

PHASE FIELD APPROACH TO OPTIMAL PACKING PROBLEMS AND RELATED CHEEGER CLUSTERS

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ABSTRACT. In a fixed domain of \mathbb{R}^N we study the asymptotic behaviour of optimal clusters associated to α -Cheeger constants and natural energies like the sum or maximum: we prove that, as the parameter α converges to the “critical” value $\left(\frac{N-1}{N}\right)_+$, optimal Cheeger clusters converge to solutions of different packing problems for balls, depending on the energy under consideration. As well, we propose an efficient phase field approach based on a multiphase Gamma convergence result of Modica-Mortola type, in order to compute α -Cheeger constants, optimal clusters and, as a consequence of the asymptotic result, optimal packings. Numerical experiments are carried over in two and three space dimensions.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let $N \geq 2$ be the space dimension and $\alpha > \frac{N-1}{N}$ be a fixed constant. For every bounded measurable subset E of \mathbb{R}^N , we define its α -Cheeger constant by

$$(1) \quad h_\alpha(E) := \min \left\{ \frac{\mathcal{H}^{N-1}(\partial^* \Omega)}{|\Omega|^\alpha} : \Omega \subseteq E, \Omega \text{ measurable} \right\}.$$

Above, \mathcal{H}^{N-1} denotes the $(N-1)$ -dimensional Hausdorff measure and, if Ω^* has finite perimeter, $\partial^* \Omega$ is its reduced boundary, in the measure theoretical sense. If Ω is the empty set or it has positive measure but does not have finite perimeter, the ratio $\frac{\mathcal{H}^{N-1}(\partial^* \Omega)}{|\Omega|^\alpha}$ is assumed by convention to be equal to $+\infty$.

For $\alpha = 1$, definition (1) corresponds to the classical Cheeger constant, which was thoroughly studied in the last years, see for instance the review papers [19, 23]; for $\alpha \neq 1$, and strictly larger than the scale invariance exponent $\frac{N-1}{N}$, the notion of α -Cheeger constant is a variant which has appeared in the literature more recently, we refer in particular to [24] and references therein.

Object of this paper are the following optimal partition problems

$$(2) \quad \min \left\{ \max_{i=1, \dots, k} h_\alpha(E_i) : (E_1, \dots, E_k) \in \mathcal{P}_k(D) \right\}$$

$$(3) \quad \min \left\{ \sum_{i=1}^k h_\alpha(E_i) : (E_1, \dots, E_k) \in \mathcal{P}_k(D) \right\},$$

where D is a given open bounded Lipschitz subset of \mathbb{R}^N , and

$$\mathcal{P}_k(D) = \left\{ (E_1, \dots, E_k) : \forall i, j = 1, \dots, k, E_i \subseteq D, E_i \cap E_j = \emptyset, E_i \text{ measurable} \right\}.$$

Notice that the condition that the union of the sets E_i covers the given box D (up to a negligible set) is not required in the definition $\mathcal{P}_k(D)$, so that there is some abuse of notation in adopting the usual epithet of optimal “partitions” as done above, and in the sequel we prefer to speak rather about “clusters”. Thus, solutions of (2)-(3) will be generically called α -Cheeger clusters.

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For $\alpha = 1$, problem (3) has been firstly studied by Caroccia in the paper [10], where the existence of solutions and some regularity results for the free boundaries are obtained. In fact, for arbitrary $\alpha > \frac{N-1}{N}$, the existence of solutions (E_1, \dots, E_k) for both problems (2)-(3) is quite immediate, and for convenience of the reader it will be briefly discussed in Section 2 below. On the other hand, the analysis of their qualitative properties may require some more attention (in particular for the *maximum* problem (2)), but it is not our purpose to discuss here regularity issues.

Let us also mention that the asymptotics as $k \rightarrow +\infty$ of the energies in (2)-(3) has received a lot of attention in some recent works focused on the honeycomb conjecture. Actually this celebrated conjecture, which was proved by Hales [15] for cells of equal area minimizing the total perimeter, was already formulated in the context of the classical Cheeger constant in [10], in relationship with a conjecture by Caffarelli and Lin involving optimal partitions of spectral type [8]. A proof of the honeycomb conjecture for the α -Cheeger constant when $\alpha = 1$ and $\alpha = 2$ has been obtained very recently, respectively in [7] and in [6], under the restriction the admissible clusters are made by convex cells.

In this paper, we are interested in the study of α -Cheeger clusters. Our aim is twofold. A first purpose is to get qualitative results describing the behaviour of α -Cheeger clusters, in the limit when $\alpha \rightarrow \left(\frac{N-1}{N}\right)_+$ or $\alpha \rightarrow +\infty$. A second purpose is to give an efficient phase field numerical approach for the computation of optimal α -Cheeger clusters and, as a consequence of their asymptotic behaviour, of optimal packings of balls in arbitrary boxes D . The theoretical results are presented and discussed in the two subsections hereafter. Proofs and numerical results are then given respectively in Sections 2 and 3.

1.1. Limiting behaviour of α -Cheeger clusters. Our main asymptotical results show that, as $\alpha \rightarrow \left(\frac{N-1}{N}\right)_+$, solutions to problems (2) and (3) converge to solutions to two different optimal packing problems for balls. More precisely: solutions of problem (2) converge to a solution of the classical packing problem, which consists in finding k mutually disjoint equal balls with maximal radius in D ; solutions of problem (3) converge to a solution of a more peculiar packing problem, which consists in finding k mutually disjoint balls in D maximizing the product of their volumes. The statements read as follows, where the $L^1(D, \mathbb{R}^k)$ -convergence has to be meant as the convergence of the characteristic functions $\{1_{\Omega_1^\alpha}, \dots, 1_{\Omega_k^\alpha}\}$ of solutions of (4) or (6) to the characteristic functions of balls $\{1_{B(x_1, r_1)}, \dots, 1_{B(x_k, r_k)}\}$ solving (5) or (7).

Theorem 1. As $\alpha \rightarrow \left(\frac{N-1}{N}\right)_+$, up to subsequences, a solution to

$$(4) \quad \min \left\{ \max_{i=1, \dots, k} \frac{\mathcal{H}^{N-1}(\partial^* \Omega_i)}{|\Omega_i|^\alpha} : \{\Omega_i\} \in \mathcal{P}_k(D) \right\}$$

converges in $L^1(D, \mathbb{R}^k)$ to a family of balls solving the following optimal packing problem

$$(5) \quad \max \left\{ r : \exists \{x_i, r_i \geq r\}_{i=1, \dots, k}, B(x_i, r_i) \subset D, B(x_i, r_i) \cap B(x_j, r_j) = \emptyset \right\}.$$

Theorem 2. As $\alpha \rightarrow \left(\frac{N-1}{N}\right)_+$, up to subsequences, a solution to

$$(6) \quad \min \left\{ \sum_{i=1}^k \frac{\mathcal{H}^{N-1}(\partial^* \Omega_i)}{|\Omega_i|^\alpha} : \{\Omega_i\} \in \mathcal{P}_k(D) \right\}$$

converges in $L^1(D, \mathbb{R}^k)$ to a family of balls solving the following optimal packing problem

$$(7) \quad \max \left\{ \prod_{i=1}^k r_i : \exists \{x_i, r_i\}_{i=1, \dots, k}, B(x_i, r_i) \subset D, B(x_i, r_i) \cap B(x_j, r_j) = \emptyset \right\}.$$

When $\alpha \rightarrow +\infty$, we are not able to give a complete description of the asymptotic behaviour of α -Cheeger clusters. Nevertheless we observe that, in case of problem (4), such behaviour seems to be related to an optimal partition problem into cells of equal volume which minimize the product of their perimeters. This question is naturally linked to the possible validity of a stronger version of the classical honeycomb conjecture solved by Hales. The picture is detailed in the statement given hereafter, along with a conjecture. For every $k \in \mathbb{N}$, we call k -cell a connected region obtained as the union of k unit area regular hexagons taken from the hexagonal tiling of \mathbb{R}^2 .

