A Note Regarding Second-Order Γ-limits for the Cahn–Hilliard Functional

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Abstract

This note completely resolves the asymptotic development of order 2 by Γ-convergence of the mass-constrained Cahn–Hilliard functional, by showing that one of the critical assumptions of the authors' previous work [10] is unnecessary.

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1 Introduction

In the recent paper [10] we have solved in most cases a long standing open problem, namely, the asymptotic development by Γ-convergence of order 2 of the Modica–Mortola or Cahn–Hilliard functional (see [8, 13, 17])

$$F_\varepsilon(u) := \int_\Omega W(u) + \varepsilon^2 |\nabla u|^2 \, dx, \qquad u \in H^1(\Omega), \tag{1.1}$$

subject to the mass constraint

$$\int_\Omega u \, dx = m. \tag{1.2}$$

Here $\Omega \subset \mathbb{R}^n$, $2 \le n \le 7$, is an open, connected, bounded set with

$$\mathcal{L}^n(\Omega) = 1 \quad \text{and} \quad \partial\Omega \text{ is of class } C^{2,\alpha}, \quad \alpha \in (0,1], \tag{1.3}$$

and the double-well potential $W : \mathbb{R} \to [0,\infty)$ satisfies:

$$W \text{ is of class } C^2(\mathbb{R}\backslash\{a,b\}) \text{ and has precisely two zeros at } a < b, \tag{1.4}$$

$$\lim_{s\to a} \frac{W''(s)}{|s-a|^{q-1}} = \lim_{s\to b} \frac{W''(s)}{|s-b|^{q-1}} := \ell > 0, \quad q \in (0,1], \tag{1.5}$$

$$W' \text{ has exactly 3 zeros at } a < c < b, \quad W''(c) < 0, \tag{1.6}$$

$$\liminf_{|s|\to\infty} |W'(s)| > 0. \tag{1.7}$$

We assume that the mass m in (1.2) satisfies

$$a < m < b. \tag{1.8}$$

We recall that given a metric space X and a family of functions $\mathcal{F}_\varepsilon : X \to \overline{\mathbb{R}}$, $\varepsilon > 0$, an *asymptotic development* of *order* k

$$\mathcal{F}_\varepsilon = \mathcal{F}^{(0)} + \varepsilon \mathcal{F}^{(1)} + \cdots + \varepsilon^k \mathcal{F}^{(k)} + o(\varepsilon^k)$$

holds if there exist functions $\mathcal{F}^{(i)} : X \to \overline{\mathbb{R}}$, $i = 0, 1, \ldots, k$, such that the functions

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf_X \mathcal{F}^{(i-1)}}{\varepsilon} \tag{1.9}$$

are well-defined and

$$\mathcal{F}_\varepsilon^{(i)} \xrightarrow{\Gamma} \mathcal{F}^{(i)}, \tag{1.10}$$

where $\mathcal{F}_\varepsilon^{(0)} := \mathcal{F}_\varepsilon$ and $\overline{\mathbb{R}}$ is the extended real line (see [2]).

In our case $X := L^1(\Omega)$ and we define

$$\mathcal{F}_\varepsilon(u) := \begin{cases} F_\varepsilon(u) & \text{if } u \in H^1(\Omega) \text{ and (1.2) holds,} \\ \infty & \text{otherwise in } L^1(\Omega), \end{cases} \tag{1.11}$$

where F_ε is the functional in (1.1). It is well-known (see [3], [13], [17]) that, under appropriate assumptions on Ω and W, the Γ-limit $\mathcal{F}^{(1)}$ of order 1 (see (1.9) and (1.10)) of (1.11) is given by

$$\mathcal{F}^{(1)}(u) := \begin{cases} 2c_W \operatorname{P}(\{u = a\}; \Omega) & \text{if } u \in BV(\Omega; \{a, b\}) \text{ and (1.2) holds,} \\ \infty & \text{otherwise in } L^1(\Omega), \end{cases} \tag{1.12}$$

where $\operatorname{P}(\cdot; \Omega)$ is the perimeter in Ω (see [1, 5, 19]), a, b are the wells of W, and

$$c_W := \int_a^b W^{1/2}(s)\, ds. \tag{1.13}$$

Hence, $u \in BV(\Omega; \{a, b\})$ is a minimizer of the functional $\mathcal{F}^{(1)}$ in (1.12) if and only if the set $\{u = a\}$ is a solution of the classical *partition problem*

$$\mathcal{I}_\Omega(\mathfrak{v}) := \min\{P(E; \Omega) : E \subset \Omega, \mathcal{L}^n(E) = \mathfrak{v}\} \tag{1.14}$$

at the value $\mathfrak{v} = \mathfrak{v}_m$, where (see (1.3))

$$\mathfrak{v}_m := \frac{b - m}{b - a}. \tag{1.15}$$

When Ω is bounded and of class C^2, minimizers E of (1.14) exist, have constant generalized mean curvature κ_E, intersect the boundary of Ω orthogonally, and their singular set is empty if $n \le 7$, and has dimension at most $n - 8$ if $n \ge 8$ (see [6, 7, 11, 18]). Here and in what follows we use the convention that κ_E is the average of the principal curvatures taken with respect to the outward unit normal to ∂E.

Under the hypothesis that the *isoperimetric function* $\mathfrak{v} \mapsto \mathcal{I}_\Omega(\mathfrak{v})$ satisfies the Taylor expansion

$$\mathcal{I}_\Omega(\mathfrak{v}) = \mathcal{I}_\Omega(\mathfrak{v}_m) + \mathcal{I}_\Omega'(\mathfrak{v}_m)(\mathfrak{v} - \mathfrak{v}_m) + O(|\mathfrak{v} - \mathfrak{v}_m|^{1+\beta}) \tag{1.16}$$

for all $\mathfrak{v}$ close to $\mathfrak{v}_m$ and for some $\beta \in (0, 1]$, in [10] we proved the following theorems (see also [4]).

Theorem 1.1 ([10]). *Assume that Ω, m, W satisfy hypotheses* (1.3)-(1.8) *with $q = 1$ and that* (1.16) *holds. Then*

$$\mathcal{F}^{(2)}(u) = \frac{2c_W^2 (n-1)^2}{W''(a)(b-a)^2} \kappa_u^2 + 2(c_{\mathrm{sym}} + c_W \tau_u)(n-1)\kappa_u \operatorname{P}(\{u = a\}; \Omega)$$

if $u \in BV(\Omega; \{a, b\})$ is a minimizer of the functional $\mathcal{F}^{(1)}$ in (1.11) *and $\mathcal{F}^{(2)}(u) = \infty$ otherwise in $L^1(\Omega)$.*

Theorem 1.2 ([10]). *Assume that* Ω, m, W *satisfy hypotheses* (1.3)-(1.8) *with* $q \in (0,1)$ *and that* (1.16) *holds. Then*

$$\mathcal{F}^{(2)}(u) = 2(c_{\text{sym}} + c_W \tau_u)(n-1)\kappa_u \, \mathrm{P}(\{u=a\};\Omega)$$

if $u \in BV(\Omega;\{a,b\})$ *is a minimizer of the functional* $\mathcal{F}^{(1)}$ *and* $\mathcal{F}^{(2)}(u) = \infty$ *otherwise in* $L^1(\Omega)$.

