BALANCED VISCOSITY SOLUTIONS TO A RATE-INDEPENDENT SYSTEM FOR DAMAGE

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ABSTRACT. This article is the third one in a series of papers by the authors on vanishing-viscosity solutions to rate-independent damage systems. While in the first two papers [KRZ13, KRZ15] the assumptions on the spatial domain Ω were kept as general as possible (i.e. nonsmooth domain with mixed boundary conditions), we assume here that $\partial\Omega$ is smooth and that the type of boundary conditions does not change. This smoother setting allows us to derive enhanced regularity spatial properties both for the displacement and damage fields. Thus, we are in a position to work with a stronger solution notion at the level of the viscous approximating system. The vanishing-viscosity analysis then leads us to obtain the existence of a stronger solution concept for the rate-independent limit system. Furthermore, in comparison to [KRZ13, KRZ15], in our vanishing-viscosity analysis we do not switch to an artificial arc-length parameterization of the trajectories but we stay with the true physical time. The resulting concept of Balanced Viscosity solution to the rate-independent damage system thus encodes a more explicit characterization of the system behavior at time discontinuities of the solution.

1. Introduction

We consider in a three-dimensional spatial domain Ω the rate-independent system for damage evolution

$$-\operatorname{div}\big(g(z)\mathbb{C}\varepsilon(u+u_D)\big) = \ell \qquad \text{in } \Omega \times (0,T), \tag{1.1a}$$

$$\partial\mathcal{R}_1(z_t) + A_q z + f'(z) + \tfrac{1}{2}g'(z)\mathbb{C}\varepsilon(u+u_D) : \varepsilon(u+u_D) \ni 0 \qquad \text{in } \Omega \times (0,T), \tag{1.1b}$$

with $q > 3$, A_q the q-Laplacian type operator

$$A_q z = -\operatorname{div}((1+|\nabla z|^2)^{(q/2)-1}\nabla z),$$

and the 1-homogeneous dissipation potential

$$\mathcal{R}_1(v) = \int_\Omega \mathrm{R}_1(v)\,\mathrm{d}x \qquad \text{with } \mathrm{R}_1(v) = \begin{cases} |v| & \text{if } v \le 0, \\ \infty & \text{otherwise.} \end{cases}$$

Here, $u : [0,T] \times \Omega \to \mathbb{R}^3$ denotes the displacement field and $z : [0,T] \times \Omega \to \mathbb{R}$ characterizes the time and space-dependent damage state in the body $\Omega \subset \mathbb{R}^3$. The natural state spaces for u and z are $\mathcal{U} = H^1_0(\Omega;\mathbb{R}^3)$ and $\mathcal{Z} = W^{1,q}(\Omega)$. The energy potential is of the form

$$\mathcal{E}(t,u,z) = \int_\Omega g(z)\frac{1}{2}\mathbb{C}(x)\varepsilon(u+u_D(t)) : \varepsilon(u+u_D(t)) + f(z) + \frac{1}{q}(1+|\nabla z|^2)^{\frac{q}{2}}\,\mathrm{d}x - \langle \ell(t), u\rangle,$$

where $\varepsilon(w) = \frac{1}{2}(\nabla w + \nabla w^T)$ ($w \in \mathcal{U}$) is the strain tensor and u_D denotes the Dirichlet datum. Since the underlying energy $\mathcal{E}(t,\cdot,\cdot)$ in general is nonconvex and since $\mathcal{R}_1$ is of linear growth, solutions to (1.1) might be discontinuous in time. In order to select reasonable jump discontinuities we adopt here the vanishing-viscosity approach to the weak solvability of rate-independent systems, pioneered in [EM06] and developed both for abstract rate-independent systems, cf. e.g. [MRS12a, Mie11, MRS16], and for applied problems in fracture and plasticity, see for instance [KMZ08, DDS11, BFM12, CL17]. In the context of damage, in addition to the previously mentioned [KRZ13, KRZ15] we quote the recent [CL16, Neg16]. Let us stress that, in all of these papers the vanishing-viscosity analysis is performed by suitably adapting the original reparameterization technique of [EM06]. In [KN17], a time-incremental alternate minimization scheme for a damage model of

Ambrosio-Tortoreli type (without viscous regularization) was investigated. It turned out that in the time-continuous limit this procedure results in a class of solutions that is closely related (but not identical) to those obtained by vanishing viscosity limits. Also here, the reparameterization technique of [EM06] was applied.

Hence, we approximate the rate-independent flow rule for the damage parameter by its viscous regularization, and thus address the *rate-dependent* system

$$-\operatorname{div}(g(z)\mathbb{C}\varepsilon(u+u_D) = \ell \qquad \text{in } \Omega \times (0,T), \tag{1.2a}$$

$$\partial \mathcal{R}_1(z_t) + \epsilon z_t + A_q z + f'(z) + \tfrac{1}{2} g'(z)\mathbb{C}\varepsilon(u+u_D) : \varepsilon(u+u_D) \ni 0 \qquad \text{in } \Omega \times (0,T), \tag{1.2b}$$

where the underlying regularized dissipation potential is given by

$$\mathcal{R}_\epsilon : L^2(\Omega) \to [0,+\infty] \text{ given by } \mathcal{R}_\epsilon(v) := \mathcal{R}_1(v) + \frac{\epsilon}{2}\|v\|^2_{L^2(\Omega)}\,, \tag{1.3}$$

and $\epsilon > 0$ is the viscosity parameter. The goal is to perform the limit passage as $\epsilon \downarrow 0$ from (1.2) to (1.1), without switching to an artificial arc-length reparameterization of the trajectories, but *staying with the true physical time*. The basics for this approach to the construction of the resulting concept of *Balanced Viscosity* (BV) solutions to the limit rate-independent system were set in [MRS12a, MRS16] for abstract rate-independent systems in finite-dimensional and infinite-dimensional Banach spaces, respectively. A notable feature of this vanishing-viscosity technique is that it allows for a *direct* limit passage from the *time discrete* version of (1.2) to (1.1), as the viscosity parameter ϵ and the time discretization step τ *simultaneously* tend to zero with $\frac{\epsilon}{\tau} \to \infty$. This provides a *constructive approach* to Balanced Viscosity solutions of system (1.1) which could also be further advanced from a numerical viewpoint.

While the techniques applied here have been developed in an abstract context in [MRS16], let us emphasize that the existence and convergence results therein, (in particular [MRS16, Thms. 3.11 and 3.12]), are not directly applicable to the present damage system. The main point is that, in contrast to [MRS16] in our setting the dissipation potential $\mathcal{R}_1$ may take the value $+\infty$ to enforce the unidirectionality of the damaging process. This causes additional technical difficulties for the derivation of uniform a priori bounds. Moreover, the definition of BV solution has to be carefully tailored to accomodate this irreversibility constraint. Further analytical difficulties occur due to the presence of the quadratic term on the right-hand side of the differential inclusion (1.1b), which at a first glance belongs to $L^1(\Omega)$, only. This necessitates a careful study of the spatial regularity properties of the displacement and the damage fields, which was already initiated in [KRZ13, KRZ15].

The main results of this paper are the following:

Regularity: Thanks to the assumed smoothness of $\partial\Omega$ (made precise in Section 2.1) and the assumption $q > 3$ on the q-Laplacian regularization in (1.1b), which ensures enough spatial regularity for the coefficient $g(z)$ of the elasticity operator in (1.1a), solutions $u = u(t,z)$ of (1.1a) belong to $H^2(\Omega) \cap W^{1,p}(\Omega)$ for every $p \geq 1$ if the external data ℓ, u_D are smooth enough. We derive explicit bounds for the corresponding norms of u in terms of z by adapting arguments from [BM13] to our situation. These results improve the integrability properties of the quadratic term in (1.1b) and in (1.2b) and allow us to test (a regularized version of) (1.2b) by $\partial_t A_q z$, which ultimately guarantees that $\mathrm{D}_z\mathcal{E}(t,u(t,z),z) \in L^2(\Omega)$, again with uniform bounds, see Section 3.1. Let us mention that, in the case of the standard Laplacian regularization (i.e. $q = 2$), this regularity estimate was first proposed in [BFL00] for doubly nonlinear differential inclusions in phase change modeling.

Based on the improved integrability property of $\mathrm{D}_z\mathcal{E}(t,u(t,z),z)$ we may consider subdifferentials and convex conjugate functions of the dissipation potentials with respect to the $L^2(\Omega)$ duality, instead of the $\mathcal{Z} - \mathcal{Z}^*$ duality. Furthermore, based on these results we derive a (generalized) λ-convexity property of the energy functional, (cf. Corollary 2.13), and a chain rule identity (cf. Lemma 2.16). The latter is essential for the existence proof of BV solutions for the damage system.

This chain rule identity was not available in the earlier [KRZ15], which still addressed the case of a q-Laplacian regularization in the damage flow rule, whereas in [KRZ13] some technical difficulties were smeared out by taking as regularizing operator a (less physical) fractional Laplacian. Hence, in [KRZ15] we had to deal with a weaker notion of vanishing-viscosity solution compared to the present paper. In particular, in [KRZ15] it could be shown that the vanishing-viscosity limits satisfied an energy-dissipation inequality but, due to the lack of an appropriate chain rule this could not be improved to an energy-dissipation identity.

Existence and approximation of BV solutions: The concept of BV solution to the rate-independent system (1.1) consists of a (local) stability condition and of an energy-dissipation balance that encodes the (possible) onset of viscous behavior in the jump regime. More precisely, let $u(t,z) \in \mathcal{U}$ be the unique solution of (1.1a) and $\mathcal{I}(t,z) := \mathcal{E}(t,u(t,z),z)$ the reduced energy. We call a curve $z \in L^\infty(0,T;\mathcal{Z}) \cap \mathrm{BV}([0,T];L^2(\Omega))$ with $\mathrm{D}_z\mathcal{I}(\cdot,z(\cdot)) \in L^\infty(0,T;L^2(\Omega))$ a *Balanced Viscosity* solution to (1.1) if z satifies the local stability ($\mathrm{S_{loc}}$) and the energy-dissipation balance (ED)

$$-\mathrm{D}_z\mathcal{I}(t,z(t)) \in \partial\mathcal{R}_1(0) \quad \text{for all } t \in [0,T]\backslash \mathrm{J}_z, \tag{$\mathrm{S_{loc}}$}$$

$$Var_{\mathfrak{f}}(z;[0,t]) + \mathcal{I}(t,z(t)) = \mathcal{I}(0,z(0)) + \int_0^t \partial_t\mathcal{I}(r,z(r))\,\mathrm{d}r \quad \text{for all } t \in [0,T], \tag{ED}$$

where J_z denotes the countable jump set of z. The quantity $Var_{\mathfrak{f}}(\cdot;[0,t])$ is a total variation functional that encompasses both the dissipation, with respect to the 1-homogeneous potential $\mathcal{R}_1$, in continuous parts of the solution, as well as the dissipation at jump discontinuities. At jump discontinuities it reflects the viscous regularization term from (1.2b). While referring to Section 5.1 for its precise definition (and to [MRS16] for more comments on it), we may mention here its structure at a jump from z_- to z_+ for $t \in \mathrm{J}_z$. Indeed, the *jump contribution* $\Delta_{\mathfrak{f}}(t;z_-,z_+)$ to $Var_{\mathfrak{f}}(z;[0,t])$ is given by

$$\Delta_{\mathfrak{f}}(t;z_-,z_+) := \inf_{\vartheta \in \mathcal{T}_t^{\varrho}(z_-,z_+)} \int_0^1 \mathfrak{f}_t(\vartheta(r),\vartheta'(r))\,\mathrm{d}r\,, \tag{1.4}$$

$$\mathfrak{f}_t(\vartheta,\vartheta') = \mathcal{R}_1(\vartheta') + \|\vartheta'\|_{L^2(\Omega)} \inf_{\xi\in\partial\mathcal{R}_1(0)} \|-\mathrm{D}_z\mathcal{I}(t,\vartheta) - \xi\|_{L^2(\Omega)}\,, \tag{1.5}$$

where $\mathcal{T}_t^{\varrho}(z_-,z_+)$ denotes the set of admissible transition curves connecting z_- with z_+ and satisfying certain properties.

The appearance of the term from (1.4) in the vanishing-viscosity limit of (1.2) can be motivated by a comparison with the energy-dissipation balance that is valid for solutions of the viscous system (1.2). In fact, we will show in Theorem 4.1 that solutions to (1.2) exist and that they satisfy for all $t \in [0,T]$ the relation

$$\int_0^t \mathcal{R}_\epsilon(\dot{z}_\epsilon) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(r,z_\epsilon(r)))\,\mathrm{d}r + \mathcal{I}(t,z_\epsilon(t)) = \mathcal{I}(0,z(0)) + \int_0^t \partial_t\mathcal{I}(r,z_\epsilon(r))\,\mathrm{d}r \tag{1.6}$$

with $\mathcal{R}_\epsilon^*(\eta) = \frac{1}{2\epsilon}\inf_{\xi\in\partial\mathcal{R}_1(0)} \|\eta-\xi\|^2_{L^2(\Omega)}$ provided that $\eta \in L^2(\Omega)$. It turns out that

$$\mathfrak{f}_t(t,z,v) = \inf_{\varepsilon>0}\left(\mathcal{R}_\epsilon(v) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(t,z))\right).$$

The challenge here is to perform a sharp limit analysis for $\epsilon \to 0$ in order to show that the dissipation integral in (1.6) tends to $Var_{\mathfrak{f}}(z;[0,t])$ as $\epsilon \to 0$.

The **main result of this paper**, Theorem 5.7, states the existence of Balanced Viscosity solutions to the damage system (1.1) under suitable assumptions on the data z_0, u_D and ℓ. They are obtained from a vanishing-viscosity analysis of the time discretized version of the viscous system (1.2) as the time step size τ, the viscosity parameter ϵ and the ratio τ/ϵ tend to zero. The convergence of discrete solutions of corresponding numerical schemes to BV solutions is an immediate consequence. Let us stress that, with the techniques from [MRS16] we could prove the existence of BV solutions also by taking the vanishing-viscosity analysis of the *time-continuous* system in (1.2), as standardly done in works on the vanishing-viscosity approach to rate-independent systems. Here we have opted for this simultaneous limit passage to highlight the constructive character of this approach.

The paper is organized as follows: In Section 2 we collect and prove the basic regularity and differentiability properties of the reduced energy $\mathcal{J}$ and prove the chain rule identity. Some of the arguments are taken from the earlier paper [KRZ15] but are adapted to the enhanced smoothness assumptions on the boundary $\partial\Omega$. In Section 3 we study a time-discrete version of the viscous damage system (1.2), derive the necessary a priori estimates and provide an energy-dissipation inequality for suitable interpolants of the time incremental solutions. The main part of Section 3 is devoted to proving that $A_q z_k \in L^2(\Omega)$ for time incremental solutions z_k. In Section 4 we shortly address the existence of viscous solutions to the system (1.2). The main focus of the paper lies on the analysis of the vanishing-viscosity limit as both the viscosity parameter and the time step size tend to zero simultaneously (Sections 5 & 6). The notion of BV solutions is introduced and explained at length in Section 5, where also the main existence theorem is formulated and where further properties of BV solutions are discussed. The corresponding proofs are collected in Section 6. A short Appendix collects some elliptic regularity results that are key for our analysis.

We conclude by fixing some notation that will be used throughout the paper.

Notation 1.1. Throughout the paper, for a given Banach space X, we will by $\|\cdot\|_X$ denote its norm; in the case of product spaces $X\times\ldots\times X$, we will often write $\|\cdot\|_X$ in place of $\|\cdot\|_{X\times\ldots\times X}$. We will denote by $\langle\cdot,\cdot\rangle_X$ the duality pairing between X^* and X, using the symbol $(\cdot,\cdot)_X$ for the scalar product in X, if X is a Hilbert space.