Proposition 3. *As $\alpha \rightarrow +\infty$, up to subsequences a solution to problem (4) converges in $L^1(D, \mathbb{R}^k)$ to a partition of D into k mutually disjoint subsets of equal measure. Moreover, if $N = 2$, D is a k -cell, and Conjecture 4 below holds true, this partition is the hexagonal one.*

Conjecture 4. Let D be a k -cell. Then we expect that Hales' celebrated result [15] can be strengthened into the following product-version of the honeycomb conjecture:

$$\min \left\{ \prod_{i=1}^k \mathcal{H}^1(\partial^* \Omega_i) : \{\Omega_i\} \in \mathcal{P}_k(D), |\Omega_i| = 1 \right\} = (\mathcal{H}^1(\partial H))^k.$$

The proof of this new conjecture seems to be non-trivial, as it would require a product form of the hexagonal isoperimetric inequality in the spirit of Hales. It is not a purpose of this paper to discuss such issue in detail. We limit ourselves to support the validity of the conjecture by performing a few simulations using algorithms similar to those introduced in [22], the difference coming from the presence of the logarithmic function. In fact, in order to avoid numerical instabilities, we minimize the logarithm of the product of perimeters:

$$\min \left\{ \sum_{i=1}^k \log(\mathcal{H}^1(\partial^* \Omega_i)) : \{\Omega_i\} \in \mathcal{P}_k(D), |\Omega_i| = 1 \right\}.$$

The numerical results presented in Figure 1 are performed for 8, 16 and 32 cells in the torus (the choice of the torus instead of a k -cell is based on the observation that, if Conjecture 4 would fail on some k -cell which tiles the plane, then it will also fail on the torus).

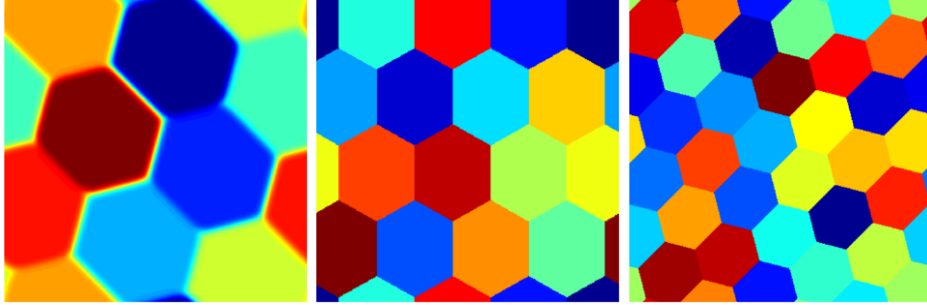


FIGURE 1. Illustration of Conjecture 4 for 8, 16, 32 cells in the torus

Concerning the asymptotic behaviour of α -Cheeger clusters for problem (6) when $\alpha \rightarrow +\infty$, the same arguments used to obtain Proposition 3 can be used to show that, up to subsequences, a solution to such problem converges as well in L^1 to a partition of D into k mutually disjoint subsets of equal measure. Anyway in this case we have no conjecture about the optimization problem solved by this limit configuration.

1.2. Phase field approach for computing α -Cheeger clusters and optimal packings.

There exist various works in the literature dealing with the computation of the Cheeger sets of a given domain D (which corresponds to take $\alpha = 1$ and $k = 1$), using techniques related to convex optimization as, for instance, [11]. In [9] the authors exploit a projection algorithm in order to solve a more general weighted Cheeger problem, while in [18] a convex hull method is used for the optimization in the class of convex bodies. In [16] a characterization of the Cheeger set for convex domains is given, which can easily provide an algorithm for the computation

of the Cheeger set for convex polygons. To our knowledge, no algorithm was implemented for the numerical computation of Cheeger clusters. An algorithm for computing numerically partitions of minimal length was introduced in [22], based on the approximation of perimeter by Γ -convergence using the classical Modica-Mortola theorem [21].

Our approach is as well of Γ -convergence type and relies on the Modica-Mortola theorem for the approximation of the perimeter term, at the numerator in the α -Cheeger ratio. Nevertheless, two difficulties arise: on one hand, even for the computation of the α -Cheeger constant of a set (corresponding to the case $k = 1$), one has to handle the approximation of the measure term in the denominator. This is done by considering a sufficiently high L^p -norm of the phase field function, as the more natural L^1 or L^2 norms are not suitable. Precisely, the measure of the set is efficiently approximated by the $L^{\frac{2N}{N-1}}$ -norm, this choice being available for every α . On the other hand, one has to cope with the collective behaviour of the different phases and the empty regions, this being handled in the spirit of the partitioning algorithm introduced in [5], with the novelty of the presence of the empty regions. The outcoming Γ -convergence result reads:

Theorem 5. *For any fixed $\alpha > \frac{N-1}{N}$ and $p > 1$, the functional*

$$F(\Omega_1, \dots, \Omega_k) := \sum_{i=1}^k \left(\frac{\mathcal{H}^{N-1}(\partial^* \Omega_i)}{|\Omega_i|^\alpha} \right)^p$$

is the Γ -limit as $\varepsilon \rightarrow 0$ in $L^1(D, \mathbb{R}^k)$ of the sequence

$$F_\varepsilon(u_1, \dots, u_k) := \sum_{i=1}^k \left(\frac{\varepsilon \int_D |\nabla u_i|^2 dx + \frac{9}{\varepsilon} \int_D u_i^2 (1 - u_i)^2 dx}{\left(\int_D |u_i|^{\frac{2N}{N-1}} dx \right)^\alpha} \right)^p$$

if $u_i \in H_0^1(D)$, $u_i \geq 0$, $\sum_{i=1}^k u_i \leq 1$, and $+\infty$ if not.

In the above statement, the separation of phases is related to the constraints $u_i \geq 0$, $\sum_{i=1}^k u_i \leq 1$, and to the fact that each phase will naturally converge to a bang-bang configuration. In order to avoid handling the constraint $\sum_{i=1}^k u_i \leq 1$, we use a penalization approach inspired by the paper of Caffarelli and Lin [8]. Namely, we slightly change our functional by introducing the penalty term $\frac{1}{\varepsilon} \sum_{1 \leq i < j \leq k} \int_D u_i^2 u_j^2 dx$. This term enhances the separation of phases during the computational process and avoids (at least the) local minima consisting on flat functions at an intermediate level between 0 and 1. At the same time, the presence of this term, together with the bounds $0 \leq u_i \leq 1$, still ensure that the Γ -convergence holds. Thus we obtain the following corollary, which is the main practical tool for the numerical implementation our phase field approach:

Corollary 6. *For any fixed $\alpha > \frac{N-1}{N}$ and $p > 1$, the functional*

$$F(\Omega_1, \dots, \Omega_k) := \sum_{i=1}^k \left(\frac{\mathcal{H}^{N-1}(\partial^* \Omega_i)}{|\Omega_i|^\alpha} \right)^p$$

is the Γ -limit as $\varepsilon \rightarrow 0$ in $L^1(D, \mathbb{R}^N)$ of the sequence

$$(8) \quad F_\varepsilon(u_1, \dots, u_k) := \sum_{i=1}^k \left(\frac{\varepsilon \int_D |\nabla u_i|^2 dx + \frac{9}{\varepsilon} \int_D u_i^2 (1 - u_i)^2 dx}{\left(\int_D |u_i|^{\frac{2N}{N-1}} dx \right)^\alpha} \right)^p + \frac{1}{\varepsilon} \sum_{1 \leq i < j \leq k} \int_D u_i^2 u_j^2 dx$$

if $u_i \in H_0^1(D)$, $0 \leq u_i \leq 1$ and $+\infty$ if not.

