Existence and nonexistence results for anisotropic quasilinear elliptic equations

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Abstract

We consider a new class of quasilinear elliptic equations with a power-like reaction term: the differential operator weights partial derivatives with different powers, so that the underlying functional-analytic framework involves anisotropic Sobolev spaces. Critical exponents for embeddings of these spaces into L^q have two distinct expressions according to whether the anisotropy is "concentrated" or "spread out". Existence results in the subcritical case are influenced by this phenomenon. On the other hand, nonexistence results are obtained in the at least critical case in domains with a geometric property which modifies the standard notion of starshapedness.

Résumé

Nous considérons une nouvelle classe d'équations elliptiques quasilinéaires avec un terme de réaction de type puissance: les dérivées partielles ont des puissances différentes dans l'opérateur différentiel, de façon que l'espace fonctionnel naturel devient un espace de Sobolev anisotrope. Les exposants critiques pour les injections de ces espaces dans L^q ont des expressions différentes qui dépendent de la "concentration" de l'anisotropie. Nos résultats d'existence dans le cas sous-critique sont influencés par ce phenomène. D'autre part, nos résultats de non existence dans le cas critique et sur-critique sont obtenus dans des domaines ayant une propriété qui modifie la notion usuelle d'ensemble étoilé.

2000MSC : 35J70, 46E35, 35B65.

Keywords : anisotropic Sobolev spaces, critical exponents, minimax methods, Pohožaev identity.

1 Introduction

We are interested in existence and nonexistence results for the following anisotropic quasilinear elliptic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{n} \partial_i \left(|\partial_i u|^{m_i-2} \partial_i u \right) = \lambda u^{p-1} & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\Omega \subset \mathbf{R}^n$ ($n \geq 2$) is a smooth bounded domain, $m_i > 1$ for all i, $\lambda > 0$ and $p > 1$. Note that if $m_i = 2$ for all i, then (1) reduces to the well-known semilinear equation $-\Delta u = \lambda u^{p-1}$.

There is by now a large number of papers and an increasing interest about anisotropic problems. With no hope of being complete, let us mention some pioneering works on anisotropic Sobolev spaces [14, 19, 24, 25, 27] and some more recent regularity results for minimizers of anisotropic functionals [1, 5, 10, 17, 18, 28].

Historically, the study of the semilinear problem $-\Delta u = f(x,u)$ started by settling the background of a rigorous functional-analytic framework (Sobolev spaces) and by establishing the existence of solutions in a variational way, that is, minimizing suitable functionals. But then, the following step was to find solutions by means of minimax methods such as Birkhoff theory, Ljusternik-Schnirelmann category, mountain-pass and linking theorems. As far as we are aware, minimax methods have not yet been used for problems like (1), so the present paper is a first contribution in this direction.

A further motivation for the study of (1) is given by the necessity of an explanation of the link between quasilinear elliptic equations and embedding inequalities. It is known that for quasilinear elliptic equations involving the m-Laplace operator Δ_m $(m > 1)$, power-like reaction terms exhibit several critical exponents, see [9] and references therein. More precisely, critical exponents of suitable embedding inequalities are also the borderline between existence and nonexistence results for solutions of such equations. Therefore, one may wonder if these results may be obtained only using functional analysis, without exploiting the typical features of elliptic operators such as regularity theory, maximum principles, homogeneous eigenvalue problems. And the elliptic operator in (1) precisely fails to possess these properties.

Our starting point is the observation that embedding theorems for anisotropic Sobolev spaces occur below a critical exponent which has a different value if the anisotropy is spread out or concentrated. More precisely, let $m = (m_1, ..., m_n)$ and denote by $W_0^{1,m}(\Omega)$ the closure of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1,m} = \sum_{i=1}^{n} \|\partial_i u\|_{m_i} \ .$$

When the exponents m_i are not "too far apart", the critical exponent m^* for the embedding $W_0^{1,m}(\Omega) \subset L^q(\Omega)$ is just the usual critical exponent corresponding to the harmonic mean of the m_i; on the other hand, if the m_i are "too much spread out" it coincides with the maximum m_+ of the m_i. Therefore the effective critical exponent is in fact the maximum of these two values, $m_\infty = \max\{m^*, m_+\}$, see Theorem 1 below. Existence results for (1) are quite different in the two mentioned situations.

However, before wondering about existence results, due to the lack of a satisfactory regularity theory, one must be careful in describing what is meant by a *solution* of (1). In next section (Definition 1), we introduce three different kinds of solutions, weak, mild, and strong, according to their summability. In Theorem 2 we prove that, in the subcritical case, weak solutions of (1) are actually strong, namely they are summable at any power. In order to prove this fact, due to the anisotropy of the differential operator, we need several essential modifications of the method developed by Brezis & Kato in [4].

Once the different kinds of solutions are clarified, we may turn to existence results. In Theorems 3 and 4 we apply respectively constrained minimization methods and the mountain-pass Theorem in order to prove the existence of strong solutions of (1) in the "compact" case. It turns out that also the application of these by now standard tools is not straightforward. First of all, the "kinetic functional" (which coincides with the Dirichlet integral when $m_i \equiv 2$) is not homogeneous and rescaling is not allowed. Therefore, the minimization method merely enables us to find some λ for which (1) admits nontrivial solutions. Moreover, it is not clear which exponents p yield a resonance situation, *i.e.* eigenvalue problems, see Problem 2 in Section 8.3. On the other hand, the application of the mountain-pass Theorem requires further restrictions on the exponents m_i, see (5) below and the remarks in Section 8.1.

In order to prove nonexistence results for at least critical growth problems, the most common tool is the celebrated Pohožaev identity [21, 22]. However, even its weaker formulations require solutions of class $C^1(\overline{\Omega})$ in order to have well-defined boundary terms, see [7, 8, 12]. And it seems a challenging problem to obtain such regularity for weak solutions of (1), see [10]. To overcome this difficulty we introduce a sequence of "doubly approximating" problems inspired by a nice idea of Otani [20]. This procedure turns out to be quite delicate, due to the anisotropy of the operator. Indeed, we need to prove a strong regularity result for the approximating problems, see Theorem 5. When the approximation procedure is over, we are able to prove our main nonexistence result, see Theorem 6. It states that, in the at least critical case, (1) admits no mild solutions other than $u \equiv 0$. This result requires two assumptions of different kind. First, the domain Ω must have a new geometrical feature, which modifies the classical notion of starshapedness according to the anisotropy of the operator; we call this property α-starshapedness and we feel that it sheds some light on the interplay between the structure of the differential operator and the geometry of the domain. Second, the exponents m_i must be sufficiently concentrated: this technical assumption, which might probably be relaxed (see Problem 3), guarantees the regularity of solutions to the coercive approximating problem.

The precise statements of the results are given in Section 2, and their proofs are postponed to the subsequent sections. Finally in Section 8, we collect some further remarks and we address some related open problems.

2 Results

2.1 Functional setting and summability of solutions

Throughout the paper we assume that Ω is an open bounded domain with (at least) Lipschitz boundary $\partial\Omega$, and we denote by $(\ ,\ )$ the Euclidean scalar product on $\mathbf{R}^n$. We also always assume, without recalling it at each statement, that the exponents p and m_i appearing in (1) satisfy the conditions

$$p > 1 \ , \qquad m_i > 1 \qquad \forall i = 1, ..., n \ , \qquad\qquad \sum_{i=1}^{n} \frac{1}{m_i} > 1 \ . \tag{2}$$

The last condition in (2) ensures that the anisotropic Sobolev space $W_0^{1,m}(\Omega)$ embeds into some Lebesgue spaces $L^q(\Omega)$; if it is violated, one has embeddings into Orlicz's or Hölder's spaces. Embeddings of the kind $W_0^{1,m}(\Omega) \subset L^q(\Omega)$ are in fact a fundamental tool to study the existence of solutions for the boundary value problem (1). Let us set

$$m^* = \frac{n}{\sum_{i=1}^{n} \frac{1}{m_i} - 1} \ , \qquad m_+ := \max\{m_1, \dots, m_n\} \ , \qquad m_\infty = \max\{m_+, m^*\} \ . \tag{3}$$

Note that m^* is well-defined thanks to (2), and that it coincides with the usual critical exponent $\overline{m}^* := n\overline{m}/(n - \overline{m})$ for the harmonic mean $\overline{m}$ of the m_i. Note also that it may well happen that $m_+ > m^*$ (this occurs for instance if $n = 4$, $m_1 = m_2 = m_3 = 2$, $m_+ = m_4 = 100$), thus it is meaningful to define the maximal exponent m_∞. Actually, m_∞ turns out to be the "true" critical exponent. In Section 1 we prove the following result, which we could not find in the literature.

