MINIMALITY VIA SECOND VARIATION FOR A FUNCTIONAL INVOLVING FRACTIONAL PERIMETER

DAYANA PAGLIARDINI

Abstract. We discuss the local minimality of some configurations for a functional involving fractional perimeter and a prescribed curvature term. We show that critical configurations with positive second variation are L^1-local minimizers of our functional. Then we prove that we can obtain a sequence of L^1-minimizers for the fractional Allen–Cahn energy, knowing an isolated L^1-minimizer of its Γ-limit. Finally we find minimizers for the fractional Allen–Cahn energy starting from critical configurations with positive second variation.

1. Introduction

In recent years fractional operators have received considerable attention both in pure and applied mathematics. The motivations are multiple: they appear in biological observations (for example when a predator decide that a nonlocal dispersive strategy to hunt its preys is more efficient), in finance, crystal dislocation, minimal surfaces, and digital image reconstruction. For instance, in the latter case, it was computed that a square pixel of side ε has an error along the diagonal of ε with respect to the classical perimeter, but the error is ε^{1-s} when considering the fractional perimeter (see [26], [14], [6], [8], [16]). In particular, from a probabilistic point of view, the fractional Laplacian is an infinitesimal generator of Lévi processes, see [5].

In this paper we want to study a functional involving fractional perimeter and a prescribed curvature term, i.e.

$$J_s(E) = P_s(E) + \int_E g \, \mathrm{d}x \tag{1.1}$$

where $E \subset \mathbb{R}^N$ is a bounded open set of fixed volume $m > 0$, $g \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is a given function, $s \in (0, 1/2)$ and $P_s(E)$ is the well known fractional perimeter defined as

$$P_s(E) = \int_E \int_{E^C} \frac{\mathrm{d}x \, \mathrm{d}y}{|x-y|^{N+2s}}, \tag{1.2}$$

with E^C the complement of E.

Functionals involving fractional perimeter are largely studied in the existing literature. For example, in [11], Cesaroni and Novaga are interested in the existence and the regularity properties of minimizers of the isoperimetric problem

$$\min_{|E|=m} \mathcal{F}(E) = \min_{|E|=m} \left(P_s(E) - \int_E g(x) \, \mathrm{d}x \right).$$

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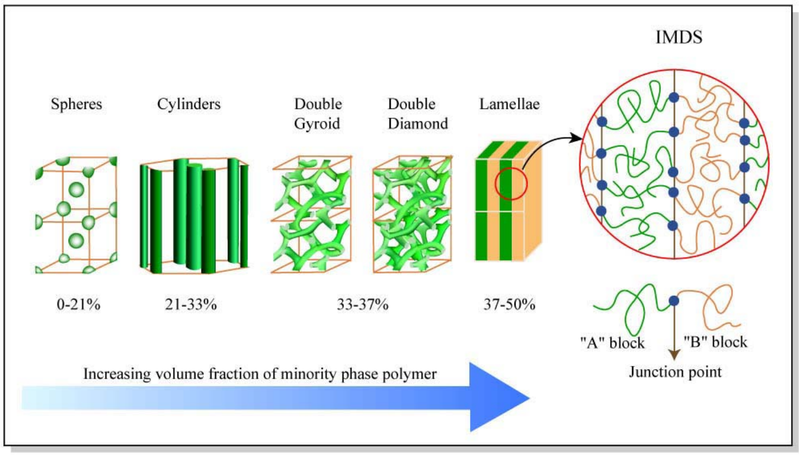

Figure 1. The relative lenghts of each block produce different morphologies. Figure by MIT OpenCourseWare [30].

Here $E \subset \mathbb{R}^N$ is a measurable set, $P_s(E)$ is defined as in (1.2) and $g : \mathbb{R}^N \to \mathbb{R}$ is a locally Lipschitz continuous function bounded from below, periodic or coercive.

In [9], Caffarelli, Savin and Valdinoci considered a minimization problem combining the Dirichlet energy with the nonlocal perimeter of a level set, namely

$$\int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x + P_s(\{u > 0\}, \Omega)$$

with $s \in (0, 1)$. They obtained regularity results for the minimizers and for their free boundaries $\partial\{u > 0\}$, density estimates, monotonicity formulas, Euler–Lagrange equations and extension properties.

In [3], Acerbi, Fusco and Morini studied the functional

$$J(E) = P_\Omega(E) + \gamma \int_\Omega \int_\Omega G(x, y)(u(x) - m)(u(y) - m) \, \mathrm{d}x \, \mathrm{d}y, \tag{1.3}$$

where $\Omega = \Pi^N$ is the N-dimensional flat torus of unit volume, $u = \chi_E - \chi_{E^C}$ with $E = \{x \in \Omega : u(x) = 1\}$ and E^C its complement, $P_\Omega(E)$ the De Giorgi's perimeter of E in Ω, see [18], $\gamma \geq 0$ a constant depending on the structural properties of the materials, $m = \fint_\Omega u$ and G the Green's function associated to $-\Delta$.

The functional in (1.3) represents the variational limit of Otha–Kawasaki energy, see [27], and models the microphase separation for A/B diblock copolymer melts, intensively studied in engineering nanostructure for their properties and rich pattern formation with different morphologies depending on the relative lengths of each block. Precisely, sufficiently different block lengths lead to spheres; using less different block lengths we obtain a "hexagonally packed cylinder" geometry, while blocks of similar length form lamellae, see Figure 1.

They showed that critical configurations having positive second variation are L^1-local minimizers of the nonlocal area functional.

Recently, in [19], Julin proved that regular critical sets of the functional in (1.3) are analytic and, moreover, that the ball is the unique possible stable critical set when the strength of the nonlocal part is suitably small.

Our main goal is to prove the result of [3] for our fractional functional (1.1). More precisely, in the first part of this paper, we prove the following result.

Theorem 1.1. *Let $s \in (0,1/2)$, $E \subset \mathbb{R}^N$ be a bounded open set of class C^3 with fixed volume $m > 0$ satisfying the Euler–Lagrange equation corresponding to J_s and such that*

$$\partial^2 J_s(E)[\varphi] > 0 \quad \textit{for all} \quad \varphi \in \tilde{H}^s(\partial E) \setminus \{0\}.$$

Then there exist $\delta > 0$, $C_0 > 0$ such that

$$J_s(F) \geq J_s(E) + C_0 |E \Delta F|^2$$

for all $F \subset \mathbb{R}^N$, with $|F| = |E|$ and $|E \Delta F| \leq \delta$.

We refer to Theorem 5.8 for a precise statement.

In the second part of this paper, we want to generalize an important theorem of Kohn-Sternberg, ([20, Theorem 2.1]). We consider a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with complement Ω^C, we define

$$\mathcal{K}(u,\Omega) := \frac{1}{2}\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \int_{\Omega^C} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y,$$

and we study the functionals of the form $F_\varepsilon : X := \{u \in L^\infty(\mathbb{R}^N) : \|u\|_{L^\infty(\mathbb{R}^N)} \leq 1\} \to \mathbb{R} \cup \{+\infty\}$ given by

$$F_\varepsilon(u) = \mathcal{K}(u,\Omega) + \frac{1}{\varepsilon^{2s}}\int_\Omega W(u)\,\mathrm{d}x + \int_\Omega gu\,\mathrm{d}x, \quad \text{if } s \in (0,1/2), \tag{1.4}$$

$$F_\varepsilon(u) = \frac{1}{|\log\varepsilon|}\mathcal{K}(u,\Omega) + \frac{1}{|\varepsilon\log\varepsilon|}\int_\Omega W(u)\,\mathrm{d}x + \int_\Omega gu\,\mathrm{d}x, \quad \text{if } s = 1/2, \tag{1.5}$$

$$F_\varepsilon(u) = \varepsilon^{2s-1}\mathcal{K}(u,\Omega) + \frac{1}{\varepsilon}\int_\Omega W(u)\,\mathrm{d}x + \int_\Omega gu\,\mathrm{d}x, \quad \text{if } s \in (1/2,1), \tag{1.6}$$

where $u \in H^s(\Omega;\mathbb{R})$, $g \in C^1(\Omega)$ is a given function, $\varepsilon \in \mathbb{R}^+$ is a positive parameter and W is the well known double-well potential, that is, an even function such that

$$\begin{aligned} &W : \mathbb{R} \to [0,+\infty), \qquad W \in C^2(\mathbb{R},\mathbb{R}^+), \quad W(\pm 1) = 0, \\ &W > 0 \text{ in } (-1,1), \qquad W'(\pm 1) = 0, \quad W''(\pm 1) > 0. \end{aligned} \tag{1.7}$$

F_ε is the energy of the fractional Allen–Cahn equation and is the fractional counterpart of the functional studied by Modica and Mortola in [24, 25]. Valdinoci and Savin in [29] proved that the functionals (1.4), (1.5), (1.6) Γ-converge to $F_0 : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined as

$$F_0(u) = \begin{cases} \mathcal{K}(u,\Omega) + \int_\Omega gu\,\mathrm{d}x & \text{if} \quad u|_\Omega = \chi_E - \chi_{E^C}, \text{ for some set } E \subset \Omega \\ +\infty & \text{otherwise,} \end{cases} \tag{1.8}$$

where $s \in (0,1/2)$, E^C is the complement of the set E, and

$$F_0(u) = \begin{cases} c^* P_\Omega(E) + \int_\Omega gu\,\mathrm{d}x & \text{if} \quad u|_\Omega = \chi_E - \chi_{E^C}, \text{ for some set } E \subset \Omega \\ +\infty & \text{otherwise,} \end{cases} \tag{1.9}$$

where $s \in [1/2, 1)$, c^* is a constant depending on N and s (see [29, Theorem 4.2] for more details) and $P_\Omega(E)$ denotes the perimeter of E in Ω. So we prove the following generalization of Theorem 2.1 in [20] in the following way

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, $s \in (0, 1/2)$ and suppose that u_0 is an isolated L^1-local minimizer of F_0. Then there exists $\varepsilon_0 > 0$ and a family $\{u_\varepsilon\}_{\varepsilon<\varepsilon_0}$ such that*

$$u_\varepsilon \text{ is an } L^1\text{-local minimizer of } F_\varepsilon, \tag{1.10}$$

and

$$\|u_\varepsilon - u_0\|_{L^1(\Omega)} \to 0 \quad as \quad \varepsilon \to 0. \tag{1.11}$$

Remark 1.3. We require that u_0 is isolated in order to avoid the possibility of the situation obtained in the classical case in which Kohn and Sternberg proved (see [20], Section 3.3) that if u_0 is not isolated critical point, Theorem 1.2 can fail. We conjecture that you can build a counterexample also in our setting, but the Kohn-Sterberg's one is not obviously generalizable.

Remark 1.4. In Theorem 1.2, thought u_0 is an isolated local minimizer, we are unable to conclude that u_ε is isolated. Nevertheless, it seems reasonable to expect that also u_ε is isolated for any ε sufficiently small if u_0 is nondegenerate in some suitable sense.

Finally, we prove the following result, which can be seen as a link between the two previous theorems:

Corollary 1.5. *Suppose $s \in (0, 1/2)$. Let E be as in Theorem 1.1 and $u = \chi_E$. Then there exist $\varepsilon_0 > 0$ and a family $\{u_\varepsilon\}_{\varepsilon<\varepsilon_0}$ of local minimizers of F_ε such that $u_\varepsilon \to u$ in $L^1(\mathbb{R}^N)$ as $\varepsilon \to 0$.*

The paper is organized as follows. In Section 2 we give some preliminary definitions and results. In Section 3 we compute the first and the second variation of (1.1) and we show that regular local minimizers have nonnegative second variation, that is a sort of viceversa of our main theorem. Section 4 is devoted to the proof of the $W^{1+2s',p}$-local minimality of critical configurations with positive second variation. In the most important section of this work, Section 5, we prove our principal theorem, i.e. that any $W^{1+2s',p}$-local minimizer is an L^1-local minimizer. Finally, Section 6 is dedicated to the proof of Theorem 1.2 and Corollary 1.5.

2. Notation and preliminary results

In this section we introduce the framework that we will be used throughout this work.

Definition 2.1. Let $E \subset \mathbb{R}^N$ be an open set with ∂E of class C^1, $s \in (0, 1)$ and for any $p \in [1, +\infty)$ we define

$$W^{s,p}(\partial E) := \left\{ u \in L^p(\partial E) : \frac{|u(x) - u(y)|}{|x - y|^{s+(N-1)/p}} \in L^p(\partial E) \times L^p(\partial E) \right\}$$

i.e. an intermediary Banach space between $L^p(\partial E)$ and $W^{1,p}(\partial E)$, endowed with the natural norm

$$\|u\|_{W^{s,p}(\partial E)} := \Big(\int_{\partial E} |u|^p \, \mathrm{d}\mathcal{H}^{N-1} + \int_{\partial E} \int_{\partial E} \frac{|u(x) - u(y)|^p}{|x - y|^{N-1+sp}} \, \mathrm{d}\mathcal{H}^{N-1}_x \, \mathrm{d}\mathcal{H}^{N-1}_y \Big).$$

For $0 < s < 1/2$ the s-perimeter of a measurable set $E \subset \mathbb{R}^N$ is defined as

$$P_s(E) = \int_E \int_{E^C} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{N+2s}}, \tag{2.1}$$

where E^C is the complement of E. The s-perimeter corresponds to the usual semi-norm of the characteristic function χ_E of E in the fractional Sobolev space $H^s(\mathbb{R}^N)$, that is

$$P_s(E) = [\chi_E]^2_{H^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\chi_E(x) - \chi_E(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

We say that a set $E \subset \mathbb{R}^N$ has finite s-perimeter if $P_s(E) < \infty$.

We recall that, by [2, Theorem 2] the s-perimeter Γ-converges to De Giorgi's perimeter as $s \to 1/2$. Precisely, it holds

$$\Gamma - \lim_{s \uparrow 1/2} (1-2s) P_s(E) = \omega_{N-1} P(E), \tag{2.2}$$

where ω_{N-1} denote the volume of the unit ball in $\mathbb{R}^{N-1}$.

The s-perimeter can be localized to a bounded open set $\Omega \subset \mathbb{R}^N$ by taking away the contribution of points of E and E^C outside Ω, i.e.

$$P_s(E, \Omega) = \int_{E\cap\Omega} \int_{E^C} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{N+2s}} + \int_{E\cap\Omega^C} \int_{E^C\cap\Omega} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{N+2s}}. \tag{2.3}$$

Again, as $s \to 1/2$, the usual notion of perimeter is recovered, because we still have

$$\Gamma - \lim_{s \uparrow 1/2} (1-2s) P_s(E, \Omega) = \omega_{N-1} P(E, \Omega),$$

where $P(E, \Omega)$ is the perimeter of E inside Ω.

