

# Fundamental solution for the $Q$ -Laplacian and sharp Moser-Trudinger inequality in Carnot groups\*

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## Abstract

For a general Carnot group  $G$  with homogeneous dimension  $Q$  we prove the existence of a fundamental solution of the  $Q$ -Laplacian whose exponential is a homogeneous norm on  $G$ . This implies a representation formula for smooth functions on  $G$  which is used to prove the sharp Carnot group version of the celebrated Moser-Trudinger inequality.

## 1 Introduction

Let  $G$  be a Carnot group i.e. a simply connected stratified Lie group with homogeneous dimension  $Q$ . It has been known since the work of Varopoulos [21], [22] and Saloff-Coste [17], that the following version of the Sobolev inequality holds on  $G$

$$(1.1) \quad \left( \int_G |f(x)|^q dx \right)^{1/q} \leq C_{p,q} \left( \int_G |\nabla_0 f(x)|^p dx \right)^{1/p},$$

provided that  $1 \leq p < Q$  and  $\frac{1}{p} - \frac{1}{q} = \frac{1}{Q}$ , where  $|\nabla_0 f|$  stands for the *horizontal gradient* of a function  $f \in C_0^\infty(G)$ . By completion of  $C_0^\infty(G)$  under the norm  $\|f\|_p + \|\nabla_0 f\|_p$  the above inequality

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holds for functions in the *horizontal* Sobolev space  $HW_0^{1,p}(G)$ . We refer to the Section 2 for a more detailed account on this terminology and background results of analysis on Carnot groups.

In the case  $p = Q$  the Sobolev inequality (1.1) turns into the Trudinger inequality stated as follows. There exist constants  $A_Q > 0$  and  $c_0 > 0$  such that for any domain  $\Omega \subset G$ ,  $|\Omega| < \infty$  and  $f \in HW_0^{1,Q}(\Omega)$  the following inequality holds:

$$(1.2) \quad \frac{1}{|\Omega|} \int_{\Omega} \exp \left( A_Q \frac{|f|^{Q'}(u)}{\|\nabla_0 f\|_Q^{Q'}} \right) du \leq c_0 .$$

where  $Q' = Q/(Q - 1)$  is the dual exponent of  $Q$ .

This statement has been first established by Trudinger [20] in the Euclidean space  $\mathbb{R}^n$ . In the setting of Carnot groups (1.2) was proven by Saloff-Coste in [18].

It is by now known that appropriate versions of the Sobolev inequality (1.1) and Trudinger's inequality (1.2) hold even in general metric spaces. We refer to [10] for a comprehensive recent account on this subject.

However, to find the values of the best constants  $C_{p,q}$  in (1.1) and  $A_Q$  and  $c_0$  in (1.2) is a much more delicate matter. For the Sobolev inequalities (1.1) the value of the best constant in  $\mathbb{R}^n$  was found by Talenti in [19]. Moser [14] was the one who proved Trudinger's inequality (1.2) in  $\mathbb{R}^n$  with sharp exponent  $A_n = n\omega_{n-1}^{n-1}$ .

In the setting of Carnot groups not much is known about sharp constants. The only results that have so far been proven are in the case of the Heisenberg group  $H^n$  - the simplest non-trivial Carnot group. For the Sobolev inequality (1.1) in the case  $p = 2$  the value of

$$C_{2, \frac{2n+2}{n}} = (4\pi)^{-1} n^{-2} (\Gamma(n+1))^{\frac{1}{n+1}}$$

has been determined by Jerison and Lee [12]. The value of the best constant in (1.1) in the remaining cases  $p \neq 2$  is still open even for the Heisenberg group.

Concerning the Moser-Trudinger inequality (1.2) we note the recent paper of Cohn and Lu [6] who found the value of the sharp exponent  $A_Q$  in the Heisenberg group  $H^n$  to be

$$A_Q = Q(2\pi^n \Gamma(1/2) \Gamma((Q-1)/2) \Gamma(Q/2)^{-1} \Gamma(n)^{-1})^{Q'-1} .$$

Here  $Q = 2n + 2$  and as before  $Q' = Q/(Q - 1)$ .

As main result of this paper we establish the Moser-Trudinger inequality (1.2) with a sharp constant  $A_Q$  for general Carnot groups. The best exponent  $A_Q$  is given in terms of an integral on a "unit sphere" of the horizontal gradient of a certain homogeneous norm. We compute explicitly the value of  $A_Q$  for H-type groups. Inspired by the proof of Cohn and Lu [6], our approach is based on the method of Adams [1]. The method uses an estimate of the one dimensional non-increasing rearrangement of the convolution of two functions due to O'Neil [15]. This reasoning avoids the difficult problem of studying the behaviour under symmetrization of the  $L_p$  norm of the horizontal gradient. Our starting point is to prove a representation formula for  $C_0^\infty$  functions in terms of their horizontal gradient and fundamental solution of the  $Q$ -Laplace equation which has independent interest. The proof is based on recent results from [2] about the existence of singular solutions of the  $Q$ -Laplace equation with some additional nice properties.

