FRACTURE MECHANICS IN PERFORATED DOMAINS: A VARIATIONAL MODEL FOR BRITTLE POROUS MEDIA

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Abstract. This paper deals with fracture mechanics in periodically perforated domains. Our aim is to provide a variational model for *brittle porous media*. For the sake of simplicity we will restrict our analysis to the case of anti-planar elasticity.

Given the perforated domain $\Omega_\varepsilon \subset \mathbf{R}^N$ (ε being an internal scale representing the size of the periodically distributed perforations), we will consider a total energy of the type

$$\mathcal{F}_\varepsilon(u) := \int_{\Omega_\varepsilon} |\nabla u(x)|^2\, dx + \mathcal{H}^{N-1}(S_u).$$

Here u is in $SBV(\Omega_\varepsilon)$ (the space of *special functions of bounded variation*), S_u is the set of discontinuities of u, which is identified with a macroscopic crack in the porous medium Ω_ε, and $\mathcal{H}^{N-1}(S_u)$ stands for the $(N-1)$-Hausdorff measure of the crack S_u. Critical points of the total energy are, according to Griffith's theory, stable configurations of the cracked body.

We study the asymptotic behavior of the functionals $\mathcal{F}_\varepsilon$ in terms of Γ-convergence as $\varepsilon \to 0$. As a first (non-trivial) step we show that the domain of any limit functional is $SBV(\Omega)$ despite the degeneracies introduced by the perforations. Then we provide explicit formula for the bulk and surface energy densities of the Γ-limit, representing in our model the effective elastic and brittle properties of the porous medium, respectively.

Keywords: Brittle fracture, Homogenization, Perforated domains, Composite and mixture properties, Cavities, Variational methods, Special functions of bounded variation.

2000 Mathematics Subject Classification: 74R10, 74Q15, 74E30, 35R05, 76B10, 35J25, 26A45.

Contents

1. Introduction

There is a huge mathematical literature concerning the dependence of solutions of partial differential equations, as well as minimum problems, on their domain of definition. In particular it has been largely studied the asymptotic behavior for minimizers u_n defined in varying domains Ω_n with homogenizing small holes, usually referred to as *perforated domains* (we refer to the books [10], [12], [19]). Typically the integral functionals to be minimized depend on u and on its gradient, and on the perforations it is imposed either a Dirichlet type boundary condition (see [9], [17], [20], [25], [35], [38], [39] and the references therein) or a Neumann type boundary condition (see [1], [2], [3], [14], [15], [18], [21], [37] and the references therein). Under standard growth assumptions this kind of minimization problems can be settled in the framework of Sobolev spaces.

The aim of this paper is to study the problem of periodic homogenization of small holes in the framework of fracture mechanics, i.e., for total energies involving not only a bulk term, but also a surface term, obtaining in the homogenized limit a variational model for *brittle porous media*. The homogenizing holes represent traction free micro-cracks in the body, so that our analysis will focus on natural Neumann boundary conditions on the perforations. The case of Dirichlet conditions has been considered in [32] in connection with the study of the asymptotic limit of obstacle problems for Mumford-Shah type functionals (see [36]) in perforated domains.

From a mathematical point of view, the minimization of total energies involving both bulk and surface terms can be settled within the theory of SBV-deformations. The functional space SBV of *special functions of bounded variation* has been introduced by De Giorgi and Ambrosio [26] to deal with free discontinuity problems arising in image segmentation (see [36]), and was proposed by Ambrosio and Braides [4] as a suitable framework for fracture mechanics.

Variational models to describe equilibria of brittle hyperelastic bodies have been largely developed in the recent years. Inspired by Griffith's theory of crack propagation, these models in fracture mechanics are based on the assumption that the cracked deformed configuration of the body is reached through a minimization process driven by the competition of surface and bulk energies. The surface energy represents the energy dissipated to break atomic bonds, and hence spent to enlarge the crack, while the bulk energy represents the elastic energy stored in the body, and partially released during the crack growth.

Let us consider for a while a non-porous brittle body (i.e., without perforations). We will consider for simplicity the case of *generalized antiplanar elasticity*, in which $\Omega \subset \mathbf{R}^N$ represents a section of a cylindrical body (in the relevant physical case we have $N = 2$), the displacement function $u \in SBV(\Omega)$ is assumed to be scalar, and the crack is implicitly identified with the set S_u of discontinuities of u. Concerning the total energy, we will consider for simplicity the following model case

$$E(u) := \int_\Omega |\nabla u(x)|^2 \, dx + \mathcal{H}^{N-1}(S_u). \tag{1.1}$$

Here $\mathcal{H}^{N-1}$ stands for the $(N-1)$-Hausdorff measure, so that if $N = 2$ and S_u is a smooth curve $\mathcal{H}^{N-1}(S_u)$ is just the usual length of the crack. More general energies could be considered, as for instance surface energies eventually depending also on the normal ν of the crack, due to anisotropy of the body, while the dependence on the opening of the crack for cohesive models would require a specific analysis. Critical points (and in particular minimum points) of the total energy (1.1) represent stable configurations of the cracked domain according to Griffith's theory.

To study the effect that the perforations have on the variational problem, let us begin by discussing the case of a single crack K present in the body. Assume that K is a closed subset of Ω and that in $\Omega \setminus K$ the elastic behavior of the body is unperturbed, so that the density of the elastic energy remains the same in $\Omega \setminus K$, while the surface energy will be dissipated only to enlarge the pre-existing crack K. We have that the total energy is given now by

$$E(K,u) := \int_{\Omega\setminus K} |\nabla u(x)|^2\, dx + \mathcal{H}^{N-1}(S_u \setminus K). \tag{1.2}$$

This kind of energy plays an essential role in the variational approaches to quasi-static crack growth as proposed by Francfort and Marigo [30] and developed in many subsequent papers in the framework of SBV-functions (we refer to [29], [23], [33] and to the references therein).

We model the presence of homogenizing cracks considering a sequence $K_\varepsilon := \varepsilon(K + \mathbf{Z}^N)$, with $\varepsilon \to 0$ and K closed, and studying the asymptotic behavior in terms of Γ-convergence of the corresponding energy functionals $\mathcal{F}_\varepsilon(u) := E(K_\varepsilon, u)$[1]. The bulk and surface energy densities of the Γ-limit $\mathcal{F}_{hom}$ will represent the *effective* elastic and brittle properties of the porous brittle medium. Since we do not prescribe the shape of the holes, we will refer to them as *micro-cracks*.

A similar mathematical problem has been considered in [33] in connection to stability properties of equilibria in fracture mechanics for sequences of $(N-1)$-rectifiable sets satisfying $\mathcal{H}^{N-1}(K_n) \le c$. In that case they prove that the Γ-limit of the functionals $E(K_n, \cdot)$ has still the form (1.2), where K is a suitable $(N-1)$-rectifiable set which represents the limit fracture, in a suitable sense, corresponding to the sequence K_n. In that model the fractures K_n represent the cracks created in the body during a load process. Therefore the assumption $\mathcal{H}^{N-1}(K_n) \le c$ is very natural in their setting, meaning that K_n have finite energy according to Griffith's theory. Our situation of periodically distributed cracks $K_\varepsilon = \varepsilon(K + \mathbf{Z}^N)$ is very different, having K_ε by definition diverging area as $\varepsilon \to 0$. Indeed, in our case the homogenizing micro-cracks will affect both bulk and surface energies in the Γ-convergence process.

Our main result is two-fold: in the first part of the paper we deal with the natural lack of coercivity of the problem, establishing a compactness property for sequences with equi-bounded energies, under some natural assumptions ensuring that K_ε does not disconnect the body. Furthermore we prove that the energy functionals $\mathcal{F}_\varepsilon$ Γ-converge (with respect to a suitable topology) to the functional $\mathcal{F}_{hom}$ given by

$$\mathcal{F}_{hom}(u) := \int_\Omega f_{hom}(\nabla u)\, dx + \int_{S_u} g_{hom}(\nu_u)\; d\mathcal{H}^{N-1}, \tag{1.3}$$

where f_{hom} and g_{hom} are defined through formulas below, and represent the material properties of the porous medium.

Concerning f_{hom} we have

$$f_{hom}(\xi) := \inf\left\{ \int_{Q\setminus K} |\nabla w + \xi|^2 dx : \; w \in W^{1,2}_\sharp(Q \setminus K) \right\}, \tag{1.4}$$

where Q is the unit cube and $W^{1,2}_\sharp(Q\setminus K)$ denotes the class of Q-periodic functions in $W^{1,2}(Q\setminus K)$, i.e. Sobolev functions on $Q \setminus K$ whose traces on opposite faces of Q coincide.

This homogenization formula is well known in the context of periodic homogenization in Sobolev spaces and represents the effective energy density in perforated domains subject to Neumann conditions (see for instance [1]). It turns out that the same formula represents the effective bulk

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[1] In our anti-planar setting both the crack S_u and the perforations K_ε are defined in a horizontal section Ω of the cylindrical body and they are assumed to be invariant with respect to the vertical direction of the body. This assumption has to be understood as a mere mathematical simplification of the problem.

energy density also for brittle materials. In this respect we conclude that there is no interaction between macroscopic cracks and micro-cracks for the elastic properties of a brittle porous medium.

Passing to the density of the homogenized surface energy g_{hom}, for all $(a,b,\nu) \in \mathbf{R} \times \mathbf{R} \times \mathbf{S}^{N-1}$ let $u_{a,b,\nu} : \mathbf{R}^N \to \mathbf{R}$ be given by

$$u_{a,b,\nu}(y) := \begin{cases} b & \text{if } y \cdot \nu \geq 0 \\ a & \text{if } y \cdot \nu < 0. \end{cases} \tag{1.5}$$

The surface energy density $g_{hom} : \mathbf{R}^N \to [0, +\infty)$ is given by

$$g_{hom}(\nu) := \lim_{\varepsilon \to 0^+} \inf_{w \in P(Q^\nu \setminus K_\varepsilon)} \{\mathcal{H}^{N-1}(S_w \setminus K_\varepsilon) : \, w = u_{0,1,\nu} \text{ on a neighbourhood of } \partial Q^\nu\}. \tag{1.6}$$

Above Q^ν is any unit cube centered at the origin with one face orthogonal to ν, and $P(Q^\nu \setminus K_\varepsilon)$ is the family of characteristic functions (see (2.3)). We show the existence of the limit in (1.6) in Lemma 5.8. Note that formula (1.6) involves only locally constant functions. We deduce that the toughness of the porous medium does not depend on the elastic properties of the corresponding non-porous material.

Let us finally discuss our result under a slight different perspective. The porous brittle material in our model has been obtained by homogenizing a constituent material with holes. The problem can be settled in the framework of homogenization of composite materials, in which one of the constituent materials is the void. From a mathematical point of view, we deal with energy densities fast oscillating with respect to x, taking values in 0 and 1, and the presence of the coefficient 0 (that is of the void) brings high degeneracy into the problem. Homogenization problems in SBV spaces have been largely studied in the last years, as for instance in [5], [6], [13]. Our homogenization formulas extend those given in the mentioned papers to our context, in which the homogenizing coefficients do not satisfy standard ellipticity conditions. The lack of ellipticity produces many specific difficulties in our analysis, the most remarkable being in the proof of suitable compactness properties for minimizers. In this respect, our approach has been to provide new Poincaré type inequalities in SBV in dimension two, which allow us to truncate the minimizers at suitable levels around each perforation. In view of this, we can extend the minimizers by means of standard cut-off techniques inside the perforations, filling the holes with good control of the total energy. Finally we are in a position to use Ambrosio's compactness results for sequences in SBV with bounded energy. The general N-dimensional case is then recovered by a slicing argument, using the results obtained in dimension two. A different approach to the problem, based on excision techniques introduced by De Giorgi, Carriero and Leaci [27] (see also [8, Chapter 7]), has been developed in the recent paper [16]. Their approach provides, as for the Sobolev setting, a family of uniformly bounded extension operators to fill the holes.

The paper is organized as follows. In Section 2 we provide some preliminary results used in the rest of the paper. In Section 3 we set the mathematical framework to study the asymptotic behavior of energy functionals $\mathcal{F}_\varepsilon$. In Section 4 we provide a Poincaré type inequality in SBV in dimension two, and we prove suitable compactness properties for sequences with bounded energy. In Section 5 we prove the Γ-convergence result of the functionals $\mathcal{F}_\varepsilon$, and in Section 6 we give the analogous Γ-convergence result for energy functionals taking into account Dirichlet boundary conditions on $\partial\Omega$. Finally, in Section 7 we will discuss the validity of our results for more general energy functionals.

2. Preliminaries

In this section we will fix some notation and introduce some notions of geometric measure theory we will need in the sequel.

For every r, s with $0 < r < s$ we set

$$Q_r := \{x \in \mathbf{R}^N : \|x\|_\infty < r/2\}, \qquad Q_{r,s} := \{x \in \mathbf{R}^N : r/2 < \|x\|_\infty < s/2\}, \tag{2.1}$$

and, for simplicity the unitary cube Q_1 by Q.

Throughout the paper Ω is a bounded open subset of $\mathbf{R}^N$ with Lipschitz boundary and $\mathcal{A}(\Omega)$ denotes the family of all open subsets of Ω. Let $A \in \mathcal{A}(\Omega)$. We denote by $SBV(A)$ the space of *special functions of bounded variation*, and by $SBV^2(A)$ the subspace

$$SBV^2(A) := \{u \in SBV(A) \, : \, \nabla u \in L^2(A), \mathcal{H}^{N-1}(S_u) < +\infty\}.$$

Here $\mathcal{H}^{N-1}$ stands for the $(N-1)$-dimensional Hausdorff measure, and S_u denotes the jump set of u. For the notations and the general theory concerning the function space $SBV(A)$ we refer the reader to [8]. We indicate by $SBV_0(A)$ the subset of piecewise constant functions in $SBV(A)$ defined by

$$SBV_0(A) := \{u \in SBV(A) : \nabla u = 0 \text{ for } \mathcal{L}^N \text{ a.e. } x \in A\}. \tag{2.2}$$

Moreover let us consider the family of sets with finite perimeter in A, which will be identified by the functional space $P(A)$ defined by

$$P(A) = \{u \in SBV_0(A) : u(x) \in \{0, 1\} \text{ for } \mathcal{L}^N \text{ a.e. } x \in A\}. \tag{2.3}$$

2.1. **Rectifiable Sets and Coarea Formula.** In this subsection we recall the definition of rectifiable sets and several notions dealing with the tangential calculus which can be developed on them (see [8, Definition 2.57] and [8, Proposition 2.76]).

Definition 2.1. *Let $E \subset \mathbf{R}^N$ be an $\mathcal{H}^m$-measurable set. We say that E is* countably $\mathcal{H}^m$-rectifiable *if $E = N \cup \bigcup_{i \geq 1} \Gamma_i$ where $\mathcal{H}^m(N) = 0$ and each Γ_i is the graph of a function $f_i \in C^1(\mathbf{R}^m, \mathbf{R}^N)$.*

Countably $\mathcal{H}^m$-rectifiable sets E have nice tangential properties. In particular, they can be endowed with a tangent space $\mathrm{Tan}^m(E, x)$, called *approximate tangent space*, for $\mathcal{H}^m$ a.e. $x \in E$. Essentially, this follows from the locality property of the tangent space of C^1 graphs and the decomposition of E into such sets (see [8, Proposition 2.85] and [8, Definition 2.86]).