Given an open set $\Omega \subset \mathbb{R}^N$ and a vector field $X \in C_c^\infty(\Omega, \mathbb{R}^N)$ we denote by $\{\Phi_t\}_{t\in\mathbb{R}}$ the flow induced by X, that is, the smooth solution $\Phi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ to the ODE

$$\begin{cases} \partial_t \Phi_t(x) = X(\Phi_t(x)), & t \in \mathbb{R} \\ \Phi_0(x) = x. & x \in \mathbb{R}^N \end{cases} \tag{2.4}$$

Recall that, by the implicit function Theorem, there always exists $\varepsilon > 0$ such that $\{\Phi_t\}_{|t|<\varepsilon} : \mathbb{R}^N \to \mathbb{R}^N$ is a smooth family of diffeomorphisms.

Let $E \subset \mathbb{R}^N$ be an open set with C^2 boundary. Assume that $P_s(E, \Omega) < \infty$, and let $E_t = \Phi_t(E)$ for any $|t| < \varepsilon$. By the area formula the function $t \longmapsto P_s(E_t, \Omega)$ is smooth for $|t| < \varepsilon$. Moreover, if ν_E is the outer unit normal to E and if $\zeta = X \cdot \nu_E$ denotes the normal component of X, the first variation of $P_s(\cdot, \Omega)$ at E along X is (see [17, Theorem 6.1])

$$\delta P_s(E, \Omega)[X] = \frac{\mathrm{d}}{\mathrm{d}t} P_s(E_t, \Omega)_{|t=0} = \int_{\partial E} H_s \zeta \,\mathrm{d}\mathcal{H}^{N-1}, \tag{2.5}$$

where H_s is the fractional mean curvature of ∂E, defined as

$$H_s(p) := \int_{\mathbb{R}^N} \frac{\chi_E(x) - \chi_{E^C}(x)}{|x-p|^{N+2s}}\,\mathrm{d}x \quad \text{for any } p \in \partial E. \tag{2.6}$$

The integral in (2.6) is understood in the principal value sense, i.e. we define

$$H_s^\delta(p) = \int_{\mathbb{R}^N \setminus B_\delta(p)} \frac{\chi_E(x) - \chi_{E^C}(x)}{|x-p|^{N+2s}}\,\mathrm{d}x \tag{2.7}$$

and we let

$$H_s(p) = \lim_{\delta\to 0} H_s^\delta(p).$$

The fractional mean curvature $H_s(p)$ is well-defined provided that ∂E is regular near p and, in this case, it agrees with usual mean curvature in the limit as $s \to 1/2$ by the relation

$$\lim_{s\to 1/2}(1-2s)H_s(p) = \omega_{N-1}H(p),$$

where H denotes the classical mean curvature of ∂E, see [4, Theorem 12].

3. First and Second variation of J_s

From now on we consider the following functional

$$J_s(E) = P_s(E) + \int_E g\,\mathrm{d}x \tag{3.1}$$

where $E \subset \mathbb{R}^N$ is a measurable set of fixed volume $m > 0$, $g \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is a given function, $s \in (0, 1/2)$ and $P_s(E)$ is defined in (1.2).

Let $X : \mathbb{R}^N \to \mathbb{R}^N$ be a C_c^∞ vector field and let $\Phi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be the associated flow as in (2.4) that is volume preserving, i. e.

$$|\Phi(\cdot, t)(E)| = |E| \quad \text{for all} \quad t \in [0, 1]. \tag{3.2}$$

Let E be an open set of class C^3. By (2.5), the first variation of J_s at E with respect to the flow Φ is

$$\frac{d}{dt}J_s(E_t)_{|t=0} = \int_{\partial E} H_s(X\cdot\nu_E)\,\mathrm{d}\mathcal{H}^{N-1} + \int_{\partial E} g(X\cdot\nu_E)\,\mathrm{d}\mathcal{H}^{N-1}, \tag{3.3}$$

where $E_t = \Phi_t(E)$ and ν_E is the outer unit normal to E.

Then, given E a sufficiently smooth (local) minimizer of the functional (3.1), we have the following Euler-Lagrange equation

$$H_s(x) + g(x) = \lambda \quad \text{for all} \quad x \in \partial E, \tag{3.4}$$

with λ that is a constant Lagrange multiplier associated to m.

We now give the following three definitions.

Definition 3.1. A set $E \subset \mathbb{R}^N$ is a regular critical set for the functional (3.1) if it is of class C^3 and (3.4) holds on ∂E in the weak sense, that is

$$\begin{aligned}&\int_{\partial E} H_s(\xi\cdot\nu_E)\,\mathrm{d}\mathcal{H}^{N-1} = -\int_{\partial E} g(\xi\cdot\nu_E)\,\mathrm{d}\mathcal{H}^{N-1}, \quad \text{for all} \quad \xi \in C_c^1(\mathbb{R}^N,\mathbb{R}^N)\\ &\text{s.t.} \int_{\partial E}\xi\cdot\nu_E\,\mathrm{d}\mathcal{H}^{N-1} = 0.\end{aligned} \tag{3.5}$$

Definition 3.2. A set $E \subset \mathbb{R}^N$ of finite s-perimeter is a local minimizer for the functional (3.1) if there exists $\delta > 0$ such that

$$J_s(E) \le J_s(F)$$

for all $F \subset \mathbb{R}^N$ with $|E| = |F|$ and $|E\Delta F| \le \delta$. If the inequality is strict whenever $|E\Delta F| > 0$, we say that E is an isolated local minimizer.

Moreover, if the local minimizer E is a regular critical set according to Definition 3.1, then E is called a regular local minimizer.

Definition 3.3. Given $\omega > 0$, we say that a set $E \subset \mathbb{R}^N$ of finite s-perimeter is an ω-minimizer for the s-area functional if for any ball $B_r(x_0) \subset \mathbb{R}^N$ and any set of finite s-perimeter $F \subset \mathbb{R}^N$ such that $E\Delta F \subset\subset B_r(x_0)$ we have

$$P_s(E) \leq P_s(F) + \omega r^N.$$

As an easy consequence of these definitions we have the following

Lemma 3.4. *If E is a local minimizer for* (3.1)*, then E is an ω-minimizer for the s-area functional.*

Proof. Since E is a minimizer for (3.1), proceeding as in [11, Lemma 3.4], we obtain that exist $R > 0$ and μ_0 depending on E such that $E \subseteq B_{R/2}$ and it is a solution to

$$\min_{F\subset B_R} \Big(P_s(F) + \int_F g \, \mathrm{d}x + \mu \big||F| - |E|\big| \Big), \tag{3.6}$$

for every $\mu \geq \mu_0$. Now, for any $x_0 \in \partial E$, consider a ball $B_r(x_0) \subset \mathbb{R}^N$ such that $\omega_N r^N \leq \delta/2$, with δ as in Definition 3.2. So, for all F of finite s-perimeter such that $E\Delta F \subset\subset B_r(x_0)$, we have

$$\begin{aligned} P_s(E) &\leq P_s(F) + \|g\|_{L^\infty(\mathbb{R}^N)}|E\Delta F| + \mu_0\big||E| - |F|\big| \\ &\leq P_s(F) + (\|g\|_{L^\infty(\mathbb{R}^N)} + \mu_0)|E\Delta F| \leq P_s(F) + \omega r^N. \end{aligned} \tag{3.7}$$

Therefore, E is an ω-minimizer for some $\omega \geq (\|g\|_{L^\infty(\mathbb{R}^N)} + \mu_0)\omega_N$. □

Let us fix some notation. Given a vector field X, we define $X_\tau := X - (X \cdot \nu_E)\nu_E$ its tangential part on ∂E. In particular, denoting with ∇_τ the tangential gradient operator given by $\nabla_\tau \varphi := (\nabla\varphi)_\tau$, we recall that the second fundamental form $B_{\partial E}$ of ∂E is given by $D_\tau \nu_E$.

Before proving the representation formula for the second variation of J_s at E with respect to the flow Φ, we recall that of the s-perimeter, calculated in [17, Theorem 6.1]:

$$\begin{aligned} \frac{d^2}{dt^2} P_s(E_t)_{|t=0} = &\int_{\partial E\times\partial E} \frac{|(X\cdot\nu_E)(x) - (X\cdot\nu_E)(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}\mathcal{H}^{N-1}_x \, \mathrm{d}\mathcal{H}^{N-1}_y \\ &- \int_{\partial E\times\partial E} \frac{|\nu_E(x) - \nu_E(y)|^2}{|x-y|^{N+2s}} (X\cdot\nu_E)^2 \, \mathrm{d}\mathcal{H}^{N-1}_x \, \mathrm{d}\mathcal{H}^{N-1}_y \\ &+ \int_{\partial E} H_s\Big((\mathrm{div}X)(X\cdot\nu_E) - \mathrm{div}_\tau X_\tau (X\cdot\nu_E)\Big) \, \mathrm{d}\mathcal{H}^{N-1} \end{aligned} \tag{3.8}$$

where H_s is the nonlocal mean curvature defined in (2.6).

Theorem 3.5. *If E, X and Φ are as above, we have*

$$\begin{aligned} \frac{d^2}{dt^2} J_s(E_t)_{|t=0} = \frac{d^2}{dt^2} P_s(E_t)_{|t=0} &+ \int_{\partial E} g\Big((X\cdot\nu_E)\,\mathrm{div}X \\ &- \mathrm{div}_\tau(X_\tau(X\cdot\nu_E))\Big) \, \mathrm{d}\mathcal{H}^{N-1} + \int_{\partial E} (\nabla g\cdot\nu_E)(X\cdot\nu_E)^2 \, \mathrm{d}\mathcal{H}^{N-1}. \end{aligned} \tag{3.9}$$

Proof. Since (3.8) holds, from (3.3) we have only to compute

$$\frac{d}{dt}_{|t=0} \int_{\partial E} g(X\cdot\nu_E) \, \mathrm{d}\mathcal{H}^{N-1}.$$

Proceeding as in the proof of [10, Theorem 3.6, Step 3] and using [23, Proposition 17.1], we have

$$\begin{aligned}\frac{d}{dt}_{|t=0}\int_{\partial E_t} g(X\cdot\nu_{E_t})\,\mathrm{d}\mathcal{H}^{N-1} &= \frac{d}{dt}_{|t=0}\Big(\int_{\partial E}(g\circ\Phi)(\dot\Phi(\nu_E\circ\Phi))J_\Phi\,\mathrm{d}\mathcal{H}^{N-1}\Big)\\ &= \int_{\partial E}\frac{\partial}{\partial t}(g\circ\Phi)_{|t=0}(X\cdot\nu_E)\,\mathrm{d}\mathcal{H}^{N-1} + \int_{\partial E} g\frac{\partial}{\partial t}(\dot\Phi(\nu_E\circ\Phi)J_\Phi)_{|t=0}\,\mathrm{d}\mathcal{H}^{N-1}\\ &= \mathcal{I}_1+\mathcal{I}_2.\end{aligned} \tag{3.10}$$

The first integral $\mathcal{I}_1$ can be written as

$$\begin{aligned}\mathcal{I}_1 = \int_{\partial E}(\nabla g\cdot X)(X\cdot\nu_E)\,\mathrm{d}\mathcal{H}^{N-1} &= \int_{\partial E}(\nabla g\cdot\nu_E)(X\cdot\nu_E)^2\,\mathrm{d}\mathcal{H}^{N-1}\\ &+ \int_{\partial E}(\nabla_\tau g\cdot X_\tau)(X\cdot\nu_E)\,\mathrm{d}\mathcal{H}^{N-1}.\end{aligned} \tag{3.11}$$

By property (g) of [10, Lemma 3.8], the other integral turns out to be

$$\begin{aligned}\mathcal{I}_2 = &\int_{\partial E} g\Big(Z\cdot\nu_E - 2X_\tau\cdot\nabla_\tau(X\cdot\nu_E) + B_\tau[X_\tau,X_\tau]\Big)\,\mathrm{d}\mathcal{H}^{N-1}\\ &+ \int_{\partial E} g\,\mathrm{div}_\tau((X\cdot\nu_E)X)\,\mathrm{d}\mathcal{H}^{N-1},\end{aligned} \tag{3.12}$$

where $Z = \dfrac{\partial^2\Phi}{\partial t^2}_{|t=0}$.

Then we note that, from [10, Equation (2.4)],

$$\begin{aligned}\int_{\partial E}(\nabla_\tau g\cdot X_\tau)(X\cdot\nu_E)\,\mathrm{d}\mathcal{H}^{N-1} &+ \int_{\partial E} g\,\mathrm{div}_\tau((X\cdot\nu_E)X)\,\mathrm{d}\mathcal{H}^{N-1}\\ &= \int_{\partial E}\mathrm{div}_\tau(g(X\cdot\nu_E)X)\,\mathrm{d}\mathcal{H}^{N-1} = \int_{\partial E} gH(X\cdot\nu_E)^2\,\mathrm{d}\mathcal{H}^{N-1}\end{aligned} \tag{3.13}$$

and, from [3, Equation (7.5)],

$$\begin{aligned}H(X\cdot\nu_E)^2 + Z\cdot\nu_E - 2X_\tau\cdot D_\tau(X\cdot\nu_E) + B_\tau[X_\tau,X_\tau]&\\ = (X\cdot\nu_E)\,\mathrm{div}X - \mathrm{div}_\tau(X_\tau(X\cdot\nu_E)),&\end{aligned} \tag{3.14}$$

so, combining (3.8), (3.10), (3.11), (3.12), (3.13) and (3.14), we conclude the proof of the theorem. □

Remark 3.6. Notice that in the case of a regular critical set E, the second variation of J_s at E with respect to the flow Φ is reduced to

$$\begin{aligned}\frac{d^2}{dt^2}J_s(E_t)_{|t=0} = &\int_{\partial E\times\partial E}\frac{|(X\cdot\nu_E)(x)-(X\cdot\nu_E)(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y\\ &-\int_{\partial E\times\partial E}\frac{|\nu_E(x)-\nu_E(y)|^2}{|x-y|^{N+2s}}(X\cdot\nu_E)^2\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y\\ &+\int_{\partial E}(\nabla g\cdot\nu_E)(X\cdot\nu_E)^2\,\mathrm{d}\mathcal{H}^{N-1}.\end{aligned} \tag{3.15}$$

Proof. If Φ satisfies $|\Phi(\cdot,t)(E)| = |E|$ for every $|t|\in[0,1]$, it follows that

$$\frac{d}{dt}|E_t|_{|t=0} = \int_{\partial E} X\cdot\nu_E\,\mathrm{d}\mathcal{H}^{N-1} = 0 \tag{3.16}$$

and

$$\frac{d^2}{dt^2}|E_t|_{|t=0} = \int_{\partial E} (\mathrm{div} X)(X \cdot \nu_E)\, \mathrm{d}\mathcal{H}^{N-1} = 0. \tag{3.17}$$

Then we have that

$$\int_{\partial E} \mathrm{div}_\tau\Big((X \cdot \nu_E) X_\tau\Big)\, \mathrm{d}\mathcal{H}^{N-1} = 0 \tag{3.18}$$

by the tangential divergence theorem (since ∂E has no boundary). So, being E a regular critical set for J_s, from (3.9), (3.16), (3.17), (3.18) and (3.5) we obtain (3.15). □

Definition 3.7. Given any sufficiently smooth open set $E \subset \mathbb{R}^N$, we denote by

$$\tilde{H}^s(\partial E) = \Big\{ \varphi \in H^s(\partial E) : \int_{\partial E} \varphi \, \mathrm{d}\mathcal{H}^{N-1} = 0 \Big\}$$

endowed with the norm $\|\varphi\|_{H^s(\partial E)}$. To E we associate the quadratic form $\partial^2 J_s(E) : \tilde{H}^s(\partial E) \to \mathbb{R}$, defined as

$$\begin{aligned} \partial^2 J_s(E)[\varphi] = & \int_{\partial E \times \partial E} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{N+2s}}\, \mathrm{d}\mathcal{H}^{N-1}_x\, \mathrm{d}\mathcal{H}^{N-1}_y \\ & - \int_{\partial E \times \partial E} \frac{|\nu_E(x) - \nu_E(y)|^2}{|x-y|^{N+2s}} \varphi^2\, \mathrm{d}\mathcal{H}^{N-1}_x\, \mathrm{d}\mathcal{H}^{N-1}_y \\ & + \int_{\partial E} (\nabla g \cdot \nu_E) \varphi^2 \, \mathrm{d}\mathcal{H}^{N-1}. \end{aligned} \tag{3.19}$$

Observe that if E is a regular critical set and the flow Φ is volume-preserving, then $\partial^2 J_s(E)[X \cdot \nu_E]$ is exactly the second variation of J_s at E with respect to Φ.

