This is a preprint of the article published in
J. reine angew. Math. **643** (2010), 39–57
DOI 10.1515/CRELLE.2010.044

Monotone volume formulas for geometric flows

Reto Müller

Abstract

We consider a closed manifold M with a Riemannian metric $g_{ij}(t)$ evolving by $\partial_t\, g_{ij} = -2S_{ij}$ where $S_{ij}(t)$ is a symmetric two-tensor on $(M, g(t))$. We prove that if S_{ij} satisfies the tensor inequality $\mathcal{D}(S_{ij}, X) \geq 0$ for all vector fields X on M, where $\mathcal{D}(S_{ij}, X)$ is defined in (1.6), then one can construct a forwards and a backwards reduced volume quantity, the former being non-increasing, the latter being non-decreasing along the flow $\partial_t\, g_{ij} = -2S_{ij}$. In the case where $S_{ij} = R_{ij}$, the Ricci curvature of M, the result corresponds to Perelman's well-known reduced volume monotonicity for the Ricci flow presented in [12]. Some other examples are given in the second section of this article, the main examples and motivation for this work being List's extended Ricci flow system developed in [8], the Ricci flow coupled with harmonic map heat flow presented in [11], and the mean curvature flow in Lorentzian manifolds with nonnegative sectional curvatures. With our approach, we find new monotonicity formulas for these flows.

1 Introduction and formulation of the main result

Let M be a closed manifold with a time-dependent Riemannian metric $g_{ij}(t)$. Let $\mathcal{S}(t)$ be a symmetric two tensor on $(M, g(t))$ with components $S_{ij}(t)$ and trace $S(t) := \operatorname{tr}_{g(t)} \mathcal{S}(t) = g^{ij}(t)S_{ij}(t)$. Assume that $g(t)$ evolves according to the flow equation

$$\partial_t\, g_{ij}(t) = -2S_{ij}(t). \tag{1.1}$$

A typical example would be the case where $\mathcal{S}(t) = \operatorname{Ric}(t)$ is the Ricci tensor of $(M, g(t))$ and the metric $g(t)$ is a solution to the Ricci flow, introduced by Richard Hamilton in [3]. Other examples are given in Section 2 of this article.

In analogy to Perelman's $\mathcal{L}$-distance for the Ricci flow defined in [12], we will now introduce forwards and backwards reduced distance functions for the flow (1.1), as well as a forwards and a backwards reduced volume.

Definition 1.1 (forwards reduced distance and volume)
Suppose that (1.1) has a solution for $t \in [0, T]$. For $0 \leq t_0 \leq t_1 \leq T$ and a curve $\gamma : [t_0, t_1] \to M$, we define the $\mathcal{L}_f$-length of $\gamma(t)$ by

$$\mathcal{L}_f(\gamma) := \int_{t_0}^{t_1} \sqrt{t}\left(S(\gamma(t)) + |\partial_t\, \gamma(t)|^2\right) dt.$$

For a fixed point $p \in M$ and $t_0 = 0$, we define the forwards reduced distance

$$\ell_f(q, t_1) := \inf_{\gamma \in \Gamma} \left\{ \frac{1}{2\sqrt{t_1}} \int_0^{t_1} \sqrt{t}\left(S + |\partial_t\, \gamma|^2\right) dt \right\}, \tag{1.2}$$

where $\Gamma = \{\gamma : [0, t_1] \to M \mid \gamma(0) = p,\, \gamma(t_1) = q\}$, i.e. the forwards reduced distance is the $\mathcal{L}_f$-length of an $\mathcal{L}_f$-shortest curve times $\frac{1}{2\sqrt{t_1}}$. Existence of such $\mathcal{L}_f$-shortest curves will be discussed in the fourth section. Finally, the forwards reduced volume is defined to be

$$V_f(t) := \int_M (4\pi t)^{-n/2} e^{\ell_f(q,t)} dV(q). \tag{1.3}$$

In order to define the backwards reduced distance and volume, we need a backwards time $\tau(t)$ with $\partial_t \tau(t) = -1$. Without loss of generality, one may assume (possibly after a time shift) that $\tau = -t$.

Definition 1.2 (backwards reduced distance and volume)
If (1.1) has a solution for $\tau \in [0, \bar{\tau}]$ we define the $\mathcal{L}_b$-length of a curve $\gamma : [\tau_0, \tau_1] \to M$ by

$$\mathcal{L}_b(\gamma) := \int_{\tau_0}^{\tau_1} \sqrt{\tau} \left(S(\gamma(\tau)) + |\partial_\tau \gamma(\tau)|^2 \right) d\tau.$$

Again, we fix the point $p \in M$ and $\tau_0 = 0$ and define the backwards reduced distance by

$$\ell_b(q, \tau_1) := \inf_{\gamma \in \Gamma} \left\{ \frac{1}{2\sqrt{\tau_1}} \int_0^{\tau_1} \sqrt{\tau} \left(S + |\partial_\tau \gamma|^2 \right) d\tau \right\}, \tag{1.4}$$

where now $\Gamma = \{\gamma : [0, \tau_1] \to M \mid \gamma(0) = p,\, \gamma(\tau_1) = q\}$. The backwards reduced volume is defined by

$$V_b(\tau) := \int_M (4\pi\tau)^{-n/2} e^{-\ell_b(q,\tau)} dV(q). \tag{1.5}$$

Next, we define an evolving tensor quantity $\mathcal{D}$ associated to the tensor $\mathcal{S}$.

Definition 1.3
Let $g(t)$ evolve by $\partial_t g_{ij} = -2S_{ij}$ and let S be the trace of $\mathcal{S}$ as above. Let $X \in \Gamma(TM)$ be a vector field on M. We set

$$\begin{aligned} \mathcal{D}(\mathcal{S}, X) := \partial_t S - \triangle S - 2\,|S_{ij}|^2 + 4(\nabla_i S_{ij})X_j - 2(\nabla_j S)X_j \\ + 2R_{ij}X_iX_j - 2S_{ij}X_iX_j, \end{aligned} \tag{1.6}$$

Remark. The quantity $\mathcal{D}$ consists of three terms. The first term, $\partial_t S - \triangle S - 2\,|S_{ij}|^2$, captures the evolution properties of $S = g^{ij}S_{ij}$ under the flow (1.1). The second one, $4(\nabla_i S_{ij})X_j - 2(\nabla_j S)X_j$, is a multiple of the error term E that appears in the twice traced second Bianchi type identity $\nabla_i S_{ij} = \frac{1}{2}\nabla_j S + E$ for the symmetric tensor $\mathcal{S}$. Finally, the last term directly compares the tensor S_{ij} with the Ricci tensor.

We can now state our main result.

Theorem 1.4 (monotonicity of forwards and backwards reduced volume)
Suppose that $g(t)$ evolves by (1.1) and the quantity $\mathcal{D}(\mathcal{S}, X)$ is nonnegative for all vector fields $X \in \Gamma(TM)$ and all times t for which the flow exists. Then the forwards *reduced volume $V_f(t)$ is non-increasing in t along the flow. Moreover, the* backwards *reduced volume $V_b(\tau)$ is non-increasing in τ, i.e. non-decreasing in t.*

The remainder of this work is organized as follows. In the next section, we consider some examples where Theorem 1.4 can be applied. In Section 3, we start the proof of the theorem by showing that the quantity $\mathcal{D}(\mathcal{S},X)$ is the difference between two differential Harnack type quantities for the tensor $\mathcal{S}$ defined as follows.

Definition 1.5
For two tangent vector fields $X, Y \in \Gamma(TM)$ on M, we define

$$\begin{aligned}\mathcal{H}(\mathcal{S},X,Y) &:= 2(\partial_t\,\mathcal{S})(Y,Y) + 2\left|\mathcal{S}(Y,\cdot)\right|^2 - \nabla_Y\nabla_Y S + \tfrac{1}{t}\mathcal{S}(Y,Y)\\ &\quad - 4(\nabla_X\mathcal{S})(Y,Y) + 4(\nabla_Y\mathcal{S})(X,Y) - 2\left\langle \mathrm{Rm}(X,Y)X,Y\right\rangle,\\ \mathcal{H}(\mathcal{S},X) &:= \partial_t\,S + \tfrac{1}{t}S - 2\left\langle \nabla S, X\right\rangle + 2\mathcal{S}(X,X).\end{aligned}$$

Lemma 1.6
The quantity $\mathcal{D}(\mathcal{S},X)$ is the difference between the trace of $\mathcal{H}(\mathcal{S},X,Y)$ with respect to the vector field Y and the expression $\mathcal{H}(\mathcal{S},X)$, i.e. for an orthonormal basis $\{e_i\}$, we have

$$\mathcal{D}(\mathcal{S},X) = \sum_i \mathcal{H}(\mathcal{S},X,e_i) - \mathcal{H}(\mathcal{S},X).$$

In Section 4, after introducing a notation which makes it possible to deal with the forwards and the backwards case at the same time, we study geodesics for the $\mathcal{L}$-length functionals and some regularity properties for the corresponding distances. In the last section finally, we prove Theorem 1.4, following the proof for the Ricci flow case by Perelman from [12], Section 7. See also Kleiner and Lott [7] or Müller [10] for more details.