Theorem 1. *Let $\Omega \subset \mathbf{R}^n$ be an open bounded domain with Lipschitz boundary. If* (2) *holds, then for all $q \in [1, m_\infty]$ there is a continuous embedding $W_0^{1,m}(\Omega) \subset L^q(\Omega)$. For $q < m_\infty$, the embedding is compact.*

Remark 1. Theorem 1 is no longer true if the zero trace condition on the boundary is removed. More precisely, denote by $W^{1,m}(\Omega)$ the closure of the restrictions to Ω of functions in $C_c^\infty(\mathbf{R}^n)$ with respect to the norm $\|\cdot\|_{1,m} + \|\cdot\|_1$. Then, even for smooth domains Ω, in order to have the embedding $W^{1,m}(\Omega) \subset L^{m^*}(\Omega)$ some geometric restrictions on Ω are needed, see e.g. [14, 19, 24, 25].

We are now going to characterize three different kinds of solutions to the boundary value problem (1). To this end, we also need to consider the smallest exponent

$$m_- := \min\{m_1, \ldots, m_n\} \ ,$$

and, for given $q \in [1, +\infty]$, we denote by $q' := q/(q-1)$ its conjugate exponent.

Definition 1. We say that $u \in W_0^{1,m}(\Omega) \cap L^{(p-1)m'_\infty}(\Omega)$ is a *weak* solution of (1) if $u \geq 0$ a.e. in Ω and

$$\sum_{i=1}^n \int_\Omega |\partial_i u|^{m_i-2} \partial_i u \partial_i v = \lambda \int_\Omega u^{p-1} v \qquad \forall v \in W_0^{1,m}(\Omega) \ . \tag{4}$$

If in addition $u \in L^{(p-1)m'_-}(\Omega)$, we say that u is a *mild* solution. Finally, if $u \in L^\infty(\Omega)$ we say that u is a *strong* solution.

Clearly, every strong solution is also a mild solution, and the latter is also a weak solution. In some cases, we may prove the converse implications:

Theorem 2. *If one of the two following situations occurs*

$$(i)\ p < m_\infty \qquad\qquad (ii)\ p = m_\infty \ \text{ and } \ m_\infty > m_+$$

then every weak solution of (1) *is also a strong solution.*

A proof of Theorem 2 is given in Section 4. For related results concerning local minimizers, we refer to [5, Theorem 2]. We believe that Theorem 2 holds under the mere assumption $p \leq m_\infty$, see Problem 1 in Section 8.3. In Section 8.2 we discuss an example which suggests the kind of solutions we should expect, according to the value of p. We also stress that, in the semilinear case (*i.e.* $m_i \equiv 2$), elliptic theory enables one to show that a strong solution of (1) in a smooth domain is a classical solution in $C^2(\overline{\Omega})$, but for general m_i this regularity seems out of reach.

2.2 Existence results

First of all, we remark that it is not clear which p yields the so-called resonance for (1). Namely, is there some p which gives rise to a "generalized eigenvalue" problem? Obviously, if $m_+ = m_-$, the resonance problem corresponds to $p = m_-$, see [3]. In the general case, we have

Theorem 3. *Assume that $p < m_\infty$. Then, for any $\gamma > 0$ there exist $\lambda_\gamma > 0$ and $u_\gamma \in W_0^{1,m}(\Omega)$ such that $\|u_\gamma\|_p = \gamma$ and u_γ is a strong solution of* (1) *when $\lambda = \lambda_\gamma$.*

In other words, there exists a continuum of pairs $(\lambda_\gamma, u_\gamma) \in (0, \infty) \times W_0^{1,m}(\Omega)$ which solve (1), seen as an eigenvalue problem. We point out that Theorem 3 cannot be used to deduce the existence of a solution to problem (1) for a given λ. In fact, unless all the m_i are equal, rescaling methods fail due to the lack of homogeneity of the differential operator.

Then, to recover an existence result for fixed λ, we apply the mountain-pass Theorem [2]. In order to deal with a "superlinear" subcritical problem we need to assume that

$$m_+ < m^*. \tag{5}$$

Note that if $m_i = m_+$ for $n-1$ indices i, then (5) is automatically fulfilled; in particular, it holds if $n = 2$.

Then we prove:

Theorem 4. *Assume that the exponents m_i satisfy assumption* (5) *and let $p \in (m_+, m^*)$. Then, for all $\lambda > 0$ problem* (1) *admits a nontrivial strong solution.*

Due to assumption (5), this statement is probably not optimal, but it seems not clear at all which are the sharp assumptions that ensure both a mountain-pass geometric structure and the Palais-Smale condition for the involved functional, see Problem 2 in Section 8.3. In Section 8.1 we exhibit two examples where the assumptions of Theorem 4 are violated and the mountain-pass Theorem cannot be applied.

2.3 Regularity and nonexistence results

We now require that the m_i satisfy the additional assumption

$$m_i \geq 2 \tag{6}$$

and the "not too far apart condition"

$$m_+ < \frac{n+2}{n} m_- \ . \tag{7}$$

Note that if (6) and (7) hold, we necessarily have $n \geq 3$ and (5), so that $m_\infty = m^*$. In order to establish our main nonexistence result, we consider some approximating problems, which are coercive and uniformly elliptic, and we prove that they admit a unique and smooth solution.

Theorem 5. *Assume that $\partial\Omega \in C^{2,\gamma}$, and that the exponents m_i satisfy assumptions* (6) *and* (7)*. Let $p > 1$, $\lambda > 0$ and $f \in C_c^\infty(\Omega)$. Then, for all $\varepsilon > 0$, the problem*

$$\begin{cases} -\displaystyle\sum_{i=1}^{n} \partial_i \left[(|\partial_i w|^{m_i - 2} + \varepsilon(1 + |Dw|^2)^{(m_- - 2)/2}) \partial_i w \right] + \lambda |w|^{p-2} w = f & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{8}$$

admits a unique (classical) solution $w \in C^2(\overline{\Omega})$.

We finally turn to the at least critical case $p \geq m^*$. We prove nonexistence results in domains which have $C^{2,\gamma}$ boundary and satisfy the following geometrical condition.

Definition 2. Let $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbf{R}^n$ with $\alpha_i > 0$ for all i. We say that a bounded smooth domain $\Omega \subset \mathbf{R}^n$ is α-starshaped with respect to the origin if

$$\sum_{i=1}^{n} \alpha_i x_i \nu_i \geq 0 \qquad \text{on } \partial\Omega , \tag{9}$$

with $\nu = (\nu_1, \ldots, \nu_n)$ denoting the outer normal to $\partial\Omega$. We say that Ω is *strictly* α-starshaped with respect to the origin if (9) holds with strict inequality. If these inequalities hold after replacing x_i by $x_i - P_i$, we say that Ω is (strictly) α-starshaped with respect to the center $P = (P_1, \ldots P_n)$. If Ω is (strictly) α-starshaped with respect to some of its points, we simply say that Ω is (strictly) α-starshaped.

Several remarks about this notion of "anisotropic starshapedness" are in order. We collect them in Section 2.5.

Since the solution in Theorem 5 is smooth up to the boundary, we may write the Pohožaev identity, see Proposition 1. Then, thanks to a suitable double passage to the limit, we prove:

Theorem 6. *Assume that $\partial\Omega \in C^{2,\gamma}$, and that the exponents m_i satisfy assumptions* (6) *and* (7). *Let $\alpha = (\alpha_1, ..., \alpha_n)$ with*

$$\alpha_i = n\left(\frac{1}{m_i} - \frac{1}{m^*}\right) .$$

Assume that either $p > m^*$ and Ω is α-starshaped, or $p = m^*$ and Ω is strictly α-starshaped. Then, for every $\lambda > 0$, the unique mild solution of (1) *is $u \equiv 0$.*

Note that by (5) the α_i in Theorem 6 are all strictly positive. If $m_+ = m_-$, then $\alpha_i = 1$ for all i, and α-starshapedness reduces to standard starshapedness.

2.4 Miscellaneous consequences

Thanks to Theorem 2, we can slightly improve Theorem 6 when $p = m^*$.

Corollary 1. *Assume that $\partial\Omega \in C^{2,\gamma}$, and that the exponents m_i satisfy assumptions* (6) *and* (7). *Let $\alpha = (\alpha_1, ..., \alpha_n)$ be as in Theorem 6. Assume that $p = m^*$ and Ω is strictly α-starshaped. Then, for every $\lambda > 0$, the unique weak solution of* (1) *is $u \equiv 0$.*

As already mentioned, existence results are strongly affected by the validity of condition (5); notice indeed that, as a consequence of Theorems 3 and 6, there holds

Corollary 2. *If $p = m^* < m_+$, then for any domain Ω there exists $\lambda > 0$ such that* (1) *admits a nontrivial strong solution. If $p = m^* > m_+$, and* (6) *and* (7) *hold, then there exist domains Ω such that for every $\lambda > 0$ the unique weak solution of* (1) *is $u \equiv 0$.*

By analyzing the proof of Theorem 6, we realize that in some cases we may state a stronger result, which excludes also the existence of sign-changing solutions. Indeed we have:

Corollary 3. *Assume that $\partial\Omega \in C^{2,\gamma}$, and that the exponents m_i satisfy assumptions* (6) *and* (7). *Let $\alpha = (\alpha_1, ..., \alpha_n)$ be as in Theorem 6. Assume that $p > m^*$ and Ω is α-starshaped. Then $u \equiv 0$ is the unique mild solution to the problem*

$$\begin{cases} -\displaystyle\sum_{i=1}^{n} \partial_i \left(|\partial_i u|^{m_i - 2} \partial_i u \right) = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 \qquad \text{on } \partial\Omega \ . \end{cases}$$

2.5 About α-starshapedness

The notion of α-starshapedness with respect to the center P may be reformulated in a more geometric way as

$$(T_\alpha(x - P), \nu) = (x - P, T_\alpha \nu) \geq 0 \qquad \text{on } \partial\Omega \ , \tag{10}$$

where T_α denotes the second order tensor $\sum_{i=1}^{n} \alpha_i e_i \otimes e_i$. In the following, by starshapedness we mean the classical notion, which corresponds to our definition when all the α_i coincide, namely when the tensor T_α is a positive multiple of the identity matrix. Some basic differences between α-starshapedness and (ordinary) starshapedness as well as the relationship between the two notions reveal themselves by looking at simple examples in dimension $n = 2$.