From Remark 3.6 it follows

Corollary 3.8. *Let E be a bounded open set of class $C^{3,\alpha}$ for some $\alpha \in (0,1)$ and a local minimizer of J_s according to Definition 3.2. Then*

$$\partial^2 J_s(E)[\varphi] \geq 0 \quad \text{for all} \quad \varphi \in \tilde{H}^s(\partial E).$$

Proof. Let $\varphi \in C^\infty(\partial E) \cap H^s(\partial E)$. We set $X := \nabla u$ where u solves

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^N \setminus \partial E, \\ \partial_{\nu_E} u = \varphi & \text{on} \quad \partial E, \end{cases} \tag{3.20}$$

and ν_E is the outer unit normal on ∂E. Let $\mathcal{N}(\partial E)$ be a tubular neighborhood of ∂E. Since $E \in C^{3,\alpha}$ for some $\alpha \in (0,1)$ and $\mathrm{div} X$=0 on $\mathbb{R}^N \setminus \partial E$, it results $X \cdot \nu \in C^{2,\alpha}$ separately in $\overline{E} \cap \mathcal{N}(\partial E)$ and $\overline{\mathbb{R}^N \setminus E} \cap \mathcal{N}(\partial E)$, and globally Lipschitz continuous in $\mathcal{N}(\partial E)$. Here ν denotes a $C^{2,\alpha}$ extension of the outer normal field ν_E from ∂E to $\mathcal{N}(\partial E)$. We now set

$$X_\varepsilon(x) := \int_{\mathbb{R}^N} \rho_\varepsilon(z) X(x+z)\, \mathrm{d}z,$$

where ρ_ε is a standard mollifier. Since $\mathrm{div} X_\varepsilon$=0, the associated flow is volume preserving, so local minimality and (3.15) imply that $\partial^2 J_s(E)[\varphi_\varepsilon] \geq 0$, where $\varphi_\varepsilon := X_\varepsilon \cdot \nu$. Observing that we can write

$$\begin{aligned} (X_\varepsilon \cdot \nu)(x) &= (X \cdot \nu)_\varepsilon(x) - \int_{\mathbb{R}^N} \rho_\varepsilon(z) X(x+z) \cdot [\nu(x+z) - \nu(x)]\, \mathrm{d}z \\ &=: (X \cdot \nu)_\varepsilon(x) - R_\varepsilon(x), \end{aligned} \tag{3.21}$$

and that $X \cdot \nu$ is continuous in $\mathcal{N}(\partial E)$, we get that $X_\varepsilon \cdot \nu \to X \cdot \nu$ uniformly in $\mathcal{N}(\partial E)$. In particular $\varphi_\varepsilon \to \varphi$ uniformly on ∂E and so we can check that $\partial^2 J_s(E)[\varphi] = \lim_{\varepsilon\to 0} \partial^2 J_s(E)[\varphi_\varepsilon] \geq 0$. Indeed, considering

(3.22)
$$\begin{aligned}\partial^2 J_s(E)[\varphi_\varepsilon] &= \int_{\partial E\times\partial E} \frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^2}{|x-y|^{N+2s}} \,\mathrm{d}\mathcal{H}^{N-1}_x \,\mathrm{d}\mathcal{H}^{N-1}_y \\ &\quad - 2\int_{\partial E}\int_{\partial E} \varphi_\varepsilon^2(x)\frac{1-\nu(x)\nu(y)}{|x-y|^{N+2s}} \,\mathrm{d}\mathcal{H}^{N-1}_x \,\mathrm{d}\mathcal{H}^{N-1}_y + \int_{\partial E} (\nabla g\cdot\nu)\varphi_\varepsilon^2 \,\mathrm{d}\mathcal{H}^{N-1},\end{aligned}$$

we observe that, since $\varphi_\varepsilon \to \varphi$ uniformly on ∂E and $g \in C^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, the first and the third term pass to the limit. So, being E compact, if we show that

$$\int_{\partial E} \frac{1-\nu(x)\nu(y)}{|x-y|^{N+2s}} \,\mathrm{d}y \leq C, \tag{3.23}$$

for some $C>0$, we have proved that $\partial^2 J_s(E)[\varphi] = \lim_\varepsilon \partial^2 J_s(E)[\varphi_\varepsilon]$ because $\varphi_\varepsilon \to \varphi$ uniformly on ∂E, hence

$$\int_{\partial E}\int_{\partial E} \varphi_\varepsilon^2(x)\frac{1-\nu(x)\nu(y)}{|x-y|^{N+2s}} \,\mathrm{d}\mathcal{H}^{N-1}_x \,\mathrm{d}\mathcal{H}^{N-1}_y \to \int_{\partial E}\int_{\partial E} \varphi^2(x)\frac{1-\nu(x)\nu(y)}{|x-y|^{N+2s}} \,\mathrm{d}\mathcal{H}^{N-1}_x \,\mathrm{d}\mathcal{H}^{N-1}_y.$$

First of all we note that it is sufficient to prove (3.23) when y is very close to x. Thus, since $E \in C^{3,\alpha}$ for some $\alpha\in(0,1)$, we can write

$$\partial E = \{(y,u(y)) : y \in B_1\} \quad \text{with} \quad u \in C^{3,\alpha}(B_1), \quad u(0)=0 \quad \text{and} \quad \nabla u(0)=0. \tag{3.24}$$

For brevity we set $Y=(y,u(y))$ and $X=(x,u(x))$. We recall the expression of the outer unit normal, i.e.

$$\nu(y,u(y)) = \frac{(-\nabla u(y),1)}{\sqrt{1+|\nabla u(y)|^2}}, \quad \text{for all} \quad y\in\partial E.$$

By (3.24) we have $X=(0,0)$ and $\nu(X)=(0,1)$, therefore

$$\begin{aligned}&\int_{\partial E} \frac{1-\nu(X)\nu(Y)}{|X-Y|^{N+2s}} \,\mathrm{d}\mathcal{H}^{N-1}(Y) \\ &= \int_{B_1} \frac{1}{(y^2+u(y)^2)^{(N+2s)/2}}\left(1-\frac{1}{\sqrt{1+|\nabla u(y)|^2}}\right)\sqrt{1+|\nabla u(y)|^2}\,\mathrm{d}y \\ &= \int_{B_1} \frac{\sqrt{1+|\nabla u(y)|^2}-1}{(y^2+u(y)^2)^{(N+2s)/2}}\,\mathrm{d}y.\end{aligned} \tag{3.25}$$

At this point we observe that if $|\nabla u(y)|$ is sufficiently small (3.23) is proved, otherwise, being $u\in C^{1,1}(B_1)$, we obtain

$$\begin{aligned}\left|\int_{B_1} \frac{\sqrt{1+|\nabla u(y)|^2}-1}{(y^2+u(y)^2)^{(N+2s)/2}}\,\mathrm{d}y\right| &\leq \int_{B_1} \frac{|\nabla u(y)|}{(y^2+u(y)^2)^{(N+2s)/2}}\,\mathrm{d}y \\ &\leq c\int_{B_1} \frac{|y|}{|y|^{N+2s}}\,\mathrm{d}y \leq C\end{aligned} \tag{3.26}$$

where $c>0$ and $C>0$ are constants.

Now if φ is any function in $\tilde{H}^s(\partial E)$, we construct a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ of functions in $C^\infty(\partial E)\cap\tilde{H}^s(\partial E)$ such that $\varphi_n\to\varphi$ in $\tilde{H}^s(\partial E)$. Then we claim that all the terms in the expression of $\partial^2 J_s(E)$ are continuous with respect to the H^s-convergence. Indeed the first term is exactly the H^s- Gagliardo seminorm of φ_n. Then, since $\tilde{H}^s(\partial E)$ is embedded in $L^2(\partial E)$ with compactness, $g\in C^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, and (3.23) holds, all the terms in $\partial^2 J_s(E)$ pass to the limit, so the claim is proved and the proof is complete.

□

4. Strict stability implies $W^{1+2s',p}$ minimality

In this section we want to show a first important result that will be necessary to obtain the main theorem of this work: we prove that strict stability implies $W^{1+2s',p}$-minimality with $s'<s$. In the next section we will show that a $W^{1+2s',p}$-local minimizer is a L^1-local minimizer and this will allow us to obtain Theorem 1.1.

We start with this

Definition 4.1. We say that J_s has positive second variation at the regular critical set E if

$$\partial^2 J_s(E)[\varphi]>0 \quad \text{for all} \quad \varphi\in\tilde{H}^s(\partial E)\setminus\{0\}.$$

Lemma 4.2. *Suppose that J_s has positive second variation at a regular critical set E. Then*

$$m_0:=\inf\{\partial^2 J_s(E)[\varphi]:\varphi\in\tilde{H}^s(\partial E),\|\varphi\|_{\tilde{H}^s(\partial E)}=1\}>0 \tag{4.1}$$

and

$$\partial^2 J_s(E)[\varphi]\geq m_0\|\varphi\|^2_{\tilde{H}^s(\partial E)} \quad \textit{for all} \quad \varphi\in\tilde{H}^s(\partial E).$$

Proof. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a minimizing sequence for the infimum in (4.1) and assume that $\varphi_n\rightharpoonup\varphi_0$ weakly in $\tilde{H}^s(\partial E)$. If $\varphi_0\neq 0$, by (3.19) it follows that

$$m_0=\lim_n\partial^2 J_s(E)[\varphi_n]\geq\partial^2 J_s(E)[\varphi_0]>0.$$

If $\varphi_0=0$, then

$$\begin{aligned} m_0=\lim_n\partial^2 J_s(E)[\varphi_n]&=\lim_n\Big[\int_{\partial E\times\partial E}\frac{|\varphi_n(x)-\varphi_n(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y\\ &+2\int_{\partial E}\int_{\partial E}\varphi_n^2(x)\frac{\nu_E(x)\nu_E(y)-1}{|x-y|^{N+2s}}\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y+\int_{\partial E}(\nabla g\cdot\nu_E)\varphi_n^2\,\mathrm{d}\mathcal{H}^{N-1}\Big]. \end{aligned} \tag{4.2}$$

Now, we can proceed as in Corollary 3.8 observing that $\|\varphi_n\|_{\tilde{H}^s(\partial E)}=1$ so, by Sobolev embedding, it results $\varphi_n\to 0$ in $L^2(\partial E)$. Hence the second and the third term of (4.2) vanish and we conclude that

$$m_0=\lim_n\partial^2 J_s(E)[\varphi_n]\geq 1.$$

□

The next theorem allows us to construct a volume-preserving flow connecting any two regular and close sets in $\mathbb{R}^N$. If $E\subset\mathbb{R}^N$ is an open set at least of class C^3, we denote by $\mathcal{N}_r(\partial E)$ the tubular neighborhood of ∂E of thickness $2r$ and we let $d\in C^3$ the signed distance from ∂E.

Theorem 4.3. *Let $E \subset \mathbb{R}^N$ be a bounded open set of class C^3 with fixed volume $m > 0$, $s' < s$ and $p > \max\{2, N/2s'\}$. For all $\varepsilon > 0$ there exist a tubular neighborhood $\mathcal{N}_r(\partial E)$ and two constants δ, $C > 0$ with the following properties. If $\psi \in C^2(\partial E)$ and $\|\psi\|_{W^{1+2s',p}(\partial E)} \leq \delta$, then there exists a field $X \in C^2$ with $\mathrm{div}X = 0$ in $\mathcal{N}_r(\partial E)$ such that*

$$\|X - \psi\nu_E\|_{L^2(\partial E)} \leq \varepsilon\|\psi\|_{L^2(\partial E)}. \tag{4.3}$$

Moreover, the associated flow

$$\begin{cases} \dfrac{\partial \Phi}{\partial t} = X(\Phi), & (t,x) \in \mathbb{R}\times\mathbb{R}^N \\ \Phi(x,0) = x, & x \in \mathbb{R}^N \end{cases} \tag{4.4}$$

satisfies $\Phi(\partial E, 1) = \{x + \psi(x)\nu_E : x \in \partial E\}$, and for every $t \in [0,1]$

$$\|\Phi(\cdot,t) - \mathrm{Id}\|_{W^{1+2s',p}(\partial E)} \leq C\|\psi\|_{W^{1+2s',p}(\partial E)}, \tag{4.5}$$

where Id *is the identity map. If in addition $|E_1| = |E|$, for every t we have $|E_t| = |E|$ and*

$$\int_{\partial E_t} X\cdot\nu_{E_t}\,\mathrm{d}\mathcal{H}^{N-1} = 0.$$

Proof. Consider $\sigma > 0$ and set $d_\sigma := \rho_\sigma * d$, where ρ_σ is a standard mollifier. Since E is of class C^3, we can find a neighborhood $\mathcal{N}_r(\partial E)$ and σ_ε such that, if $0 < \sigma < \sigma_\varepsilon$, then

$$\|d_\sigma - d\|_{C^3(\mathcal{N}_r(\partial E))} \leq \varepsilon. \tag{4.6}$$

For such σ let Ψ be the flow associated to ∇d_σ, i.e.

$$\Psi(x,0) = x \qquad \frac{\partial\Psi}{\partial t} = \nabla d_\sigma(\Psi).$$