Acknowledgements: I thank Gerhard Huisken and Klaus Ecker for their invitation for a four month research visit in Potsdam and Berlin. During this time I developed the $\mathcal{L}_b$-functional and the monotonicity of the backwards reduced volume for the case of List's extended Ricci flow system [8] discussed in Section 2 in joint work with Valentina Vulcanov. This was part of Valentina Vulcanov's master thesis [13]. The present work is a natural generalization of this result. Last but not least, I also thank Michael Struwe for valuable suggestions and the Swiss National Science Foundation for financial support.

2 Some examples

i) The static case. Let (M,g) be a Riemannian manifold and set $S_{ij}=0$ so that g is fixed. Then the quantity $\mathcal{D}$ reduces to $\mathcal{D}(0,X) = 2R_{ij}X_iX_j = 2\,\mathrm{Ric}(X,X)$. In the case where M has nonnegative Ricci curvature, i.e. $\mathcal{D}(0,X)\geq 0$ for all vector fields X on M, Theorem 1.4 can be applied. For example the *backwards* reduced volume

$$V_b(\tau) = \int_M (4\pi\tau)^{-n/2}e^{-\ell_b(q,\tau)}dV$$

is non-increasing in τ, where

$$\ell_b(q,\tau_1) := \inf_{\gamma\in\Gamma}\left\{\frac{1}{2\sqrt{\tau_1}}\int_0^{\tau_1}\sqrt{\tau}\left|\partial_\tau\,\gamma\right|^2 d\tau\right\}.$$

Note that the assumption $\mathrm{Ric}\geq 0$ is necessary for the monotonicity, a result which we already proved in [10], page 72.

ii) The Ricci flow. Let $(M, g(t))$ be a solution to the Ricci flow, i.e. let $S_{ij} = R_{ij}$ be the Ricci and $S = R$ the scalar curvature tensor on M. Since the scalar curvature evolves by $\partial_t R = \triangle R + 2|R_{ij}|^2$, and because of the twice traced second Bianchi identity $\nabla_i R_{ij} = \frac{1}{2}\nabla_j R$, we see from (1.6) that the quantity $\mathcal{D}(\mathrm{Ric}, X)$ vanishes identically on M. Hence the theorem can be applied. Note that $\mathcal{H}(\mathrm{Ric}, X, Y)$ and $\mathcal{H}(\mathrm{Ric}, X)$ denote Hamilton's matrix and trace Harnack quantities for the Ricci flow from [4]. The backwards reduced volume corresponds to the one defined by Perelman in [12], the forwards reduced volume and the proof of its monotonicity were developed by Feldman, Ilmanen and Ni in [2].

iii) Bernhard List's flow. In his dissertation [8], Bernhard List introduced a system closely related to the Ricci flow, namely

$$\begin{cases} \partial_t g = -2\,\mathrm{Ric} + 4\,\nabla\psi \otimes \nabla\psi, \\ \partial_t \psi = \triangle_g \psi, \end{cases} \tag{2.1}$$

where $\psi : M \to \mathbb{R}$ is a smooth function. His motivation came from general relativity theory: for static vacuum solutions, the Einstein evolution problem – which is in general a hyperbolic system of partial differential equations describing a Lorentzian 4-manifold – reduces to a weakly elliptic system on a 3-dimensional Riemannian manifold M, the space slice in the so-called $3+1$ split of space-time (cf. [8], [9]). The remaining freedom for solutions consists of the Riemannian metric g on M and the lapse function, which measures the speed of the space slice in time direction. If we let ψ be the logarithm of the lapse function, the static Einstein vacuum equations read

$$\begin{cases} \mathrm{Ric}(g) = 2\,\nabla\psi \otimes \nabla\psi, \\ \quad \triangle_g \psi = 0. \end{cases} \tag{2.2}$$

Clearly the solutions of (2.2) are exactly the stationary solutions of (2.1).

If we set $S_{ij} = R_{ij} - 2\nabla_i\psi\nabla_j\psi$ with $S = R - 2|\nabla\psi|^2$, the first of the flow equations in (2.1) is again of the form $\partial_t g_{ij} = -2S_{ij}$. List proved ([8], Lemma 2.11) that under this flow

$$\partial_t S = \triangle S + 2|S_{ij}|^2 + 4|\triangle\psi|^2 .$$

Moreover, a direct computation shows that

$$4(\nabla_i S_{ij}) - 2(\nabla_j S) = -8\nabla_i(\nabla_i\psi\nabla_j\psi) + 4\nabla_j(\nabla_i\psi\nabla_i\psi) = -8\triangle\psi\nabla_j\psi,$$

and plugging this into (1.6) yields

$$\mathcal{D}(S_{ij}, X) = 4|\triangle\psi|^2 - 8\triangle\psi\nabla_j\psi X_j + 4\nabla_i\psi\nabla_j\psi X_i X_j = 4|\triangle\psi - \nabla_X\psi|^2 \geq 0 \tag{2.3}$$

for all vector fields X on M. Hence, we can apply the main theorem, i.e. the backwards and forwards reduced volume monotonicity results hold for List's flow.

iv) The Ricci flow coupled with harmonic map heat flow. This flow is introduced in [11]. Let M be closed and fix a Riemannian manifold (N, γ). The couple $(g(t), \phi(t))_{t\in[0,T)}$ consisting of a family of smooth metrics $g(t)$ on M and a family of smooth maps $\phi(t)$ from M to N is called a solution to the Ricci flow coupled with harmonic map heat flow with coupling function $\alpha(t) \geq 0$, if it satisfies

$$\begin{cases} \partial_t g = -2\,\mathrm{Ric} + 2\alpha\,\nabla\phi \otimes \nabla\phi, \\ \partial_t \phi = \tau_g \phi, \end{cases} \tag{2.4}$$

Here, $\tau_g\phi$ denotes the tension field of the map ϕ with respect to the evolving metric g. Note that Bernhard List's flow above corresponds to the special case where $\alpha = 2$ and $N = \mathbb{R}$ (and thus $\tau_g = \triangle_g$ is the Laplace-Beltrami operator). We now show that the monotonicity of the reduced volumes holds for this more general flow. To this end, we set $S_{ij} = R_{ij} - \alpha\nabla_i\phi\nabla_j\phi$ with trace $S = R - 2\alpha\, e(\phi)$, where $e(\phi) = \frac{1}{2}\left|\nabla\phi\right|^2$ denotes the standard local energy density of the map ϕ. In [11], we prove the evolution equation

$$\partial_t\, S = \triangle S + 2\left|S_{ij}\right| + 2\alpha\left|\tau_g\phi\right|^2 - 2\dot{\alpha}e(\phi).$$

Using $4(\nabla_i S_{ij})X_j - 2(\nabla_j S)X_j = -4\alpha\,\tau_g\phi\nabla_j\phi X_j$ and plugging into (1.6), we get

$$\mathcal{D}(S_{ij}, X) = 2\alpha\left|\tau_g\phi - \nabla_X\phi\right|^2 - 2\dot{\alpha}e(\phi) \tag{2.5}$$

for all X on M. Thus, we can again apply Theorem 1.4 if $\alpha(t) \geq 0$ is non-increasing.

v) The mean curvature flow. Let $M^n(t) \subset \mathbb{R}^{n+1}$ denote a family of hypersurfaces evolving by mean curvature flow. Then the induced metrics evolve by $\partial_t\, g_{ij} = -2HA_{ij}$, where A_{ij} denote the components of the second fundamental form A on M and $H = g^{ij}A_{ij}$ denotes the mean curvature of M. Letting $S_{ij} = HA_{ij}$ with trace $S = H^2$, the expression $\mathcal{H}(\mathcal{S}, X)$ from Definition 1.5 becomes

$$\begin{aligned}\mathcal{H}(\mathcal{S}, X) &= \partial_t\, H^2 + \tfrac{1}{t}H^2 - 2\big\langle\nabla H^2, X\big\rangle + 2HA(X, X)\\ &= 2H\big(\,\partial_t\, H + \tfrac{1}{2t}H - 2\left\langle\nabla H, X\right\rangle + A(X, X)\big),\end{aligned}$$