The paper is organized as follows: in Section 2 we recall terminology and some background results. In Section 3 we prove the existence of fundamental solution of the  $Q$ -Laplacian whose

exponential defines a homogenous norm. In Section 4 we give the proof of the Moser-Trudinger inequality with best constant  $A_Q$ . In Section 5 we calculate the value of  $A_Q$  for H-type groups.

## 2 Background results

Let us start by introducing some notation and terminology related to analysis on Carnot groups.

A *Carnot group* is a connected, simply connected, nilpotent Lie group  $G$  of dimension at least two with graded Lie algebra  $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$  so that  $[V_1, V_i] = V_{i+1}$  for  $i = 1, 2, \dots, r-1$  and  $[V_1, V_r] = 0$ . The integer  $r \geq 1$  is called the *step* of  $G$ . We denote the neutral element of  $G$  by 0 and we identify elements of  $\mathfrak{g}$  with left-invariant vector fields on  $G$  in the usual manner.

We fix throughout this paper an inner product  $\langle \cdot, \cdot \rangle_0$  in  $V_1$  with associated orthonormal basis  $X_1, \dots, X_k$ . Relative to this basis, we construct the *horizontal tangent subbundle*  $HTG$  of the tangent bundle  $TG$  with fibers  $HT_x G = \text{span}\{X_1(x), \dots, X_k(x)\}$ ,  $x \in G$ . A left-invariant vector field  $X$  on  $G$  is said to be *horizontal* if it is a section of the horizontal tangent bundle.

As a simply connected nilpotent group,  $G$  is diffeomorphic with  $\mathfrak{g} = \mathbb{R}^m$ ,  $m = \sum_{i=1}^r \dim V_i$ , via the exponential map  $\exp : \mathfrak{g} \rightarrow G$ . We identify an element  $g \in G$  with  $(x_1, \dots, x_k, t_{k+1}, \dots, t_m) \in \mathbb{R}^m$  by the formula

$$(2.1) \quad g = \exp \left( \sum_{i=1}^k x_i X_i + \sum_{i=k+1}^m t_i T_i \right)$$

where  $T_{k+1}, \dots, T_m$  denotes a set of nonhorizontal vectors extending  $X_1, \dots, X_k$  to a basis for all of  $\mathfrak{g}$ .

The Haar measure on  $G$  is induced by the exponential mapping from the Lebesgue measure on  $\mathfrak{g} = \mathbb{R}^m$ . Throughout this paper, statements involving measure theory are always understood to be with respect to Haar measure.

The *horizontal divergence* of a vector field

$$\eta = \sum_{i=1}^k \varphi_i X_i + \sum_{i=k+1}^m \psi_i T_i$$

is given by

$$(2.2) \quad \text{div}_0 \eta = \sum_{i=1}^k X_i(\varphi_i).$$

Note that the (Euclidean) divergence of a horizontal vector field agrees with its horizontal divergence; this can be seen by calculating the divergence relative to the basis  $X_1, \dots, X_k, T_{k+1}, \dots, T_m$  for  $\mathfrak{g}$ . We refer to [3] for the details of this calculation.

Let  $U$  be a domain in  $G$ . For  $f \in L^1_{\text{loc}}(U)$  we say that the horizontal gradient of  $f$  exists in the sense of distributions if there exists a horizontal vector field  $v = \sum_i v_i X_i$ ,  $v_i \in L^1_{\text{loc}}(U)$ , so that

$$\int_U \langle v, \eta \rangle_0 dx = - \int_U f \text{div}_0 \eta dx$$

for all smooth compactly supported vector fields  $\eta$ . We write  $\nabla_0 f := v$  for the horizontal gradient of  $f$ . When  $f \in C^1(U)$ ,  $\nabla_0 f$  is the unique horizontal vector field in  $U$  defined by the equation

$$\langle \nabla_0 f, X \rangle_0 = X(f)$$

for all horizontal vector fields  $X$ . For  $p \geq 1$ , we say that  $u : U \rightarrow \mathbb{R}$  is in the *horizontal Sobolev space*  $HW^{1,p}(U)$  if  $u \in L^p(U)$  and  $\nabla_0 u$  exists in the sense of distributions and  $\|\nabla_0 u\|_0 \in L^p(U)$ .

