Convergence of an algorithm for anisotropic mean curvature motion

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Abstract

We give a simple proof of convergence of the anisotropic variant of a well-known algorithm for mean curvature motion, introduced in 1992 by Merriman, Bence and Osher. The algorithm consists in alternating the resolution of the (anisotropic) heat equation, with initial datum the characteristic function of the evolving set, and a thresholding at level 1/2.

1 Introduction: the algorithm

More than ten years ago, Merriman, Bence and Osher [23] proposed the following algorithm for the computation of the motion by mean curvature of a surface. Given a closed set $E \subset \mathbb{R}^N$, they let $T_h E = \{u(\cdot,h) \geq 1/2\}$, where u solves the following heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \Delta u(x,t) & t>0,\ x \in \mathbb{R}^N , \\ u(\cdot,0) = \chi_E & (t=0)\,. \end{cases} \tag{1}$$

Then, they let $E_h(t) = T_h^{[t/h]}E$ (with $[t/h]$ the integer part of t/h), and conjecture that $\partial E_h(t)$ converges to $\partial E(t)$, as $h \to 0$, where $\partial E(t)$ is the (generalized) evolution by mean curvature starting from ∂E.

The proof of convergence of this scheme was established by Evans [15], Barles and Georgelin [2]. Other proofs were given by H. Ishii [19] and Cao [10], where the evolution in (1) was replaced by the convolution of χ_E with a more general symmetric kernel. This was generalized by H. Ishii, Pires and Souganidis [20] to the case of the convolution with an arbitrary kernel (with some growth assumptions). This approach was also studied by Ruuth and Merriman [26] (see also [25]). Vivier [31] and Leoni [22] have considered other generalizations with (1) replaced with a time and space dependent anisotropic heat equation, with a lower order term. The space-dependence is an additional difficulty and it is not clear how what we will present could be adapted to such situations; on the other hand, in the two latter papers,

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"only" the case of Riemannian anisotropies is considered, in contrast to what we will study here.

We propose here to study the generalization of this algorithm to the so-called *anisotropic* and *crystalline curvature motion*, as defined in [18, 30, 29, 28]. We follow the definition in [8]: we consider (ϕ, ϕ°) a pair of mutually polar convex one homogeneous functions in $\mathbb{R}^N$ (i.e., $\phi^\circ(\xi) = \sup_{\phi(\eta)\le 1} \xi\cdot\eta$, $\phi(\eta) = \sup_{\phi^\circ(\xi)\le 1}\xi\cdot\eta$, see [24]). These are assumed to be locally finite, and, to simplify, even. The pair (ϕ, ϕ°) will be referred as *the anisotropy*. The local finiteness implies that there is a constant $c > 1$ such that

$$c^{-1}|\eta| \le \phi(\eta) \le c|\eta| \quad\text{and}\quad c^{-1}|\xi| \le \phi^\circ(\xi) \le c|\xi|$$

for any η and ξ in $\mathbb{R}^N$. We refer to [7, 8] for the main properties of ϕ and ϕ°.

Being convex and 1-homogeneous, ϕ° (and ϕ) is also subadditive, so that the function $(x, y) \mapsto \phi(x-y)$ defines a distance, the "ϕ-distance". For $E \subset \mathbb{R}^N$ and $x \in \mathbb{R}^N$, we denote by $\mathrm{dist}^\phi(x,E) := \inf_{y\in E}\phi(x-y)$ the ϕ-distance of x to the set E, and by

$$d^\phi_E(x) \;:=\; \mathrm{dist}^\phi(x,E) \;-\; \mathrm{dist}^\phi(x,\mathbb{R}^N\setminus E)$$

the signed ϕ-distance to ∂E, negative in the interior of E and positive outside its closure. One easily checks that

$$|d^\phi_E(x) - d^\phi_E(y)| \;\le\; \phi(x-y) \;\le\; c|x-y|$$

for any $x, y \in \mathbb{R}^N$, so that d^ϕ_E is differentiable a.e. in $\mathbb{R}^N$. The former inequality shows moreover that $\nabla d^\phi_E(x)\cdot h \le \phi(h)$ for any $h \in \mathbb{R}^N$, if x is a point of differentiability: hence $\phi^\circ(\nabla d^\phi_E(x)) \le 1$. If ϕ and ϕ° are smooth, one shows quite easily that d^ϕ_E is differentiable at each point x which has a unique ϕ-projection $y \in \partial E$ (solving $\min_{y\in\partial E}\phi(x-y)$). In this case, $\nabla d^\phi_E(x)$ is given by $\nabla\phi((x-y)/d^\phi_E(x))$, so that $\phi^\circ(\nabla d^\phi_E(x)) = 1$. If ϕ, ϕ° are just Lipschitz-continuous, one still shows that $\phi^\circ(\nabla d^\phi_E(x)) = 1$ a.e. in $\mathbb{R}^N$, see [7, 8] for details.