that is $2H$ times Hamilton's differential Harnack expression for the mean curvature flow defined in [5]. Moreover, the quantity $\mathcal{D}(\mathcal{S}, X)$ again has a sign for all vector fields X, but unfortunately the wrong one for our purpose. Indeed, one finds $\mathcal{D}(\mathcal{S}, X) = -2\left|\nabla H - A(X,\cdot)\right|^2 \leq 0$, $\forall X \in \Gamma(TM)$, and Theorem 1.4 can't be applied. But fortunately the sign changes if we consider mean curvature flow in Minkowski space, as suggested by Mu-Tao Wang at a conference in Oberwolfach. More general, let $M^n(t) \subset L^{n+1}$ be a family of spacelike hypersurfaces in an ambient Lorentzian manifold, evolving by Lorentzian mean curvature flow. Then the induced metric solves $\partial_t\, g_{ij} = 2HA_{ij}$, i.e. we have $S_{ij} = -HA_{ij}$ and $S = -H^2$. Marking the curvature with respect to the ambient manifold with a bar, we have the Gauss equation

$$R_{ij} = \bar{R}_{ij} - HA_{ij} + A_{i\ell}A_{\ell j} + \bar{R}_{i0j0},$$

the Codazzi equation

$$\nabla_i A_{jk} - \nabla_k A_{ij} = \bar{R}_{0jki},$$

as well as the evolution equation for the mean curvature

$$\partial_t\, H = \triangle H - H(\left|A\right|^2 + \overline{\mathrm{Ric}}(\nu, \nu)),$$

cf. Section 2.1 and 4.1 of Holder [6]. Here, ν denotes the future-oriented timelike normal vector, represented by 0 in the index-notation. Combining the three equations above, we find

$$\mathcal{D}(\mathcal{S}, X) = 2\left|\nabla H - A(X,\cdot)\right|^2 + 2\overline{\mathrm{Ric}}(H\nu - X, H\nu - X) + 2\left\langle\overline{\mathrm{Rm}}(X, \nu)\nu, X\right\rangle. \tag{2.6}$$

In particular, if L^{n+1} has nonnegative sectional curvatures, we get $\mathcal{D}(\mathcal{S}, X) \geq 0$ and our main theorem can be applied.

3 Proof of Lemma 1.6

This is just a short computation. First, note that since the metric evolves by $\partial_t g_{ij} = -2S_{ij}$ its inverse evolves by $\partial_t g^{ij} = 2S^{ij} := 2g^{ik}g^{j\ell}S_{k\ell}$. As a consequence

$$\partial_t S = \partial_t(g^{ij}S_{ij}) = 2\,|S_{ij}|^2 + \sum_i (\partial_t \mathcal{S})(e_i, e_i), \tag{3.1}$$

where $\{e_i\}$ is an orthonormal basis. Therefore, by tracing and rearranging the terms, we find

$$\begin{aligned}
\sum_i \mathcal{H}(\mathcal{S}, X, e_i) &= \sum_i \big(2(\partial_t \mathcal{S})(e_i,e_i) + 2\,|\mathcal{S}(e_i,\cdot)|^2 - \nabla_{e_i}\nabla_{e_i} S + \tfrac{1}{t}\mathcal{S}(e_i,e_i)\big)\\
&\quad + \sum_i \big(-4(\nabla_X \mathcal{S})(e_i,e_i) + 4(\nabla_{e_i}\mathcal{S})(X,e_i) - 2\,\langle \mathrm{Rm}(X,e_i)X, e_i\rangle \big)\\
&= 2\big(\,\partial_t S - 2\,|S_{ij}|^2\,\big) + 2\,|S_{ij}|^2 - \triangle S + \tfrac{1}{t}S\\
&\quad - 4(\nabla_j S)X_j + 4(\nabla_i S_{ij})X_j + 2\,\mathrm{Ric}(X,X)\\
&= \partial_t S - 2\,|S_{ij}|^2 - \triangle S - 2(\nabla_j S)X_j + 4(\nabla_i S_{ij})X_j + 2R_{ij}X_iX_j\\
&\quad - 2S_{ij}X_iX_j + \partial_t S + \tfrac{1}{t}S - 2(\nabla_j S)X_j + 2S_{ij}X_iX_j\\
&= \mathcal{D}(\mathcal{S}, X) + \mathcal{H}(\mathcal{S}, X).
\end{aligned}$$

This proves the lemma. □

4 $\mathcal{L}_f$-geodesics and $\mathcal{L}_b$-geodesics

Obviously, letting τ play the role of the forwards time, the backwards reduced distance as defined in Definition 1.2 corresponds to the forwards reduced distance for the flow $\partial_\tau g_{ij} = +2S_{ij}$. Thus the computations in the forwards and the backwards case differ only by the change of some signs and we find it convenient to do them only for the forwards case. However, we mark all the signs that change in the backwards case with a hat. We illustrate this with an example. Equation (5.5) below reads

$$t^{3/2}\tfrac{d}{dt}\big(S + |X|^2\,\big) = \hat{+}t^{3/2}\mathcal{H}(\mathcal{S}, \hat{-}X) - \sqrt{t}\big(S + |X|^2\,\big),$$

with $\mathcal{H}(\mathcal{S}, \hat{-}X)$ evaluated at time t. For the forwards case, we simply neglect the hats and interpret $\mathcal{H}(\mathcal{S}, -X)$ as in Definition 1.5. For the backwards case, we change all t, ∂_t into τ, ∂_τ etc. and change all the signs with a hat, i.e. the statement is

$$\tau^{3/2}\tfrac{d}{d\tau}\big(S + |X|^2\,\big) = -\tau^{3/2}\mathcal{H}(\mathcal{S}, X) - \sqrt{\tau}\big(S + |X|^2\,\big),$$

where $\mathcal{H}(\mathcal{S}, X)$ is now evaluated at $\tau = -t$, i.e.

$$\mathcal{H}(\mathcal{S}, X) = -\,\partial_\tau S - \tfrac{1}{\tau}S - 2\,\langle \nabla S, X\rangle + 2\mathcal{S}(X,X). \tag{4.1}$$

Similarly, the matrix Harnack type expression $\mathcal{H}(\mathcal{S}, X, Y)$ from Definition 1.5 has to be interpreted as

$$\begin{aligned}
\mathcal{H}(\mathcal{S}, X, Y) = -2(\partial_\tau \mathcal{S})(Y,Y) + 2\,|\mathcal{S}(Y,\cdot)|^2 - \nabla_Y\nabla_Y S - \tfrac{1}{\tau}\mathcal{S}(Y,Y)\\
- 4(\nabla_X \mathcal{S})(Y,Y) + 4(\nabla_Y \mathcal{S})(X,Y) - 2\,\langle \mathrm{Rm}(X,Y)X, Y\rangle
\end{aligned} \tag{4.2}$$

in the backwards case.

For the Ricci flow there exist various references where the following computations can be found in detail for the backwards case, for example Kleiner and Lott [7], Müller [10] or Chow et al. [1]. The forwards case for the Ricci flow can be found in Feldman, Ilmanen and Ni [2]. This and the following section follow these sources closely.

The geodesic equation. Let $0 < t_0 \leq t_1 \leq T$ and let $\gamma_s(t)$ be a variation of the path $\gamma(t) : [t_0, t_1] \to M$. Using Perelman's notation, we set $Y(t) = \partial_s \gamma_s(t)|_{s=0}$ and $X(t) = \partial_t \gamma_s(t)|_{s=0}$. The first variation of $\mathcal{L}_f(\gamma)$ in the direction of $Y(t)$ can then be computed as follows.

$$\begin{aligned}
\delta_Y \mathcal{L}_f(\gamma) := \partial_s \mathcal{L}_f(\gamma_s)|_{s=0} &= \int_{t_0}^{t_1} \sqrt{t}\, \partial_s \big(S(\gamma_s(t)) + \langle \partial_t \gamma_s, \partial_t \gamma_s \rangle \big)|_{s=0}\, dt \\
&= \int_{t_0}^{t_1} \sqrt{t} \big(\nabla_Y S + 2 \langle \nabla_Y X, X \rangle \big) dt = \int_{t_0}^{t_1} \sqrt{t} \big(\nabla_Y S + 2 \langle \nabla_X Y, X \rangle \big) dt \\
&= \int_{t_0}^{t_1} \sqrt{t} \big(\langle Y, \nabla S \rangle + 2\, \partial_t \langle Y, X \rangle - 2 \langle Y, \nabla_X X \rangle \mathbin{\hat{+}} 4\mathcal{S}(Y, X) \big) dt \\
&= 2\sqrt{t}\, \langle Y, X \rangle \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \sqrt{t}\, \big\langle Y, \nabla S - \tfrac{1}{t} X - 2\nabla_X X \mathbin{\hat{+}} 4\mathcal{S}(X, \cdot) \big\rangle\, dt,
\end{aligned}$$

using a partial integration in the last step. An $\mathcal{L}_f$-*geodesic* is a critical point of the $\mathcal{L}_f$-length with respect to variations with fixed endpoints. Hence, the above first variation formula implies that the $\mathcal{L}_f$-geodesic equation reads

$$G_f(X) := \nabla_X X - \tfrac{1}{2} \nabla S + \tfrac{1}{2t} X \mathbin{\hat{-}} 2\mathcal{S}(X, \cdot) = 0. \tag{4.3}$$

Changing the variable $\lambda = \sqrt{t}$ in the definition of $\mathcal{L}_f$-length, we get

$$\mathcal{L}_f(\gamma(\lambda)) = \int_{\lambda_0}^{\lambda_1} \big(2\lambda^2 S(\gamma(\lambda)) + \tfrac{1}{2} |\partial_\lambda \gamma(\lambda)|^2 \big) d\lambda,$$

and the Euler-Lagrange equation (4.3) becomes

$$G_f(\tilde{X}) := \nabla_{\tilde{X}} \tilde{X} - 2\lambda^2 \nabla S \mathbin{\hat{-}} 4\lambda \mathcal{S}(\tilde{X}, \cdot) = 0, \tag{4.4}$$

where $\tilde{X} = \partial_\lambda \gamma(\lambda) = 2\lambda X$.