We will denote most of the positive constants occurring in the calculations, and depending on known quantities, by the symbols c, c', C, C', ..., whose meaning may vary even within the same line. Furthermore, the symbols I_i, $i = 0, 1, \ldots$, will be used as abbreviations for several integral terms appearing in the various estimates: we warn the reader that we will not be self-consistent with the numbering, so that, for instance, I_1 will appear several times with different meanings.

2. Preliminaries and properties of the reduced energy

We start by collecting our standing assumptions on the reference domain Ω and on the energy functional $\mathcal{E}$ in Section 2.1. Combining these requirements, in Sec. 2.2 we will obtain two regularity results for the Euler-Lagrange equation associated with the minimization of the elastic energy. In Sec. 2.3, such results will have a pivotal role in deriving a series of properties of the reduced energy $\mathcal{J}$, at the core of our subsequent analysis.

2.1. Setup.

Throughout the paper, we shall suppose that

Assumption 2.1 (Regularity of the domain). *$\Omega \subset \mathbb{R}^3$ is a bounded $\mathrm{C}^{1,1}$-domain with Dirichlet boundary $\Gamma_D = \partial\Omega$.*

From now on, we shall denote the state spaces for the variables u and z by

$$\mathcal{U} := H^1_0(\Omega;\mathbb{R}^3), \qquad \mathcal{Z} := W^{1,q}(\Omega) \quad \text{with } q > 3.$$

We will denote by

$$W^{-1,p}(\Omega) \text{ the dual space of } W^{1,p'}_0(\Omega) \text{ with } \frac{1}{p}+\frac{1}{p'}=1.$$

For later use, we recall here two crucial properties of the elliptic operator A_q holding for all z_1, z_2, $w \in \mathcal{Z}$:

$$\langle A_q z_1 - A_q z_2, z_1 - z_2\rangle_{\mathcal{Z}} \geq c_q \int_\Omega (1+|\nabla z_1|^2+|\nabla z_2|^2)^{\frac{q-2}{2}} |\nabla(z_1-z_2)|^2 \,\mathrm{d}x, \tag{2.1}$$

$$|\langle A_q z_1 - A_q z_2, w\rangle_{\mathcal{Z}}| \leq c_q' \int_\Omega (1+|\nabla z_1|^2+|\nabla z_2|^2)^{(q-2)/2} |\nabla(z_1-z_2)||\nabla w| \,\mathrm{d}x. \tag{2.2}$$

These inequalities rely on the corresponding estimates for the function $G_q : \mathbb{R}^3 \to \mathbb{R}$ defined by $G_q(A) := \frac{1}{q}(1+|A|^2)^{q/2}$ and its gradient. In particular the following monotonicity estimate is valid

$$(\nabla G_q(A) - \nabla G_q(B))\cdot(A-B) \geq c_q(1+|A|^2+|B|^2)^{(q-2)/2}|A-B|^2 \qquad \text{for all } A,\, B \in \mathbb{R}^3 \tag{2.3}$$

with $c_q > 0$ the same constant as in (2.1), which is in fact a consequence of the estimates provided in [Giu03, Lemma 8.3].

The energy functional $\mathcal{E} : [0,T] \times \mathcal{U} \times \mathcal{Z} \to \mathbb{R}$ consists of two contributions. The first one, $\mathcal{I}_1$, only depends on the damage variable. The second one, $\mathcal{E}_2 = \mathcal{E}_2(t,u,z)$, is given by the sum of an elastic energy of the type $\int_\Omega g(z) W(\varepsilon(x, u + u_D(t)))\,\mathrm{d}x$ with u_D a Dirichlet datum, and of the external loading term.

Assumption 2.2 (The energy functional)**.** *We consider*

$$\mathcal{I}_1 : \mathcal{Z} \to \mathbb{R} \ \ \textit{defined by} \ \ \mathcal{I}_1(z) := \mathcal{I}_q(z) + \int_\Omega f(z)\,\mathrm{d}x \ \ \textit{with}\ \mathcal{I}_q(z) := \frac{1}{q}\int_\Omega (1+|\nabla z|^2)^{\frac{q}{2}}\,\mathrm{d}x,\ q>3,$$

and f fulfilling

$$f \in \mathrm{C}^2(\mathbb{R}) \quad \textit{and} \quad \exists\, K_1,\, K_2 > 0 \quad \forall\, x \in \mathbb{R}: \qquad f(x) \geq K_1|x| - K_2. \tag{2.4}$$

As for $\mathcal{E}_2$, linearly elastic materials are considered with an elastic energy density

$$W(x,\eta) = \frac{1}{2}\mathbb{C}(x)\eta : \eta \qquad \textit{for } \eta \in \mathbb{R}^{3\times3}_{sym} \textit{ and almost every } x \in \Omega.$$

Hereafter, we shall suppose for the elasticity tensor that

$$\mathbb{C} \in \mathrm{C}^0_{\mathrm{lip}}(\overline{\Omega}; \mathrm{Lin}(\mathbb{R}^{3\times3}_{sym}, \mathbb{R}^{3\times3}_{sym})) \textit{ with } \mathbb{C}(x)\xi_1 : \xi_2 = \mathbb{C}(x)\xi_2 : \xi_1 \textit{ for all } x \in \Omega, \xi_i \in \mathbb{R}^{3\times3}_{sym}, \tag{2.5a}$$

$$\exists\, \gamma_0 > 0 \quad \textit{for all } \xi \in \mathbb{R}^{3\times3}_{sym} \textit{ and almost all } x \in \Omega: \quad \mathbb{C}(x)\xi : \xi \geq \gamma_0|\xi|^2. \tag{2.5b}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a further constitutive function such that

$$g \in \mathrm{C}^2(\mathbb{R}) \text{ with } g',\, g'' \in L^\infty(\mathbb{R}), \quad \text{and } \exists\, \gamma_1,\, \gamma_2 > 0\ \forall\, z \in \mathbb{R}:\ \gamma_1 \leq g(z) \leq \gamma_2. \tag{2.6}$$

Then, we take the elastic energy

$$\mathcal{E}_2 : [0,T] \times \mathcal{U} \times \mathcal{Z} \to \mathbb{R} \ \ \text{defined by} \ \ \mathcal{E}_2(t,u,z) := \int_\Omega g(z) W(x, \varepsilon(u + u_D(t)))\,\mathrm{d}x - \langle \ell(t), u\rangle_{\mathcal{U}}$$

where $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetrized strain tensor and $\ell \in \mathrm{C}^0([0,T], \mathcal{U}^*)$ an external loading. Further requirements on ℓ and u_D will be specified in Assumption 2.8 ahead. For $u \in \mathcal{U}$ and $z \in \mathcal{Z}$ the stored energy is then defined by

$$\mathcal{E}(t,u,z) := \mathcal{I}_1(z) + \mathcal{E}_2(t,u,z). \tag{2.7}$$

Minimizing the functional $\mathcal{E}$ with respect to the displacements we obtain the *reduced energy*

$$\mathcal{I} : [0,T] \times \mathcal{Z} \to \mathbb{R} \ \ \text{given by} \ \ \mathcal{I}(t,z) := \mathcal{I}_1(z) + \mathcal{I}_2(t,z) \text{ with } \mathcal{I}_2(t,z) := \inf\{\mathcal{E}_2(t,v,z) \ :\ v \in \mathcal{U}\}. \tag{2.8}$$

2.2. **Preliminary regularity results.** We focus on the regularity properties of the operator $L_{g(z)} : H^1_0(\Omega;\mathbb{R}^3) \to W^{-1,2}(\Omega;\mathbb{R}^3)$ associated with the following bilinear form describing linear elasticity, i.e.,

$$\langle L_{g(z)} u, v\rangle := \int_\Omega g(z)\mathbb{C}\varepsilon(u) : \varepsilon(v)\,\mathrm{d}x \quad \text{for all } u,v \in H^1_0(\Omega;\mathbb{R}^3), \tag{2.9}$$

where $\mathbb{C}$ is from (2.5), g from (2.6), and z is a fixed element in $\mathcal{Z} = W^{1,q}(\Omega)$, with $q > 3$. Our first result extends [KRZ15, Lemma 2.3] to a wider range of exponents, cf. Remark 2.4 below.

Lemma 2.3. *Under Assumption 2.1, let $\mathbb{C}$ and g comply with* (2.5) *and* (2.6)*, respectively. Then, there holds*

- *(a) For every $p > 1$ and $z \in W^{1,q}(\Omega)$ the operator $L_{g(z)} : W^{1,p}_0(\Omega) \to W^{-1,p}(\Omega)$ is a topological isomorphism.*
- *(b) Uniform estimate: For every $p_* > 2$ there exists a constant $c_{q,p_*} > 0$ such that for all $z \in W^{1,q}(\Omega)$ and $p \in [p_*', p_*]$ it holds*

$$\|L^{-1}_{g(z)}\|_{W^{-1,p}(\Omega;\mathbb{R}^3)\to W^{1,p}_0(\Omega;\mathbb{R}^3)} \leq c_{q,p_*}(1 + \|\nabla z\|_{L^q(\Omega)})^{\hat{k}_* \frac{p_*|p-2|}{p(p_*-2)}}, \tag{2.10}$$

where $\hat{k}_* \in \mathbb{N}$ is the smallest integer with $\hat{k}_* > \frac{3q}{2(q-3)}$.

Proof. The first statement is a consequence of [Val78, Theorem 3], see also [MR03, Theorem 7.1]. The uniform estimate follows along the same lines as in the proof of [KRZ15, Lemma 2.3], relying on a recursion argument developed in [BM13]. □

Remark 2.4. Lemma 2.3 enhances [KRZ15, Lemma 2.3] thanks to the stronger regularity condition on the reference domain Ω, which in [KRZ15] was only required to fulfill these properties:

(i) The spaces $W^{1,p}_{\Gamma_D}(\Omega;\mathbb{R}^d) = \{u \in W^{1,p}(\Omega;\mathbb{R}^d) \, : \, u|_{\Gamma_D} = 0\}$, $p \in (1,\infty)$ (and Γ_D with positive Hausdorff measure, but possibly different from $\partial\Omega$, was allowed in [KRZ15]), form an interpolation scale.

(ii) There exists $p_* > 3$ such that for all $p \in [2, p_*]$ the operator $L : W^{1,p}_{\Gamma_D}(\Omega;\mathbb{R}^d) \to W^{-1,p}_{\Gamma_D}(\Omega;\mathbb{R}^d)$ is an isomorphism.

It was for such $p_* > 3$, in fact, that the isomorphism property (a) and the uniform estimate (2.10) were obtained in [KRZ15, Lemma 2.3]. Let us highlight that, instead, in Lemma 2.3 property (a) is guaranteed for all $p > 1$, and (2.10) is shown for *every* $p_* > 2$.

The most relevant consequence of Assumption 2.1 for our analysis, though, is given by this second, enhanced, elliptic regularity result, which is to be compared with [BM13, Lemma A.1], holding for homogeneous Neumann boundary conditions.

Lemma 2.5. *Under Assumption 2.1, let $\mathbb{C}$ and g comply with (2.5) and (2.6), respectively. Then, for all $z \in W^{1,q}(\Omega)$ the operator $L_{g(z)} : \mathcal{U} \to \mathcal{U}^*$ fulfills*

$$L^{-1}_{g(z)}(h) \in H^2(\Omega;\mathbb{R}^3) \quad \text{for all } h \in L^2(\Omega;\mathbb{R}^3)$$

and there exists $c_0 > 0$ such that for all $z \in W^{1,q}(\Omega)$ and all $h \in L^2(\Omega;\mathbb{R}^3)$

$$\|u\|_{H^2(\Omega)} \le c_0 (1 + \|\nabla z\|_{L^q(\Omega)})^\alpha (\|h\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}), \tag{2.11}$$

where $u = L^{-1}_{g(z)}(h)$ and $\alpha \ge 2$ is the smallest integer bigger than or equal to $q/(q-3)$.

Proof. The proof of [BM13, Lemma A.1] can be directly transferred to our situation having in mind that for every $p \in (1,\infty)$ the operator

$$L_{\mathbb{C}} = L_{g(1)} : W^{1,p}_0(\Omega) \cap W^{2,p}(\Omega) \to L^p(\Omega), \quad u \mapsto -\mathrm{div}\mathbb{C}\varepsilon(u)$$

is a continuous isomorphism, cf. Theorem A.3. □

Remark 2.6. Observe that $\sup_{p\in[p_*',p_*]} \frac{p_*|p-2|}{p(p_*-2)} \le 1$, hence we can estimate from above the right-hand side of (2.10) by $(1 + \|\nabla z\|_{L^q(\Omega)})^{\hat{k}_*}$. That is why, in what follows, whenever applying estimates (2.10) and (2.11), possibly with two different elements z_1, $z_2 \in \mathcal{Z}$, we will simply use the quantity

$$P(z_1, z_2) := (1 + \|\nabla z_1\|_{L^q(\Omega)} + \|\nabla z_2\|_{L^q(\Omega)})^{k_*}\,, \tag{2.12}$$

where $k_* := \max\{\hat{k}_*, \alpha\} + 1$ with $\hat{k}_*$ from Lemma 2.3 and α from (2.11). With this, (2.11) can be rewritten in terms of the quantity P as

$$\|u\|_{H^2(\Omega)} \le c_0 P(z,0)(\|h\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}).$$

In the sequel we will frequently use the following regularity result from [Sav98, Theorem 2 & Remark 3.5] for solutions of the q-Laplace equation:

Proposition 2.7. *For every $q > 2$ there exists a constant $C_q > 0$ such that for all $f \in L^{q'}(\Omega)$ it holds: If $z \in W^{1,q}(\Omega)$ satisfies $\langle A_q z, \widetilde{z}\rangle = \langle f, \widetilde{z}\rangle$ for all $\widetilde{z} \in W^{1,q}(\Omega)$, then for all $\sigma \in (0, \frac{1}{q})$ the function z belongs to $W^{1+\sigma,q}(\Omega)$ and*

$$\|z\|_{W^{1+\sigma,q}(\Omega)} \le C_q(\|f\|_{L^{q'}(\Omega)} + \|z\|_{L^q(\Omega)}). \tag{2.13}$$

Note that on the right-hand side of (2.13) the L^q-norm of z appears since A_q is not bijective on $W^{1,q}(\Omega)$.

2.3. **Properties of the reduced energy.** Relying on Lemmas 2.3 and 2.5, we will show that the reduced energy functional $\mathcal{I}$ enjoys a series of differentiability properties, which in fact improve those obtained in [KRZ15, Sec. 2.3], under the additional

Assumption 2.8 (The external loadings)**.** *From now on, we will suppose that* ℓ *and* u_D *comply with the following requirements*

$$\begin{aligned} &\ell \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)) \cap \mathrm{C}^{1,1}([0,T];W^{-1,3}(\Omega;\mathbb{R}^3)),\\ &u_D \in L^\infty(0,T;H^2(\Omega;\mathbb{R}^3)) \cap \mathrm{C}^{1}([0,T];W^{1,3}(\Omega;\mathbb{R}^3)). \end{aligned} \tag{2.14}$$

The starting point is the following result, which improves [KRZ15, Lemmas 2.6, 2.7].

