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On the structure of singular measures in the Euclidean space

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Abstract. We prove that for any singular measure μ on $\mathbb{R}^n$ it is possible to cover μ-almost every point with n families of Lipschitz slabs of arbitrarily small total width. More precisely, up to a rotation, for every $\delta > 0$ there are n countable families of 1-Lipschitz functions $\{f_i^1\}_{i\in\mathbb{N}}, \dots, \{f_i^n\}_{i\in\mathbb{N}}$, $f_i^j : \{x_j = 0\} \subset \mathbb{R}^n \to \mathbb{R}$, and n sequences of positive real numbers $\{\varepsilon_i^1\}_{i\in\mathbb{N}}, \dots, \{\varepsilon_i^n\}_{i\in\mathbb{N}}$ such that, denoting $\hat{x}_j$ the orthogonal projection of the point x onto $\{x_j = 0\}$ and

$$I_i^j := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : f_i^j(\hat{x}_j) - \varepsilon_i^j < x_j < f_i^j(\hat{x}_j) + \varepsilon_i^j\},$$

it holds $\sum_{i,j} \varepsilon_i^j \leq \delta$ and $\mu(\mathbb{R}^n \setminus \bigcup_{i,j} I_i^j) = 0$.
We apply this result to show that it is possible to approximate the identity with a sequence g_h of smooth equi-Lipschitz maps satisfying

$$\limsup_{h\to\infty} \int_{\mathbb{R}^n} \det(\nabla g_h) d\mu < \mu(\mathbb{R}^n).$$

From this, we deduce a simple proof of the fact that every top-dimensional Ambrosio-Kirchheim metric current in $\mathbb{R}^n$ is a Federer-Fleming flat chain.

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1. Introduction

Fix an orthonormal basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$. For $j = 1, \dots, n$, and for $x \in \mathbb{R}^n$, we denote $\hat{x}_j \in \mathbb{R}^{n-1}$ the orthogonal projection of x onto $\{x_j = 0\}$. Given a function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ and $\varepsilon > 0$ we consider the set

$$I_\varepsilon^j(f) := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : f(\hat{x}_j) - \varepsilon < x_j < f(\hat{x}_j) + \varepsilon\}$$

and we call it the *open slab* around f, of *width* ε, in direction e_j.

Given a family $\mathcal{F}$ of slabs, we denote $w(\mathcal{F})$ its *total width*, i.e. the sum of the widths of the corresponding slabs. For a fixed sequence $\{(f_i^j, \varepsilon_i^j)\}_{(i,j)\in\mathbb{N}\times\{0,\dots,n\}}$ with $f_i^j : \mathbb{R}^{n-1} \to \mathbb{R}^n$ and ε_i^j positive real numbers, we use the short notation I_i^j to denote the slab $I_{\varepsilon_i}^j(f_i^j)$.

Given a measure μ on $\mathbb{R}^n$ and a Borel function $\rho : \mathbb{R}^n \to \mathbb{R}^n$ we denote by $\rho_\sharp\mu$ the push forward of μ via ρ, i.e. the measure defined by

$$\rho_\sharp\mu(A) := \mu(\rho^{-1}(A)),$$

for every Borel set A.

The main result of this note is the following theorem.

1.1. Theorem. *Let* μ *be a finite Borel measure on* $\mathbb{R}^n$*,* $n \geq 2$*, which is singular with respect to the Lebesgue measure. Then there exists a rotation* $\rho : \mathbb{R}^n \to \mathbb{R}^n$ *with the following property. For every* $\delta > 0$ *there is a sequence* $\{(f_i^j, \varepsilon_i^j)\}_{(i,j)\in\mathbb{N}\times\{0,\dots,n\}}$ *where* $f_i^j : \mathbb{R}^{n-1} \to \mathbb{R}$ *are* 1*-Lipschitz functions and* ε_i^j *are positive real numbers such that the family of slabs* $\mathcal{F} := \{I_i^j\}_{(i,j)\in\mathbb{N}\times\{0,\dots,n\}}$ *has total width* $w(\mathcal{F}) \leq \delta$ *and*

$$\rho_\sharp\mu\left(\mathbb{R}^n \setminus \bigcup_{i,j} I_i^j\right) = 0.$$

1.2. Remark.

(i) For $n = 2$, Theorem 1.1 is a straightforward consequence of a covering result for nullsets, which will appear in [3]. Actually a weaker version of such covering result, proved in [1] and [2] (i.e. for compact nullsets), would also suffice to our purpose.

(ii) For $n > 2$, the theorem follows from a stronger result, announced by M. Csörnyei and P. Jones (see [15]). The proof we present here is considerably simpler. We remark that all the “ingredients” for the proof were already available in the literature, indeed the proof is achieved combining a corollary of the main result in [10] with some results obtained in [4] and some important ideas from [3], also used in [18].

(iii) In Lemma 4.1, we prove that the set of rotations ρ for which the conclusion of Theorem 1.1 holds, has full measure in $SO(n)$. In particular, one can chose a rotation which is arbitrarily close to the identity map, and then reparametrize the graphs of the Lipschitz functions f_i^j with respect to the tilted coordinates. Hence one can get rid of the rotation ρ in the statement, at the price of increasing the Lipschitz constant of an arbitrarily small quantity.