Our algorithm is very efficient in computing clusters or just Cheeger sets in arbitrary boxes D , in two or three space dimensions. In Section 3, we give several numerical experiments with different values of α and k , in \mathbb{R}^2 and in \mathbb{R}^3 . In particular, for α close to the critical value $\left(\frac{N-1}{N} \right)_+$, we recover some classical results for optimal circle/sphere packings, underlining the interest of Theorems 1 and 2. The literature related to circle packings is quite vast, and efficient algorithms are known to work for thousands of cells. However, they rely on heuristic

combinatorial and geometric ideas in order to search for the optimal configuration (see for instance [14, 20]). Our approach is a global one: we initialize each cell with a random density function and we perform a direct gradient based optimization to reach the final configuration. This approach works rather well in the cases we considered. We underline the fact that the search of optimal circle packings is a non-smooth problem. The desired configuration corresponds to maximizing the minimal distance between centers while minimizing the size of the container. Any standard gradient descent algorithm will get stuck as soon as the pairwise distances between centers contain repeated minimal values. The method we present can consistently get near an optimal candidate, or even produce a good starting point for other algorithms.

2. PROOFS

Throughout this section, we set $\omega_N := \mathcal{H}^{N-1}(\partial B)/|B|^{\frac{N-1}{N}}$, being B a ball in \mathbb{R}^N .

Existence of α -Cheeger clusters. Before giving the proofs of our results, for the benefit of the reader we briefly discuss the existence of solutions to problems (2) and (3) for a fixed $\alpha > \frac{N-1}{N}$. This question relies on classical compactness results in BV -spaces and, for $\alpha = 1$ and problem (3), has been extensively discussed in [10]. For shortness, we only deal with problem (2) (the arguments being the same for problem (3)). Assume that $\{\Omega_1^n, \dots, \Omega_k^n\}$ are elements of $\mathcal{P}_k(D)$ such that

$$\max_{i=1, \dots, k} \frac{\mathcal{H}^{N-1}(\partial^* \Omega_i^n)}{|\Omega_i^n|^\alpha} \longrightarrow \inf \left\{ \max_{i=1, \dots, k} \frac{\mathcal{H}^{N-1}(\partial^* \Omega_i)}{|\Omega_i|^\alpha} : \{\Omega_i\} \in \mathcal{P}_k(D) \right\} \quad \text{as } n \rightarrow +\infty.$$

One has first to observe that

$$(9) \quad \liminf_{n \rightarrow +\infty} |\Omega_i^n| > 0 \quad \forall i = 1, \dots, k.$$

Indeed, we can assume that there exists some $M > 0$ such that $\max_{i=1, \dots, k} \frac{\mathcal{H}^{N-1}(\partial^* \Omega_i^n)}{|\Omega_i^n|^\alpha} \leq M$. Then, using the isoperimetric inequality on each cell Ω_i^n we get

$$\omega_N \frac{1}{|\Omega_i^n|^{\alpha - \frac{N-1}{N}}} \leq \frac{\mathcal{H}^{N-1}(\partial^* \Omega_i^n)}{|\Omega_i^n|^\alpha} \leq M,$$

hence, (9) is true. Exploiting also the fact that

$$\mathcal{H}^{N-1}(\partial^* \Omega_i^n) \leq M|D|^k,$$

we get that the sequence $(1_{\Omega_i^n})_n$ is bounded in $BV(D)$. By standard compactness results, we can assume that, up to a subsequence, it converges strongly in $L^1(D)$ to some limit which can be written as 1_{Ω_i} . For $i \neq j$, the sets Ω_i and Ω_j are disjoint and, in view of (9), they have strictly positive measure. Moreover, by lower semicontinuity of the perimeter, we have

$$\liminf_{n \rightarrow +\infty} \mathcal{H}^{N-1}(\partial^* \Omega_i^n) \geq \mathcal{H}^{N-1}(\partial^* \Omega_i),$$

hence $\{\Omega_1, \dots, \Omega_k\}$ is a solution to problem (2).

Proof of Theorem 1. We write $\alpha = \frac{N-1}{N} + \delta$ for some $\delta > 0$, and we consider the auxiliary problems

$$\begin{aligned} M_{k,\delta}(D) &:= \min \left\{ \max_{i=1, \dots, k} \left(\frac{\mathcal{H}^{N-1}(\partial^* \Omega_i)}{\omega_N |\Omega_i|^{\frac{N-1}{N} + \delta}} \right)^{\frac{1}{\delta}} : \{\Omega_i\} \in \mathcal{P}_k(D) \right\} \\ &= \min \left\{ \max_{i=1, \dots, k} \left(\frac{\mathcal{H}^{N-1}(\partial^* \Omega_i)}{\omega_N |\Omega_i|^{\frac{N-1}{N}}} \right)^{\frac{1}{\delta}} \frac{1}{|\Omega_i|} : \{\Omega_i\} \in \mathcal{P}_k(D) \right\}, \end{aligned}$$

which have the same optimal clusters as our initial problems (4).

If D contains k mutually disjoint balls of radius r_D , we have

$$(10) \quad M_{k,\delta}(D) \leq \frac{1}{|B_{r_D}|}.$$

Then, if $\{\Omega_\delta^1, \dots, \Omega_\delta^k\}$ is an optimal cluster for $M_k^\delta(D)$, we have

$$(11) \quad \left(\frac{\mathcal{H}^{N-1}(\partial^* \Omega_i^\delta)}{\omega_N |\Omega_i^\delta|^{\frac{N-1}{N}}} \right)^{\frac{1}{\delta}} \frac{1}{|\Omega_i^\delta|} \leq \frac{1}{|B_{r_D}|} \quad \forall i = 1, \dots, k.$$

Using (11) and the isoperimetric inequality, we get

$$(12) \quad |\Omega_i^\delta| \geq |B_{r_D}| \quad \forall i = 1, \dots, k.$$

Moreover, using (11) and the upper bound $|\Omega_i^\delta| \leq |D|$, we get

$$(13) \quad \mathcal{H}^{N-1}(\partial^* \Omega_i^\delta) \leq \omega_N |D|^{\frac{N-1}{N}} \left(\frac{|D|}{|B_{r_D}|} \right)^\delta \quad \forall i = 1, \dots, k.$$

We deduce that a (not relabeled) subsequence of $\{\Omega_1^\delta, \dots, \Omega_k^\delta\}$ converges in $L^1(D, \mathbb{R}^k)$ to a limit cluster, that we denote by $\{\Omega_1, \dots, \Omega_k\}$, satisfying $|\Omega_i| \geq |B_{r_D}|$. Let us show first that all the sets Ω_i 's are balls, and then that they solve the optimal packing problem (5).