For  $t > 0$  we define  $\delta_t : \mathfrak{g} \rightarrow \mathfrak{g}$  by setting  $\delta_t(X) = t^i X$  if  $X \in V_i$  and extending by linearity. Via conjugation with the exponential map,  $\delta_t$  induces an automorphism of  $G$  onto itself which we also denote by  $\delta_t$ . Then  $(\delta_t)_{t>0}$  is the one-parameter family of so-called *dilations* of  $G$  and

$$d_C(\delta_t(x), \delta_t(y)) = td_C(x, y), \quad x, y \in G.$$

The Jacobian determinant of  $\delta_t$  (relative to Haar measure) is everywhere equal to  $t^Q$ , where

$$Q = \sum_{i=1}^r i \dim V_i$$

is the so-called *homogeneous dimension* of  $G$ .

By a *homogeneous norm* on  $G$  we mean any continuous and positive function  $f$  on  $G \setminus \{0\}$  which satisfies the conditions  $f(\delta_t(g)) = tf(g)$  for all  $t > 0$  and  $f(g^{-1}) = f(g)$ . Any homogeneous norm  $f$  can be extended continuously to all of  $G$  by setting  $f(0) = 0$ . For example, the Carnot norm  $|\cdot|_C$  is a homogeneous norm.

Let us assume that  $G$  is a Carnot group of homogeneous dimension  $Q \geq 3$ . (This restriction rules out only the cases  $G = \mathbb{R}$  and  $G = \mathbb{R}^2$ , in which case our results are classical.) Let  $U$  be a domain in  $G$  and let  $1 < p < \infty$ . A function  $f \in HW^{1,p}(U)$  is said to be a (*weak*) *solution to the  $p$ -sub-Laplace equation* in  $U$  if

$$(2.3) \quad \int_U \|\nabla_0 f\|_0^{p-2} \langle \nabla_0 f, \nabla_0 \phi \rangle_0 dx = 0$$

for all test functions  $\phi \in C_0^\infty(U)$ . In case  $f \in C^2(U)$  standard methods show that (2.3) is equivalent with the partial differential equation

$$(2.4) \quad \Delta_{0,p} f := \operatorname{div}(\|\nabla_0 f\|_0^{p-2} \nabla_0 f) = \sum_{i=1}^k X_i(\|\nabla_0 f\|_0^{p-2} X_i f) = 0$$

which is the Euler-Lagrange equation for the variational integral

$$(2.5) \quad f \mapsto \int_U \|\nabla_0 f\|_0^p dx.$$

(Note that we have used the inner product structure on  $V_1$  in (2.4) to identify the horizontal cotangent space  $HT_0^*G = (HT_0G)^* = V_1^*$  with the horizontal tangent space  $HT_0G = V_1$ .)

We call  $f$   *$p$ -harmonic* if it satisfies (2.3) in  $U$ ; the operator  $\Delta_{0,p}$  is called the  *$p$ -sub-Laplace operator*.<sup>1</sup> Section 4 of [11] contains the basic nonlinear potential theory of  $p$ -harmonic functions in arbitrary Carnot groups.

<sup>1</sup>As  $f$  is a solution of the  $p$ -sub-Laplacian equation, it would be more precise to call  $f$   *$p$ -sub-harmonic*. We avoid this terminology because of the obvious confusion that would arise with the usual notion of subharmonicity.

In the linear case  $p = 2$  we write  $\Delta_0 = \Delta_{0,2} = \sum_{i=1}^k X_i X_i$ . This is *Kohn's sub-Laplacian operator* on  $G$ , which represents a subelliptic Carnot analog of the classical Laplacian; the harmonic analysis associated with  $\Delta_0$  has been a subject of considerable investigation, see, e.g., [8], [7], [9], [16], [13], [4]. Note in particular that by a result of Folland [7, Theorem 2.1] in any Carnot group  $G$  there exists a unique fundamental solution  $u_2$  to the equation for the 2- Laplace operator which is smooth away from 0 and homogeneous of degree  $2 - Q$ :  $u_2 \circ \delta_t = t^{2-Q} u_2$ .