(iv) This result can be used (see [18, Chapter 4]) to prove the weak converse of Rademacher's theorem, namely that for every singular measure μ on $\mathbb{R}^n$ there exist a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ which is μ-a.e. non-differentiable. This was later improved in [19], where it is proved that it is possible to find a Lipschitz function which admits any pointwise prescribed blowup, provided it is linear along the decomposability bundle of μ (see §2.5), at every point except for a set of arbitrarily small measure μ. The converse of Rademacher's theorem has also important consequences in the study of Lipschitz differentiability metric measure spaces and of spaces with Ricci curvature bounded from below (see e.g. [7, 16, 9, 14]).

In §6, we apply Theorem 1.1 to obtain a simple proof of the case $k = n$ of the “flat chain conjecture” stated in [6, Section 11]. Namely we prove that for any Ambrosio-Kirchheim metric current T of dimension n in $\mathbb{R}^n$, the measure $\|T\|$ is absolutely continuous (see Theorem 6.1).

This result hes been proved in [10, Theorem 1.15], relying on results from [22]. Our proof is a direct consequence of the following theorem, of independent interest, which we obtain as a corollary of Theorem 1.1.

1.3. Theorem. *Let μ be a finite Borel measure on $\mathbb{R}^n$ and assume that μ is not absolutely continuous with respect to Lebesgue. Then there exists a sequence of continuously differentiable, equi-Lipschitz functions $\{g_h\}_{h\in\mathbb{N}}$ converging pointwise to the identity and such that*

$$\limsup_{h\to\infty}\int_{\mathbb{R}^n}\det(\nabla g_h)d\mu<\mu(\mathbb{R}^n).$$

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2. Notation and preliminaries

We begin this section by introducing some general notations about measures. Then we define the notion of cone-null set (see [3]) and we recall some properties of the decomposability bundle of a measure, defined in [4]. Lastly we recall a fact from [4]: a measure is supported on a C-null set, for some closed cone C, whenever its decomposability bundle intersects C only at the origin.

2.1. **General notation.** Through this note, sets and functions on $\mathbb{R}^n$ are assumed to be Borel measurable, and measures on $\mathbb{R}^n$ are positive, finite, Radon measures on the Borel σ-algebra, with the obvious exception of the Lebesgue measure $\mathcal{L}^n$ and the Hausdorff measures $\mathcal{H}^k, (k\leq n)$. We say that a measure μ on $\mathbb{R}^n$ is supported on the (Borel) set E if $\mu(\mathbb{R}^n\setminus E)=0$. We say that a measure μ is absolutely continuous with respect to a measure ν, and we write $\mu\ll\nu$, if $\mu(E)=0$ for every Borel set E with $\nu(E)=0$. We say that μ is singular with respect to ν if μ supported on a Borel set E with $\nu(E)=0$. If we do not specify what is the corresponding measure ν, we always implicitly refer to the Lebesgue measure. If μ is a measure and E is a Borel set, we denote $\mu\llcorner E$ the measure defined by

$$\mu\llcorner E(A)=\mu(A\cap E),\qquad\text{for every Borel set } A.$$

2.2. **Rectifiable sets.** Given $m=1,2,\dots$, a subset $E\subset\mathbb{R}^n$ is called m-*rectifiable* if $\mathcal{H}^m(E)<\infty$ and E can be covered, except for an $\mathcal{H}^m$-null subset, by countably many m-dimensional surfaces of class C^1. If E is m-rectifiable, then one can define for $\mathcal{H}^m$-a.e. $x\in E$ a notion of m-dimensional *approximate tangent space* to E. Such a tangent space will be denoted $\mathrm{Tan}(E,x)$ and it coincides with the classical tangent space if E is a piece of an m-surface of class C^1.

2.3. **Cone-null sets.** For $j=1,\dots,n$, we introduce the positive closed cones

$$C_j^+:=\{x=(x_1,\dots,x_n)\in\mathbb{R}^n:x_j\geq 2^{-\frac{1}{2}}|x|\}.$$

For every $j=1,\dots,n$, we denote also the cones $C_j:=C_j^+\cup(-C_j^+)$. Notice that any k-tuple $(k\leq n)$ of vectors lying in the interior of different cones is linearly independent.

Given a cone C_j we call C_j-*curve* any set of the form $G=\gamma(J)$, where J is a compact interval in $\mathbb{R}$ and $\gamma:J\to\mathbb{R}^n$ is Lipschitz and satisfies $\gamma'(s)\in C_j$ for a.e.

$s \in J$. It is important to observe that the condition of being a C_j-curve is closed under uniform convergence of the corresponding Lipschitz functions (when the curves are parametrized on the same interval J). Following [3], we say that a set E in $\mathbb{R}^n$ is *C_j-null* if

$$\mathcal{H}^1(E \cap G) = 0,$$

for every C_j-curve G, where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure.

One of the main tools that we need from [4] is the following lemma (see [4, Lemma 7.3]), which is a corollary of the general result of [21]. We refer the reader to [4, Section 2.3] for a formal definition on the notion of integral of a parametrized family of measures.