Lemma 2.9 (Existence of minimizers for $\mathcal{E}(t,\cdot,z)$ & their continuous dependence on the data)**.** *Under Assumptions 2.1, 2.2, and 2.8, for every* $(t,z) \in [0,T]\times\mathcal{Z}$ *there exists a unique minimizer* $u_{\min}(t,z) \in \mathcal{U}$ *for the stored energy* $\mathcal{E}(t,\cdot,z)$ (2.7)*. In fact,* $u_{\min}(t,z) \in H^2(\Omega;\mathbb{R}^3)$*. Moreover, there exist positive constants* c_1 *and* c_2 *such that for all* (t,z), (t_1,z_1), $(t_2,z_2) \in [0,T]\times\mathcal{Z}$ *and for all* $p_*>2$

$$\|u_{\min}(t,z)\|_{H^2(\Omega)} \le c_1 P(z,0)\left(\|\ell(t)\|_{L^2(\Omega)} + \|u_D(t)\|_{H^2(\Omega)}\right); \tag{2.15}$$

$$\begin{aligned} &\|u_{\min}(t_1,z_1) - u_{\min}(t_2,z_2)\|_{W^{1,p}(\Omega)}\\ &\le c_2 P(z_1,z_2)^2\left(|t_1-t_2| + \|z_1-z_2\|_{L^{6p/(6-p)}(\Omega)}\right)\left(\|\ell\|_{\mathrm{C}^1([0,T];W^{-1,p}(\Omega))} + \|u_D(t)\|_{\mathrm{C}^1([0,T];W^{1,p}(\Omega))}\right) \end{aligned} \tag{2.16}$$

for all $p\in[p_*',\min\{p_*,3\}]$*, with* $P(\cdot,\cdot)$ *defined by* (2.12)*. In particular, there holds*

$$\begin{aligned} &\|u_{\min}(t_1,z_1) - u_{\min}(t_2,z_2)\|_{W^{1,3}(\Omega)}\\ &\le c_2 P(z_1,z_2)^2\left(|t_1-t_2| + \|z_1-z_2\|_{L^{6}(\Omega)}\right)\left(\|\ell\|_{\mathrm{C}^1([0,T];W^{-1,3}(\Omega))} + \|u_D(t)\|_{\mathrm{C}^1([0,T];W^{1,3}(\Omega))}.\right), \end{aligned} \tag{2.17}$$

Finally, the reduced energy $\mathcal{I}$ *from* (2.8) *is bounded from below and in particular satisfies the following coercivity estimate:*

$$\exists\, c_3,\, c_4>0 \quad \forall\,(t,z)\in[0,T]\times\mathcal{Z}: \qquad \mathcal{I}(t,z) \ge c_3\left(\|\nabla z\|_{L^q(\Omega)}^q + \|z\|_{L^1(\Omega)} + \|u_{\min}(t,z)\|_{H^1(\Omega;\mathbb{R}^3)}^2\right) - c_4. \tag{2.18}$$

Proof. We refer to [KRZ13, Lemma 2.1] for the proof of the existence and uniqueness of $u_{\min}(t,z)$, as well as for estimate (2.18). Clearly, $u_{\min}(t,z)$ satisfies $L_{g(z)}u_{\min}(t,z) = -L_{g(z)}u_D(t) - \ell(t)$. Observe that $L_{g(z)}u_D(t) \in L^2(\Omega)$. Indeed, by the assumptions on g, $\mathbb{C}$ and since $u_D(t)\in H^2(\Omega)$, we have $g(z)\mathrm{div}(\mathbb{C}\varepsilon(u_D(t)) \in L^2(\Omega)$. On the other hand, $\mathbb{C}\varepsilon(u_D(t))\nabla_x g(z) = g'(z)\mathbb{C}\varepsilon(u_D(t))\nabla z \in L^2(\Omega)$, which follows by Hölder's inequality taking into account that $H^1(\Omega)\subset L^6(\Omega)$ and that $q>3$. Moreover, it holds $\|L_{g(z)}u_D(t)\|_{L^2(\Omega)} \le c(1+\|\nabla z\|_{L^q(\Omega)})\|u_D(t)\|_{H^2(\Omega)}$. Hence, it follows from (2.11), cf. also Remark 2.6, and (2.10) with $p=2$ that

$$\begin{aligned} \|u_{\min}(t,z)\|_{H^2(\Omega)} &\le c_0(1+\|\nabla z\|_{L^q(\Omega)})^\alpha(\|\ell(t)\|_{L^2(\Omega)} + \|\mathrm{div}(g(z)\mathbb{C}\varepsilon(u_D(t))\|_{L^2(\Omega)} + \|u_{\min}(t,z)\|_{H^1(\Omega)})\\ &\le c(1+\|\nabla z\|_{L^q(\Omega)})^\alpha\big(\|\ell(t)\|_{L^2(\Omega)} + (1+\|\nabla z\|_{L^q(\Omega)})\|u_D(t)\|_{H^2(\Omega)}\big)\\ &\le c_1 P(z,0)\big(\|\ell(t)\|_{L^2(\Omega)} + \|u_D(t)\|_{H^2(\Omega)}\big). \end{aligned}$$

All in all, we conclude (2.15).

Finally, in order to show (2.16) we mimic the argument from the proofs of [KRZ13, Lemma 2.2] & [KRZ15, Lemma 2.7]. Namely, for $i=1,2$, let $u_i := u_{\min}(t_i,z_i) \in H^2(\Omega;\mathbb{R}^3)$. From the corresponding Euler-Lagrange equations we obtain that u_1-u_2 satisfies for all $v\in\mathcal{U}$

$$\begin{aligned} \int_\Omega g(z_1)\mathbb{C}\varepsilon(u_1-u_2) \colon \varepsilon(v)\,\mathrm{d}x &= \int_\Omega \big(g(z_2)-g(z_1)\big)\mathbb{C}\varepsilon(u_2)\colon\varepsilon(v)\,\mathrm{d}x\\ &\quad - \int_\Omega \big(g(z_1)\mathbb{C}\varepsilon(u_D(t_1)) - g(z_2)\mathbb{C}\varepsilon(u_D(t_2))\big)\colon\varepsilon(v)\,\mathrm{d}x + \int_\Omega (\ell(t_1)-\ell(t_2))\,v\,\mathrm{d}x. \end{aligned} \tag{2.19}$$

Taking into account that, for $i,j\in\{1,2\}$ $g(z_i)\varepsilon(u_j)\in L^6(\Omega;\mathbb{R}^{3\times3})$ in view of (2.6) and of the fact that $u_j\in H^2(\Omega;\mathbb{R}^3)$, giving $\varepsilon(u_j)\in L^6(\Omega;\mathbb{R}^{3\times3})$, and exploiting condition (2.14) on ℓ and u_D, via a density argument we see that (2.19) extends to test functions $v \in W_0^{1,6/5}(\Omega;\mathbb{R}^3)$. Hence, $u_1 - u_2$ fulfills for all $v \in W_0^{1,6/5}(\Omega;\mathbb{R}^3)$ the relation

$$\int_\Omega g(z_1)\mathbb{C}\varepsilon(u_1-u_2):\varepsilon(v)\,\mathrm{d}x = \langle \tilde{\ell}_{1,2}, v\rangle_{W_0^{1,6/5}(\Omega;\mathbb{R}^3)},$$

where $\tilde{\ell}_{1,2} \in W^{-1,6}(\Omega;\mathbb{R}^3)$ subsumes the terms on the right-hand side of (2.19). We now fix an arbitrary $p_* > 2$ and apply estimate (2.10) with $p \in [p_*', \min\{p_*, 3\}]$ (indeed, the restriction $p \le 3$ is in view of conditions (2.14) on ℓ and u_D). We thus obtain $\|u_1-u_2\|_{W^{1,p}(\Omega;\mathbb{R}^3)} \le c_{q,p_*} P(z_1,0)\|\tilde{\ell}_{1,2}\|_{W^{-1,p}(\Omega;\mathbb{R}^3)}$. Hence,

$$\begin{aligned}\|u_1-u_2\|_{W^{1,p}(\Omega;\mathbb{R}^3)} \le c_{p_*,q} P(z_1,0)\big(\|\ell(t_1)-\ell(t_2)\|_{W^{-1,p}(\Omega;\mathbb{R}^3)} + \|(g(z_1)-g(z_2))\mathbb{C}\varepsilon(u_2)\|_{L^p(\Omega;\mathbb{R}^3)} \\ + \|g(z_1)\mathbb{C}\varepsilon(u_D(t_1)) - g(z_2)\mathbb{C}\varepsilon(u_D(t_2))\|_{L^p(\Omega;\mathbb{R}^3)}\big). \end{aligned} \tag{2.20}$$

Now, the Lipschitz continuity of g (with Lipschitz constant C_g) and Hölder's inequality imply that

$$\begin{aligned}\|(g(z_1)-g(z_2))\mathbb{C}\varepsilon(u_2)\|_{L^p(\Omega;\mathbb{R}^3)} &\le C_g\|z_1-z_2\|_{L^{6p/(6-p)}(\Omega)}\|\varepsilon(u_2)\|_{L^6(\Omega;\mathbb{R}^3)} \\ &\le CP(z_2,0)\left(\|\ell(t)\|_{L^2(\Omega)} + \|u_D(t)\|_{H^2(\Omega)}\right)\|z_1-z_2\|_{L^{6p/(6-p)}(\Omega)}\end{aligned} \tag{2.21}$$

where the second estimate follows from (2.15) and from the fact that $\|\varepsilon(u_2)\|_{L^6(\Omega;\mathbb{R}^3)} \le C\|u_2\|_{H^2(\Omega;\mathbb{R}^3)}$ by Sobolev embeddings. Moreover,

$$\begin{aligned}&\|g(z_1)\mathbb{C}\varepsilon(u_D(t_1)) - g(z_2)\mathbb{C}\varepsilon(u_D(t_2))\|_{L^p(\Omega)} \\ &\le \|g(z_1)(\mathbb{C}\varepsilon(u_D(t_1)) - \mathbb{C}\varepsilon(u_D(t_2)))\|_{L^p(\Omega)} + \|(g(z_1)-g(z_2))\mathbb{C}\varepsilon(u_D(t_2))\|_{L^p(\Omega)} \\ &\le \gamma_2|t_1-t_2|\|u_D(t)\|_{\mathrm{C}^1([0,T];W^{1,p}(\Omega))} + C\|u_D\|_{L^\infty(0,T;H^2(\Omega))}\|z_1-z_2\|_{L^{6p/(6-p)}(\Omega)},\end{aligned}$$

where the last estimate follows from the fact that $\|g(z_1)\|_{L^\infty(\Omega)} \le \gamma_2$ by (2.6), as well as the the fact that, for $p \le 6$, $\|\varepsilon(u_D(t_2))\|_{L^p(\Omega)} \le C\|u_D\|_{L^\infty(0,T;H^2(\Omega))}$. All in all, we conclude (2.16), whence (2.17) observing that, for $p = 3$ one has $\frac{6p}{6-p} = 6$. □

Concerning the differentiability in time, we have the following analogue of [KRZ15, Lemma 2.9], [KRZ13, Lemma 2.3],

Lemma 2.10. *Under Assumptions 2.1, 2.2, and 2.8, for every $z \in \mathcal{Z}$ the map $t \mapsto \mathcal{I}(t,z)$ is in $\mathrm{C}^1([0,T];\mathbb{R})$ with*

$$\partial_t\mathcal{I}(t,z) = \int_\Omega g(z)\mathbb{C}\varepsilon(u_{\min}(t,z)+u_D(t))\colon \varepsilon(\dot{u}_D(t))\,\mathrm{d}x - \langle \dot{\ell}(t), u_{\min}(t,z)\rangle_{H_0^1(\Omega;\mathbb{R}^3)}. \tag{2.22}$$

Moreover, there exists a constant $c_5 > 0$ such that for all $t \in [0,T]$, $z \in \mathcal{Z}$ we have

$$|\partial_t\mathcal{I}(t,z)| \le c_5\big(\|u_D\|^2_{\mathrm{C}^1([0,T];H1(\Omega;\mathbb{R}^3))} + \|\ell\|^2_{\mathrm{C}^1([0,T];W^{-1,2}(\Omega;\mathbb{R}^3))}\big). \tag{2.23}$$

Finally, there exists a constant $c_6 > 0$ depending on $\|\ell\|_{\mathrm{C}^{1,1}([0,T];W^{-1,3}(\Omega;\mathbb{R}^3))}$ and $\|u_D\|_{\mathrm{C}^{1,1}([0,T];W^{1,3}(\Omega))}$ such that for all $t_i \in [0,T]$ and $z_i \in \mathcal{Z}$ we have

$$|\partial_t\mathcal{I}(t_1,z_1) - \partial_t\mathcal{I}(t_2,z_2)| \le c_6 P(z_1,z_2)^2\big(|t_1-t_2| + \|z_1-z_2\|_{L^2(\Omega)}\big). \tag{2.24}$$

Let us stress that the quantity on the right-hand side of estimate (2.23), whose proof is developed in [KRZ13, Lemma 2.3], is *independent* of $z \in \mathcal{Z}$.

Proof. We will only develop the proof of (2.24), referring to the proof of [KRZ13, Lemma 2.3] for (2.22) and (2.23). We have

$$
\begin{aligned}
&\partial_t \mathcal{I}(t_1,z_1) - \partial_t \mathcal{I}(t_2,z_2)\\
&= \int_\Omega (g(z_1){-}g(z_2))\mathbb{C}(\varepsilon(u_{\min}(t_1,z_1)+u_D(t_1))) : \varepsilon(\dot u_D(t_1))\,\mathrm{d}x\\
&\quad + \int_\Omega g(z_2)\mathbb{C}(\varepsilon(u_{\min}(t_1,z_1)+u_D(t_1)){-}\varepsilon(u_{\min}(t_2,z_2)+u_D(t_2))) : \varepsilon(\dot u_D(t_1))\,\mathrm{d}x\\
&\quad + \int_\Omega g(z_2)\mathbb{C}(\varepsilon(u_{\min}(t_2,z_2)+u_D(t_2))) : (\varepsilon(\dot u_D(t_1)){-}\varepsilon(\dot u_D(t_2)))\,\mathrm{d}x\\
&\quad - \langle \dot\ell(t_1){-}\dot\ell(t_2), u_{\min}(t_1,z_1)\rangle + \langle \dot\ell(t_2), u_{\min}(t_2,z_2){-}u_{\min}(t_1,z_1)\rangle \doteq I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

To estimate I_1, and I_3 we rely on the fact that $g,\, g' \in L^\infty(\mathbb{R})$, and on (2.15). To estimate I_2 we additionally use the boundedness of g and Hölder's inequality as follows

$$
\begin{aligned}
I_2 &\leq c\|\varepsilon(u_{\min}(t_1,z_1)+u_D(t_1)){-}\varepsilon(u_{\min}(t_2,z_2)+u_D(t_2))\|_{L^{3/2}(\Omega)}\|\varepsilon(\dot u_D(t_1))\|_{L^3(\Omega)}\\
&\leq cP(z_1,z_2)^2\left(|t_1{-}t_2| + \|z_1{-}z_2\|_{L^2(\Omega)}\right)\left(\|\ell\|_{\mathrm{C}^1([0,T];W^{-1,3/2}(\Omega))} + \|u_D(t)\|_{\mathrm{C}^1([0,T];W^{1,3/2}(\Omega))},\right)
\end{aligned}
$$

where the second estimate ensues from (2.16) with $p = 3/2$ (which yields $6p/(6-p) = 2$), and from (2.14). By (2.14) and (2.16) we also estimate I_4 and I_5. □

We now discuss the differentiability of $\mathcal{I}$ with respect to z; we shall denote by $\mathrm{D}_z\mathcal{I}(t,\cdot) : \mathcal{Z} \to \mathcal{Z}^*$ the Gâteaux-differential of the functional $\mathcal{I}(t,\cdot)$. For the proof of the following result, we refer to [KRZ15, Lemma 2.10], [KRZ13, Lemma 2.4].