Example 1. The simplest example of α-starshaped domain is a ball B: it is immediate that it is α-starshaped with respect to its center O for any choice of α. Nevertheless, if $\alpha_1 \neq \alpha_2$, and we move the center P of α-starshapedness away from the the center O of the ball, B may result not α-starshaped with respect to P (recalling (10), see Figure 1). This shows that, as it happens for starshapedness, the notion of α-starshapedness is sensitive to the choice of the center. But, in contrast to starshapedness, even a convex domain may be not α-starshaped with respect to some of its points.

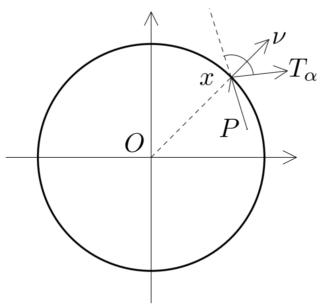

Figure 1

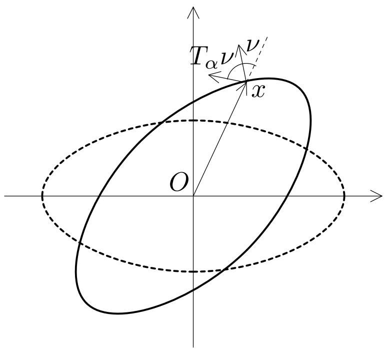

Figure 2

Example 2. Consider an ellipse E with equation $ax^2 + y^2 < 1$ ($a > 0$). One can check that E is α-starshaped with respect to $O = (0, 0)$ for every α. Now rotate E clockwise by an angle $\pi/4$: the rotated domain E' may be no longer α-starshaped with respect to O (recalling (10), see Figure 2). Thus, in contrast to starshapedness, α-starshapedness is not invariant under rotations.

Example 3. For fixed α with $\alpha_1 \neq \alpha_2$, it may happen that a domain is starshaped with respect to some center, but *not* α-starshaped with respect to *any* center. For instance, consider the set M represented in Figure 3. Clearly, it is starshaped with respect to O. However, take α with $\alpha_1 \gg \alpha_2$, and let $P \in M$. To make the sum $\alpha_1(x_1 - P_1)\nu_1 + \alpha_2(x_2 - P_2)\nu_2$ positive on ∂M, the point P should belong to both shadowed subsets of M. The intersection between such regions is empty.

Example 4. Again for fixed α with $\alpha_1 \neq \alpha_2$, the converse situation with respect to the previous example may occur, that is, it may happen that a domain is α-starshaped with respect to some center, but *not* starshaped with respect to *any* center. Consider for instance the set N represented in Figure 4. Since the product $x_1\nu_1$ remains positive on the whole boundary of N, choosing α with $\alpha_1 \gg \alpha_2$ the condition $\alpha_1 x_1 \nu_1 + \alpha_2 x_2 \nu_2 > 0$ is satisfied on ∂N, so N is strictly α-starshaped with respect to O. On the other hand, N cannot be starshaped with respect to any P, because such a P should belong to the intersection of the two disjoint shadowed subsets of N.

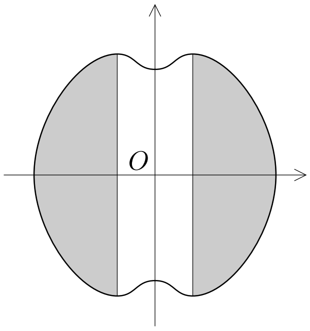

Figure 3

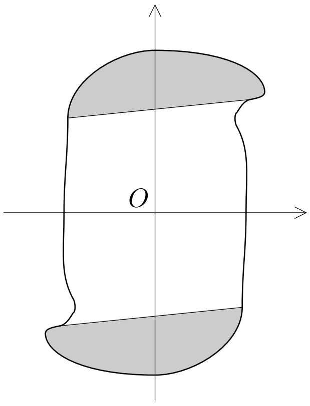

Figure 4

3 Proof of Theorem 1

The continuity of the embedding $W_0^{1,m}(\Omega) \subset L^{m_+}(\Omega)$ relies on a well-known Poincaré-type inequality. More precisely, denoting by $\{e_1, ..., e_n\}$ the canonical basis of $\mathbf{R}^n$, assume that Ω has width $a > 0$ in the direction of e_i, namely $\sup_{x,y\in\Omega} (x - y, e_i) = a$. We claim that, for every $q \geq 1$, we have

$$\|u\|_q \leq \frac{aq}{2} \|\partial_i u\|_q \qquad \forall u \in C_c^1(\Omega). \tag{11}$$

We prove (11) in the case $q > 1$, the case $q = 1$ being simpler. Assume without loss of generality that $\Omega \subset \{x \in \mathbf{R}^n;\ 0 < x_i < a\}$, and, for all $x \in \mathbf{R}^n$, set $x = (x_i, x')$ in order to emphasize its i-th component. Let $u \in C_c^1(\Omega)$ and let $v(x) = u(x)\partial_i u(x)$. We consider u (and v) as defined on the whole $\mathbf{R}^n$, set to 0 outside $\mathrm{spt}(u)$. Denote by v^+ (resp. v^-) the positive part (resp. negative part) of v. Then, we have

$$0 = \frac{|u(a, x')|^q - |u(0, x')|^q}{q}$$

$$= \int_0^a |u(t, x')|^{q-2} v(t, x')dt = \int_0^a |u(t, x')|^{q-2} v^+(t, x')dt + \int_0^a |u(t, x')|^{q-2} v^-(t, x')dt$$

and

$$\int_0^a |u(t,x')|^{q-2}v^+(t,x')dt - \int_0^a |u(t,x')|^{q-2}v^-(t,x')dt = \int_0^a |u(t,x')|^{q-2}|v(t,x')|dt$$

which show that

$$\int_0^a |u(t,x')|^{q-2}v^+(t,x')dt = \frac{1}{2}\int_0^a |u(t,x')|^{q-2}|v(t,x')|dt.$$

Therefore, we also have

$$|u(x_i,x')|^q = q\int_0^{x_i} |u(t,x')|^{q-2}v(t,x')dt \le q\int_0^{x_i} |u(t,x')|^{q-2}v^+(t,x')dt$$

$$\le q\int_0^a |u(t,x')|^{q-2}v^+(t,x')dt = \frac{q}{2}\int_0^a |u(t,x')|^{q-1}|\partial_i u(t,x')|dt.$$

Hence, by integrating first with respect to $x_i \in (0,a)$ and then with respect to $x' \in \mathbf{R}^{n-1}$ we obtain

$$\|u\|_q^q \le \frac{aq}{2}\|u\|_q^{q-1}\|\partial_i u\|_q$$

via Hölder's inequality and (11) follows. Hence, by density, the embedding $W_0^{1,m}(\Omega) \subset L^{m_+}(\Omega)$ is continuous. On the other hand, for the continuity of the embedding $W_0^{1,m}(\Omega) \subset L^{m^*}(\Omega)$, we refer to [27, Teorema 1.2] and [29, Corollary 2]. Thus, the embedding $W_0^{1,m}(\Omega) \subset L^{m_\infty}(\Omega)$ is also continuous.

In order to show the compactness of the embedding $W_0^{1,m}(\Omega) \subset L^q(\Omega)$ for $q < m_\infty$, we combine the continuous embedding $W_0^{1,m}(\Omega) \subset W_0^{1,m_-}(\Omega)$ with the compact embedding $W_0^{1,m_-}(\Omega) \subset L^1(\Omega)$ to deduce the compact embedding $W_0^{1,m}(\Omega) \subset L^1(\Omega)$. Then we conclude by interpolation between $L^1(\Omega)$ and $L^{m_\infty}(\Omega)$. □

4 Proof of Theorem 2

We first show that any weak solution of (1) belongs to $L^q(\Omega)$ for all $q \in [1,\infty)$. We have two different proofs under assumptions (i) and (ii), and we begin with

(ii) *The case* $m_+ < m^* = p$.