A *Cahn-Hoffman* vector field n_ϕ is a vector field on ∂E such that $n_\phi(x) \in \partial\phi^\circ(\nu_E(x)) = \partial\phi^\circ(\nabla d^\phi_E(x))$ a.e. on ∂E, where $\partial\phi^\circ$ is the (zero-homogeneous) subgradient of ϕ° (see [24, 14]), and ν_E is the (Euclidean) exterior normal to ∂E. If such a field is given in a neigborhood of ∂E, then it is characterized by

$$\phi^\circ(n_\phi(x)) \;=\; 1 \quad\text{and}\quad n_\phi(x)\cdot\nabla d^\phi_E(x) = 1 \quad \text{a.e.}$$

This follows from Euler's identity, since ϕ° is 1-homogeneous. In this case, $\kappa_\phi = \mathrm{div}\, n_\phi$ is a *ϕ-curvature* of ∂E. The ϕ-curvature flow is then an evolution $E(t)$ such that at each time, the velocity of $\partial E(t)$ is given by

$$V \;=\; -\kappa_\phi\, n_\phi\,, \tag{2}$$

where n_ϕ is a Cahn-Hoffman vector field and κ_ϕ is the associated curvature. If ϕ, ϕ° are smooth (e.g., in $C^2(\Omega\setminus\{0\})$) then n_ϕ, κ_ϕ are uniquely defined, whereas if ϕ,

ϕ° are merely Lipschitz (when, for instance, the *Wulff shape* $\{\phi \le 1\}$ is a convex polytope), then n_ϕ can be nonunique and the anisotropy is called *crystalline* [30, 7].

As easily shown by formal asymptotic expansion, the natural anisotropic generalization of the Merriman-Bence-Osher algorithm is as follows. Given E a closed set with compact boundary in $\mathbb{R}^N$, we let $T_h(E) = \{x \,:\, u(x,h) \ge 1/2\}$ where $u(x,t)$ is the solution of

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) \;\in\; \operatorname{div}\Big(\phi^\circ(\nabla u)\partial\phi^\circ(\nabla u)\Big)(x,t) & t>0,\; x\in\mathbb{R}^N\,, \\ u(\cdot,0) \;=\; \chi_E & (t=0)\,. \end{cases} \tag{3}$$

The funtion $u(x,t)$ is well defined and unique by classical results on contraction semigroups [9]: if E is compact, it corresponds to the flow in $L^2(\mathbb{R}^N)$ of the subdifferential of the functional $u \mapsto \int_{\mathbb{R}^N} \phi^\circ(\nabla u)^2/2\,dx$ if $u \in H^1(\mathbb{R}^N)$, and $+\infty$ otherwise. On the other hand, if $\mathbb{R}^N \setminus E$ is compact, one defines u by simply letting $u = 1+v$ where v solves the same equation with initial data $\chi_E - 1$.

We are interested in the limit of the discrete evolutions $t \mapsto E_h(t) = T_h^{[t/h]}E$, as $h \to 0$. Our main result is a result of consistency with suitable "regular" evolutions: it states that if there exists a regular evolution starting from E in the sense of our Definition 2.1 (which is a variant of a definition first introduced in [7] and includes smooth evolutions when the anisotropy is smooth), then $E_h(t)$ converges to this evolution. This consistency result, together with the monotonicity of the scheme ($E \subseteq F \Rightarrow T_hE \subseteq T_hF$, as follows from the comparison principle for equation (3)), yields also convergence to all generalized solutions defined (in the smooth case) using barriers [5, 6] or, equivalently, viscosity solutions [12, 13, 3], as long as these are unique. Also, it yields the convergence of the scheme to crystalline evolutions, when the initial set is convex. Existence and uniquenesse of such (regular and generalized) evolutions is established, in the convex case, in [4].

Another important consequence of our consistency result is a comparision principle for the regular evolutions of Definition 2.1, which follows from the monotonicity of the scheme. It gives an alternative proof of uniqueness for the convex crystalline evolutions studied in [4] (the original proof is based on [7]).

We observe that the case of the evolution law (2) is not be covered by the anisotropic motions of Ishii, Pires and Souganidis [20], generated by convolution. The possible anisotropic laws that may be obtained by convolution generated motion have been studied, in 2D, by Ruuth and Merriman [26, 27].

