger, Reilly: simmetry and stability by integral identities,* Bruno Pini Math. Anal. Semin. (to appear).

[11] K.-O. WIDMAN: *Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations,* Math. Scand. **21** (1967), 17-37.

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