Existence of $\mathcal{L}_f$-geodesics. From standard existence theory for ordinary differential equations, we see that for $\lambda_0 = \sqrt{t_0}$, $p \in M$ and $v \in T_pM$ there is a unique solution $\gamma(\lambda)$ to (4.4) on an interval $[\lambda_0, \lambda_0 + \varepsilon]$ with $\gamma(\lambda_0) = p$ and $\partial_\lambda \gamma(\lambda)|_{\lambda = \lambda_0} = \lim_{t \to t_0} 2\sqrt{t} X = v$. If C is a bound for $|\mathcal{S}|$ and $|\nabla S|$ on $M \times [0, T]$ and $\tilde{X}(\lambda) \neq 0$, we find for $\mathcal{L}_f$-geodesics

$$\begin{aligned}
\partial_\lambda |\tilde{X}| = \tfrac{1}{2|\tilde{X}|}\, \partial_\lambda |\tilde{X}|^2 &= \hat{+} 2\lambda |\tilde{X}|\, \mathcal{S}\Big(\tfrac{\tilde{X}}{|\tilde{X}|}, \tfrac{\tilde{X}}{|\tilde{X}|} \Big) + 2\lambda^2 \Big\langle \nabla S, \tfrac{\tilde{X}}{|\tilde{X}|} \Big\rangle \\
&\leq 2\lambda C |\tilde{X}| + 2\lambda^2 C.
\end{aligned} \tag{4.5}$$

Hence, by a continuity argument, the unique $\mathcal{L}_f$-geodesic $\gamma(\lambda)$ can be extended to the whole interval $[\lambda_0, \sqrt{T}]$, i.e. for any $p \in M$ and $t_1 \in [t_0, T]$ we get a globally defined smooth $\mathcal{L}_f$-exponential map, taking $v \in T_pM$ to $\gamma(t_1)$, where $\lim_{t \to t_0} 2\sqrt{t}\, \partial_t \gamma(t) = v$. Moreover,

$\tilde{X} = 2\sqrt{t}X(t)$ has a limit as $t \to 0$ for $\mathcal{L}_f$-geodesics and the definition of $\mathcal{L}_f(\gamma)$ can be extended to $t_0 = 0$.

For all (q, t_1) there exists a minimizing $\mathcal{L}_f$-geodesic from $p = \gamma(0)$ to $q = \gamma(t_1)$. To see this, we can either show that $\mathcal{L}_f$-geodesics minimize for a short time and then use the broken geodesic argument as in the standard Riemannian case, or alternatively we can use the direct method of calculus of variations. There exists a minimizer of $\mathcal{L}_f(\gamma)$ among all Sobolev curves, which then has to be a solution of (4.3) and hence a smooth $\mathcal{L}_f$-geodesic.

In the following, we fix $p \in M$ and $t_0 = 0$ and denote by $L_f(q, t_1)$ the $\mathcal{L}_f$-length of a shortest $\mathcal{L}_f$-geodesic $\gamma(t)$ joining $p = \gamma(0)$ with $q = \gamma(t_1)$, i.e. the reduced length is

$$\ell_f(q, t_1) = \tfrac{1}{2\sqrt{t_1}} L_f(q, t_1).$$

Technical issues about $L_f(q, t_1)$. We first prove lower and upper bounds for $L_f(q, t_1)$. Since M is closed, there is a positive constant C_0 such that $-C_0 g(t) \leq \mathcal{S}(t) \leq C_0 g(t)$ (and thus $-C_0 n \leq S(t) \leq C_0 n$) for all $t \in [0, T]$. We can then obtain the following estimates.

Lemma 4.1
Denote by $d(p, q)$ the standard distance between p and q at time $t = 0$, i.e. the Riemannian distance with respect to $g(0)$. Then the reduced distance $L_f(q, t_1)$ satisfies

$$\frac{d^2(p,q)}{2\sqrt{t_1}} e^{-2C_0 t_1} - \frac{2nC_0}{3} t_1^{3/2} \leq L_f(q, t_1) \leq \frac{d^2(p,q)}{2\sqrt{t_1}} e^{2C_0 t_1} + \frac{2nC_0}{3} t_1^{3/2}. \tag{4.6}$$

Proof. The bounds for $\mathcal{S}(t)$ imply $-2C_0 g(t) \leq \hat{-} 2\mathcal{S}(t) = \partial_t\, g(t) \leq 2C_0 g(t)$ and thus

$$e^{-2C_0 t} g(0) \leq g(t) \leq e^{2C_0 t} g(0). \tag{4.7}$$

Using $\lambda = \sqrt{t}$ as above, we can estimate

$$\begin{aligned} \mathcal{L}_f(\gamma) &= \int_0^{\sqrt{t_1}} \Big(\tfrac{1}{2} \left|\partial_\lambda\, \gamma(\lambda)\right|^2 + 2\lambda^2 S(\gamma(\lambda)) \Big) d\lambda \\ &\geq \tfrac{1}{2} e^{-2C_0 t_1} \int_0^{\sqrt{t_1}} \left|\partial_\lambda\, \gamma(\lambda)\right|^2_{g(0)} d\lambda - \tfrac{2}{3} n C_0 \lambda^3 \Big|_0^{\sqrt{t_1}} \\ &\geq \frac{d^2(p,q)}{2\sqrt{t_1}} e^{-2C_0 t_1} - \frac{2nC_0}{3}\, t_1^{3/2}. \end{aligned}$$

With $L_f(q, t_1) = \inf_{\gamma \in \Gamma} \mathcal{L}_f(\gamma)$ we get the lower bound in (4.6). For the upper bound, let $\eta(\lambda) : [0, \sqrt{t_1}] \to M$ be a minimal geodesic from p to q with respect to $g(0)$. Then

$$\begin{aligned} L_f(q, t_1) \leq \mathcal{L}_f(\eta) &= \int_0^{\sqrt{t_1}} \Big(\tfrac{1}{2} \left|\partial_\lambda\, \eta(\lambda)\right|^2 + 2\lambda^2 S(\eta(\lambda)) \Big) d\lambda \\ &\leq \tfrac{1}{2} e^{2C_0 t_1} \int_0^{\sqrt{t_1}} \left|\partial_\lambda\, \eta(\lambda)\right|^2_{g(0)} d\lambda + \tfrac{2}{3} n C_0 \lambda^3 \Big|_0^{\sqrt{t_1}} \\ &= \frac{d^2(p,q)}{2\sqrt{t_1}} e^{2C_0 t_1} + \frac{2nC_0}{3}\, t_1^{3/2}, \end{aligned}$$

which proves the claim. ∎

Lemma 4.2
The distance $L_f : M \times (0,T) \to \mathbb{R}$ is locally Lipschitz continuous with respect to the metric $g(t)+dt^2$ on space-time and smooth outside of a set of measure zero.

Proof. For any $0 < t_* < T$, $q_* \in M$ and small $\varepsilon > 0$, let $t_1 < t_2$ be in $(t_* - \varepsilon, t_* + \varepsilon)$ and $q_1, q_2 \in B_{g(t_*)}(q_*, \varepsilon) = \{q \in M \mid d_{g(t_*)}(q_*, q) < \varepsilon\}$, where $d_{g(t_*)}(\cdot,\cdot)$ denotes the Riemannian distance with respect to the metric $g(t_*)$. Since

$$|L_f(q_1,t_1) - L_f(q_2,t_2)| \le |L_f(q_1,t_1) - L_f(q_1,t_2)| + |L_f(q_1,t_2) - L_f(q_2,t_2)|\,,$$

it suffices for the Lipschitz continuity with respect to $g(t)+dt^2$ to show that $L_f(q_1,\cdot)$ is locally Lipschitz in the time variable uniformly in $q_1 \in B_{g(t_*)}(q_*, \varepsilon)$ and $L_f(\cdot, t)$ is locally Lipschitz in the space variable uniformly in $t \in (t_* - \varepsilon, t_* + \varepsilon)$. Our proof is related to the proofs of Lemma 7.28 and Lemma 7.30 in [1]. In the following, $C = C(C_0, n, t_*, \varepsilon)$ denotes a generic constant which might change from line to line.