In the non-linear case  $p \neq 2$  there are existence results but there is no theory to give us smoothness of solutions of the  $p$ -Laplacian except for the particular cases of Heisenberg or H-type groups [5], [11], [3]. In this paper we are primarily interested in the conformal case  $p = Q$ . We know from Proposition 4.16 in [11] that there exists a weak solution  $u_Q$  of the  $Q$ -Laplace equation (the so-called *singular solution*) that is continuous on  $G \setminus 0$ , has a prescribed singularity  $\lim_{x \rightarrow 0} u_Q(x) = \infty$  at  $0 \in G$  and asymptotic behaviour  $\lim_{x \rightarrow \infty} u_Q(x) = -\infty$  at  $\infty$ . According to a recent result in [2] this singular solution has the additional property that its exponential is a homogeneous norm. To be more precise let us recall the exact statement from [2] as:

**Theorem 2.6.** *Let  $G$  be a Carnot group with homogeneous dimension  $Q$  and let  $u_Q$  be a singular solution for the  $Q$ -laplacian with pole at  $0 \in G$ . There exists a constant  $a_Q > 0$  such that the function  $N(x) = \exp(-a_Q u_Q(x))$  is a homogeneous norm on  $G$  (i.e. it satisfies  $N(\delta_t x) = tN(x)$ ).*

Let us denote by  $S = \{N = 1\}$  be the “unit sphere” with respect to the homogeneous norm  $N$  from Theorem 2.6. We shall need the following version of the integration in polar coordinates from [9].

**Proposition 2.7.** *There exists a Radon measure  $d\sigma$  on  $S$  such that for any  $f \in L^1(G)$  we have*

$$(2.8) \quad \int_G f(x) dx = \int_0^\infty \int_S f(\delta_s u) s^{Q-1} d\sigma(u) ds .$$

The following lemma will be important in the next section:

**Lemma 2.9.**

$$|\nabla_0 N|_{|_S} \in L^Q(S, d\sigma) .$$

*Proof.* From Theorem 2.6  $u_Q(x) = \frac{1}{a_Q} \log \frac{1}{N(x)}$ . One can easily check that composition of a smooth function with a horizontal Sobolev function is again a horizontal Sobolev function and the chain rule holds almost everywhere. This gives:

$$(2.10) \quad |\nabla_0 u_Q(x)| = \frac{|\nabla_0 N(x)|}{N(x)} \text{ for a.e. } x \in G .$$

By assumption  $u_Q \in HW_{loc}^{1,Q}(G \setminus \{0\})$  and therefore:

$$(2.11) \quad \int_{B(0,2) \setminus B(0,1)} |\nabla_0 u_Q|^Q(x) dx < \infty ,$$

where  $B(0, r) = \{N < r\}$ .

Applying Proposition 2.7 we see from (2.10) and (2.11) that

$$(2.12) \quad \int_1^2 \int_S \frac{|\nabla_0 N(\delta_s u)|^Q}{N^Q(\delta_s u)} s^{Q-1} d\sigma(u) ds < \infty .$$

On the other hand by Theorem 2.6 we have that  $N$  is homogeneous of degree 1:  $N(\delta_s u) = s$  for  $s > 0, u \in S$ . This implies that  $|\nabla_0 N|$  is homogeneous of degree 0 (see eg. [9]) which gives  $|\nabla_0 N(\delta_s u)| = |\nabla_0 N(u)|$  for  $s > 0$  and a.e.  $u \in S$ . Using these relations (2.12) takes the form:

$$\int_1^2 \int_S |\nabla_0 N(u)|^Q d\sigma(u) \frac{1}{s} ds = \log 2 \int_S |\nabla_0 N(u)|^Q d\sigma(u) < \infty ,$$

proving the lemma. □

**Remark 2.13.** In the Moser-Trudinger inequality the following constant

$$(2.14) \quad c_Q := \int_S |\nabla_0 N(u)|^Q d\sigma(u)$$

will play a crucial role.

### 3 Fundamental solution for the $Q$ -Laplacian

In this section we prove the existence of the fundamental solution of the  $Q$ -Laplacian with some additional properties that are crucial in the proof of the Moser-Trudinger inequality. The main result of this section is the following theorem.

**Theorem 3.1.** *Let  $G$  be a Carnot group with homogeneous dimension  $Q$  and let  $u_Q$  be the singular solution of the  $Q$ -Laplacian from Theorem 2.6. Then up to a constant multiple  $u_Q$  is a fundamental solution of the  $Q$ -Laplacian, i.e. for some  $b_Q \in \mathbb{R}$*

$$(3.2) \quad \operatorname{div}(|\nabla_0 u_Q|^{Q-2} \nabla_0 u_Q) = b_Q \cdot \delta$$

*in the sense of distributions.*

*Proof.* We shall use Theorem 2.6 so we consider  $N(x) = \exp(-a_Q u_Q(x))$  which is a homogeneous norm. Then (3.2) is equivalent to

$$\operatorname{div} \left( \frac{|\nabla_0 N|^{Q-2}}{N^{Q-1}} \cdot \nabla_0 N \right) = -d_Q \cdot \delta .$$

In fact we shall show that

$$(3.3) \quad \operatorname{div} \left( \frac{|\nabla_0 N|^{Q-2}}{N^{Q-1}} \cdot \nabla_0 N \right) = -c_Q \cdot \delta ,$$

where  $c_Q$  is given in (2.14).