Furthermore, any Lipschitz function $f : \mathbf{R}^N \to \mathbf{R}^k$ exhibits good differentiability properties on E. Indeed, it turns out that the restriction of f to the affine space $x + \mathrm{Tan}^m(E, x)$ is differentiable for $\mathcal{H}^m$ a.e. $x \in E$. The *tangential differential* of f on E at x, $d^E f_x$, is then defined as the restriction of the differential df_x to $\mathrm{Tan}^m(E, x)$ for $\mathcal{H}^m$ a.e. $x \in E$ (see [8, Definition 2.89] and [8, Theorem 2.90]).

Given this, we can state a version of the Coarea formula valid on countably rectifiable sets (see [8, Theorem 2.93]).

Theorem 2.2. *Let $f : \mathbf{R}^N \to \mathbf{R}^k$ be a Lipschitz function and let $E \subset \mathbf{R}^N$ be a countably $\mathcal{H}^m$-rectifiable set, with $m \geq k$. Then the function $t \to \mathcal{H}^{m-k}(E \cap f^{-1}(t))$ is $\mathcal{L}^k$ measurable in $\mathbf{R}^k$, $E \cap f^{-1}(t)$ is countably $\mathcal{H}^{m-k}$-rectifiable for $\mathcal{L}^k$ a.e. $t \in \mathbf{R}^k$ and*

$$\int_E \mathbf{C}_k(d^E f_x) d\mathcal{H}^m(x) = \int_{\mathbf{R}^k} \mathcal{H}^{m-k}(E \cap f^{-1}(t)) \, dt. \tag{2.4}$$

In the formula above $\mathbf{C}_k(d^E f_x)$ is the k-dimensional coarea factor *of $d^E f_x$ defined by*

$$\mathbf{C}_k(d^E f_x) = \sqrt{\det((d^E f_x) \circ (d^E f_x)^*)}$$

where $(d^E f_x)^* : \mathbf{R}^k \to (\mathrm{Tan}^m(E, x))^*$ is the transpose operator.

3. Formulation of the problem

In this section we will introduce the perforated domains Ω_ε and the energy functionals $\mathcal{F}_\varepsilon$.

3.1. **The perforated domain.** We consider a closed subset K of the open unitary cube Q such that $Q \setminus K$ is connected. For every $\varepsilon > 0$ set

$$K_\varepsilon := \varepsilon(K + \mathbf{Z}^N),$$

and

$$\Omega_\varepsilon := \Omega \setminus K_\varepsilon.$$

The sets K_ε represent the ε-perforations, while Ω_ε the perforated domains.

3.2. **The energy functionals.** Let us fix a boundary datum ψ (which is the trace of a function) in $W^{1,2}(\Omega) \cap L^\infty(\Omega)$ and introduce the functionals $\mathcal{F}_\varepsilon^\psi : L^1(\Omega) \to [0, +\infty]$ defined by

$$\mathcal{F}_\varepsilon^\psi(u) := \begin{cases} \displaystyle\int_{\Omega_\varepsilon} |\nabla u|^2\, dx + \mathcal{H}^{N-1}(S_u^{\psi,\varepsilon}) & \text{if } u \in SBV^2(\Omega_\varepsilon) \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases} \tag{3.1}$$

where

$$S_u^{\psi,\varepsilon} := (S_u \cap \Omega_\varepsilon) \cup \{x \in \partial\Omega \cap \partial\Omega_\varepsilon : \psi(x) \neq u(x)\},$$

and the inequality is intended in the sense of traces. The set $S_u^{\psi,\varepsilon}$ takes into account the crack formed inside Ω_ε, and the part of $\partial\Omega_\varepsilon \cap \partial\Omega$ where u does not agree with the imposed deformation ψ (which is thus considered as part of the crack which has reached the boundary).

Our aim is to study the asymptotic behavior of the energy functionals $\mathcal{F}_\varepsilon^\psi$ defined in (3.1) as $\varepsilon \to 0$ in terms of Γ-convergence with respect to a suitable topology and to prove compactness properties for sequences of corresponding of minimizers.

Remark 3.1. The choice of the L^1 setting is rather natural since it provides suitable compactness properties for minimizers (see Section 4). In this respect we notice that compactness for sequences of functions with bounded energy cannot hold in general since the energy functionals are not affected by the values of the functions inside the holes K_ε. Nevertheless we will see that we can assign the values inside K_ε for sequences with bounded energy in order to gain compactness. Furthermore any limit point obtained with this procedure is uniquely determined by the values outside the holes K_ε. Indeed, it is easy to prove that if $(u_\varepsilon), (v_\varepsilon) \subset L^1(\Omega)$ are such that $u_\varepsilon \to u$ in $L^1(\Omega)$, $v_\varepsilon \to v$ in $L^1(\Omega)$ and $u_\varepsilon \equiv v_\varepsilon$ in Ω_ε, then $u = v$ $\mathcal{L}^N$ a.e. in Ω.

4. Compactness

The main aim of this section is to prove a compactness result in SBV^2 for suitable extensions of sequences of functions in $L^1(\Omega_{\varepsilon_n})$ with bounded energy, where (ε_n) is a fixed vanishing sequence. This result will allow us to identify the domain of any Γ-limit of the functionals $\mathcal{F}_\varepsilon^\psi$ defined in (3.1) and to take advantage of integral representation techniques.

We will consider only sequences uniformly bounded in L^∞. This framework is not restrictive in our setting of the problem, since the boundary datum ψ is in L^∞, and the energy functionals decrease by truncation. Therefore we can assume the minimizing sequences to have L^∞ norm bounded by that of ψ.

First we focus on sequences of functions defined in more regular perforated domains obtained substituting the original reference set K with the larger one $Q_{1-2\delta}$, defined according to (2.1) where $0 < \delta <$ dist $(K, \partial Q)$ is a fixed parameter (see Figure 1). In addition, let us set

$$R := \{x \in \overline{Q} : \text{ dist } (x, \partial Q) < \delta\}, \qquad R_n := \varepsilon_n(R + \mathbf{Z}^N) \cap \Omega. \tag{4.1}$$

Notice that $R = Q_{1-2\delta,1}$. Throughout the section (v_n) will be a sequence in $L^1(R_n)$ bounded in energy and in L^∞, i.e., satisfying

$$\int_{R_n} |\nabla v_n|^2\,dx + \mathcal{H}^{N-1}(S_{v_n}) \le c, \quad \|v_n\|_{L^\infty(R_n)} \le \|\psi\|_{L^\infty(R_n)}, \tag{4.2}$$

where c is a constant independent of n.

In our applications the functions v_n will be given by the restriction to R_n of functions $u_n \in L^1(\Omega)$ with uniformly bounded energy. In view of Remark 3.1 the cluster points of (u_n) in $L^1(\Omega)$ (suitably modified on K_{ε_n}) are determined by those of (v_n) (suitably extended on Ω).

For these sequences (v_n) we provide the existence of suitable BV and SBV^2 extensions (these last ones only in the 2 dimensional case) preserving an uniform bound of the corresponding energy. By a slicing argument and taking advantage of Remark 3.1 we will then prove that, up to subsequences, we have convergence in $L^1(\Omega)$ to a function belonging to $SBV^2(\Omega)$ (see subsection 4.3).

The desired compactness result for sequences defined on general perforated sets Ω_{ε_n} will be then achieved by an approximation argument (see subsection 4.4).

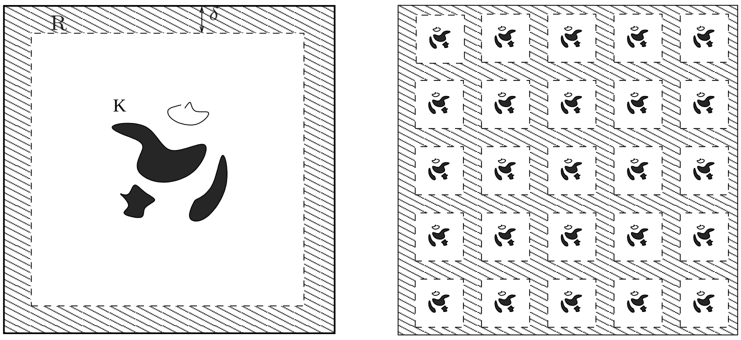

FIGURE 1. The "larger" obstacle $Q_{1-2\delta}$ and the related perforated domain R_n

Under assumption (4.2) we establish the following results (see subsections 4.1, 4.2, and 4.3, respectively):

(1) *BV-compactness*: there exist functions $\widetilde{v}_n, v \in BV(\Omega)$ such that $\widetilde{v}_n \equiv v_n$ in R_n and, up to a subsequence, $\widetilde{v}_n \to v$ in $L^1(\Omega)$;
(2) *2-dimensional SBV^2-compactness*: if $N = 2$ there exist functions $\widehat{v}_n, v \in SBV^2(\Omega)$ such that $\widehat{v}_n \equiv v_n$ in R_n and, up to a subsequence, $\widehat{v}_n \to v$ in $L^1(\Omega)$;
(3) *N-dimensional SBV^2-closure*: if $\widetilde{v}_n \to v$ in $L^1(\Omega)$ and $\widetilde{v}_n \equiv v_n$ in R_n then v is in $SBV^2(\Omega)$.

The most difficult part of this program is to prove the 2-dimensional SBV^2-extension result in (2). The hypothesis on the dimension comes into play only into a technical result, Lemma 4.3, where the *smallness* of a set in terms of area and perimeter implies some estimate on the diameter of its "connected components". In view of this estimate we are able to prove a Poincaré type inequality in SBV (see Theorem 4.8), which allows us to perform the construction of the functions $\widehat{v}_n$ in (2) without creating new jumps. Moreover if the original sequence (v_n) belongs to $W^{1,2}(R_n)$ or to $SBV_0(R_n)$ then $(\widehat{v}_n)$ belongs to $W^{1,2}(\Omega)$, $SBV_0(\Omega)$, respectively.

4.1. BV**-compactness.** Here we prove a compactness result in $BV(\Omega)$.

Proposition 4.1 (Compactness in $BV(\Omega)$)**.** *Let $(v_n) \subset L^1(R_n)$ be a sequence such that*

$$\sup_n (|Dv_n|(R_n) + \|v_n\|_{L^\infty(R_n)}) < +\infty,$$

then there exist functions $\widetilde{v}_n \in BV(\Omega)$ such that

$$\widetilde{v}_n \equiv v_n \text{ on } R_n \quad \text{and} \quad |D\widetilde{v}_n|(\Omega) + \|\widetilde{v}_n\|_{L^\infty(\Omega)} \leq c(|Dv_n|(R_n) + \|v_n\|_{L^\infty(\Omega)})$$

for a constant c independent of n. In particular, up to a subsequence, $(\widetilde{v}_n)$ converges to v in $L^1(\Omega)$ for some $v \in BV(\Omega)$.

Proof. Let us fix some notation: for $i \in \mathbf{Z}^N$ set $Q_n^i := \varepsilon_n(i + Q)$, $R_n^i := \varepsilon_n(i + R) \cap \Omega$. Let also $\mathcal{I}_n = \{i \in \mathbf{Z}^N : Q_n^i \cap \partial\Omega \neq \emptyset\}$, and for every $Q_n^i \subset \Omega$ set

$$m_n^i := \frac{1}{|R_n^i|} \int_{R_n^i} v_n(x)\, dx, \tag{4.3}$$

and

$$\widetilde{v}_n(x) := \begin{cases} v_n(x) & \text{if } x \in R_n^i, \\ m_n^i & \text{if } x \in Q_n^i \setminus R_n^i,\ i \notin \mathcal{I}_n \\ 0 & \text{elsewhere in } \Omega. \end{cases} \tag{4.4}$$

We claim that $(\widetilde{v}_n)$ defined above satisfies the thesis. Indeed, by construction $\widetilde{v}_n \equiv v_n$ on R_n and $\|\widetilde{v}_n\|_{L^\infty(\Omega)} \leq \|v_n\|_{L^\infty(R_n)}$. Standard trace results in BV (see [8, Theorems 3.84, 3.87]), yield that the function $\widetilde{v}_n$ belongs to $BV(\Omega)$ with distributional derivative

$$D\widetilde{v}_n = Dv_n \llcorner R_n + \sum_{i \in \mathbf{Z}^N} D\widetilde{v}_n \llcorner (\partial R_n^i \cap Q_n^i),$$

and

$$D\widetilde{v}_n \llcorner (\partial R_n^i \cap Q_n^i) = ((\mathrm{tr}(v_n) - m_n^i)\nu_{\partial R_n^i})\mathcal{H}^{N-1} \llcorner (\partial R_n^i \cap Q_n^i),$$

where $\mathrm{tr}(v_n)$ is the trace left by v_n on the boundary ∂R_n^i. Since by hypothesis $\sup_n |Dv_n|(R_n) < +\infty$, to conclude it suffices to give an uniform estimate of the total variation of $D\widetilde{v}_n$ concentrated on the union of $\partial R_n^i \cap Q_n^i$.

To this aim notice that $\#\mathcal{I}_n \leq c/\varepsilon_n^{N-1}$, with c depending only on $\mathcal{H}^{N-1}(\partial\Omega)$ since $\partial\Omega$ is Lipschitz. Here $\#$ denotes the cardinality of the relevant set. Thus, taking into account that $\mathcal{H}^{N-1}(\partial R_n^i) \leq c\varepsilon_n^{N-1}$ and that $\sup_n \|\widetilde{v}_n\|_{L^\infty(R_n)} < +\infty$, we deduce $\sup_n \sum_{i \in \mathcal{I}_n} |D\widetilde{v}_n|(\partial R_n^i) < +\infty$. Furthermore, to control $|D\widetilde{v}_n|(\partial R_n^i \cap Q_n^i)$ for $i \notin \mathcal{I}_n$ we use a scaling argument and the continuity of the Trace Operator on R (see [8, Theorem 3.87]). For $i \notin \mathcal{I}_n$ let $w_n^i : R \to \mathbf{R}$ be defined as $w_n^i(y) = v_n(\varepsilon_n(i + y))$. It is easy to check that $w_n^i \in BV(R)$, the mean value of w_n^i on R equals m_n^i, and $|Dw_n^i|(R) = \varepsilon_n^{1-N}|Dv_n|(R_n^i)$. Moreover, there exists a positive constant $c(R)$ independent of n such that

$$\int_{\partial R \cap Q} |\mathrm{tr}(w_n^i) - m_n^i|\, d\mathcal{H}^{N-1} \leq c(R)|Dw_n^i|(R). \tag{4.5}$$

A scaling argument gives

$$\int_{\partial R_n^i \cap Q_n^i} |\mathrm{tr}(v_n) - m_n^i|\, d\mathcal{H}^{N-1} = \varepsilon_n^{N-1} \int_{\partial R \cap Q} |\mathrm{tr}(w_n^i) - m_n^i|\, d\mathcal{H}^{N-1},$$

from which we infer that for every $i \notin \mathcal{I}_n$

$$|D\widetilde{v}_n|(\partial R_n^i \cap Q_n^i) = \int_{\partial R_n^i \cap Q_n^i} |\mathrm{tr}(v_n) - m_n^i|\, d\mathcal{H}^{N-1} \leq c(R)|Dv_n|(R_n^i)$$

and this gives the desired estimate. The rest of the statement is a direct consequence of the BV compactness theorem (see [8, Theorem 3.23]). $\square$

Remark 4.2. The BV compactness result still holds if we replace the δ-neighbourhood R_n with any connected neighborhood C of ∂Q with Lipschitz continuous boundary. It is also possible to consider varying domains C_n, provided they ensure the continuity of the trace operator together with a uniform estimate on the relative constants.