By using the definitions of $M_{k,\delta}(D)$ and r_D , we have the following estimate:

$$\frac{\mathcal{H}^{N-1}(\partial^* \Omega_i^\delta)}{|\Omega_i^\delta|^{\frac{N-1}{N} + \delta}} \leq \omega_N (M_{k,\delta}(D))^\delta \leq \frac{\omega_N}{|B_{r_D}|^\delta} \quad \forall i = 1, \dots, k.$$

Then, passing to the liminf as $\delta \rightarrow 0$, we obtain

$$\frac{\mathcal{H}^{N-1}(\partial^* \Omega_i)}{|\Omega_i|^{\frac{N-1}{N}}} \leq \omega_N \quad \forall i = 1, \dots, k,$$

which implies that the sets Ω_i 's are balls.

Let us show that they solve the optimal packing problem (5). Denote by r_1, \dots, r_k the radii of $\Omega_1, \dots, \Omega_k$, set $r_{\min} := \min\{r_1, \dots, r_k\}$ and let r_* denote the maximal radius in the optimal packing problem (5). Clearly, by the definition of r_* , it holds $r_* \geq r_{\min}$. On the other hand, since D contains k mutually disjoint balls of radius r_* , by definition of $M_{k,\delta}(D)$ we have

$$M_{k,\delta}(D) \leq \frac{1}{|B_{r_*}|}.$$

Hence the inequalities (11) and (12) are in force with r_* in place of r_D and therefore, passing to the limit as $\delta \rightarrow 0$, we obtain

$$|\Omega_i| \geq |B_{r_*}| \quad \forall i = 1, \dots, k.$$

Then $r_{\min} \geq r_*$ and we conclude that $r_{\min} = r_*$. \square

Proof of Theorem 2. We write $\alpha = \frac{N-1}{N} + \delta$ for some $\delta > 0$, and we set

$$m_{k,\delta}(D) := \min \left\{ \sum_{i=1}^k \frac{\mathcal{H}^{N-1}(\partial^* \Omega_i)}{\omega_N |\Omega_i|^{\frac{N-1}{N} + \delta}} : \{\Omega_i\} \in \mathcal{P}_k(D) \right\},$$

(where we have introduced just for convenience the constant ω_N with respect to our initial problems (6)).

Let $(\Omega_1^\delta, \dots, \Omega_k^\delta)$ be an optimal cluster for $m_{k,\delta}(D)$. From the inclusion $\Omega_i^\delta \subset D$, we know that

$$(14) \quad \limsup_{\delta \rightarrow 0} |\Omega_i^\delta|^\delta \leq \limsup_{\delta \rightarrow 0} |D|^\delta = 1 \quad \forall i = 1, \dots, k.$$

We claim that

$$(15) \quad \lim_{\delta \rightarrow 0} |\Omega_i^\delta|^\delta = 1 \quad \forall i = 1, \dots, k.$$

Indeed, if D contains k mutually disjoint balls of radius r_D , we have

$$(16) \quad \sum_{i=1}^k \frac{1}{|\Omega_i^\delta|^\delta} \leq \sum_{i=1}^k \frac{\mathcal{H}^{N-1}(\partial^* \Omega_i^\delta)}{\omega_N |\Omega_i^\delta|^{\frac{N-1}{N}}} \frac{1}{|\Omega_i^\delta|^\delta} = m_{k,\delta}(D) \leq \frac{k}{|B_{r_D}|^\delta}.$$

In the limit as $\delta \rightarrow 0$, up to passing to a subsequence such that $\{|\Omega_i^\delta|^\delta\}$ converges, we find

$$\sum_{i=1}^k \frac{1}{\lim_{\delta \rightarrow 0} |\Omega_i^\delta|^\delta} \leq k.$$

In view of (14), the above inequality implies that necessarily (15) is satisfied.

Now we observe that, by the isoperimetric inequality, there holds

$$\begin{aligned} & \frac{\mathcal{H}^{N-1}(\partial^* \Omega_1^\delta)}{\omega_N |\Omega_1^\delta|^{\frac{N-1}{N}}} \frac{1}{|\Omega_1^\delta|^\delta} + \sum_{i=2}^k \frac{1}{|\Omega_i^\delta|^\delta} \leq \\ & \frac{\mathcal{H}^{N-1}(\partial^* \Omega_1^\delta)}{\omega_N |\Omega_1^\delta|^{\frac{N-1}{N}}} \frac{1}{|\Omega_1^\delta|^\delta} + \sum_{i=2}^k \frac{\mathcal{H}^{N-1}(\partial^* \Omega_i^\delta)}{\omega_N |\Omega_i^\delta|^{\frac{N-1}{N}}} \frac{1}{|\Omega_i^\delta|^\delta} = m_{k,\delta}(D) \leq \frac{k}{|B_{r_D}|^\delta}. \end{aligned}$$

In the limit as $\delta \rightarrow 0_+$, thanks to (15), we deduce that

$$(17) \quad \limsup_{\delta \rightarrow 0} \frac{\mathcal{H}^{N-1}(\partial^* \Omega_1^\delta)}{\omega_N |\Omega_1^\delta|^{\frac{N-1}{N}}} \leq 1,$$

which implies in particular

$$(18) \quad \limsup_{\delta \rightarrow 0} \mathcal{H}^{N-1}(\partial^* \Omega_1^\delta) \leq \omega_N |D|^{\frac{N-1}{N}}.$$

By repeating the same argument for every $i = 2, \dots, k$, we obtain that the perimeters of Ω_i^δ are bounded from above for every $i = 1, \dots, k$.

Next, let us show that the measures of Ω_i^δ are bounded from below for every $i = 1, \dots, k$. We exploit once again the isoperimetric inequality to write

$$\sum_{i=1}^k \exp\left(\delta \log \frac{1}{|\Omega_i^\delta|}\right) = \sum_{i=1}^k \frac{1}{|\Omega_i^\delta|^\delta} \leq \sum_{i=1}^k \frac{\mathcal{H}^{N-1}(\partial^* \Omega_i^\delta)}{\omega_N |\Omega_i^\delta|^{\frac{N-1}{N}}} \frac{1}{|\Omega_i^\delta|^\delta} = m_{k,\delta}(D) \leq \frac{k}{|B_{r_D}|^\delta}.$$

By using the elementary inequality $e^t \geq 1 + t$ for every $t \in \mathbb{R}$, we deduce that

$$k - \delta \log \left(\prod_{i=1}^k |\Omega_i^\delta| \right) \leq \frac{k}{(\gamma_N r_D^N)^\delta},$$

or equivalently

$$\log \left(\prod_{i=1}^k |\Omega_i^\delta| \right) \geq k \frac{1}{\delta} \left(1 - \frac{1}{(\gamma_N r_D^N)^\delta} \right),$$

where γ_N denotes the measure of the unit ball in \mathbb{R}^N .

Since the right hand side of the above inequality remains bounded as a function of δ in a neighbourhood of $\delta = 0$, and since logarithm is monotone increasing, we infer that the product of the measures of Ω_i^δ 's remains bounded from below. Since each of these measures is bounded from above (by the measure of D), we conclude that $|\Omega_i^\delta|$ is bounded below for every $i = 1, \dots, k$.

So far, we have obtained that, for every $i = 1, \dots, k$, the sets Ω_i^δ have perimeters bounded from above and measures bounded from below *uniformly* as $\delta \rightarrow 0_+$; hence, up to a (not relabeled) subsequence, they admit a k -uple of (non trivial) limit sets, and from inequalities (17) we see that these limit sets are necessarily balls. Let us denote them by (B_1^0, \dots, B_k^0) , and let us show that they solve the optimal packing problem (7).