Lemma 2.11. *Under Assumptions 2.1, 2.2, and 2.8, for all $t \in [0,T]$ the functional $\mathcal{I}(t,\cdot) : \mathcal{Z} \to \mathbb{R}$ is Gâteaux-differentiable at all $z \in \mathcal{Z}$, and for all $\eta \in \mathcal{Z}$ we have*

$$
\langle \mathrm{D}_z\mathcal{I}(t,z),\eta\rangle_{\mathcal{Z}} = \langle A_q z,\eta\rangle_{\mathcal{Z}} + \int_\Omega f'(z)\eta\,\mathrm{d}x + \int_\Omega g'(z)\widetilde{W}(t,\nabla u_{\min}(t,z))\eta\,\mathrm{d}x, \tag{2.25}
$$

where we use the abbreviation $\widetilde{W}(t,\nabla v) = W(x,\varepsilon(v+\nabla u_D(t))) = \frac12\mathbb{C}\varepsilon(v+u_D(t)){:}\varepsilon(v+u_D(t))$. In particular, the following estimate holds with a constant c_7 that depends on the data ℓ, u_D, but is independent of t and z:

$$
\forall\,(t,z)\in[0,T]\times\mathcal{Z}: \quad \|\mathrm{D}_z\mathcal{I}(t,z)\|_{\mathcal{Z}^*} \leq c_7\left(\|z\|_{\mathcal{Z}}^{q-1} + \|f'(z)\|_{L^\infty(\Omega)} + 1\right). \tag{2.26}
$$

Hereafter, we will use the short-hand notation

$$
\widetilde{\mathcal{I}}(t,z) := \mathcal{I}_2(t,z) + \int_\Omega f(z)\,\mathrm{d}x \quad \text{for all } (t,z)\in[0,T]\times\mathcal{Z} \tag{2.27}
$$

with $\mathcal{I}_2$ from (2.8) as the part of the reduced energy collecting all lower order terms. Accordingly, $\mathrm{D}_z\mathcal{I}$ from (2.25) decomposes as

$$
\mathrm{D}_z\mathcal{I}(t,z) = A_q z + \mathrm{D}_z\widetilde{\mathcal{I}}(t,z) \quad \text{for all } (t,z)\in[0,T]\times\mathcal{Z}. \tag{2.28}
$$

In view of (2.25), and taking into account the $H^2(\Omega;\mathbb{R}^3)$-regularity of $u_{\min}$ from Lemma 2.5, the term $\mathrm{D}_z\widetilde{\mathcal{I}}(t,z)$ can be identified with an element of $L^2(\Omega)$. In Lemma 2.12 below we will even show that the map $(t,z)\mapsto \mathrm{D}_z\widetilde{\mathcal{I}}(t,z)$ is Lipschitz continuous w.r.t. a suitable *Lebesgue* norm. Therefore, with the symbol $\mathrm{D}_z\widetilde{\mathcal{I}}$ we shall denote both the derivative of $\widetilde{\mathcal{I}}$ as an operator, and the corresponding density in $L^2(\Omega)$. Accordingly, we shall write

$$
\text{for a given } v\in L^2(\Omega) \qquad \int_\Omega \mathrm{D}_z\widetilde{\mathcal{I}}(t,z)v\,\mathrm{d}x \quad \text{ in place of } \quad \langle \mathrm{D}_z\widetilde{\mathcal{I}}(t,z),v\rangle_{L^2(\Omega)}. \tag{2.29}
$$

For $h\in\mathrm{C}^0(\mathbb{R})$ and $z_1,z_2\in\mathcal{Z}$ let

$$
C_h(z_1,z_2) = \max\{\,|h(s)| \;:\; |s| \leq \|z_1\|_{L^\infty(\Omega)} + \|z_2\|_{L^\infty(\Omega)}\,\}. \tag{2.30}
$$

This notation will be used along the proof of the following lemma.

Lemma 2.12. *Under Assumptions 2.1, 2.2, and 2.8, there exists a constant $c_8 > 0$ that depends on the norms $\|\ell\|_{\mathrm{C}^{1,1}([0,T];W^{-1,3}(\Omega;\mathbb{R}^3))}$ and $\|u_D\|_{\mathrm{C}^1([0,T];W^{1,3}(\Omega;\mathbb{R}^3))}$ such that for all $t_i \in [0,T]$ and all $z_i \in \mathcal{Z}$ it holds*

$$\left|\widetilde{\mathcal{I}}(t_1,z_1) - \widetilde{\mathcal{I}}(t_2,z_2)\right| \leq c_8(1 + C_{f'}(z_1,z_2) + P(z_1,z_2)^3)\left(|t_1-t_2| + \|z_1-z_2\|_{L^3(\Omega)}\right), \tag{2.31}$$

with $C_{f'}(z_1,z_2)$ as in (2.30)*, corresponding to $h = f'$. Further,*

$$\begin{aligned} &\|\mathrm{D}_z\widetilde{\mathcal{I}}(t_1,z_1) - \mathrm{D}_z\widetilde{\mathcal{I}}(t_2,z_2)\|_{L^2(\Omega)} \\ &\leq c_8\big(1 + C_{f'}(z_1,z_2) + P(z_1,z_2)^3\big)\big(|t_1-t_2| + \|z_1-z_2\|_{L^6(\Omega)}\big), \end{aligned} \tag{2.32}$$

$$\begin{aligned} &\|\mathrm{D}_z\widetilde{\mathcal{I}}(t_1,z_1) - \mathrm{D}_z\widetilde{\mathcal{I}}(t_2,z_2)\|_{L^{4/3}(\Omega)} \\ &\leq c_8\big(1 + C_{f'}(z_1,z_2) + P(z_1,z_2)^3\big)\big(|t_1-t_2| + \|z_1-z_2\|_{L^4(\Omega)}\big), \end{aligned} \tag{2.33}$$

and

$$\|\mathrm{D}_z\widetilde{\mathcal{I}}(t,z)\|_{L^2(\Omega)} \leq c_8(1 + \|f'(z)\|_{L^\infty(\Omega)} + P(z,0)^2) \qquad \text{for all } (t,z) \in [0,T]\times\mathcal{Z}. \tag{2.34}$$

Proof. Although the proof follows the same lines as that of [KRZ15, Lemma 2.12], let us briefly see how the improved estimates (2.15) and (2.17) lead to (2.31), (2.34), and (2.33), while we will omit the calculations for (2.34). As for (2.31), we observe that

$$\begin{aligned} \left|\widetilde{\mathcal{I}}(t_1,z_1) - \widetilde{\mathcal{I}}(t_2,z_2)\right| \leq & \int_\Omega |f(z_1)-f(z_2)|\,\mathrm{d}x + \int_\Omega |g(z_1)-g(z_2)||\widetilde{W}(t_1,\nabla u_1)|\,\mathrm{d}x \\ & + \int_\Omega |g(z_2)||\widetilde{W}(t_1,\nabla u_1) - \widetilde{W}(t_2,\nabla u_2)|\,\mathrm{d}x + |\,\langle \ell(t_1)-\ell(t_2), u_1\rangle_{\mathcal{U}}\,| \\ & + |\,\langle \ell(t_2), u_1-u_2\rangle_{\mathcal{U}}\,| \doteq I_1 + I_2 + I_3 + I_4 + I_5, \end{aligned}$$

where $u_i := u_{\min}(t_i,z_i) \in H^2(\Omega;\mathbb{R}^3)$ and, as above, $\widetilde{W}(t_i,\nabla u_i) = \frac{1}{2}\mathbb{C}\varepsilon(u_i+u_D(t_i)){:}\varepsilon(u_i+u_D(t_i))$ for $i=1,2$. We observe that (cf. notation (2.30))

$$\begin{aligned} I_1 &\leq C_{f'}(z_1,z_2)\|z_1-z_2\|_{L^1(\Omega)}, \\ I_2 &\leq C\|z_1-z_2\|_{L^2(\Omega)}\|\varepsilon(u_1+u_D(t_1))\|_{L^3(\Omega)}\|\varepsilon(u_1+u_D(t_1))\|_{L^6(\Omega)} \\ &\leq C'P(z_1,0)^2\|z_1-z_2\|_{L^2(\Omega)}, \\ I_3 &\leq C\|\varepsilon(u_1+u_D(t_1)) + \varepsilon(u_2+u_D(t_2))\|_{L^2(\Omega)}\|\varepsilon(u_1+u_D(t_1)) - \varepsilon(u_2+u_D(t_2))\|_{L^2(\Omega)} \\ &\leq CP(z_1,z_2)P(z_1,z_2)^2(|t_1-t_2| + \|z_1-z_2\|_{L^3(\Omega)}), \\ I_4 &\leq C|t_1-t_2|\|u_1\|_{H^1(\Omega)} \leq C'|t_1-t_2|, \\ I_5 &\leq C\|u_1-u_2\|_{H^1(\Omega)} \leq CP(z_1,z_2)^2(|t_1-t_2| + \|z_1-z_2\|_{L^3(\Omega)})\,, \end{aligned}$$

where, in the estimate for I_2 we have exploited (2.15), while in the estimates for I_3 and I_5 we have also resorted to (2.16) with $p=2$. All in all, we conclude (2.31). The estimate for I_4 follows from (2.14).

As for (2.32), we have that

$$\begin{aligned} \|\mathrm{D}_z\widetilde{\mathcal{I}}(t_1,z_1) - \mathrm{D}_z\widetilde{\mathcal{I}}(t_2,z_2)\|_{L^2(\Omega)} \leq & \|f'(z_1)-f'(z_2)\|_{L^2(\Omega)} + \|(g'(z_1){-}g'(z_2))\widetilde{W}(t_1,\nabla u_1)\|_{L^2(\Omega)} \\ & + \|g'(z_2)(\widetilde{W}(t_1,\nabla u_1){-}\widetilde{W}(t_2,\nabla u_2))\|_{L^2(\Omega)} \doteq I_6 + I_7 + I_8\,. \end{aligned}$$

We observe that $I_6 \leq C_{f'}(z_1,z_2)\|z_1-z_2\|_{L^2(\Omega)}$, while

$$\begin{aligned} I_7 &\leq C\|z_1-z_2\|_{L^3(\Omega)}\|\varepsilon(u_1+u_D(t_1))\|_{L^6(\Omega)} \leq C'\|z_1-z_2\|_{L^3(\Omega)}P(z_1,0), \\ I_8 &\leq C\|\varepsilon(u_1+u_D(t_1)) + \varepsilon(u_2+u_D(t_2))\|_{L^6(\Omega)}\|\varepsilon(u_1+u_D(t_1)) - \varepsilon(u_2+u_D(t_2))\|_{L^3(\Omega)} \\ &\leq C'P(z_1,z_2)^3(|t_1-t_2| + \|z_1-z_2\|_{L^6(\Omega)})\,. \end{aligned}$$

thanks to estimates (2.15) and (2.17) and the fact that g', $g'' \in L^\infty(\mathbb{R})$. The proof of (2.33) follows the very same lines: we estimate $\|f'(z_1)-f'(z_2)\|_{L^{4/3}(\Omega)}$ by means of $C_{f'}(z_1,z_2)\|z_1-z_2\|_{L^{4/3}(\Omega)}$, while we have with

Hölder's inequality

$$\|(g'(z_1)-g'(z_2))\widetilde{W}(t_1,\nabla u_1)\|_{L^{4/3}(\Omega)} \leq C\|z_1-z_2\|_{L^4(\Omega)}\|\varepsilon(u_1+u_D(t_1))\|^2_{L^4(\Omega)} \leq C'\|z_1-z_2\|_{L^4(\Omega)},$$

where the last estimate follows from (2.10) with $p=4$. Finally,

$$\begin{aligned}&\|g'(z_2)(\widetilde{W}(t_1,\nabla u_1)-\widetilde{W}(t_2,\nabla u_2))\|_{L^{4/3}(\Omega)}\\&\leq C\|\varepsilon(u_1+u_D(t_1))+\varepsilon(u_2+u_D(t_2))\|_{L^4(\Omega)}\|\varepsilon(u_1+u_D(t_1))-\varepsilon(u_2+u_D(t_2))\|_{L^2(\Omega)}\\&\leq C'P(z_1,z_2)^3(|t_1-t_2|+\|z_1-z_2\|_{L^3(\Omega)}).\end{aligned}$$

This concludes the proof. □

From all of the above results, and in particular from Lemma 2.12, we now draw a series of consequences on which our subsequent analysis will rely. First of all, we observe the Fréchet differentiability of the functional $z\in\mathcal{Z}\mapsto\mathcal{I}(t,z)$. This is due to the continuity of the mapping $z\in\mathcal{Z}\mapsto\mathrm{D}_z\mathcal{I}(t,z)\in\mathcal{Z}^*$, in turn due to the continuity of $z\mapsto A_qz$ and of $z\mapsto\mathrm{D}_z\widetilde{\mathcal{I}}(t,z)$. If restricted to bounded sets in $\mathcal{Z}$, the latter mapping is even continuous with values in $L^2(\Omega)$ w.r.t. to $L^6(\Omega)$-convergence for z, cf. (2.32) (and the restriction of the power functional $\partial_t\mathcal{I}$ is continuous w.r.t. $L^2(\Omega)$-convergence for z). Taking into account that $\mathcal{Z}\Subset L^6(\Omega)$, we may then claim the continuity of $\mathrm{D}_z\widetilde{\mathcal{I}}$ and $\partial_t\mathcal{I}$ w.r.t. weak convergence in $\mathcal{Z}$.

Corollary 2.13 (Fréchet differentiability of $\mathcal{I}$)**.** *Under Assumptions 2.1, 2.2, and 2.8, the functional $\mathcal{I}$ is Fréchet differentiable on $[0,T]\times\mathcal{Z}$ and*

$$t_n\to t \text{ and } z_n\to z \text{ strongly in } \mathcal{Z} \text{ implies } \mathrm{D}_z\mathcal{I}(t_n,z_n)\to\mathrm{D}_z\mathcal{I}(t,z) \text{ strongly in } \mathcal{Z}^*. \tag{2.35}$$

Furthermore,

$$\begin{aligned}&t_n\to t \text{ and } z_n\rightharpoonup z \text{ in } \mathcal{Z} \text{ implies}\\&\liminf_{n\to\infty}\mathcal{I}(t_n,z_n)\geq\mathcal{I}(t,z),\quad \widetilde{\mathcal{I}}(t_n,z_n)\to\widetilde{\mathcal{I}}(t,z),\quad \partial_t\mathcal{I}(t_n,z_n)\to\partial_t\mathcal{I}(t,z),\\&\mathrm{D}_z\widetilde{\mathcal{I}}(t_n,z_n)\to\mathrm{D}_z\widetilde{\mathcal{I}}(t,z) \text{ strongly in } L^2(\Omega).\end{aligned} \tag{2.36}$$

We now observe a sort of (generalized) λ-convexity property for $\mathcal{I}(t,\cdot)$, (2.38) below, involving the $H^1(\Omega)$ and the $L^1(\Omega)$-norm, valid on bounded sets in $\mathcal{Z}$ (indeed, note that the constant modulating the $L^1(\Omega)$-norm in (2.38) depends on the radius of a $\mathcal{Z}$-ball).