4.2. **Compactness in $SBV^2(\Omega)$: the case $N = 2$.** This subsection is focused on SBV compactness properties in dimension two. In this setting given a sequence $(v_n) \subset L^1(R_n)$ with bounded energy (see (4.2)) we construct an $SBV^2(\Omega)$ extension with uniform control on the increase of the energy. In order to do that, we first extend any function $v \in SBV^2(R)$ with quantified small jump set (see Proposition 4.9) to a function $\widehat{v} \in SBV^2(Q)$ such that $v \equiv \widehat{v}$ in R and

$$\int_Q |\nabla \widehat{v}|^2 + \mathcal{H}^{N-1}(S_{\widehat{v}}) \leq c \left(\int_R |\nabla v|^2 + \mathcal{H}^{N-1}(S_v) \right),$$

with c independent of v and depending only on the geometry of R. Then, the extension for v_n is obtained by exploiting the periodicity of the problem by repeating the construction in each ε_n-square contained in Ω in which v_n has small jump set (up to the usual scaling argument) and using the BV-extension $\widetilde{v}_n$ in the holes of the remaining squares (see Proposition 4.11 for more details).

To describe briefly the idea to accomplish the extension in the case of fixed geometry consider a function $v \in SBV^2(R)$, and by a standard argument based on composition with bilipschitz functions we assume that $v \in SBV^2(Q_{r_0,1})$, with $1 - 2\mathrm{dist}(K, \partial Q) < r_0 < 1 - 2\delta$. Set now $r_2 = 1 - 2\delta$ and let $r_1 \in (r_0, r_2)$ be arbitrarily chosen. In Theorem 4.8 and Proposition 4.9 we will show that if the jump set of v is sufficiently small we are able to modify v in a region containing Q_{r_0,r_1} and contained in Q_{r_0,r_2}. The construction acts by truncating v at suitable levels, in such a way that this truncated function has oscillation on Q_{r_0,r_1} controlled in terms of $|D^a v|(Q_{r_0,r_2})$, and above all without creating any new jump. In view of this Poincaré type inequality the extension of v to the whole Q is obtained by joining it smoothly to a suitable constant through a cut-off function (see Figure 2 for a sketch of the construction).

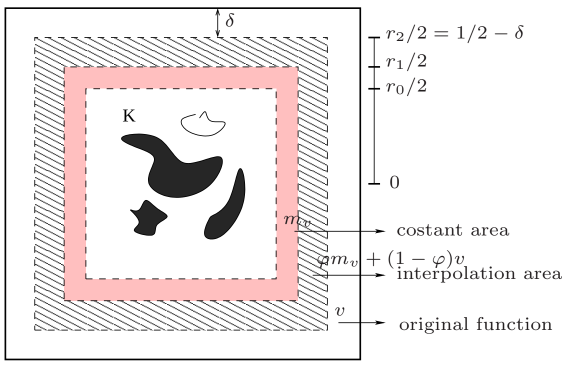

FIGURE 2. Definition of $\widehat{v}$ in different areas

Let us begin with the truncation procedure that we set primarily for functions in SBV_0. Let us fix some more preliminary notation. As already mentioned we fix positive radii r_0, r_1, r_2 as follows

$$r_0 \in (1 - 2\mathrm{dist}(K, \partial Q), 1 - 2\delta), \ r_2 = 1 - 2\delta, \ r_1 \in (r_0, r_2). \tag{4.6}$$

Moreover, for every $s \in \mathbf{R}$ we denote by E_s the s sub-level of v in Q_{r_1,r_2}, i.e.,

$$E_s := \{x \in Q_{r_1,r_2} : v(x) \leq s\}, \tag{4.7}$$

and by $\mathrm{med}(v)$ a *median* of v in Q_{r_1,r_2}, namely

$$\mathrm{med}(v) := \sup\{s \in \mathbf{R} : \, |E_s| \le |Q_{r_1,r_2}|/2\}. \tag{4.8}$$

In formula above, the 2-dimensional Lebesgue measure $\mathcal{L}^2$ has been indicated with $|\cdot|$, a notation that we will use for the rest of the subsection.

Lemma 4.3 (Truncation Lemma in $SBV_0(Q_{r_1,r_2})$). *For every $v \in SBV_0(Q_{r_1,r_2})$ with $\mathcal{H}^1(S_u) < (r_2 - r_1)/2$, the set $I = \{r \in (r_1, r_2) : \mathcal{H}^0(\partial Q_r \cap S_v) = 0\}$ is such that $\mathcal{L}^1(I) > 0$. In particular, for $\mathcal{L}^1$ a.e. $r \in I$ the trace of v on ∂Q_r is constant.*

Proof. Set

$$J := \{r \in (r_1, r_2) : \mathcal{H}^0(\partial Q_r \cap S_v) \ge 1\},$$

then the thesis is equivalent to proving that $\mathcal{L}^1(J) < r_2 - r_1$ (see [8, Theorem 3.87]).

In order to estimate $\mathcal{L}^1(J)$ we use the Coarea Formula 2.2 applied with $k = 1$, $m = 1$, $f(y) = \|y\|_\infty$ and $E = S_v$. A simple computation shows that $d^{S_v} f_x = \langle \nabla f(x), \nu_v^\perp(x)\rangle \nu_v^\perp(x)$ for $\mathcal{H}^1$ a.e. $x \in S_u$, so that

$$\mathbf{C}_1(d^{S_v} f_x) = |\langle \nabla f, \nu_v^\perp(x)\rangle|.$$

Since $|\nabla f| = 1$ $\mathcal{L}^2$ a.e. in $\mathbf{R}^2$, by (2.4) we infer

$$\mathcal{L}^1(J) \le \int_J \mathcal{H}^0(\partial Q_r \cap S_v)\, dr \le 2\mathcal{H}^1(S_v) < r_2 - r_1,$$

from which the result follows. □

Remark 4.4. The same inequality can also be obtained by using classical slicing results (separately in suitable sectors of Q_{r_1,r_2}), instead of the Coarea Formula.

In the following we will deal with one-dimensional sections of a set of finite perimeter E. To make the framework rigorous we could fix a $\mathcal{L}^2$-representant of E, e.g., $E^+ = \{x \in \mathbf{R}^2 : \limsup_{r\to 0^+} r^{-2}|B_r(x) \cap E| > 0\}$. A careful reading shows anyway that all the statements below are independent of the $\mathcal{L}^2$ representant of E.

Given a set of finite perimeter E, we denote by $\partial^* E$ the *essential boundary* of E [8, Definition 3.60]. By applying Lemma 4.3 to the characteristic function of a set with finite perimeter we immediately deduce the following corollary.

Corollary 4.5. *For any set of finite perimeter $E \subseteq Q_{r_1,r_2}$ with $\mathcal{H}^1(\partial^* E) < (r_2 - r_1)/2$ there exists a set of positive $\mathcal{L}^1$ measure in (r_1, r_2) such that for any r in this set either $\mathcal{H}^1(E \cap \partial Q_r) = 0$ or $\mathcal{H}^1(E \cap \partial Q_r) = \mathcal{H}^1(\partial Q_r)$.*

Under some additional conditions on the smallness of both $|E|$ and $\mathcal{H}^1(\partial^* E)$ the previous result can be refined.

Lemma 4.6. *There exists a constant $C_1 \in (1, +\infty)$ depending only on r_1 and r_2 such that the following holds true. For any set of finite perimeter $E \subseteq Q_{r_1,r_2}$, with $|E| \le |Q_{r_1,r_2}|/2$ and $\mathcal{H}^1(\partial^* E) \le (r_2 - r_1)/C_1$, there exists a set $\mathcal{I} \subseteq (r_1, r_2)$ of positive $\mathcal{L}^1$ measure such that $\mathcal{H}^1(E \cap \partial Q_r) = 0$ for $\mathcal{L}^1$ a.e. $r \in \mathcal{I}$.*

Proof. Let us set

$$I := \{r \in (r_1, r_2) : \mathcal{H}^1(\partial Q_r \cap E) = \mathcal{H}^1(\partial Q_r)\}$$

$$J := \{r \in (r_1, r_2) : 0 < \mathcal{H}^1(\partial Q_r \cap E) < \mathcal{H}^1(\partial Q_r)\}.$$

We have to prove that if C_1 is large enough, then $\mathcal{L}^1(I \cup J) < r_2 - r_1$.

By the Coarea Formula we have

$$\int_{r_1}^{r_2} \mathcal{H}^1(\partial Q_r \cap E)\,dr = 2|E|.$$

If $\tilde{c}$ is the constant of the Relative Isoperimetric Inequality in Q_{r_1,r_2} (see formula (3.43) in [8]), an elementary rearrangement argument gives

$$\begin{aligned}\int_{r_1}^{r_1+\mathcal{L}^1(I)} 4r\,dr &\le \int_I \mathcal{H}^1(\partial Q_r)\,dr = \int_I \mathcal{H}^1(\partial Q_r \cap E)\,dr\\ &\le 2|E| \le 2\tilde{c}\big(\mathcal{H}^1(\partial^* E)\big)^2 \le 2\frac{\tilde{c}}{C_1^2}(r_2-r_1)^2,\end{aligned}$$

from which we immediately obtain

$$\mathcal{L}^1(I) \le \frac{\sqrt{\tilde{c}}}{C_1}(r_2-r_1). \tag{4.9}$$

In order to estimate $\mathcal{L}^1(J)$ we notice that $\mathcal{H}^0(\partial Q_r \cap \partial^* E) \ge 1$ for $\mathcal{L}^1$ a.e. $r \in J$, so that by the Coarea Formula we infer

$$\mathcal{L}^1(J) \le \int_J \mathcal{H}^0(\partial Q_r \cap \partial^* E)\,dr \le 2\mathcal{H}^1(\partial^* E) \le 2\frac{r_2-r_1}{C_1}. \tag{4.10}$$

From (4.9) and (4.10) we easily conclude. □

Remark 4.7. In dimension greater than 2 the result of Lemma 4.3 is no more true. Indeed, one can exhibit sets with small perimeter intersecting the boundary of each Q_r in a set of positive $\mathcal{H}^{N-1}$ measure. In this case an analogous of Lemma 4.3 should deal with a suitable quantification of the measure of the subset intersecting the boundary of each cube. We didn't investigate further this kind of result since our tecniques allow us to prove the closure and compactness result in any dimension arguing by sections, taking advantage of the 2 dimensional case.

From Lemma 4.3 we can deduce a (localized) Poincaré type inequality for functions in $SBV(Q_{r_0,1})$.

Theorem 4.8 (A Poincaré type inequality in $SBV(Q_{r_0,1})$)**.** *Let C_1 be as in Lemma 4.3 and let $v \in SBV(Q_{r_0,1})$ with $\mathcal{H}^1(S_v) \le (r_2-r_1)/4C_1$. Then there exist a function denoted by $T(v)$ in $SBV(Q_{r_0,1})$ and a constant $m_v \in \mathbf{R}$ satisfying*

i) *$T(v) = v$ in R;*
ii) *$|DT(v)| \le |Dv|$ in $Q_{r_0,1}$ in the sense of measures;*
iii) *$\|T(v)-m_v\|_{L^\infty(Q_{r_0,r_1})} \le 4C_1|D^a v|(Q_{r_1,r_2})/(r_2-r_1)$.*

Proof. If $|D^a v|(Q_{r_1,r_2}) = 0$ we apply Lemma 4.3 and select $\bar{r} \in (r_1,r_2)$ such that the trace of v on $\partial Q_{\bar{r}}$ is constant. In particular, choosing m_v equal to such a constant and setting $T(v) := m_v$ in $Q_{\bar{r}}$ all the conditions of the theorem are satisfied.

Otherwise we have $|D^a v|(Q_{r_1,r_2}) > 0$, then the BV Coarea Formula (see [8, Theorem 3.40]) implies

$$\int_{\mathrm{med}(v)-4C_1|D^a v|(Q_{r_1,r_2})/(r_2-r_1)}^{\mathrm{med}(v)} \mathcal{H}^1(\partial^* E_s \setminus S_v)\,ds \le \int_{\mathbf{R}} \mathcal{H}^1(\partial^* E_s \setminus S_v)\,ds = |D^a v|(Q_{r_1,r_2}),$$

where E_s is the sub-level of v in Q_{r_1,r_2} defined in (4.7) and $\mathrm{med}(v)$ is defined in (4.8). By the Mean Value Theorem, there exists $s' \in (\mathrm{med}(v)-4C_1|D^a v|(Q_{r_1,r_2})/(r_2-r_1), \mathrm{med}(v))$ such that

$$\mathcal{H}^1(\partial^* E_{s'} \setminus S_v) \le (r_2-r_1)/4C_1$$

and so

$$\mathcal{H}^1(\partial^* E_{s'}) \le \mathcal{H}^1(\partial^* E_{s'} \setminus S_v) + \mathcal{H}^1(S_v) \le (r_2 - r_1)/2C_1. \tag{4.11}$$

Analogously, we can find $s'' \in (\mathrm{med}(v), \mathrm{med}(v) + 4C_1|D^a v|(Q_{r_1,r_2})/(r_2 - r_1))$ such that

$$\mathcal{H}^1(\partial^* E_{s''}) \le (r_2 - r_1)/2C_1. \tag{4.12}$$

Set $E = E_{s'} \cap (Q_{r_1,r_2} \setminus E_{s''})$, then the definition of median (4.8) and the choice $s' < \mathrm{med}(v)$ yield $|E| \le |Q_{r_1,r_2}|/2$. In addition, $\mathcal{H}^1(\partial^* E) \le (r_2 - r_1)/C_1$ by (4.11) and (4.12). Apply Lemma 4.6 to the set E defined above and find $\bar{r} \in (r_1, r_2)$ with $\mathcal{H}^1(E \cap \partial Q_{\bar{r}}) = 0$. Set $T(v) := s' \vee v \wedge s''$ in $Q_{r_0,\bar{r}}$, $T(v) = v$ in $Q \setminus Q_{\bar{r}}$ and $m_v := \mathrm{med}(v)$. The thesis follows easily by construction. □

Proposition 4.9 (An Extension result)**.** *There exists a constant $C_2 > 0$ depending only on r_0, r_1, r_2 such that for any $v \in SBV^2(R)$ with $\mathcal{H}^1(S_v) \le (r_2 - r_1)/4C_1$ there exists $\widehat{v} \in SBV^2(Q)$ such that $\widehat{v} = v$ in R, $\|\nabla \widehat{v}\|_{L^2(Q)} \le C_2 \|\nabla v\|_{L^2(R)}$ and $\mathcal{H}^{N-1}(S_{\widehat{v}}) \le C_2 \mathcal{H}^{N-1}(S_v)$. Moreover, if $v \in W^{1,2}(R)$ ($v \in SBV_0(R)$), then $\widehat{v} \in W^{1,2}(Q)$ ($v \in SBV_0(Q)$).*

Proof. Thanks to the regularity of the sets $Q_{r,s}$, by a standard tecnique that relies on inner composition with bilipschitz functions (see [8] and [7]) we may assume the function v to be extended in $Q_{r_0,1}$ in such a way that

$$\int_{Q_{r_0,1}} |\nabla v|^2 \, dx \le c \int_R |\nabla v|^2 \, dx, \qquad \mathcal{H}^{N-1}(S_v) \le c\mathcal{H}^{N-1}(S_v \cap R),$$

for a universal constant $c > 0$ depending only on the geometry of R.