Claim 1: $L_f(q_1,t_2) \le L_f(q_1,t_1) + C(t_2 - t_1)$.

Proof. Let $\gamma : [0,t_1] \to M$ be a minimal $\mathcal{L}_f$-geodesic from p to q_1 and define $\eta : [0,t_2] \to M$ by

$$\eta(t) := \begin{cases} \gamma(t) & \text{if } t \in [0,t_1], \\ q_1 & \text{if } t \in [t_1,t_2]. \end{cases} \tag{4.8}$$

We compute

$$\begin{aligned} L_f(q_1,t_2) &\le \mathcal{L}_f(\eta) = \mathcal{L}_f(\gamma) + \int_{t_1}^{t_2} \sqrt{t} S(q_1,t)dt \\ &\le L_f(q_1,t_1) + \tfrac{2}{3} n C_0 \Big(t_2^{3/2} - t_1^{3/2}\Big) \\ &\le L_f(q_1,t_1) + C(t_2 - t_1), \end{aligned}$$

which proves Claim 1. □

Claim 2: $L_f(q_1,t_1) \le L_f(q_1,t_2) + C(t_2 - t_1)$.

Proof. Let $\gamma : [0,t_2] \to M$ be a minimal $\mathcal{L}_f$-geodesic from p to q_1 and define $\eta : [0,t_1] \to M$ by

$$\eta(t) := \begin{cases} \gamma(t) & \text{if } t \in [0, 2t_1 - t_2], \\ \gamma(\phi(t)) & \text{if } t \in [2t_1 - t_2, t_1], \end{cases} \tag{4.9}$$

where $\phi(t) := 2t + t_2 - 2t_1 \geq t$ on $[2t_1 - t_2, t_1]$ with $\partial_t \phi(t) \equiv 2$. We compute

$$\begin{aligned}
L_f(q_1, t_1) \leq \mathcal{L}_f(\eta) = \mathcal{L}_f(\gamma) &- \int_{2t_1-t_2}^{t_2} \sqrt{t}\Big(S(\gamma(t), t) + |\partial_t \gamma(t)|^2\Big) dt \\
&+ \int_{2t_1-t_2}^{t_1} \sqrt{t}\Big(S(\gamma(\phi(t)), t) + |\partial_t \gamma(\phi(t)) \cdot \partial_t \phi(t)|^2\Big) dt \\
&\leq L_f(q_1, t_2) + \tfrac{2}{3} n C_0 \Big(t_2^{3/2} - (2t_1 - t_2)^{3/2}\Big) + \tfrac{2}{3} n C_0 \Big(t_1^{3/2} - (2t_1 - t_2)^{3/2}\Big) \\
&+ 2 \int_{2t_1-t_2}^{t_2} \sqrt{\phi^{-1}(t)}\, |\partial_t \gamma(t)|^2_{g(\phi^{-1}(t))}\, dt \\
&\leq L_f(q_1, t_1) + C(t_2 - t_1) + 2 \int_{2t_1-t_2}^{t_2} \sqrt{\phi^{-1}(t)}\, |\partial_t \gamma(t)|^2_{g(\phi^{-1}(t))}\, dt.
\end{aligned}$$

Since $\phi^{-1}(t) \leq t$ and $t - \phi^{-1}(t) \leq 2\varepsilon$ on $[2t_1 - t_2, t_2]$, we can estimate the very last term via (4.7) by

$$\int_{2t_1-t_2}^{t_2} \sqrt{\phi^{-1}(t)}\, |\partial_t \gamma(t)|^2_{g(\phi^{-1}(t))}\, dt \leq e^{4C_0\varepsilon} \int_{2t_1-t_2}^{t_2} \sqrt{t}\, |\partial_t \gamma(t)|^2_{g(t)}\, dt.$$

As a consequence of the upper bound from Lemma 4.1 and the growth condition (4.5), $|\partial_t \gamma(t)|^2_{g(t)}$ must be uniformly bounded on $[2t_1 - t_2, t_2]$ by a constant C_1. Thus

$$\int_{2t_1-t_2}^{t_2} \sqrt{\phi^{-1}(t)}\, |\partial_t \gamma(t)|^2_{g(\phi^{-1}(t))}\, dt \leq e^{4C_0\varepsilon} C_1 \big(t_2^{3/2} - (2t_1 - t_2)^{3/2}\big) \leq C(t_2 - t_1).$$

Together with the computation above, this proves the claim. □

Claim 3: $L_f(q_1, t_2) \leq L_f(q_2, t_2) + C d_{g(t_2)}(q_1, q_2)$.

Proof. Let $\gamma : [0, t_2] \to M$ be a minimal $\mathcal{L}_f$-geodesic from p to q_2 and define the curve $\eta : [0, t_2 + d_{g(t_2)}(q_1, q_2)] \to M$ by

$$\eta(t) := \begin{cases} \gamma(t) & \text{if } t \in [0, t_2], \\ \alpha(t) & \text{if } t \in [t_2, t_2 + d_{g(t_2)}(q_1, q_2)], \end{cases} \tag{4.10}$$

where $\alpha : [t_2, t_2 + d_{g(t_2)}(q_1, q_2)] \to M$ is a minimal geodesic of constant unit speed with respect to $g(t_2)$, joining q_2 to q_1. Then, using $|\partial_t \alpha(t)|^2_{g(t)} \leq e^{4C_0\varepsilon} |\partial_t \alpha(t)|^2_{g(t_2)} = e^{4C_0\varepsilon}$, we obtain

$$\begin{aligned}
L_f(q_1, t_2 + d_{g(t_2)}(q_1, q_2)) &\leq \mathcal{L}_f(\eta) \\
&= L_f(q_2, t_2) + \int_{t_2}^{t_2 + d_{g(t_2)}(q_1, q_2)} \sqrt{t}\Big(S(\alpha(t), t) + |\partial_t \alpha(t)|^2\Big) dt \\
&\leq L_f(q_2, t_2) + \tfrac{2}{3}\Big(C_0 n + e^{4C_0\varepsilon}\Big)\Big((t_2 + d_{g(t_2)}(q_1, q_2))^{3/2} - t_2^{3/2}\Big) \\
&\leq L_f(q_2, t_2) + C d_{g(t_2)}(q_1, q_2).
\end{aligned}$$

Finally, using Claim 2 from above, we find

$$\begin{aligned}
L_f(q_1, t_2) &\leq L_f(q_1, t_2 + d_{g(t_2)}(q_1, q_2)) + C d_{g(t_2)}(q_1, q_2) \\
&\leq L_f(q_2, t_2) + C d_{g(t_2)}(q_1, q_2),
\end{aligned}$$

which proves Claim 3. □

The Lipschitz continuity in the time variable follows from Claim 1 and Claim 2. The Lipschitz continuity in the space variable follows from Claim 3 and the symmetry between q_1 and q_2.

From the definition of $L_f : M \times (0,T) \to \mathbb{R}$, we see that it is smooth outside of the set $\bigcup_t (C(t) \times \{t\})$, where for a fixed time t_1 the *cut locus* $C(t_1)$ is defined to be the set of points $q \in M$ such that either there is more than one minimal $\mathcal{L}_f$-geodesic $\gamma : [0,t_1] \to M$ from $p = \gamma(0)$ to $q = \gamma(t_1)$ or q is conjugate to p along γ. A point q is called conjugate to p along γ if there exists a nontrivial $\mathcal{L}_f$-Jacobi field J along γ with $J(0) = J(t_1) = 0$.

As in the standard Riemannian geometry, the set $C_1(t_1)$ of conjugate points to $(p,0)$ is contained in the set of critical values for the $\mathcal{L}_f$-exponential map from $(p,0)$ defined above. Hence it has measure zero by Sard's theorem. If there exist more than one minimal $\mathcal{L}_f$-geodesic from p to q, then $L(q,t_1)$ is not differentiable at q. But since $L_f(q,t_1)$ is Lipschitz, it has to be differentiable almost everywhere by Rademacher's theorem and thus the set $C_2(t_1)$ consisting of points for which there exist more than one minimal $\mathcal{L}_f$-geodesic also has to have measure zero. Combining this, $C(t_1) = C_1(t_1) \cup C_2(t_1)$ has measure zero for all $t_1 \in (0,T)$ and so $\bigcup_t (C(t) \times \{t\})$ is of measure zero, too. This finishes the proof of the lemma. □

5 Proof of Theorem 1.4

Making use of Lemma 4.2, we first pretend that $L_f(q,t_1)$ is smooth everywhere and derive formulas for $|\nabla L_f|^2$, $\partial_{t_1} L_f$ and $\triangle L_f$ under this assumption.