Then there exists $t_0 > 0$ such that $\Psi_{|\partial E\times(-t_0,t_0)}$ is a C^∞ diffeomorphism onto some neighborhood U of ∂E. So we want to construct a C^∞ vector field $\tilde{X} : U \to \mathbb{R}^N$ such that

$$\mathrm{div}\tilde{X} = 0 \quad \text{in } U, \qquad \tilde{X} = \nabla d_\sigma \quad \text{on } \partial E.$$

To do this, for every $y \in U$ we set

$$\zeta(y) = \zeta(\Psi(x,t)) := \exp\Big(-\int_0^t \Delta d_\sigma(\Psi(x,s))\,\mathrm{d}s\Big). \tag{4.7}$$

Thus $\mathrm{div}(\zeta\nabla d_\sigma) = 0$ in U holds by construction. We define $\tilde{X} : U \to \mathbb{R}^N$ any C^∞-vector field coinciding with $\zeta\nabla d_\sigma$ on U, and denote by $\tilde{\Phi}$ the associated flow. Note that $\tilde{\Phi}$ and Ψ have the same trajectories in U, then we consider two functions $\pi_\sigma : U \to \partial E$, $t_\sigma : U \to \mathbb{R}$ defined by

$$\tilde{\Phi}(\pi_\sigma(y), t_\sigma(y)) = y.$$

If t is small, we have $t_\sigma(\tilde{\Phi}(x,t)) = t$, for all $x \in \partial E$. Hence, $\nabla t_\sigma(\tilde{\Phi}(x,t))\cdot\dfrac{\partial}{\partial t}\tilde{\Phi}(x,t) = 1$ and in particular $\nabla t_\sigma\cdot\nabla d_\sigma = 1$ on ∂E. Therefore, since $t_\sigma = 0$ on ∂E, we have

$$\nabla t_\sigma = \frac{\nabla d}{\nabla d\cdot\nabla d_\sigma} \quad \text{on } \partial E.$$

Thus, for $\sigma < \sigma_\varepsilon$ sufficiently small it results $\|\nabla t_\sigma - \nabla d_\sigma\|_{L^\infty(\partial E)} \leq \varepsilon$. Taking r smaller enough, it may be assumed that $\mathcal{N}_r(\partial E) \subset U$ and for all $y \in \mathcal{N}_r(\partial E)$

$$|t_\sigma(y) - d_\sigma(y)| \leq 2\varepsilon|d(y)|.$$

That is to say, there exists a function $a_\sigma \in C^3(\mathcal{N}_r(\partial E))$, with $\|a_\sigma\|_{L^\infty(\mathcal{N}_r(\partial E))} \le 2\varepsilon$ such that

$$t_\sigma(y) = d(y)(1 + a_\sigma(y)). \tag{4.8}$$

Let us now take $\psi \in C^2(\partial E)$. If $\|\psi\|_{L^\infty(\partial E)}$ is small, we set

$$S(x) := \pi_\sigma(x + \psi(x)\nu_E(x))$$

for $x \in \partial E$. Since E is of class C^3 we have that S is of class C^2 and

$$D_\tau S(x) = (D_\tau \pi_\sigma)(x + \psi(x)\nu_E(x)) + R(x),$$

where $|R(x)| \le C\|\psi\|_{C^1(\partial E)}$. Therefore, since $\pi_\sigma(x) = x$ on ∂E, we deduce that S is a C^2-diffeomorphism, provided that $\|\psi\|_{C^1(\partial E)}$ is small. Moreover, fixed $s' < s$, it is checked that if $\|\psi\|_{W^{1+2s',p}(\partial E)} \le 1$ then

$$\|S^{-1}\|_{W^{1+2s',p}(\partial E)} \le C \tag{4.9}$$

for some constant $C > 0$ independent of ψ. Note also that

$$|S^{-1}(x) - x| = |S^{-1}(x) - S^{-1}(\pi_\sigma(x + \psi(x)\nu_E(x)))| \le C|x - \pi_\sigma(x + \psi(x)\nu_E(x))| \le C|\psi(x)|. \tag{4.10}$$

Now for $y \in \mathcal{N}_r(\partial E)$ we set

$$G(y) := (S^{-1} \circ \pi_\sigma)(y) + \nu_E((S^{-1} \circ \pi_\sigma)(y))\psi((S^{-1} \circ \pi_\sigma)(y)). \tag{4.11}$$

In this way, $G(y)$ is the unique point of the trajectory of $\tilde{\Phi}$ passing through y that belongs to the graph of ψ. Finally, let us define

$$X(y) := t_\sigma(G(y))\tilde{X}(y) \tag{4.12}$$

for $y \in \mathcal{N}_r(\partial E)$. Notice that $X \in C^2(\mathcal{N}_r(\partial E), \mathbb{R}^N)$ and we call X any C^2-extension of the vector field to $\mathbb{R}^N$.

Note that $t_\sigma \circ G$ is constant along the trajectories of $\tilde{\Phi}$, such that we have $\mathrm{div} X = 0$ in $\mathcal{N}_r(\partial E)$. Denoting by Φ the flow associated to X, and since $t_\sigma(G(x))$ is the time needed to go from x to $G(x)$ along the trajectory of $\tilde{\Phi}$, we have $\Phi(x, 1) = G(x)$. Thus, we may conclude that $\Phi(\partial E, 1)$ is the graph of ψ. Now observe that from (4.8) and (4.11) follow that

$$X(y) = \psi((S^{-1} \circ \pi_\sigma)(y))(1 + a_\sigma(G(y)))\zeta(y)\nabla d_\sigma(y). \tag{4.13}$$

Thus, from (4.9) we have

$$\|X\|_{W^{1+2s',p}(\mathcal{N}_r(\partial E))} \le C\|\psi\|_{W^{1+2s',p}(\partial E)} \tag{4.14}$$

for a constant $C > 0$ independent of ψ.

At this point we can show (4.3). Indeed from (4.13), (4.9), and (4.10), for every $x \in \partial E$, we have

$$\begin{aligned} |X(x) - \psi(x)\nu_E(x)| &= |\psi((S^{-1} \circ \pi_\sigma)(x))(1 + a_\sigma(G(x)))\zeta(x)\nabla d_\sigma(x) - \psi(x)\nabla d(x)| \\ &\le |\psi(S^{-1}(x))||(1 + a_\sigma(G(x)))\zeta(x)\nabla d_\sigma(x) - \nabla d(x)| + |(\psi(S^{-1}(x) - \psi(x))\nabla d(x)| \\ &\le C\varepsilon|\psi(S^{-1}(x))| + |\psi(S^{-1}(x)) - \psi(x)| \\ &\le C\varepsilon|\psi(S^{-1}(x))| + \|\psi\|_{C^1(\partial E)}|S^{-1}(x) - x| \\ &\le C\varepsilon(|\psi(S^{-1}(x))| + |\psi(x)|) \end{aligned} \tag{4.15}$$

provided that $\|\psi\|_{C^1(\partial E)}$ is small. This leads to (4.3).

To prove (4.5), first we observe that Φ is close to Id in L^∞ thanks to (4.4) and (4.14). Then, differentiating (4.4) and solving the resulting equation, since $p > N/2s'$, one easily gets from ([1, Theorem 7.57]) and (4.14)

(4.16)

$$\begin{aligned}\|\nabla_x \Phi - \mathrm{Id}\|_{L^p(\mathcal{N}_{\varepsilon_0}(\partial E))} &\leq \|\nabla_x \Phi - \mathrm{Id}\|_{L^\infty(\mathcal{N}_{\varepsilon_0}(\partial E))} \leq C(\varepsilon_0)\|\nabla X\|_{L^\infty(\mathcal{N}_{\varepsilon_0}(\partial E))} \\ &\leq C(\varepsilon_0)\|\nabla X\|_{W^{2s',p}(\mathcal{N}_{\varepsilon_0}(\partial E))} \leq C(\varepsilon_0)\|\psi\|_{W^{1+2s',p}(\partial E)} \leq C(\varepsilon_0)\varepsilon.\end{aligned}$$

In a similar way we have

$$[\nabla_x \Phi - \mathrm{Id}]_{2s',p} \leq C(\varepsilon_0)[\nabla X]_{2s',p} \leq C\|X\|_{W^{1+2s',p}(\partial E)} \leq C\|\phi\|_{W^{1+2s',p}(\partial E)} \leq C(\varepsilon_0)\varepsilon,$$

where $[\cdot]_{2s',p}$ is the (s,p)-Gagliardo seminorm (see [15, Section 2]), and (4.5) follows.

Let now suppose that $|E_1| = |E|$ and recall that by [13, Equation (2.30)], we have

$$\frac{d^2}{dt^2}|E_t| = \int_{\partial E_t} (\mathrm{div} X)(X \cdot \nu_{E_t}) = 0 \quad \text{for all } t \in [0,1].$$

Hence the function $t \to |E_t|$ is affine in $[0,1]$ and since $|E_0| = |E| = |E_1|$, it is constant. Therefore

$$0 = \frac{d}{dt}|E_t| = \int_{E_t} \mathrm{div} X \,\mathrm{d}x = \int_{\partial E_t} X \cdot \nu_{E_t}\,\mathrm{d}\mathcal{H}^{N-1} \quad \text{for all } t \in [0,1].$$

This proves the theorem. □

With this theorem in hand, we can prove the first of minimality results, i.e. $W^{1+2s',p}$-minimality for $s' < s$.

Theorem 4.4. *Let $s' < s$, $p > \max\{2, N/2s'\}$ and $E \subset \mathbb{R}^N$ be a bounded open regular critical set for J_s with fixed volume $m > 0$ and positive second variation. There exist $\delta > 0$, $C_0 > 0$ such that*

$$J_s(F) \geq J_s(E) + C_0|E\Delta F|^2,$$

whenever $F \subset \mathbb{R}^N$ satisfies $|F| = |E|$ and $\partial F = \{x + \psi(x)\nu_E(x) : x \in \partial E\}$ for some $\psi : \partial E \to \mathbb{R}$ with $\|\psi\|_{W^{1+2s',p}(\partial E)} \leq \delta$.

Proof. Since we need estimates that depend on $\|\psi\|_{W^{1+2s',p}(\partial E)}$ only, we can assume that ψ is of class C^∞ by a standard approximation argument.

Step 1. Let $s' < s$. We claim there exists $\delta_1 > 2\mathcal{H}^{N-1}(\partial E)^{1/2} > 0$ such that, if $\partial F = \{x + \psi(x)\nu_E(x) : x \in \partial E\}$ with $|F| = |E|$ and $\|\psi\|_{W^{1+2s',p}(\partial E)} \leq \delta_1$, then

$$\inf\left\{\partial^2 J_s(F)[\varphi] : \varphi \in \tilde{H}^s(\partial F), \|\varphi\|_{\tilde{H}^s(\partial F)} = 1\right\} \geq \frac{m_0}{2}, \tag{4.17}$$

where $m_0 > 0$ is the constant defined in (4.1). By contradiction, let us assume that there exists a sequence $\partial F_n = \{x + \psi_n(x)\nu_E(x) : x \in \partial E\}$, $n \in \mathbb{N}$, with $|F_n| = |E|$ and $\|\psi_n\|_{W^{1+2s,p}(\partial E)} \to 0$, a sequence $\varphi_n \in \tilde{H}^s(\partial F_n)$, with $\|\varphi_n\|_{\tilde{H}^s(\partial F_n)} = 1$ such that

$$\partial^2 J_s(F_n)[\varphi_n] < \frac{m_0}{2}. \tag{4.18}$$

Let Φ_n be a family of diffeomorphisms from E to F_n converging to the identity in $W^{1+2s',p}(\partial E)$, (see Theorem 4.3), which exists thanks to the fact that ψ_n converges to 0.

Set

$$a_n = \fint_{\partial E} \varphi_n \circ \Phi_n \, \mathrm{d}\mathcal{H}^{N-1} \qquad \text{and} \qquad \tilde{\varphi}_n := \varphi_n \circ \Phi_n - a_n.$$

Since $\nu_{F_n} \circ \Phi_n \to \nu_E$ in $C^{0,\alpha}(\partial E)$ all terms in the expression (3.19) of $\partial^2 J_s(F_n)[\varphi_n]$ are asymptotically close to the corresponding terms of $\partial^2 J_s(E)[\tilde{\varphi}_n]$, i.e. $\partial^2 J_s(F_n)[\varphi_n] - \partial^2 J_s(E)[\tilde{\varphi}_n] \to 0$. Since $\|\tilde{\varphi}_n\|_{\tilde{H}^s(\partial E)} = 1$, from Lemma 4.2 we contradict (4.18).

Step 2. Let us fix F so that $\|\psi\|_{W^{1+2s',p}(\partial E)} \le \delta_2 < \delta_1$, where δ_2 is to be chosen. Let us consider the field X and the flow Φ constructed in Theorem 4.3. We claim that

$$\left| \int_{\partial E_t} (X \cdot \nu_{E_t}) \nu_{E_t} \, \mathrm{d}\mathcal{H}^{N-1} \right| \le \delta_1 \|X \cdot \nu_{E_t}\|_{L^2(\partial E_t)} \tag{4.19}$$

for all $t \in [0,1]$ (where $E_t = \phi(\cdot,t)(E)$ as usual). To this aim, we write

$$\begin{aligned}
&\int_{\partial E_t} (X \cdot \nu_{E_t}) \nu_{E_t} \, \mathrm{d}\mathcal{H}^{N-1} \\
&\quad = \int_{\partial E} \Big(X(\Phi(x,t)) \cdot \nu_{E_t}(\Phi(x,t)) \Big) \nu_{E_t}(\Phi(x,t)) J_{N-1}(d^{\partial E}\Phi(\cdot,t))(x) \, \mathrm{d}\mathcal{H}^{N-1} \\
&\quad = \int_{\partial E} (X(\Phi(x,t)) \cdot \nu_E(x)) \nu_E(x) \, \mathrm{d}\mathcal{H}^{N-1} + R_1 \\
&\quad = \int_{\partial E} (X(x) \cdot \nu_E(x)) \nu_E(x) \, \mathrm{d}\mathcal{H}^{N-1} + R_1 + R_2 \\
&\quad = \int_{\partial E} \psi(x) \nu_E(x) \, \mathrm{d}\mathcal{H}^{N-1} + R_1 + R_2 + R_3,
\end{aligned} \tag{4.20}$$

where $d^{\partial E}$ is the signed distance from the boundary of E,

$$\begin{aligned}
R_1 = &\int_{\partial E} \Big(X(\Phi(x,t)) \cdot \nu_{E_t}(\Phi(x,t)) \Big) \nu_{E_t}(\Phi(x,t)) J_{N-1}(d^{\partial E}\Phi(\cdot,t))(x) \, \mathrm{d}\mathcal{H}^{N-1} \\
&- \int_{\partial E} (X(\Phi(x,t)) \cdot \nu_E(x)) \nu_E(x) \, \mathrm{d}\mathcal{H}^{N-1};
\end{aligned} \tag{4.21}$$

$$R_2 = \int_{\partial E} (X(\Phi(x,t)) \cdot \nu_E(x)) \nu_E(x) \, \mathrm{d}\mathcal{H}^{N-1} - \int_{\partial E} (X(x) \cdot \nu_E(x)) \nu_E(x) \, \mathrm{d}\mathcal{H}^{N-1}$$

and

$$R_3 = \int_{\partial E} (X(x) \cdot \nu_E(x)) \nu_E(x) \, \mathrm{d}\mathcal{H}^{N-1} - \int_{\partial E} \psi(x) \nu_E(x) \, \mathrm{d}\mathcal{H}^{N-1}.$$