Let now $T(v)$ and m_v be as in Theorem 4.8. If $v \in SBV_0(Q_{r_0,1})$ define $\widehat{v}$ simply by extending $T(v)$ to the whole of Q with constant value m_v. Otherwise, we consider a cut-off function $\varphi \in C^1(Q, [0,1])$ such that $\varphi \equiv 0$ on Q_{r_0}, $\varphi \equiv 1$ on $Q_{r_1,1}$. Define the function $\widehat{v}$ on Q as $\widehat{v} := \varphi T(v) + (1 - \varphi) m_v$. A straightforward computation shows that

$$\int_Q |\nabla \widehat{v}|^2 dx \le \int_{Q_{r_0,1}} |\nabla v|^2 dx + c \int_{Q_{r_0,r_1}} |T(v) - m_v|^2 \, dx. \tag{4.13}$$

Taking into account iii) of Theorem 4.8 and using Jensen inequality we obtain

$$\int_{Q_{r_0,r_1}} |T(v) - m_v|^2 \, dx \le c \int_{Q_{r_1,r_2}} |\nabla v|^2 dx. \tag{4.14}$$

From (4.13) and (4.14), noticing that $S_{\widehat{v}} \subseteq S_v$, the thesis follows. □

Remark 4.10. We notice that with the same techniques used in the proof of Proposition 4.9 one can prove an extension result for functions v in $SBV(R)$ with $\mathcal{H}^1(S_v) \le (r_2 - r_1)/4C_1$ to functions which are in $SBV(Q)$.

We are now in a position to prove the compactness of sequences (v_n) satisfying (4.2).

Proposition 4.11 (Compactness in $SBV^2(\Omega)$, $\Omega \subset \mathbf{R}^2$)**.** *Let $(v_n) \in L^1(R_n)$ be satisfying (4.2). Then there exist functions $\widehat{v}_n \in SBV^2(\Omega)$ satisfying $\widehat{v}_n \equiv v_n$ in R_n such that (up to a subsequence) $(\widehat{v}_n)$ converges in $L^1(\Omega)$ to some $v \in SBV^2(\Omega)$.*

Proof. Set $\mathcal{J}_n = \{i \in \mathbf{Z}^N : \text{ either } Q_n^i \not\subset \Omega \text{ or } \mathcal{H}^1(S_{v_n} \cap Q_n^i) > \varepsilon_n(r_2 - r_1)/4C_1\}$ with C_1 as in Lemma 4.3. For every $i \in \mathcal{J}_n$ we define $\widehat{v}_n$ on Q_n^i to be equal to the BV extension $\widetilde{v}_n$ defined in Proposition 4.1. By the Lipschitz regularity of $\partial\Omega$ and the fact that $\sup_n \mathcal{H}^1(S_{v_n}) < +\infty$ we deduce that $\#(\mathcal{J}_n) \le c/\varepsilon_n$. This together with the assumption $\sup \|v_n\|_{L^\infty(R_n)} < +\infty$ provides the estimate

$$\int_{\cup_{i \in \mathcal{J}_n} Q_n^i} |\nabla \hat{v}_n|^2 \, dx + \mathcal{H}^{N-1}(S_{\hat{v}_n} \cap \cup_{i \in \mathcal{J}_n} Q_n^i) \le c$$

for some c independent of n.

Let us now consider a square Q_n^i contained in Ω and satisfying $\mathcal{H}^1(S_{v_n}\cap Q_n^i)\le \varepsilon_n(r_2-r_1)/4C_1$. Let $v_n^i : R\to \mathbf{R}$ be defined as $v_n^i(y)=v_n(\varepsilon_n(i+y))$. It can be checked that v_n^i satisfies the hypotheses of Proposition 4.9. Let $\widehat{v}_n^i\in SBV^2(Q)$ be its extension provided by Proposition 4.9 and define $\widehat{v}_n$ as $\widehat{v}_n^i$ scaled back to Q_n^i. Using a standard scaling argument, we obtain

$$\|\nabla\widehat{v}_n\|_{L^2(Q_n^i)}\le C_2\|\nabla v_n\|_{L^2(R_n^i)},\quad \mathcal{H}^1(S_{\widehat{v}_n}\cap Q_n^i)\le C_2\mathcal{H}^1(S_{v_n}\cap R_n^i),\quad \|\widehat{v}_n\|_{L^\infty(\Omega)}\le\|v_n\|_{L^\infty(R_n)}.$$

The compactness then follows by Ambrosio's SBV Theorem (see [8, Theorem 4.8]). □

4.3. **Compactness in $SBV^2(\Omega)$: the general case.** Let us turn our attention to prove that in dimension greater than 2 the L^1 limit of any (extension of) (v_n) as in (4.2) is actually in $SBV^2(\Omega)$. We argue by a slicing procedure that allow us to infer the result from Proposition 4.11.

Proposition 4.12 (SBV^2 closure)**.** *Let $(v_n)\subset L^1(R_n)$ be a sequence as in* (4.2) *and let v be the L^1 limit of some sequence $(\widetilde{v}_n)\subset L^1(\Omega)$, with $\widetilde{v}_n\equiv v_n$ in R_n. Then $v\in SBV^2(\Omega)$.*

Proof. First note that by Remark 3.1 v is also the L^1 limit of the sequence of extensions constructed in Proposition 4.1, thus we deduce that $v\in BV(\Omega)$.

We argue by a slicing procedure that allows us to use the result in Proposition 4.11. Let $V_{i,j}$ be the 2-dimensional subspace in $\mathbf{R}^N$ generated by the vectors e_i, e_j of the canonical base. We use the standard notation $V_{i,j}^\perp$ to denote the space orthogonal to $V_{i,j}$.

Given $z\in V_{i,j}^\perp$ we denote by $v^{i,j,z}$ the restriction of the function v to the planar set $\Omega^{i,j,z}:=(V_{i,j}+z)\cap\Omega$. We claim that for $\mathcal{H}^{N-2}$ a.e $z\in V_{i,j}^\perp$ the function $v^{i,j,z}$ belongs to $SBV^2(\Omega^{i,j,z})$, and

$$\int_{V_{i,j}^\perp}\left(\int_{\Omega^{i,j,z}}|\nabla v^{i,j,z}|^2 d\mathcal{H}^2+\mathcal{H}^1(S_{v^{i,j,z}})\right)d\mathcal{H}^{N-2}(z)<+\infty. \tag{4.15}$$

Once claim (4.15) is proved we conclude the proof of the Proposition as follows. Fix $1\le i,\ j\le N$, and let $z\in V_{i,j}^\perp$ be such that

$$\int_{\Omega^{i,j,z}}|\nabla v^{i,j,z}|^2 d\mathcal{H}^2+\mathcal{H}^1(S_{v^{i,j,z}}):=M(z)<+\infty. \tag{4.16}$$

For every fixed $t\in\mathbf{R}$ let us set $L^{i,j,z,t}:=\Omega\cap\{te_j+se_i+z,\ s\in\mathbf{R}\}$, and let $v^{i,j,z,t}$ be the restriction of v to $L^{i,j,z,t}$. By (4.16) and standard one-dimensional slicing theory, we have that for almost every $t\in\mathbf{R}$ the function $v^{i,j,z,t}$ belongs to $SBV^2(L^{i,j,z,t})$, and moreover

$$\int_{\mathbf{R}}\left(\int_{L^{i,j,z,t}}|\nabla v^{i,j,z,t}|^2\,d\mathcal{H}^1+\mathcal{H}^0(S_{v^{i,j,z,t}})\right)dt\le M(z). \tag{4.17}$$

Integrating (4.17) with respect to z and taking into account (4.15), (4.16) we conclude that for almost all $\xi\in e_i^\perp$, setting $L^{i,\xi}:=\Omega\cap\{s\,e_i+\xi, s\in\mathbf{R}\}$, the restriction $v^{i,\xi}$ of v to $L^{i,\xi}$ belongs to $SBV^2(L^{i,\xi})$, and again by one-dimensional slicing theory

$$\int_{e_i^\perp}\left(\int_{L^{i,\xi}}|\nabla v^{i,\xi}|^2\,d\mathcal{H}^1+\mathcal{H}^0(S_{v^{i,j,z,t}})\right)d\mathcal{H}^{N-1}(\xi)<+\infty. \tag{4.18}$$

Since the choice of the direction e_i is arbitrary, we have that (4.18) holds true for all $1\le i\le N$. By standard slicing argument we deduce that $v\in SBV^2(\Omega)$, that concludes the proof of the proposition using the claim.

It remains to prove the claim (4.15). Let us set $R_n^z:=R_n\cap\Omega^{i,j,z}$ and

$$M_n(z):=\int_{R_n^z}|\nabla v_n^{i,j,z}|^2 d\mathcal{H}^2+\mathcal{H}^1(S_{v_n^{i,j,z}}).$$

In view of (4.2), by Fubini Theorem and standard slicing arguments we have that

$$\int_{V_{i,j}^{\perp}} M_n(z)\, d\mathcal{H}^{N-2}(z) \leq c. \tag{4.19}$$

Hence for $\mathcal{H}^{N-2}$-a.e. $z \in V_{i,j}^{\perp}$ the values $\liminf_n M_n(z)$ are finite and the restriction $v_n^{i,j,z}$ of v_n to R_n^z belongs to $SBV^2(R_n^z)$.

Let us fix $z \in V_{i,j}^{\perp}$ such that, up to a subsequence not relabeled, $M_n(z)$ is bounded uniformly in n. We observe that for given n the set R_n^z either coincides with $\Omega^{i,j,z}$, or with the two dimensional δ-neighbourhood of the grid

$$\varepsilon_n(([-1/2, 1/2]^2 \setminus [-1/2+\delta, 1/2-\delta]^2) + \mathbf{Z}^2) \cap \Omega^{i,j,z}$$

which we label as $R_n(\Omega^{i,j,z})$. In both cases we can apply Proposition 4.11 to the sequence $(v_n^{i,j,z})$ on $R_n(\Omega^{i,j,z})$ and get functions $w_n^{i,j,z}$ with $w_n^{i,j,z} \equiv v_n^{i,j,z}$ on $R_n(\Omega^{i,j,z})$ satisfying

$$\int_{\Omega^{i,j,z}} |\nabla w_n^{i,j,z}|^2 d\mathcal{H}^2 + \mathcal{H}^1(S_{w_n^{i,j,z}}) \leq c' \int_{R_n(\Omega^{i,j,z})} |\nabla v_n^{i,j,z}|^2 d\mathcal{H}^2 + c'\mathcal{H}^1(S_{v_n^{i,j,z}} \cap R_n(\Omega^{i,j,z})) \leq c' M_n(z). \tag{4.20}$$

where c' is a constant depending only on δ and the fixed geometry of the perforations. In particular, a two-dimensional argument analogous to Lemma 3.1 implies that $w_n^{i,j,z}$ converge to $v^{i,j,z}$ for $\mathcal{H}^{N-2}$ a.e. $z \in V^{\perp}$. Finally, (4.20) and Ambrosio's SBV theorem yield

$$\int_{\Omega^{i,j,z}} |\nabla v^{i,j,z}|^2 d\mathcal{H}^2 + \mathcal{H}^1(S_{v^{i,j,z}}) \leq c' \liminf_n M_n(z).$$

Integrating with respect to z, in view of (4.19) and using Fatou Lemma we conclude

$$\begin{aligned} &\int_{V_{i,j}^{\perp}} \left(\int_{\Omega^{i,j,z}} |\nabla v^{i,j,z}|^2\, d\mathcal{H}^2 + \mathcal{H}^1(S_{v^{i,j,z}}) \right) d\mathcal{H}^{N-2}(z) \\ &\quad \leq c' \int_{V_{i,j}^{\perp}} \liminf_n M_n(z) d\mathcal{H}^{N-2}(z) \leq c' \liminf_n \int_{V_{i,j}^{\perp}} M_n(z) d\mathcal{H}^{N-2}(z) < +\infty. \end{aligned}$$

This concludes the proof of the claim (4.15) and of the Proposition. □

4.4. L^1**-compactness.** In this section we will state the compactness result for sequences of functions on the perforated domains bounded in energy. In the sequel we will need the following Lemma

Lemma 4.13. *Let K be a closed set in Q. Then there exists a sequence of sets (C^m) in Q that are closures of open sets with Lipschitz boundary such that $C^{m+1} \subset\subset C^m$, and $\cap_{m\geq 1} C^m = K$. In particular the sequence (C^m) converges to K in the Hausdorff metric on $\bar{Q}$, and (χ_{C^m}) converges to χ_K in $L^1(Q)$.*

Moreover, if $Q \setminus K$ is connected we can choose the sets C^m such that $Q \setminus C^m$ is connected.

Proof. For every $m \in \mathbf{N}$ consider an open set A^m with Lipschitz boundary such that

$$\{x \in Q : \operatorname{dist}(x, K) > 1/m\} \subset\subset A^m \subset\subset \{x \in Q : \operatorname{dist}(x, K) > 1/(m+1)\}. \tag{4.21}$$

The existence of such as a set can be justified by taking a finite covering of the set $\{x \in Q : \operatorname{dist}(x, K) > 1/(m+1)\}$ made of balls compactly contained in $\{x \in Q : \operatorname{dist}(x, K) > 1/m\}$ and slightly traslating them in order to avoid cusp singularities. Then the first part of the statement is proved choosing C^m as the closure of the complementary of A^m.

Assume now that $Q \setminus K$ is connected. Let B^m denote the connected component of A^m whose closure contains ∂Q, and set $C^m = Q \setminus B^m$. Clearly C^m is the closure of an open set with Lipschitz boundary and $Q \setminus C^m = B^m$ is connected. Moreover since B^m, B^{m+1} are two connected components intersecting in a "neighbourhood" of ∂Q, by construction we have $B^m \subset\subset B^{m+1}$, and thus $C^{m+1} \subset\subset C^m \subset\subset Q$.