To that aim, we are going to estimate from above and from below the quotient

$$\frac{m_{k,\delta}(D) - k}{\delta}.$$

On one hand, if (B_1^*, \dots, B_k^*) are balls which solve the optimal packing problem (7), taking them as a test cluster in the definition of $m_{k,\delta}(D)$, we get the upper bound

$$\frac{m_{k,\delta}(D) - k}{\delta} \leq \frac{1}{\delta} \sum_{i=1}^k \left(\frac{1}{|B_i^*|^\delta} - 1 \right) = \sum_{i=1}^k \frac{1}{\delta} \left(\exp \left(\delta \log \frac{1}{|B_i^*|} \right) - 1 \right).$$

Hence,

$$(19) \quad \limsup_{\delta \rightarrow 0} \frac{m_{k,\delta}(D) - k}{\delta} \leq \log \left(\prod_{i=1}^k \frac{1}{|B_i^*|} \right).$$

On the other hand, by applying as usual the isoperimetric inequality, we have

$$\frac{m_{k,\delta}(D) - k}{\delta} \geq \frac{1}{\bar{\delta}} \sum_{i=1}^k \left(\frac{1}{|\Omega_i^\delta|^\delta} - 1 \right).$$

Now we exploit the fact that $|\Omega_i^\delta| \rightarrow |B_i^0|$ as $\delta \rightarrow 0_+$, so that, for every fixed $\eta > 0$, we can find $\bar{\delta} = \bar{\delta}(\eta)$ such that

$$|\Omega_i^\delta| \leq |(1 + \eta)B_i^0| \quad \forall \delta \leq \bar{\delta}(\eta), \quad \forall i = 1, \dots, k.$$

Therefore,

$$\frac{m_{k,\delta}(D) - k}{\delta} \geq \frac{1}{\bar{\delta}} \sum_{i=1}^k \left(\frac{1}{|(1 + \eta)B_i^0|^\delta} - 1 \right).$$

Then, by arguing as above and using the arbitrariness of η , we arrive at

$$(20) \quad \limsup_{\delta \rightarrow 0} \frac{m_{k,\delta}(D) - k}{\delta} \leq \log \left(\prod_{i=1}^k \frac{1}{|B_i^0|} \right).$$

Eventually, combining (19) and (20), we obtain

$$\prod_{i=1}^k \frac{1}{|B_i^0|} \geq \prod_{i=1}^k \frac{1}{|B_i^*|}.$$

By the definition of B_i^* , the above inequality holds necessarily with equality sign, which amounts to say that the limit balls B_i^0 maximize the product of volumes of a family of k mutually disjoint balls contained into D , namely they solve problem (7). \square

Proof of Proposition 3. Let $(\Omega_1^\alpha, \dots, \Omega_k^\alpha)$ be a solution to problem (4), and assume $\alpha \rightarrow +\infty$. Assuming without loss of generality that $|D| = k$, we consider a k -uple $(\widehat{\Omega}_1, \dots, \widehat{\Omega}_k) \in \mathcal{P}_k(D)$ of sets with finite perimeter, such that $|\widehat{\Omega}_i| = 1$ for every $i = 1, \dots, k$. Then we have

$$(21) \quad \frac{\mathcal{H}^{N-1}(\partial^* \Omega_i^\alpha)}{|\Omega_i^\alpha|^\alpha} \leq C := \max_{i=1, \dots, k} |\partial^* \widehat{\Omega}_i| \quad \forall i = 1, \dots, k.$$

By applying the isoperimetric inequality, we obtain

$$\frac{\omega_N |\Omega_i^\alpha|^{\frac{N-1}{N}}}{|\Omega_i^\alpha|^\alpha} \leq C \quad \forall i = 1, \dots, k,$$

or equivalently

$$|\Omega_i^\alpha| \geq \left(\frac{\omega_N}{C} \right)^{\frac{1}{\alpha - \frac{N-1}{N}}} \quad \forall i = 1, \dots, k.$$

Hence,

$$(22) \quad \liminf_{\alpha \rightarrow +\infty} |\Omega_i^\alpha| \geq 1 \quad \forall i = 1, \dots, k.$$

Moreover, from inequality (21), we know that

$$\sum_{i=1}^k \left(\frac{\mathcal{H}^{N-1}(\partial^* \Omega_i^\alpha)}{C} \right)^{\frac{1}{\alpha}} \leq \sum_{i=1}^k |\Omega_i^\alpha| \leq k.$$

Thus, by using the elementary inequality $e^t \geq 1 + t$ for every $t \in \mathbb{R}$, we arrive at

$$\frac{1}{\alpha} \sum_{i=1}^k \log \left(\frac{\mathcal{H}^{N-1}(\partial^* \Omega_i^\alpha)}{C} \right) \leq 0,$$

which means that

$$(23) \quad \prod_{i=1}^k \mathcal{H}^{N-1}(\partial^* \Omega_i^\alpha) \leq C^k.$$

We observe that, by (22), $\mathcal{H}^{N-1}(\partial^* \Omega_i^\alpha)$ is bounded from below for every $i = 1, \dots, k$; then inequality (23) ensures that $\mathcal{H}^{N-1}(\partial^* \Omega_i^\alpha)$ is also bounded from above for every $i = 1, \dots, k$. We conclude that $(\Omega_1^\alpha, \dots, \Omega_k^\alpha)$ admits a limit in L^1 , hereafter denoted by $(\Omega_1^\infty, \dots, \Omega_k^\infty)$, and in view of (22) this limit turns out to be a partition of D .

Finally, if $N = 2$ and D is a k -cell, by taking the sets $\widehat{\Omega}_i$'s in (21) equal to the k -copies of H which compose the k -cell, inequality (23) implies

$$(24) \quad \prod_{i=1}^k \mathcal{H}^1(\partial^* \Omega_i^\infty) \leq (\mathcal{H}^1(\partial H))^k.$$

Therefore, if Conjecture 4 is satisfied, we conclude that

$$\prod_{i=1}^k \mathcal{H}^1(\partial^* \Omega_i^\infty) \leq \min \left\{ \prod_{i=1}^k \mathcal{H}^1(\partial^* \Omega_i) : \{\Omega_i\} \in \mathcal{P}_k(D), |\Omega_i| = 1 \right\}.$$

Hence the above inequality holds with equality sign, and the partition $\{\Omega_i^\infty\}$ is optimal for the minimization problem at the right hand side. \square

Proof of Theorem 5. We prove separately the so-called Γ -liminf and Γ -limsup inequalities (see the monograph [12] for an introduction to Γ -convergence).