Let u be a weak solution to (1). The assertion that u belongs to $L^q(\Omega)$ for all $q < \infty$ may be equivalently reformulated as

$$u \in L^{(a+1)m^*}(\Omega) \qquad \text{for all } a > 0\ . \tag{12}$$

By Theorem 1, to have (12) it is enough to show that $u^{a+1} \in W_0^{1,m}(\Omega)$, which is in turn equivalent to

$$\lim_{L\to+\infty} \sum_{i=1}^n \left(\int_\Omega \Big|\partial_i(u \cdot \min[u^a, L])\Big|^{m_i} \right)^{1/m_i} < +\infty\ . \tag{13}$$

In any case (*i.e.* if the l.h.s. of (13) is bounded or unbounded), as $L \to \infty$, up to a subsequence, there exists at least one index j such that

$$\sum_{i=1}^{n}\left(\int_\Omega \left|\partial_i(u\cdot\min[u^a,L])\right|^{m_i}\right)^{1/m_i} \le C\left(\int_\Omega \left|\partial_j(u\cdot\min[u^a,L])\right|^{m_j}\right)^{1/m_j}\ , \tag{14}$$

where C denotes some positive constant independent of L. Fix such an index j, and, for every $L>0$, set $\varphi_L := u\cdot\min[u^{am_j},L^{m_j}] \in W_0^{1,m}(\Omega)$. Note that

$$|\partial_i u|^{m_i-2}\partial_i u\partial_i\varphi_L \ge \min[u^{am_j},L^{m_j}]\,|\partial_i u|^{m_i} \qquad \text{for a.e. } x\in\Omega\ ,\ \forall\, i=1,\dots,n\ , \tag{15}$$

and

$$\left|\partial_i(u\cdot\min[u^a,L])\right|^{m_i} \le (a+1)^{m_i}\min[u^{am_i},L^{m_i}]|\partial_i u|^{m_i} \qquad \text{for a.e. } x\in\Omega\ ,\ \forall\, i=1,\dots,n. \tag{16}$$

Test (1) with φ_L, integrate by parts and use (15), Hölder's inequality and Theorem 1 to obtain (for any $k>0$)

$$\begin{aligned}
\sum_{i=1}^{n}\int_\Omega \min[u^{am_j},L^{m_j}]\cdot|\partial_i u|^{m_i} &\le \sum_{i=1}^{n}\int_\Omega |\partial_i u|^{m_i-2}\partial_i u\partial_i\varphi_L = \lambda\int_\Omega u^{m^*}\cdot\min[u^{am_j},L^{m_j}]\\
&= C_k + \lambda\int_{u\ge k} u^{m^*-m_j}\,u^{m_j}\cdot\min[u^{am_j},L^{m_j}]\\
&\le C_k + \lambda\left(\int_{u\ge k} u^{m^*}\right)^{(m^*-m_j)/m^*}\cdot\left(\int_{u\ge k}\left(u^{m_j}\cdot\min[u^{am_j},L^{m_j}]\right)^{m^*/m_j}\right)^{m_j/m^*}\\
&\le C_k + \varepsilon_k\left(\int_\Omega (u\cdot\min[u^a,L])^{m^*}\right)^{m_j/m^*} \le C_k + \varepsilon_k\left[\sum_{i=1}^{n}\left(\int_\Omega \left|\partial_i(u\cdot\min[u^a,L])\right|^{m_i}\right)^{1/m_i}\right]^{m_j}\ ,
\end{aligned}$$

where $C_k \to \infty$ and $\varepsilon_k \to 0$ as $k\to\infty$ and they may denote different constants from line to line (with C_k independent of L provided one takes $L>k^a$). We also stress that in applying the Hölder's inequality, we have used the assumption $m_+ < m^*$). ¿From the last inequality and from (16) we infer (for $L>k^a$)

$$\int_\Omega \left|\partial_j(u\cdot\min[u^a,L])\right|^{m_j} \le C_k + \varepsilon_k\left[\sum_{i=1}^{n}\left(\int_\Omega \left|\partial_i(u\cdot\min[u^a,L])\right|^{m_i}\right)^{1/m_i}\right]^{m_j} \tag{17}$$

Inserting (14) into (17), we get

$$\int_\Omega \left|\partial_j(u\cdot\min[u^a,L])\right|^{m_j} \le C_k + \varepsilon_k\int_\Omega \left|\partial_j(u\cdot\min[u^a,L])\right|^{m_j}\ .$$

Choosing k sufficiently large (*i.e.* ε_k sufficiently small), this shows that the r.h.s. of (14) remains bounded as $L\to+\infty$ and (13) follows.

(i) *The case* $p < m_\infty$.

Let u be a weak solution to (1). We claim that, if the implication

$$u\in L^{am_+ + p}(\Omega) \implies u\in L^{(a+1)m_\infty}(\Omega) \tag{18}$$

holds for all $a > 0$, then $u \in L^q(\Omega)$ for all $q < \infty$. Indeed, define the sequence $\{a_k\}$ by setting

$$a_0 = \frac{m_\infty - p}{m_+}\ , \qquad a_{k+1} = \frac{m_\infty}{m_+} a_k + \frac{m_\infty - p}{m_+}\ .$$

Since $a_k \to +\infty$ (thanks to the assumption $p < m_\infty$), applying (18) with $a = a_k$, we deduce that $u \in L^q(\Omega)$ for every $q < \infty$.

Let us prove (18). By arguing as in the case $m_+ < m^* = p$, with $m_j = m_+$, we arrive at

$$\sum_{i=1}^n \int_\Omega \min[u^{am_+}, L^{m_+}] \cdot |\partial_i u|^{m_i} \le \lambda \int_\Omega u^p \cdot \min[u^{am_+}, L^{m_+}]\ \le C\ , \tag{19}$$

where C is a positive constant independent of L because $u \in L^{am_+ + p}(\Omega)$.

Assume that $L \ge 1$, let $\Omega_1 = \{x \in \Omega;\ u(x) \le 1\}$ and note that

$$\min[u^{am_+}, L^{m_+}] \ge \min[u^{am_i}, L^{m_i}] \qquad \text{a.e. in } \Omega \setminus \Omega_1\ , \quad \forall i = 1, ..., n. \tag{20}$$

Then, by (16), (19) and (20) we obtain (for constants C indepedent of L)

$$\sum_{i=1}^n \int_{\Omega\setminus\Omega_1} \Big|\partial_i(u \cdot \min[u^a, L])\Big|^{m_i} \le C \sum_{i=1}^n \int_{\Omega\setminus\Omega_1} \min[u^{am_i}, L^{m_i}] \cdot |\partial_i u|^{m_i} \le C\ . \tag{21}$$

On the other hand,

$$\sum_{i=1}^n \int_{\Omega_1} \Big|\partial_i(u \cdot \min[u^a, L])\Big|^{m_i} = \sum_{i=1}^n (a+1)^{m_i} \int_{\Omega_1} u^{am_i} \cdot |\partial_i u|^{m_i} \le C\ \sum_{i=1}^n \int_\Omega |\partial_i u|^{m_i} < +\infty. \tag{22}$$

Hence, combining (21) and (22) and letting $L \to \infty$ we obtain that

$$\sum_{i=1}^n \int_\Omega \Big|\partial_i(u^{a+1})\Big|^{m_i} < +\infty\ .$$

By Theorem 1, $u \in L^{(a+1)m_\infty}(\Omega)$, and (18) is proved.

Conclusion. Put $f(x) := \lambda u^{p-1}(x)$ so that also $f \in L^q(\Omega)$ for all $q < \infty$. Then, (1) reads

$$\begin{cases} -\displaystyle\sum_{i=1}^n \partial_i \left(|\partial_i u|^{m_i - 2} \partial_i u\right) = f \qquad \text{in } \Omega \\ \\ u = 0 \qquad \text{on } \partial\Omega\ . \end{cases}$$

In view of [13, Theorem 2], we obtain $u \in L^\infty(\Omega)$. □

5 Proof of Theorems 3 and 4

By Theorem 2, for both statements it suffices to prove the existence of weak solutions.

We first prove Theorem 3. Let $\gamma > 0$ and consider the minimization problem

$$\min\left\{\sum_{i=1}^{n}\int_{\Omega}\frac{|\partial_i u|^{m_i}}{m_i};\ u \in W_0^{1,m}(\Omega),\ \|u\|_p = \gamma\right\}. \tag{23}$$

Consider a minimizing sequence $\{u_k\}$ for (23). Since it is bounded in $W_0^{1,m}(\Omega)$, by Theorem 1 up to a subsequence $\{u_k\}$ converges in $L^p(\Omega)$ to some u. Clearly, $\|u\|_p = \gamma$, so that $u \neq 0$. By weak lower semicontinuity of the norm, we also infer

$$\liminf_{k\to\infty}\|\partial_i u_k\|_{m_i} \geq \|\partial_i u\|_{m_i} \qquad \forall i = 1, ..., n.$$

Therefore, u is a minimizer for (23) and there exists a Lagrange multiplier $\lambda_\gamma > 0$ satisfying the requirements of the statement. Moreover, u may be taken nonnegative since $|u|$ has the same norms as u. $\square$

The proof of Theorem 4 is obtained as a consequence of the mountain-pass Theorem [2] in its simplest form. Therefore, we just quickly outline it.