Lemma 5.1
The reduced distance $L_f(q,t_1)$ has the gradient properties

$$|\nabla L_f(q,t_1)|^2 = -4t_1 S \hat{+} \tfrac{4}{\sqrt{t_1}} K + \tfrac{2}{\sqrt{t_1}} L_f(q,t_1), \tag{5.1}$$

$$\partial_{t_1} L_f(q,t_1) = 2\sqrt{t_1} S \hat{-} \tfrac{1}{t_1} K - \tfrac{1}{2t_1} L_f(q,t_1), \tag{5.2}$$

where

$$K := \int_0^{t_1} t^{3/2} \mathcal{H}(\mathcal{S}, \hat{-}X) dt$$

and $\mathcal{H}(\mathcal{S}, \hat{-}X)$ is the Harnack type expression from Definition 1.5, evaluated at time t. Remember that in the backwards case we interpret $\mathcal{H}(\mathcal{S}, X)$ as in (4.1).

Proof. A minimizing curve satisfies $G_f(X) = 0$, hence the first variation formula above yields

$$\delta_Y L_f(q,t_1) = 2\sqrt{t_1} \langle X(t_1), Y(t_1) \rangle = \langle \nabla L_f(q,t_1), Y(t_1) \rangle .$$

Thus, the gradient of L_f must be $\nabla L_f(q,t_1) = 2\sqrt{t_1} X(t_1)$. This yields

$$|\nabla L_f|^2 = 4t_1 |X|^2 = -4t_1 S + 4t_1 \big(S + |X|^2\big). \tag{5.3}$$

Moreover, we compute

$$\begin{aligned} \partial_{t_1} L_f(q,t_1) &= \tfrac{d}{dt_1} L_f(q,t_1) - \nabla_X L_f(q,t_1) = \sqrt{t_1}\big(S + |X|^2\big) - \langle \nabla L_f(q,t_1), X \rangle \\ &= \sqrt{t_1}\big(S + |X|^2\big) - 2\sqrt{t_1} |X|^2 = 2\sqrt{t_1} S - \sqrt{t_1}\big(S + |X|^2\big). \end{aligned} \tag{5.4}$$

Note that ∂_{t_1} denotes the partial derivative with respect to t_1 keeping the point q fixed, while $\frac{d}{dt_1}$ refers to differentiation along an $\mathcal{L}$-geodesic, i.e. simultaneously varying the time t_1 and the point q. Next, we determine $\big(S+|X|^2\big)$ in terms of L_f. With the Euler-Lagrange equation (4.3), we get

$$\begin{aligned}
\tfrac{d}{dt}\big(S(\gamma(t))+|X(t)|^2\big) &= \partial_t S + \nabla_X S + 2\langle \nabla_X X, X\rangle \hat{-} 2\mathcal{S}(X,X)\\
&= \partial_t S + 2\langle \nabla S, X\rangle - \tfrac{1}{t}|X|^2 \hat{+} 2\mathcal{S}(X,X)\\
&= \hat{+}\mathcal{H}(\mathcal{S}, \hat{-}X) - \tfrac{1}{t}\big(S+|X|^2\big).
\end{aligned}$$

From this we obtain

$$t^{3/2}\tfrac{d}{dt}\big(S+|X|^2\big) = \hat{+}t^{3/2}\mathcal{H}(\mathcal{S},\hat{-}X) - \sqrt{t}\big(S+|X|^2\big) \tag{5.5}$$

and thus by integrating and using the notation $K=\int_0^{t_1} t^{3/2}\mathcal{H}(\mathcal{S},\hat{-}X)dt$, we conclude

$$\begin{aligned}
\hat{+}K - L_f(q,t_1) &= \int_0^{t_1} t^{3/2}\tfrac{d}{dt}\big(S+|X|^2\big)dt\\
&= t_1^{3/2}\big(S(\gamma(t_1))+|X(t_1)|^2\big) - \int_0^{t_1}\tfrac{3}{2}\sqrt{t}\big(S+|X|^2\big)dt\\
&= t_1^{3/2}\left(S+|X|^2\right) - \tfrac{3}{2}L_f(q,t_1).
\end{aligned}$$

Hence, we have

$$t_1^{3/2}\big(S+|X|^2\big) = \hat{+}K + \tfrac{1}{2}L_f(q,t_1). \tag{5.6}$$

If we insert this into (5.3) and (5.4), we get (5.1) and (5.2), respectively. □

To compute the second variation of $\mathcal{L}_f(\gamma)$, we use the following claim.

Claim 1: Under the flow $\partial_t g_{ij} = \hat{-}2S_{ij}$, we have

$$\begin{aligned}
\partial_t\langle\nabla_Y Y, X\rangle = \langle \nabla_X\nabla_Y Y, X\rangle + \langle\nabla_Y Y, \nabla_X X\rangle \hat{-} 2\mathcal{S}(\nabla_Y Y, X)\\
\hat{-}2(\nabla_Y\mathcal{S})(Y,X) \hat{+} (\nabla_X\mathcal{S})(Y,Y).
\end{aligned} \tag{5.7}$$

Proof. We start with

$$\partial_t\langle\nabla_Y Y, X\rangle = \langle\nabla_X\nabla_Y Y, X\rangle + \langle\nabla_Y Y,\nabla_X X\rangle \hat{-} 2\mathcal{S}(\nabla_Y Y, X) + \langle\dot{\nabla}_Y Y, X\rangle, \tag{5.8}$$

where $\dot{\nabla} := \partial_t\nabla$. From [10], page 21, we know that under the flow $\partial_t g = h$, we have

$$\langle\dot{\nabla}_U V, W\rangle = \tfrac{1}{2}(\nabla_U h)(V,W) - \tfrac{1}{2}(\nabla_W h)(U,V) + \tfrac{1}{2}(\nabla_V h)(U,W).$$

Hence, with $U=V=Y$, $W=X$ and $h=\hat{-}2\mathcal{S}$, we get

$$\langle\dot{\nabla}_Y Y, X\rangle = \hat{-}2(\nabla_Y\mathcal{S})(Y,X) \hat{+} (\nabla_X\mathcal{S})(Y,Y).$$

Inserting this into (5.8) proves the claim. □

Using Claim 1, we can now write $2\langle\nabla_Y\nabla_X Y, X\rangle$ as

$$\begin{aligned}
2\langle\nabla_Y\nabla_X Y, X\rangle &= 2\langle\nabla_X\nabla_Y Y, X\rangle + 2\langle\mathrm{Rm}(Y,X)Y, X\rangle\\
&= 2\,\partial_t\langle\nabla_Y Y, X\rangle - 2\langle\nabla_Y Y,\nabla_X X\rangle \hat{+} 4\mathcal{S}(\nabla_Y Y, X)\\
&\quad \hat{+} 4(\nabla_Y\mathcal{S})(Y,X) \hat{-} 2(\nabla_X\mathcal{S})(Y,Y) + 2\langle\mathrm{Rm}(Y,X)Y,X\rangle,
\end{aligned}$$

and a partial integration yields

$$
\begin{aligned}
\int_0^{t_1} 2\sqrt{t}\, \langle \nabla_Y \nabla_X Y, X\rangle \, dt = 2\sqrt{t}\, \langle \nabla_Y Y, X\rangle \big|_0^{t_1} &- \int_0^{t_1} \sqrt{t}\, \tfrac{1}{t}\, \langle \nabla_Y Y, X\rangle \, dt \\
&- \int_0^{t_1} 2\sqrt{t}\, \big\langle \nabla_Y Y, \nabla_X X \,\hat{-}\, 2\mathcal{S}(X,\cdot)\big\rangle \, dt \\
&\hat{+} \int_0^{t_1} \sqrt{t}\, (4(\nabla_Y \mathcal{S})(Y,X) - 2(\nabla_X \mathcal{S})(Y,Y))\, dt \\
&+ \int_0^{t_1} 2\sqrt{t}\, \langle \mathrm{Rm}(Y,X)Y, X\rangle \, dt.
\end{aligned}
\tag{5.9}
$$

If the geodesic equation (4.3) holds, we can write the first two integrals on the right hand side of (5.9) as

$$
-2\int_0^{t_1} \sqrt{t}\, \big\langle \nabla_Y Y, \tfrac{1}{2t}X + \nabla_X X \,\hat{-}\, 2\mathcal{S}(X,\cdot)\big\rangle \, dt = -\int_0^{t_1} \sqrt{t}\, \langle \nabla_Y Y, \nabla S\rangle \, dt,
$$

and equation (5.9) becomes

$$
\begin{aligned}
\int_0^{t_1} 2\sqrt{t}\, \langle \nabla_Y \nabla_X Y, X\rangle \, dt = 2\sqrt{t}\, \langle \nabla_Y Y, X\rangle \big|_0^{t_1} &- \int_0^{t_1} \sqrt{t}\, \langle \nabla_Y Y, \nabla S\rangle \, dt \\
&\hat{+} \int_0^{t_1} \sqrt{t}\, (4(\nabla_Y \mathcal{S})(Y,X) - 2(\nabla_X \mathcal{S})(Y,Y))\, dt \\
&+ \int_0^{t_1} 2\sqrt{t}\, \langle \mathrm{Rm}(Y,X)Y, X\rangle \, dt.
\end{aligned}
\tag{5.10}
$$