Γ -liminf inequality. Let $u_i^\varepsilon \in H_0^1(D) \setminus \{0\}$, such that $(u_1^\varepsilon, \dots, u_k^\varepsilon) \xrightarrow{L^1(D, \mathbb{R}^k)} (u_1, \dots, u_k)$ and

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_1^\varepsilon, \dots, u_k^\varepsilon) < +\infty.$$

In a first step, we notice the existence of a constant M , such that for small ε and for every i we have

$$\varepsilon \int_D |\nabla u_i^\varepsilon|^2 dx + \frac{9}{\varepsilon} \int_D (u_i^\varepsilon)^2 (1 - u_i^\varepsilon)^2 dx \leq M |D|^\alpha.$$

From the Modica-Mortola theorem, we get that $u_i \in BV(D, \{0, 1\})$, hence $u_i = 1_{\Omega_i}$, where Ω_i are pairwise disjoint and satisfy

$$\mathcal{H}^{N-1}(\partial^* \Omega) \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon \int_D |\nabla u_i^\varepsilon|^2 dx + \frac{9}{\varepsilon} \int_D (u_i^\varepsilon)^2 (1 - u_i^\varepsilon)^2 dx.$$

Let us prove that $\Omega_i \neq \emptyset$. We know that

$$\varepsilon \int_D |\nabla u_i^\varepsilon|^2 dx + \frac{9}{\varepsilon} \int_D (u_i^\varepsilon)^2 (1 - u_i^\varepsilon)^2 dx \leq M \left(\int_D (u_i^\varepsilon)^{\frac{2N}{N-1}} dx \right)^\alpha.$$

Setting $\delta := \frac{2N}{N-1} \alpha - 2 > 0$ and using the Cauchy-Schwartz inequality on the left hand side, we get

$$6 \int_D |\nabla u_i^\varepsilon| u_i^\varepsilon (1 - u_i^\varepsilon) dx \leq M \|u_i^\varepsilon\|_{L^{\frac{2N}{N-1}}(D)}^{2+\delta},$$

or, by the chain rule,

$$6 \int_D \left| \nabla \left(\frac{(u_i^\varepsilon)^2}{2} - \frac{(u_i^\varepsilon)^3}{3} \right) \right| dx \leq M \|u_i^\varepsilon\|_{L^{\frac{2N}{N-1}}(D)}^{2+\delta}.$$

Using the Sobolev inequality with a dimensional constant S_N , we get

$$6 S_N \left\| \frac{(u_i^\varepsilon)^2}{2} - \frac{(u_i^\varepsilon)^3}{3} \right\|_{L^{\frac{N}{N-1}}(D)} \leq M \|u_i^\varepsilon\|_{L^{\frac{2N}{N-1}}(D)}^{2+\delta}.$$

Since $0 \leq u_i^\varepsilon \leq 1$ we get $\frac{(u_i^\varepsilon)^2}{2} - \frac{(u_i^\varepsilon)^3}{3} \geq \frac{(u_i^\varepsilon)^2}{6}$ so

$$S_N \|u_i^\varepsilon\|_{L^{\frac{2N}{N-1}}(D)}^2 \leq M \|u_i^\varepsilon\|_{L^{\frac{2N}{N-1}}(D)}^{2+\delta},$$

or

$$\frac{S_N}{M} \leq \|u_i^\varepsilon\|_{L^{\frac{2N}{N-1}}(D)}^\delta.$$

Passing to the limit, we get $|\Omega_i| \geq \frac{S_N}{M}$. Since $\int_D (u_i^\varepsilon)^{\frac{2N}{N-1}} dx \rightarrow |\Omega_i| \neq 0$, the Γ -liminf property occurs.

Γ -limsup inequality. For one single set, the Modica-Mortola theorem gives the procedure of constructing the recovering sequence. For partition problems we refer to the paper by Baldo [1], where the recovering sequence requires more attention because of the exact partition requirement which is obtained via the constraint $\forall x \in D, \sum_{i=1}^k u_i(x) = 1$. Since in our problem we do not have a complete partition of the set D , we can give a direct proof as follows.

Let $(\Omega_1, \dots, \Omega_k)$ be pairwise disjoint measurable subsets of D with finite perimeter. Let us fix a positive constant $\delta > 0$. For a usual convolution kernel $(\rho_\eta)_\eta$, we have

$$(25) \quad \left(\sum_{i=1}^k 1_{\Omega_i} \right) * \rho_\eta \leq 1,$$

and we claim that we can choose η small enough such that for every $i = 1, \dots, k$ we can find $t_i \in (\frac{1}{2}, 1)$ such that the set $A_i^\delta := \{1_{\Omega_i} * \rho_\eta > t_i\}$ is smooth and

$$\begin{aligned} \mathcal{H}^{N-1}(\partial A_i^\delta) &\leq \mathcal{H}^{N-1}(\partial^* \Omega_i) + \delta, \\ |A_i^\delta \Delta \Omega_i| &\leq \delta. \end{aligned}$$

Indeed, from the Sard theorem, we know that almost all levels are smooth. From the strict convergence $\rho_\eta * 1_{\Omega_i} \xrightarrow{BV} 1_{\Omega_i}$, we get that

$$\int_0^1 |\nabla \rho_\eta * 1_{\Omega_i}| dx \rightarrow \mathcal{H}^{N-1}(\partial^* \Omega_i),$$

and the co-area formula gives the existence of a level $t > \frac{1}{2}$ such that

$$\mathcal{H}^{N-1}(\partial^* \Omega_i) \geq \limsup_{\eta \rightarrow 0} \mathcal{H}^{N-1}(\partial \{\rho_\eta * 1_{\Omega_i} > t\}).$$

On the other hand, from (25) and the choice of $t_i > \frac{1}{2}$, the sets A_i^δ are disjoint, at positive distance. Using now the Modica-Mortola theorem, we can find recovering sequences $u_i^\varepsilon \xrightarrow{L^1(D)} 1_{A_i^\delta}$ with

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \int_D |\nabla u_i^\varepsilon|^2 dx + \frac{9}{\varepsilon} \int_D (u_i^\varepsilon)^2 (1 - u_i^\varepsilon)^2 dx \leq \mathcal{H}^{N-1}(\partial A_i^\delta).$$

Choosing, a sequence $\delta_n \rightarrow 0$, by a diagonal procedure we can find ε_n small enough such that the Γ -limsup property holds, and moreover $u_i^{\varepsilon_n} u_j^{\varepsilon_n} = 0$. \square

Proof of Corollary 6. The proof is implicitly contained into the one of Theorem 5. In particular, we point out that the choice of the recovering sequence for the Γ -limsup property is also suitable for the penalized functionals. \square

3. NUMERICAL RESULTS

In order to discretize the functional (8), we consider a rectangular box D in \mathbb{R}^2 or \mathbb{R}^3 endowed with a finite differences uniform grid with M discretization points along each axis direction. A function u will be numerically represented by its values at the grid points. We use basic order 1 centered finite differences in order to compute the gradient terms $|\nabla u_i|$, and basic quadrature formulas to compute all integrals. Similar approaches were already used in [22] and [4]. Moreover, in [4, Section 4] detailed expressions of the gradient with respect to each of the grid point variables of (8) are given for the Modica-Mortola term. The other integrals, like the denominator of F_ε and the penalization terms, are approximated by their arithmetic mean along the grid. The optimization is done using a LBFGS quasi-Newton method implemented in Matlab

[25]. The algorithm uses information on a number of previous gradients, 5 in our computations, in order to build an approximation of the Hessian. In addition to being more efficient in avoiding local minima than a simple gradient descent algorithm, the LBFGS algorithm recalled above allows us to impose pointwise bounds for every variable, fact which is important in our approach. This also motivated us to use a penalized approach, rather than a projected gradient approach. The optimization procedure is presented in Algorithm 1.