2.4. Lemma. *Let $j \in \{1, \dots, n\}$. For every measure μ on $\mathbb{R}^n$, one of the following (mutually incompatible) alternatives holds:*

(i) μ is supported on a Borel set E which is C_j-null;

(ii) there exists a non-trivial measure of the form $\mu' = \int_0^1 \mu_t \, dt$ where μ' is absolutely continuous w.r.t. μ, each μ_t is the restriction of $\mathcal{H}^1$ to some 1-rectifiable set E_t, and

$$\mathrm{Tan}(E_t, x) \subset C_j, \qquad \text{for } \mu_t\text{-a.e. } x \text{ and a.e. } t.$$

2.5. **Decomposability bundle of a measure.** In [4], to any Radon measure μ is assigned a Borel map $x \mapsto V(\mu, x)$, called the *decomposability bundle* of the measure μ, which associates to every point $x \in \mathbb{R}^n$ a vector subspace of $\mathbb{R}^n$. Roughly speaking, one constructs such vector subspace writing "pieces" of μ as an integral of a parametrized family of 1-dimensional rectifiable measures and collecting all the corresponding tangential directions at every point. We refer the reader to [4, Section 2.6] for the precise definition. Here we recall only a property which we strictly need in the present note. Even if here we state it as a lemma, indeed such property follows from the very definition of decomposability bundle.

2.6. Lemma. *Let μ be a measure on $\mathbb{R}^n$. Assume there exists a non-trivial measure of the form $\mu' = \int_0^1 \mu_t \, dt$ where μ' is absolutely continuous w.r.t. μ and each μ_t is the restriction of $\mathcal{H}^1$ to some 1-rectifiable set E_t. Then*

$$\mathrm{Tan}(E_t, x) \subset V(\mu, x), \qquad \text{for } \mu_t\text{-a.e. } x \text{ and a.e. } t.$$

Combining Lemma 2.4 and Lemma 2.6 we immediately get the following proposition.

2.7. Proposition. *Let $j \in \{1, \dots, n\}$ and let μ be a measure on $\mathbb{R}^n$. If $V(\mu, x) \cap C_j = \{0\}$ for μ-a.e. $x \in \mathbb{R}^n$, then μ is supported on a Borel set E, which is C_j-null.*

We remark that the reverse implication holds true as well, nevertheless we will not need this fact in the present note.

3. Structure of cone-null sets

One of the main ideas for the proof of Theorem 1.1 is borrowed from [1], where the main result is deduced from a geometric interpretation of the classical combinatorial result of [11], due to Dilworth (see also [12]).
In a partially ordered set $(\mathcal{S}, \leq)$, with the term *chain* we denote a totally ordered subset of $\mathcal{S}$. An *antichain* is a subset $S \subset \mathcal{S}$ such that for every $(s,t) \in S \times S$ with $s \leq t$ it holds $s = t$. The following theorem (see [20]) is a dual version of Dilworth's theorem. For the reader's convenience we include its short proof. We denote by $\sharp(S)$ the number of elements of the set S.

3.1. Theorem. *Let $(\mathcal{S}, \leq)$, be a partially ordered finite set. Then the maximal cardinality of a chain in $\mathcal{S}$ equals the smallest number of antichains into which $\mathcal{S}$ can be partitioned.*

Proof. For every $s \in \mathcal{S}$, let

$$l(s) := \sup\{\sharp(S) : \ S \text{ is a chain and } s \text{ is a maximal element of } S\}.$$

Let $L := \max\{l(s) : s \in \mathcal{S}\}$. Clearly, for every $j = 1, \ldots, L$, the set

$$A_j = \{s \in \mathcal{S} : l(s) = j\}$$

is an antichain and

$$\mathcal{S} = \bigcup_{j=1}^{L} A_j.$$

It is not possible to find a partition with a smaller number of antichains, since every two elements of the chain of maximal cardinality necessarily belong to different antichains. □

The next proposition follows from the previous theorem, considering on any finite subset of $\mathbb{R}^n$ the partial order induced by the closed cones C_j^+. More precisely, for fixed $j \in \{1, \ldots, n\}$ and $\mathcal{S}$ a finite subset of $\mathbb{R}^n$, we introduce the following partial order on $\mathcal{S}$:

$$s \leq t \quad \text{ if } s,t \in \mathcal{S} \text{ satisfy } t = s + v, \text{ for some } v \in C_j^+. \tag{3.1}$$

A crucial (although elementary) observation regarding such partial order is that every antichain A is the graph of a 1-Lipschitz function $f : \pi_j A \subset \{x_j = 0\} \to \mathbb{R}$, where π_j is the orthogonal projection onto $\{x_j = 0\}$.

3.2. Proposition. *Let E be a compact set in $\mathbb{R}^n$ which is C_j-null. Then for every $\delta > 0$, there are (finitely many) piecewise affine, 1-Lipschitz functions $f_1, \ldots, f_N : \mathbb{R}^{n-1} \to \mathbb{R}$, such that*

$$E \subset \bigcup_{i=1}^{N} I^j_{\delta/N}(f_i).$$

Proof. Without loss of generality, we may assume $E \subset [0,1]^n$. For every $k \in \mathbb{N}$, let G_k be the orthogonal grid obtained dividing each side of $[0,1]^n$ into k equal parts. Let E_k be the set of the centers of the cells of G_k which have non-empty intersection with the set E. Consider on G_k the partial order defined in (3.1).