On the space $W_0^{1,m}(\Omega)$ consider the functional

$$J(u) = \sum_{i=1}^{n}\int_{\Omega}\frac{|\partial_i u|^{m_i}}{m_i} - \frac{\lambda}{p}\int_{\Omega}|u|^p.$$

Theorem 1 ensures that $J \in C^1(W_0^{1,m}(\Omega))$. Its Fréchet derivative J' is defined by

$$\langle J'(u), v\rangle = \sum_{i=1}^{n}\int_{\Omega}|\partial_i u|^{m_i-2}\partial_i u\partial_i v - \lambda\int_{\Omega}|u|^{p-2}uv \qquad \forall v \in W_0^{1,m}(\Omega).$$

By elementary calculus, it is not difficult to show that there exists a constant $C > 0$ independent of ℓ such that

$$a_i > 0\ \forall i\ , \quad \sum_{i=1}^{n} a_i = \ell \in (0,1) \implies \sum_{i=1}^{n}\frac{a_i^{m_i}}{m_i} \geq C\ell^{m_+}. \tag{24}$$

By the embedding $W_0^{1,m}(\Omega) \subset L^p(\Omega)$ we obtain

$$J(u) \geq \sum_{i=1}^{n}\int_{\Omega}\frac{|\partial_i u|^{m_i}}{m_i} - c\|u\|_{1,m}^p \qquad \forall u \in W_0^{1,m}(\Omega).$$

This, combined with (24) by taking $a_i = \|\partial_i u\|_{m_i}$, proves that there exists $\alpha, \beta > 0$ such that

$$J(u) \geq \alpha \qquad \forall \|u\|_{1,m} = \beta. \tag{25}$$

Moreover, if $u \in W_0^{1,m}(\Omega) \setminus \{0\}$ and $t > 0$ is sufficiently large, then $v := tu$ satisfies

$$J(v) < 0 \qquad \text{and} \qquad \|v\|_{1,m} > \beta \ . \tag{26}$$

Conditions (25) and (26) show that the functional J has a mountain-pass geometry.

Consider now a Palais-Smale sequence $\{u_k\}$ for J. It satisfies (for some c)

$$J(u_k) \to c \qquad \text{and} \qquad J'(u_k) \to 0 \text{ in } [W_0^{1,m}(\Omega)]' \qquad \text{as } k \to \infty.$$

To derive a contradiction, assume that $\|u_k\|_{1,m}$ diverges; then,

$$o(\|u_k\|_{1,m}) = J(u_k) - \frac{1}{p}\langle J'(u_k), u_k\rangle = \sum_{i=1}^n \left(\frac{1}{m_i} - \frac{1}{p}\right) \int_\Omega |\partial_i u_k|^{m_i}$$

against the assumption $m_i < p$ for all i. Therefore, up to a subsequence, $\{u_k\}$ converges weakly in $W_0^{1,m}(\Omega)$ to some u. By the compact embedding stated in Theorem 1, we also have $u_k \to u$ in $L^p(\Omega)$. Hence, since both $\langle J'(u_k), u_k\rangle \to 0$ and $\langle J'(u_k), u\rangle \to 0$, we have

$$\sum_{i=1}^n \int_\Omega |\partial_i u_k|^{m_i} \to \lambda \int_\Omega |u|^p = \sum_{i=1}^n \int_\Omega |\partial_i u|^{m_i}. \tag{27}$$

By weak lower semicontinuity of the norms, for all i we have $\liminf_k \|\partial_i u_k\|_{m_i} \geq \|\partial_i u\|_{m_i}$; this, together with (27) and weak convergence, shows that $u_k \to u$ in $W_0^{1,m}(\Omega)$. Therefore, the Palais-Smale condition holds.

As a straightforward consequence of the mountain-pass Theorem, we deduce that J admits a critical point. Since $J(u) = J(|u|)$ for all u, we may assume that such critical point is nonnegative. This concludes the proof of Theorem 4. $\square$

6 Proof of Theorem 5

It is not restrictive to assume that $\lambda = 1$. The proof consists of four steps.

Step 1. There exists a unique function $w \in X := W_0^{1,m}(\Omega) \cap L^p(\Omega)$ which solves (8). For all test functions $\varphi \in X$ it satisfies

$$\int_\Omega \left\{ \sum_{i=1}^n \left[(|\partial_i w|^{m_i - 2} + \varepsilon(1 + |Dw|^2)^{(m_- - 2)/2})\partial_i w \right] \partial_i \varphi + |w|^{p-2} w \varphi \right\} = \int_\Omega f\varphi \ . \tag{28}$$

In fact, equation (28) holds if and only if $J'(w) = 0$, where J is the integral functional

$$J(u) = \int_\Omega \left[j(Du) + \frac{1}{p}|u|^p - fu \right] \ , \qquad u \in X \ ,$$

with $j(\xi) := \sum_{i=1}^n \frac{|\xi_i|^{m_i}}{m_i} + \frac{\varepsilon}{m_-}(1 + |\xi|^2)^{m_-/2}$. If we endow X with the weak $W_0^{1,m}(\Omega)$ topology, the functional J is lower semicontinuous, because j is convex and coercive. Thus, by the direct method of the calculus of variations, J admits at least one minimizer $w \in X$, which satisfies (28). Finally, the uniqueness is gained by the strict convexity of the functional J.

Step 2. The weak solution w found in Step 1 belongs to $L^\infty(\Omega)$.
Set $k := (\sup|f|)^{1/(p-1)}$, and $\Omega_k := \{x \in \Omega \,:\, |w(x)| > k\}$. By taking $\varphi = (\mathrm{sign} w)\max\{|w| - k, 0\}$ as a test function in (28), we have:

$$\begin{aligned}\int_{\Omega_k} \sum_{i=1}^n |\partial_i w|^{m_i} \quad &\le \int_{\Omega_k} \Big[\sum_{i=1}^n |\partial_i w|^{m_i} + \varepsilon(1+|Dw|^2)^{(m_- -2)/2}|Dw|^2 \Big] \\ &= \int_{\Omega_k} (|w|-k)[f\cdot(\mathrm{sign} w) - |w|^{p-1}] \le 0 \ ,\end{aligned}$$

which shows that $\|w\|_\infty \le k$.

Step 3. The weak solution w found in Step 1 belongs to $W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap H^2_{\mathrm{loc}}(\Omega)$.
The first equation in (8) may be written in the form $\sum_{i=1}^n \partial_i a_i(Du) = b(x)$, by setting

$$a_i(\xi) := \big[|\xi_i|^{m_i-2} + \varepsilon\big(1+|\xi|^2\big)^{(m_- -2)/2}\big]\xi_i \ , \qquad b(x) := -f(x) + |w|^{p-2}w \ . \tag{29}$$

The functions a_i satisfy assumptions (2.3)-(2.6) of [18] (taking therein $p = m_-$ and $q = m_+$). The function $b(x)$ is in $L^\infty(\Omega)$ (using the global boundedness of w already proved in Step 2). Finally, (4.2) in [18] is valid in view of (6) and (7). Hence, by Theorem 4.1 of [18], there exists a function $\tilde w \in W^{1,q}_{\mathrm{loc}}(\Omega)$ which satisfies, for every $\Omega' \subset \Omega$,

$$\int_\Omega \big[\sum_{i=1}^n a_i(D\tilde w)\partial_i\varphi + b(x)\varphi\big] = 0 \qquad \forall \varphi \in W^{1,m_+}_0(\Omega') \ . \tag{30}$$

We notice that also w satisfies (30); then, since the functional J is strictly convex, we deduce that $w = \tilde w$. Using again Theorem 4.1 of [18], we deduce that $w \in W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap H^2_{\mathrm{loc}}(\Omega)$.