We can now compute the second variation of $\mathcal{L}_f(\gamma)$ for $\mathcal{L}_f$-geodesics γ where $G_f(X) = 0$ is satisfied. Using the first variation

$$
\delta_Y \mathcal{L}_f(\gamma) = \int_{t_0}^{t_1} \sqrt{t}(\nabla_Y S + 2\,\langle \nabla_Y X, X\rangle)dt
$$

from the last section, we compute

$$
\begin{aligned}
\delta_Y^2 \mathcal{L}_f(\gamma) &= \int_0^{t_1} \sqrt{t}\big(\, \partial_s \langle \nabla S, Y\rangle + 2\,\langle \nabla_Y \nabla_Y X, X\rangle + 2\,|\nabla_Y X|^2 \,\big)dt \\
&= \int_0^{t_1} \sqrt{t}\big(\, \langle \nabla S, \nabla_Y Y\rangle + \nabla_Y \nabla_Y S + 2\,|\nabla_X Y|^2 + 2\,\langle \nabla_Y \nabla_X Y, X\rangle \,\big)dt \\
&= 2\sqrt{t}\, \langle \nabla_Y Y, X\rangle \big|_0^{t_1} + \int_0^{t_1} \sqrt{t}\, \Big(\nabla_Y \nabla_Y S + 2\,|\nabla_X Y|^2\Big)\, dt \\
&\quad \hat{-} \int_0^{t_1} \sqrt{t}(2(\nabla_X \mathcal{S})(Y,Y) - 4(\nabla_Y \mathcal{S})(Y,X))dt \\
&\quad + \int_0^{t_1} 2\sqrt{t}\, \langle \mathrm{Rm}(Y,X)Y, X\rangle \, dt,
\end{aligned}
\tag{5.11}
$$

where we used (5.10) in the last step. Now choose the test variation $Y(t)$ such that

$$
\nabla_X Y = \hat{+}\mathcal{S}(Y,\cdot) + \tfrac{1}{2t}Y,
\tag{5.12}
$$

which implies $\partial_t |Y|^2 = \hat{-}2\mathcal{S}(Y,Y) + 2\langle \nabla_X Y, Y\rangle = \frac{1}{t}|Y|^2$ and hence $|Y(t)|^2 = t/t_1$, in particular $Y(0)=0$. We have

$$\begin{aligned}\mathrm{Hess}_{L_f}(Y,Y) &= \nabla_Y\nabla_Y L_f = \delta_Y^2(L_f) - \langle \nabla_Y Y, \nabla L_f\rangle \\ &\le \delta_Y^2\mathcal{L}_f - 2\sqrt{t_1}\,\langle \nabla_Y Y, X\rangle\,(t_1),\end{aligned} \tag{5.13}$$

where the Y in $\mathrm{Hess}_{L_f}(Y,Y) = \nabla_Y\nabla_Y L_f$ denotes a vector $Y(t_1)\in T_qM$, while in $\delta_Y^2\mathcal{L}_f$ it denotes the associated variation of the curve, i.e. the vector field $Y(t)$ along γ which solves the above ODE (5.12). Note that (5.13) holds with equality if Y is an $\mathcal{L}_f$-Jacobi field. We obtain

$$\begin{aligned}\mathrm{Hess}_{L_f}(Y,Y) \le& \int_0^{t_1}\sqrt{t}\big(\nabla_Y\nabla_Y S + 2|\nabla_X Y|^2 + 2\langle \mathrm{Rm}(Y,X)Y,X\rangle\big)dt \\ &\hat{-}\int_0^{t_1}\sqrt{t}(2(\nabla_X\mathcal{S})(Y,Y) - 4(\nabla_Y\mathcal{S})(Y,X))dt.\end{aligned} \tag{5.14}$$

Lemma 5.2
For K defined as in Lemma 5.1, and under the assumption $\mathcal{D}(\mathcal{S},Z)\ge 0$, $\forall Z\in\Gamma(TM)$, the distance function $L_f(q,t_1)$ satisfies

$$\triangle L_f(q,t_1) \le \frac{n}{\sqrt{t_1}} \hat{+} 2\sqrt{t_1}S - \frac{1}{t_1}K. \tag{5.15}$$

Proof. Note that with (5.12) we find

$$\begin{aligned}|\nabla_X Y|^2 &= |\mathcal{S}(Y,\cdot)|^2 \hat{+} \tfrac{1}{t}\mathcal{S}(Y,Y) + \tfrac{1}{4t^2}|Y(t)|^2 \\ &= |\mathcal{S}(Y,\cdot)|^2 \hat{+} \tfrac{1}{t}\mathcal{S}(Y,Y) + \tfrac{1}{4t\,t_1},\end{aligned} \tag{5.16}$$

as well as

$$\begin{aligned}\tfrac{d}{dt}\mathcal{S}(Y(t),Y(t)) &= (\partial_t\,\mathcal{S})(Y,Y) + (\nabla_X\mathcal{S})(Y,Y) + 2\mathcal{S}(\nabla_X Y,Y) \\ &= (\partial_t\,\mathcal{S})(Y,Y) + (\nabla_X\mathcal{S})(Y,Y) + \tfrac{1}{t}\mathcal{S}(Y,Y) \hat{+} 2|\mathcal{S}(Y,\cdot)|^2.\end{aligned} \tag{5.17}$$

Using (5.16), a partial integration and then (5.17), we get from (5.14)

$$\begin{aligned}\mathrm{Hess}_{L_f}(Y,Y) \le& \int_0^{t_1}\sqrt{t}\big(\nabla_Y\nabla_Y S + 2\langle \mathrm{Rm}(Y,X)Y,X\rangle\big)dt \\ &\hat{-}\int_0^{t_1}\sqrt{t}\big(2(\nabla_X\mathcal{S})(Y,Y) - 4(\nabla_Y\mathcal{S})(Y,X)\big)dt \\ &+\int_0^{t_1}\sqrt{t}\big(2|\mathcal{S}(Y,\cdot)|^2 \hat{+} \tfrac{2}{t}\mathcal{S}(Y,Y) + \tfrac{1}{2t\,t_1}\big)dt \\ =& \frac{1}{\sqrt{t_1}} - \int_0^{t_1}\sqrt{t}\,\mathcal{H}(\mathcal{S},\hat{-}X,Y)dt \hat{+} \int_0^{t_1}\sqrt{t}\big(2(\nabla_X\mathcal{S})(Y,Y) \hat{+} 4|\mathcal{S}(Y,\cdot)|^2\big)dt \\ &\hat{+}\int_0^{t_1}\sqrt{t}\big(\tfrac{3}{t}\mathcal{S}(Y,Y) + 2(\partial_t\,\mathcal{S})(Y,Y)\big)dt \\ =& \frac{1}{\sqrt{t_1}} - \int_0^{t_1}\sqrt{t}\,\mathcal{H}(\mathcal{S},\hat{-}X,Y)dt \hat{+} \int_0^{t_1}\sqrt{t}\big(2\tfrac{d}{dt}\mathcal{S}(Y,Y) + \tfrac{1}{t}\mathcal{S}(Y,Y)\big)dt \\ =& \frac{1}{\sqrt{t_1}} \hat{+} 2\sqrt{t_1}\mathcal{S}(Y,Y) - \int_0^{t_1}\sqrt{t}\,\mathcal{H}(\mathcal{S},\hat{-}X,Y)dt.\end{aligned}$$

Here, $\mathcal{H}(\mathcal{S}, \mp X, Y)$ denotes the Harnack type expression from Definition 1.5 evaluated at time t. Remember that in the backwards case $\mathcal{H}(\mathcal{S}, X, Y)$ has to be interpreted as in (4.2). Now let $\{Y_i(t_1)\}$ be an orthonormal basis of T_qM, and define $Y_i(t)$ as above, solving the ODE (5.12). We compute

$$\begin{aligned}
\partial_t \langle Y_i, Y_j\rangle &= \mp 2\mathcal{S}(Y_i, Y_j) + \langle \nabla_X Y_i, Y_j\rangle + \langle Y_i, \nabla_X Y_j\rangle \\
&= \mp 2\mathcal{S}(Y_i, Y_j) + \left\langle \pm\mathcal{S}(Y_i, \cdot) + \tfrac{1}{2t} Y_i, Y_j \right\rangle + \left\langle Y_i, \pm\mathcal{S}(Y_j, \cdot) + \tfrac{1}{2t} Y_j \right\rangle \\
&= \tfrac{1}{t}\langle Y_i, Y_j\rangle .
\end{aligned}$$