Denote by ℓ_k the maximal cardinality of a chain in E_k. Our first aim is to prove that

$$\lim_{k\to\infty} \frac{\ell_k}{k} = 0, \tag{3.2}$$

in order to deduce from Theorem 3.1 that E_k can be covered with $o(k)$ antichains.

Assume by contradiction that there exist $l > 0$ such that, for infinitely many indexes k, there is a chain $C_k := (c_1^k, \dots, c_{m_k}^k)$ in E_k of cardinality at least lk. For $i = 1, \dots, m_k$, denote $t_i := c_i^k \cdot e_j$ and consider a function $g_k : \{0, t_1, \dots, t_{m_k}, 1\} \to [0,1]^n$, defined by

$$g_k(t_i) := c_i^k, \qquad \text{for every } i = 1, \dots, m_k$$

and

$$g_k(0) := c_1^k - t_1 e_j, \quad g_k(1) := c_{m_k}^k + (1 - t_{m_k}) e_j.$$

Extend g_k to a curve $\gamma_k : [0,1] \to [0,1]^n$ which is affine on $[0, t_1]$, on $[t_{m_k}, 1]$ and on $[t_i, t_{i+1}]$ for every $i = 1, \dots, m_{k-1}$. Clearly $\gamma_k([0,1])$ is a C_j-curve, and by construction, γ_k is $\sqrt{2}$-Lipschitz. Hence, up to a (non-relabelled) subsequence, γ_k converges to a Lipschitz function γ as $k \to \infty$ and $\gamma(I)$ is a C_j-curve, as observed in §2.3. We want to show that $\mathcal{H}^1(\gamma(I) \cap E) > 0$, which would be a contradiction, since E is C_j-null. For every k define a function $\phi_k : [0,1] \to \mathbb{R}$ by

$$\phi_k(t) := \operatorname{dist}(\gamma_k(t), E).$$

Since γ_k uniformly converges to γ, then ϕ_k uniformly converges to the continuous function

$$\phi := t \mapsto \operatorname{dist}(\gamma(t), E).$$

Observe that for every k and for every $t \in [0,1]$ such that $\gamma_k(t)$ belongs to a cell of G_k which contains one of the c_i^k it holds $\phi_k(t) \le k^{-1}\sqrt{n}$. The set $I_k \subset [0,1]$ of such parameters t has length $|I_k| \ge l$, by the contradiction assumption, hence we have

$$\phi_k \le k^{-1}\sqrt{n}, \qquad \text{on a set of length at least } l, \text{ for every } k.$$

Fix now $\varepsilon > 0$, and let k be such that $\|\phi_k - \phi\|_\infty \le \varepsilon$ and $k^{-1}\sqrt{n} \le \varepsilon$. Then by triangular inequality $\phi \le 2\varepsilon$ on a set of length at least l. This proves that $\phi \equiv 0$ on a set of length at least l, hence, since E is compact, we have the contradiction that the C_j-curve $\gamma(I)$ satisfies

$$\mathcal{H}^1(\gamma(I) \cap E) \ge l.$$

This proves (3.2). Now by Theorem 3.1, E_k can be covered by ℓ_k antichains. As we observed after (3.1), every antichain A is the graph of a 1-Lipschitz function h_A, defined on a discrete set contained in $\{x_j = 0\}$, with values in $[0,1]$. For every antichain A, let f_A be a (piecewise affine) 1-Lipschitz extension of h_A to

$\{x_j = 0\}$. The open slab $I^j_{2k^{-1}\sqrt{n}}(f_A)$ of width $2k^{-1}\sqrt{n}$ around f_A contains every cell intersected by the graph of f_A. Therefore E can be covered by ℓ_k slabs of total width $2\ell_k k^{-1}\sqrt{n}$, which, in view of (3.2), completes the proof of the proposition. □

4. Proof of Theorem 1.1

We begin with the following lemma. For $m \le n$, by $\gamma_{n,m}$ we denote the Haar measure on the Grasmannian $\mathrm{Gr}_{n,m}$ of (unoriented) m-planes in $\mathbb{R}^n$ (see [17, Section 2.1.4]) and by σ we denote the Haar measure on the special orthogonal group $SO(n)$. Moreover we denote

$$S := \bigcup_{j=1}^{n} C_j.$$

For $n \ge 3$ and for $j = 1, \dots, n$ we say that a hyperplane $v \in \mathrm{Gr}_{n,n-1}$ is *tangent* to C_j if $C_j \cap v$ is an $(n-2)$-plane. We say that v is tangent to S if it is tangent to C_j for some $j = 1, \dots, n$. Notice that if v is not tangent to C_j, but $v \cap C_j \ne \{0\}$, then v intersects the interior of C_j.

4.1. Lemma. *Let $n \ge 3$, let μ be a finite measure on $\mathbb{R}^n$, and let $V : \mathbb{R}^n \to \mathrm{Gr}_{n,n-1}$ be a Borel map. Then for σ-almost every rotation $\rho \in SO(n)$ it holds*

$$\rho(V(x)) \quad \text{is not tangent to } S\text{, for } \mu\text{-a.e. } x \in \mathbb{R}^n. \tag{4.1}$$

Proof. Firstly we observe that $\gamma_{n,n-1}$-almost every $v \in \mathrm{Gr}_{n,n-1}$, is not tangent to S. Indeed for every $j \in \{1, \dots, n\}$ the set of $v \in \mathrm{Gr}_{n,n-1}$ which are tangent to C_j has $\gamma_{n,n-1}$-measure zero. In particular for every $v \in \mathrm{Gr}_{n,n-1}$, $\rho(v)$ is not tangent to C_j, for σ-a.e. $\rho \in SO(n)$. Hence, $\rho(v)$ is not tangent to S, for σ-a.e. $\rho \in SO(n)$.