Step 4. The weak solution w is of class $C^2(\overline{\Omega})$.
The coefficients a_i and a defined in (29) are differentiable in their variables, bounded with their first derivatives on every compact region of $\Omega \times \mathbf{R} \times \mathbf{R}^n$, and satisfy a uniform ellipticity condition ($\sum_{i,j} \partial_j a_i(\xi)\eta_i\eta_j \ge \varepsilon|\eta|^2$). Therefore, the interior C^2 regularity of w is obtained in a standard way, by applying the theory of uniformly elliptic equations (see *e.g.* [11, Chapters 13, 14, 15] or Theorems 6.2 and 6.3 in Chapter 4 of [15]). In order to obtain gradient bounds up to the boundary (which entail $w \in C^2(\overline{\Omega})$), one may either combine the interior gradient bound with the boundary Lipschitz estimate in [11, Theorem 14.1], or adapt the technique used by Lieberman in [16] or by Tolksdorf in [26]. □

7 Proof of Theorem 6

The proof of Theorem 6 is inspired by a work of Otani [20], and is based on the construction of a sequence of “doubly approximating” problems for (1). More precisely, let u be a mild solution to (1). Let $u_k = g_k(u)$, where the g_k ($k \in \mathbf{N}$) are $C^1(\mathbf{R}^+)$ functions such that

$$g_k(s) = s \quad \forall\, s \le k \ , \qquad g_k(s) = k+1 \quad \forall\, s \ge k+2 \ , \qquad 0 \le g_k'(s) \le 1 \quad \forall\, s \ge 0.$$

Then, for all $k \in \mathbf{N}$ and all $\varepsilon \in (0,1)$ there exists a function $f^\varepsilon \in C^\infty_c(\Omega)$ such that

$$\|f^\varepsilon\|_\infty \le C_k \quad \forall \varepsilon > 0 \ , \qquad f^\varepsilon \to 2\lambda u_k^{p-1} \quad \text{in } L^r(\Omega), \quad \forall r \in [1,\infty) \tag{31}$$

for some constant $C_k > 0$ independent of ε. By Theorem 5, we know that for all $\varepsilon \in (0,1)$ there exists a unique solution $w_\varepsilon^k \in C^2(\overline{\Omega})$ to:

$$\begin{cases} -\sum_{i=1}^{n} \partial_i \left[\left[|\partial_i w|^{m_i-2} + \varepsilon(1+|Dw|^2)^{(m_- -2)/2} \right] \partial_i w \right] + \lambda |w|^{p-2} w = f^\varepsilon & \text{in } \Omega \\ w = 0 \qquad \text{on } \partial\Omega \ . \end{cases} \tag{32}$$

The structure of the proof will be the following. We first establish some a priori estimates satisfied by w_ε^k for fixed k (see (33)). Then we apply to w_ε^k a generalized Pohožaev identity for solutions to variational equations, see Proposition 1. Thanks to the a priori estimates we deduce an integral inequality satisfied by the limit w_k of w_ε^k as $\varepsilon \to 0$ (Lemma 1). In the next step, we pass to the limit in k, proving that the limit w_0 of w_k as $k \to +\infty$ coincides with u (the initial mild solution to (1)), and that u satisfies in turn an integral inequality. Finally, we conclude the proof by showing that, when Ω is α-starshaped (resp. strictly α-starshaped), and p is strictly supercritical (resp. critical), such integral inequality is fulfilled if and only if u is identically zero.

We now begin with the a priori estimates. We drop the index k since we maintain it fixed, so we simply denote by w_ε the unique solution to (32). By using (31) and by arguing as in the proof of Step 2 in Theorem 5 we infer that $\|w_\varepsilon\|_\infty \le C_k^{1/(p-1)}$ for every $\varepsilon \in (0,1)$. Then, multiplying (32) by w_ε and integrating over Ω gives

$$\sum_{i=1}^{n} \int_\Omega |\partial_i w_\varepsilon|^{m_i} \le \int_\Omega f^\varepsilon w_\varepsilon \le |\Omega| C_k^{p/(p-1)} \ .$$

We have thus obtained that, for some $C > 0$ and all $\varepsilon \in (0,1)$,

$$\|w_\varepsilon\|_{1,m} \le C \ , \qquad \|w_\varepsilon\|_\infty \le C \ . \tag{33}$$

By (33), Theorem 1 and interpolation, up to a subsequence, there exists w_k such that

$$w_\varepsilon \rightharpoonup w_k \quad \text{in } W_0^{1,m}(\Omega) \ , \qquad w_\varepsilon \to w_k \quad \text{in } L^r(\Omega) \quad \forall r \in [1,\infty). \tag{34}$$

Test (32) with some $v \in W_0^{1,m}(\Omega)$ and let $\varepsilon \to 0$. By (34), we know that $|\partial_i w_\varepsilon|^{p-2}\partial_i w_\varepsilon$ remains bounded in $L^{m_i'}(\Omega)$ for all $i = 1, \dots, n$, thus Theorem 1 in [6] gives $|\partial_i w_\varepsilon|^{p-2}\partial_i w_\varepsilon \rightharpoonup |\partial_i w_k|^{p-2}\partial_i w_k$ in $L^{m_i'}(\Omega)$. Hence, using also (31), we obtain

$$\sum_{i=1}^{n} \int_\Omega |\partial_i w_k|^{m_i-2} \partial_i w_k \partial_i v = -\lambda \int_\Omega |w_k|^{p-2} w_k v + 2\lambda \int_\Omega u_k^{p-1} v \qquad \forall v \in W_0^{1,m}(\Omega) \ . \tag{35}$$

In particular, taking $v = w_k$ in (35), yields

$$\sum_{i=1}^{n} \int_\Omega |\partial_i w_k|^{m_i} = -\lambda \int_\Omega |w_k|^p + 2\lambda \int_\Omega u_k^{p-1} w_k. \tag{36}$$

Next, multiply (32) by w_ε and integrate by parts. Letting $\varepsilon \to 0$ and taking into account (31) and (34), gives

$$\sum_{i=1}^{n} \int_\Omega |\partial_i w_\varepsilon|^{m_i} \to -\lambda \int_\Omega |w_k|^p + 2\lambda \int_\Omega u_k^{p-1} w_k.$$

This, together with (36) and (34), shows that

$$w_\varepsilon \to w_k \quad \text{in } W_0^{1,m}(\Omega). \tag{37}$$

Without loss of generality, in the sequel we assume that the center of α-starshapedness is the origin, that is (10) holds (with strict inequality if $p = m^*$).

In order to derive an integral inequality for w_k, we shall apply to w_ε^k the generalized Pohožaev identity [22], as stated in [23, §1].

Proposition 1. *Let Ω be a smooth bounded open set in $\mathbf{R}^n$, and let u be a function in $C^2(\Omega)\cap C^1(\overline{\Omega})$ with $u = 0$ on $\partial\Omega$. Assume that u solves the Euler-Lagrange equation*

$$\mathrm{div}\mathcal{F}_\xi(x, u, Du) = \mathcal{F}_s(x, u, Du) \ ,$$

where the integrand $\mathcal{F} = \mathcal{F}(x, s, \xi)$ is supposed to be of class C^1 on $\overline{\Omega} \times \mathbf{R} \times \mathbf{R}^n$ together with $\mathcal{F}_\xi$. Then, for any scalar function a and vector field h of class $C^1(\Omega) \cap C(\overline{\Omega})$ the function u satisfies the identity

$$\begin{aligned}\int_{\partial\Omega} [\mathcal{F}(x, 0, Du) - (Du, \mathcal{F}_\xi(x, 0, Du))](h, \nu)\, ds = \int_\Omega \mathcal{F}(x, u, Du)\mathrm{div}h + \int_\Omega (h, \mathcal{F}_x(x, u, Du))\\ - \int_\Omega (DuDh + D(au), \mathcal{F}_\xi(x, u, Du)) - \int_\Omega au\mathcal{F}_s(x, u, Du) \ .\end{aligned}$$

We are now ready to prove

Lemma 1. *Let m_i, α, Ω, and p satisfy the assumptions of Theorem 6. Assume that $w_k \in W_0^{1,m}(\Omega)$ satisfies* (35). *Then*

$$\frac{n}{m^*} \sum_{i=1}^n \int_\Omega |\partial_i w_k|^{m_i} + \frac{n\lambda}{p} \int_\Omega |w_k|^p + 2\lambda \sum_{i=1}^n \alpha_i \int_\Omega u_k^{p-1} x_i \partial_i w_k + R_k \le 0 \ ,$$

where $R_k := \limsup_{\varepsilon\to 0} \sum_{i=1}^n \left(1 - \frac{1}{m_i}\right) \int_{\partial\Omega} |\partial_i w_\varepsilon|^{m_i} (x, T_\alpha \nu)\, ds.$

Proof. Let $w_\varepsilon \in C^2(\overline{\Omega})$ be the unique solution of (32), see Theorem 5. We observe that (32) is the Euler-Lagrange equation of the integral functional with integrand

$$\mathcal{F}(x, s, \xi) := \sum_{i=1}^n \frac{|\xi_i|^{m_i}}{m_i} + \frac{\varepsilon}{m_-} \left(1 + |\xi|^2\right)^{m_-/2} + \frac{\lambda}{p} |s|^p - f^\varepsilon(x)s \ .$$

Then, we apply Proposition 1, by choosing as a scalar function $a(x)$ the constant $a \equiv n/m^*$ and as a vector function $h(x)$ the field x deformed through the tensor T_α, namely $h(x) = (\alpha_1 x_1, \dots, \alpha_n x_n)$.