Thus the $\{Y_i(t)\}$ are orthogonal with $\langle Y_i(t), Y_j(t)\rangle = \frac{t}{t_1}\langle Y_i(t_1), Y_j(t_1)\rangle = \frac{t}{t_1}\delta_{ij}$. In particular, there exist orthonormal vector fields $e_i(t)$ along γ with $Y_i(t) = \sqrt{t/t_1}\, e_i(t)$. Summing over $\{e_i\}$ yields

$$\begin{aligned}
\triangle L_f(q, t_1) &\le \sum_i \Big(\frac{1}{\sqrt{t_1}} \pm 2\sqrt{t_1}\mathcal{S}(Y_i, Y_i) - \int_0^{t_1} \sqrt{t}\,\mathcal{H}(\mathcal{S}, \mp X, Y_i) dt \Big) \\
&= \frac{n}{\sqrt{t_1}} \pm 2\sqrt{t_1} S - \frac{1}{t_1}\int_0^{t_1} t^{3/2} \sum_i \mathcal{H}(\mathcal{S}, \mp X, e_i) dt \\
&= \frac{n}{\sqrt{t_1}} \pm 2\sqrt{t_1} S - \frac{1}{t_1}\int_0^{t_1} t^{3/2} \big(\mathcal{H}(\mathcal{S}, \mp X) + \mathcal{D}(\mathcal{S}, \mp X)\big) dt \\
&\le \frac{n}{\sqrt{t_1}} \pm 2\sqrt{t_1} S - \frac{1}{t_1} K,
\end{aligned}$$

using Lemma 1.6 and the assumption $\mathcal{D}(\mathcal{S}, \mp X) \ge 0$. □

The three formulas from Lemma 5.1 and Lemma 5.2 can now be combined to one evolution inequality for the reduced distance function $\ell_f(q, t_1) = \frac{1}{2\sqrt{t_1}} L_f(q, t_1)$. From (5.1), (5.2) and (5.15), we get

$$\begin{aligned}
|\nabla \ell_f|^2 &= \frac{1}{4t_1}|\nabla L_f|^2 = -S + \frac{1}{t_1}\ell_f \pm \frac{1}{t_1^{3/2}} K, \\
\partial_{t_1} \ell_f &= -\frac{1}{4t_1^{3/2}} L_f + \frac{1}{2\sqrt{t_1}}\partial_{t_1} L_f = -\frac{1}{t_1}\ell_f + S \mp \frac{1}{2t_1^{3/2}} K, \\
\triangle \ell_f &= \frac{1}{2\sqrt{t_1}} \triangle L_f \le \frac{n}{2t_1} \pm S - \frac{1}{2t_1^{3/2}} K,
\end{aligned}$$

and thus

$$\triangle \ell_f \pm \partial_{t_1}\ell_f \pm |\nabla \ell_f|^2 \mp S - \tfrac{n}{2t} \le 0. \tag{5.18}$$

This is equivalent to

$$(\partial_t \pm \triangle \mp S) v_f(q, t) \le 0, \tag{5.19}$$

where $v_f(q, t) := (4\pi t)^{-n/2} e^{\pm \ell_f(q,t)}$ is the density function for the reduced volume $V_f(t)$.

Note that so far we pretended that $L_f(q, t_1)$ is smooth. In the general case, it is obvious that the inequality (5.18) holds in the classical sense at all points where L_f is smooth. But what happens at all the other points? This question is answered by the following lemma.

Lemma 5.3
The inequality (5.18) holds on $M \times (0, T)$ in the barrier sense*, i.e. for all $(q_*, t_*) \in M \times (0, T)$*

there exists a neighborhood U of q_* in M, some $\varepsilon > 0$ and a smooth upper barrier $\tilde{\ell}_f$ defined on $U \times (t_* - \varepsilon, t_* + \varepsilon)$ with $\tilde{\ell}_f \geq \ell_f$ and $\tilde{\ell}_f(q_*, t_*) = \ell_f(q_*, t_*)$ which satisfies (5.18). Moreover, (5.18) holds on $M \times (0, T)$ in the distributional sense.

Proof. Given $(q_*, t_*) \in M \times (0, T)$, let $\gamma : [0, t_*] \to M$ be a minimal $\mathcal{L}_f$-geodesic from p to q_*, so that $\ell_f(q_*, t_*) = \frac{1}{2\sqrt{t_*}} \mathcal{L}_f(\gamma)$. Extend γ to a smooth $\mathcal{L}_f$-geodesic $\gamma : [0, t_* + \varepsilon] \to M$ for some $\varepsilon > 0$. For a given orthonormal basis $\{Y_i(t_*)\}$ of $T_{q_*}M$, solve the ODE (5.12) on $[0, t_* + \varepsilon]$ and let $\gamma_i(s, t)$ be a variation of $\gamma(t)$ in the direction of Y_i, i.e. $\gamma_i(0, t) = \gamma(t)$ and $\partial_s \gamma_i(s, t)|_{s=0} = Y_i(t)$. Finally, for a small neighborhood U of q_* we choose a smooth family of curves $\eta_{q,t_1} : [0, t_1] \to M$ from $\eta_{q,t_1}(0) = p$ to $\eta_{q,t_1}(t_1) = q \in U$, $t_1 \in (t_* - \varepsilon, t_* + \varepsilon)$, with the following property:

$$\eta_{\gamma_i(s,t),t} = \gamma_i(s, \cdot)|_{[0,t]}, \quad \forall t \in (t_* - \varepsilon, t_* + \varepsilon) \text{ and } |s| < \varepsilon.$$

Define $\tilde{L}_f(q, t_1) := \mathcal{L}_f(\eta_{q,t_1})$ and $\tilde{\ell}_f(q, t_1) = \frac{1}{2\sqrt{t_1}} \tilde{L}_f(q, t_1)$. By construction $\eta_{q_*,t_*} = \gamma|_{[0,t_*]}$ and hence $\tilde{L}_f(q, t_1)$ is a smooth upper barrier for $L_f(q, t_1)$ with $\tilde{L}_f(q_*, t_*) = L_f(q_*, t_*)$. Moreover, $\tilde{L}_f$ satisfies the formulas in Lemma 5.1 and Lemma 5.2. Thus $\tilde{\ell}_f(q, t_1)$ is a smooth upper barrier for $\ell_f(q, t_1)$ that satisfies (5.18).

To see that (5.18) holds in the distributional sense, we use the general fact that if a differential inequality of the type (5.15) holds in the barrier sense and we have a bound on $|\nabla L_f|$, then the inequality also holds in the distributional sense, see for example [1], Lemma 7.125. Obviously (5.1) and (5.2) also hold in the distributional sense, since they hold in the barrier sense. Combining this, the claim from the lemma follows. $\square$

Proof of Theorem 1.4. Since (5.18) and hence also (5.19) hold in the distributional sense, we simply compute, using $\partial_t \, dV = \hat{-} S dV$,

$$\begin{aligned} \partial_t V_f(t) &= \int_M v_f(q,t) \, \partial_t \, dV + \int_M \partial_t \, v_f(q,t) dV \\ &\leq \int v_f(q,t) \cdot (\hat{-} S) dV \hat{+} \int_M (S - \triangle) v_f(q,t) dV \\ &= \hat{-} \int_M \triangle v_f(q,t) dV = 0. \end{aligned} \tag{5.20}$$

Thus, the reduced volume $V_f(t)$ is non-increasing in t. $\square$

References

[1] B. Chow et al. *The Ricci flow: Techniques and applications: Part I: Geometric aspects.* Mathematical Surveys and Monographs. AMS, 2007.

[2] M. Feldman, T. Ilmanen, and L. Ni. Entropy and reduced distance for Ricci expanders. *J. Geom. Anal.*, 15:49–62, 2005.

[3] R. S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Diff. Geom.*, 17:255–306, 1982.

[4] R. S. Hamilton. The Harnack estimate for the Ricci flow. *J. Diff. Geom.*, 37:215–226, 1993.

[5] R. S. Hamilton. Harnack estimate for the mean curvature flow. *J. Diff. Geom.*, 41:225–243, 1995.

[6] M. Holder. *Geometrische Evolutionsgleichungen in Kosmologischen Raumzeiten.* PhD thesis, University of Tübingen, 1999.

[7] B. Kleiner and J. Lott. Notes on Perelman's papers. *Geometry & Topology*, 12:2587–2855, 2008.

[8] B. List. *Evolution of an extended Ricci flow system.* PhD thesis, AEI Potsdam, 2005.

[9] C. W. Misner, C. W. Thorne, and J. A. Wheeler. *Gravitation.* Freeman, 1973.

[10] R. Müller. *Differential Harnack inequalities and the Ricci flow.* Series of Lectures in Mathematics. EMS, 2006.

[11] R. Müller. *Ricci flow coupled with harmonic map heat flow.* PhD thesis, ETH Zürich, No. 18290, 2009.

[12] G. Perelman. The entropy formula for the Ricci flow and its geometric applications. *ArXiv:math/0211159*, 2002.

[13] V. Vulcanov. $\mathcal{L}$-functional of Perelman. Master's thesis, Universitatea de Vest din Timişoara, 2007.

Reto Müller
Scuola Normale Superiore di Pisa, 56126 Pisa, Italy