Here κ_u is the constant mean curvature of the set $\{u=a\}$,

$$c_{\text{sym}} := \int_{\mathbb{R}} W(z(t))t \, dt, \tag{1.17}$$

where z is the solution to the Cauchy problem

$$\begin{cases} z'(t) = \sqrt{W(z(t))} & \text{for } t \in \mathbb{R}, \\ z(0) = c, & z(t) \in [a,b], \end{cases} \tag{1.18}$$

with c being the central zero of W' (see (1.6)), and $\tau_u \in \mathbb{R}$ is a constant such that

$$\mathrm{P}(\{u=a\};\Omega) \int_{\mathbb{R}} z(t-\tau_u) - \operatorname{sgn}_{a,b}(t) \, dt = \frac{2c_W(n-1)}{W''(a)(b-a)} \kappa_u$$

if $q = 1$ in (1.5) and

$$\int_{\mathbb{R}} z(t-\tau_u) - \operatorname{sgn}_{a,b}(t) \, dt = 0$$

if $q \in (0,1)$ in (1.5), where

$$\operatorname{sgn}_{a,b}(t) := \begin{cases} a & \text{if } t \le 0, \\ b & \text{if } t > 0. \end{cases}$$

The assumption (1.16) is known to hold at a.e. $\mathfrak{v}_m$, or, equivalently, for a.e. mass $m \in (a,b)$, since $\mathcal{I}_\Omega$ is semi-concave [15]. However, in the case that the isoperimetric function is differentiable at $\mathfrak{v}_m$ the mean curvature of the interface of minimizers is completely determined since (see Chapter 17 in[11])

$$\mathcal{I}'_\Omega(\mathfrak{v}_m) = (n-1)\kappa_E$$

for every minimizer E of (1.14) with $\mathfrak{v} = \mathfrak{v}_m$. Hence Theorems (1.1) and (1.2) *do not provide a selection criteria for minimizers.* Indeed, the case of two global minimizers of the partition problem (1.14) with different mean curvatures is excluded by (1.16).

The purpose of this note is to *remove the assumption that* $\mathcal{I}_\Omega$ *is regular at* $\mathfrak{v}_m$. Specifically, the theorem that we prove is the following:

Theorem 1.3. *Theorems* (1.1) *and* (1.2) *continue to hold without assuming* (1.16).

The lim sup portion of this theorem was already established in [10], see Remark 5.5 in the same. Thus this work focuses on proving the Γ-lim inf inequality.

Besides its intrinsic interest, Theorem 1.3 has important applications in the study of the speed of motion of the mass-preserving Allen–Cahn equation

$$\begin{cases} \partial_t u_\varepsilon = \varepsilon^2 \Delta u_\varepsilon - W'(u_\varepsilon) + \varepsilon \lambda_\varepsilon & \text{in } \Omega \times [0,\infty), \\ \frac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega \times [0,\infty), \\ u_\varepsilon = u_{0,\varepsilon} & \text{on } \Omega \times \{0\} \end{cases} \tag{1.19}$$

and of the Cahn–Hilliard equation

$$\begin{cases} \partial_t u_\varepsilon = -\Delta(\varepsilon^2 \Delta u_\varepsilon - W'(u_\varepsilon)) & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = \dfrac{\partial}{\partial \nu}(\varepsilon^2 \Delta u_\varepsilon - W'(u_\varepsilon)) = 0 & \text{on } \partial\Omega \times [0,\infty), \\ u_\varepsilon = u_{0,\varepsilon} & \text{on } \Omega \times \{0\} \end{cases} \tag{1.20}$$

in dimension $n \geq 2$.

In what follows we say that a measurable set $E_0 \subset \Omega$ is a *volume–constrained local perimeter minimizer* of $P(\cdot, \Omega)$ if there exists $\rho > 0$ such that

$$P(E_0; \Omega) = \inf \{P(E; \Omega) : E \subset \Omega \text{ Borel}, \, \mathcal{L}^n(E_0) = \mathcal{L}^n(E), \, \mathcal{L}^n(E_0 \ominus E) < \rho\},$$

where $\ominus$ denotes the symmetric difference of sets. We define

$$u_{E_0} := a\chi_{E_0} + b\chi_{\Omega \setminus E_0}. \tag{1.21}$$

The following theorem significantly improves Theorems 1.2, 1.4, 1.7, and 1.9 in [15]. In particular, the assumption that the local minimizer E_0 has positive second variation (see Theorem 1.9 in [15]) is no longer needed.

Theorem 1.4. *Assume that Ω, m, W satisfy hypotheses* (1.3)-(1.8) *with $q = 1$, and let E_0 be a volume–constrained local perimeter minimizer with $\mathcal{L}^n(E_0) = \mathfrak{v}_m$. Assume that $u_{0,\varepsilon} \in L^\infty(\Omega)$ satisfy*

$$\int_\Omega u_{0,\varepsilon} \, dx = m, \qquad u_{0,\varepsilon} \to u_{E_0} \text{ in } L^2(\Omega) \text{ as } \varepsilon \to 0^+$$

and

$$\mathcal{F}_\varepsilon(u_{0,\varepsilon}) \leq \varepsilon \mathcal{F}^{(1)}(u_{E_0}) + C\varepsilon^2$$

for some $C > 0$. Let u_ε be a solution to (1.19)*. Then, for any $M > 0$,*

$$\sup_{0 \leq t \leq M\varepsilon^{-1}} ||u_\varepsilon(t) - u_{E_0}||_{L^1(\Omega)} \to 0 \text{ as } \varepsilon \to 0^+.$$

The following theorem improves upon Theorem 1.4 in [15].

Theorem 1.5. *Assume that Ω, m, W satisfy hypotheses* (1.3)-(1.8) *with $q = 1$ and that there exists a constant $C_1 > 0$ so that*

$$|W'(s)| \leq C_1 |s|^p + C_1,$$

where $p = \frac{n}{n-2}$ for $n \geq 3$, and $p > 0$ for $n = 1, 2$. Let E_0 be a volume–constrained global perimeter minimizer with $\mathcal{L}^n(E_0) = \mathfrak{v}_m$. Assume that $u_{0,\varepsilon} \in L^2(\Omega)$ satisfy

$$\int_\Omega u_{0,\varepsilon} \, dx = m, \qquad u_{0,\varepsilon} \to u_{E_0} \text{ in } (H^1(\Omega))' \text{ as } \varepsilon \to 0^+$$

and

$$\mathcal{F}_\varepsilon(u_{0,\varepsilon}) \leq \mathcal{F}_0(u_{E_0})\varepsilon + C\varepsilon^2$$

for some $C > 0$. Let u_ε be a solution to (1.20)*. Then, for any $M > 0$,*

$$\sup_{0 \leq t \leq M\varepsilon^{-1}} ||u_\varepsilon - u_{E_0}||_{(H^1(\Omega))'} \to 0 \text{ as } \varepsilon \to 0^+.$$

2 Localized Isoperimetric Function

One of the central ideas in [10] was the development and use of a generalized Pólya–Szegő inequality to reduce the second-order Γ-lim inf of (1.1) to a one-dimensional problem. This generalized Pólya–Szegő inequality relied on comparing the perimeter of the level sets of arbitrary functions with values of $\mathcal{I}_\Omega$. On the one hand, this approach is simple and very general. On the other hand, it is clearly not sharp in our setting because the minimizers of (1.14) may be widely separated in L^1, while the transition layers we are considering are known to converge in L^1. Hence, the isoperimetric function may be too pessimistic in estimating the perimeter of the level sets of transition layers.

In light of this, following [15], we use instead a localized version of the isoperimetric function. Specifically, given a set E_0, and some $\delta > 0$, we define the *local isoperimetric function* of parameter δ about the set E_0 to be

$$\mathcal{I}_\Omega^{E_0,\delta}(\mathfrak{v}) := \inf\{P(E,\Omega) : E \subset \Omega \text{ Borel}, \mathcal{L}^n(E) = \mathfrak{v}, \alpha(E_0, E) \leq \delta\}, \tag{2.1}$$

where

$$\alpha(E_1, E_2) := \min\{\mathcal{L}^n(E_1 \setminus E_2), \mathcal{L}^n(E_2 \setminus E_1)\} \tag{2.2}$$

for any Borel sets $E_1, E_2 \subset \Omega$.