Now, denote by $f(x,\rho) : \mathbb{R}^n \times SO(n) \to \{0,1\}$ the Borel function

$$f(x,\rho) := \begin{cases} 1 & \text{if } \rho(V(x)) \text{ is tangent to } S, \\ 0 & \text{otherwise.} \end{cases}$$

By Fubini's theorem

$$\int_{x\in\mathbb{R}^n}\int_{\rho\in SO(n)} f(x,\rho)\, d\sigma(\rho)\, d\mu(x) = \int_{\rho\in SO(n)}\int_{x\in\mathbb{R}^n} f(x,\rho)\, d\mu(x)\, d\sigma(\rho).$$

The inner integral in the LHS being zero for every x, implies that the inner integral in the RHS is zero for σ-a.e. ρ, which proves the lemma. □

Proof of Theorem 1.1. Let $x \mapsto V(\mu, x)$ be the decomposability bundle of the measure μ. Since μ is singular, by [10, Corollary 1.12] and [4, Corollary 6.5] it holds

$$V(\mu, x) \ne \mathbb{R}^n, \quad \text{for } \mu\text{-a.e. } x. \tag{4.2}$$

Firstly we want to prove that, up to a rotation $\rho : \mathbb{R}^n \to \mathbb{R}^n$, the set E of points $x \in \mathbb{R}^n$ such that $V(\mu, x)$ has non trivial intersection with every cone C_j, for $j = 1, \dots, n$, has measure $\mu(E) = 0$. This is trivial for $n = 2$, because $C_1 \cap C_2$ is just the union of 2 lines. Let then $n \geq 3$.

By (4.2) we can find a Borel measurable map $V' : \mathbb{R}^n \to \mathrm{Gr}_{n,n-1}$, such that

$$V(\mu, x) \subset V'(x), \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n.$$

Let ρ be any rotation satisfying (4.1), where we applied Lemma 4.1 to the map $V := V'$. For the sake of simplicity we will assume that ρ is the identity map.

Since for μ-a.e. x, $V'(x)$ is not tangent to S, then by definition of E, for μ-a.e. $x \in E$, $V'(x)$ must have non-trivial intersection with the interior of every C_j, for $j = 1, \dots, n$. Then $\mu(E) = 0$, because $\dim(V'(x)) = n - 1$ for every $x \in \mathbb{R}^n$, whereas, as observed in §2.3, vectors in the interior of different cones are linear independent.

For $j = 1, \dots, n$, denote $\mu_j := \mu \llcorner E_j$, where

$$E_j := \{x \in \mathbb{R}^n : V(\mu, x) \cap C_j = \{0\} \text{ and } V(\mu, x) \cap C_k \neq \{0\} \text{ for } k < j\}. \tag{4.3}$$

Observe that the union over j of the sets E_j covers $\mathbb{R}^n \setminus E$, hence, by the previous discussion, it covers μ-a.e. point of $\mathbb{R}^n$.

Since by definition of E_j, for μ_j-a.e. x it holds $V(\mu_j, x) \cap C_j = \{0\}$, then by Proposition 2.7, μ_j is supported on a C_j-null Borel set F_j.

The conclusion then follows by decomposing each set F_j as the union of a μ-negligible set and a countable union of compact C_j-null sets $\{K_i^j\}_{i \in \mathbb{N}}$ (clearly the property of being C_j-null is preserved by subsets) and applying Proposition 3.2 to each compact set K_i^j, choosing the parameter δ_i^j in the proposition so that $\sum_{i,j} \delta_i^j \leq \delta$. $\square$

5. Covering with disjoint slabs

In some circumstances it could be important that the slabs I_i^j in $\mathcal{F}$ of Theorem 1.1, corresponding to the same superscript j, are disjoint. Moreover it is also possible to require that the corresponding functions f_i^j are of class C^1, slightly increasing their Lipschitz constant. We state this result in the following corollary. The complete proof can be found in [18, Corollary 4.1.3] and it is obtained modifying the slabs of Proposition 3.2 a posteriori. See e.g. the use made in [13] of this type of covering in the plane, for an interesting application where it is important to have disjoint slabs.

5.1. Corollary. *Let E be a compact set in $\mathbb{R}^n$ which is C_j-null and let μ be a finite Borel measure supported on E. Then for every $\varepsilon_0 > 0$, there exists $\varepsilon \leq \varepsilon_0$ and finitely many 1-Lipschitz functions $f_1, \dots, f_N : \mathbb{R}^{n-1} \to \mathbb{R}$ such that the slabs $I^j_{\varepsilon/N}(f_1), \dots, I^j_{\varepsilon/N}(f_N)$ are disjoint and satisfy*

$$\mu\left(E \setminus \bigcup_{i=1}^{N} I^j_{\varepsilon/N}(f_i)\right) = 0.$$

5.2. Remark. The interested reader is referred to [18, Proposition 4.1.15] for the details on how to make the slabs disjoint and at the same time requiring that the corresponding functions are of class C^1. The price to pay is a small increase in the Lipschitz constant.