To conclude we prove that $\cap_{m\geq 1} C^m = K$. As $C^m \supseteq \{x \in Q : \mathrm{dist}(x, K) \leq 1/m\}$, therefore $\cap_{m\geq 1} C^m \supseteq K$. On the other hand, with fixed $x_0 \in Q \setminus K$ and $x_1 \in B^1$, there exists a continuous curve $\gamma : [0,1] \to Q \setminus K$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$ by the connectedness of $Q \setminus K$. Let $\eta =$ dist $(K, \gamma([0,1]))$, then $\eta > 0$ and $x_0 \in B^m$ for any $m > [1/\eta]$. Hence, $x_0 \notin \cap_{m\geq 1} C^m$, and this yields the claim. $\square$

Theorem 4.14 (L^1-Compactness for (u_n))**.** *Let $(u_n) \subset L^1(\Omega_n)$ be a sequence satisfying*

$$\int_{\Omega_n} |\nabla u_n|^2 \, dx + \mathcal{H}^{N-1}(S_{u_n}) + \|u_n\|_{L^\infty(\Omega_n)} \leq c \tag{4.22}$$

for some constant c independent of n. Then there exists $u \in SBV^2(\Omega)$ and a sequence $(w_n) \subset L^1(\Omega)$, with $w_n \equiv u_n$ in Ω_n, such that (up to a subsequence) (w_n) converges to u in $L^1(\Omega)$.

Proof. For any $m \in \mathbf{N}$ let C^m be as in Lemma 4.13, that is a closed set with Lipschitz continuous boundary containing K such that $Q \setminus C^m$ is connected. Set

$$C_n^m := \bigcup_{z\in\mathbf{Z}} \varepsilon_n(z + C^m), \qquad \Omega_n^m := \Omega \setminus C_n^m.$$

By applying Remark 4.2 to the perforated domain Ω_n^m we deduce that there exists a subsequence (not relabeled for convenience) $(\tilde{u}_n^m)$ converging in $L^1(\Omega)$ to some $u^m \in BV(\Omega)$, with $\tilde{u}_n^m \equiv u_n$ on Ω_n^m. Actually, by Remark 3.1 we infer that the limit function u^m does not depend on m; and thus we drop the superscript m and denote it only by u.

A diagonalization argument allows us to find a sequence $(\tilde{u}_{n(m)}^m)$ which converges to u in $L^1(\Omega)$. Finally set

$$w_m(x) := \begin{cases} \tilde{u}_{n(m)}^m(x) & \text{if } x \in K_{n(m)}; \\ u_{n(m)}(x) & \text{if } x \in \Omega_{n(m)}. \end{cases}$$

To conclude notice that the set $\{x \in \Omega : w_m(x) \neq \tilde{u}_{n(m)}^m(x)\}$ is contained in $C_{n(m)}^m \setminus K_{n(m)}$ so that (w_m) converges to u in measure and hence, since w_m are uniformly bounded in L^∞, $w_m \to u$ in $L^1(\Omega)$. Finally, in view of Proposition 4.12 we conclude that $u \in SBV^2(\Omega)$. $\square$

Remark 4.15. It is clear that if we remove the assumption that $Q\setminus K$ is connected the compactness result does not hold true anymore. For instance, it suffices to consider $K = Q_{1/4,1/2}$ and u_n to be equal to 1 in all the inner squares (rescaled and translated copies of $Q_{1/4}$) and 0 otherwise.

Nevertheless the compactness still stands in a weaker form. Indeed, let $\tilde{\Omega}_n$ be the connected component of Ω_n containing $\varepsilon_n \mathbf{Z}^N$, then it is possible to prove that for any $(u_n) \subset L^1(\Omega_n)$ as in (4.22) there exists a subsequence (w_n) with $w_n \equiv u_n$ on $\tilde{\Omega}_n$, and locally constant in $\Omega_n \setminus \tilde{\Omega}_n$, such that (up to a subsequence) (w_n) converges to u in $L^1(\Omega)$ for some $u \in SBV^2(\Omega)$.

5. The Γ-convergence result

In the sequel we study the asymptotics as $\varepsilon \to 0$ of the family of functionals $\mathcal{F}_\varepsilon^\psi$ defined in (3.1). In order to apply the direct methods of Γ-convergence we localize the energy functionals and for simplicity we first neglect the boundary conditions. We will set the problem in the ambient space $L^2(\Omega)$ and we represent the Γ-limit of $\mathcal{F}_\varepsilon$ with respect to the L^2-topology only on $SBV^2(\Omega) \cap L^2(\Omega)$. This formulation fits with the study of asymptotic behavior of minimizers of the functionals $\mathcal{F}_\varepsilon^\psi$ taking into account a L^∞ boundary datum ψ (see Section 6 and the related discussion therein).

For every $A \in \mathcal{A}(\Omega)$ and $\varepsilon > 0$ we set $A_\varepsilon = A \setminus K_\varepsilon$ and we introduce the functionals $\mathcal{F}_\varepsilon : L^2(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ defined for every $A \in \mathcal{A}(\Omega)$ by

$$\mathcal{F}_\varepsilon(u, A) := \begin{cases} \displaystyle\int_{A_\varepsilon} |\nabla u|^2 \, dx + \mathcal{H}^{N-1}(S_u \cap A_\varepsilon) & \text{if } u \in SBV^2(A), \\ +\infty & \text{otherwise in } L^2(\Omega). \end{cases} \tag{5.1}$$

Theorem 5.1. *For every $A \in \mathcal{A}(\Omega)$ the family $(\mathcal{F}_\varepsilon(\cdot, A))$ Γ-converges to some functional $\mathcal{F}_{hom}(\cdot, A)$ with respect to the L^2 topology. Moreover the functional $\mathcal{F}_{hom}(\cdot, A)$ restricted to $SBV^2(A)$ is given by*

$$\mathcal{F}_{hom}(u, A) := \int_A f_{hom}(\nabla u) \, dx + \int_{S_u \cap A} g_{hom}(\nu_u) \, d\mathcal{H}^{N-1}, \tag{5.2}$$

where f_{hom} and g_{hom} are defined in (1.4) *and* (1.6)*, respectively.*

The proof of Theorem 5.1 will be a consequence of several preliminary results (see Propositions 5.3, 5.4, 5.6 and 5.9). The first step is to show the compactness property in the sense of Γ-convergence of $\mathcal{F}_\varepsilon$ and the integral representation of its Γ-limit $\mathcal{F}$. These results follow by standard arguments in Γ-convergence; we will limit ourselves to provide the related references.

We start with the so called Fundamental Estimate (see [13, Proposition 3.1]).

Lemma 5.2. *Let $(\mathcal{F}_\varepsilon)$ be defined as in* (5.1)*. For every $\eta > 0$ and for every A', A, $B \in \mathcal{A}(\Omega)$, with $A' \subset\subset A$, there exists a constant $M > 0$ such that: for every $\varepsilon > 0$ and for every $u \in SBV^2(A), v \in SBV^2(B)$ there exists a function $\varphi \in C_0^\infty(A)$ with $\varphi = 1$ on A', $0 \le \varphi \le 1$ and*

$$\begin{aligned} &\mathcal{F}_\varepsilon(\varphi u + (1 - \varphi)v, A' \cup B) \\ &\qquad \le (1 + \eta)(\mathcal{F}_\varepsilon(u, A) + \mathcal{F}_\varepsilon(v, B)) + M\|u - v\|^2_{L^2((A \setminus A') \cap B)}. \end{aligned} \tag{5.3}$$

The Fundamental Estimate and standard arguments of the localization methods of Γ-convergence imply the following result (see [24] and [13]).

Proposition 5.3. *Let (ε_n) be a positive vanishing sequence. Then there exists a subsequence (ε_{j_n}) of (ε_n) and a functional $\mathcal{F} : L^2(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ such that for every $A \in \mathcal{A}(\Omega)$*

$$\mathcal{F}(\cdot, A) = \Gamma\text{-}\lim_n \mathcal{F}_{\varepsilon_{j_n}}(\cdot, A).$$

Moreover $\mathcal{F}$ satisfies the following properties

(a) *the set function $\mathcal{F}(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure on Ω for every fixed $u \in SBV^2(\Omega) \cap L^2(\Omega)$, and the functional $\mathcal{F}(\cdot, A)$ is local and $L^2(A)$ lower semicontinuous for every $A \in \mathcal{A}(\Omega)$;*
(b) *for every $A \in \mathcal{A}(\Omega)$ with $A \subset\subset \Omega$ and for every $y \in \mathbf{R}^N$ such that $y + A \subset \Omega$ and $u \in SBV^2(A)$ we have $\mathcal{F}(u(\cdot - y), A + y) = \mathcal{F}(u, A)$;*
(c) *for every $z \in \mathbf{R}$, $A \in \mathcal{A}(\Omega)$ and $u \in SBV^2(A) \cap L^2(A)$ we have $\mathcal{F}(u + z, A) = \mathcal{F}(u, A)$.*

By taking into account the integral representation results of [11] we get the following result.

Proposition 5.4. *Assume that $(\mathcal{F}_{\varepsilon_n}(\cdot, A))$ Γ-converges to a functional $\mathcal{F}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$. Then there exist Borel functions $f : \mathbf{R}^N \to [0, +\infty]$ and $g : \mathbf{R} \times \mathbf{S}^{N-1} \to [0, +\infty]$ such that for every $A \in \mathcal{A}(\Omega)$ and $u \in SBV^2(A)$*

$$\mathcal{F}(u, A) = \int_A f(\nabla u) dx + \int_{S_u \cap A} g(u^+ - u^-, \nu) \, d\mathcal{H}^{N-1}. \tag{5.4}$$

Proof. To prove the result we apply the integral representation Theorem 1 [11]. In order to match the assumptions of that result we need to extend $\mathcal{F}(\cdot, A)$, $A \in \mathcal{A}(\Omega)$, to $SBV^2(A)$ by relaxation with respect to the L^1 topology and then to use a perturbation argument to enforce the growth condition from below.

In this respect let us consider the functional $\mathcal{F}(\cdot, A)$ extended to $SBV^2(A)$ as follows

$$\mathcal{F}(u, A) := \inf\{\liminf_n \mathcal{F}(u_n, A),\ u_n \to u \text{ in } L^1(A)\}.$$

By a truncation argument it possible to check that this relaxation procedure does not change the value of $\mathcal{F}$ on $SBV^2(A) \cap L^2(A)$. Thanks to Proposition 5.3, conditions $(H1)$-$(H3)$ in Theorem 1 [11] are satisfied, namely $\mathcal{F}$ is a variational semicontinuous functional on $SBV^2(\Omega) \times \mathcal{A}(\Omega)$ with respect to the L^1 topology.

In order to enforce the growth condition from below $(H4)$ let us fix $\delta > 0$ and consider the functional

$$\mathcal{F}^\delta(u, A) = \mathcal{F}(u, A) + \delta \int_A |\nabla u|^2 dx + \delta \int_{S_u \cap A} (1 + |u^+ - u^-|)\, d\mathcal{H}^{N-1}.$$

According to Theorem 1 of [11] there exist Borel functions $f^\delta : A \times \mathbf{R} \times \mathbf{R}^N \to [0, +\infty]$, $g^\delta : A \times \mathbf{R} \times \mathbf{R} \times \mathbf{S}^{N-1} \to [0, +\infty]$ for which

$$\mathcal{F}^\delta(u, A) = \int_A f^\delta(x, u, \nabla u) dx + \int_{S_u \cap A} g^\delta(x, u^+, u^-, \nu_u)\, d\mathcal{H}^{N-1}$$

for every $A \in \mathcal{A}(\Omega)$ and $u \in SBV^2(A)$.

Thanks to properties (b) and (c) in Proposition 5.3 we conclude that both f^δ and g^δ are independent of x, that f^δ does not depend on u, and that g^δ depends on (u^+, u^-) only through their difference so that we may write $g^\delta = g^\delta(u^+ - u^-, \nu)$. By construction the families $(f^\delta)_{\delta>0}$, $(g^\delta)_{\delta>0}$ are increasing in δ, hence we can set $f = \lim_{\delta\to 0^+} f^\delta$, $g = \lim_{\delta \to 0^+} g^\delta$. To conclude we use the pointwise convergence of $(\mathcal{F}^\delta(\cdot, A))_{\delta>0}$ to $\mathcal{F}(\cdot, A)$ and the Monotone Convergence Theorem. □

Remark 5.5. A more refined argument actually shows that g is independent of (u^+, u^-), so that (5.4) rewrites as

$$\mathcal{F}(u, A) = \int_A f(\nabla u) dx + \int_{S_u \cap A} g(\nu_u)\, d\mathcal{H}^{N-1}.$$

We will derive directly such a result in Proposition 5.9 where we prove the equality $g = g_{hom}$.

In the next proposition we identify the bulk density of all Γ-cluster points of $(\mathcal{F}_\varepsilon)$ to be f_{hom}. We will use the standard notation $[\cdot]$ for the integer part.

Proposition 5.6. *Assume that $(\mathcal{F}_{\varepsilon_n}(\cdot, A))$ Γ-converges to a functional $\mathcal{F}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$. Then for every $\xi \in \mathbf{R}^N$*

$$f(\xi) = f_{hom}(\xi), \tag{5.5}$$

where f is the bulk energy density of $\mathcal{F}(\cdot, A)$, and f_{hom} is defined in (1.4).

Proof. For the sake of simplicity we assume that the unitary cube Q is contained in Ω. Fix $\xi \in \mathbf{R}^N$, we begin with proving inequality $f(\xi) \le f_{hom}(\xi)$. To this aim consider any $w \in W^{1,2}_\sharp(Q \setminus K)$, extend it to 0 on K and define $w_n(x) = \varepsilon_n w(x/\varepsilon_n)$. We have $(w_n) \in L^2(Q) \cap W^{1,2}(Q \setminus K_{\varepsilon_n})$ and (w_n) converges to 0 in $L^2(Q)$. Moreover, setting $v_n(x) = w_n(x) + \xi \cdot x$, by periodicity and a change of variables it follows

$$\mathcal{F}_{\varepsilon_n}(v_n, Q) = \int_{Q\setminus K_{\varepsilon_n}} |\nabla w(x/\varepsilon_n) + \xi|^2 dx \le \varepsilon_n^N \left(1 + \left[\frac{1}{\varepsilon_n}\right]\right)^N \int_{Q \setminus K} |\nabla w + \xi|^2 dx.$$

Since (v_n) converges to $\xi \cdot x$ in $L^2(Q)$ we deduce

$$f(\xi) = \mathcal{F}(\xi \cdot x, Q) \leq \liminf_n \mathcal{F}_{\varepsilon_n}(v_n, Q) \leq \int_{Q \setminus K} |\nabla w + \xi|^2 dx,$$

taking the infimum with respect to w we conclude.