The following proposition, which connects the definition of $\mathcal{I}_\Omega^{E_0,\delta}$ with L^1 convergence, can be found in [15]. We present its proof for the convenience of the reader.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $E_0 \subset \Omega$ be a Borel set and let u_{E_0} be as in* (1.21). *Then*

$$\alpha(E_0, \{u \leq s\}) \leq \delta \tag{2.3}$$

for all $u \in L^1(\Omega)$ such that

$$\|u - u_{E_0}\|_{L^1} \leq (b-a)\delta, \tag{2.4}$$

and for every $s \in \mathbb{R}$, where the function α is given in (2.2).

Proof. Fix $\delta > 0$, and for $s \in \mathbb{R}$ define $F_s := \{x \in \Omega : u(x) \leq s\}$. If $a < s < b$, then by (2.4),

$$\begin{aligned}(b-a)\delta &\geq \int_{F_s \setminus E_0} |u - u_{E_0}|\, dx + \int_{E_0 \setminus F_s} |u - u_{E_0}| dx \\ &\geq (b-s)\mathcal{L}^n(F_s \setminus E_0) + (s-a)\mathcal{L}^n(E_0 \setminus F_s) \geq (b-a)\alpha(E_0, F_s),\end{aligned}$$

so that (2.3) is proved in this case. If $s \geq b$, again by (2.4),

$$(b-a)\delta \geq \int_{E_0 \setminus F_s} |u - u_{E_0}| dx \geq (s-a)\mathcal{L}^n(E_0 \setminus F_s) \geq (b-a)\alpha(E_0, F_s).$$

The case $s \leq a$ is analogous. □

By construction, we know that $\mathcal{I}_\Omega^{E_0,\delta} \geq \mathcal{I}_\Omega$. Furthermore, by BV compactness and lower-semicontinuity, and the fact that α is continuous in L^1, we have that $\mathcal{I}_\Omega^{E_0,\delta}$ is lower semi-continuous. The next proposition establishes a stronger type of regularity for $\mathcal{I}_\Omega^{E_0,\delta}$.

Proposition 2.2. *Assume that Ω satisfies* (1.3) *and let $E_0 \subset \Omega$ be a local volume-constrained perimeter minimizer, with $\mathcal{L}^n(E_0) = \mathfrak{v}_m$. Then for δ small enough there exists a neighborhood J_δ of $\mathfrak{v}_m$ so that $\mathcal{I}_\Omega^{E_0,\delta}$ is semi-concave on J_δ.*

Before proving this proposition, we state and prove a technical lemma. In what follows we say that an open set $U \subset \mathbb{R}^n$ has *piecewise C^2 boundary* if ∂U can be written as the union of finitely many connected $(n-1)$-dimensional manifolds with boundary of class C^2 up to the boundary, with pairwise disjoint relatively interiors.

Lemma 2.3. *Let $U = \Omega \backslash \overline{E_0}$ for some volume constrained perimeter minimizer E_0. Given $\tau > 0$, let*

$$U_\tau := \{x \in U : d(x, \mathbb{R}^n \backslash U) > \tau\}. \tag{2.5}$$

Then there exist a $A > 0$ and $C_1, C_2 > 0$ so that for all τ sufficiently small and all $\mathfrak{v} \in (C_1\tau, A)$,

$$\mathcal{I}_{U_\tau}(\mathfrak{v}) \geq C_2 \mathfrak{v}^{(n-1)/n}.$$

Proof. We remark that the the boundary of U will have piecewise C^2 with components that meet transversally. Furthermore the components of the boundary of U can be locally extended without intersecting U.

Step 1: We begin by constructing a C^1 vector field T which points into the domain U.

Let M_i, $i = 1, \ldots, m$, be the finitely many connected $(n-1)$-dimensional manifolds of class C^2 with boundary whose union gives ∂U. Extend each M_i in such a way that M_i is a subset of the boundary of an open set V_i of class C^2 with $V_i \cap U = \emptyset$. Set $F_i = \partial V_i$. Next we extend the normal vector field ν_{F_i} to a vector field $T_i \in C^1(\mathbb{R}^n; \mathbb{R}^n)$. If M_i and M_j intersect transversally, then for $x \in \partial M_i \cap \partial M_j$ we have $T_i(x) \cdot T_j(x) = \nu_{F_i}(x) \cdot \nu_{F_j}(x) = 0$ and thus by continuity we can find $\tilde{\rho} > 0$ such that $|T_i(x) \cdot T_j(x)| \leq \frac{1}{2m}$ for all x in a $\tilde{\rho}$-neighborhood (denoted by $U_{i,j}$) of $\partial M_i \cap \partial M_j$. By taking $\tilde{\rho}$ even smaller, if necessary, we can assume that the same $\tilde{\rho}$ works for all i and j such that M_i and M_j intersect transversally. Next, set

$$d_0 := \min_{i \neq j} d(M_i \backslash U_{i,j}, M_j \backslash U_{i,j}) > 0$$

and let $\rho := 1/2 \min(\tilde{\rho}, d_0) > 0$.

We then choose smooth cutoff functions φ_i which are valued 1 on M_i and 0 at distance $\rho/2$ from the same sets and consider the vector field $T := \sum_{i=1}^m \varphi_i T_i$. Note that $T \in C^1(\mathbb{R}^n; \mathbb{R}^n)$, with $\|T\|_\infty \leq C$ and $\|\nabla T\|_\infty \leq C$ for some constant $C > 0$.

We claim that

$$T(x) \cdot \nabla d_{V_i}(x) \geq 1/4 \tag{2.6}$$

for all points $x \in U$ in a ρ_0-neighborhood of F_i, where d_{V_i} is the signed distance to the set V_i enclosed by F_i. By Theorem 3 in [9] we have that d_{V_i} is a C^2 function in a neighborhood of F_i.

Suppose $x \in M_i$. Then $T_i(x) = \nu_{F_i}(x) = \nabla d_{V_i}(x)$, and so

$$T(x) \cdot \nabla d_{V_i}(x) = 1 + \sum_{j \neq i}^m \varphi_j(x) T_j(x) \cdot T_i(x). \tag{2.7}$$

If x is in ρ-neighborhood of $\partial M_i \cap \partial M_j$, then $T_j(x) \cdot T_i(x) \geq -\frac{1}{2m}$ otherwise $\varphi_j(x) = 0$. Thus, in both cases $T(x) \cdot \nabla d_{V_i}(x) \geq \frac{1}{2}$. By continuity of T and ∇d_{V_i}, the inequality (2.7) implies that (2.6) holds in a neighborhood of M_i.

Step 2: We consider the flow along T, meaning that for $x \in \mathbb{R}^n$ we take the initial value problem

$$\begin{cases} \frac{d\Psi}{dt}(t) = T(\Psi(t)), \\ \Psi(0) = x. \end{cases}$$

Since T is Lipschitz continuous, there exists a unique global solution Ψ defined for all $t \in \mathbb{R}$. To highlight the dependence on x we write $\Psi(\cdot, x)$ and we define $\Psi_t(x) := \Psi(t, x)$. Let $U^t := \Psi_t(U)$. By construction Ψ_t satisfies

$$(1 - Ct)|x - y| \leq |\Psi_t(x) - \Psi_t(y)| \leq (1 + Ct)|x - y|.$$

This implies that for any set $E \subset U^t$ of finite perimeter,

$$\begin{aligned} (1 - Ct)^n \mathcal{L}^n(E) \leq \mathcal{L}^n(\Psi_t^{-1}(E)) \leq (1 + Ct)^n \mathcal{L}^n(E), \\ (1 - Ct)^{n-1} P(E; U^t) \leq P(\Psi_t^{-1}(E); U) \leq (1 + Ct)^{n-1} P(E; U^t). \end{aligned} \tag{2.8}$$

We claim that $U_\tau \subset U^{c_3\tau}$, where $c_3 := 1/\|T\|_\infty$. To see this, let $y \in U_\tau$. For every $t \in \mathbb{R}$ we have $|\Psi_t(y) - y| \le |t|\|T\|_\infty$, and so

$$d(\Psi_t(y), \mathbb{R}^n \setminus U) \ge d(y, \mathbb{R}^n \setminus U) - |\Psi_t(y) - y| > \tau - |t|\|T\|_\infty \ge 0$$

provided $|t| \le \tau/\|T\|_\infty$. In turn, $\Psi_t(y) \in U$ for $|t| \le \tau/\|T\|_\infty$. Define $x_\tau := \Psi_{-c_3\tau}(y)$. and consider the function $\Psi(\cdot, x_\tau)$. Since the system of differential equations is autonomous and solutions are unique, we have that $\Psi_{c_3\tau}(x_\tau) = \Psi_{c_3\tau}(\Psi_{-c_3\tau}(y)) = y$, which shows that $y \in U^{c_3\tau} = \Psi_{c_3\tau}(U)$.