Now fix $\varepsilon > 0$. Recalling (4.13), (4.8), (4.7), and (4.9), we have

$$\int_{\partial E} |X(\Phi(x,t))| \, \mathrm{d}\mathcal{H}^{N-1} \le C \|\psi\|_{L^2(\partial E)}. \tag{4.22}$$

Observing that by (4.5)

$$\|\nu_E - \nu_{E_t}(\Phi(\cdot,t))\|_{L^\infty(\partial E)}, \qquad \|J_{N-1}(d^{\partial E}\Phi(\cdot,t)) - 1\|_{L^\infty(\partial E)}$$

are arbitrarily small, from (4.22), (4.3) and (4.14) we obtain

$$|R_1| + |R_2| + |R_3| \le \varepsilon \|\psi\|_{L^2(\partial E)},$$

provided that δ_2 is sufficiently small. This and (4.20) prove that

$$\begin{aligned}
\left| \int_{\partial E_t} (X \cdot \nu_{E_t}) \nu_{E_t} \, \mathrm{d}\mathcal{H}^{N-1} \right| &\le \left| \int_{\partial E} \psi \nu_E \, \mathrm{d}\mathcal{H}^{N-1} \right| + \varepsilon \|\psi\|_{L^2(\partial E)} \le \Big(\mathcal{H}^{N-1}(\partial E)^{1/2} + \varepsilon \Big) \|\psi\|_{L^2(\partial E)} \\
&\le \Big(\frac{\delta_1}{2} + \varepsilon \Big) \|\psi\|_{L^2(\partial E)}.
\end{aligned} \tag{4.23}$$

A similar argument shows that

$$\|X \cdot \nu_{E_t}\|_{L^2(\partial E_t)} \geq (\mathcal{H}^{N-1}(\partial E)^{1/2} - \varepsilon)\|\psi\|_{L^2(\partial E)}, \tag{4.24}$$

and (4.19) follows, if ε and, in turn, δ_2 are suitably chosen.

Since E is a critical set for J_s, recalling (3.9), (3.15), and thanks to an integration by parts, we can write

(4.25)

$$\begin{aligned}
J_s(F) - J_s(E) = J_s(E_1) - J_s(E) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} J_s(E_t)\,\mathrm{d}t = \\
&= -\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(1-t)\frac{\mathrm{d}}{\mathrm{d}t} J_s(E_t)\,\mathrm{d}t = \int_0^1 (1-t)\frac{\mathrm{d}^2}{\mathrm{d}t^2} J_s(E_t)\,\mathrm{d}t \\
&- \Big[(1-t)\frac{\mathrm{d}}{\mathrm{d}t} J_s(E_t)\Big]_{t=0}^{t=1} = \int_0^1 (1-t)\Big(\partial^2 J_s(E_t)[X\cdot\nu_{E_t}] \\
&- \int_{\partial E_t} H_s^{E_t}\,\mathrm{div}_\tau(X_\tau(X\cdot\nu_{E_t}))\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\partial E_t} g\,\mathrm{div}_{\tau_t}(X_{\tau_t}(X\cdot\nu_{E_t}))\,\mathrm{d}\mathcal{H}^{N-1}\Big)\,\mathrm{d}t,
\end{aligned}$$

where div_{τ_t} is the tangential divergence on ∂E_t, $H_s^{E_t}$ is the nonlocal mean curvature of ∂E_t and $X_{\tau_t} := X - (X\cdot\nu_{E_t})\nu_{E_t}$. Since

$$\int_{\partial E_t} X\cdot\nu_{E_t}\,\mathrm{d}\mathcal{H}^{N-1} = -\int_{E_t} \mathrm{div}X\,\mathrm{d}x = 0,$$

by (4.19) and (4.17), we obtain that

$$\begin{aligned}
J_s(F) - J_s(E) &\geq \frac{m_0}{2}\int_0^1 (1-t)\|X\cdot\nu_{E_t}\|^2_{H^s(\partial E_t)}\,\mathrm{d}t \\
&\quad - \int_0^1 (1-t)\int_{\partial E_t} (H_s^{E_t} + g)\,\mathrm{div}_{\tau_t}(X_{\tau_t}(X\cdot\nu_{E_t}))\,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t.
\end{aligned} \tag{4.26}$$

We claim that, if δ_2 is sufficiently small,

$$I_t := \Big|\int_{\partial E_t} (H_s^{E_t} + g)\,\mathrm{div}_{\tau_t}(X_{\tau_t}(X\cdot\nu_{E_t}))\,\mathrm{d}\mathcal{H}^{N-1}\Big| \leq \frac{m_0}{4}\|X\cdot\nu_{E_t}\|^2_{H^s(\partial E_t)} \tag{4.27}$$

for all $t \in [0,1]$. Indeed, since

$$\int_{\partial E_t} \mathrm{div}_{\tau_t}(X_{\tau_t}(X\cdot\nu_{E_t}))\,\mathrm{d}\mathcal{H}^{N-1} = 0,$$

we get

$$\begin{aligned}
I_t &= \Big|\int_{\partial E_t} \Big[(H_s^{E_t} + g)\Big]\,\mathrm{div}_{\tau_t}(X_{\tau_t}(X\cdot\nu_{E_t}))\,\mathrm{d}\mathcal{H}^{N-1}\Big| \\
&= \Big|\int_{\partial E_t} \Big[(H_s^{E_t} + g) - \lambda\Big]\,\mathrm{div}_{\tau_t}(X_{\tau_t}(X\cdot\nu_{E_t}))\,\mathrm{d}\mathcal{H}^{N-1}\Big|.
\end{aligned} \tag{4.28}$$

Now, since $E \in C^3$, it follows that H_s is of class C^1 (see [4]) so,

$$\begin{aligned}
I_t &= \Big|\int_{\partial E_t} \nabla_\tau(H_s^{E_t} + g - \lambda)X_{\tau_t}(X\cdot\nu_{E_t})\,\mathrm{d}\mathcal{H}^{N-1}\Big| \\
&\leq \|\nabla_\tau(H_s^{E_t} + g - \lambda)\|_{L^\infty(\partial E_t)}\|X_{\tau_t}\|_{L^2(\partial E_t)}\|X\cdot\nu_{E_t}\|_{L^2(\partial E_t)}.
\end{aligned} \tag{4.29}$$

Given $\varepsilon > 0$, if δ_2 is sufficiently small, since E satisfies (3.4) and $E_t = \Phi_t(E)$ with $\Phi \in C^3$, for t small, the first norm on the right-hand side of (4.28) can be taken smaller that ε. Hence, proceeding as in [3, Lemma 7.1] we have

$$\|X_\tau\|_{L^2(\partial E_t)} \leq C\|X \cdot \nu_{E_t}\|_{L^2(\partial E_t)}$$

and, from (4.29), we get

$$I_t \leq c\varepsilon \|X \cdot \nu_{E_t}\|^2_{H^s(\partial E_t)},$$

so (4.27) follows.

Then we observe that from (4.26) and (4.27) we obtain

$$\begin{aligned} J_s(F) &\geq J_s(E) + \frac{m_0}{4}\int_0^1 (1-t)\|X \cdot \nu_{E_t}\|^2_{H^s(\partial E_t)}\,\mathrm{d}t \\ &\geq J_s(E) + \frac{m_0}{4}\int_0^1 (1-t)\|X \cdot \nu_{E_t}\|^2_{L^2(\partial E_t)}\,\mathrm{d}t. \end{aligned} \tag{4.30}$$

Finally, recalling (4.24) we have

$$J_s(F) \geq J_s(E) + \frac{m_0}{16}\|\psi\|^2_{L^2(\partial E)} \geq J_s(E) + C_0|E\Delta F|^2,$$

and the theorem is proved. □

5. $W^{1+2s',p}$-local Minimality implies L^1-local Minimality

In this section, with $W^{1+2s',p}$-local minimality at hand, we want to switch to L^1-local minimality.

First of all we prove a result showing that if E is a regular isolated $W^{1+s',p}$-local minimizer of J_s, in the sense of Theorem 4.4, then E is a minimizer among all sets sufficiently close in the Hausdorff distance. In the proof of this theorem we need several known results; for the reader's convenience we state them below.

The first of them is a corollary of an important regularity result concerning sequences of w-minimizers for the s-area functional. This is proved as in [17, Corollary 3.6]:

Theorem 5.1. *If $N \geq 2$, $\omega > 0$, $s_0 \in (0,1)$, E_n, $n \in \mathbb{N}$, is an ω-minimizer of the s_n-perimeter for some $s_n \in [s_0, 1)$, and E_n converges to E in L^1, for some set E of class C^2, then there exists a bounded sequence $\{u_n\}_{n\in\mathbb{N}} \subset C^{1,\alpha}(\partial E)$ for some $\alpha \in (0,1)$ independent of n such that*

$$\partial E_n = \{x + u_n(x)\nu_E(x) : x \in \partial E\}, \qquad \lim_{n\to\infty} \|u_n\|_{C^1(\partial E)} = 0.$$

Then, thanks to [7, Theorem 2.3], we have an analogous result for minimizers of a problem with an obstacle:

Proposition 5.2. *Let $\mathcal{O} \subset \mathbb{R}^N$ be a C^2 domain (obstacle), with $\alpha > s+1/2$. Assume that E_n, $n \in \mathbb{N}$, is fixed outside B_1 and minimizes the s-perimeter in B_1 among all sets that contain $\mathcal{O} \cap B_1$. If $0 \in \partial E$, then there exists a bounded sequence $\{u_n\}_{n\in\mathbb{N}} \subset C^{1,\alpha}(\partial E \cap B_1)$ for some $\alpha \in (0,1)$ independent of n such that*

$$\partial E_n = \{x + u_n(x)\nu_E(x) : x \in \partial E\}, \qquad \lim_{n\to\infty} \|u_n\|_{C^1(\partial E)} = 0.$$

Proof. From [7, Theorem 2.3] we obtain that ∂E_n is graph of a function $u_n \in C^{1,\alpha}(\partial E \cap B_1)$ for some $\alpha \in (0,1)$. Then, thanks to the compact embedding of $C^{1,\alpha}$ in C^1 we have the thesis. □

Moreover, arguing similarly to [21, Proposition 4.25] and to [12, Corollary 5.3], we prove the following result:

Theorem 5.3. *Let $E \subset \mathbb{R}^N$ be of class C^3 and let $F \subset \mathbb{R}^N$ of class $C^{2,\alpha}$ for some $\alpha \in (0,1)$ such that $\partial F = \{x + \eta(x)\nu_E(x) : x \in \partial E\}$ with $\eta \in C^2(\partial E)$. If $\partial G_t = \{x + t\eta(x)\nu_E(x) : x \in \partial E\}$ for some $t \in [0,1]$ and $H_s(x) = H_s^{G_{t(x)}}(x)$ where $t(x)$ is such that $x \in \partial G_{t(x)}$, then for every $W \subset \mathbb{R}^N$ such that $\partial W \subseteq E\Delta F$*

$$P_s(W) \geq P_s(E) - \int_{W\Delta E} \Big|H_s(x)\Big| \,\mathrm{d}x. \tag{5.1}$$

Proof. We observe that for all $t \in [0,1]$,

$$\|H_s^{G_t}\|_{L^\infty(\partial G_t)} \leq \max\Big(\|H_s^F\|_{L^\infty(\partial F)}, \|H_s^E\|_{L^\infty(\partial E)}\Big),$$

so $H_s(x)$ is well defined for every $x \in F\Delta E$ and continuous there.

On the other hand denoting with $H_s^\delta(x) = H_s^{\delta,\partial G_{t(x)}}(x)$, it is well defined for every $x \in \mathbb{R}^N$.

Let $A \subset \mathbb{R}^N$ be a bounded set. Then

$$\begin{aligned}\int_A H_s^\delta(x)\,\mathrm{d}x &= \int_A \left(\int_{\mathbb{R}^N\setminus B_\rho(x)} \frac{\chi_{G_{t(x)}}(y) - \chi_{G^C_{t(x)}}(y)}{|x-y|^{N+2s}}\,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_{\mathbb{R}^N\times\mathbb{R}^N} \chi_A(x)\Big(\chi_{G_{t(x)}}(y) - \chi_{G^C_{t(x)}}(y)\Big)\frac{\chi_{(0,\rho)^C}(|x-y|)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} \Big(\chi_A(x) - \chi_A(y)\Big)\Big(\chi_{G_{t(x)}}(y) - \chi_{G^C_{t(x)}}(y)\Big)\frac{\chi_{(0,\rho)^C}(|x-y|)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned} \tag{5.2}$$

where $G^C_{t(x)}$ and $(o,\rho)^C$ are respectively the complements of $G_{t(x)}$ and $(0,\rho)$, and for the last equality, note that

$$\chi_{G_{t(x)}}(y) - \chi_{G^C_{t(x)}}(y) = -\Big(\chi_{G_{t(y)}}(x) - \chi_{G^C_{t(y)}}(x)\Big),$$

for almost every $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.

For a general bounded set A this gives

$$-\int_A \Big|H_s^\delta(x)\Big|\,\mathrm{d}x \leq \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} |\chi_A(x) - \chi_A(y)|\frac{\chi_{(0,\rho)^C}(|x-y|)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y,$$

while if we take $A = G_t$ for some $t \in [0,1]$, as we can deduce from the following picture, Figure 2, we have

$$\Big(\chi_{G_t}(x) - \chi_{G_t}(y)\Big)\Big(\chi_{G_{t(x)}}(y) - \chi_{G^C_{t(x)}}(y)\Big) = -|\chi_{G_t}(x) - \chi_{G_t}(y)| \tag{5.3}$$

for almost every $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.

Hence $\forall\, t \in [0,1]$,

$$-\int_{G_t} H_s^\delta(x)\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} |\chi_{G_t}(x) - \chi_{G_t}(y)|\frac{\chi_{(0,\rho)^C}(|x-y|)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

Now let W be such that $\partial W \subseteq E\Delta F$. If $P_s(W) = \infty$, then (5.1) is clear.

(a) $y \in G_t$ and $x \in G_t^C$ **(b)** $x \in G_t$ and $y \in G_t^C$

Figure 2. The two different situations that we have in (5.3).