Proof of Corollary 5.1. Step 1: ordering the slabs. Let $f_1, \dots, f_N$ be the functions obtained applying Proposition 3.2 to the set E, with $\delta := \varepsilon_0/2$. Firstly we define a new set of 1-Lipschitz functions $f_1^1, \dots, f_N^1$ such that

$$f_i^1 \le f_j^1, \text{ for } i < j \quad \text{and} \quad E \subset \bigcup_{i=1}^N I^j_{\delta/N}(f_i^1) = \bigcup_{i=1}^N I^j_{\delta/N}(f_i). \tag{5.1}$$

To get this, we define, for every $x \in \mathbb{R}^{n-1}$,

$$f_i^1(x) := f_{\sigma(i,x)}(x),$$

where $\sigma(i,x)$ are defined inductively as follows: let

$$\sigma(1,x) := \min\{j : f_j(x) \le f_k(x), \text{ for every } k = 1, \dots, N\},$$

and $I_1(x) := \{\sigma(1,x)\}$; moreover, for $i = 2, \dots, N$, let

$$\sigma(i,x) := \min\{j \notin I_{i-1}(x) : f_j(x) \le f_k(x), \text{ for every } k \notin I_{i-1}(x)\},$$

and

$$I_i(x) = \{\sigma(j,x) : j \le i\}.$$

Observe that the first property in (5.1) follows directly from the definition of $\sigma(i,x)$ and the second property follows from the simple observation that for every x it holds $I_N(x) = \{1, \dots, N\}$. Moreover, denoting $E^i_j := \{x : \sigma(i,x) = j\}$, for every $i = 1, \dots, N$ it holds

$$f_i^1 = \sum_{j=1}^n \chi_{E^i_j} f_j,$$

hence f_i^1 is 1-Lipschitz on each E^i_j. Moreover $f_k = f_i^1 = f_j$ on $\partial E^i_j \cap \partial E^i_k$. This suffices to prove that f_i^1 is 1-Lipschitz for every $i = 1, \dots, N$.

Step 2: separating the slabs. Fix $\varepsilon \in [\delta, 2\delta]$ to be chosen later. We define another set of 1-Lipschitz functions $f_1^2, \dots f_N^2$ such that

$$f_i^2 \le f_{i+1}^2 - 2\varepsilon/N, \quad \text{for } i = 1, \dots, N-1 \quad \text{and } \mu\left(E \setminus \bigcup_{i=1}^N I^j_{\varepsilon/N}(f_i^2)\right) = 0, \tag{5.2}$$

which completes the proof of the corollary. Again, we construct the functions inductively. Let $f_1^2 := f_1^1$ and for $i = 2, \dots, N$ let

$$f_i^2 := \max\{f_{i-1}^2 + 2\varepsilon/N, f_i^1\}.$$

The first property of (5.2) holds by definition. Regarding the second property, we observe that $\bigcup_{i=1}^N I^j_{\varepsilon/N}(f_i^2)$ covers the set

$$E \setminus \bigcup_{i=1}^{N} \operatorname{graph}(f_i^1 + \varepsilon/N).$$

To conclude, it is sufficient to choose $\varepsilon \in [\delta, 2\delta]$ satisfying

$$\mu\left(\bigcup_{i=1}^{N} \operatorname{graph}(f_i^1 + \varepsilon/N)\right) = 0.$$

□

6. Flat chain conjecture

In this section, we assume the reader to be familiar with the work [6]. We refer to [6] also for notation and definitions. As an application of Theorem 1.1, we provide a simple proof of the following theorem. We remark that the result has been proved already in [10, Theorem 1.15], using more technical results from [22].

6.1. Theorem. *Let $T \in \mathbf{M}_n(\mathbb{R}^n)$ be top-dimensional Ambrosio-Kirchheim metric current. Then $\|T\| \ll \mathcal{L}^n$.*

As it was observed in the proof of [6, Theorem 3.8], Theorem 6.1 is a direct consequence of Theorem 1.3. For the reader's convenience, we include the short proof of this fact at the end of the section. The existence of the maps $\{g_h\}_{h\in\mathbb{N}}$ in Theorem 1.3 can be obtained with the clever technique used in [4, Lemma 4.12], which on the other hand is a particular case of a result contained in [3]. Here we show a proof which we find slightly more geometrically transparent, using the slabs given by Theorem 1.1.

Proof of Theorem 1.3. Assume without loss of generality that μ is supported on $[0,1]^n$. We denote by μ_{ac} and μ_{sing} respectively the absolutely continuous and the singular measures given by the Radon Nikodým decomposition of μ (see [5, Theorem 2.22]). Remember that by assumptions $\mu_{sing} \neq 0$. Let ρ be the rotation given by Theorem 1.1 applied to the measure μ_{sing}. Up to a change of coordinates, we can assume that ρ is the identity map. For arbitrarily small $\delta > 0$ we will construct a smooth $2n$-Lipschitz map $g_\delta : \mathbb{R}^n \to \mathbb{R}^n$ such that, denoting Id: $\mathbb{R}^n \to \mathbb{R}^n$ the identity map, it holds $|g_\delta - \mathrm{Id}| \leq \delta$ and

$$\int_{\mathbb{R}^n} \det(\nabla g_\delta) d\mu_{sing} \leq \delta, \tag{6.1}$$

which clearly implies the theorem, by the well-known w^*-continuity property of determinants in the Sobolev space $W^{1,\infty}$ (see e.g. [8]).