Next, we claim that $U^{c_3\tau} \subset U_{c_4\tau}$ for all τ sufficiently small, and for some constant c_4. Let $x \in U$ be in a $\rho_0/2$ neighborhood of M_i, where ρ_0 was given in Step 1. Since d_{V_i} is C^2, by the chain rule we may write

$$\begin{aligned} d_{V_i}(\Psi_t(x)) &= d_{V_i}(x) + \int_0^t \nabla d_{V_i}(\Psi_s(x)) \cdot T(\Psi_s(x))\, ds \\ &\ge d_{V_i}(x) + \frac{t}{4} \ge d(x, \mathbb{R}^n \backslash U) + \frac{t}{4}, \end{aligned}$$

where we have used (2.6), and where we have assumed that $t < \frac{\rho_0}{2\|T\|_\infty}$. As this is true for all i, and as $d(x, \mathbb{R}^n \backslash U) = \min_i d_{V_i}(x)$ for $x \in U$, we find that

$$d(\Psi_t(x), \mathbb{R}^n \backslash U) \ge d(x, \mathbb{R}^n \backslash U) + t/4$$

for x near ∂U and for t sufficiently small. This proves the claim for x close to the boundary, and for x far away from the boundary there is nothing to prove.

In summary, we know that $U_\tau \subset U^{c_3\tau} \subset U_{c_4\tau}$, as along as τ is sufficiently small, for c_3, c_4 independent of τ. These two inclusions, along with (2.8), imply that for any set E of finite perimeter we have that

$$P(E; U_\tau) \ge P(E; U^{c_3/c_4\tau})$$

and that $\mathcal{L}^n(U_\tau \backslash U^{c_3/c_4\tau}) \le c_5\tau$, with $c_5 > 0$ independent of τ.

Finally, let $E \subset U_\tau$ be a set of finite perimeter satisfying $\mathcal{L}^n(E) > 2c_5\tau$. By (2.8), the previous inequalities, and the isoperimetric inequality (which applies as U must be Lipschitz) we have that

$$\begin{aligned} P(E; U_\tau) &\ge P(E; U^{c_3/c_4\tau}) \\ &\ge C P(\Psi^{-1}_{c_3/c_4\tau}(E \cap U^{c_3/c_4\tau}); U) \\ &\ge C \left(\mathcal{L}^n(\Psi^{-1}_{c_3/c_4\tau}(E \cap U^{c_3/c_4\tau}))\right)^{(n-1)/n} \\ &\ge C \left(\mathcal{L}^n(E \cap U^{c_3/c_4\tau})\right)^{(n-1)/n} \\ &\ge C \left(\mathcal{L}^n(E) - c_5\tau\right)^{(n-1)/n} \ge C\mathcal{L}^n(E)^{(n-1)/n}. \end{aligned}$$

This completes the proof.

□

Now we prove Proposition 2.2.

Proof of Proposition 2.2. Let $E_{\hat{\mathfrak{v}}}$ be a minimizer of

$$\min\{P(E; \Omega) : E \subset \Omega \text{ Borel}, \mathcal{L}^n(E) = \hat{\mathfrak{v}}, \alpha(E_0, E) \le \delta\},$$

with $\hat{\mathfrak{v}} \in J_\delta = (\mathfrak{v}_m - \delta/2, \mathfrak{v}_m + \delta/2)$. Since ∂E_0 is regular and intersects the boundary of Ω orthogonally, we know that

$$\mathcal{I}_{\Omega \cap E_0}(\mathfrak{v}) \ge C\mathfrak{v}^{\frac{n-1}{n}}, \qquad \mathcal{I}_{\Omega \setminus E_0}(\mathfrak{v}) \ge C\mathfrak{v}^{\frac{n-1}{n}} \tag{2.9}$$

for all $\mathfrak{v}$ sufficiently small (see, e.g., Corollary 3 in Section 5.2.1 of [12]). We pick δ small enough that (2.9) holds for $\mathfrak{v} \in (0, 2\delta)$.

Next, we claim that we can construct a smooth function $\phi_{\hat{\mathfrak{v}}}$ defined on a neighborhood of $\hat{\mathfrak{v}}$ so that

$$\phi_{\hat{\mathfrak{v}}}(\hat{\mathfrak{v}}) = \mathcal{I}_\Omega^{E_0,\delta}(\hat{\mathfrak{v}}), \quad \phi_{\hat{\mathfrak{v}}}(\mathfrak{v}) \geq \mathcal{I}_\Omega^{E_0,\delta}(\mathfrak{v}), \quad \phi''_{\hat{\mathfrak{v}}} \leq C, \tag{2.10}$$

where C does not depend on $\hat{\mathfrak{v}}$, but may depend on δ.

To prove this claim, we consider two different cases. First, suppose that $\alpha(E_{\hat{\mathfrak{v}}}, E_0) < \delta$. Then by (2.1), $E_{\hat{\mathfrak{v}}}$ is actually a volume-constrained local perimeter minimizer, and hence we can prove (2.10) by using a normal perturbation and the fact that $\partial\Omega$ is smooth, see Lemma 4.3 in [15] for details.

Now suppose that $\alpha(E_{\hat{\mathfrak{v}}}, E_0) = \delta$. In view of (2.2) we may assume, without loss of generality, that $\alpha(E_{\hat{\mathfrak{v}}}, E_0) = \mathcal{L}^n(E_0 \backslash E_{\hat{\mathfrak{v}}})$ (the opposite case is analogous). Hence, we may locally perturb $E_{\hat{\mathfrak{v}}}$ inside the set $\Omega\backslash E_0$ without increasing the value of $\alpha(E_{\hat{\mathfrak{v}}}, E_0)$. In particular, by (2.1), $E_{\hat{\mathfrak{v}}}\backslash E_0$ is a local minimizer of the problem

$$\min\{P(E; \Omega\backslash E_0) : E \subset \Omega\backslash E_0 \text{ Borel}, \mathcal{L}^n(E) = \mathcal{L}^n(E_{\hat{\mathfrak{v}}}\backslash E_0)\}.$$

Hence, by [6], $\partial E_{\hat{\mathfrak{v}}} \cap (\Omega\backslash E_0)$ is analytic.

We note that

$$\begin{aligned}\delta - \mathcal{L}^n(E_{\hat{\mathfrak{v}}}\backslash E_0) &= \mathcal{L}^n(E_0\backslash E_{\hat{\mathfrak{v}}}) - \mathcal{L}^n(E_{\hat{\mathfrak{v}}}\backslash E_0) \\ &= \mathcal{L}^n(E_0) - \mathcal{L}^n(E_{\hat{\mathfrak{v}}}) = \mathfrak{v}_m - \hat{\mathfrak{v}} \in (-\delta/2, \delta/2).\end{aligned}$$

Hence, we know that $\mathcal{L}^n(E_{\hat{\mathfrak{v}}}\backslash E_0) \in [\delta, \frac{3\delta}{2}]$. Since E_0 is a local volume constrained perimeter minimizer by [6] and [7], its boundary is smooth inside Ω and intersects $\partial\Omega$ transversally. In particular, it may only have finitely many connected components, and hence by selecting δ sufficiently small we may assume that $\partial E_{\hat{\mathfrak{v}}} \cap (\Omega\backslash E_0)$ is non-empty.