Corollary 2.14 (λ-convexity of $\mathcal{I}$)**.** *Under Assumptions 2.1, 2.2, and 2.8, there exists a constant $\alpha>0$ and for every $M>0$ there exists $\Lambda_M>0$ such that for every $t\in[0,T]$, z_1, $z_2\in\mathcal{Z}$ with $\|z_1\|_{\mathcal{Z}}+\|z_2\|_{\mathcal{Z}}\leq M$ and for every $\theta\in[0,1]$ the functional $\mathcal{L}$ with*

$$\mathcal{L}(t,z):=\mathcal{I}(t,z)+\frac{1}{2}\|z\|^2_{L^2(\Omega)} \tag{2.37}$$

complies with

$$\begin{aligned}\mathcal{L}(t,(1-\theta)z_1+\theta z_2)\leq&(1-\theta)\mathcal{L}(t,z_1)+\theta\mathcal{L}(t,z_2)\\&-\theta(1-\theta)(\alpha\|z_1-z_2\|^2_{H^1(\Omega)}-\Lambda_M\|z_1-z_2\|^2_{L^1(\Omega)}).\end{aligned} \tag{2.38}$$

Proof. From (2.3) it follows that the mapping $A\in\mathbb{R}^3\mapsto G_q(A)-\frac{c_q}{2}|A|^2$ is convex, which entails that $A\mapsto G_q(A)$ is c_q-convex, i.e. there holds $G_q((1-\theta)A_1+\theta A_2)\leq(1-\theta)G_q(A_1)+\theta G_q(A_2)-\frac{c_q}{2}\theta(1-\theta)|A_1-A_2|^2$ for every A_1, $A_2\in\mathbb{R}^3$ and $\theta\in[0,1]$. As a consequence, we have that

$$\mathcal{I}_q((1-\theta)z_1+\theta z_2)\leq(1-\theta)\mathcal{I}_q(z_1)+\theta\mathcal{I}_q(z_2)-\frac{c_q}{2}\theta(1-\theta)\int_\Omega|\nabla(z_1-z_2)|^2\,\mathrm{d}x\,. \tag{2.39}$$

As for $\widetilde{\mathcal{I}}$, with trivial calculations we have that

$$\begin{aligned}&\widetilde{\mathcal{I}}(t,(1-\theta)z_1+\theta z_2)-(1-\theta)\widetilde{\mathcal{I}}(t,z_1)-\theta\widetilde{\mathcal{I}}(t,z_2)\\&=(1-\theta)\left(\widetilde{\mathcal{I}}(t,(1-\theta)z_1+\theta z_2)-\widetilde{\mathcal{I}}(t,z_1)\right)+\theta\left(\widetilde{\mathcal{I}}(t,(1-\theta)z_1+\theta z_2)-\widetilde{\mathcal{I}}(t,z_2)\right)\doteq I_1+I_2.\end{aligned}$$

There holds

$$
\begin{aligned}
I_1 &= (1-\theta)\int_0^1\int_\Omega \mathrm{D}_z\widetilde{\mathcal{I}}(t,(1-s)z_1+s((1-\theta)z_1+\theta z_2))\theta(z_2-z_1)\,\mathrm{d}x\,\mathrm{d}s\\
&= (1-\theta)\theta\int_0^1\int_\Omega \Big(\mathrm{D}_z\widetilde{\mathcal{I}}(t,(1-s)z_1+s((1-\theta)z_1+\theta z_2))-\mathrm{D}_z\widetilde{\mathcal{I}}(t,z_1)\Big)(z_2-z_1)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad -(1-\theta)\theta\int_\Omega \mathrm{D}_z\widetilde{\mathcal{I}}(t,z_1)(z_1-z_2)\,\mathrm{d}x \doteq I_{1,1}+I_{1,2}\,.
\end{aligned}
$$

We now estimate $I_{1,1}$ by using Hölder's inequality and inequality (2.33), taking into account that $(1-s)z_1+s((1-\theta)z_1+\theta z_2)-z_1 = s\theta(z_2-z_1)$. Therefore,

$$
\begin{aligned}
|I_{1,1}| &\leq c_8\theta(1-\theta)\int_0^1 (1+C_{f'}(z_1,\zeta_{1,2})+P(z_1,\zeta_{1,2})^3)\|s\theta(z_2-z_1)\|_{L^4(\Omega)}\|z_2-z_1\|_{L^4(\Omega)}\,\mathrm{d}s\\
&\leq \widetilde{C}_1(M)(1-\theta)\theta\|z_2-z_1\|^2_{L^4(\Omega)},
\end{aligned}
$$

where we have used the place-holder $\zeta_{1,2}:=(1-s)z_1+s((1-\theta)z_1+\theta z_2)$, and where $\widetilde{C}_1(M)>0$ depends on the constant M that bounds $\|z_1\|_{\mathcal{Z}}$ and $\|z_1\|_{\mathcal{Z}}$. With analogous calculations one has that

$$
I_2 \leq \widetilde{C}_1(M)(1-\theta)\theta\|z_2-z_1\|^2_{L^4(\Omega)} + \underbrace{(1-\theta)\theta\int_\Omega \mathrm{D}_z\widetilde{\mathcal{I}}(t,z_2)(z_1-z_2)\,\mathrm{d}x}_{I_{2,2}}\,.
$$

Therefore, estimating $I_{1,2}+I_{2,2}\leq \widetilde{C}_2(M)(1-\theta)\theta\|z_2-z_1\|^2_{L^4(\Omega)}$ with the same arguments as above, we conclude that

$$
\widetilde{\mathcal{I}}(t,(1-\theta)z_1+\theta z_2) \leq (1-\theta)\widetilde{\mathcal{I}}(t,z_1)+\theta\widetilde{\mathcal{I}}(t,z_2)+\frac{\widetilde{C}(M)}{2}(1-\theta)\theta\|z_2-z_1\|^2_{L^4(\Omega)} \tag{2.40}
$$

for some $\widetilde{C}(M)>0$. We now combine (2.39) with (2.40). Adding to this the trivial identity

$$
\frac12\|(1-\theta)z_1+\theta z_2\|^2_{L^2(\Omega)} = \frac{(1-\theta)}{2}\|z_1\|^2_{L^2(\Omega)}+\frac{\theta}{2}\|z_2\|^2_{L^2(\Omega)}-\frac{(1-\theta)\theta}{2}\|z_1-z_2\|^2_{L^2(\Omega)},
$$

and using Ehrling's Lemma, cf. e.g. [RR04, Thm. 7.30], to estimate $\|\eta\|^2_{L^4(\Omega)}\leq\delta\|\eta\|^2_{H^1(\Omega)}+C(\delta)\|\eta\|^2_{L^1(\Omega)}$ for arbitrary $\delta>0$, finally results in (2.38). □

A slight generalization of property (2.38) was proposed in [MRS16, Sec. 3.4, (3.63)] as a sufficient condition for a sort of "uniform differentiability" condition for $\mathcal{I}(t,\cdot)$, cf. (2.41) ahead, which was in turn introduced in [MRS16, Sec. 2.1, (E.3)]. As we will see, (2.41) is at the core of key chain rule properties for viscous solutions to (1.2) and for Balanced Viscosity solutions to (1.1), cf. Lemma 2.16 and Theorem 5.8 ahead. As a trivial consequence of (2.41), we have a monotonicity property for the Fréchet subdifferential $\mathcal{D}_z\mathcal{I}$, which will allow us to prove the (crucial, for our analysis) uniqueness of solutions for the time-incremental problems giving rise to discrete solutions.

Corollary 2.15. *Under Assumptions 2.1, 2.2, and 2.8, for every $M>0$ there exist constants c_9, $c_{10}(M)>0$ such that for all $t\in[0,T]$, $z_i\in\mathcal{Z}$, $i=1,2$, with $\|z_1\|_{\mathcal{Z}}+\|z_2\|_{\mathcal{Z}}\leq M$, we have*

$$
\mathcal{L}(t,z_2)-\mathcal{L}(t,z_1)\geq \langle \mathrm{D}_z\mathcal{L}(t,z_1),z_2-z_1\rangle_{\mathcal{Z}}+\alpha\|z_1-z_2\|^2_{H^1(\Omega)}-\Lambda_M\|z_1-z_2\|^2_{L^1(\Omega)}\,. \tag{2.41}
$$

As a consequence, there holds

$$
\|z_1-z_2\|^2_{L^2(\Omega)}+\langle \mathrm{D}_z\mathcal{I}(t,z_1)-\mathrm{D}_z\mathcal{I}(t,z_2),z_1-z_2\rangle_{\mathcal{Z}}\geq c_9\|z_1-z_2\|^2_{H^1(\Omega)}-c_{10}(M)\|z_1-z_2\|^2_{L^2(\Omega)}\,. \tag{2.42}
$$

Note that, in accordance with (2.38) and (2.41), only the constant c_{10} depends on M.

Proof. Estimate (2.41) can be deduced from (2.38) by the very same calculations as in the proof of [MRS16, Lemma 3.26], while (2.42) can be obtained by adding (2.41) with the estimate obtained exchanging z_1 with z_2, and observing that $-\|z_1-z_2\|^2_{L^1(\Omega)}\geq -C\|z_1-z_2\|^2_{L^2(\Omega)}$. □

A key ingredient for the proof of energy identities in the context of solutions to the *viscous* damage system (1.2) (cf. Section 4), and of BV solutions to the rate-independent (1.1) (cf. Section 5), is the validity of the chain rule identity (but, indeed, a chain rule *inequality* would suffice)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t,z(t)) - \partial_t\mathcal{I}(t,z(t)) = \langle \mathrm{D}_z\mathcal{I}(t,z(t)), z'(t)\rangle_{L^2(\Omega)} \qquad \text{for a.a. } t\in(0,T), \tag{2.43}$$

along solution curves $z:[0,T]\to\mathcal{Z}$ with $\mathrm{D}_z\mathcal{I}(t,z(t))\in L^2(\Omega)$. Since $\mathcal{I}\in \mathrm{C}^1([0,T]\times\mathcal{Z})$, the validity of (2.43) with the duality pairing $\langle\cdot,\cdot\rangle_{\mathcal{Z}}$ is guaranteed along any curve $z\in\mathrm{AC}([0,T];\mathcal{Z})$. The following result extends (2.43) to curves z with weaker regularity and summability properties.

Lemma 2.16 (Chain rule for $\mathcal{I}$ in $L^2(\Omega)$)**.** *Under Assumptions 2.1, 2.2, and 2.8, for every curve*

$$z\in L^\infty(0,T;\mathcal{Z})\cap H^1(0,T;L^2(\Omega)),\ \textit{with } A_q z\in L^2(0,T;L^2(\Omega)), \tag{2.44}$$

the map $t\mapsto\mathcal{I}(t,z(t))$ is absolutely continuous on $[0,T]$, and (2.43) holds.

Remark 2.17. Due to estimate (2.34) for $\mathrm{D}_z\widetilde{\mathcal{I}}$, it follows from (2.44) that the function $t\mapsto\mathrm{D}_z\mathcal{I}(t,z(t))$ belongs to $L^2(0,T;L^2(\Omega))$. Therefore, $\mathrm{D}_z\mathcal{I}(t,z(t)) = A_q(z(t)) + \mathrm{D}_z\widetilde{\mathcal{I}}(t,z(t))$ belongs to $L^2(0,T;L^2(\Omega))$ as well and the integral on the r.h.s. of (2.43) is well defined for almost all $t\in(0,T)$.

In fact, for later use in Sec. 5, let us point out that, in alternative to (2.44), in Lemma 2.16 we might as well suppose

$$z\in L^\infty(0,T;\mathcal{Z})\cap W^{1,1}(0,T;L^2(\Omega)),\ \text{with } A_q z\in L^\infty(0,T;L^2(\Omega)). \tag{2.45}$$

Proof. First of all, we show the absolute continuity of $t\mapsto\mathcal{I}(t,z(t))$. We will in fact show that $t\mapsto\mathcal{L}(t,z(t))$ is absolutely continuous, with $\mathcal{L}$ from (2.37). With this aim, for every $0\le s\le t\le T$ we estimate

$$\mathcal{L}(t,z(t))-\mathcal{L}(s,z(s)) = \mathcal{L}(t,z(t))-\mathcal{L}(s,z(t))+\mathcal{L}(s,z(t))-\mathcal{L}(s,z(s)) \doteq I_1+I_2\,.$$

Since $\partial_t\mathcal{L}=\partial_t\mathcal{I}$, we have

$$|I_1|\le\int_s^t\partial_t\mathcal{I}(r,z(t))\,\mathrm{d}r \overset{(1)}{\le} C(t-s) \tag{2.46}$$

with (1) due to (2.23). As for I_2, from the uniform differentiability property (2.41) we deduce that

$$I_2\ge\int_\Omega \mathrm{D}_z\mathcal{L}(t,z(s))(z(t)-z(s))\,\mathrm{d}x+\alpha\|z(t)-z(s)\|^2_{H^1(\Omega)}-\Lambda_M\|z(t)-z(s)\|^2_{L^1(\Omega)} \tag{2.47}$$

(cf. notation (2.29)), where we have used that, by (2.44) and estimate (2.34) for $\mathrm{D}_z\widetilde{\mathcal{I}}$ that the function $s\mapsto \mathrm{D}_z\mathcal{I}(s,z(s))$ belongs to $L^2(0,T;L^2(\Omega))$, and so does $s\mapsto\mathrm{D}_z\mathcal{L}(t,z(s))$, with $t\in[0,T]$ fixed, due to (2.32). All in all we arrive at

$$\begin{aligned}|\mathcal{L}(s,z(s))-\mathcal{L}(t,z(t))|&\le 2\Lambda_M\|z(t)-z(s)\|^2_{L^1(\Omega)}+2c|t-s|\\&\quad+\left(\|\mathrm{D}_z\mathcal{L}(t,z(t))\|_{L^2(\Omega)}+\|\mathrm{D}_z\mathcal{L}(s,z(s))\|_{L^2(\Omega)}\right)\|z(t)-z(s)\|_{L^2(\Omega)}.\end{aligned} \tag{2.48}$$

Up to a suitable reparameterization, cf. [AGS08, Lemma 1.1.4], we can suppose that $z\in W^{1,\infty}(0,\widetilde{T};L^2(\Omega))$ with Lipschitz constant 1. With [AGS08, Lemma 1.2.6] we finally conclude from (2.48) the absolute continuity of $t\mapsto\mathcal{L}(t,z(t))$, which gives the same property for $t\mapsto\mathcal{I}(t,z(t))$. For the proof of identity (2.43), we refer to [MRS13, Prop. 2.4]. □

3. A priori estimates for the time-discrete solutions

We construct time-discrete solutions to the Cauchy problem for the viscous damage system (1.2) by solving the following time incremental minimization problems: for fixed $\epsilon>0$, we consider a uniform partition $\{0=t_0^\tau<\ldots<t_N^\tau=T\}$ of the time interval $[0,T]$ with fineness $\tau=t_{k+1}^\tau-t_k^\tau=T/N$. The elements $(z_k^\tau)_{0\le k\le N}$ are determined through $z_0^\tau:=z_0\in\mathcal{Z}$ and

$$z_{k+1}^\tau\in\mathrm{Argmin}\Big\{\mathcal{I}(t_{k+1}^\tau,z)+\tau\mathcal{R}_\epsilon\left(\frac{z-z_k^\tau}{\tau}\right)\,:\,z\in\mathcal{Z}\Big\},\qquad k\in\{0,\ldots,N-1\}. \tag{3.1}$$

Our first result, Prop. 3.1 below, states the existence of minimizers for problem (3.1), which is an immediate outcome of classical variational arguments, as well as the uniqueness of solutions to the associated Euler-Lagrange equation (3.2) below. This will be a key ingredient in the proof of the main result of this section, Proposition 3.2 ahead. Indeed, in order to obtain some of the a priori estimates stated therein, we shall have to perform calculations on an approximate version of (3.2). Then, the above mentioned uniqueness property will ensure that those a priori estimates also hold for the solutions to (3.2), i.e. for the minimizers from (3.1).

Proposition 3.1. *Under Assumptions 2.1, 2.2, and 2.8, for every ϵ, $\tau > 0$ and for every $k \in \{1, \dots, N-1\}$ the minimum problem* (3.1) *admits a solution z_{k+1}^τ satisfying the Euler-Lagrange equation*

$$\omega + \epsilon \frac{z - z_k^\tau}{\tau} + \mathrm{D}_z \mathcal{I}(t_{k+1}^\tau, z) = 0 \qquad \text{in } \mathcal{Z}^*, \qquad \text{with } \omega \in \partial_{\mathcal{Z},\mathcal{Z}^*}\mathcal{R}_1\left(\frac{z - z_k^\tau}{\tau}\right), \tag{3.2}$$

where $\partial_{\mathcal{Z},\mathcal{Z}^*}\mathcal{R}_1 : \mathcal{Z} \rightrightarrows \mathcal{Z}^*$ is the convex analysis subdifferential of $\mathcal{R}_1$. Moreover, for every $\epsilon > 0$ and for every $M > 0$, there exists $\tau(\epsilon, M) > 0$ such that for all $0 < \tau \le \tau(\epsilon, M)$ the Euler-Lagrange equation (3.2) *admits at most one solution in the closed ball $\overline{B}_M(0)$ of $\mathcal{Z}$.*

Suppose in addition that f and g comply with the following condition

$$f(0) \le f(z), \quad g(0) \le g(z) \quad \text{for all } z \le 0, \tag{3.3}$$

and that the initial datum z_0 fulfills $z_0(x) \in [0,1]$ for all $x \in \Omega$. Then, the minimizer z_{k+1}^τ from (3.1) *also fulfills $z_{k+1}^\tau(x) \in [0,1]$ for all $x \in \Omega$ and all $k \in \{0, \dots, N-1\}$.*

Proof. The existence of minimizers can be checked via the direct method in the calculus of variations. Observe that every minimizer fulfills (3.2), where we have used that the convex analysis subdifferential $\partial_{\mathcal{Z},\mathcal{Z}^*}\mathcal{R}_\epsilon : \mathcal{Z} \rightrightarrows \mathcal{Z}^*$ is given by $\partial_{\mathcal{Z},\mathcal{Z}^*}\mathcal{R}_\epsilon(\eta) = \partial_{\mathcal{Z},\mathcal{Z}^*}\mathcal{R}_1(\eta) + \epsilon\eta$ for every $\eta \in \mathcal{Z}$ (here and in what follows, for notational simplicity we write η in place of $J(\eta)$, with $J : \mathcal{Z} \to \mathcal{Z}^*$ the Riesz isomorphism).