Our evolution is also different from the evolutions considered by Leoni [22] (or Vivier [31]): in her paper, the heat equation (1) is replaced with an equation of the form $u_t = A(x,t) : D^2u + H(x,t,Du)$. The resulting surface motion is a variant of the Mean Curvature Motion, with a (x,t)-dependent velocity which is a function of a Riemannian curvature (depending on A) plus a lower order forcing term.

It would be interesting to prove a similar consistency result for the variational variant of (3), which is somehow simpler to solve numerically (in the truly nonlinear anisotropic cases): for $E \subset \mathbb{R}^N$ bounded, one would define $T_h E = \{u_h \geq 1/2\}$ where u_h is the solution of (with $\Omega \Supset E$ bounded or $\Omega = \mathbb{R}^N$)

$$\min_{u \in H^1(\Omega)} \int_\Omega \phi^\circ(\nabla u(x))^2 \;+\; \frac{1}{h}(u(x) - \chi_E(x))^2 \, dx \,. \tag{4}$$

Although it is likely that this variant produces the same evolution as the original scheme (it is true in the isotropic case, since u_h is given by the convolution of χ_E with a radially symmetric kernel), we could not extend our proof in all cases to this new scheme (in the smooth cases, a proof can be given, which is slightly more complicated than for the original algorithm).

Our proof follows the same idea as our recent proof of consistency [11] for (a generalization of) the variational algorithm of Almgren, Taylor and Wang [1]. However, we have just learned that K. Ishii [21] has recently given an optimal estimate on the rate of convergence of Merriman-Bence-Osher's algorithm, in the isotropic case, by means of a new proof of convergence which is very similar to the proof we give here.

2 The consistency result and some consequences

If $E \subset \mathbb{R}^N$ we say that E satisfies the interior rW_ϕ-condition if and only if for any $x \in \partial E$, there exists $y \in E$ with $\phi(x-y) = r$ and $\phi(x'-y) \geq r$ for any $x' \in \mathbb{R}^N \setminus E$. We say that E satisfies the exterior rW_ϕ-condition if $\mathbb{R}^N \setminus E$ satisfies the interior rW_ϕ-condition.

We will show a consistency result with regular evolutions of (2), in the sense of the following definition:

Definition 2.1 *We say that $t \mapsto E(t)$ is a rW_ϕ-regular ϕ-curvature flow on $[t_0, t_1]$, $t_0 < t_1$, if and only if*

(i.) for any $t \in [t_0, t_1]$, $E(t)$ satisfies the interior and exterior rW_ϕ-conditions;

(ii.) there exists a bounded and relatively open neighborhood A of $\bigcup_{t_0 \leq t \leq t_1} \partial E(t) \times \{t\}$ in $\mathbb{R}^N \times [t_0, t_1]$ such that $d(x,t) := d^\phi_{E(t)}(x)$ is Lipschitz in A;

(iii.) there exists a vector field $n : A \to \mathbb{R}^N$ with $n \in \partial\phi^\circ(\nabla d)$ a.e. in A, and $\operatorname{div} n \in L^\infty(A)$;

(iv.) there exists $\bar{c} > 0$ such that $|\partial d/\partial t - \operatorname{div} n| \leq \bar{c}|d|$ a.e. in A.

This definition, up to the additional requirement that $E(t)$ satisfies an interior and exterior rW_ϕ-condition, is due to Bellettini and Novaga [7, Def. 2.2].

Such evolutions are known to exist if ϕ, ϕ° and ∂E are smooth enough (for instance, in $C^{3,\alpha}(\mathbb{R}^N \setminus \{0\})$ [1]), or for any ϕ, ϕ°, when the initial set E is convex and satisfies an interior rW_ϕ-condition (exterior is always true in the case of convex sets) [4]. They also exist in the purely crystalline case, i.e., when both ϕ and ϕ° are piecewise linear, in dimension $N = 2$ [16, 17, 28] (see Section 4 for an example).

Our main theorem states that the anisotropic Merriman-Bence-Osher scheme is consistent with such evolutions.

Theorem 1 *Let E be a regular flow in the sense of Definition 2.1, on a time interval $[t_0, t_1]$. Then, for any t and τ with $t_0 \le t < t + \tau \le t_1$, $\partial T_h^{[\tau/h]} E(t)$ converges to $\partial E(t+\tau)$ in the Hausdorff sense, as $h \to 0$.*

The following corollary, also shown in [7], is obvious.