Next, let $U := \Omega\backslash\overline{E_0}$. Let $\tilde{d} \in C^\infty(\mathbb{R}^n\backslash\partial U)$ be a regularized distance function from $\mathbb{R}^n\backslash U$, satisfying the properties

$$C_1 \leq \frac{\tilde{d}(x)}{d(x, \mathbb{R}^n\backslash U)} \leq C_2 \quad \text{for } x \in U, \quad \|\nabla\tilde{d}\|_\infty \leq C, \tag{2.11}$$

where $d(x, \mathbb{R}^n\backslash U)$ is the signed distance function. Such a regularized distance function, as well as the aforementioned properties, is constructed in [16].

Let $\phi_\tau : \mathbb{R} \to \mathbb{R}^+$ be a smooth function satisfying $\phi_\tau(s) = 0$ for all $s < \tau/2$, $\phi_\tau(s) = 1$ for all $s > \tau$, with ϕ_τ strictly increasing for $\tau/2 < s < \tau$, and $\|\phi'_\tau\|_\infty \leq \frac{C}{\tau}$ with τ to be chosen. We define $\Phi_\tau(x) := \phi_\tau(\tilde{d}(x))$.

Let $T \in C_c^\infty(U; \mathbb{R}^n)$ be an extension of the vector field $\Phi_\tau \nu_{\partial E_{\hat{\mathfrak{v}}}}$. Define a one parameter family of diffeomorphisms given by $f_t(x) = x + tT(x)$, where t is sufficiently small. Note that $f_t(x) = x$ for all $x \in \overline{E_0}$ and all t sufficiently small. Hence by (2.1) the sets $f_t(E_{\hat{\mathfrak{v}}})$ satisfy $P(f_t(E_{\hat{\mathfrak{v}}}); \Omega) \geq \mathcal{I}_\Omega^{E_0,\delta}(\mathcal{L}^n(f_t(E_{\hat{\mathfrak{v}}})))$ since $\alpha(E_{\hat{\mathfrak{v}}}, E_0) = \mathcal{L}^n(E_0\backslash E_{\hat{\mathfrak{v}}})$. Using the formulas in Chapter 17 of [11], there exists a function $\phi_{\hat{\mathfrak{v}}} = P(f_{t(\mathfrak{v})}(E_{\hat{\mathfrak{v}}}); \Omega)$ such that for all $\mathfrak{v}$ in a neighborhood of $\hat{\mathfrak{v}}$:

$$\begin{aligned}&\phi_{\hat{\mathfrak{v}}}(\hat{\mathfrak{v}}) = \mathcal{I}_\Omega^{E_0,\delta}(\hat{\mathfrak{v}}), \qquad \phi_{\hat{\mathfrak{v}}}(\mathfrak{v}) \geq \mathcal{I}_\Omega^{E_0,\delta}(\mathfrak{v}), \\ &\phi''_{\hat{\mathfrak{v}}}(\hat{\mathfrak{v}}) = \frac{\int_{\partial E_{\hat{\mathfrak{v}}}} |\nabla_{\partial E_{\hat{\mathfrak{v}}}} \Phi_\tau|^2 - \Phi_\tau^2 \|A_{E_{\hat{\mathfrak{v}}}}\|^2 \, d\mathcal{H}^{n-1}}{\left(\int_{\partial E_{\hat{\mathfrak{v}}}} \Phi_\tau \, d\mathcal{H}^{n-1}\right)^2},\end{aligned}$$

where $\|A_{E_{\hat{\mathfrak{v}}}}\|$ is the Frobenius norm of the second fundamental form of the boundary of $E_{\hat{\mathfrak{v}}}$, and where the mapping $t(\mathfrak{v}) \to \mathfrak{v}$ is a smooth, increasing map with $t(0) = 0$. The second derivative formula can be proved as in [11], [18].

In order to prove (2.10) we thus only need to prove that

$$\frac{\int_{\partial E_{\hat{\mathfrak{v}}}} |\nabla_{\partial E_{\hat{\mathfrak{v}}}} \Phi_\tau|^2 \, d\mathcal{H}^{n-1}}{\left(\int_{\partial E_{\hat{\mathfrak{v}}}} \Phi_\tau \, d\mathcal{H}^{n-1}\right)^2} \le C. \tag{2.12}$$

To this end, using (2.11) and the fact that $\|\phi'_\tau\|_\infty \le \frac{C}{\tau}$ we have that

$$\int_{\partial E_{\hat{\mathfrak{v}}}} |\nabla_{\partial E_{\hat{\mathfrak{v}}}} \Phi_\tau|^2 \, d\mathcal{H}^{n-1} \le \frac{C}{\tau^2} P(E_{\hat{\mathfrak{v}}}; \Omega) \le \frac{C}{\tau^2}. \tag{2.13}$$

On the other hand, denoting the set $\tilde{U} := \{\Phi_\tau \ge 1\} = \{\tilde{d} \ge \tau\}$, we have that

$$\int_{\partial E_{\hat{\mathfrak{v}}}} \Phi_\tau \, d\mathcal{H}^{n-1} \ge \int_{\partial E_{\hat{\mathfrak{v}}} \cap \tilde{U}} d\mathcal{H}^{n-1} = P(\partial E_{\hat{\mathfrak{v}}}; \tilde{U}). \tag{2.14}$$

By (2.11) and the fact that U has Lipschitz boundary, we have that

$$\mathcal{L}^n(U \backslash \tilde{U}) \le \mathcal{L}^n(\{x : 0 \le d(x, \mathbb{R}^n \backslash U) \le C_2 \tau\}) \le C\tau.$$

Using the notation (2.5) we also have, by (2.11), that $U_{\tau/C_2} \subset \tilde{U}$, and that $\mathcal{L}^n(\tilde{U} \backslash U_{\tau/C_2}) \le C_4 \tau$. Hence using (1.14) and Lemma 2.3 we find that

$$\mathcal{I}_{\tilde{U}}(\mathfrak{v}) \ge \inf_{\eta \le C_4 \tau} \mathcal{I}_{U_{\tau/C_2}}(\mathfrak{v} - \eta) \ge C(\mathfrak{v} - C_4 \tau)^{(n-1)/n}$$

as long as $\mathfrak{v} - C_4 \tau \in (C\tau, A)$.

Again recalling that $\mathcal{L}^n(E_{\hat{\mathfrak{v}}} \backslash E_0) \in [\delta, \frac{3\delta}{2}]$ we find that, for δ sufficiently small and $\tau = c\delta$ with sufficiently small $c > 0$,

$$P(E_{\hat{\mathfrak{v}}}; \tilde{U}) \ge C\delta^{(n-1)/n}.$$

This inequality, together with (2.13) and (2.14), proves (2.12).

By then using an argument as in the proof of Lemma 2.7 in [18] (see also [15]) we find that $\mathcal{I}_\Omega^{E_0,\delta}$ is semi-concave on J_δ, which is the desired conclusion. □

As $\mathcal{I}_\Omega^{E_0,\delta}$ is semi-concave, it has a left derivative $(\mathcal{I}_\Omega^{E_0,\delta})'_-$ and a right derivative $(\mathcal{I}_\Omega^{E_0,\delta})'_+$ at every point $\mathfrak{v}$ in J_δ, with $(\mathcal{I}_\Omega^{E_0,\delta})'_-(\mathfrak{v}) \ge (\mathcal{I}_\Omega^{E_0,\delta})'_+(\mathfrak{v})$. Furthermore, by considering a normal perturbation of E_0, we have that $(n-1)\kappa_{E_0} \in [(\mathcal{I}_\Omega^{E_0,\delta})'_-(\mathfrak{v}_0), (\mathcal{I}_\Omega^{E_0,\delta})'_+(\mathfrak{v}_0)]$. The following result gives a simple, yet important observation.