In order to check that the Euler-Lagrange equation (3.2) has a unique solution, let $M > 0$ and z_1, $z_2 \in \mathcal{Z}$ be solutions to (3.2) such that $\|z_1\|_{\mathcal{Z}} + \|z_2\|_{\mathcal{Z}} \le M$. Subtracting the equation for z_2 from that for z_1 and testing the obtained relation by $z_1 - z_2$, we obtain

$$\begin{aligned} 0 = &\ \langle \omega_1 - \omega_2, z_1 - z_2 \rangle_{\mathcal{Z}} + \frac{\epsilon}{\tau}\|z_1 - z_2\|_{L^2(\Omega)}^2 + \langle \mathrm{D}_z\mathcal{I}(t_{k+1}^\tau, z_1) - \mathrm{D}_z\mathcal{I}(t_{k+1}^\tau, z_2), z_1 - z_2 \rangle_{\mathcal{Z}} \\ \ge &\left(\frac{\epsilon}{\tau} - c_{10}(M) - 1\right)\|z_1 - z_2\|_{L^2(\Omega)}^2 + c_9\|z_1 - z_2\|_{H^1(\Omega)}^2 \end{aligned}$$

where $\omega_i \in \partial\mathcal{R}_1\left(\frac{z_i - z_k^\tau}{\tau}\right)$ for $i = 1, 2$, and the second inequality follows from the monotonicity estimate (2.42). Hence, for $\tau \le \tau(\epsilon, M) := \frac{\epsilon}{(c_{10}(M)+1)}$, we conclude that $\|z_1 - z_2\|_{L^2(\Omega)}^2 \le 0$, whence $z_1 = z_2$.

For the proof of the property $z_k^\tau \in [0,1]$ in Ω under (3.3), we refer to [KRZ13, Prop. 4.5]. □

The following piecewise constant and piecewise linear interpolation functions will be used:

$$\overline{z}_\tau(t) = z_{k+1}^\tau \text{ for } t \in (t_k^\tau, t_{k+1}^\tau], \ \ \underline{z}_\tau(t) = z_k^\tau \text{ for } t \in [t_k^\tau, t_{k+1}^\tau), \ \ \widehat{z}_\tau(t) = z_k^\tau + \frac{t - t_k^\tau}{\tau}(z_{k+1}^\tau - z_k^\tau) \text{ for } t \in [t_k^\tau, t_{k+1}^\tau].$$

Furthermore, we shall use the notation

$$\begin{array}{lll} \tau(r) & = \tau & \text{for } r \in (t_k^\tau, t_{k+1}^\tau), \\ \overline{t}_\tau(r) & = t_{k+1}^\tau & \text{for } r \in (t_k^\tau, t_{k+1}^\tau], \\ \underline{t}_\tau(r) & = t_k^\tau & \text{for } r \in [t_k^\tau, t_{k+1}^\tau), \\ \overline{u}_\tau(r) & = u_{\min}(\overline{t}_\tau(r), \overline{z}_\tau(r)) & \text{for } r \in (t_k^\tau, t_{k+1}^\tau], \\ \underline{u}_\tau(r) & = u_{\min}(\underline{t}_\tau(r), \underline{z}_\tau(r)) & \text{for } r \in [t_k^\tau, t_{k+1}^\tau), \\ \widehat{u}_\tau(r) & = \underline{u}_\tau(r) + \frac{r - \underline{t}_\tau(r)}{\tau}(\overline{u}_\tau(r) - \underline{u}_\tau(r)) & \text{for } r \in [t_k^\tau, t_{k+1}^\tau]. \end{array}$$

Clearly,

$$\overline{t}_\tau(t), \underline{t}_\tau(t) \to t \quad \text{as } \tau \to 0 \text{ for all } t \in (0,T), \text{ and } \underline{t}_\tau(0) = 0, \ \overline{t}_\tau(T) = T. \tag{3.4}$$

We will also denote by $\overline{\ell}_\tau$ and $\overline{u}_{D,\tau}$ the (left-continuous) piecewise constant interpolants of the values $(\ell_k^\tau := \ell(t_k^\tau))_{k=0}^N$, $(u_{D,k}^\tau := u_D(t_k^\tau))_{k=0}^N$ and, for a given N-uple $\{v_\tau^k\}_{k=0}^N$, use the short-hand notation

$$\triangle_k^\tau(v) := v_{k+1}^\tau - v_k^\tau.$$

In view of (3.2) and of formula (2.28) for $\mathrm{D}_z\mathcal{I}$, the above interpolants fulfill for almost all $t \in (0,T)$

$$\overline{\omega}_\tau(t) + \epsilon\widehat{z}_\tau'(t) + A_q\overline{z}_\tau(t) + \mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_\tau(t), \overline{z}_\tau(t)) = 0 \quad \text{in } \mathcal{Z}^*, \quad \text{with } \overline{\omega}_\tau(t) \in \partial_{\mathcal{Z},\mathcal{Z}^*}\mathcal{R}_1\left(\widehat{z}_\tau'(t)\right). \tag{3.5}$$

The following result collects all the a priori estimates on the functions $(\overline{z}_\tau, \widehat{z}_\tau, \overline{u}_\tau, \widehat{u}_\tau)_\tau$, uniform w.r.t. the parameters ϵ, $\tau > 0$, that are at the core of the existence of solutions of the viscous system, cf. Theorem 4.1 ahead, and of its vanishing-viscosity analysis developed in Section 5. In fact, let us mention that the estimates for $(\overline{u}_\tau, \widehat{u}_\tau)_\tau$ have to be understood as side results, while the really relevant bounds for the limit passage are those for $(\overline{z}_\tau, \widehat{z}_\tau)$. We also prove that the Euler-Lagrange equation (3.5) holds in $L^2(\Omega)$, with $\partial_{\mathcal{Z},\mathcal{Z}^*}\mathcal{R}_1$ replaced by the subdifferential operator $\partial_{L^2(\Omega)}\mathcal{R}_1 : L^2(\Omega) \rightrightarrows L^2(\Omega)$. From now on, we will denote the latter operator by $\partial\mathcal{R}_1$.

Proposition 3.2. *Under Assumptions 2.1, 2.2, and 2.8, suppose that the initial datum $z_0 \in \mathcal{Z}$ fulfills in addition*

$$A_q z_0 \in L^2(\Omega). \tag{3.6}$$

Then, for every $\epsilon > 0$ there exists $\overline{\tau}_\epsilon > 0$, only depending on ϵ and on the problem data (cf. (3.14) *ahead), such that for every $\tau \in (0, \overline{\tau}_\epsilon)$ there holds*

$$A_q\overline{z}_\tau \in L^\infty(0,T;L^2(\Omega)) \text{ and } \overline{\omega}_\tau \in L^\infty(0,T;L^2(\Omega)), \tag{3.7}$$

with $\overline{\omega}_\tau$ a selection in $\partial_{\mathcal{Z},\mathcal{Z}^*}\mathcal{R}_1(\widehat{z}_\tau')$ which fulfills (3.5)*. Therefore, the functions $(\overline{t}_\tau, \overline{z}_\tau, \widehat{z}_\tau)$ satisfy*

$$\partial\mathcal{R}_1(\widehat{z}_\tau'(t)) + \epsilon\widehat{z}_\tau'(t) + \mathrm{D}_z\mathcal{I}(\overline{t}_\tau(t), \overline{z}_\tau(t)) \ni 0 \qquad \text{in } L^2(\Omega) \text{ for a.a. } t \in (0,T). \tag{3.8}$$

Furthermore, there exist constants C, $C(\epsilon)$, $C(\sigma) > 0$, with $C(\epsilon) \uparrow +\infty$ as $\epsilon \downarrow 0$, such that for all $\epsilon > 0$ and $\tau \in (0, \overline{\tau}_\epsilon)$ the following estimates hold:

$$\sup_{t\in[0,T]} \left|\mathcal{I}(\overline{t}_\tau(t), \overline{z}_\tau(t))\right| \leq C, \tag{3.9a}$$

$$\|\overline{z}_\tau\|_{L^\infty(0,T;W^{1,q}(\Omega))} + \|\widehat{z}_\tau\|_{L^\infty(0,T;W^{1,q}(\Omega))} \leq C, \tag{3.9b}$$

$$\|\overline{z}_\tau\|_{L^\infty(0,T;W^{1+\sigma,q}(\Omega))} \leq C(\sigma) \text{ for all } 0 < \sigma < \tfrac{1}{q}, \tag{3.9c}$$

$$\|\widehat{z}_\tau'\|_{L^2(0,T;H^1(\Omega))} + \|\widehat{z}_\tau'\|_{L^\infty(0,T;L^2(\Omega))} \leq C(\epsilon), \tag{3.9d}$$

$$\|\widehat{z}_\tau\|_{W^{1,1}(0,T;H^1(\Omega))} \leq C, \tag{3.9e}$$

$$\|A_q(\overline{z}_\tau)\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \tag{3.9f}$$

$$\|\overline{\omega}_\tau\|_{L^\infty(0,T;L^2(\Omega))} \leq C(\epsilon), \tag{3.9g}$$

$$\|\overline{u}_\tau\|_{L^\infty(0,T;H^2(\Omega))} \leq C, \tag{3.9h}$$

$$\|\widehat{u}_\tau'\|_{L^2(0,T;W^{1,3}(\Omega))} \leq C(\epsilon), \tag{3.9i}$$

$$\|\widehat{u}_\tau\|_{W^{1,1}(0,T;W^{1,3}(\Omega))} \leq C. \tag{3.9j}$$

Therefore,

$$\|\mathrm{D}_z\mathcal{I}(\overline{t}_\tau, \overline{z}_\tau)\|_{L^\infty(0,T;L^2(\Omega))} \leq C. \tag{3.9k}$$

Based on Proposition 3.2 we derive a discrete energy inequality, cf. (3.11) below, involving the Fenchel-Moreau conjugate of the functional $\mathcal{R}_\epsilon$ w.r.t. the scalar product in $L^2(\Omega)$, namely the functional

$$\mathcal{R}_\epsilon^* : L^2(\Omega) \to [0, +\infty) \qquad \text{defined by } \mathcal{R}_\epsilon^*(\xi) := \frac{1}{2\epsilon} \min_{\eta \in \partial\mathcal{R}_1(0)} \|\xi - \eta\|_{L^2(\Omega)}^2 . \tag{3.10}$$

Observe that we are in a position to work with this Legendre transform of $\mathcal{R}_\epsilon$, and not with the one w.r.t. the $(\mathcal{Z}, \mathcal{Z}^*)$-duality, relying on the fact that $\mathrm{D}_z\mathcal{I}(\overline{t}_\tau(t), \overline{z}_\tau(t)) \in L^2(\Omega)$ for almost all $t \in (0,T)$, thanks to (3.7).

Let us mention in advance that (3.11) will be the starting point of the vanishing-viscosity analysis developed in Sec. 6. We postpone the proof of Corollary 3.3 to the end of this section.

Corollary 3.3. *Under Assumptions 2.1, 2.2, and 2.8, suppose that the initial datum z_0 fulfills* (3.6).

Then, there exists $C>0$ such that for every $\epsilon>0$ and $\tau\in(0,\overline{\tau}_\epsilon)$ the functions $\overline{z}_\tau$, $\widehat{z}_\tau$ comply with the discrete energy-dissipation inequality *for every $0\le s\le t\le T$*

$$\begin{aligned}&\int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)}\left(\mathcal{R}_\epsilon(\widehat{z}_\tau'(r))+\mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\overline{t}_\tau(r),\overline{z}_\tau(r)))\right)\,\mathrm{d}r+\mathcal{I}(t,\widehat{z}_\tau(t))\\&\le\mathcal{I}(s,\widehat{z}_\tau(s))+\int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)}\partial_t\mathcal{I}(r,\widehat{z}_\tau(r))\,\mathrm{d}r\\&+C\sup_{t\in[0,T]}\|\overline{z}_\tau(t)-\widehat{z}_\tau(t)\|_{L^2(\Omega)}\int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)}\left(|\overline{t}_\tau(r)-r|+\|\overline{z}_\tau(r)-\widehat{z}_\tau(r)\|_{L^6(\Omega)}\right)\,\mathrm{d}r.\end{aligned}\tag{3.11}$$

Therefore, there exists a constant $C>0$ such that for every $\epsilon>0$ and $\tau\in(0,\overline{\tau}_\epsilon)$

$$\sup_{t\in[0,T]}|\mathcal{I}(t,\widehat{z}_\tau(t))|\le C,\tag{3.12a}$$

$$\int_0^T\left(\mathcal{R}_\epsilon(\widehat{z}_\tau'(r))+\mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\overline{t}_\tau(r),\overline{z}_\tau(r)))\right)\,\mathrm{d}r\le C\,.\tag{3.12b}$$

Let us now comment on the proof of Prop. 3.2: Estimates (3.9) (and the related enhanced spatial regularity (3.7), which leads to (3.8) as a subdifferential inclusion in $L^2(\Omega)$) will be proved by performing on equation (3.5) the following a priori estimates:

Energy estimate: based on the energy-dissipation inequality

$$\mathcal{I}(\overline{t}_\tau(t),\overline{z}_\tau(t))+\int_0^{\overline{t}_\tau(t)}\mathcal{R}_\epsilon(\widehat{z}_\tau'(s))\,\mathrm{d}s\le\mathcal{I}(0,z_0)+\int_0^{\overline{t}_\tau(t)}\partial_t\mathcal{I}(s,\underline{z}_\tau(s))\,\mathrm{d}s\tag{3.13}$$

for every $t\in[0,T]$, it leads to the uniform bounds (3.9a)–(3.9b). Observe that the proof of (3.13) works for *every* $\tau>0$.

We then choose

$$\overline{\tau}_\epsilon:=\tau(\epsilon,M)\text{ according to Proposition 3.1, with }M\ge\sup_{\tau>0}\|\overline{z}_\tau\|_{L^\infty(0,T;W^{1,q}(\Omega))}\,.\tag{3.14}$$

First regularity estimate: In view of estimate (2.15), from the estimate for $\overline{z}_\tau$ in $L^\infty(0,T;W^{1,q}(\Omega))$ we deduce (3.9h).

Enhanced energy estimate: it consists in (formally) differentiating (3.5) w.r.t. time, and testing it by $\widehat{z}_\tau'$. In view of the coercivity property (2.1) of the elliptic operator A_q, this gives estimates (3.9d) & (3.9e) for $\widehat{z}_\tau'$.

Second regularity estimate: Estimates (3.9i) & (3.9j) for $\widehat{u}_\tau'$ derive from (3.9d) & (3.9e), respectively, via the continuous dependence estimate (2.17).

Third regularity estimate: it consists in testing (3.5) by (the formally written term) $\partial_t A_q\overline{z}_\tau$. This gives rise to estimate (3.9f), which induces the spatial regularity (3.9c) by applying Proposition 2.7, and it induces (3.9g) by a comparison argument in (3.5).