Fix $\delta > 0$ and for $j = 1, \dots, n$, let E_j be the sets defined as in (4.3). Observe that, since the decomposability bundle of a measure ν which is absolutely continuous with respect to $\mathcal{L}^n$ coincides with $\mathbb{R}^n$, ν-almost everywhere, we could have used μ_{sing} in place of μ in (4.3). By [10, Corollary 1.12], [4, Corollary 6.5], and the previous discussion, it holds

$$\mu_{sing}(\mathbb{R}^n) = \sum_{j=1}^{n} \mu(E_j) > 0,$$

and by Proposition 2.7, for every j there is a C_j-null compact set $K_j \subset E_j$ such that

$$\sum_{j=1}^{n} \mu_{sing}(E_j \setminus K_j) \leq \frac{\delta}{2(2n)^n}. \tag{6.2}$$

For fixed j, let

$$\mathcal{F} := \{I_i := I^j_{\varepsilon/N}(f^j_i)\}_{i\in\{1,\ldots,N\}}$$

be the family of disjoint slabs given by Corollary 5.1 applied to the compact set K_j and the measure $\mu \llcorner K_j$, with $\varepsilon_0 := \delta/(2n)$. Denote by A_j the open set

$$A_j := \bigcup_{i=1}^{N} I^j_{\varepsilon/N}(f^j_i)$$

and by F_j a compact subset of $A_j \cap K_j$ such that

$$\sum_{j=1}^{n} \mu_{sing}(K_j \setminus F_j) \leq \frac{\delta}{2(2n)^n}. \tag{6.3}$$

Denote by η the positive quantity

$$\eta := \min_j \{\mathrm{dist}(F_j, \mathbb{R}^n \setminus A_j)\}.$$

We denote by $f_j : \mathbb{R}^n \to \mathbb{R}$ the function

$$f_j(z_1, \ldots, z_n) := z_j - \mathcal{H}_1(\{x \in A_j : \hat{x}_j = \hat{z}_j, x_j \leq z_j\}).$$

We claim that f_j has the following properties, for every j:

(i) $0 \leq z_j - f_j(z) \leq \delta/n$, for every $z \in \mathbb{R}^n$;
(ii) $f_j(z + te_j) = f_j(z) + t$, if the segment $[z, z + te_j]$ is contained in A_j;
(iii) f_j is 2-Lipschitz.

Property (i) follows from the fact that the total width of $\mathcal{F}$ is at most $\delta/(2n)$. Property (ii) follows directly from the definition of f_j. To check property (iii), observe firstly that, by definition $|f_j(z + te_j) - f_j(z)| \leq |t|$, for every z and for every t. To estimate the Lipschitz constant of f_j along $e_j^\perp$, fix $w \in e_j^\perp$ and $z \in \mathbb{R}^n$. Assume without loss of generality that $f_j(z + w) \geq f_j(z)$. Hence

$$\mathcal{H}_1(\{x \in A_j : \hat{x}_j = \hat{z}_j, x_j \leq z_j\}) \geq \mathcal{H}_1(\{x \in A_j : \hat{x}_j = \hat{z}_j + tw, x_j \leq z_j\}).$$

Let t be the smallest non-negative real number such that

$$\mathcal{H}_1(\{x \in A_j : \hat{x}_j = \hat{z}_j, x_j \leq z_j - t\}) = \mathcal{H}_1(\{x \in A_j : \hat{x}_j = \hat{z}_j + tw, x_j \leq z_j\}). \tag{6.4}$$

It holds $t \leq |w|$, because the slabs in $\mathcal{F}$ are disjoint and the corresponding functions are 1-Lipschitz. By (6.4) we have $f_j(z - te_j) = f(z + w) - t$. Since f_j is 1-Lipschitz in the direction e_j, the previous estimate and the fact that $t \leq |w|$

is sufficient to prove that f_j is 1-Lipschitz along $e_j^\perp$, which concludes the proof of (iii).

Let now ϕ be a radial mollifier with support on the ball $B(0,\eta)$ and consider the convolutions $g_j := f_j * \phi$. Eventually, define $g_\delta : \mathbb{R}^n \to \mathbb{R}^n$, by

$$g_\delta(z) := (g_1(z), \dots, g_n(z)).$$

Observe that g_δ is smooth and it has the following properties:

(i)' $|g_\delta - \mathrm{Id}| \le \delta$;
(ii)' $\nabla g_\delta(e_j) = 0$ on F_j;
(iii)' g_δ is $2n$-Lipschitz.

From the symmetry of ϕ with respect to the axis $\{x_j = 0\}$ and from (i) it follows that $0 \le z_j - g_j(z) \le \delta/n$, for every $z \in \mathbb{R}^n$ and for every $j = 1, \dots, n$, which implies (i)'. Property (ii) and the definition of η imply (ii)'. Property (iii)' follows from (iii).

Combining (ii)', (iii)' and the estimates (6.2) and (6.3), we get (6.1).