Proposition 2.4. *Assume that Ω satisfies (1.3) and let $E_0 \subset \Omega$ be a volume-constrained local perimeter minimizer in Ω, with $\mathcal{L}^n(E_0) = \mathfrak{v}_m$. Then as $\delta \to 0$, $(\mathcal{I}_\Omega^{E_0,\delta})'_-(\mathfrak{v}_m) \to (n-1)\kappa_{E_0}$ and $(\mathcal{I}_\Omega^{E_0,\delta})'_+(\mathfrak{v}_m) \to (n-1)\kappa_{E_0}$, where κ_{E_0} is the mean curvature of E_0.*

Proof. We will prove the result for the left derivative. For any fixed δ, pick a sequence of points $\mathfrak{v}_k \uparrow \mathfrak{v}_0$ at which $\mathcal{I}_\Omega^{E_0,\delta}$ is differentiable. This is possible as $\mathcal{I}_\Omega^{E_0,\delta}$ is semi-concave. Also, as $\mathcal{I}_\Omega^{E_0,\delta}$ is semi-concave we have that $(\mathcal{I}_\Omega^{E_0,\delta})'(\mathfrak{v}_k) \to (\mathcal{I}_\Omega^{E_0,\delta})'_-(\mathfrak{v}_0)$. Let $E_{\mathfrak{v}_k}$ be a minimizer of

$$\min\{P(E; \Omega) : \, E \subset \Omega \text{ Borel}, \, \mathcal{L}^n(E) = \mathfrak{v}_k, \, \alpha(E_0, E) \le \delta\}.$$

We claim that there exists a volume-constrained perimeter minimizer E_0^δ, satisfying $\alpha(E_0^\delta, E_0) \le \delta$, $\mathcal{L}^n(E_0^\delta) = \mathfrak{v}_0$, and with mean curvature $\kappa_0^\delta = (\mathcal{I}_\Omega^{E_0,\delta})'_-(\mathfrak{v}_0)$.

First, suppose that we can pick a subsequence of $\mathfrak{v}_k$ (not relabeled), such that

$$\min\{\mathcal{L}^n(E_{\mathfrak{v}_k} \backslash E_0), \mathcal{L}^n(E_0 \backslash E_{\mathfrak{v}_k})\} = \mathcal{L}^n(E_{\mathfrak{v}_k} \backslash E_0).$$

Suppose furthermore that $\liminf_{k\to\infty} \mathcal{L}^n(E_0 \backslash E_{\mathfrak{v}_k}) \geq \delta$.

Under these assumptions, and as long as δ is small enough and $\mathfrak{v}_k$ is close enough to $\mathfrak{v}_0$, we have that $\partial E_{\mathfrak{v}_k} \cap E_0$ is a non-empty set. Furthermore, taking local variations with support in E_0 will not increase the value of $\alpha(E_{\mathfrak{v}_k}, E_0)$. Hence, the mean curvature of $E_{\mathfrak{v}_k}$ inside the set E_0, which we will denote $\kappa^*_{\delta,k}$, will satisfy (see Chapter 17 in [11])

$$(n-1)\kappa^*_{\delta,k} = (\mathcal{I}_\Omega^{E_0,\delta})'(\mathfrak{v}_k).$$

We remark that since $(\mathcal{I}_\Omega^{E_0,\delta})'(\mathfrak{v}_k) \to (\mathcal{I}_\Omega^{E_0,\delta})'_-(\mathfrak{v}_0)$, we immediately have that $\kappa^*_{\delta,k}$ is bounded.

By BV compactness, $\chi_{E_{\mathfrak{v}_k}} \to \chi_{E_0^\delta}$ in $L^1(\Omega)$, for some set E_0^δ with finite perimeter. By lower semi-continuity of the perimeter, we have that $P(E_0^\delta;\Omega) = \mathcal{I}_\Omega^{E_0,\delta}(r_0) \leq P(E_0;\Omega)$. As E_0 is a local volume-constrained perimeter minimizer, for δ small enough we have that E_0^δ is a local volume-constrained perimeter minimizer as well. In particular, ∂E_0^δ is a surface of constant mean curvature. Furthermore, by the assumption that $\liminf_{k\to\infty} \mathcal{L}^n(E_0 \backslash E_{\mathfrak{v}_k}) \geq \delta$ we know that $\partial E_0^\delta \cap E_0$ is a set with positive perimeter.

By using the uniform bound on the curvatures, along with elliptic regularity, we then have that $E_{\mathfrak{v}_k} \to E_0^\delta$ in C^∞ in compact subsets of E_0 (see the proof of Theorem 1.9 in [15]). Hence the mean curvature κ_0^δ of E_0^δ satisfies $(n-1)\kappa_0^\delta = (\mathcal{I}_\Omega^{E_0,\delta})'_-(\mathfrak{v}_0)$.

The case where $\liminf_{k\to\infty} \mathcal{L}^n(E_0 \backslash E_{\mathfrak{v}_k}) < \delta$ is in fact simpler, because the α-constraint will not be saturated and any local perturbation is permissible. On the other hand, if we cannot pick a subsequence of $\mathfrak{v}_k$ satisfying $\min\{\mathcal{L}^n(E_{\mathfrak{v}_k} \backslash E_0), \mathcal{L}^n(E_0 \backslash E_{\mathfrak{v}_k})\} = \mathcal{L}^n(E_{\mathfrak{v}_k} \backslash E_0)$, then we must be able to pick a subsequence satisfying $\min\{\mathcal{L}^n(E_{\mathfrak{v}_k} \backslash E_0), \mathcal{L}^n(E_0 \backslash E_{\mathfrak{v}_k})\} = \mathcal{L}^n(E_0 \backslash E_{\mathfrak{v}_k})$. We then conduct the same steps, but this time in $\Omega \backslash E_0$. This proves the claim.

Finally, we recall that $\alpha(E_0^\delta, E_0) \leq \delta$. Hence we have that $\chi_{E_0^\delta} \to \chi_{E_0}$ in $L^1(\Omega)$ as $\delta \to 0$. By again using the same argument, E_0^δ must in fact converge in C^∞ to E_0, and hence $\kappa_0^\delta \to \kappa_{E_0}$, or in other words, $(\mathcal{I}_\Omega^{E_0,\delta})'_- \to (n-1)\kappa_{E_0}$. This concludes the proof. $\square$

3 Rearrangements and Weighted Problem

Let $I = (A, B)$ for some $A < B$ and consider a function $\eta : I \to [0, \infty)$ which satisfies the following:

$$\eta \in C(I) \cap C^1((A, t_0]) \cap C^1([t_0, B)), \qquad \eta > 0 \quad \text{in } I \tag{3.1}$$

$$d_1(t-A)^{\frac{n-1}{n}} \leq \eta(t) \leq d_2(t-A)^{\frac{n-1}{n}} \text{ for } t \in (A, A+t^\star), \tag{3.2}$$

$$d_3(B-t)^{\frac{n-1}{n}} \leq \eta(t) \leq d_4(B-t)^{\frac{n-1}{n}} \text{ for } t \in (B-t^\star, B), \tag{3.3}$$

$$|\eta'(t)| \leq \frac{d_5\eta(t)}{\min\{B-t, t-A\}} \quad \text{for } t \in I \setminus \{t_0\}, \quad \eta'_-(t_0) \geq \eta'_+(t_0), \tag{3.4}$$

$$\int_A^{t_0} \eta\, dt = \mathfrak{v}_m, \qquad \int_I \eta\, dt = 1, \tag{3.5}$$

for some $A < t_0 < B$ and for some constants $d_1, \ldots, d_5 > 0$ and $t^\star > 0$.