The energy & the enhanced energy estimates can be rendered rigorously on the discrete equation (3.5), cf. Lemma 3.4 below. In its proof, we shall revisit the calculations developed in [KRZ15, Sec. 5], relying on the novel estimates provided by Lemmas 2.10 and 2.12. Instead, the third regularity estimate obtained in Lemma 3.5 ahead cannot be performed directly on (3.5). In fact, it would involve testing the subdifferential inclusion (3.5), set in $\mathcal{Z}^*$, by the difference $\frac{1}{\tau}(A_q\overline{z}_\tau(t)-A_q\underline{z}_\tau(t))$ which then should belong to $\mathcal{Z}$. This cannot be rigorous, since $A_q\overline{z}_\tau(t)$ is in principle an element of $\mathcal{Z}^*$, only. Therefore, in the proof of Lemma 3.5 we shall perform all the calculations on an approximate version of (3.5), featuring a regularized version of the dissipation potential $\mathcal{R}_1$.

Lemma 3.4. *Under Assumptions 2.1, 2.2, and 2.8, and the condition that the initial datum $z_0 \in \mathcal{Z}$ fulfills* (3.6)*, estimates* (3.9a)–(3.9b)*,* (3.9d)–(3.9e)*, and* (3.9h)–(3.9j) *hold true for every $\tau > 0$.*

Proof. The discrete energy-dissipation inequality (3.13) can be derived by choosing the competitor $z = z_k^\tau$ in the minimum problem (3.1), which leads to

$$\mathcal{I}(t_{k+1}^\tau, z_{k+1}^\tau) + \tau_k \mathcal{R}_\epsilon\left(\frac{z_{k+1}^\tau - z_k^\tau}{\tau_k}\right) \leq \mathcal{I}(t_{k+1}^\tau, z_k^\tau) = \mathcal{I}(t_k^\tau, z_k^\tau) + \int_{t_k^\tau}^{t_{k+1}^\tau} \partial_t \mathcal{I}(s, z_k^\tau)\, \mathrm{d}s.$$

Then, (3.13) follows upon summing over the index k. In view of estimate (2.23) on the power functional $\partial_t \mathcal{I}$, and Assumption 2.8, the right-hand side of (3.13) is uniformly bounded. Since the second term on its l.h.s. is non-negative, we immediately conclude estimate (3.9a). Then, the coercivity property (2.18), combined with Poincaré's inequality, gives (3.9b) for $\overline{z}_\tau$. The bound for $\widehat{z}_\tau$ then trivially follows. From the bound for $\int_0^T \mathcal{R}_\epsilon(\widehat{z}_\tau'(t))\,\mathrm{d}t$ we also infer that $\epsilon^{1/2}\|\widehat{z}_\tau'\|_{L^2(0,T;L^2(\Omega))} \leq C$.

Thanks to (2.15), we have that

$$\|\overline{u}_\tau\|_{L^\infty(0,T;H^2(\Omega))} \leq c_1 \sup_{t\in(0,T)} P(0, \overline{z}_\tau(t)) \left(\|\overline{\ell}_\tau\|_{L^\infty(0,T;L^2(\Omega))} + \|\overline{u}_{D,\tau}\|_{L^\infty(0,T;H^2(\Omega))}\right) \leq C',$$

where we have used estimate (3.9b), as well as Assumption 2.8. Then, (3.9h) follows.

In order to derive estimates (3.9d) and (3.9e), we follow the proof of [KRZ15, Lemma 5.3] and observe that, by the 1-homogeneity of $\mathcal{R}_1$, (3.5) rewrites as

$$\begin{cases} \langle \overline{h}_\tau(\rho), \widehat{z}_\tau'(\rho)\rangle_{\mathcal{Z}} = \mathcal{R}_1(\widehat{z}_\tau'(\rho)) & \text{for all } \rho \in (t_k^\tau, t_{k+1}^\tau) \\ \langle \overline{h}_\tau(r), \widehat{z}_\tau'(\rho)\rangle_{\mathcal{Z}} \leq \mathcal{R}_1(\widehat{z}_\tau'(\rho)) & \text{for all } r \in [0,T] \setminus \{t_0^\tau, \ldots, t_N^\tau\}, \end{cases}$$

where we have used the place-holder $\overline{h}_\tau(\rho) := -(\epsilon \widehat{z}_\tau'(\rho) + A_q \overline{z}_\tau(\rho) + \mathrm{D}_z \widetilde{\mathcal{I}}(\overline{t}_\tau(\rho), \overline{z}_\tau(\rho)))$. Subtracting the second relation from the first one gives $\tau^{-1}\langle \overline{h}_\tau(\rho) - \overline{h}_\tau(r), \widehat{z}_\tau'(\rho)\rangle_{\mathcal{Z}} \geq 0$ for $\rho \in (t_k^\tau, t_{k+1}^\tau)$ and $r \in (t_{k-1}^\tau, t_k^\tau)$. Hence, we get

$$\begin{aligned} &\epsilon\tau^{-1} \underbrace{\int_\Omega (\widehat{z}_\tau'(\rho) - \widehat{z}_\tau'(r))\widehat{z}_\tau'(\rho)\,\mathrm{d}x}_{=I_1} + \underbrace{\tau^{-1}\langle A_q \overline{z}_\tau(\rho) - A_q \overline{z}_\tau(r), \widehat{z}_\tau'(\rho)\rangle_{\mathcal{Z}}}_{=I_2} \\ &\qquad \leq \underbrace{-\tau^{-1}\int_\Omega (\mathrm{D}_z \widetilde{\mathcal{I}}(\overline{t}_\tau(\rho), \overline{z}_\tau(\rho)) - \mathrm{D}_z \widetilde{\mathcal{I}}(\overline{t}_\tau(r), \overline{z}_\tau(r)))\widehat{z}_\tau'(\rho)\,\mathrm{d}x}_{=I_3}\,. \end{aligned} \tag{3.15}$$

Observe that $I_1 \geq \frac{1}{2}\int_\Omega (|\widehat{z}_\tau'(\rho)|^2 - |\widehat{z}_\tau'(r)|^2)\,\mathrm{d}x$, whereas it follows from estimate (2.1) that

$$I_2 \geq c_q \int_\Omega \left(1 + |\nabla \overline{z}_\tau(\rho)|^2 + |\nabla \overline{z}_\tau(r)|^2\right)^{(q-2)/2} |\nabla \widehat{z}_\tau'(\rho)|^2\,\mathrm{d}x \geq C_q \int_\Omega \left(1 + |\nabla \widehat{z}_\tau(\rho)|^2\right)^{(q-2)/2} |\nabla \widehat{z}_\tau'(\rho)|^2\,\mathrm{d}x \tag{3.16}$$

for some positive constant C_q, where we have used that $|\nabla \widehat{z}_\tau(\rho)|^2 \leq 2(|\nabla \overline{z}_\tau(\rho)|^2 + |\nabla \overline{z}_\tau(r)|^2)$. As for I_3, by the Hölder inequality

$$|I_3| \leq C\tau^{-1}\|\mathrm{D}_z \widetilde{\mathcal{I}}(\overline{t}_\tau(\rho), \overline{z}_\tau(\rho)) - \mathrm{D}_z \widetilde{\mathcal{I}}(\overline{t}_\tau(r), \overline{z}_\tau(r))\|_{L^2(\Omega)} \|\widehat{z}_\tau'(\rho)\|_{L^2(\Omega)}\,.$$

Relying on (2.32), we then find

$$|I_3| \leq C(1 + \|\widehat{z}_\tau'(\rho)\|_{L^6(\Omega)})\|\widehat{z}_\tau'(\rho)\|_{L^2(\Omega)}\,, \tag{3.17}$$

where we have also used that $\sup_{t\in[0,T]} C_{f''}(\overline{z}_\tau(\rho),\overline{z}_\tau(r)) + P(\overline{z}_\tau(\rho),\overline{z}_\tau(r))) \leq C$ thanks to the previously proved estimate (3.9b). Hence, multiplying (3.15) by τ, we infer

$$\begin{aligned} &\frac{\epsilon}{2}\|\widehat{z}'_\tau(\rho)\|^2_{L^2(\Omega)} + C_q\tau\int_\Omega \left(1+|\nabla\widehat{z}_\tau(\rho)|^2\right)^{(q-2)/2}|\nabla\widehat{z}'_\tau(\rho)|^2\,\mathrm{d}x \\ &\leq \frac{\epsilon}{2}\|\widehat{z}'_\tau(r)\|^2_{L^2(\Omega)} + \tau C(1+\|\widehat{z}'_\tau(\rho)\|_{L^6(\Omega)})\|\widehat{z}'_\tau(\rho)\|_{L^2(\Omega)}\,, \end{aligned} \tag{3.18}$$

which leads, upon summation, to the following estimate on the time interval (t_0,t), with $t_0\in(0,t_1^\tau)$ and $t\in(t_k^\tau,t_{k+1}^\tau)$:

$$\begin{aligned} &\frac{\epsilon}{2}\|\widehat{z}'_\tau(t)\|^2_{L^2(\Omega)} + C_q\int_{t_1^\tau}^{\bar{t}_\tau(t)}\int_\Omega \left(1+|\nabla\widehat{z}_\tau(\rho)|^2\right)^{(q-2)/2}|\nabla\widehat{z}'_\tau(\rho)|^2\,\mathrm{d}x\,\mathrm{d}\rho \\ &\qquad\leq \frac{\epsilon}{2}\|\widehat{z}'_\tau(t_0)\|^2_{L^2(\Omega)} + \frac{C_q}{4}\int_{t_1^\tau}^{\bar{t}_\tau(t)}(1+\|\widehat{z}'_\tau(\rho)\|^2_{H^1(\Omega)})\,\mathrm{d}\rho + C\int_{t_1^\tau}^{\bar{t}_\tau(t)}\|\widehat{z}'_\tau(\rho)\|^2_{L^2(\Omega)}\,\mathrm{d}\rho, \end{aligned} \tag{3.19}$$

where we have used Young's inequality, and the continuous embedding $H^1(\Omega)\subset L^6(\Omega)$, to handle the last term on the r.h.s. of (3.18). For the first time step with $t_0\in(0,t_1^\tau)$, following the very same calculations as in the proof of [KRZ15, Lemma 5.3], we obtain

$$\begin{aligned} &\epsilon\|\widehat{z}'_\tau(t_0)\|^2_{L^2(\Omega)} + C_q\tau\int_\Omega(1+|\nabla\widehat{z}_\tau(t_0)|^2)^{(q-2)/2}|\nabla\widehat{z}'_\tau(t_0)|^2\,\mathrm{d}x \\ &\leq \frac{\epsilon}{2}\|\widehat{z}'_\tau(t_0)\|^2_{L^2(\Omega)} + \epsilon^{-1}\|\mathrm{D}_z\mathcal{I}(0,z_0)\|^2_{L^2(\Omega)} + \frac{C_q\tau}{4}(1+\|\widehat{z}'_\tau(t_0)\|^2_{H^1(\Omega)}) + C\tau\|\widehat{z}'_\tau(t_0)\|^2_{L^2(\Omega)}\,. \end{aligned} \tag{3.20}$$

Summing (3.19) with (3.20), and adding the term $\frac{C_q\tau}{2}\int_0^{\bar{t}_\tau(t)}\|\widehat{z}'_\tau(\rho)\|^2_{L^2(\Omega)}\,\mathrm{d}\rho$ to both sides, we thus end up with the following estimate

$$\begin{aligned} &\frac{\epsilon}{2}\|\widehat{z}'_\tau(t)\|^2_{L^2(\Omega)} + C_q\int_0^{\bar{t}_\tau(t)}\|\widehat{z}'_\tau(\rho)\|^2_{H^1(\Omega)}\,\mathrm{d}\rho \\ &\leq C + \epsilon^{-1}\|\mathrm{D}_z\mathcal{I}(0,z_0)\|^2_{L^2(\Omega)} + \frac{C_q}{4}\int_0^{\bar{t}_\tau(t)}\|\widehat{z}'_\tau(\rho)\|^2_{H^1(\Omega)}\,\mathrm{d}\rho + C\int_0^{\bar{t}_\tau(t)}\|\widehat{z}'_\tau(\rho)\|^2_{L^2(\Omega)}\,\mathrm{d}\rho\,. \end{aligned} \tag{3.21}$$

Applying the discrete Gronwall Lemma we get estimate (3.9d).

In order to prove (3.9e), which is uniform w.r.t. ϵ, we follow the proof of [KRZ15, Lemma 5.5]. Since $\widehat{z}'_\tau$ is not defined in the points t_k^τ, we write (3.15) for $\rho=m_k$ and $\sigma=m_{k-1}$, with $m_k := \frac{1}{2}(t_{k-1}^\tau+t_k^\tau)$, $k\in\{2,\ldots,N\}$, and set $\widehat{z}'_\tau(m_0):=0$. For all $k\in\{1,\ldots,N\}$ we have (cf. [KRZ15, Formula (5.26)])

$$\begin{aligned} &\frac{\epsilon}{\tau}\int_\Omega(\widehat{z}'_\tau(m_k)-\widehat{z}'_\tau(m_{k-1}))\,\widehat{z}'_\tau(m_k)\,\mathrm{d}x + \tau^{-1}\langle A_q\overline{z}_\tau(m_k)-A_q\underline{z}_\tau(m_k),\widehat{z}'_\tau(m_k)\rangle_{\mathcal{Z}} + \|\widehat{z}'_\tau(m_k)\|^2_{L^2(\Omega)} \\ &\qquad\leq -\frac{1}{\tau}\int_\Omega\Big(\mathrm{D}_z\widetilde{\mathcal{I}}(t_k^\tau,\overline{z}_\tau(m_k))-\mathrm{D}_z\widetilde{\mathcal{I}}(t_{k-1}^\tau,\underline{z}_\tau(m_k))\Big)\,\widehat{z}'_\tau(m_k)\,\mathrm{d}x + \|\widehat{z}'_\tau(m_k)\|^2_{L^2(\Omega)} \\ &\qquad\qquad + \frac{\delta_{1,k}}{\tau}\left|\int_\Omega \mathrm{D}_z\mathcal{I}(0,z_0)\widehat{z}'_\tau(m_1)\,\mathrm{d}x\right|, \end{aligned} \tag{3.22}$$

with the Kronecker symbol $\delta_{i,j}$. Arguing as in the proof of [KRZ15, Lemma 5.5], by estimate (2.1) and the fact that $|\nabla\widehat{z}_\tau(m_k)|^2\leq 2|\nabla\overline{z}_\tau(m_k)|^2+2|\nabla\overline{z}_\tau(m_{k-1})|^2$, it follows that the left-hand side of (3.22) can be bounded from below by

$$\text{L.H.S.} \geq \frac{\epsilon}{2\tau}\|\widehat{z}'_\tau(m_k)\|_{L^2(\Omega)}\left(\|\widehat{z}'_\tau(m_k)\|_{L^2(\Omega)} - \|\widehat{z}'_\tau(m_{k-1})\|_{L^2(\Omega)}\right) + \mathrm{M}_k^2, \tag{3.23}$$

with the abbreviation

$$\mathrm{M}_k^2 := C_q\int_\Omega(1+|\nabla\widehat{z}_\tau(m_k)|^2)^{\frac{q-2}{2}}|\nabla\widehat{z}'_\tau(m_k)|^2\,\mathrm{d}x + \|\widehat{z}'_\tau(m_k)\|^2_{L^2(\Omega)}$$

and C_q from (3.16). As for the first term of the right-hand side of (3.22), instead of (3.17) we shall use