We have to show that for any  $f \in C_0^\infty(G)$

$$(3.4) \quad f(0) = -c_Q^{-1} \int_G \langle \nabla_0 f(x), \nabla_0 N(x) \rangle_0 \frac{|\nabla_0 N(x)|^{Q-2}}{N(x)^{Q-1}} dx .$$

We know that for  $h \in C_0^\infty(G \setminus \{0\})$

$$(3.5) \quad \int_G \langle \nabla_0 h(x), \nabla_0 N(x) \rangle_0 \frac{|\nabla_0 N(x)|^{Q-2}}{N(x)^{Q-1}} dx = 0 ,$$

moreover, by a standard density argument (3.5) is true for all functions  $h \in HW_0^{1,Q}(G \setminus \{0\})$ .

For  $0 < r < 1/2$  consider the following auxiliary function

$$\varphi_r(x) = \begin{cases} 1, & N(x) \leq r , \\ \frac{1}{\log 2} \log \frac{2r}{N(x)}, & r < N(x) < 2r , \\ 0 & N(x) > 2r . \end{cases}$$

Define  $f_r = f \cdot \varphi_r$ . We check first that  $f - f_r \in HW_0^{1,Q}(G \setminus \{0\})$ .

To continue the proof we apply (3.5) to  $h = f - f_r$  and obtain

$$(3.6) \quad \begin{aligned} & \int_G \langle \nabla_0 f(x), \nabla_0 N(x) \rangle_0 \frac{|\nabla_0 N(x)|^{Q-2}}{N(x)^{Q-1}} dx \\ &= \int_{B(0,2r) \setminus B(0,r)} \langle \nabla_0 f_r(x), \nabla_0 N(x) \rangle_0 \frac{|\nabla_0 N(x)|^{Q-2}}{N(x)^{Q-1}} dx . \end{aligned}$$

Let us write the right hand side of the above equality as  $\alpha_r + \beta_r$ , where

$$\begin{aligned} \alpha_r &= \int_{B(0,2r) \setminus B(0,r)} \varphi_r(x) \langle \nabla_0 f(x), \nabla_0 N(x) \rangle_0 \frac{|\nabla_0 N(x)|^{Q-2}}{N(x)^{Q-1}} dx \\ \beta_r &= \int_{B(0,2r) \setminus B(0,r)} f(x) \langle \nabla_0 \varphi_r(x), \nabla_0 N(x) \rangle_0 \frac{|\nabla_0 N(x)|^{Q-2}}{N(x)^{Q-1}} dx . \end{aligned}$$

We show now that  $\alpha_r \rightarrow 0$  and  $\beta_r \rightarrow -c_Q \cdot f(0)$  as  $r \rightarrow 0$ . To check the first claim we use integration in polar coordinates:

$$(3.7) \quad \begin{aligned} |\alpha_r| &\leq \|\nabla_0 f\|_\infty \int_{B(0,2r) \setminus B(0,r)} \frac{|\nabla_0 N(x)|^{Q-1}}{N^{Q-1}(x)} dx \\ &= \|\nabla_0 f\|_\infty \int_r^{2r} \int_S |\nabla_0 N(u)|^{Q-1} d\sigma(u) ds \\ &= \left( \|\nabla_0 f\|_\infty \int_S |\nabla_0 N(u)|^{Q-1} d\sigma(u) \right) r \rightarrow 0 \text{ as } r \rightarrow 0 , \end{aligned}$$

where we used that

$$\int_S |\nabla_0 N(u)|^{Q-1} d\sigma(u) < \infty$$

which follows by Lemma 2.9. For the second claim we notice that

$$\nabla_0 \varphi_r(x) = \begin{cases} -\frac{1}{\log 2} \frac{\nabla_0 N(x)}{N(x)}, & x \in B(0, 2r) \setminus B(0, r), \\ 0, & x \in G \setminus (B(0, 2r) \setminus B(0, r)), \end{cases}$$

and so

$$\begin{aligned} & \int_{B(0, 2r) \setminus B(0, r)} \langle \nabla_0 \varphi_r(x), \nabla_0 N(x) \rangle_0 \frac{|\nabla_0 N(x)|^{Q-2}}{N(x)^{Q-1}} dx \\ &= -\frac{1}{\log 2} \int_{B(0, 2r) \setminus B(0, r)} \frac{|\nabla_0 N(x)|^Q}{N^Q(x)} dx \\ &= -\frac{1}{\log 2} \int_r^{2r} \int_S |\nabla_0 N(u)|^Q d\sigma(u) \frac{1}{s} ds = -c_Q. \end{aligned}$$