On the other hand, if $P_s(W) < \infty$, then Lebesgue's dominated convergence theorem implies that

$$\lim_{\rho\to 0}\frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N}|\chi_W(x)-\chi_W(y)|\frac{\chi_{(0,\rho)^C}(|x-y|)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y = P_s(W) \tag{5.4}$$

and analogously, since $E \in C^3$, we have

$$\lim_{\rho\to 0}\frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N}|\chi_E(x)-\chi_E(y)|\frac{\chi_{(0,\rho)^C}(|x-y|)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y = P_s(E). \tag{5.5}$$

Therefore

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N}|\chi_W(x)-\chi_W(y)|\frac{\chi_{(0,\rho)^C}(|x-y|)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y &\geq -\int_W H_s^\delta(x)\,\mathrm{d}x\\ &\geq -\int_{W\Delta E}H_s^\delta(x)\,\mathrm{d}x-\int_E H_s^\delta(x)\,\mathrm{d}x\\ &= -\int_{W\Delta E}H_s^\delta(x)\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N}|\chi_E(x)-\chi_E(y)|\frac{\chi_{(0,\rho)^C}(|x-y|)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{5.6}$$

Since $W\Delta E \subset F\Delta E$, it results

$$\lim_{\rho\to 0}\int_{W\Delta E}H_s^\delta(x)\,\mathrm{d}x = \int_{W\Delta E}H_s(x)\,\mathrm{d}x, \tag{5.7}$$

and passing to the limit $\rho \to 0$ in (5.6), combining (5.4), (5.5) and (5.6), we obtain (5.1). □

Thanks to this result we can show the following useful lemma.

Lemma 5.4. *Let $E \subset \mathbb{R}^N$ be of class C^3 and let $F \subset \mathbb{R}^N$ be a set of finite s-perimeter. Then there exists $C = C(E) > 0$ such that*

$$P_s(F) - P_s(E) \geq -C|E\Delta F|.$$

Proof. First of all we observe that

$$P_s(F) - P_s(E) \geq P_s(G_\Lambda) - P_s(E) + \Lambda|E\Delta G_\Lambda| - \Lambda|E\Delta F|, \tag{5.8}$$

where G_Λ is a minimizer of

$$P_s(G) + \Lambda|E\Delta G|. \tag{5.9}$$

By Remark 3.4, it follows that G_Λ is an ω-minimizer for the s-area functional; as $\Lambda \to \infty$, we have that $G_\Lambda \in C^{2,\alpha}$ for some $\alpha \in (0,1)$ uniformly in Λ and $\partial G_\Lambda \to \partial E$ in C^1 by Theorem 5.1. So

$$\partial G_\Lambda = \{x + \eta(x)\nu_E(x) : x \in \partial E\},$$

where $\eta \in C^2(\partial E)$; if $x \in G_\Lambda \Delta E$, denoting with π the projection of $G_\Lambda \Delta E$ in ∂E, it follows that $\eta(\pi(x)) \neq 0$ and $x \in \partial G_t = \{x + t\eta(x)\nu_E(x) : x \in \partial E\}$ for $t \in [0,1]$. Then we observe that, since E and G_Λ have regular boundaries,

$$\|H_s\|_{L^\infty(\partial G_t)} \leq \max\Big(\|H_s\|_{L^\infty(\partial E)}, \|H_s\|_{L^\infty(\partial G_\Lambda)}\Big) \tag{5.10}$$

and since G_Λ is a C^2-minimizer of (5.9), from the Euler-Lagrange equation for (5.9), it follows

$$\|H_s\|_{L^\infty(\partial G_\Lambda)} \leq \Lambda.$$

Therefore applying Theorem 5.3, we have

$$P_s(G_\Lambda) \geq P_s(E) - \int_{E\Delta G_\Lambda} \Big|H_s(x)\Big| \,\mathrm{d}x, \tag{5.11}$$

with $H_s(x) = H_s^{G_t}(x)$, and using (5.10), it results

$$P_s(G_\Lambda) \geq P_s(E) - \max\Big(\|H_s\|_{L^\infty(\partial E)}, \|H_s\|_{L^\infty(\partial G_\Lambda)}\Big)|E\Delta G_\Lambda|,$$

that is, if $\Lambda > \|H_s\|_{L^\infty(\partial E)}$,

$$P_s(G_\Lambda) \geq P_s(E) - \Lambda|E\Delta G_\Lambda|. \tag{5.12}$$

Combining (5.8) and (5.12), we have

$$P_s(F) - P_s(E) \geq -\Lambda|E\Delta F| = -C(E)|E\Delta F|.$$

□

Now, we want to generalize a key lemma that in the classical setting is a consequence of L^p elliptic theory:

Theorem 5.5. *Let E be a set of class C^2 and let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of sets of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$ such that*

$$\partial E_n = \{x + t\psi_n(x)\nu_E(x) : x \in \partial E\},$$

where $\psi_n \to 0$ in $C^1(\partial E)$. Suppose also that $H_{s,\partial E_n} \in L^p(\partial E_n)$ and

$$H_{s,\partial E_n}(\cdot + \psi_n(\cdot)\nu_E(\cdot)) \to H_{s,\partial E}(\cdot) \quad \text{in } L^p(\partial E). \tag{5.13}$$

Then $\psi_n \to 0$ in $W^{1+2s',p}(\partial E)$, for all $s' < s$.

Proof. First of all we prove that H_s behaves as

$$(-\Delta)^{(1+2s)/2} + l.o.t.$$

From (2.6), for $p \in \partial E_t$, we have

$$\begin{aligned} H_s^{E_t}(p) &= \int_{\mathbb{R}^N} \frac{\chi_{E_t^C}(x) - \chi_{E_t}(x)}{|x-p|^{N+2s}} \,\mathrm{d}x = \lim_{\delta\to 0} \int_{\mathbb{R}^N\setminus B_\delta(p)} \frac{\chi_{E_t^C}(x) - \chi_{E_t}(x)}{|x-p|^{N+2s}} \,\mathrm{d}x \\ &= \int_{\mathbb{R}^N\setminus B_\rho(p)} \frac{\chi_{E_t^C}(x) - \chi_{E_t}(x)}{|x-p|^{N+2s}} \,\mathrm{d}x + \lim_{\delta\to 0} \int_{B_\rho(p)\setminus B_\delta(p)} \frac{\chi_{E_t^C}(x) - \chi_{E_t}(x)}{|x-p|^{N+2s}} \,\mathrm{d}x, \end{aligned} \tag{5.14}$$

for $\rho > \delta > 0$. Since $E \in C^2$ the first term integrand is bounded, so we focus on the second term. W.l.o.g. we can suppose that $p = 0$ and, unless you rotate and translate that

$$\partial E_t = \{x + tu(x)\nu_E(x) : x \in \partial E\},$$

hence

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{B_\rho(p)\setminus B_\delta(p)}\frac{\chi_{E_t^C}(x)-\chi_{E_t}(x)}{|x-p|^{N+2s}}\,\mathrm{d}x=\lim_{\delta\to 0}\int_{B_\rho(0)\setminus B_\delta(0)}\frac{\chi_{E_t^C}(x)-\chi_{E_t}(x)}{|x|^{N+2s}}\,\mathrm{d}x\\
&=\lim_{\delta\to 0}\int_{C_\rho(0)\setminus C_\delta(0)}\frac{\chi_{E_t^C}(x)-\chi_{E_t}(x)}{|x|^{N+2s}}\,\mathrm{d}x,
\end{aligned}\tag{5.15}$$

where $C_\rho = D_\rho\times[-r,r]$, D_ρ is the disc of $\mathbb{R}^{N-1}$ of centre 0 and radius ρ and $r>0$. Now, by Fubini's Theorem

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{C_\rho(0)\setminus C_\delta(0)}\frac{\chi_{E_t^C}(x)-\chi_{E_t}(x)}{|x|^{N+2s}}\,\mathrm{d}x=\\
&\lim_{\delta\to 0}\int_{D_\rho(0)\setminus D_\delta(0)}\Big(\int_{-\rho}^{u(x')}\frac{1}{|(x',x_N)|^{N+2s}}\,\mathrm{d}x_N-\int_{u(x')}^{\rho}\frac{1}{|(x',x_N)|^{N+2s}}\,\mathrm{d}x_N\Big)\,\mathrm{d}x',
\end{aligned}\tag{5.16}$$

where we use the writing $x=(x',x_N)$, with $x'\in\mathbb{R}^{N-1}$ and $x_N\in\mathbb{R}$. Thanks to a change of variables we obtain

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{D_\rho(0)\setminus D_\delta(0)}\Big(\int_{-\rho}^{u(x')}\frac{1}{|(x',x_N)|^{N+2s}}\,\mathrm{d}x_N-\int_{u(x')}^{\rho}\frac{1}{|(x',x_N)|^{N+2s}}\,\mathrm{d}x_N\Big)\,\mathrm{d}x'\\
&=\lim_{\delta\to 0}\int_{D_\rho(0)\setminus D_\delta(0)}\Big(\int_{-u(x')}^{\rho}\frac{1}{|(x',x_N)|^{N+2s}}\,\mathrm{d}x_N-\int_{u(x')}^{\rho}\frac{1}{|(x',x_N)|^{N+2s}}\,\mathrm{d}x_N\Big)\,\mathrm{d}x'\\
&=\lim_{\delta\to 0}\int_{D_\rho(0)\setminus D_\delta(0)}\int_{-u(x')}^{u(x')}\frac{1}{|(x',x_N)|^{N+2s}}\,\mathrm{d}x_N\,\mathrm{d}x'.
\end{aligned}\tag{5.17}$$

From a Taylor expansion and since $u(x')\simeq O(|x'|^2)$, we have that

$$\begin{aligned}
H_s^{E_t}(p)&=2\int_{D_\rho(0)}\frac{u(x')}{|x'|^{N+2s}}\,\mathrm{d}x'+\frac{1}{3}\int_{D_\rho(0)}\frac{u(x')^3}{|x'|^{N+2s+2}}\,\mathrm{d}x'\\
&\simeq 2\int_{D_\rho(0)}\frac{u(x')}{|x'|^{N+2s}}\,\mathrm{d}x'+\frac{1}{3}\int_{D_\rho(0)}|x'|^{4-N-2s}\,\mathrm{d}x'\\
&=2(-\Delta)^\sigma u(0)+Err(\|u\|_{C^2}),
\end{aligned}\tag{5.18}$$

where $\sigma=\dfrac{1+2s}{2}$ and $Err(\|u\|_{C^2})$ is the error term, depending only on $\|u\|_{C^2}$.

So we proved the claim.

At this point, hypothesis (5.13) and a density argument imply that

$$(-\Delta)^\sigma(\psi_n-\psi)\to 0\in L^p$$

and, from [22, Theorem 3.2], we can conclude that

$$\psi_n-\psi\to 0\in W^{2\sigma',p}\quad\text{for all}\quad \sigma'<\sigma,$$

that is our thesis. □

Remark 5.6. Consider the Lemma 5.5 with (5.13) replaced by

$$\sup_n\|H_{s,\partial E_n}\|_{L^p(\partial E_n)}<\infty.$$

Following the same line of reasoning, we obtain that the functions ψ_n are equibounded in $W^{1+2s',p}(\partial E)$.

At this point we denote with d the signed distance to a set E, and for all $\delta \in \mathbb{R}$, we define

$$\mathcal{I}_\delta(E) = \{x : d(x) < \delta\},$$

thus we are ready to prove the L^∞-local minimality result.

Theorem 5.7. *Let $E \subset \mathbb{R}^N$ be a bounded open set of class C^3 with fixed volume $m > 0$, $p > 1$ and $s' < s$. Suppose that there exists $\delta > 0$ such that*

$$J_s(F) \geq J_s(E) \tag{5.19}$$

for all $F \subset \mathbb{R}^N$ with $|F| = |E|$ and such that $\partial F = \{x + \psi(x)\nu_E(x) : x \in \partial E\}$ for some function ψ with $\|\psi\|_{W^{1+2s',p}(\partial E)} \leq \delta$. Then there exists $\delta_0 > 0$ such that (5.19) holds for all $F \subset \mathbb{R}^N$ of finite s-perimeter, with $|F| = |E|$ and $\mathcal{I}_{-\delta_0}(E) \subset F \subset \mathcal{I}_{\delta_0}(E)$.

Proof. Arguing by contradiction, we assume that there exist two sequences $\delta_n \to 0$ and $E_n \subset \mathbb{R}^N$ such that $|E_n| = |E|$, $\mathcal{I}_{-\delta_n}(E) \subset E_n \subset \mathcal{I}_{\delta_n}(E)$, and

$$J_s(E_n) < J_s(E)$$

for all $n \in \mathbb{N}$. For every n we call F_n a minimizer of the following penalized obstacle problem

$$\min\{J_s(F) + \Lambda\Big||F| - |E|\Big| : \mathcal{I}_{-\delta_n}(E) \subset F \subset \mathcal{I}_{\delta_n}(E)\}, \tag{5.20}$$

with $\Lambda > 1$ to be chosen later. It results

$$J_s(F_n) \leq J_s(E_n) < J_s(E). \tag{5.21}$$

We split the proof into three steps.

Step 1. We claim that for for $\Lambda > \|g\|_{L^\infty(\mathbb{R}^N)}$ and n sufficiently large

$$|F_n| = |E|. \tag{5.22}$$

Assume by contradiction that $|F_n| \neq |E|$. In particular we consider the case $|F_n| > |E|$.

Without loss of generality, for simplicity, we let $|E| = 1$. We observe that

$$\Big|\int_{F_n} g \,\mathrm{d}x\Big| \leq \|g\|_{L^\infty(\mathbb{R}^N)}|F_n| \leq \|g\|_{L^\infty(\mathbb{R}^N)}\Big||F_n| - 1\Big| + \|g\|_{L^\infty(\mathbb{R}^N)}. \tag{5.23}$$

Using this computation and minimality of F_n, say (5.21), we get that there exists C independent of n such that

$$\begin{aligned} P_s(F_n) &\leq P_s(E) + \int_E g \,\mathrm{d}x - \int_{F_n} g \,\mathrm{d}x - \Lambda\Big||F_n| - 1\Big| \\ &\leq P_s(E) + \int_E g \,\mathrm{d}x - \Big(\Lambda - \|g\|_{L^\infty(\mathbb{R}^N)}\Big)\Big||F_n| - 1\Big| + \|g\|_{L^\infty(\mathbb{R}^N)} \\ &\leq C \end{aligned} \tag{5.24}$$

and

$$\Lambda - \|g\|_{L^\infty(\mathbb{R}^N)} \leq P_s(E) + \int_E g \,\mathrm{d}x + \|g\|_{L^\infty(\mathbb{R}^N)} \leq C.$$

In particular this implies that $|F_n| \to |E|$ if Λ is sufficiently large.