Algorithm 1 Optimization procedure

Require:

- M : initial discretization parameter ($M = 20$ in 2D, $M = 10$ in 3D)
- k : number of cells
- d : dimension
- $maxit$: maximal number of iterations
- $\varepsilon \in [1/M, 4/M]$: Modica-Mortola parameter
- tol : stopping criterion
- n : number of refinements chosen so that the final resolution is as good as we want (in our case $2^{n-1}M > 300$ in 2D and $2^{n-1}M > 100$ in 3D)

- 1: Initialize densities u_1, \dots, u_k as k random vectors of M^d elements
- 2: $step = 1$
- 3: **repeat**
- 4: **if** $step > 1$ **then**
- 5: $M \leftarrow 2M$
- 6: interpolate linearly the previous densities $(u_i)_{i=1}^k$ on the new grid
- 7: **end if**
- 8: Run the LBFGS optimization procedure [25] with the following parameters
 - starting point $(u_i)_{i=1}^k$ (random for first step, interpolated from the previous optimization result for the next steps)
 - tolerances and maximum number of iterations: $tol = 10^{-8}$, $maxit = 10000$
 - function to optimize: (8). Value and gradient are computed at each iteration
 - pointwise upper and lower bounds $0 \leq u_i \leq 1$.
- 9: The previous algorithm returns the optimized densities $(u_i)_{i=1}^n$
- 10: $step = step + 1$ (go to next step)
- 11: **until** $step > n$

return the k density functions

Since we perform an optimization of a non-convex functional, there is no guarantee of convergence to a global minimum. In order to avoid local minima, we choose as initialization some random densities for each of the functions u_i . Moreover, in order to avoid the rapid convergence to a characteristic function and to diminish the effects of the non-convex potential, we choose the parameter ε equal to $1/M$, where M is the number of discretization points along each axis direction. The parameter ε dictates the width of the interface where the functions u_i go from 0 to 1. Sometimes it is useful to consider larger values of ε , $2/M$ or $4/M$, to allow the cells to move more freely.

In order to reduce the number of iterations needed to reach an optimum, we propose a grid refinement procedure as already noted in [5] and [4]. We perform an initial optimization on a grid of rather low size with $M \in [20, 50]$. Then we interpolate the result on a finer grid, usually doubling the discretization parameter, and we continue the optimization on this refined grid. We continue until we reach the desired level of accuracy. In 2D we can easily perform computations on grids of size up to 400×400 while in 3D we go up to $100 \times 100 \times 100$. Our current algorithm works well if the number of cells is not too large. When considering many phases, computations become more costly and memory costs become larger. It is possible to use techniques from [2] in order to address these issues. The idea is that, when dealing with partitioning or multiphase problems in the phase-field context, one could restrict the computation in a neighborhood of the significant part of each cell, for example $\{u_i > \delta\}$, for a given threshold $\delta > 0$. This could

significantly reduce the computational and memory cost of the computations and could allow the use of the algorithm for many cells. However, this goes beyond the scope of this article.

Even if we choose to work on finite differences grid we may still compute α -Cheeger sets and α -Cheeger clusters corresponding to non-rectangular domains. For a general domain D we consider a rectangular box $D' \supset D$ on which we construct the finite differences grid. We set all functions involved in the computations to be equal to zero on grid points outside D and set the gradient to be equal to zero on the same points lying outside D . In this way the optimization is made only on points inside the desired domain D .

It is also possible to use a finite element framework in order to minimize (8) on general domains. In [4, Section 3] you can find a detailed presentation of such a finite element framework in the context of Modica-Mortola functionals. Once the mass and rigity matrices for the Lagrange P1 finite elements are obtained, all functionals needed in our computations can be expressed as vector matrix products.

Now we exemplify the use of the proposed algorithm for computing α -Cheeger sets, α -Cheeger clusters and optimal packings. We make available an implementation of the algorithm described above which can be found online at the following link: https://github.com/bbogo/Cheeger_patch. This implementation uses the finite element framework for the optimization of (8). As detailed below it is also possible, for convex domains, to compare the Cheeger sets found by minimizing (8) with the exact Cheeger sets obtained by using the representation formula provided by Kawohl and Lachand-Robert in [16].

- **Computation of α -Cheeger sets.** In this case, corresponding to $k = 1$, there is no need to use the penalization term. We optimize directly the non-penalized ratio between the Modica-Mortola ratio and the volume term with constraints $0 \leq u \leq 1$. Examples can be seen in Figure 2 for a domain in \mathbb{R}^2 and in Figure 3 for a domain in \mathbb{R}^3 .

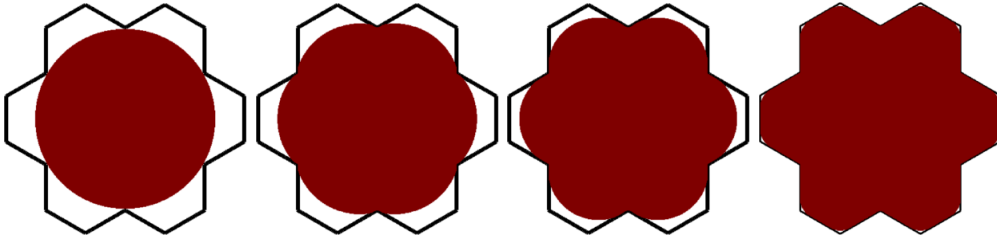


FIGURE 2. The α -Cheeger set for a non-convex set in $2D$, for $\alpha \in \{0.5001, 0.75, 1, 2\}$.

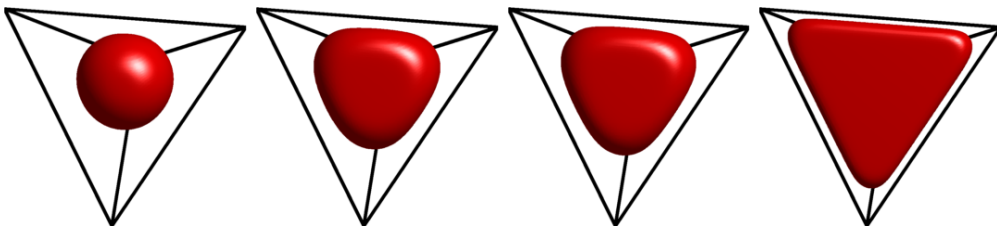


FIGURE 3. The α -Cheeger set for a regular tetrahedron in $3D$, for $\alpha \in \{0.667, 0.9, 1, 2\}$.

In order to test the accuracy of our method we compare our algorithm with an implementation of the Kawohl & Lachand-Robert explicit formula for finding Cheeger sets ($\alpha = 1$) associated to convex sets in 2D [16]. As can be seen in Figure 4 the relaxation algorithm we propose is quite precise. We represent with red the ε -level set of the result obtained when minimizing 8 and with dotted blue the result obtained using the algorithm described in [16]. We justify the choice of the ε -level set by the fact that we impose the boundary condition $u = 0$ on the density. Choosing a level set corresponding to a larger value would correspond to a contour which does not touch the boundary of D , contrary to the known behaviour of Cheeger sets. In the test cases presented below the results given by the two algorithms are almost indistinguishable. The

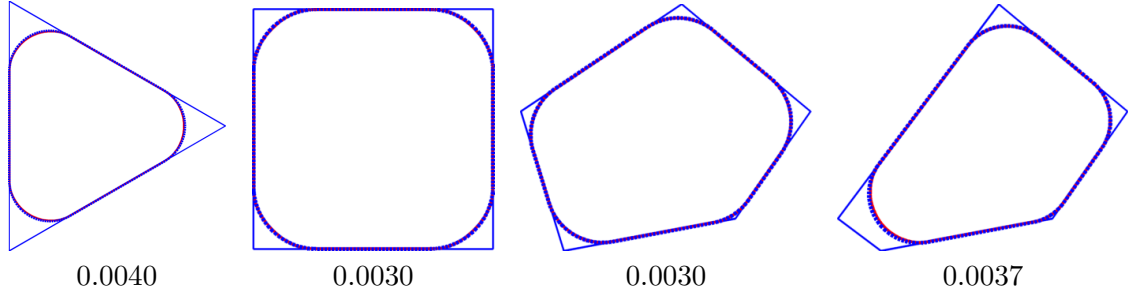


FIGURE 4. Comparison between results obtained when minimizing (8) (red) and the Kawohl & Lachand-Robert formula (dotted-blue). Relative errors for the Cheeger constants are also given.

relative errors, i.e. $|h(\omega_{ap}) - h(\omega)|/h(\omega)$ are also provided. Here $h(\omega)$ is the Cheeger constant of ω and ω_{ap} is the approximated Cheeger set obtained when minimizing 8. The computed errors are small in view of the fact that the Cheeger sets are computed with a relaxed formulation. As expected, working on finer meshes leads to better approximations both of the Cheeger sets and of the Cheeger constants.