Corollary 2.2 *Let E, F be two flows in the sense of Definition 2.1, on $[t_0, t_1]$, and assume $E(t_0) \subseteq F(t_0)$. Then $E(t) \subseteq F(t)$ for all $t \in [t_0, t_1]$. In particular, if $E(t_0) = F(t_0)$, then $E(t) = F(t)$ for all $t \in [t_0, t_1]$.*

The next corollary follows, with a standard proof, from the monotonicity and consistency of the scheme.

Corollary 2.3 *Assume $E \subset \mathbb{R}^N$ is a closed set with compact boundary such that the generalized ϕ-curvature flow $E(t)$, starting from E, is uniquely defined on a time interval $[0, T)$ (e.g., ϕ, $\phi^\circ \in C^2(\mathbb{R}^N \setminus \{0\})$, and no fattening occurs [12]). Then $\partial T_h^{[t/h]} E(t) \to \partial E(t)$ in the Hausdorff sense for any $t < T$, as $h \to 0$. The same conclusion holds for any ϕ, ϕ° if E is convex, by the uniqueness result in [4].*

Let us observe that this result follows easily from Theorem 1 when evolutions according to Definition 2.1 are known to exist. If not (e.g., if ϕ, ϕ° are merely C^2) this is still true, however the proof relies on a comparison with appropriate strict super- and subsolutions, defined according to obvious modifications of Definition 2.1 (as in [11]).

Remark 2.4 In case ϕ, ϕ° are not even, Theorem 1 still holds, but (i) the signed distance to the interface $d^{\phi}_{E(t)}(x)$ must be defined, in Definition 2.1, in a non symmetric way, (ii) the term $\partial\phi^\circ(\nabla u)$ in equation (3) must be replaced with $-\partial\phi^\circ(-\nabla u)$ (since ∇u has a reverse orientation with respect to the outer normal to the set E).

3 Proof of Theorem 1

The proof of Theorem 1 is divided in several steps. The idea is to build appropriate sub- and super-solutions to equation (3), by means of the function $d(x,t)$, and to compare $T_h E(t)$ with $E(t+h)$.

These barriers will be built by means of the function $\gamma : \mathbb{R} \times [0, +\infty) \to [0, 1]$ that solves the following heat equation

$$\begin{cases} \dfrac{\partial \gamma}{\partial \tau}(\xi, \tau) = \dfrac{\partial^2 \gamma}{\partial \xi^2}(\xi, \tau)\,, & \xi \in \mathbb{R},\ \tau > 0\,, \\ \gamma(\xi, 0) = Y(\xi)\,, & \xi \in \mathbb{R},\ (\tau = 0)\,. \end{cases} \tag{5}$$

where $Y = \chi_{[0,+\infty)}$ is the Heavyside function. It is well known that γ is given by

$$\gamma(\xi,\tau) \;=\; \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\xi} e^{-\frac{s^2}{4\tau}}\,ds\,.$$

In particular, one readily sees that it is self-similar: indeed, the change of variables $s' = s/\sqrt{\tau}$ yields

$$\gamma(\xi,\tau) \;=\; \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\frac{\xi}{\sqrt{\tau}}} e^{-\frac{s'^2}{4}}\,ds' \;=\; \gamma\left(\frac{\xi}{\sqrt{\tau}},1\right) \;=:\; \gamma_1\left(\frac{\xi}{\sqrt{\tau}}\right)\,.$$

We first show the following (obvious) result.

Lemma 3.1 *For any $\varepsilon > 0$, there exists $\tau_0 > 0$ such that if $0 \le \tau \le \tau_0$, then $\gamma(\varepsilon,\tau) \ge 1-\tau$.*

Proof. We just need to observe that $\tau \mapsto \gamma(\varepsilon,\tau)$ is C^1 with derivative 0 at 0. This derivative is indeed given by $(-\varepsilon/\tau^{3/2})\gamma_1'(\varepsilon/\sqrt{\tau}) = (-\varepsilon/\tau^{3/2})\exp(-\varepsilon^2/(4\tau))$. There exists τ_0 such that it is in $[-1,0]$ for $\tau \le \tau_0$, hence $\gamma(\varepsilon,\tau) \ge \gamma(\varepsilon,0) - \tau$ if $\tau \in [0,\tau_0]$, which shows the lemma. $\square$

Let us now consider E, $r > 0$, $t_0 \le t_1$, A and the functions $d(x,t)$, $n(x,t)$, as in Definition 2.1. Possibly reducing r, we can assume that $\{|d| \le r\} \subset A$. Let us fix $t \in [t_0,t_1)$, $\delta \in [0,r/2]$ and let $F = \{d(\cdot,t) \le \delta\}$. Let u be the solution of

$$\begin{cases} \dfrac{\partial u}{\partial \tau}(x,\tau) \;\in\; \mathrm{div}\left(\phi^\circ(\nabla u)\partial\phi^\circ(\nabla u)\right)(x,\tau) & \tau > 0,\ x \in \mathbb{R}^N\,, \\ u(\cdot,0) \;=\; \chi_F \;=\; Y(-d(\cdot,t)+\delta) & (\tau = 0)\,. \end{cases} \tag{6}$$

We first show the following result.