Then we define $\tilde{F}_n = \lambda_n F_n$ with λ_n such that $|\tilde{F}_n| = |E|$, that is

$$|\tilde{F}_n| = \lambda_n^N |F_n| = |E|,$$

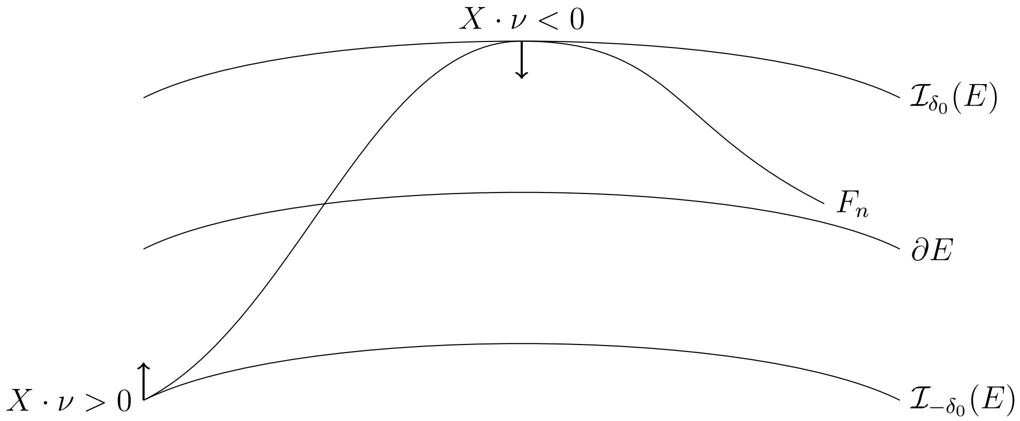

Figure 3. F_n is a solution of the obstacle problem (5.20), so we can only consider vector fields directed towards the interior of the obstacle.

that implies

$$\lambda_n = \left(\frac{|E|}{|F_n|}\right)^{1/N}.$$

From this, we observe that $\lambda_n \to 1$ as $n \to \infty$.

So

$$P_s(\tilde{F}_n) = \lambda_n^{N-2s} P_s(F_n)$$

and in particular for n large

$$\mathcal{I}_{-\delta} \subset \tilde{F}_n \subset \mathcal{I}_\delta$$

that is a contradiction.

The case $|F_n| < |E|$ is proved similarly.

Step 2. We claim that for n large enough F_n is of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$ and

$$\partial F_n = \{x + \psi_n(x)\nu_E(x) : x \in \partial E\},$$

for some ψ_n such that $\psi_n \to 0$ in $C^1(\partial E)$. In order to show this we observe that F_n solves (5.20), thus by Proposition 5.2, the claim follows.

Step 3. We claim that $\psi_n \to 0$ in $W^{1+2s',p}$ for all $s' < s$ and $p > 1$. For this purpose, we observe that since by Step 2 F_n is a C^1 solution of the minimum problem (5.20), we claim that

$$\sup_n \|H_{s,\partial F_n}\|_{L^\infty(\partial E)} \le C < +\infty, \tag{5.25}$$

where $C = \max\Big(\|H_{s,\partial \mathcal{I}_{\delta_0}}\|_{L^\infty(\partial E)}, \|H_{s,\partial \mathcal{I}_{-\delta_0}}\|_{L^\infty(\partial E)}, \|g\|_{L^\infty(\partial E)} + \Lambda\Big)$.

Indeed, if we consider $X : F_n \to \mathbb{R}^N$ a C_c^∞ vector field, $\|X\|_{L^\infty(F_n)} = 1$ such that, being ν the normal vector field to ∂F_n exterior to F_n, we have $X \cdot \nu < 0$ on $\mathcal{I}_{\delta_0}(E)$ and $X \cdot \nu > 0$ on $\mathcal{I}_{-\delta_0}(E)$ as in Figure 3.

Then we consider the associated flow $\Phi : F_n \times (-1,1) \to F_n$ defined by $\dfrac{\partial \Phi}{\partial t} = X(\Phi)$, $\Phi(x,0) = x$ and we denote with $F_{n,t} = \Phi(\cdot,t)(F_n)$ and with $\tilde{J}_s$ the functional in (5.20).

So by minimality of F_n we have

$$\frac{d}{dt}\tilde{J}_s(F_{n,t})_{|t=0} = \frac{d}{dt}J_s(F_{n,t})_{|t=0} + \frac{d}{dt}_{|t=0}\Big(\Lambda\Big||F_{n,t}| - |E|\Big|\Big) = 0 \tag{5.26}$$

that is, recalling (3.3),

$$\int_{\partial F_n} H_s(X\cdot\nu)\,\mathrm{d}\mathcal{H}^{N-1} + \int_{\partial F_n} g(X\cdot\nu)\,\mathrm{d}\mathcal{H}^{N-1} + \Lambda\frac{d}{dt}_{|t=0}|F_{n,t}| = 0. \tag{5.27}$$

From [13, Equations (2.28) and (2.29)],

$$\frac{d}{dt}_{|t=0}|F_{n,t}| = \int_{F_n} \mathrm{div}X,$$

hence (5.27) is equivalent to

$$\int_{\partial F_n} H_s(X\cdot\nu)\,\mathrm{d}\mathcal{H}^{N-1} + \int_{\partial F_n} g(X\cdot\nu)\,\mathrm{d}\mathcal{H}^{N-1} + \Lambda\int_{\partial F_n} X\cdot\nu\,\mathrm{d}\mathcal{H}^{N-1} = 0. \tag{5.28}$$

Now, recalling that $\mathcal{I}_{\pm\delta_0}(E)$ have smooth boundary, from (5.28) we obtain (5.25). Since the functions ψ_n are equibounded in C^1, the previous estimate on the nonlocal curvatures implies that the functions ψ_n are equibounded in $W^{1+2s',p}(\partial E)$ for all $p>1$ thanks to Remark 5.6. Therefore, by (5.22), each F_n is a solution of the obstacle problem (5.20) under the volume constraint. Since F_n is of class $W^{1+2s',p}$, we have that $H_{s,\partial F_n} = f_n$, where

$$f_n = \begin{cases} \lambda_n - g & \text{in } A_n := \partial F_n \cap N_{\delta_n}(\partial E), \\ \lambda - g + \rho_n & \text{otherwise,} \end{cases} \tag{5.29}$$

with λ_n, λ being Lagrange multipliers corresponding respectively to F_n and E, and ρ_n is a reminder term converging uniformly to 0.

We claim that

$$H_{s,\partial F_n}\Big(\,\cdot + \psi_n(\cdot)\nu_E(\cdot)\Big) \to H_{s,\partial E}(\cdot) \ \text{ in } \ L^p(\partial E) \ \text{ for all } \ p>1. \tag{5.30}$$

From (5.29) and (5.25) we obtain that the sequence λ_n is bounded.

If $\mathcal{H}^{N-1}(A_n)\to 0$, then (5.30) follows immediately.

Otherwise, we can assume with no loss of generality that $\mathcal{H}^{N-1}(A_n) \geq c > 0$. Thus, by a compactness argument we may find a cyinder $C = B'\times(-L,L)$, with $B'\subset\mathbb{R}^{N-1}$ that is a ball centered at the origin, and functions φ_n, $\varphi\in W^{1+2s',p}(B',(-L,L))$ such that we have, eventually rotating and relabelling the coordinate axes,

$$E\cap C = \{(x',x_N)\in B'\times(-L,L) : x_N < \varphi(x')\},$$

$$F_n\cap C = \{(x',x_N)\in B'\times(-L,L) : x_N < \varphi_n(x')\}, \quad \text{and} \quad \mathcal{H}^{N-1}(A_n\cap C) \geq c' > 0 \tag{5.31}$$

for all n. Moreover, recalling Step 2 we also have

$$\varphi_n \to \varphi \quad \text{in} \quad C^1(\overline{B'}). \tag{5.32}$$

Denote by A'_n the projection of $A_n \cap C$ over B'. Then from (5.29) follows

$$\begin{aligned}\lambda_n \mathcal{H}^{N-1}(A'_n) - \int_{A'_n} g(x', \varphi_n(x'))\, \mathrm{d}\mathcal{H}^{N-1}(x') \\ + \lambda \mathcal{H}^{N-1}(B' \setminus A'_n) - \int_{B'\setminus A'_n} g(x', \varphi(x'))\, \mathrm{d}\mathcal{H}^{N-1}(x') + \omega_n\end{aligned} \tag{5.33}$$

with $\omega_n \to 0$, that is equivalent, proceeding as Lemma 5.5, to

$$\int_{B'} (-\Delta)^{(1+2s)/2} \varphi_n \, \mathrm{d}\mathcal{H}^{N-1}(x') + \text{l.o.t.} \tag{5.34}$$

Thanks to an integration by parts ([28, Theorem 1.5]) and by (5.32), we obtain

$$\begin{aligned}\int_{B'} (-\Delta)^{(1+2s)/2} \varphi_n \, \mathrm{d}\mathcal{H}^{N-1}(x') + \text{l.o.t.} &\to \int_{B'} (-\Delta)^{(1+2s)/2} \varphi \, \mathrm{d}\mathcal{H}^{N-1}(x') + \text{l.o.t.} \\ &= \lambda \mathcal{H}^{N-1}(B') - \int_{B'} g(x', \varphi(x')\, \mathrm{d}\mathcal{H}^{N-1}(x').\end{aligned} \tag{5.35}$$

So we arrive to

$$(\lambda_n - \lambda)\mathcal{H}^{N-1}(A'_n) \to 0.$$

As $\mathcal{H}^{N-1}(A'_n) \geq c'' > 0$ by (5.31), we obtain (5.30). Therefore, by Lemma 5.5 we conclude that $\psi_n \to 0$ in $W^{1+2s',p}(\partial E)$ for all $s' < s$ and $p > 1$. Thus, recalling (5.22), by Theorem 4.4 we have that $J_s(F_n) \geq J_s(E)$ for all n sufficiently large, that is a contradiction to (5.21). □

At this point we recall our main result, Theorem 1.1, and we prove it:

Theorem 5.8. *Let $s \in (0, 1/2)$, $E \subset \mathbb{R}^N$ a bounded open regular critical set for J_s with fixed volume $m > 0$ such that*

$$\partial^2 J_s(E)[\varphi] > 0 \quad \textit{for all } \varphi \in \tilde{H}^s(\partial E) \setminus \{0\}.$$

Then, there exist δ, $C_0 > 0$ such that

$$J_s(F) \geq J_s(E) + C_0 |E\Delta F|^2$$

for all $F \subset \mathbb{R}^N$, with $|F| = |E|$ and $|E\Delta F| \leq \delta$.

Proof. Arguing by contradiction, suppose that there exists a sequence $E_n \subset \mathbb{R}^N$, with $|E_n| = |E|$, such that $|E_n \Delta E| \to 0$ and

$$J_s(E_n) \leq J_s(E) + \frac{C_0}{4} |E_n \Delta E|^2, \tag{5.36}$$

where $C_0 > 0$ is the constant appearing in Theorem 4.4. We may suppose that $\chi_{E_n} \to \chi_E$ in $L^1(\mathbb{R}^N)$, then we replace the sequence E_n with a sequence of minimizers F_n of the following penalized functional:

$$J_s(F) + \Lambda_1 \sqrt{(|F\Delta E| - \varepsilon_n)^2 + \varepsilon_n} + \Lambda_2 \Big| |F| - |E| \Big|, \tag{5.37}$$

where $\varepsilon_n := |E_n \Delta E|$, and the constants Λ_1, $\Lambda_2 > 0$ will be chosen later. It is assumed up to a subsequence that $\chi_{F_n} \to \chi_{F_0}$ in L^1, where $F_0 \subset \mathbb{R}^N$ minimizes

$$J_s(F) + \Lambda_1 |F\Delta E| + \Lambda_2 \Big| |F| - |E| \Big|. \tag{5.38}$$

Using Lemma 5.4 it easy to check that if Λ_1 is sufficiently large (independently of Λ_2) E is the unique minimizer of (5.38). Thus, $F_0 = E$. Now we observe that if Λ_2 is sufficiently large, proceeding as in Theorem 5.7, Step 1, we have $|F_n| = |E|$ for all n. Moreover it can be proved that for all n the set F_n is a Λ-minimizer of the s-area for some $\Lambda > 0$ independent of n. Therefore, Theorem 5.1 yields that $F_n \to E$ in C^1. More precisely,

$$\partial F_n = \{x + \psi_n(x)\nu_E(x) : x \in \partial E\},$$

where $\psi_n \to 0$ in $C^1(\partial E)$.

We aim to show that $\psi_n \to 0$ in $W^{1+2s',p}(\partial E)$ for all $s' < s$ and $p > 1$. To this purpose, we claim that

$$\varepsilon_n^{-1}|F_n \Delta E| \to 1. \tag{5.39}$$

Indeed, if $\Big||F_n\Delta E| - \varepsilon_n\Big| \geq \sigma\varepsilon_n$ for some $\sigma > 0$ and for infinitely many n, since F_n minimizes the functional (5.37) and (5.36) holds, we get

$$\begin{aligned} J_s(F_n) + \Lambda_1\sqrt{\sigma^2\varepsilon_n^2 + \varepsilon_n} &\leq J_s(E_n) + \Lambda_1\sqrt{(|E_n\Delta E| - \varepsilon_n)^2 + \varepsilon_n} \\ &= J_s(E_n) + \Lambda_1\sqrt{\varepsilon_n} \leq J_s(E) + \frac{C_0}{4}|E_n\Delta E|^2 + \Lambda_1\sqrt{\varepsilon_n} \\ &= J_s(E) + \frac{C_0}{4}\varepsilon_n^2 + \Lambda_1\sqrt{\varepsilon_n} \leq J_s(F_n) + \frac{C_0}{4}\varepsilon_n^2 + \Lambda_1\sqrt{\varepsilon_n}, \end{aligned} \tag{5.40}$$

where in the last inequality we have used the local minimality of E with respect to L^∞ perturbations proved in Theorem 5.7. Recalling the above chain of inequalities we have that

$$\Lambda_1\sqrt{\sigma^2\varepsilon_n^2 + \varepsilon_n} \leq \frac{C_0}{4}\varepsilon_n^2 + \Lambda_1\sqrt{\varepsilon_n}.$$

The fact that this is impossible for n large proves the claim. Next, we set $f_n(t) := \sqrt{(t-\varepsilon_n)^2 + \varepsilon_n}$ and we observe that

$$|f_n'(t)| \leq 3\sqrt{\varepsilon_n} \quad \text{if} \quad |t - \varepsilon_n| \leq 3\varepsilon_n. \tag{5.41}$$