Using this calculation we can write

$$\begin{aligned} \beta_r &= -c_Q f(0) + \int_{B(0, 2r) \setminus B(0, r)} (f(x) - f(0)) \langle \nabla_0 \varphi_r(x), \nabla_0 N(x) \rangle_0 \frac{|\nabla_0 N(x)|^{Q-2}}{N(x)^{Q-1}} dx \\ &= -c_Q f(0) + \omega_r, \end{aligned}$$

where

$$|\omega_r| \leq \sup_{x \in B(0, 2r) \setminus B(0, r)} |f(x) - f(0)| \cdot c_Q \rightarrow 0, \text{ as } r \rightarrow 0.$$

□

The crucial ingredient in the proof of the Moser-Trudinger inequality is the following representation formula.

**Theorem 3.8.** *Let  $G$  be a Carnot group of homogeneous dimension  $Q$ . There exists a homogeneous norm  $N$  with  $\log N \in HW_{loc}^{1, Q}(G \setminus \{0\})$  such that for any  $f \in C_0^\infty(G)$  the following formula holds:*

$$(3.9) \quad f(v) = -c_Q^{-1} \int_G \langle \nabla_0 f(vu^{-1}), \nabla_0 N(u) \rangle_0 \frac{|\nabla_0 N(u)|^{Q-2}}{N(u)^{Q-1}} du.$$

*Proof.* For  $v = 0$  formula (3.9) is just formula (3.4) from the proof of Theorem 3.1. For a general  $v \in G$  (3.9) is obtained by group translation. □

**Remark 3.10.** Formula (3.9) was recently obtained by Cohn and Lu in the case of the Heisenberg group [6] by a different method using a direct calculation with the explicit formula of  $N$ . It is remarkable that (3.9) holds in general Carnot groups and its proof uses just the properties (and not the explicit formula) of the homogeneous norm  $N$ .



## 4 Moser-Trudinger inequality with sharp constant

For a domain  $\Omega \subset G$  we denote by  $HW_0^{1,Q}(\Omega)$  the horizontal Sobolev space of functions  $f \in L^Q(\Omega)$  supported in  $\Omega$  such that  $|\nabla_0 f| \in L^Q$ . We use the notation  $|\Sigma|$  for the Haar measure of a measurable set  $\Sigma \subset G$ .

**Theorem 4.1.** *Let  $G$  be a Carnot group with homogeneous dimension  $Q$ . Denote by  $A_Q := Q \cdot c_Q^{Q'-1}$ , where  $Q' = Q/(Q-1)$  and  $c_Q$  is given in (2.14). There exists a constant  $c_0$  such that for any domain  $\Omega \subset G$ ,  $|\Omega| < \infty$  and  $f \in HW_0^{1,Q}(\Omega)$  the following inequality holds:*

$$(4.2) \quad \frac{1}{|\Omega|} \int_{\Omega} \exp \left( A_Q \frac{|f|^{Q'}(u)}{\|\nabla_0 f\|_{L^Q}^{Q'}} \right) du \leq c_0 .$$

If  $A_Q$  is replaced by any greater number the statement is false.

*Proof.* The proof uses ideas from [6] and [1] and representation formula 3.9.

It is enough to prove Theorem 4.1 for  $f \in C_0^\infty(\Omega)$ . Representation formula (3.9) implies

$$(4.3) \quad f(v) \leq c_Q^{-1} |\nabla_0 f| * g(v)$$

where

$$g(u) = \frac{|\nabla_0 N(u)|^{Q-1}}{N(u)^{Q-1}} ,$$

and  $h * g$  stands for the convolution of two functions on  $G$  given by

$$h * g(v) = \int_G h(vu^{-1})g(u) du .$$

Let us introduce some notation. For a non-negative function  $F$  defined on  $G$  we consider its non-increasing rearrangement

$$F^*(t) = \inf \{s > 0 : \alpha_F(s) \leq t\} ,$$

where

$$\alpha_F(s) = |\{u \in G : F(u) > s\}| .$$

Then for any measurable function  $\rho : [0, \infty) \rightarrow [0, \infty)$

$$\int_G \rho(F(u)) du = \int_0^\infty \rho(F^*(s)) ds .$$

We also consider the average of the rearrangement

$$F^{**}(t) = \frac{1}{t} \int_0^t F^*(s) ds .$$

To estimate the rearrangement of the convolution we shall apply O'Neil's lemma [15] which is valid in Carnot groups (even in more general setting)

$$(4.4) \quad (h * g)^*(t) \leq (h * g)^{**}(t) \leq th^{**}(t)g^{**}(t) + \int_t^\infty h^*(s)g^*(s) ds .$$