The proof of the opposite inequality $f_{hom}(\xi) \leq f(\xi)$ will be split into several steps. Let us first deal with regular perforations K, namely we assume that K is the closure of an open set with Lipschitz boundary (with $Q \setminus K$ connected).

Consider a sequence $(w_n) \in L^2(Q)$ converging to $\xi \cdot x$ in $L^2(Q)$ and such that

$$f(\xi) = \mathcal{F}(\xi \cdot x, Q) = \lim_n \mathcal{F}_{\varepsilon_n}(w_n, Q).$$

Since $\mathcal{F}_{\varepsilon_n}$ decreases by truncation we may also suppose $\|w_n\|_{L^\infty(Q)} \leq \|\xi \cdot x\|_{L^\infty(Q)}$ for every $n \in \mathbf{N}$. We first use a blow-up type argument in order to get from (w_n) a new sequence whose energy has not increased and whose jump set is vanishing (see Step 1 in Proposition 5.2 [13]).

Step 1. *Reduction to a recovery sequence with vanishing jumps.* More precisely, we prove that there exist a diverging sequence $(j_n) \subseteq \mathbf{N}$ and $(v_n) \subseteq L^2(Q)$ such that

(a) (v_n) converges to $\xi \cdot x$ in $L^2(Q)$;
(b) $\|v_n\|_{L^\infty(Q)} \leq \|\xi \cdot x\|_{L^\infty(Q)}$ for every $n \in \mathbf{N}$;
(c) $\lim_n \mathcal{H}^{N-1}(S_{v_n} \cap (Q \setminus K_{1/j_n})) = 0$;
(d) $\limsup_n F_{1/j_n}(v_n, Q) \leq f(\xi)$.

Fix a sequence $(j_n) \subset \mathbf{N}$ to be chosen later, let $Q_n^{\mathtt{i}} = j_n \varepsilon_n(\mathtt{i} + Q)$ be a cube among those of type $j_n \varepsilon_n(i + Q) \subset Q$, $i \in \mathbf{Z}^N$, which satisfies

$$\left[\frac{1}{j_n \varepsilon_n}\right]^N \mathcal{F}_{\varepsilon_n}(w_n, Q_n^{\mathtt{i}}) \leq \mathcal{F}_{\varepsilon_n}(w_n, Q). \tag{5.6}$$

Define $v_n \in L^2(Q)$ to be

$$v_n(x) = \frac{1}{j_n \varepsilon_n} w_n(j_n \varepsilon_n(\mathtt{i} + x)) - \xi \cdot \mathtt{i},$$

then a simple change of variables entails

$$\|v_n - \xi \cdot x\|_{L^2(Q)} = (j_n \varepsilon_n)^{-(1+N/2)} \|w_n - \xi \cdot x\|_{L^2(Q_n^{\mathtt{i}})}. \tag{5.7}$$

It is easy to check that we may choose (j_n) in such a way that $j_n \to +\infty$, $j_n \varepsilon_n \to 0$ and (5.7) vanishes as $n \to +\infty$. So that (a) is established.

Moreover, the choice of $Q_n^{\mathtt{i}}$ in (5.6) implies by changing variables

$$\begin{aligned}&\mathcal{H}^{N-1}(S_{v_n} \setminus K_{1/j_n}) \\ &\quad = (j_n \varepsilon_n)^{1-N} \mathcal{H}^{N-1}(S_{w_n} \cap (Q_n^{\mathtt{i}} \setminus K_{\varepsilon_n})) \leq (j_n \varepsilon_n)^{1-N} \left[\frac{1}{j_n \varepsilon_n}\right]^{-N} \mathcal{F}_{\varepsilon_n}(w_n, Q)\end{aligned}$$

and

$$\int_{Q \setminus K_{1/j_n}} |\nabla v_n|^2 dx = (j_n \varepsilon_n)^{-N} \int_{Q_n^{\mathtt{i}} \setminus K_{\varepsilon_n}} |\nabla w_n|^2 dx \leq (j_n \varepsilon_n)^{-N} \left[\frac{1}{j_n \varepsilon_n}\right]^{-N} \mathcal{F}_{\varepsilon_n}(w_n, Q),$$

from which we deduce (c) and (d), respectively.

Eventually, statement (b) follows by truncating v_n at levels $\pm \|\xi \cdot x\|_{L^\infty(Q)}$.

Next we refine the recovery sequence to obtain a sequence with Sobolev regularity. To do this we employ by now standard techniques to truncate gradients.

Step 2. *Reduction to a recovery sequence in Sobolev spaces.* In this step we prove that for every fixed cube $Q' \subset\subset Q$ there exists $(u_n) \subseteq W^{1,2}(Q')$ such that

(a′) (u_n) converges to $\xi \cdot x$ in $L^2(Q')$;
(b′) $\|u_n\|_{L^\infty(Q')} \leq \|\xi \cdot x\|_{L^\infty(Q)}$ for every $n \in \mathbf{N}$;
(d′) $\limsup_n F_{1/j_n}(u_n, Q') \leq f(\xi)$.

Following an argument of Larsen [34, Lemma 2.1] we can modify v_n in order to construct a function $\widetilde{v}_n \in W^{1,\infty}(Q \setminus K_{1/j_n})$ such that

$$\lim_n \mathcal{L}^N(\{x \in Q \setminus K_{1/j_n} : \widetilde{v}_n(x) \neq v_n(x)\}) = 0 \tag{5.8}$$

and

$$\sup_n \int_{Q \setminus K_{1/j_n}} |\nabla \widetilde{v}_n|^2 dx < +\infty.$$

Up to a truncation argument, thanks to Step 1 (b), we may assume also that $\|\widetilde{v}_n\|_{L^\infty(Q)} \leq \|\xi \cdot x\|_{L^\infty(Q)}$. Furthermore, by taking advantage of the connectedness of $Q \setminus K$ and of the Lipschitz regularity assumption on K we employ classical extension results to fill the holes (see [1, Theorem 2.1], and also [22]). More precisely, with fixed $Q' \subset\subset Q$ we extend $\widetilde{v}_n$ to the full Q' (we keep the notation $\widetilde{v}_n$ for the extended function) with $\widetilde{v}_n \in W^{1,2}(Q')$ and $\sup_n \|\widetilde{v}_n\|_{W^{1,2}(Q')} < +\infty$. Then [31, Lemma 1.2] provides a sequence $(u_n) \in W^{1,2}(Q')$ such that

$$\lim_n \mathcal{L}^N(\{x \in Q' : \widetilde{v}_n(x) \neq u_n(x)\}) = 0, \tag{5.9}$$

and $(|\nabla u_n|^2)$ is equi-integrable on Q'. Up to the usual truncation argument we may assume also that $\|u_n\|_{L^\infty(Q')} \leq \|\xi \cdot x\|_{L^\infty(Q)}$.

By collecting (5.8) and (5.9) we infer

$$\lim_n \mathcal{L}^N(\{x \in Q' \setminus K_{1/j_n} : u_n(x) \neq v_n(x)\}) = 0. \tag{5.10}$$

Since $(|\nabla u_n|^2)$ is equi-integrable, by Step 1 (c) and (d) we get

$$\begin{aligned}\limsup_n \int_{Q' \setminus K_{1/j_n}} |\nabla u_n|^2 dx &= \limsup_n \int_{(Q' \setminus K_{1/j_n}) \setminus \{u_n \neq v_n\}} |\nabla u_n|^2 dx \\ &= \limsup_n \int_{(Q' \setminus K_{1/j_n}) \setminus \{u_n \neq v_n\}} |\nabla v_n|^2 dx \leq \limsup_n \int_{Q \setminus K_{1/j_n}} |\nabla v_n|^2 dx \leq f(\xi),\end{aligned}$$

so that (d′) is established.

Let us pass to the proof of (a′). Given any subsequence of (u_n) by Sobolev embedding we may extract a further subsequence (u_{j_n}) converging to a function u in $L^2(Q')$. Set $\varphi_n = \chi_{(Q' \setminus K_{1/j_n}) \setminus \{u_{j_n} \neq v_{j_n}\}}$, then by (5.10) (see also Remark 3.1) (φ_n) converges to $1 - \mathcal{L}^N(K)$ weak $*$ $L^\infty(Q')$. By taking into account Step 1 (a), $(\varphi_n(u_{j_n} - v_{j_n}))$ converges to $(1 - \mathcal{L}^N(K))(u - \xi \cdot x)$ weak $L^1(Q')$, and since $\varphi_n(u_{j_n} - v_{j_n}) = 0$ $\mathcal{L}^N$ a.e. on Q' we deduce that $u = \xi \cdot x$ $\mathcal{L}^N$ a.e. on Q'. Furthermore, Urysohn property implies (a′), i.e. the whole sequence (u_n) converges to $\xi \cdot x$ in $L^2(Q')$. This concludes the proof of Step 2.

Step 3. *Conclusion.* Let us first prove $f_{hom}(\xi) \leq f(\xi)$ for K Lipschitz regular. In this case the classical homogenization result for Sobolev spaces in perforated domains (see [12, Theorem 19.1]) and Step 2 entail

$$\mathcal{L}^N(Q') f_{hom}(\xi) \leq \liminf_n F_{1/j_n}(u_n, Q') \leq f(\xi).$$

The thesis follows as $\mathcal{L}^N(Q') \to 1^-$.

Finally we recover the general case (without assuming further regularity on K) through an approximation argument. More precisely consider a generic closed set K (with $Q \setminus K$ connected) and let (C^m) be a sequence as in Lemma 4.13. Let $f^m_{hom} : \mathbf{R}^N \to [0, +\infty]$ be defined as f_{hom} in (1.4) with K there substituted by C^m, i.e.

$$f^m_{hom}(\xi) = \inf \left\{ \int_{Q\setminus C^m} |\nabla w + \xi|^2 : \, w \in W^{1,2}_\sharp(Q \setminus C^m) \right\}.$$

It is clear that $f^m_{hom} \le f^{m+1}_{hom} \le f_{hom}$, we claim that

$$\sup_m f^m_{hom} = f_{hom}. \tag{5.11}$$

Indeed, for every $m \in \mathbf{N}$ let $w_m \in W^{1,2}_\sharp(Q \setminus C^m)$, with $\fint_{Q\setminus C^1} w_m dx = 0$, be such that

$$\int_{Q\setminus C^m} |\nabla w_m + \xi|^2 dx \le f^m_{hom}(\xi) + \frac{1}{m}.$$

Note that for every fixed $M > 0$

$$\sup_{m \ge M} \int_{Q\setminus C^M} |\nabla w_m + \xi|^2 dx \le f_{hom}(\xi) + 1 < +\infty.$$

In particular, the sequence $(w_m)_{m\ge M}$ is bounded in $W^{1,2}(Q \setminus C^M)$ by Poincaré-Wirtinger inequality for every M. Then a diagonal argument implies the existence of a subsequence (w_{j_m}) weakly precompact in $W^{1,2}(Q \setminus C^M)$ for every M. Denote by w a cluster point, then $w \in W^{1,2}_\sharp(Q \setminus C^M)$ for every M and

$$\int_{Q\setminus C^M} |\nabla w + \xi|^2 dx \le \liminf_m \int_{Q\setminus C^M} |\nabla w_{j_m} + \xi|^2 dx \le \sup_m f^m_{hom}(\xi).$$

By letting $M \to +\infty$ we infer that actually $w \in L^2_{\rm loc}(Q \setminus K)$, $\nabla w \in L^2(Q \setminus K, \mathbf{R}^N)$ and

$$\int_{Q\setminus K} |\nabla w + \xi|^2 dx \le \sup_m f^m_{hom}(\xi). \tag{5.12}$$

In particular, it is easy to check that the truncated functions $w^j = (w \wedge j) \vee (-j)$ belong to $W^{1,2}_\sharp(Q \setminus K)$ and for every M

$$\begin{aligned} f_{hom}(\xi) &\le \int_{Q\setminus K} |\nabla w^j + \xi|^2 dx \\ &= \int_{(Q\setminus K)\setminus\{|w|\ge j\}} |\nabla w + \xi|^2 dx + |\xi|^2 \mathcal{L}^N((Q \setminus K) \cap \{|w| \ge j\}) \\ &\le \int_{Q\setminus K} |\nabla w + \xi|^2 dx + |\xi|^2 \left(\mathcal{L}^N((Q \setminus C^M) \cap \{|w| \ge j\}) + \mathcal{L}^N(C^M \setminus K) \right). \end{aligned}$$

Since $w \in L^2(Q \setminus C^M)$ we have $\mathcal{L}^N((Q \setminus C^M) \cap \{|w| \ge j\}) \to 0$ as $j \to +\infty$, so that

$$\limsup_j \int_{Q\setminus K} |\nabla w^j + \xi|^2 dx \le \int_{Q\setminus K} |\nabla w + \xi|^2 dx + |\xi|^2 \mathcal{L}^N(C^M \setminus K). \tag{5.13}$$

Eventually from (5.12) and (5.13) we deduce

$$f_{hom}(\xi) \le \sup_m f^m_{hom}(\xi) + |\xi|^2 \mathcal{L}^N(C^M \setminus K),$$

and inequality (5.11) follows as $M \to +\infty$.

Finally denote by $\mathcal{F}^m_{\varepsilon_n}$ the functional defined in (5.1) with C^m in place of K, then $\mathcal{F}^m_{\varepsilon_n} \le \mathcal{F}_{\varepsilon_n}$. Up to extracting a further subsequence we assume that $(\mathcal{F}^m_{\varepsilon_n}(\cdot, A))$ Γ-converges to a functional $F^m(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$. By Step 1 and 2 we know that the bulk energy density of F^m is

f^m_{hom}, and by construction $f^m_{hom}(\xi) \le f(\xi)$ for every $m \in \mathbf{N}$. Hence, we derive $f_{hom}(\xi) \le f(\xi)$ from (5.11). $\square$

Remark 5.7. The argument above entails the existence of a minimizer for the minimum problem defining f_{hom} in (1.4) in a suitable Deny-Lions type space (see [28]).

In order to prove the counterpart of Proposition 5.6 for the surface term we first show that the limit defining g_{hom} exists. To this aim we introduce some extension procedure.

Given any $\nu \in \mathbf{S}^{N-1}$, let $\{\nu_1, \dots, \nu_{N-1}\}$ any collection of unitary vectors such that $\{\nu_1, \dots, \nu_{N-1}, \nu\}$ form an orthonormal basis of $\mathbf{R}^N$ with unit cell Q^ν. Given $w \in P(Q^\nu \setminus K_\varepsilon)$ such that $w = u_{0,1,\nu}$ (defined in (1.5)) on a neighbourhood of ∂Q^ν, we regard w as extended to $\mathbf{R}^N$ as follows. First we extend it on Q^ν by setting $w \equiv u_{0,1,\nu}$ in K_ε, then on the strip $\mathcal{S} = \{x \in \mathbf{R}^N : |\langle x, \nu\rangle| \le 1/2\}$ by 1-periodicity in directions $\nu_1, \dots, \nu_{N-1}$, and finally we set $w \equiv u_{0,1,\nu}$ on $\{x \in \mathbf{R}^N : |\langle x, \nu\rangle| \ge 1/2\}$.