□

Proof of Theorem 6.1. We define a (signed) measure μ by

$$\mu(B) := T(\chi_B dx_1 \wedge \cdots \wedge dx_n), \qquad \text{for every } B \subset \mathbb{R}^n \text{ Borel},$$

and we let $\mu \llcorner A + \mu \llcorner (\mathbb{R}^n \setminus A)$ be the Hahn decomposition of μ. It is sufficient to prove that both positive measures $\mu \llcorner A$ and $-\mu \llcorner (\mathbb{R}^n \setminus A)$ are absolutely continuous. Assume by contradiction that one of the two measures is not absolutely continuous (without loss of generality we assume that such measure is $\mu \llcorner A$) and let g_h be the sequence obtained applying Theorem 1.3 to $\mu \llcorner A$. Then by the continuity property [6, Definition 3.1 (ii)] and by [6, (3.2)] it holds,

$$\mu \llcorner A(\mathbb{R}^n) = T(\chi_A dx_1 \wedge \cdots \wedge dx_n) = \lim_{h\to\infty} T(\chi_A d(g_h)_1 \wedge \cdots \wedge d(g_h)_n) =$$

$$\lim_{h\to\infty} T(\chi_A \det(\nabla g_h) dx_1 \wedge \cdots \wedge dx_n) \le \limsup_{h\to\infty} \int_A \det(\nabla g_h) d\mu < \mu \llcorner A(\mathbb{R}^n),$$

which is a contradiction. □

References

[1] Alberti, Giovanni; Csörnyei, Marianna; Preiss, David. Structure of null sets in the plane and applications. *European Congress of Mathematics. Proceedings of the 4th Congress (4ECM, Stockholm, June 27-July 2, 2004)*, pp. 3–22. Edited by A. Laptev. European Mathematical Society (EMS), Zürich 2005.

[2] Alberti, Giovanni; Csörnyei, Marianna; Preiss, David. Differentiability of Lipschitz functions, structure of null sets, and other problems. *Proceedings of the international congress of mathematicians (ICM 2010, Hyderabad, India, August 19-27, 2010)*. Volume 3 (invited lectures), pp. 1379–1394. Edited by R. Bhatia et al. Hindustan Book Agency, New Delhi, and World Scientific, Hackensack (New Jersey), 2010.

[3] Alberti, Giovanni; Csörnyei, Marianna; Preiss, David. Structure of null sets, differentiability of Lipschitz functions, and other problems. Paper in preparation.

[4] Alberti, Giovanni; Marchese, Andrea. On the differentiability of Lipschitz functions with respect to measures in the Euclidean space. *Geom. Funct. Anal.* 26 (2016), no. 1, 1–66.

[5] Ambrosio, Luigi; Fusco, Nicola; Pallara, Diego. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[6] Ambrosio, Luigi; Kirchheim, Bernd. Currents in metric spaces. *Acta Math.* 185 (2000), no. 1, 1–80.

[7] Bate, David. Structure of measures in Lipschitz differentiability spaces. *J. Amer. Math. Soc.* 28 (2015), no. 2, 421–482.

[8] Dacorogna, Bernard. *Direct Methods in the Calculus of Variations.* Applied Mathematical Sciences, 78. Springer-Verlag, Berlin, 1989.

[9] De Philippis, Guido; Marchese, Andrea; Rindler; Filip. On a conjecture of Cheeger arXiv:1607.02554.

[10] De Philippis, Guido; Rindler, Filip. On the structure of $\mathcal{A}$-free measures and applications. *Ann. of Math. (2)* 184 (2016), no. 3, 1017–1039.

[11] Dilworth, Robert P. A decomposition theorem for partially ordered sets. *Ann. of Math. (2)* 51 (1950), 161–166.

[12] Erdös, Paul; Szekeres, George. A combinatorial problem in geometry. *Compositio Math.* 2 (1935), 463–470.

[13] Fischer, Julian; Kneuss, Oliver. Bi-Sobolev Solutions to the Prescribed Jacobian Inequality in the Plane with L^p Data. arXiv: 1408.1587

[14] Gigli, Nicola; Pasqualetto, Enrico. Behaviour of the reference measure on RCD spaces under charts. arxiv: 1607.05188.

[15] Jones, Peter W. Product formulas for measures and applications to analysis and geometry. *Talk given at the conference Geometric and algebraic structures in mathematics, Stony Brook University*, May 2011. Video available at: http://www.math.sunysb.edu/Videos/dennisfest/

[16] Kell, Martin; Mondino, Andrea. On the volume measure of non-smooth spaces with ricci curvature bounded below. arXiv:1607.02036.

[17] Krantz, Steven G.; Parks, Harold R. *Geometric integration theory.* Cornerstones. Birkhäuser, Boston 2008.

[18] Marchese, Andrea. Two applications of the theory of currents (PhD thesis). http://cvgmt.sns.it/paper/2332/

[19] Marchese, Andrea; Schioppa, Andrea. Lipschitz functions with prescribed blowups at many points. arXiv: 1612.05280

[20] Mirsky, Leonid. A dual of Dilworth's decomposition theorem. *Amer. Math. Monthly* 78 (1971), 876–877.

[21] Rainwater, John. A note on the preceding paper. *Duke Math. J.* 36 (1969), 799–800.

[22] Schioppa, Andrea. Metric currents and Alberti representations. *J. Funct. Anal.* 271 (2016), no. 11, 3007–3081.

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