We obtain

$$\int_{\partial\Omega}\Big[\sum_{i=1}^n\Big(\frac{1}{m_i}-1\Big)|\partial_i w_\varepsilon|^{m_i}+\frac{\varepsilon}{m_-}\Big(1+|Dw_\varepsilon|^2\Big)^{(m_--2)/2}\Big(1+(1-m_-)|Dw_\varepsilon|^2\Big)\Big]\cdot(x,T_\alpha\nu)\,ds$$

$$=n\int_\Omega\Big[\sum_{i=1}^n\frac{|\partial_i w_\varepsilon|^{m_i}}{m_i}+\frac{\varepsilon}{m_-}\left(1+|Dw_\varepsilon|^2\right)^{m_-/2}+\frac{\lambda}{p}|w_\varepsilon|^p-f^\varepsilon w_\varepsilon\Big]-\sum_{i=1}^n\alpha_i\int_\Omega w_\varepsilon x_i\partial_i f^\varepsilon$$

$$-\sum_{i=1}^n\alpha_i\int_\Omega\Big[|\partial_i w_\varepsilon|^{m_i}+\varepsilon\Big(1+|Dw_\varepsilon|^2\Big)^{(m_--2)/2}|\partial_i w_\varepsilon|^2\Big]$$

$$-\frac{n}{m^*}\int_\Omega\Big[\sum_{i=1}^n|\partial_i w_\varepsilon|^{m_i}+\varepsilon\Big(1+|Dw_\varepsilon|^2\Big)^{(m_--2)/2}|Dw_\varepsilon|^2+\lambda|w_\varepsilon|^p-f^\varepsilon w_\varepsilon\Big]\,.$$

We observe that, by the present choice (6) of the α_i and (3) of m^*, the terms containing $\int_\Omega|\partial_i w_\varepsilon|^{m_i}$ for all i cancel. We now want to send $\varepsilon\to 0$. First note that

$$\limsup_{\varepsilon\to 0}\,\varepsilon\int_{\partial\Omega}\Big(1+|Dw_\varepsilon|^2\Big)^{(m_--2)/2}\Big(1+(1-m_-)|Dw_\varepsilon|^2\Big)\cdot(x,T_\alpha\nu)\,ds\le 0\;,$$

because the map $s\mapsto(1+s^2)^{(m_--2)/2}(1+(1-m_-)s^2)$ is bounded from above on $\mathbf{R}$, and $(x,T_\alpha\nu)\ge 0$ by the α-starshapedness assumption. Moreover, integrating by parts and using (32), yields

$$-\int_\Omega w_\varepsilon x_i\partial_i f^\varepsilon=\int_\Omega f^\varepsilon x_i\partial_i w_\varepsilon+\int_\Omega\Big[\lambda|w_\varepsilon|^p+\sum_{j=1}^n|\partial_j w_\varepsilon|^{m_j}+\varepsilon\big(1+|Dw_\varepsilon|^2\big)^{(m_--2)/2}|Dw_\varepsilon|^2\Big]$$

Therefore, using (31), (34), (37) and Lemma 1,

$$0\ge\limsup_{\varepsilon\to 0}\int_{\partial\Omega}\sum_{i=1}^n\Big(1-\frac{1}{m_i}\Big)|\partial_i w_\varepsilon|^{m_i}(x,T_\alpha\nu)\,ds+n\lambda\int_\Omega\Big[\frac{1}{p}|w_k|^p-2u_k^{p-1}w_k\Big]$$

$$+2\lambda\sum_{i=1}^n\alpha_i\int_\Omega u_k^{p-1}x_i\partial_i w_k+n\int_\Omega\Big[\lambda|w_k|^p+\sum_{j=1}^n|\partial_j w_k|^{m_j}\Big]-\frac{n\lambda}{m^*}\int_\Omega\Big[|w_k|^p-2u_k^{p-1}w_k\Big]\,.$$

Finally, using (36), we obtain the result. □

Next, we let $k\to\infty$ and we obtain

Lemma 2. *Let m_i, α, Ω, and p satisfy the assumptions of Theorem 6. Assume that u is a mild solution of* (1). *Then*

$$n\lambda\Big(\frac{1}{m^*}-\frac{1}{p}\Big)\int_\Omega u^p+R\le 0$$

where $R:=\limsup\limits_{k\to+\infty}R_k$, with R_k as in Lemma 1.

Proof. By (36), Hölder's inequality, and the convergence $u_k \to u$ in $L^p(\Omega)$, we have

$$\sum_{i=1}^{n} \int_\Omega |\partial_i w_k|^{m_i} + \lambda \int_\Omega |w_k|^p \le C\Big(\int_\Omega |w_k|^p \Big)^{1/p} \qquad \forall k \in \mathbf{N} \ .$$

Hence, $\{w_k\}$ is bounded in $L^p(\Omega)$ and in $W_0^{1,m}(\Omega)$, and therefore there exists $w_0 \in W_0^{1,m}(\Omega) \cap L^p(\Omega)$ such that, up to a subsequence, $w_k \rightharpoonup w_0$ in $W_0^{1,m}(\Omega)$ and in $L^p(\Omega)$. Thus, letting $k \to +\infty$ in (36), we get

$$\lim_{k\to+\infty} \Big[\sum_{i=1}^{n} \int_\Omega |\partial_i w_k|^{m_i} + \lambda \int_\Omega |w_k|^p \Big] = 2\lambda \int_\Omega u^{p-1} w_0 \ . \tag{38}$$

On the other hand, taking $v = w_0$ in (35), and letting $k \to +\infty$, gives

$$\sum_{i=1}^{n} \int_\Omega |\partial_i w_0|^{m_i} + \lambda \int_\Omega |w_0|^p = 2\lambda \int_\Omega u^{p-1} w_0 \ ,$$

which together with (38) and weak convergence proves that $w_k \to w_0$ (strongly) in $W_0^{1,m}(\Omega) \cap L^p(\Omega)$.

We claim that $w_0 = u$. Indeed, by letting $k \to +\infty$ in (35), we infer that w_0 satisfies

$$\sum_{i=1}^{n} \int_\Omega |\partial_i w_0|^{m_i-2} \partial_i w_0 \partial_i v = -\lambda \int_\Omega |w_0|^{p-2} w_0 v + 2\lambda \int_\Omega u^{p-1} v \qquad \forall v \in W_0^{1,m}(\Omega) \cap L^p(\Omega).$$

Since u is a mild solution to (1), the above identity holds when replacing w_0 with u, and by subtracting we obtain, for all $v \in W_0^{1,m}(\Omega) \cap L^p(\Omega)$,

$$\sum_{i=1}^{n} \int_\Omega \Big(|\partial_i w_0|^{m_i-2} \partial_i w_0 - |\partial_i u|^{m_i-2} \partial_i u \Big) \partial_i v = \lambda \int_\Omega \Big(|u|^{p-2} u - |w_0|^{p-2} w_0 \Big) v \ . \tag{39}$$

Choosing $v = w_0 - u$, and taking into account that

$$\int_\Omega \Big(|\partial_i w_0|^{m_i-2} \partial_i w_0 - |\partial_i u|^{m_i-2} \partial_i u \Big) (\partial_i w_0 - \partial_i u) \ge \int_\Omega \Big(|\partial_i w_0|^{m_i-1} - |\partial_i u|^{m_i-1} \Big) (|\partial_i w_0| - |\partial_i u|) \ , \tag{40}$$

the left hand side of (39) is nonnegative. Since its right hand side is nonpositive, we get immediately $w_0 = u$.

We can now pass to the limit as $k \to +\infty$ in Lemma 1. Since u is a mild solution we have

$$\int_\Omega u_k^{p-1} x_i \partial_i w_k \to \int_\Omega u^{p-1} x_i \partial_i u \qquad \forall i = 1, ..., n \ . \tag{41}$$

Therefore, using the identities

$$\int_\Omega u^{p-1} x_i \partial_i u = -\frac{1}{p} \int_\Omega u^p \qquad \text{and} \qquad \sum_{i=1}^{n} \int_\Omega |\partial_i u|^{m_i} = \lambda \int_\Omega u^p \ ,$$

we obtain the statement. $\square$

We are now in a position to give the proof of Theorem 6.

If $p > m^*$, since $R \geq 0$, by Lemma 2 we obtain $u \equiv 0$.

If $p = m^*$, Lemma 2 yields

$$0 = R = \limsup_{k\to+\infty} \limsup_{\varepsilon\to 0} \sum_{i=1}^{n} \Big(1 - \frac{1}{m_i}\Big) \int_{\partial\Omega} |\partial_i w_\varepsilon|^{m_i}(x, T_\alpha \nu)\, ds \ . \tag{42}$$

Integrating (32) (with $w = w_\varepsilon$) over Ω, by the divergence Theorem, we obtain

$$\Big| \int_\Omega \lambda |w_\varepsilon|^{p-2} w_\varepsilon - f^\varepsilon \Big| = \Big| \sum_{i=1}^{n} \int_{\partial\Omega} \Big[|\partial_i w_\varepsilon|^{m_i-2} + \varepsilon(1 + |Dw_\varepsilon|^2)^{(m_- -2)/2} \Big] \partial_i w_\varepsilon\, \nu_i \Big| \ .$$

Letting first $\varepsilon \to 0$ and then $k \to +\infty$, we deduce

$$\lambda \int_\Omega u^{p-1} \leq \limsup_{k\to+\infty} \limsup_{\varepsilon\to 0} \sum_{i=1}^{n} \int_{\partial\Omega} \Big[|\partial_i w|^{m_i-1} + \varepsilon |\partial_i w| (1 + |Dw|^2)^{(m_- -2)/2} \Big]\, ds \ .$$

Since $\inf_{\partial\Omega}(x, T_\alpha \nu) > 0$ (by *strict* α-starshapedness), by (42), the right hand side above equals zero. We deduce that $u \equiv 0$, and the proof of Theorem 6 is complete.