Lemma 3.2 *For any $\varepsilon \in (0,r/2)$, there exists $\tau_0 > 0$ (independent of δ) such that $\tau \le \tau_0$ yields $u(x,\tau) \le \tau$ for any x such that $d(x,t) - \delta = \varepsilon$.*

Proof. Let us fix $x_0 \in \mathbb{R}^N \setminus F$ with $d(x_0,t) - \delta = \varepsilon$. Since $E(t)$ satisfies the exterior rW_ϕ-condition, the function $(d(\cdot,t) - \delta)$ is, outside F, equal to $\mathrm{dist}^\phi(\cdot,F)$. Hence, letting $W = \{x : \phi(x - x_0) < \varepsilon\}$, one sees that $W \cap F = \emptyset$. We deduce that $\chi_F \le 1 - \chi_W$ in $\mathbb{R}^N$, so that $u(\cdot,\tau) \le 1 - w(\cdot,\tau)$ where w is the solution of

$$\begin{cases} \dfrac{\partial w}{\partial \tau}(x,\tau) \;\in\; \mathrm{div}\left(\phi^\circ(\nabla w)\partial\phi^\circ(\nabla w)\right)(x,\tau) & \tau > 0,\ x \in \mathbb{R}^N\,, \\ w(\cdot,0) \;=\; \chi_W & (\tau = 0)\,. \end{cases}$$

This solution is explicitly given by $w(x,\tau) = U(\phi(x-x_0)/\varepsilon, \tau/\varepsilon^2)$ where $U(|x|,\tau) = \tilde{U}(x,\tau)$ and $\tilde{U}$ is the (radial) solution of the heat equation $\partial\tilde{U}/\partial t = \Delta\tilde{U}$ with initial datum χ_{B_1}, the characteristic function of the unit ball in $\mathbb{R}^N$. It is well-known that

$$\tilde{U}(x,\tau) = \frac{1}{\sqrt{4\pi\tau}^N}\int_{\{|y|\le 1\}} \exp\left(-\frac{|x-y|^2}{4\tau}\right)dy$$

so that

$$U(|x|,\tau) \;=\; \frac{1}{\sqrt{4\pi\tau}^N}\int_{\{|y|\le 1\}} \exp\left(-\frac{(|x|-y_1)^2 + \sum_{i=2}^N y_i^2}{4\tau}\right)\,dy\,.$$

Using arguments similar to the proof of the previous lemma (based on the fact that U is smooth near $(\xi,\tau)=0,0$ and $\partial U/\partial t(0,0)=0$), one sees that there exists $\tau_0>0$ such that if $\tau\le\tau_0$, $U(0,\tau)\ge 1-\varepsilon^2\tau$, hence $w(0,\tau)\ge 1-\tau$ if $\tau\le\tau_0'=\varepsilon^2\tau_0$. We deduce that $u(x_0,\tau)\le\tau$ if $\tau\le\tau_0'$, depending only on ε. This shows the lemma. $\square$

Let us fix $\varepsilon<r/4$ and let us look for a supersolution of (6) on a time interval $[0,h]$, h small, of the form

$$v(x,\tau) \;=\; \gamma\Big(-d(x,t+\tau)+\delta+\overline{c}\,\overline{\varepsilon}\tau,\tau\Big)+h\,,$$

in $B=\bigcup_{0\le\tau\le h}\{x\,:\,d(x,t)-\delta\le\varepsilon\,,d(x,t+\tau)-\delta\ge-\varepsilon\}\times\{\tau\}$, where the constant $\overline{\varepsilon}$ will be precised later on. We observe that since the speed of the motion is bounded at any time for τ small enough, if h is small enough (depending only on r,ε), B remains inside $\{(x,\tau)\in\mathbb{R}^N\times[0,h]\,:\,\delta-\varepsilon\le d(x,t+\tau)\le\delta+2\varepsilon\}$, and $(0,t)+B\subset A$.