Lemma 5.8. *For every $\nu \in \mathbf{S}^{N-1}$ there exists the limit as $\varepsilon \to 0^+$ of $m_\varepsilon(\nu)$, where*

$$m_\varepsilon(\nu) = \inf_{w \in P(Q^\nu \setminus K_\varepsilon)} \{\mathcal{H}^{N-1}(S_w \setminus K_\varepsilon) : w = u_{0,1,\nu} \text{ on a neighbourhood of } \partial Q^\nu\}.$$

Proof. Let $\varepsilon, \sigma, \eta > 0$ be fixed, with $\sigma \le \varepsilon$, and let $\nu_1, \dots, \nu_{N-1}$ be unitary vectors as above. Fix $w \in P(Q^\nu \setminus K_\varepsilon)$ such that $w = u_{0,1,\nu}$ on a neighbourhood of ∂Q^ν, and regard it as extended to $\mathbf{R}^N$ as explained above.

Consider the strip $\mathcal{S}_{\sigma/\varepsilon} = \{x \in \mathbf{R}^N : |\langle x, \nu\rangle| \le \sigma/(2\varepsilon)\}$ and its decomposition into cubes of the family $\Lambda = \left\{\frac{\sigma}{\varepsilon}(i + Q^\nu) : i \in \oplus_{k=1}^{N-1}\nu_k\mathbf{Z}\right\}$, where $\oplus_{k=1}^{N-1}\nu_k\mathbf{Z}$ is the $N-1$ dimensional integer lattice generated by $\nu_1, \dots, \nu_{N-1}$. Moreover let $\mathcal{I} = \left\{i \in \oplus_{k=1}^{N-1}\nu_k\mathbf{Z} : \frac{\sigma}{\varepsilon}(i + Q^\nu) \subset \eta Q^\nu\right\}$, then a simple counting argument gives

$$\#\mathcal{I} \le \left(\frac{\varepsilon\eta}{\sigma}\right)^{N-1}. \tag{5.14}$$

Define $w_\sigma : Q^\nu \to \{0, 1\}$ by $u_{0,1,\nu}$ on $Q^\nu \setminus \mathcal{S}_{\sigma/\varepsilon}$ and on each cube of the family Λ intersecting $Q^\nu \setminus \eta Q^\nu$, and let $w_\sigma(x) = w(\varepsilon x/\sigma)$ otherwise on $\mathcal{S}_{\sigma/\varepsilon}$.

By construction $w_\sigma \in P(Q^\nu \setminus K_\sigma)$ and $w_\sigma = u_{0,1,\nu}$ on $Q^\nu \setminus \eta Q^\nu$, and since $S_{w_\sigma} \cap (Q^\nu \setminus \eta Q^\nu) \subseteq \{x \in \mathbf{R}^N : \langle x, \nu\rangle = 0\} \cap (Q^\nu \setminus \eta Q^\nu)$, we have $\mathcal{H}^{N-1}(S_{w_\sigma} \cap (Q^\nu \setminus \eta Q^\nu)) \le 1 - \eta^{N-1}$. Furthermore (5.14), the 1-periodicity of w in directions $\nu_1, \dots, \nu_{N-1}$, and a scaling argument imply

$$\begin{aligned} \mathcal{H}^{N-1}(S_{w_\sigma} \setminus K_\sigma) &\le \#\mathcal{I}\left(\frac{\sigma}{\varepsilon}\right)^{N-1} \mathcal{H}^{N-1}(S_w \setminus K_\varepsilon) + 1 - \eta^{N-1} \\ &\le \eta^{N-1}\mathcal{H}^{N-1}(S_w \setminus K_\varepsilon) + 1 - \eta^{N-1}. \end{aligned} \tag{5.15}$$

Passing to the infimum on the class of admissible functions on both sides of (5.15) and then on the superior limit as $\sigma \to 0^+$ and the inferior limit as $\varepsilon \to 0^+$ we infer

$$\limsup_{\sigma \to 0^+} m_\sigma(\nu) \le \eta^{N-1} \liminf_{\varepsilon \to 0^+} m_\varepsilon(\nu) + 1 - \eta^{N-1},$$

and the thesis follows as $\eta \to 1^-$. $\square$

In the next proposition we identify the surface density of all Γ-cluster points of $(\mathcal{F}_\varepsilon)$ to be g_{hom}.

Proposition 5.9. *Assume that $(\mathcal{F}_{\varepsilon_n}(\cdot, A))$ Γ-converges to a functional $\mathcal{F}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$. Then for every $(a, b, \nu) \in \mathbf{R} \times \mathbf{R} \times \mathbf{S}^{N-1}$*

$$g(b - a, \nu) = g_{hom}(\nu), \tag{5.16}$$

where g is the surface energy density of $\mathcal{F}(\cdot, A)$ and g_{hom} is defined in (1.6).

Proof. Fix $(a,b,\nu)\in\mathbf{R}\times\mathbf{R}\times\mathbf{S}^{N-1}$. We start with inequality $g(b-a,\nu)\leq g_{hom}(\nu)$.

To this aim fixed $\varepsilon>0$ consider any $w\in P(Q^\nu\setminus K_\varepsilon)$ such that $w=u_{0,1,\nu}$ on a neighbourhood of ∂Q^ν, regarded as extended to $\mathbf{R}^N$ with the convention adopted before Lemma 5.8. Define $w_n(x)=a+(b-a)w(\varepsilon x/\varepsilon_n)$, then a simple change of variables gives

$$\|w_n-u_{a,b,\nu}\|_{L^2(Q^\nu)}\leq|b-a|\left(\frac{\varepsilon_n}{\varepsilon}\right)^{N/2}\left(1+\left[\frac{\varepsilon}{\varepsilon_n}\right]\right)^{(N-1)/2}\|w-u_{0,1,\nu}\|_{L^2(Q^\nu)},$$

so that (w_n) converges to $u_{a,b,\nu}$ in $L^2(Q^\nu)$. Moreover, a straightforward calculation implies

$$\begin{aligned}\mathcal{F}_{\varepsilon_n}(w_n,Q^\nu)&=\mathcal{H}^{N-1}(S_{w_n}\cap(Q^\nu\setminus K_{\varepsilon_n}))\\&=\mathcal{H}^{N-1}(S_{w_n}\cap\{x\in Q^\nu\setminus K_{\varepsilon_n}:|\langle x,\nu\rangle|\leq\varepsilon_n/(2\varepsilon)\})\\&\leq\left(\frac{\varepsilon_n}{\varepsilon}\right)^{N-1}\left(1+\left[\frac{\varepsilon}{\varepsilon_n}\right]\right)^{N-1}\mathcal{H}^{N-1}(S_w\cap(Q^\nu\setminus K_\varepsilon)).\end{aligned}$$

Taking the limit as $n\to+\infty$ we infer

$$\mathcal{F}(u_{a,b,\nu},Q^\nu)\leq\liminf_n\mathcal{F}_{\varepsilon_n}(w_n,Q^\nu)\leq\mathcal{H}^{N-1}(S_w\cap(Q^\nu\setminus K_\varepsilon)),$$

by passing first to the infimum on all such w's and then to the limit as $\varepsilon\to0^+$ inequality $g(b-a,\nu)\leq g_{hom}(\nu)$ follows by Lemma 5.8.

The proof of the opposite inequality $g_{hom}(\nu)\leq g(b-a,\nu)$ will be split into three steps. To fix notations we will assume $a\leq b$. Consider a sequence $(w_n)\in L^2(Q^\nu)$ converging to $u_{a,b,\nu}$ in $L^2(Q^\nu)$ and such that

$$g(b-a,\nu)=\mathcal{F}(u_{a,b,\nu},Q^\nu)=\lim_n\mathcal{F}_{\varepsilon_n}(w_n,Q^\nu).$$

By a truncation argument we may also suppose $a\leq w_n\leq b$ for every $n\in\mathbf{N}$. First we use a blow-up type argument as in [13, Proposition 6.2], in order to get from (w_n) a new sequence whose energy has not increased in the limit and whose gradient energy is vanishing.

Step 1. *Reduction to a recovery sequence with vanishing gradients.* We prove that there exist a diverging sequence $(j_n)\in\mathbf{N}$ and $(v_n)\in L^2(Q^\nu)$ such that

(a) (v_n) converges to $u_{a,b,\nu}$ in $L^2(Q^\nu)$;
(b) $a\leq v_n\leq b$ for every $n\in\mathbf{N}$;
(c) $\lim_n\int_{Q^\nu\setminus K_{1/j_n}}|\nabla v_n|^2dx=0$;
(d) $\limsup_n F_{1/j_n}(v_n,Q^\nu)\leq g(b-a,\nu)$.

Fix a sequence $(j_n)\subset\mathbf{N}$ to be chosen later, let $Q_n^{\mathtt{i}}=j_n\varepsilon_n(\mathtt{i}+Q^\nu)$ be a cube among those of type $j_n\varepsilon_n(i+Q^\nu)\subset Q^\nu$, $i\in\oplus_{k=1}^{N-1}\nu_k\mathbf{Z}$, satisfying

$$\left[\frac{1}{j_n\varepsilon_n}\right]^{N-1}\mathcal{F}_{\varepsilon_n}(w_n,Q_n^{\mathtt{i}})\leq\mathcal{F}_{\varepsilon_n}(w_n,Q^\nu).\tag{5.17}$$

Define $v_n\in L^2(Q^\nu)$ to be $v_n(x)=w_n(j_n\varepsilon_n(\mathtt{i}+x))$, then a simple change of variables entails

$$\|v_n-u_{a,b,\nu}\|_{L^2(Q^\nu)}=(j_n\varepsilon_n)^{-N/2}\|w_n-u_{a,b,\nu}\|_{L^2(Q_n^{\mathtt{i}})}.\tag{5.18}$$

It is easy to check that we may choose (j_n) in such a way that $j_n\to+\infty$, $j_n\varepsilon_n\to0$ and (5.18) vanishes as $n\to+\infty$. So that (a) is established.

Moreover, the choice of $Q_n^{\mathtt{i}}$ in (5.17) implies by changing variables

$$\begin{aligned}\mathcal{H}^{N-1}(S_{v_n}\setminus K_{1/j_n})&=(j_n\varepsilon_n)^{1-N}\mathcal{H}^{N-1}(S_{w_n}\cap(Q_n^{\mathtt{i}}\setminus K_{\varepsilon_n}))\\&\leq(j_n\varepsilon_n)^{1-N}\left[\frac{1}{j_n\varepsilon_n}\right]^{1-N}\mathcal{F}_{\varepsilon_n}(w_n,Q)\end{aligned}$$

and

$$\int_{Q\setminus K_{1/j_n}} |\nabla v_n|^2 dx = (j_n\varepsilon_n)^{2-N} \int_{Q_n^{\rm i}\setminus K_{\varepsilon_n}} |\nabla w_n|^2 dx \leq (j_n\varepsilon_n)^{2-N} \left[\frac{1}{j_n\varepsilon_n}\right]^{1-N} \mathcal{F}_{\varepsilon_n}(w_n, Q),$$

from which we deduce (c) and (d), respectively. Eventually, statement (b) follows straightforward.

In the next step the BV Co-area Formula (see [8, Theorem 3.40]) allows us to select suitable sublevels of the sequence in Step 1 whose perimeters is controlled by the energy functionals (see [13, Proposition 6.2]). Subsequently we use a geometric truncation argument, similar to that called *transfer of jump set* performed in [29], in order to obtain a sequence in SBV_0 matching the boundary conditions.

Step 2. *Reduction to a recovery sequence in $SBV_0(Q^\nu)$ satisfying the boundary conditions.* We prove that there exists $(\widehat{v}_n) \in SBV_0(Q^\nu)$ such that

(a′) $(\widehat{v}_n)$ converges to $u_{a,b,\nu}$ in $L^2(Q^\nu)$;
(b′) $\widehat{v}_n$ assumes only the values a, b for every $n \in \mathbf{N}$;
(c′) $\widehat{v}_n = u_{a,b,\nu}$ on a neighbourhood of ∂Q^ν;
(d′) $\limsup_n F_{1/j_n}(\widehat{v}_n, Q^\nu) \leq g(b-a, \nu)$.

Indeed, let us consider the sets $E_t^n = \{x \in Q^\nu : v_n(x) < t\}$, $E_t = \{x \in Q^\nu : u_{a,b,\nu}(x) < t\}$. Thanks to property (a) of Step 1 $E_t^n \to E_t$ in measure for $\mathcal{H}^1$ a.e. t and the BV Coarea Formula (see [8, Theorem 3.40]) yields

$$\int_a^b \mathcal{H}^{N-1}(\partial^* E_{s_n}^n \setminus K_{1/j_n})\, ds \leq |Dv_n|(Q^\nu \setminus K_{1/j_n}). \tag{5.19}$$

Note that the absolute continuous part of $|Dv_n|(Q^\nu \setminus K_{1/j_n})$ can be estimated by using the Hölder inequality, while for the singular part is sufficient to take into account that thanks to property (b) of Step 1 $|v_n^+ - v_n^-| \leq (b-a)$. Hence we can refine inequality (5.19) and obtain

$$\int_a^b \mathcal{H}^{N-1}(\partial^* E_{s_n}^n \setminus K_{1/j_n})\, ds \leq (b-a) F_{1/j_n}(v_n, Q^\nu) + \left(\int_{Q^\nu\setminus K_{1/j_n}} |\nabla v_n|^2 dx\right)^{1/2}. \tag{5.20}$$

By using the Mean Value Theorem in (5.19) and by using property (c) of Step 1 in (5.20), we may choose $s_n \in (a, b)$ such that we have convergence in measure of the sublevels $E_{s_n}^n$ and

$$\limsup_n \mathcal{H}^{N-1}(\partial^* E_{s_n}^n \setminus K_{1/j_n}) \leq \limsup_n F_{1/j_n}(v_n, Q^\nu). \tag{5.21}$$

Set $E_n = E_{s_n}^n$. Taking into account that $u_{a,b,\nu}$ is piecewise constant in Q^ν we easily infer that E_n tends in measure to the lower half cube. Let us now fix $\eta \in (0, 1/2)$ and set

$$Q_\eta^- := \{x \in Q^\nu : -\eta < \langle x, \nu\rangle < 0\}, \qquad Q_\eta^+ := \{x \in Q^\nu : 0 < \langle x, \nu\rangle < \eta\}.$$

Since $E_n \cap Q_\eta^-$ tends to Q_η^- in measure and $E_n \cap Q_\eta^+$ tends to zero in measure, recalling that $\mathcal{L}^N(Q_\eta^-) = \mathcal{L}^N(Q_\eta^+) = \eta$, for n large enough, we have that

$$\mathcal{L}^N(E_n \cap Q_\eta^-) > \eta - \eta^2, \qquad \mathcal{L}^N(E_n \cap Q_\eta^+) \leq \eta^2.$$

Therefore, thanks to Fubini's Theorem we may find a scalar $\bar{s}$ in a set of positive measure in $(0, \eta)$ such that

$$\mathcal{H}^{N-1}(E_n \cap \{x \in Q^\nu : \langle x, \nu\rangle = -\bar{s}\}) \geq 1 - \eta, \qquad \mathcal{H}^{N-1}(E_n \cap \{x \in Q^\nu : \langle x, \nu\rangle = \bar{s}\}) \leq \eta.$$

Finally set

$$H_n^- := \{x \in Q^\nu : \mathrm{dist}(x, \partial Q^\nu) \leq \eta,\, -\bar{s} \leq \langle x, \nu\rangle \leq 0\},$$
$$H_n^+ := \{x \in Q^\nu : \mathrm{dist}(x, \partial Q^\nu) \leq \eta,\, 0 < \langle x, \nu\rangle < \bar{s}\},$$

and consider functions $\widehat{v}_n^\eta$ defined by

$$\widehat{v}_n^\eta(x) := \begin{cases} a & \text{in } \{x \in Q^\nu : \langle x, \nu \rangle \le -\bar{s}\} \cup (Q_\eta^- \cap E_n) \cup ((Q_\eta^+ \cap E_n) \setminus H_n^+) \cup H_n^-, \\ b & \text{everywhere else in } Q^\nu. \end{cases}$$

By construction $\widehat{v}_n^\eta \in SBV_0(Q^\nu)$ and property (a′), (b′), (c′) are satisfied. In addition, by taking into account (5.21) we get

$$\begin{aligned} \limsup_n F_{1/j_n}(\widehat{v}_n^\eta, Q^\nu) &= \limsup_n \mathcal{H}^{N-1}(S_{\widehat{v}_n^\eta} \setminus K_{1/j_n}) \\ &\le \limsup_n \mathcal{H}^{N-1}(\partial^* E_{s_n}^n \setminus K_{1/j_n}) + O(\eta) \le \limsup_n F_{1/j_n}(v_n, Q^\nu) + O(\eta), \end{aligned}$$

where $O(\eta) \to 0$ as $\eta \to 0^+$. Finally we get a sequence $(\widehat{v}_n)$ satisfying property (d′), as well as (a′), (b′), (c′), by taking a positive vanishing sequence (η_n) and a standard diagonalization argument.