Remark 2. To ensure the convergence (41), it is sufficient to have $x_i u^{p-1} \in L^{m_i'}(\Omega)$ for all i. Actually, for Ω α-starshaped with respect to the origin, the condition $u \in L^{(p-1)m_-'}(\Omega)$ for mild solutions in Definition 1, may be relaxed to $x_i u^{p-1} \in L^{m_i'}(\Omega)$ for all $i = 1, \ldots, n$.

8 Concluding remarks and open problems

8.1 About Theorem 4

We show here that the assumption $p > m_+$ in Theorem 4 is necessary to apply the mountain-pass Theorem.

• *An example of an unbounded Palais-Smale sequence.*

Assume that $\Omega \subset \mathbf{R}^{10}$ is the cylinder $\Omega = (0, \pi) \times B_1$, where B_1 is the unit ball in $\mathbf{R}^9$. Take $\lambda = 1$, $m_1 = m_+ = p = 2$, $m_i = \frac{4}{3}$ for $i = 2, ..., 10$ and let $m = (m_1, ..., m_{10})$. Then, the corresponding functional reads

$$J(u) = \frac{1}{2} \int_\Omega |\partial_1 u|^2 + \frac{3}{4} \sum_{i=2}^{10} \int_\Omega |\partial_i u|^{4/3} - \frac{1}{2} \int_\Omega |u|^2.$$

For all $k \in \mathbf{N}$, consider the function $\phi_k : [0, 1] \to \mathbf{R}$ defined by

$$k\phi_k(r) = \begin{cases} k^{15} - 1 & \text{if} \quad r \in [0, k^{-3}] \\ r^{-5} - 1 & \text{if} \quad r \in [k^{-3}, 1] \end{cases} \qquad\qquad k\phi_k'(r) = \begin{cases} 0 & \text{if} \quad r \in [0, k^{-3}) \\ -5r^{-6} & \text{if} \quad r \in (k^{-3}, 1] \ . \end{cases}$$

Consider also the sequence of functions

$$u_k(x_1, x') = \sin x_1 \cdot \phi_k(|x'|) \ , \qquad x' = (x_2, ..., x_{10}).$$

Then, we have (c denotes possibly different positive constants)

$$\|\partial_1 u_k\|_2^2 = \|u_k\|_2^2 = c\int_0^\pi \sin^2 x_1 dx_1 \cdot \int_0^1 r^8 \phi_k^2(r)dr$$

$$\geq \frac{c}{k^2}\int_{k^{-3}}^1 r^8(r^{-10} - 2r^{-5} + 1)dr \geq \frac{c}{k^2}\int_{k^{-3}}^1 r^{-2}dr + o(1) \geq ck$$

so that the sequence $\{u_k\}$ is unbounded in $W_0^{1,m}(\Omega)$. On the other hand,

$$\sum_{i=2}^{10}\int_\Omega |\partial_i u_k|^{4/3} \leq c\int_{B_1} |\nabla_{x'} u_k|^{4/3} = c\int_0^1 r^8 |\phi_k'(r)|^{4/3} dr \leq \frac{c}{k^{4/3}} \ .$$

Hence, by Hölder's inequality

$$|\langle J'(u_k), v\rangle| \leq \sum_{i=2}^{10}\int_\Omega |\partial_i u_k|^{1/3}|\partial_i v| \leq o(1)\cdot \|v\|_{1,m} \qquad \forall v \in W_0^{1,m}(\Omega) \ .$$

Therefore, as $k \to \infty$ we have

$$J(u_k) \to 0 \qquad \text{and} \qquad J'(u_k) \to 0 \text{ in } [W_0^{1,m}(\Omega)]'$$

so that $\{u_k\}$ is an unbounded Palais-Smale sequence for J.

- *Failure of the mountain-pass geometry.*

Take $n = 2$, $m_1 = \frac{4}{3}$, $m_2 = 3$, and note that $m^* = 24$, so that (2) and (5) are satisfied. Take $p = 2$, then

$$J(u) = \frac{3}{4}\|\partial_1 u\|_{4/3}^{4/3} + \frac{1}{3}\|\partial_1 u\|_3^3 - \frac{\lambda}{2}\|u\|_2^2 \ .$$

Hence, by inequality (11) and interpolation we obtain

$$J(u) \geq C_1\|u\|_{4/3}^{4/3} + C_2\|u\|_3^3 - \frac{\lambda}{2}\|u\|_{4/3}^{4/5}\|u\|_3^{6/5} \ .$$

By applying Young's inequality we then obtain $J(u) > 0$ for all $u \neq 0$, provided λ is sufficiently small. Clearly, in this case J does not have the standard mountain-pass geometry.

An even simpler argument works when $p = m_+$. Take any n, any m_i and assume that $p = m_+$. Then, by applying inequality (11), we see that J does not have a mountain-pass geometry if $\lambda \leq (2/ap)^p$, where a is the width of Ω in the direction corresponding to the maximal exponent m_+.

8.2 About mild solutions

Consider the semilinear problem ($m_i \equiv 2$)

$$\begin{cases} -\Delta u = \lambda(1+u)^{p-1} & \text{in } B \\ u \geq 0 & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases} \tag{43}$$

where $\lambda > 0$, $p > 1$ and B denotes the unit ball in $\mathbf{R}^n$ $(n \geq 3)$.

It is well-known [4] (see also Theorem 2 above), that any weak solution of (43) is a strong solution whenever $p \leq \frac{2n}{n-2}$.

A simple calculation shows that, if

$$p > \frac{2n-2}{n-2} \qquad \text{and} \qquad \lambda = \frac{2}{p-2}\left(n - \frac{2p-2}{p-2}\right) ,$$

then the function

$$U(x) = |x|^{-2/(p-2)} - 1$$

satisfies (43) in $B \setminus \{0\}$. It is also not difficult to prove the following facts

$$U \in H_0^1(B) \Longleftrightarrow p > \frac{2n}{n-2} , \qquad U \in L^q(B) \Longleftrightarrow q < \frac{n(p-2)}{2} .$$

Therefore:

(i) U is a weak solution of (43) (*i.e.* $U \in H_0^1(B) \cap L^{2n(p-1)/(n+2)}(B)$) if and only if $p > \frac{2n}{n-2}$.

(ii) U is a mild solution of (43) (*i.e.* $U \in L^{2(p-1)}(B)$) if and only if $n > 4$ and $p > \frac{2n-4}{n-4}$.

These statements suggest that, in general, one cannot expect a weak solution of (1) to be a mild solution if $p > m_\infty$. Moreover, it seems more likely that a weak solution is indeed a mild solution for large values of the exponent p.

8.3 Some open problems

Problem 1. Prove Theorem 2 under the only assumption that $p \leq m_\infty$. The example in Section 8.2 shows that it is not reasonable to expect strong solutions of (1) if $p > m_\infty$. Note that our proof of Theorem 2, case (i), does not work if $p = m_\infty = m_+$ due to the failure of the step which uses Hölder's inequality: no ε_k appears. On the other hand, our proof in case (ii) cannot be followed when $p = m_\infty$, because there is no positive a_0 which can initialize (18).

Problem 2. Find sharp statements in the situation of Theorems 3 and 4. For which exponents $p < m_\infty$ does (1) admit a solution for all $\lambda > 0$? It seems that the resonance situation occurs as soon as $p \leq m_+$ (see also Section 8.1) but maybe there are some "spectral gaps", namely some $p \in (m_-, m_+)$ such that (1) admits a strong solution for all $\lambda > 0$.

Problem 3. Prove Theorems 5 and 6 under less restrictive assumptions on the exponents m_i. For instance, one could try to relax (6) and (7) with (5). In fact, (6) and (7) are used in Step 3 of the proof of Theorem 5. As suggested in [1], we actually believe that a gradient estimate for the solution of the approximating problems can be obtained under the *sole* assumption (5). Assumption (6) is also needed in some of the estimates in the proof of Theorem 6, but the case where $m_i < 2$ for some i may be handled in a similar way as in (4.22) in [20].

Acknowledgements. This research was begun in the fall of 2002, when the third author enjoyed the hospitality of S. Salsa and the Mathematics Department of the Politecnico in Milano. We thank P. Marcellini, J. Smoller, and N. Ural'tseva for their useful bibliographic help. We are also grateful to the referee for his comments and suggestions which led us to improve Theorems 2 and 5.

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