At $\tau=0$, $v(x,0)=Y(-d(x,t)+\delta)+h$ is strictly larger than $\chi_F(x)=u(x,0)$. If $0\le\tau\le h$ and $d(x,t)-\delta=\varepsilon$, by Lemma 3.2 we have $u(x,\tau)\le\tau\le h\le v(x,\tau)$, provided h is small enough. If on the other hand, $d(x,t+\tau)-\delta=-\varepsilon$, then by Lemma 3.1, still for h small enough, $v(x,\tau)=\gamma(-d(x,t+\tau)+\delta+\overline{c}\,\overline{\varepsilon}\tau,\tau)+h\ge\gamma(\varepsilon,\tau)+h\ge 1-\tau+h$, hence $v(x,\tau)\ge 1\ge u(x,\tau)$. We find that $v\ge u$ on $\{(x,\tau)\in\partial B\,:\,\tau<h\}$, which is the parabolic boundary of B (and, in fact, our proof even shows that $v\ge u$ in a neighborhood of this boundary).

Hence, to get that v is a supersolution of (6) in B, one has to show that $\partial v/\partial\tau\ge\operatorname{div}Z$, for some field $Z\in\phi^\circ(\nabla v)\partial\phi^\circ(\nabla v)$, inside B.

One has, a.e. in B,

$$\begin{aligned}\frac{\partial v}{\partial\tau}(x,\tau) \;&=\; \left(-\frac{\partial d}{\partial t}(x,t+\tau)+\overline{c}\,\overline{\varepsilon}\right)\frac{\partial\gamma}{\partial\xi}(-d(x,t+\tau)+\delta+\overline{c}\,\overline{\varepsilon}\tau,\tau)\\ &\qquad+\;\frac{\partial\gamma}{\partial\tau}(-d(x,t+\tau)+\delta+\overline{c}\,\overline{\varepsilon}\tau,\tau)\,,\end{aligned}\tag{7}$$

whereas

$$\nabla v(x,\tau) \;=\; -\frac{\partial\gamma}{\partial\xi}(-d(x,t+\tau)+\delta+\overline{c}\,\overline{\varepsilon}\tau,\tau)\nabla d(x,t+\tau)\,.$$

Using the assumption that ϕ° is even, we see that $\phi^\circ(\nabla v)=\partial\gamma/\partial\xi$ (since $\phi^\circ(\nabla d)=1$ a.e. in $\mathbb{R}^N$) and that $\partial\phi^\circ(\nabla v)=-\partial\phi^\circ(\nabla d)$ (since $\partial\gamma/\partial\xi>0$ and $\partial\phi^\circ$ is 0-homogeneous, and odd). Let now

$$Z(x,\tau) \;=\; -\frac{\partial\gamma}{\partial\xi}(-d(x,t+\tau)+\delta+\overline{c}\,\overline{\varepsilon}\tau,\tau)\,n(x,t+\tau) \;=\; \phi^\circ(\nabla v(x,\tau))(-n(x,t+\tau)).$$

Since (by assumption) $n(x,t+\tau)\in\partial\phi^\circ(\nabla d(x,t+\tau))=-\partial\phi^\circ(\nabla v(x,\tau))$ for a.e. x in $\mathbb{R}^N$ and any $\tau\in(0,h)$, one has $Z(x,\tau)\in\phi^\circ(\nabla v)\partial\phi^\circ(\nabla v)(x,\tau)$. Being ϕ° 1-homogeneous, Euler's identity yields $\nabla d\cdot n=\phi^\circ(\nabla d)=1$ as soon as $n\in\partial\phi^\circ(\nabla d)$.

We deduce

$$\begin{aligned}\operatorname{div} Z(x,\tau) &= -\operatorname{div}\left[\frac{\partial\gamma}{\partial\xi}(-d(x,t+\tau)+\delta+\overline{c}\,\overline{\varepsilon}\tau,\tau)n(x,t+\tau)\right]\\ &= \frac{\partial^2\gamma}{\partial\xi^2}(-d(x,t+\tau)+\delta+\overline{c}\,\overline{\varepsilon}\tau,\tau)\\ &\quad - \frac{\partial\gamma}{\partial\xi}(-d(x,t+\tau)+\delta+\overline{c}\,\overline{\varepsilon}\tau,\tau)(\operatorname{div} n(x,t+\tau))\,, \quad (8)\end{aligned}$$