In our case

$$h(u) = |\nabla_0 f(u)| \text{ and } g(u) = \frac{|\nabla_0 N(u)|^{Q-1}}{N(u)^{Q-1}}.$$

To calculate  $g^*(t)$  denote by  $u^*$  the  $\delta$ -projection of  $u \in G$  onto  $S$  defined by the condition  $u = \delta_{N(u)}(u^*)$ .

With this notation it is easy to see that

$$\alpha_g(s) = |\{u \in G : g(u) > s\}| = |\{u \in G : N(u) \leq s^{-\frac{1}{Q-1}} |\nabla_0 N(u^*)|\}|.$$

By integration in polar coordinates

$$\alpha_g(s) = \int_S \int_0^{s^{\frac{1}{Q-1}} |\nabla_0 N(u^*)|} r^{Q-1} dr d\sigma(u^*) = \frac{s^{-\frac{Q}{Q-1}}}{Q} c_Q.$$

This implies that

$$g^*(t) = \left(\frac{Q}{c_Q}\right)^{\frac{1}{Q}-1} t^{\frac{1}{Q}-1} \text{ and } g^{**}(t) = Qg^*(t).$$

Inequality (4.4) becomes

$$(4.5) \quad (|\nabla_0 f| * g)^*(t) \leq \left(\frac{Q}{c_Q}\right)^{-\frac{1}{Q'}} \left( Q \cdot t^{-\frac{1}{Q'}} \int_0^t |\nabla_0 f|^*(s) ds + \int_0^\infty |\nabla_0 f|^*(s) s^{-\frac{1}{Q'}} ds \right).$$

We shall estimate the right side of (4.5) by using the following lemma from [1].

**Lemma 4.6.** *Let  $a(s, t)$  be a non-negative measurable function on  $(-\infty, \infty) \times [0, \infty)$  such that*

$$a(s, t) \leq 1 \text{ whenever } 0 < s < t \text{ and}$$

$$\sup_{t>0} \left( \int_{-\infty}^0 + \int_t^\infty a(s, t)^{Q'} ds \right)^{1/Q'} = b < \infty.$$

*Then there exists a constant  $c_0 = c_0(Q, b)$  such that for  $\Phi \geq 0$  with  $\int_{-\infty}^\infty \Phi^Q(s) ds \leq 1$  it follows that*

$$\int_0^\infty \exp(-H(t)) dt \leq c_0,$$

where

$$H(t) = t - \left( \int_{-\infty}^\infty a(s, t) \Phi(s) ds \right)^{Q'}.$$

To apply Lemma 4.6 assume that  $\int_\Omega |\nabla_0 f|^Q(u) du \leq 1$ . Setting  $\Phi(s) = (|\Omega|e^{-s})^{1/Q} |\nabla_0 f|^*(|\Omega|e^{-s})$  obtain

$$\int_0^\infty \Phi^Q(s) ds = \int_0^{|\Omega|} (|\nabla_0 f|^*(s))^Q ds = \int_\Omega |\nabla_0 f|^Q(u) du \leq 1.$$

The auxiliary function  $a(s, t)$  is defined to be

$$a(s, t) = \begin{cases} 0, & \text{if } -\infty < s < 0, \\ 1, & \text{if } s < t, \\ Qe^{\frac{t-s}{Q'}}, & \text{if } t \leq s < \infty, \end{cases}$$

which gives

$$\sup_{t>0} \left( \int_{-\infty}^0 + \int_t^{\infty} a(s,t)^{Q'} ds \right)^{1/Q'} = Q.$$

By a direct computation we obtain

$$\int_{-\infty}^{\infty} a(s,t)\Phi(s) ds = Q e^{t/Q'} |\Omega|^{-1/Q'} \int_0^{|\Omega|e^{-t}} |\nabla_0 f|^*(s) ds + \int_{|\Omega|e^{-t}}^{|\Omega|} |\nabla_0 f|^*(s) s^{-1/Q'} ds.$$