Step 3. *Conclusion.* Let $u_n = (\widehat{v}_n - a)/(b-a)$. Then u_n coincides with $u_{0,1,\nu}$ on a neighborhood of ∂Q^ν and converges to $u_{0,1,\nu}$ in $L^2(Q^\nu)$. Eventually

$$g_{hom}(\nu) \le \limsup_n \mathcal{H}^{N-1}(S_{u_n} \setminus K_{1/j_n}) = \limsup_n F_{1/j_n}(\widehat{v}_n, Q^\nu) \le g(b-a, \nu).$$

□

6. Matching boundary conditions

In this section we extend our asymptotic analysis adding a Dirichlet boundary condition on the fixed boundary $\partial\Omega$. We present a Γ-convergence result for (suitable restrictions of) the functionals $\mathcal{F}_\varepsilon^\psi$ defined in (3.1) and prove the convergence of the associated minimum problems. This last result will be a consequence of standard Γ-convergence theory once the equicoercivity of the associated minimum configurations is proved (see [24, Theorem 7.4]).

Since we are interested mainly in the asymptotic behavior of minimizers we restrict ourselves to the domain $SBV^2(\Omega) \cap L^2(\Omega)$. Indeed, as already mentioned at the beginning of Section 4, the functionals $\mathcal{F}_\varepsilon^\psi$ are decreasing by truncation, and thus we can limit our analysis to functions equibounded in $L^\infty(\Omega)$. According to this, we investigate the Γ-convergence of $(\mathcal{F}_\varepsilon^\psi)$ on the L^1-subspace $SBV^2(\Omega) \cap L^2(\Omega)$. In this respect, it is also clear that the convergence property is not affected by the choice of any L^p topology in which the study of the Γ-limit is set.

We begin with the Γ-convergence analysis. It exploits the result in the unconstrained case proved in Theorem 5.1.

Theorem 6.1. *The family $(\mathcal{F}_\varepsilon^\psi)$ Γ-converges to some functional $\mathcal{F}_{hom}^\psi$ with respect to the $L^2(\Omega)$ topology. Moreover the functional $\mathcal{F}_{hom}^\psi$ restricted to $SBV^2(\Omega)$ is given by*

$$\mathcal{F}_{hom}^\psi(u) := \int_\Omega f_{hom}(\nabla u)\, dx + \int_{S_u^\psi} g_{hom}(\nu_u)\, d\mathcal{H}^{N-1}, \tag{6.1}$$

where $S_u^\psi := S_u \cup \{x \in \partial\Omega : \psi(x) \ne u(x)\}$, and f_{hom}, g_{hom} are defined in (1.4) *and* (1.6), *respectively.*

Proof. Consider an open set $\tilde{\Omega}$ with $\Omega \subset\subset \tilde{\Omega}$, and let $\tilde{\mathcal{F}}_\varepsilon : SBV^2(\tilde{\Omega}) \cap L^2(\tilde{\Omega}) \to [0, +\infty]$ be defined as in (5.1) with A replaced by $\tilde{\Omega}$. By Theorem 5.1 we have that the functionals $\tilde{\mathcal{F}}_\varepsilon$ Γ-converge to the functional $\tilde{\mathcal{F}}_{hom}$ defined as in (5.2) with A replaced by $\tilde{\Omega}$.

In order to prove the Γ-liminf inequality for the functionals $\mathcal{F}_\varepsilon^\psi$, let $u_\varepsilon \to u$ in $L^2(\Omega)$, and set $\tilde{u}_\varepsilon$ (respectively $\tilde{u}$) equal to u_ε (respectively u) in Ω, and equal to ψ in $\tilde{\Omega} \setminus \Omega$. Taking into account that $\psi \in W^{1,2}(\tilde\Omega)$, we have that

$$\tilde{\mathcal{F}}_\varepsilon(\tilde u_\varepsilon) \le \mathcal{F}^\psi_\varepsilon(\tilde u_\varepsilon) + \int_{\tilde\Omega\setminus\Omega} |\nabla\psi|^2\,dx,$$

and thus by the Γ-liminf inequality for the functionals $\tilde{\mathcal{F}}_\varepsilon$ we get

$$\mathcal{F}^\psi_{hom}(u) \le \tilde{\mathcal{F}}_{hom}(\tilde u) \le \liminf \mathcal{F}^\psi_\varepsilon(\tilde u_\varepsilon) + \int_{\tilde\Omega\setminus\Omega} |\nabla\psi|^2\,dx.$$

We deduce the Γ-liminf inequality for the family $(\mathcal{F}^\psi_\varepsilon)$ by absolute continuity of Lebesgue integral by letting $\tilde\Omega$ decrease to Ω.

Let us pass to the Γ-limsup inequality. To this aim let $u \in SBV^2(\Omega)\cap L^2(\Omega)$ and let $\tilde u$ be its extension to $\tilde\Omega$ defined to be equal to ψ in $\tilde\Omega\setminus\Omega$. Taking into account the fundamental estimate in Lemma 5.2 it is easy to infer the existence of a recovery sequence $(\tilde u_\varepsilon)$ for the functionals $\tilde{\mathcal{F}}_\varepsilon$ satisfying

$$\lim \tilde{\mathcal{F}}_\varepsilon(\tilde u_\varepsilon) = \tilde{\mathcal{F}}_{hom}(\tilde u),$$

with $\tilde u_\varepsilon \equiv \psi$ on $\tilde\Omega\setminus\Omega$. Therefore, setting u_ε to be the restriction of $\tilde u_\varepsilon$ to Ω we have

$$\limsup \mathcal{F}^\psi_\varepsilon(u_\varepsilon) \le \lim \tilde{\mathcal{F}}_\varepsilon(\tilde u_\varepsilon) = \tilde{\mathcal{F}}_{hom}(\tilde u) = \mathcal{F}^\psi_{hom}(u) + \int_{\tilde\Omega\setminus\Omega} f_{hom}(\nabla\psi)dx.$$

Again, since the term $\int_{\tilde\Omega\setminus\Omega} f_{hom}(\nabla\psi)dx$ can be chosen arbitrarily small, we deduce the Γ-limsup inequality for the functionals $\mathcal{F}^\psi_\varepsilon$. □

Before investigating the convergence of the minimum problems associated to $\mathcal{F}^\psi_\varepsilon$, we recall that for any $u \in L^1(\Omega)$ the value $\mathcal{F}^\psi_\varepsilon(u)$ (as well as $\mathcal{F}_\varepsilon(u)$) is not affected by that of u in the sets $\Omega\setminus\Omega_\varepsilon$. Due to this fact, a real compactness result for sequences of minimizers cannot hold unless K is negligible. Hence, in the general case, the next theorem can be thought as a selection principle of compact minimizing sequences in $L^1(\Omega)$. We recall also that, since the energy functionals decrease by truncations, we can always assume that the minimizers u_ε satisfy $\|u_\varepsilon\|_{L^\infty(\Omega)} \le \|\psi\|_{L^\infty(\Omega)}$.

Theorem 6.2. *For any $\varepsilon > 0$ let $u_\varepsilon \in L^1(\Omega_\varepsilon)$ be a minimizer for $\mathcal{F}^\psi_\varepsilon$ with $\|u_\varepsilon\|_{L^\infty(\Omega)} \le \|\psi\|_{L^\infty(\Omega)}$. Then there exists a family $(w_\varepsilon) \subset L^1(\Omega)$ which is compact in $L^1(\Omega)$ and such that $w_\varepsilon \equiv u_\varepsilon$ in Ω_ε for any $\varepsilon > 0$ (in particular w_ε are minimizers for $\mathcal{F}^\psi_\varepsilon$). Moreover, any cluster point u of w_ε is a minimizer for $\mathcal{F}^\psi_{hom}$.*

Proof. We can apply Theorem 4.14, obtaining the desired sequence $(w_\varepsilon) \subset L^1(\Omega)$. The fact that any cluster point u of w_ε is a minimizer for $\mathcal{F}^\psi_{hom}$ is a direct consequence of the Γ-convergence result given in Theorem 6.1 (see [24, Theorem 7.4]). □

7. Further results

In the present section we extend the asymptotic analysis performed in Sections 5, 6 for the Mumford-Shah energy in periodically perforated domains to more general free-discontinuity energies. We limit ourselves to state the generalizations of Theorems 5.1, 6.1, 6.2, being the proofs analogous and only technically more demanding (e.g., in the coercive case see [13, Section 8]).

In the following we keep the notation fixed in Sections 5, 6. Furthermore, let $p \in (1,+\infty)$ and consider $f : \mathbf{R}^N \times \mathbf{R}^N \to [0,+\infty)$, $g : \mathbf{R}^N \times \mathbf{S}^{N-1} \to [0,+\infty)$ two Borel functions. We suppose that f satisfies

(f1) $f(\cdot,\xi)$ is 1-periodic for every $\xi \in \mathbf{R}^N$, (f2) there exist two constants $c_1, c_2 > 0$ such that for every $(x,\xi) \in \mathbf{R}^N \times \mathbf{R}^N$

$$c_1|\xi|^p \le f(x,\xi) \le c_2(1+|\xi|^p),$$

and that g satisfies

(g1) $g(\cdot,\nu)$ is 1-periodic for every $\nu \in \mathbf{S}^{N-1}$,
(g2) $g(x,-\nu) = g(x,\nu)$ for every $(x,\nu) \in \mathbf{R}^N \times \mathbf{S}^{N-1}$,
(g3) there exist two constants $c_3, c_4 > 0$ such that for every $(x,\nu) \in \mathbf{R}^N \times \mathbf{S}^{N-1}$

$$c_3 \le g(x,\nu) \le c_4.$$

Then we introduce the family of functionals $\mathcal{G}^\psi_\varepsilon : L^p(\Omega) \to [0,+\infty]$ defined by

$$\mathcal{G}^\psi_\varepsilon(u) = \begin{cases} \displaystyle\int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}, \nabla u\right) dx + \int_{S_u^{\psi,\varepsilon}} g\left(\frac{x}{\varepsilon}, \nu_u\right) d\mathcal{H}^{N-1} & u \in SBV^p(\Omega), \\ +\infty & \text{otherwise in } L^p(\Omega). \end{cases} \tag{7.1}$$

We are now in a position to extend the results of Theorems 5.1, 6.1, 6.2 to the family to $(\mathcal{G}^\psi_\varepsilon)$.

Theorem 7.1. *The family $(\mathcal{G}^\psi_\varepsilon)$ Γ-converges to some functional $\mathcal{G}^\psi_{hom}$ with respect to the $L^p(\Omega)$ topology. Moreover, the functional $\mathcal{G}^\psi_{hom}$ restricted to $SBV^p(\Omega)$ is given by*

$$\mathcal{G}^\psi_{hom}(u) := \int_\Omega f_{hom}(\nabla u)\, dx + \int_{S_u^\psi} g_{hom}(\nu_u)\, d\mathcal{H}^{N-1},$$

where the bulk energy density $f_{hom} : \mathbf{R}^N \to [0,+\infty)$ is the convex function given by

$$f_{hom}(\xi) = \lim_{\varepsilon\to 0^+} \inf \left\{ \int_{Q\setminus K_\varepsilon} f\left(\frac{x}{\varepsilon}, \nabla w + \xi\right) dx : w \in W^{1,p}_\sharp(Q\setminus K_\varepsilon) \right\}, \tag{7.2}$$

and the surface energy density $g_{hom} : \mathbf{S}^{N-1} \to [0,+\infty)$ is the BV-elliptic function given by

$$g_{hom}(\nu) = \lim_{\varepsilon\to 0^+} \inf_{w\in P(Q^\nu\setminus K_\varepsilon)} \left\{ \int_{S_w\setminus K_\varepsilon} g\left(\frac{x}{\varepsilon}, \nu_w\right) d\mathcal{H}^{N-1} : w = u_{0,1,\nu} \,\text{on a neighborhood of}\, \partial Q^\nu \right\}.$$

Moreover, if u_ε are minimizers for $\mathcal{G}^\psi_\varepsilon$ satisfying $\|u_\varepsilon\|_{L^\infty(\Omega)} \le \|\psi\|_{L^\infty(\Omega)}$, then there exists a family $(w_\varepsilon) \subset L^p(\Omega)$ which is compact in $L^p(\Omega)$ and such that $w_\varepsilon \equiv u_\varepsilon$ in Ω_ε for any $\varepsilon > 0$ (in particular w_ε are minimizers for $\mathcal{G}^\psi_\varepsilon$). Any cluster point u of w_ε is a minimizer for $\mathcal{G}^\psi_{hom}$.

Remark 7.2. In case $f(x,\cdot)$ is convex for all $x \in \mathbf{R}^N$ formula (7.2) can be specialized (see [12, Remark 19.2]), and reduces to the cell minimization formula

$$f_{hom}(\xi) = \inf \left\{ \int_{Q\setminus K} f(x, \nabla w + \xi)\, dx : \; w \in W^{1,p}_\sharp(Q\setminus K) \right\}.$$

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