By (5.39) we have $\Big||F_n\Delta E| - \varepsilon_n\Big| \leq 2\varepsilon_n$ for n large enough. Thus, if $\Big||F\Delta E| - \varepsilon_n\Big| \leq \varepsilon_n$ and $|F| = |F_n|$, by the minimality of F_n and by (5.41), we get

$$\begin{aligned} J_s(F_n) &\leq J_s(F) + \Lambda_1\sqrt{(|F\Delta E| - \varepsilon_n)^2 + \varepsilon_n} - \Lambda_1\sqrt{(|F_n\Delta E| - \varepsilon_n)^2 + \varepsilon_n} \\ &\leq J_s(F) + 3\Lambda_1\sqrt{\varepsilon_n}\Big||F_n\Delta E| - |F\Delta E|\Big| \leq J_s(F) + 3\Lambda_1\sqrt{\varepsilon_n}|F_n\Delta F|. \end{aligned} \tag{5.42}$$

Let X be a smooth divergence-free vector-field in $\mathbb{R}^N$ and let $\Phi(\cdot, t)$ the corresponding volume-preserving flow. Thanks to the Coarea Formula, we can check that

$$|F_n\Delta\Phi(\cdot,t)(F_n)| = |t|\int_{\partial F_n} |X\cdot\nu|\,\mathrm{d}\mathcal{H}^{N-1} + o(t),$$

and, also by (5.42) we have

$$\begin{aligned} &J_s(\Phi(\cdot,t)(F_n)) - J_s(F_n) + 3\Lambda_1\sqrt{\varepsilon_n}|F_n\Delta\Phi(\cdot,t)(F_n)| \\ &\quad = J_s(\Phi(\cdot,t)(F_n)) - J_s(F_n) + 3\Lambda_1\sqrt{\varepsilon_n}|t|\int_{\partial F_n}|X\cdot\nu|\,\mathrm{d}\mathcal{H}^{N-1} + o(t) \geq 0 \end{aligned} \tag{5.43}$$

for t sufficiently small. Dividing the previous inequality by t, letting $t \to 0^+$ and $t \to 0^-$ and recalling (3.3), we conclude

$$\Big| \int_{\partial F_n} \Big(H_{s,\partial F_n} + g \Big)(X \cdot \nu) \, \mathrm{d}\mathcal{H}^{N-1} \Big| \leq 3\Lambda_1 \sqrt{\varepsilon_n} \int_{\partial F_n} |X \cdot \nu| \, \mathrm{d}\mathcal{H}^{N-1}.$$

Using a density argument similar to the one used in the proof of Corollary 3.8, we deduce

$$\Big| \int_{\partial F_n} \Big(H_{s,\partial F_n} + g \Big) \varphi \, \mathrm{d}\mathcal{H}^{N-1} \Big| \leq 3\Lambda_1 \sqrt{\varepsilon_n} \int_{\partial F_n} |\varphi| \, \mathrm{d}\mathcal{H}^{N-1}$$

forall $\varphi \in C^\infty(\partial F_n)$ with $\int_{\partial F_n} \varphi \, \mathrm{d}\mathcal{H}^{N-1} = 0$. This implies that

$$\|H_{s,\partial F_n} + g - \lambda_n\|_{L^\infty(\partial F_n)} \leq 3\Lambda_1 \sqrt{\varepsilon_n} \to 0.$$

We may now proceed as in Step 3 of the proof of Theorem 5.7 to deduce that

$$H_{s,\partial F_n}(\cdot + \psi_n(\cdot)\nu_E(\cdot)) \to H_{s,\partial E}(\cdot) \quad \text{in } L^\infty(\partial E)$$

and thus $\psi_n \to 0$ in $W^{1+2s',p}(\partial E)$ for all $s' < s$ and $p > 1$. Finally, since $J_s(F_n) \leq J_s(E_n)$ by the minimality of F_n and since (5.39) holds, we have that

$$J_s(F_n) \leq J_s(E_n) \leq J_s(E) + \frac{C_0}{4}|E_n \Delta E|^2 \leq J_s(E) + \frac{C_0}{2}|F_n \Delta E|^2$$

for n large. This is a contradiction to the minimality property proved in Theorem 4.4. □

Remark 5.9. The previous proof does not make use of the second variation. Indeed, we showed that any critical set E, for which the conclusion of Theorem 4.4 holds, satisfies also the conclusion of Theorem 1.1.

6. Link between L^1-minimizers of J_s and minimizers of F_ε

In this last section we want to prove a direct corollary to Theorem 1.1, but first we have to recall some results about the nonlocal Γ-convergence.

We consider a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with complement Ω^C, $s \in (0,1)$, and for $u \in H^s(\Omega, \mathbb{R})$ we define

$$\mathcal{K}(u, \Omega) := \frac{1}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y + \int_\Omega \int_{\Omega^C} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y.$$

Then we consider the following functionals $F_\varepsilon : X := \{u \in L^\infty(\mathbb{R}^N) : \|u\|_{L^\infty(\mathbb{R}^N)} \leq 1\} \to \mathbb{R} \cup \{+\infty\}$

$$F_\varepsilon(u) = \mathcal{K}(u, \Omega) + \frac{1}{\varepsilon^{2s}} \int_\Omega W(u) \, \mathrm{d}x + \int_\Omega g u \, \mathrm{d}x, \quad \text{if } s \in (0, 1/2), \tag{6.1}$$

$$F_\varepsilon(u) = \frac{1}{|\log \varepsilon|}\mathcal{K}(u, \Omega) + \frac{1}{|\varepsilon \log \varepsilon|} \int_\Omega W(u) \, \mathrm{d}x + \int_\Omega g u \, \mathrm{d}x, \quad \text{if } s = 1/2, \tag{6.2}$$

$$F_\varepsilon(u) = \varepsilon^{2s-1}\mathcal{K}(u, \Omega) + \frac{1}{\varepsilon} \int_\Omega W(u) \, \mathrm{d}x + \int_\Omega g u \, \mathrm{d}x, \quad \text{if } s \in (1/2, 1), \tag{6.3}$$

where $g \in C^1(\Omega)$ is a given function, $\varepsilon \in \mathbb{R}^+$ is a positive parameter and W is the well known double well potential, that is an even function such that:

$$\begin{aligned} &W : \mathbb{R} \to [0, +\infty), \qquad W \in C^2(\mathbb{R}, \mathbb{R}^+), \quad W(\pm 1) = 0, \\ &W > 0 \text{ in } (-1, 1), \qquad W'(\pm 1) = 0, \quad W''(\pm 1) > 0. \end{aligned} \tag{6.4}$$

Definition 6.1. The function u_ε is an L^1-local minimizer of F_ε if

$$F_\varepsilon(u_\varepsilon) \leq F_\varepsilon(v) \quad \text{whenever} \quad \|v - u_\varepsilon\|_{L^1(\Omega)} \leq \delta \tag{6.5}$$

for some $\delta > 0$.

We say that u_ε is an isolated L^1-minimizer of F_ε if the first inequality is strict whenever $0 < \|v - u_0\|_{L^1(\Omega)} \leq \delta$.

Our results are based on the relationship between F_ε and the "limiting " functional

$$F_0 : X \longrightarrow \mathbb{R} \cup \{+\infty\}$$

defined, when $s \in (0, 1/2)$, as

$$F_0(u) = \begin{cases} \mathcal{K}(u,\Omega) + \int_\Omega gu \,\mathrm{d}x & \text{if} \quad u|_\Omega = \chi_E - \chi_{E^C}, \text{ for some set } E \subset \Omega \\ +\infty \quad \text{otherwise,} \end{cases} \tag{6.6}$$

where E^C is the complement of the set E and, when $s \in [1/2, 1)$, as

$$F_0(u) = \begin{cases} c^* P_\Omega(E) + \int_\Omega gu \,\mathrm{d}x & \text{if} \quad u|_\Omega = \chi_E - \chi_{E^C}, \text{ for some set } E \subset \Omega \\ +\infty \quad \text{otherwise,} \end{cases} \tag{6.7}$$

where c^* is a constant depending on N and s (see [29, Theorem 4.2] for more details) and $P_\Omega(E)$ denotes the perimeter of E in Ω.

In particular, since $u \longmapsto \int_\Omega gu \,\mathrm{d}x$ is continuous in $L^1(\Omega)$, from [29, Theorem 1.2 and Theorem 1.3] we have that

a) if $F_\varepsilon(u_\varepsilon)$ is uniformly bounded for a sequence of $\varepsilon \to 0^+$, then there exists a convergent subsequence

$$u_\varepsilon \to u_* := \chi_E - \chi_{E^C} \quad \text{in} \quad L^1(\Omega); \tag{6.8}$$

b) if $s \in (0, 1)$, F_ε Γ-converges to F_0, i.e., for any $u \in X$,

  b1) for any u_ε converging to u in X,

$$F_0(u) \leq \liminf_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon), \tag{6.9}$$

  b2) if Ω is a Lipschitz domain, there exists u_ε converging to u in X such that

$$F_0(u) \geq \limsup_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon). \tag{6.10}$$

At this point we can prove the second fundamental result of our paper that is that for all ε sufficiently small, there exists a local minimizer of F_ε near each isolated local minimizer of the associated geometry problem:

Theorem 6.2. *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, and suppose that u_0 is an isolated L^1-local minimizer of F_0. Then there exists $\varepsilon_0 > 0$ and a family $\{u_\varepsilon\}_{\varepsilon<\varepsilon_0}$ such that*

$$u_\varepsilon \textit{ is an } L^1\textit{-local minimizer of } F_\varepsilon, \textit{ and} \tag{6.11}$$

$$\|u_\varepsilon - u_0\|_{L^1(\Omega)} \to 0 \quad \textit{as} \quad \varepsilon \to 0. \tag{6.12}$$

Proof. We will prove the theorem for F_ε relative to $s \in (0, 1/2)$ but the other cases are similar.

Since u_0 is isolated for F_0, we can choose $\delta > 0$ such that

$$F_0(u_0) < F_0(v) \quad \text{whenever} \quad 0 < \|v - u_0\|_{L^1(\Omega)} \leq \delta. \tag{6.13}$$

Let u_ε be any minimizer of F_ε on the ball

$$B = \{u : \|u - u_0\|_{L^1(\Omega)} \leq \delta\}.$$

The existence of this minimizer can be shown directly using the calculus of variations. Indeed, any minimizing sequence is bounded in $H^s(\Omega)$, so there is a subsequence converging in L^1 and almost everywhere to a point of B. It easy to check that the limit is a minimizer because of the convexity of $\dfrac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}$ and applying Fatou's lemma to $\varepsilon^{-2s} W(u)$. If ε is sufficiently small, the approximations to u_0 given by (6.9) and (6.10) lie in B, thus we have

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \leq \liminf_{\varepsilon \to 0} F_\varepsilon(u_0) = F_0(u_0). \tag{6.14}$$

We claim that for all sufficiently small ε, u_ε lies in the interior of B, so (6.11) follows, since u_ε are L^1-local minimizers of F_ε. Arguing by contradiction, assume that for some sequence $\varepsilon_j \to 0$,

$$\|u_{\varepsilon_j} - u_0\|_{L^1(\Omega)} = \delta. \tag{6.15}$$

By (6.8) there is a subsequence, still denoted ε_j such that

$$u_{\varepsilon_j} \to u_* \quad \text{in } L^1(\Omega).$$

Then

$$\|u_* - u_0\|_{L^1(\Omega)} = \delta$$

by passing to the limit in (6.15), and

$$F_0(u_*) \leq F_0(u_0)$$

by (6.9) and (6.14). This is a contradiction to (6.13). Therefore the claim is established, and (6.11) is proved.

The proof of (6.12) is almost the same. If it were to fail, then for some $\gamma > 0$ and some sequence $\varepsilon_j \to 0$ we would have

$$\gamma \leq \|u_{\varepsilon_j} - u_0\|_{L^1(\Omega)} \leq \delta.$$

We pass to the limit along an L^1-convergent sequence and we obtain a u_* such that

$$\delta \geq \|u_* - u_0\|_{L^1(\Omega)} \geq \gamma \quad \text{and} \quad F_0(u_*) \leq F_0(u_0).$$

As before, this is an absurd because u_0 is an isolated L^1 minimizer for F_0. Therefore (6.12) holds, and the proof is complete. □

Now we denote with $d_{L^1}(v, S)$ the L^1 distance between a function $v \in L^1(\mathbb{R}^N)$ and a set $S \subset L^1(\mathbb{R}^N)$, i.e.

$$d_{L^1}(v, S) := \inf_{u \in S} \|v - u\|_{L^1(\mathbb{R}^N)}$$

and, following [13, Proposition 3.2], we state a slight modification of this theorem:

Proposition 6.3. *Let $S \subset L^1(\mathbb{R}^N)$ be a set of locally minimizing critical points of F_0 in the sense that there exist positive numbers M and δ such that for all $u \in S$ one has*

$$F_0(v) > F_0(u) = M \quad \text{whenever} \quad 0 < d_{L^1}(v, S) < \delta.$$

Suppose also that S is compact in $L^1(\mathbb{R}^N)$; that is, for every sequence $\{u_j\}_{j\in\mathbb{N}} \subset S$ assume there exists a subsequence $\{u_{j_k}\}_{j,k\in\mathbb{N}}$ converging in L^1 to a limit $u \in S$. Then there exists an $\varepsilon_0 > 0$ and, for all $\varepsilon < \varepsilon_0$, a family of L^1-local minimizers $\{u_\varepsilon\}_{\varepsilon<\varepsilon_0}$ of F_ε such that $d(u_\varepsilon, S) \to 0$ as $\varepsilon \to 0$. Moreover, for any sequence $\varepsilon_j \to 0$, there exists a subsequence $\{\varepsilon_{j_k}\}_{j,k\in\mathbb{N}}$ and an element u of S such that $u_{\varepsilon_{j_k}} \to u$ in L^1.

Proof. The proof is similar to that of Theorem 6.2 replacing L^1-norm with L^1-distance from the set S. □

Finally we can prove the last important result of this work that is a link between Theorem 1.1 and Theorem 1.2:

Corollary 6.4. *Suppose $s \in (0, 1/2)$. Let E be a regular critical set for the functional* (3.1) *with positive second variation and $u = \chi_E$. Then there exist $\varepsilon_0 > 0$ and a family $\{u_\varepsilon\}_{\varepsilon<\varepsilon_0}$ of local minimizers of F_ε such that $u_\varepsilon \to u$ in $L^1(\mathbb{R}^N)$ as $\varepsilon \to 0$.*

Proof. Since the well-known Savin-Valdinoci result holds (see [29, Theorem 1.4]), and the prescribed curvature term is continuous with respect to the L^1 convergence of u, we have that F_ε relative to $s \in (0, 1/2)$ Γ-converges to the corresponding F_0. So, it is easy to check that

$$P_s(E, \Omega) + \int_E g \,\mathrm{d}x = F_0(\chi_E),$$

hence the conclusion follows from the L^1-local minimality of E obtained from Theorem 1.1, arguing as in the proof of Proposition 6.3. □

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Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy
E-mail address: dayana.pagliardini@sns.it