Next, define the energy

$$G_\varepsilon(v) := \begin{cases} \int_I (W(v) + \varepsilon^2|v'|^2)\eta\, dt & \text{if } v \in H^1_\eta(I) \text{ and } \int_I v\eta\, dt = m, \\ \infty & \text{otherwise.} \end{cases}$$

Under the hypotheses (3.1)–(3.5), following the proof of Theorem 4.4 in [10], it can be shown that $G_\varepsilon^{(1)} = \varepsilon^{-1}G_\varepsilon \xrightarrow{\Gamma} G_0^{(1)}$, where $G_0^{(1)}$ is given by

$$G_0^{(1)}(v) := \begin{cases} \frac{2c_W}{b-a}|Dv|_\eta(I) \text{ if } v \in BV_\eta(I) \text{ and } \int_I v\eta\, dt = m, \\ \infty \text{ otherwise,} \end{cases}$$

with c_W the constant given in (1.13). In view of (1.15) and (3.5), it can also be shown as in Theorem 4.6 in [10] that $v_0 = a\chi_{[A,t_0)} + b\chi_{[t_0,B]}$ is an isolated L^1-local minimizer of $G_0^{(1)}$, and hence for some $\hat{\delta}$ sufficiently small we have that v_0 is the unique limit of minimizers v_ε of the functionals

$$J_\varepsilon(v) := \begin{cases} G_\varepsilon(v) & \text{if } v \in H^1_\eta(I),\ \int_I v\eta\,dt = m \text{ and } \|v - v_0\|_{L^1_\eta} \leq \hat{\delta}, \\ \infty & \text{otherwise.} \end{cases}$$

Note that v_ε satisfies the Euler-Lagrange equation

$$2\varepsilon^2(v_\varepsilon'\eta)' - W'(v_\varepsilon)\eta = \varepsilon\lambda_\varepsilon\eta.$$

Our goal is to prove the following theorem:

Theorem 3.1. *Assume that W satisfies hypotheses* (1.4)–(1.7) *and that η satisfies* (3.1)–(3.5). *Let v_ε be a minimizer of G_ε with $v_\varepsilon \to v_0$ in L^1_η as $\varepsilon \to 0^+$. Then,*

$$\liminf_{\varepsilon\to 0^+} \frac{G_\varepsilon^{(1)}(v_\varepsilon) - 2c_W\eta(t_0)}{\varepsilon} \geq 2\eta_-'(t_0)\int_{-\infty}^0 W^{1/2}(z(s-\tau_0))z'(s-\tau_0)s\,ds$$
$$+ 2\eta_+'(t_0)\int_0^\infty W^{1/2}(z(s-\tau_0))z'(s-\tau_0)s\,ds + \begin{cases} \frac{\lambda_0^2}{2W''(a)}\int_I \eta(t)\,dt & \text{if } q = 1, \\ 0 & \text{if } q < 1, \end{cases} \tag{3.6}$$

where c_{sym} is the constant given in (1.17),

$$\lim_{j\to\infty} \lambda_{\varepsilon_j} = \lambda_0 \in \left[\frac{2c_W\eta_+'(t_0)}{(b-a)\eta(t_0)}, \frac{2c_W\eta_-'(t_0)}{(b-a)\eta(t_0)}\right] \tag{3.7}$$

for some subsequence $\varepsilon_j \to 0^+$, and the number τ_0 is given by

$$\eta(t_0)\int_{\mathbb{R}} z(s-\tau_0) - sgn_{a,b}(s)\,ds = \frac{\lambda_0}{W''(a)}\int_I \eta(t)\,dt, \tag{3.8}$$

with z the solution to (1.18).

Proof. By taking a subsequence (not relabeled), without loss of generality, we may assume that the lim inf on the left-hand side of (3.6) is actually a limit. Also, for simplicity we take $t_0 = 0$.
Step 1. We claim that (3.7) holds. This proof follows as in Theorem 4.9 in [10]. The only difference is that at the last part of the proof we can no longer use the fact that η is of class C^1 and we need to show that

$$\lim_{\varepsilon\to 0^+}\int_I W^{1/2}(v_\varepsilon)|v_\varepsilon'|\eta'\psi\,dt = c_W\psi(0)\eta_1,$$

for some $\eta_1 \in [\eta_+'(0), \eta_-'(0)]$. Following the proof cited above, we know that $W^{1/2}(v_\varepsilon)|v_\varepsilon'|\eta\mathcal{L}^1\lfloor[A,B] \overset{*}{\rightharpoonup} c_W\eta(0)\delta_0$. Hence by picking an appropriate subsequence, we have, for some $\theta \in [0,1]$,

$$W^{1/2}(v_\varepsilon)|v_\varepsilon'|\eta\mathcal{L}^1\lfloor[A,0] \overset{*}{\rightharpoonup} \theta c_W\eta(0)\delta_0,$$
$$W^{1/2}(v_\varepsilon)|v_\varepsilon'|\eta\mathcal{L}^1\lfloor[0,B] \overset{*}{\rightharpoonup} (1-\theta)c_W\eta(0)\delta_0.$$

Hence,

$$\lim_{\varepsilon\to 0^+}\int_I W^{1/2}(v_\varepsilon)|v_\varepsilon'|\eta'\psi\,dt = c_W\psi(0)(\theta\eta_-'(0) + (1-\theta)\eta_+'(0)),$$

which is the desired conclusion.

Step 2. We claim that there exists a sequence of numbers $\tau_\varepsilon \to \tau_0$, where τ_0 is given in (3.8), so that the functions $w_\varepsilon(s) := v_\varepsilon(\varepsilon s)$, $s \in (A\varepsilon^{-1}, B\varepsilon^{-1})$, converge weakly to the profile $w_0 := z(\cdot - \tau_0)$ in $H^1((-l,l))$ for any fixed $l > 0$, and satisfy

$$w_\varepsilon(\tau_\varepsilon) = c_\varepsilon,$$

where c_ε is the central zero of $W' + \varepsilon\lambda_\varepsilon$.

This follows from the proofs of Lemmas 4.18 and 4.19 in [10] (see also [4]). We note that those proofs use significant machinery from that work, including detailed decay estimates, but do not require anything more than a Lipschitz estimate on η near 0 and (3.2), (3.3), and (3.4).

Step 3. We claim that (3.6) holds. Define $\eta_\varepsilon(s) := \eta(s\varepsilon)$, $s \in (A\varepsilon^{-1}, B\varepsilon^{-1})$. After changing variables, and setting $l_\varepsilon := C|\log\varepsilon|$, we obtain

$$\begin{aligned}
G_\varepsilon^{(1)}(v_\varepsilon) &= \varepsilon^{-1}\int_{-l_\varepsilon}^{l_\varepsilon}(W^{1/2}(w_\varepsilon) - w_\varepsilon')^2\eta_\varepsilon\,ds + 2\varepsilon^{-1}\int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon'(\eta_\varepsilon - \eta(0))\,ds \\
&\quad + \varepsilon^{-1}\int_{[A\varepsilon^{-1},B\varepsilon^{-1}]\setminus(-l_\varepsilon,l_\varepsilon)}\left(W(w_\varepsilon) + (w_\varepsilon')^2\right)\eta_\varepsilon\,ds + \varepsilon^{-1}2\eta(0)\left(\int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon'\,ds - c_W\right) \\
&\geq 2\varepsilon^{-1}\int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon'(\eta_\varepsilon - \eta(0))\,ds \\
&\quad + \varepsilon^{-1}\int_{[A\varepsilon^{-1},B\varepsilon^{-1}]\setminus(-l_\varepsilon,l_\varepsilon)} W(w_\varepsilon)\eta_\varepsilon\,ds + \varepsilon^{-1}2\eta(0)\left(\int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon'\,ds - c_W\right).
\end{aligned}$$

The last term goes to zero, see 4.105 in [10]. Following the proof of 4.106 in [10], the second to last term satisfies

$$\lim_{\varepsilon\to 0^+}\varepsilon^{-1}\int_{[A\varepsilon^{-1},B\varepsilon^{-1}]\setminus(-l_\varepsilon,l_\varepsilon)} W(w_\varepsilon)\eta_\varepsilon\,ds = \begin{cases}\frac{\lambda_0^2}{2W''(a)}\int_I \eta\,dt & \text{if } q = 1,\\ 0 & \text{if } q < 1.\end{cases}$$