According to the conclusion of Lemma 4.6

$$\begin{aligned} (4.7) \quad & \int_0^{\infty} e^{-H(t)} dt \\ &= \int_0^{\infty} \exp(-t + \left( Q e^{t/Q'} |\Omega|^{-1/Q'} \int_0^{|\Omega|e^{-t}} |\nabla_0 f|^*(s) ds + \int_{|\Omega|e^{-t}}^{|\Omega|} |\nabla_0 f|^*(s) s^{-1/Q'} ds \right)^{Q'}) dt \\ &\leq c_0. \end{aligned}$$

To prove the estimate in the first statement of Theorem 4.1 we start with (4.3) and (4.5):

$$\begin{aligned} & \int_{\Omega} \exp(A_Q |f|^{Q'}(v)) dv \leq \int_{\Omega} \exp(A_Q c_Q^{-Q'} (|\nabla_0 f| * g(v))^{Q'}) dv \\ &= \int_0^{|\Omega|} \exp(A_Q c_Q^{-Q'} ((|\nabla_0 f| * g)^*(s))^{Q'}) ds \\ &\leq \int_0^{|\Omega|} \exp(A_Q (Q c_Q^{Q'-1})^{-1} \left( Q t^{-1/Q'} \int_0^t |\nabla_0 f|^*(s) ds + \int_t^{\infty} |\nabla_0 f|^*(s) s^{-1/Q'} ds \right)^{Q'}) dt. \end{aligned}$$

Since  $A_Q = Q c_Q^{Q'-1}$  we obtain

$$(4.8) \quad \begin{aligned} & \int_{\Omega} \exp(A_Q |f|^{Q'}(v)) dv \\ &\leq \int_0^{|\Omega|} \exp \left( Q t^{-1/Q'} \int_0^t |\nabla_0 f|^*(s) ds + \int_t^{\infty} |\nabla_0 f|^*(s) s^{-1/Q'} ds \right)^{Q'} dt. \end{aligned}$$

We now make in (4.7) the change of variables  $t \rightarrow |\Omega|e^{-t}$  and notice that the middle part of (4.7) coincides with right side of (4.8). This concludes the proof of the first statement of the theorem.

To prove the second statement let  $\Omega = B = \{u \in G : N(u) < 1\}$ .

Let us assume that for some  $\beta > 0$

$$(4.9) \quad \frac{1}{|B|} \int_B \exp \left( \beta \frac{|f|^{Q'}(u)}{\|\nabla_0 f\|_Q^{Q'}} \right) du \leq c_0,$$

for all  $f \in HW_0^{1,Q}(B)$ . For  $0 < r < 1$  denote by  $B_r = \{u \in G : N(u) < r\}$  and consider the function

$$f_r(u) = \begin{cases} (\log(1/r))^{-1} \log N^{-1}(u), & \text{on } B \setminus B_r, \\ 1, & \text{on } B_r. \end{cases}$$

It follows that

$$|\nabla_0 f_r|(u) = \begin{cases} (\log(1/r))^{-1} \frac{|\nabla_0 N(u)|}{N(u)}, & \text{on } B \setminus B_r, \\ 0, & \text{on } B_r. \end{cases}$$

Integrating in polar coordinates yields

$$\begin{aligned} (\|\nabla_0 f_r\|_Q)^{Q'} &= \left( \int_{r \leq N(u) \leq 1} \left[ (\log(1/r))^{-1} \frac{|\nabla_0 N(u)|}{N(u)} \right]^Q du \right)^{Q'-1} \\ &= (\log(1/r))^{-1} \left( \int_S \int_r^1 \frac{|\nabla_0 N(\delta_s u)|^Q}{N(\delta_s u)^Q} s^{Q-1} ds d\sigma(u) \right)^{Q'-1}. \end{aligned}$$

We use again the homogeneity of  $N$ :  $N(\delta_s u) = s$  which implies  $|\nabla_0 N(\delta_s u)| = |\nabla_0 N(u)|$ . We obtain

$$(\|\nabla_0 f_r\|_Q)^{Q'} = (\log(1/r))^{-1} (c_Q)^{Q'-1}.$$

Using (4.9) and the fact that  $f_r \equiv 1$  on  $B_r$  we have

$$\frac{|B_r|}{|B|} \exp\left(\frac{\beta}{c_Q^{Q'-1}} \log(1/r)\right) \leq c_0.$$

Since  $|B_r|/|B| = r^Q = \exp(-Q \log(1/r))$  we conclude that

$$\exp\left(\frac{\beta - Q c_Q^{Q'-1}}{c_Q^{Q'-1}} \log(1/r)\right) \leq c_0 \text{ for any } 0 < r < 1.$$

This implies that  $\beta \leq Q c_Q^{Q'-1} = A_Q$ .

□

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