• **Computation of α -Cheeger clusters.** Some examples of Cheeger clusters can be seen in Figure 5. One can notice immediately that the cells are not necessarily convex, for instance when D is a square and $n = 5$. The results in the periodic case are in accordance with results in [7, 6].

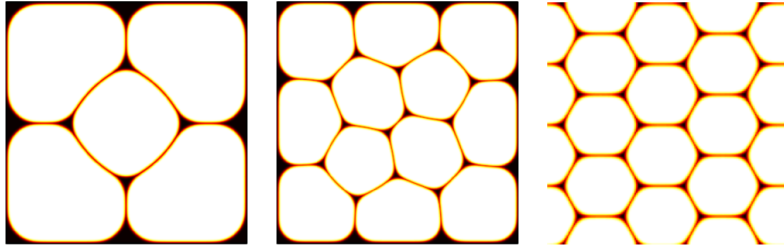


FIGURE 5. Cheeger clusters for problem (3) in a square: 5 cells, 12 cells, 16 cells (periodical).

• **Computation of optimal packings.** To this aim, we exploit Theorem 1 and we compute α -Cheeger clusters for α very close to $\frac{N-1}{N}$ and p very “large”, which in our computations means at most 100. Choosing the parameter α close to $\frac{N-1}{N}$ forces the cells in the optimal configurations to be close to disks. We choose to use a p -norm approach since this regularizes the non-smooth problem of minimizing the maximal radius of a family of disks. The minimization of a p -norm instead of the ∞ -norm is a natural idea, already used in [3] for the study of partitions of a domain which minimize the largest fundamental eigenvalue of the Dirichlet-Laplace operator.

We want to be able to quantify our results, so we use a local refinement procedure as a post-treatment. At the end of the optimization process we have access to the density functions associated to the α -Cheeger cells, which are approximately smoothed characteristic functions of disks. Using these density functions we approximate the precise location of the centers of the disks by computing the barycenter of the 0.5 level-set of each density function. Then we use a very basic local-optimization routine in order to get a precise description of the circle packing that can be compared with existing results in the literature. The algorithm just computes the pairwise distances between centers of the disks and between the centers and the boundary and uses Matlab’s algorithm `fmincon` with the option `active-set` to perform a gradient-free local optimization of the current configuration. We note that the refinement algorithm is not at all adapted for solving *alone* the problem, given random starting points for the centers. It is only useful for locally optimizing the circle packing configuration once localization is obtained by our approximation procedure.

It is possible to apply this algorithm for computing both optimal circle packings for domains in \mathbb{R}^2 or optimal sphere packings for domains in \mathbb{R}^3 . In the planar case, we present some computational results in Figure 6. In our test cases the numerical algorithm based on the Γ -convergence result combined with the post-treatment algorithm generally produce configurations which are comparable to the best known results in the literature. We recall that one of the first papers regarding the circle packings in a circle was authored by Kravitz in 1967 [17]. In this paper we can find a conjecture regarding the 19-circle packing in a disk. The optimality of this 19-packing, presented in Figure 6, was proved by Fodor in [13]. Extensive numerical results up to thousands of circles were performed and collected on the website <http://www.packomania.com/>, maintained by Eckard Specht. In all cases, we compared our results with best ones available, listed on the above cited website. The numerical algorithm manages to capture the right results in cases where the optimal circle packing configuration is unique and rigid, like the case of 19 disks in a circle or 28 disks in an equilateral triangle. Moreover, we are able to capture the best known results even in cases where the solution is not unique, like in the case when we have 18 disks in a circle.

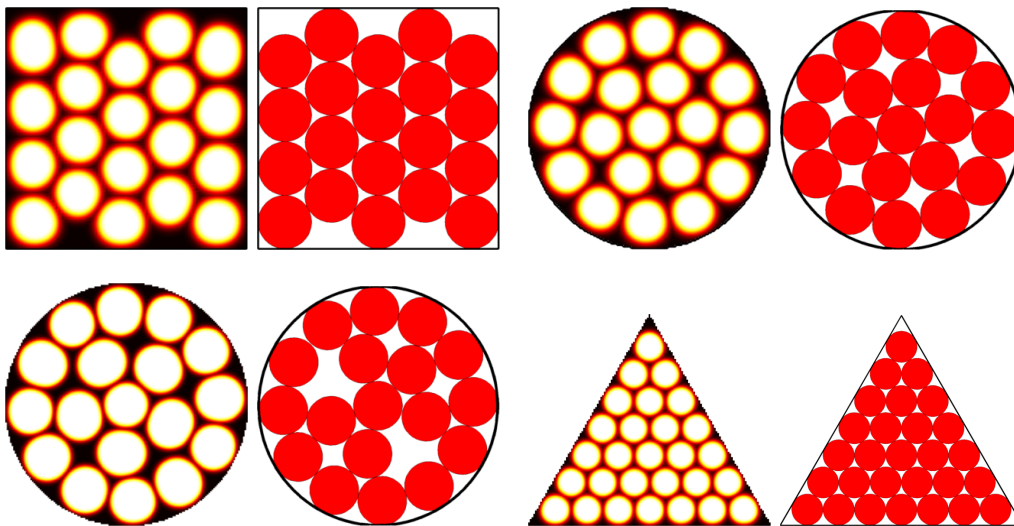


FIGURE 6. Circle packing examples in 2D for problem (5): density representation and local optimization.

Some examples of computations of optimal spherical packings for domains in \mathbb{R}^3 are presented in Figure 7. In this case, we observe again a good convergence to the best known configurations.

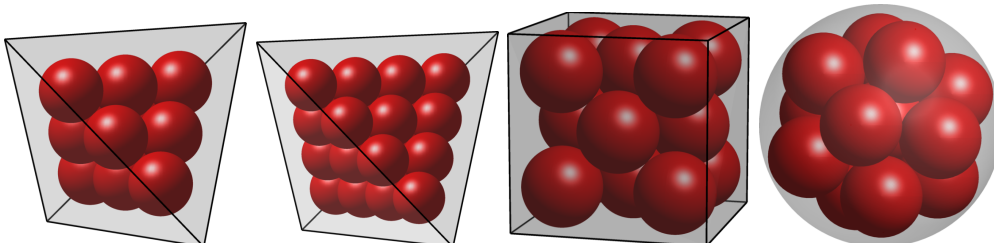


FIGURE 7. Sphere packing examples in 3D for problem (5).

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