Finally, by (3.1) the function η satisfies the following Taylor's formula:

$$\eta(t) = \eta(0) - t^-\eta_-'(0) + t^+\eta_+'(0)| + |t|R_1(t),$$

where $R_1(t) \to 0$ as $t \to 0$. Hence, we find that

$$\begin{aligned}
2\varepsilon^{-1}\int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon(s))w_\varepsilon'(s)(\eta_\varepsilon(s) - \eta(0))\,ds &= 2\int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon(s))w_\varepsilon'(s)(-\eta_-'(0)s^- + \eta_+'(0)s^+)\,ds \\
+ 2\int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon(s))w_\varepsilon'(s)|s|R_1(\varepsilon s)\,ds.
\end{aligned}$$

As in [10], we now break the integrals over $[-l_\varepsilon, -l]$, $[-l, l]$, $[l, l_\varepsilon]$ for any fixed $l > 0$. Since by Step 2, $\{w_\varepsilon\}$ converges weakly to $z(\cdot - \tau_0)$ in $H^1((-l,l))$, we can follow the computations after formula (4.106) in [10] using the exponential decay (see (4.95) and (4.96) in [10]) in $[-l_\varepsilon, -l]$ and $[l, l_\varepsilon]$ to obtain that

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} 2\int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon(s))w_\varepsilon'(s)(-\eta_-'(0)s^- + \eta_+'(0)s^+)\,ds \\
&= 2\eta_-'(0)\int_{-\infty}^{0} W^{1/2}(z(s-\tau_0))z'(s-\tau_0)s\,ds + 2\eta_+'(0)\int_0^\infty W^{1/2}(z(s-\tau_0))z'(s-\tau_0)s\,ds.
\end{aligned}$$

Similarly, using the facts that $R_1(t) \to 0$ as $t \to 0$ and $\varepsilon|s| \le \varepsilon l_\varepsilon \le C\varepsilon|\log\varepsilon|$ for $|s| \le l_\varepsilon$, we can use Step 2 to show that

$$\lim_{\varepsilon\to 0^+} 2\int_{-l}^{l} W^{1/2}(w_\varepsilon(s))w_\varepsilon'(s)|s|R_1(\varepsilon s)\,ds = 0,$$

while by Lemma 4.19 and (4.96) in [10],

$$2\int_{l}^{l_\varepsilon} W^{1/2}(w_\varepsilon(s))|w_\varepsilon'(s)||s||R_1(\varepsilon s)|\,ds \le 2\|R_1\|_{L^\infty(-\varepsilon l_\varepsilon,\varepsilon l_\varepsilon)}\|w_\varepsilon'\|_\infty \int_{l}^{l_\varepsilon} W^{1/2}(w_\varepsilon(s))|s|\,ds \to 0$$

as $\varepsilon \to 0^+$. A similar estimate holds in $[-l_\varepsilon, -l]$. This concludes the proof of (3.6).

□

4 Proof of the Main Results

Now we give a proof of Theorem 1.3.

Proof of Theorem 1.3. We only give the proof in the case $q = 1$ in (1.5), the case $q < 1$ being similar. Since $\mathcal{I}_\Omega \le \mathcal{I}_\Omega^{E_0,\delta}$ (see (1.14) and (2.1)), reasoning as in Proposition 3.1 in [10] we can construct a function $\mathcal{I} \in C(0,1) \cap C^1((0,\mathfrak{v}_m]) \cap C^1([\mathfrak{v}_m, 0))$ satisfying

$$\begin{aligned} &\mathcal{I}_\Omega^{E_0,\delta} \ge \mathcal{I} > 0 \text{ in } (0,1), \\ &\mathcal{I}(\mathfrak{v}_m) = \mathcal{I}_\Omega^{E_0,\delta}(\mathfrak{v}_m), \qquad \mathcal{I}_\pm'(\mathfrak{v}_m) = (\mathcal{I}_\Omega^{E_0,\delta})_\pm'(\mathfrak{v}_m), \\ &\mathcal{I}(\mathfrak{v}) = C_0\mathfrak{v}^{\frac{n-1}{n}} \text{ for } \mathfrak{v} \in (0,r), \qquad \mathcal{I}(\mathfrak{v}) = C_0(1-\mathfrak{v})^{\frac{n-1}{n}} \text{ for } \mathfrak{v} \in (1-r,1) \end{aligned} \tag{4.1}$$

for some constant $C_0 > 0$ and some $0 < r < 1/2$ small. Let $\eta := \mathcal{I} \circ V_\Omega$, where V_Ω satisfies

$$\frac{d}{dt}V_\Omega(t) = \mathcal{I}(V_\Omega(t)), \quad V_\Omega(0) = \mathfrak{v}_m. \tag{4.2}$$

As in the proof of Theorem 5.1 in [15] one can show that η satisfies all of the assumptions (3.1)–(3.5).

Let u_ε be a minimizer of $\mathcal{F}_\varepsilon$ and let $v_\varepsilon := f_{u_\varepsilon}$ be the increasing function given in Remark 3.11 of [10]. Following the proof of Theorem 5.1 in [10] (see also [14] or [15] for more details), we have that

$$\frac{\mathcal{F}_\varepsilon^{(1)}(u_\varepsilon) - \min \mathcal{F}_0}{\varepsilon} \ge \frac{G_\varepsilon^{(1)}(v_\varepsilon) - 2c_W\eta(t_0)}{\varepsilon},$$

Hence, by Theorem 3.1, we have that

$$\begin{aligned}\liminf_{\varepsilon\to 0^+} \frac{\mathcal{F}_\varepsilon^{(1)}(u_\varepsilon) - \min \mathcal{F}_0}{\varepsilon} \ge \frac{\lambda_0^2(\delta)}{2W''(a)}\int_I \eta(t)\,dt + 2\eta_-'(t_0)\int_{-\infty}^{0} W^{1/2}(z(s-\tau_0(\delta)))z'(s-\tau_0(\delta))s\,ds \\ + 2\eta_+'(t_0)\int_{0}^{\infty} W^{1/2}(z(s-\tau_0(\delta)))z'(s-\tau_0(\delta))s\,ds,\end{aligned}$$

where

$$\lambda_0(\delta) \in \left[\frac{2c_W\eta_+'(t_0)}{(b-a)\eta(t_0)}, \frac{2c_W\eta_-'(t_0)}{(b-a)\eta(t_0)}\right] \tag{4.3}$$

and $\tau_0(\delta)$ is given by

$$\eta(t_0)\int_{\mathbb{R}} z(s-\tau_0(\delta)) - sgn_{a,b}(s)\,ds = \frac{\lambda_0(\delta)}{W''(a)}\int_I \eta(t)\,dt. \tag{4.4}$$

By Proposition 2.4, (4.1), and (4.2), we find that as $\delta \to 0$ the quantities $\eta'_-(t_0)$ and $\eta'_+(t_0)$ converge to the same value, namely, $(n-1)\kappa_{E_0}$, and hence by (4.3) and (4.4) we have that $\lambda_0(\delta) \to \lambda_u$ and $\tau_0(\delta) \to \tau_u$ converge as well. Thus by taking $\delta \to 0$ we obtain

$$\liminf_{\varepsilon \to 0^+} \frac{\mathcal{F}_\varepsilon^{(1)}(u_\varepsilon) - \min \mathcal{F}_0}{\varepsilon} \geq \frac{2c_W^2(n-1)^2}{W''(a)(b-a)^2}\kappa_u^2 + 2(c_{sym} + c_W\tau_u)(n-1)\kappa_u P(\{u=a\};\Omega),$$

which is the desired result. □

The proofs of Theorems 1.4 and 1.5 now follow from Theorems 1.2 and 1.7 and Theorem 1.4 in [15], respectively, with the only change that we apply Theorem 1.3 of this paper in place of Theorem 1.1. in [10].

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