Since we have $\partial d/\partial t \leq \operatorname{div} n + \overline{c}|d|$ in B, we deduce using (7) and (8) that

$$\begin{aligned}\frac{\partial v}{\partial\tau}(x,\tau) \;\geq\; \operatorname{div} Z(x,\tau) \;-\; \frac{\partial^2\gamma}{\partial\xi^2}(-d(x,t+\tau)+\delta+\overline{c}\,\overline{\varepsilon}\tau,\tau)\\ +\,\overline{c}(\overline{\varepsilon}-|d(x,t+\tau)|)\frac{\partial\gamma}{\partial\xi}(-d(x,t+\tau)+\delta+\overline{c}\,\overline{\varepsilon}\tau,\tau) + \frac{\partial\gamma}{\partial\tau}(-d(x,t+\tau)+\delta+\overline{c}\,\overline{\varepsilon}\tau,\tau)\,.\end{aligned}$$

Now, γ satisfies the heat equation, so that if $|d(x,t+\tau)| \leq \overline{\varepsilon}$ a.e. in B, we get

$$\frac{\partial v}{\partial\tau}(x,\tau) \;\geq\; \operatorname{div} Z(x,\tau)\,. \quad (9)$$

We choose $\overline{\varepsilon} = \delta + 2\varepsilon$ so that $|d(x,t+\tau)| \leq \overline{\varepsilon}$ a.e. in B and (9) holds. By standard comparisons results on parabolic equations (see [9]), we deduce that $v(x,h) \geq u(x,h)$. In particular, we have shown that there exists $h_0 > 0$ (depending only on r,ε) such that if $h < h_0$,

$$\begin{aligned}T_hF &= \left\{u(\cdot,h) \geq \frac{1}{2}\right\} \;\subset\; \left\{v(\cdot,h) \geq \frac{1}{2}\right\}\\ &= \left\{x \in \mathbb{R}^N \;:\; d(x,t+h) \leq \delta + \overline{c}\,\overline{\varepsilon}h - [\gamma(\cdot,h)]^{-1}\left(\frac{1}{2} - h\right)\right\}\end{aligned}$$

Since $\gamma_1(0) = 1/2$, $\gamma_1'(0) = 1/(2\sqrt{\pi})$, we have $\gamma_1^{-1}(1/2-h) = -2\sqrt{\pi}h + o(h)$. Now, $\gamma(\xi,h) = \gamma_1(\xi/\sqrt{h})$, so that $[\gamma(\cdot,h)]^{-1} = \sqrt{h}\gamma_1^{-1}$. We find that $[\gamma(\cdot,h)]^{-1}(1/2-h) = (-2\sqrt{\pi}h + o(h))\sqrt{h}$. Hence, possibly reducing h_0, one gets that if $h < h_0$, then $[\gamma(\cdot,h)]^{-1}(1/2-h) \geq -4h^{3/2}$. Recalling that $\overline{\varepsilon} = \delta + 2\varepsilon$, we find that if $h < h_0$,

$$T_hF \;\subset\; \left\{x \in \mathbb{R}^N \;:\; d(x,t+h) \leq [\delta + (\overline{c}(\delta+2\varepsilon) + 4\sqrt{h})h]\right\}\,.$$

Now, we deduce that ($\varepsilon \in (0,r/4)$ being fixed) if $t \in [t_0,t_1)$, $h \leq h_0$ and $k \geq 1$ with $t + kh \leq t_1$, one has

$$T_h^k(E(t)) \;\subset\; \{x \in \mathbb{R}^N \;:\; d(x,t+kh) \leq \delta_k\}$$

with $\delta_0 = 0$ and $\delta_k = \delta_{k-1} + (\overline{c}(\delta_{k-1}+2\varepsilon) + 4\sqrt{h})h$, as long as $\delta_{k-1} \leq r/2$. By induction, we find

$$\delta_k \;=\; ((1+\overline{c}h)^k - 1)\left(2\varepsilon + \frac{4\sqrt{h}}{\overline{c}}\right)\,.$$

In particular, if $\tau > 0$ is fixed, with $t + \tau \leq t_1$, and $k = [\tau/h]$, we see that $\lim_{h\to 0}\delta_k = 2\varepsilon(e^{\overline{c}\tau}-1)$. If $\varepsilon < r/4$ is chosen small enough (less than $(r/4)/(e^{\overline{c}\tau}-1)$), we see that for $h > 0$ small enough, $\delta_{[\tau/h]} \leq r/2$.

We now recall that any sequence of sets in $\mathbb{R}^N$ with equibounded boudaries has a subsequence that converges in the Hausdorff sense to a closed set. If E' is the Hausdorff limit of a converging subsequence of $T_h^{[\tau/h]}E(t)$, as $h \to 0$, we deduce that $E' \subseteq \{d(\cdot, t+\tau) \le 2\varepsilon(e^{\overline{c}\tau} - 1)\}$. Since this must be true for all $\varepsilon > 0$ small enough, one sees that $E' \subseteq E(t+\tau)$. On the other hand, a symmetric argument (based on subsolutions of the equation) will yield that if $\mathbb{R}^N \setminus E''$ is the Hausdorff limit of a converging subsequence of $(\mathbb{R}^N \setminus T_h^{[\tau/h]}E(t))_{h>0}$, then $\mathbb{R}^N \setminus E'' \subseteq \overline{\mathbb{R}^N \setminus E(t+\tau)}$, that is, $\mathrm{int}(E(t+\tau)) \subseteq E''$. Without loss of generality, one can choose the same subsequence in both limits above: in this case, one can show that $E'' \subset E'$, and $E' \setminus E''$ is the Hausdorff limit of $\partial T_h^{[\tau/h]}E(t)$ (which might differ from $\partial E'$ or $\partial E''$). Since $\mathrm{int}(E(t+\tau)) \subseteq E'' \subset E' \subseteq E(t+\tau)$, we see that $E'' = \mathrm{int}(E(t+\tau))$, $E' = E(t+\tau)$, $E' \setminus E'' = \partial E(t+\tau)$, and by uniqueness of this Hausdorff limit we deduce Theorem 1. □

4 A numerical example

The algorithm is quite easy to implement, numerically. Of course, there is some difficulty in computing precisely the solution of (3) in strongly anisotropic or crystalline cases, especially when the subgradient $\partial\phi^\circ$ is multivalued. We experimented an implicit method, based on iterative resolutions of the variational problem (4). More precisely, we approximate $u(\cdot, h)$ with $w_n(x)$ where $h = nh'$, n is a fixed (small) integer, $w_0 = \chi_E$ and for $i = 0, \dots, n-1$, w_{i+1} solves (in a domain Ω "large" with respect to E)

$$\min_{w \in H^1(\Omega)} \int_\Omega \phi^\circ(\nabla w(x))^2 \; + \; \frac{1}{h'}(w(x) - w_i(x))^2 \, dx \, .$$

To solve this minimization problem in the crystalline case, we discretize (here, on a bidimensional finite differences grid) and solve the dual problem (see for instance [14])

$$\min_{\xi \in L^2(\Omega;\mathbb{R}^N)} \int_\Omega \phi(\xi(x))^2 \; + \; h'((w_i(x)/h') - \mathrm{div}\,\xi(x))^2 \, dx \, ,$$

using a conjugate-gradient method. Then, $w_{i+1} = w_i - h'\mathrm{div}\,\xi$. The thresholding at level 1/2 is replaced by a "soft thresholding" $w_n(x) \mapsto \min\{1, \max\{1/2 + \sigma(w_n(x) - 1/2), 0\}\}$, where σ is adapted to the spatial discretization step, in order to keep a precision slightly higher that the grid's. In the example shown in Figure 1, the Wulff shape $\{\phi \le 1\}$ is an hexagon.

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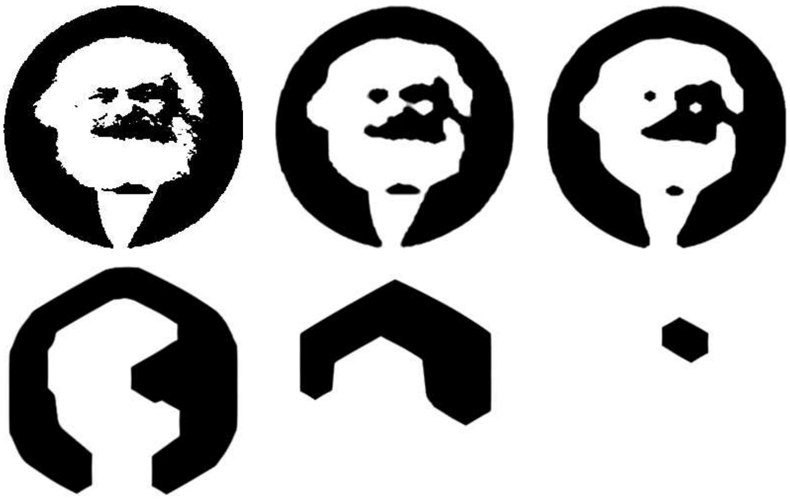

Figure 1: Evolutions at times $t = 0, 5, 25, 60, 400, 800$.

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