$$\left|\frac{1}{\tau}\int_\Omega \left(\mathrm{D}_z\widetilde{\mathcal{I}}(t_k^\tau,\overline{z}_\tau(m_k))-\mathrm{D}_z\widetilde{\mathcal{I}}(t_{k-1}^\tau,\overline{z}_\tau(m_{k-1}))\right)\widehat{z}_\tau'(m_k)\,\mathrm{d}x\right| \leq C(1+\|\widehat{z}_\tau'(m_k)\|_{L^4(\Omega)})\|\widehat{z}_\tau'(m_k)\|_{L^4(\Omega)}, \qquad (3.24)$$

which derives from estimate (2.33) for $\|\mathrm{D}_z\widetilde{\mathcal{I}}(t_k^\tau,\overline{z}_\tau(m_k))-\mathrm{D}_z\widetilde{\mathcal{I}}(t_{k-1}^\tau,\overline{z}_\tau(m_{k-1}))\|_{L^{4/3}(\Omega)}$. We then continue (3.24) by using the trivial estimate $C(1+\|\widehat{z}_\tau'(m_k)\|_{L^4(\Omega)})\|\widehat{z}_\tau'(m_k)\|_{L^4(\Omega)} \leq C\|\widehat{z}_\tau'(m_k)\|_{L^4(\Omega)}^2 + C$, and then applying the Gagliardo-Nirenberg estimate $\|\zeta\|_{L^4(\Omega)}^2 \leq c\|\zeta\|_{L^1(\Omega)}^{2(1-\theta)}\|\zeta\|_{H^1(\Omega)}^{2\theta}$, with $\theta = 9/10$, and Young's inequality, so that

$$\begin{aligned}&\left|\frac{1}{\tau}\int_\Omega \left(\mathrm{D}_z\widetilde{\mathcal{I}}(t_k^\tau,\overline{z}_\tau(m_k))-\mathrm{D}_z\widetilde{\mathcal{I}}(t_{k-1}^\tau,\overline{z}_\tau(m_{k-1}))\right)\widehat{z}_\tau'(m_k)\,\mathrm{d}x\right|\\ &\leq \frac{1}{2}\min\{C_q,1\}\|\widehat{z}_\tau'(m_k)\|_{H^1(\Omega)}^2 + C\|\widehat{z}_\tau'(m_k)\|_{L^1(\Omega)}\mathcal{R}_1(\widehat{z}_\tau'(m_k)) + C\,,\end{aligned}$$

where we have also used that $\|\widehat{z}_\tau'(m_k)\|_{L^1(\Omega)}^2 \leq \|\widehat{z}_\tau'(m_k)\|_{L^1(\Omega)}\mathcal{R}_1(\widehat{z}_\tau'(m_k))$. Therefore, the right-hand side of (3.22) can be bounded as follows

$$\text{R.H.S.} \leq \frac{1}{2}\mathrm{M}_k^2 + C\left(1+\|\widehat{z}_\tau'(m_k)\|_{L^2(\Omega)}\mathcal{R}_1(\widehat{z}_\tau'(m_k)) + \delta_{1,k}\tau^{-1}|\langle \mathrm{D}_z\mathcal{I}(0,z_0),\widehat{z}_\tau'(m_1)\rangle_Z|\right). \qquad (3.25)$$

From (3.23) and (3.25), after some algebra it results that (cf. [KRZ15, (5.28)])

$$\begin{aligned}&2\|\widehat{z}_\tau'(m_k)\|_{L^2(\Omega)}\left(\|\widehat{z}_\tau'(m_k)\|_{L^2(\Omega)} - \|\widehat{z}_\tau'(m_{k-1})\|_{L^2(\Omega)}\right) + \frac{\tau}{\epsilon}\|z_\tau'(m_k)\|_{L^2(\Omega)}^2 + \frac{\tau}{\epsilon}\mathrm{M}_k^2\\ &\leq \frac{4C\tau}{\epsilon} + \frac{4C\tau}{\epsilon}\|\widehat{z}_\tau'(m_k)\|_{L^1(\Omega)}\mathcal{R}_1(\widehat{z}_\tau'(m_k)) + 4C\frac{\delta_{1,k}}{\epsilon\tau}\left|\int_\Omega \mathrm{D}_z\mathcal{I}(0,z_0)\widehat{z}_\tau'(m_1)\,\mathrm{d}x\right|.\end{aligned}$$

At this point, we apply a suitable discrete version of the Gronwall Lemma (cf. [KRZ15, Lemma B.1]), and conclude following the very same lines of the proof of [KRZ15, Lemma 5.5]. Thus, we obtain (3.9e).

Finally, we use (2.17) and deduce that for almost all $t \in (0,T)$ there holds

$$\begin{aligned}&\|\widehat{u}_\tau'(t)\|_{W^{1,3}(\Omega)} = \frac{1}{\tau}\|u_{k+1}^\tau - u_k^\tau\|_{W^{1,3}(\Omega)}\\ &\leq \frac{c_2}{\tau}P(z_\tau^k,z_\tau^{k+1})^2\left(\tau + \|z_{k+1}^\tau - z_k^\tau\|_{L^6(\Omega)}\right)\left(\|\ell\|_{\mathrm{C}^1([0,T];W^{-1,3}(\Omega))} + \|u_D(t)\|_{\mathrm{C}^1([0,T];W^{1,3}(\Omega))}\right)\\ &\leq C(1+\|\widehat{z}_\tau'(t)\|_{L^6(\Omega)}),\end{aligned}$$

where the second inequality follows from (3.9b) and Assumption 2.8. Hence, estimates (3.9d) & (3.9e) imply (3.9i) & (3.9j), respectively. □

We postpone to Section 3.1 the proof of the forthcoming Lemma 3.5.

Lemma 3.5. *Under Assumptions 2.1, 2.2, and 2.8, and, in addition,* (3.6) *on the initial datum* z_0*, for every* $\tau \in (0,\overline{\tau}_\epsilon)$ *the enhanced regularity* (3.7) *and estimates* (3.9f)–(3.9g) *hold true, whence* (3.9k)*. Furthermore, the subdifferential inclusion* (3.8) *is satisfied.*

The **proof of Proposition 3.2** now follows from combining Lemmas 3.4 & 3.5. ■

Let us finally give the **proof of Corollary 3.3**: the very same calculations as in the proof of [KRZ15, Lemma 6.1] (cf. also the proof of Thm. 4.1 ahead), show that the interpolants $\overline{z}_\tau$, $\widehat{z}_\tau$ fulfill at every $0 \leq s \leq t \leq T$

$$\begin{aligned}&\int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \left(\mathcal{R}_\epsilon(\widehat{z}_\tau')(r) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\overline{t}_\tau(r),\overline{z}_\tau(r)))\right)\mathrm{d}r + \mathcal{I}(t,\widehat{z}_\tau(t))\\ &= \mathcal{I}(s,\widehat{z}_\tau(s)) + \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \partial_t\mathcal{I}(r,\widehat{z}_\tau(r))\,\mathrm{d}r\\ &\quad \underbrace{-\int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)}\int_\Omega \left(A_q\overline{z}_\tau(r) - A_q\widehat{z}_\tau(r)\right)\widehat{z}_\tau'(r)\,\mathrm{d}r}_{F_1} \quad \underbrace{-\int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)}\int_\Omega \left(\mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_\tau(r),\overline{z}_\tau(r)) - \mathrm{D}_z\widetilde{\mathcal{I}}(r,\widehat{z}_\tau(r))\right)\widehat{z}_\tau'(r)\,\mathrm{d}r}_{F_2}\,.\end{aligned}$$

Observe that the terms F_1 and F_2 feature integrals, instead of duality pairings between $\mathcal{Z}^*$ and $\mathcal{Z}$, thanks to (2.34) and (3.7). By monotonicity we have $F_1 \leq 0$, whereas, the very same argument leading to (3.17) yields

$$\begin{aligned} |F_2| &\leq C \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} (|(\overline{t}_\tau(r) - r| + \|\overline{z}_\tau(r) - \widehat{z}_\tau(r)\|_{L^6(\Omega)}) \|\overline{z}_\tau(r) - \widehat{z}_\tau(r)\|_{L^2(\Omega)} \,\mathrm{d}r \\ &\leq C \sup_{t\in[0,T]} \|\overline{z}_\tau(t) - \widehat{z}_\tau(t)\|_{L^2(\Omega)} \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} (|(\overline{t}_\tau(r) - r| + \|\overline{z}_\tau(r) - \widehat{z}_\tau(r)\|_{L^6(\Omega)}) \,\mathrm{d}r, \end{aligned}$$

whence (3.11).

It follows from (3.11) and (2.23) that

$$\begin{aligned} &\int_0^{\overline{t}_\tau(t)} \left(\mathcal{R}_\epsilon(\widehat{z}_\tau'(r)) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\overline{t}_\tau(r), \overline{z}_\tau(r)))\right) \mathrm{d}r + \mathcal{I}(t, \widehat{z}_\tau(t)) \\ &\leq \mathcal{I}(0, z_0) + C + C \left(\|\overline{z}_\tau\|_{L^\infty(0,T;L^2(\Omega))} + \|\widehat{z}_\tau\|_{L^\infty(0,T;L^2(\Omega))}\right) \left(1 + \int_0^{\overline{t}_\tau(t)} \|\overline{z}_\tau(r) - \widehat{z}_\tau(r)\|_{L^6(\Omega)} \,\mathrm{d}r\right) \leq C\,, \end{aligned}$$

where the very last estimate ensues from (3.9b) and (3.9e). Recalling that $\mathcal{I}$ is bounded from below (cf. (2.18)), we thus infer that $\sup_{t\in[0,T]} |\mathcal{I}(t, \widehat{z}_\tau(t))| \leq C$, i.e. (3.12a), as well as (3.12b). ∎

3.1. **Proof of Lemma 3.5.** Observe that, once estimate (3.9f) is proved, (3.9k) then follows by observing that $\mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_\tau, \overline{z}_\tau)$ is bounded in $L^\infty(0,T;L^2(\Omega))$ in view of estimate (2.34) for $\mathrm{D}_z\widetilde{\mathcal{I}}$, combined with the previously obtained (3.9b).

Hence, let us now turn to the proof of (3.9f), which is a consequence of the *Third regularity estimate*. In order to render it on the time-discrete level, we need to work on an approximate version of the discrete equation (3.5), where the dissipation metric R_1 inducing $\mathcal{R}_1$ is replaced, for technical reasons that will be apparent in the proof of Lemma 3.6 below, by a *twice-differentiable* function. Observe that the standard Yosida approximation of R_1, namely the function

$$\mathrm{R}_{1,\nu} : \mathbb{R} \to \mathbb{R} \;\; \text{defined by} \;\; \mathrm{R}_{1,\nu}(r) := \min_{y\in\mathbb{R}} \left(\frac{|y-r|^2}{2\nu} + \mathrm{R}_1(y)\right) = \begin{cases} \frac{1}{2\nu} r^2 & \text{if } r > -\nu\kappa \\ -\kappa r - \frac{\nu\kappa^2}{2} & \text{if } r \leq -\nu\kappa \end{cases} \tag{3.26}$$

with $\nu > 0$ fixed, does not enjoy this regularity, as it is only differentiable on $\mathbb{R}$, cf. [Bré73].

We will thus resort to a regularization of R_1 devised in [GR07] and defined in this way. Let $\varrho \in \mathrm{C}^\infty(\mathbb{R})$ satisfy $\mathrm{supp}(\varrho) \subset [-1.1]$ and $\|\varrho\|_{L^1(\mathbb{R})} = 1$. We then define

$$\overline{\mathrm{R}}_{1,\nu}(r) := \int_0^r \int_{-\nu^2}^{\nu^2} \mathrm{R}_{1,\nu}'(\sigma - s)\varrho_\nu(s) \,\mathrm{d}s \,\mathrm{d}\sigma \tag{3.27}$$

where $\varrho_\nu(s) = \nu^{-2}\eta(s/\nu^2)$. In [GR07] it has been proved that

$$\overline{\mathrm{R}}_{1,\nu} \in \mathrm{C}^\infty(\mathbb{R}) \text{ is convex and satisfies } -\nu|r| \leq \overline{\mathrm{R}}_{1,\nu}(r) \leq \mathrm{R}_1(r) + \nu|r| \quad \text{for all } r \in \mathbb{R}. \tag{3.28a}$$

Of course, for $r > 0$ the latter estimate is trivially satisfied, since in that case, $\mathrm{R}_1(r) = \infty$. Inequality (3.28a) in fact derives from the estimate

$$|\overline{\mathrm{R}}_{1,\nu}'(r) - \mathrm{R}_{1,\nu}'(r)| \leq \nu \quad \text{for all } r \in \mathbb{R}. \tag{3.28b}$$

Since $\mathrm{R}_{1,\nu}'$ is Lipschitz, from (3.28b) we in fact deduce that $\overline{\mathrm{R}}_{1,\nu}$ grows at most quadratically on $\mathbb{R}$. The function $\overline{\mathrm{R}}_{1,\nu}$ induces an integral functional

$$\overline{\mathcal{R}}_{1,\nu} : L^2(\Omega) \to \mathbb{R} \quad \text{defined by} \quad \overline{\mathcal{R}}_{1,\nu}(\eta) := \int_\Omega \overline{\mathrm{R}}_{1,\nu}(\eta(x)) \,\mathrm{d}x \qquad \text{for all } \eta \in L^2(\Omega). \tag{3.28c}$$

Observe that $\overline{\mathcal{R}}_{1,\nu}$ is Gâteaux-differentiable on $L^2(\Omega)$, with derivative $\mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\eta)$ defined by $\mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\eta)(x) := \overline{\mathrm{R}}_{1,\nu}'(\eta(x))$ for almost all $x \in \Omega$ (in fact, $\overline{\mathrm{R}}_{1,\nu}'(\eta) \in L^2(\Omega)$ by the linear growth of $\overline{\mathrm{R}}_{1,\nu}'$). Indeed, as soon as $\eta \in \mathcal{Z}$, $\mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\eta)$ coincides with the Gâteaux derivative $\mathrm{D}_{\mathcal{Z},\mathcal{Z}^*}\overline{\mathcal{R}}_{1,\nu}(\eta)$. For our purposes, the following closedness property relating $\mathrm{D}\overline{\mathcal{R}}_{1,\nu} : L^2(\Omega) \to L^2(\Omega)$ to the convex subdifferential $\partial\mathcal{R}_1 : L^2(\Omega) \rightrightarrows L^2(\Omega)$ will have a prominent role: for any $(t_0, t_1) \subset (0, T)$ and all sequences $(\eta_\nu)_\nu$, η, $\xi \in L^2(t_0, t_1; L^2(\Omega))$ there holds

$$\begin{cases} \eta_\nu \rightharpoonup \eta & \text{as } \nu \downarrow 0 \quad \text{in } L^2(t_0, t_1; L^2(\Omega)), \\ \mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\eta_\nu) \rightharpoonup \xi & \text{as } \nu \downarrow 0 \quad \text{in } L^2(t_0, t_1; L^2(\Omega)), \\ \limsup_{\nu\downarrow 0} \int_{t_0}^{t_1} \int_\Omega \mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\eta_\nu)\eta_\nu \,\mathrm{d}x \,\mathrm{d}t \leq \int_{t_0}^{t_1}\int_\Omega \xi\eta \,\mathrm{d}x\,\mathrm{d}t \end{cases}$$
$$\Rightarrow\ \xi(t) \in \partial\mathcal{R}_1(\eta(t)) \qquad \text{for almost all } t \in (t_0, t_1). \tag{3.28d}$$

We refer to [GR07, Prop. 3.1] for the proof of (3.28d).

For a fixed time step $\tau > 0$, given a partition $\{0 = t_0^\tau < \ldots < t_N^\tau = T\}$ of $[0, T]$, we now incrementally solve the minimum problems featuring the regularized functionals $\overline{\mathcal{R}}_{1,\nu}$. Namely, starting from $z_0^{\tau,\nu} := z_0$, we set

$$z_{k+1}^{\tau,\nu} \in \mathrm{Argmin}\Big\{ \mathcal{I}(t_{k+1}^\tau, z) + \tau\overline{\mathcal{R}}_{1,\nu}\left(\frac{z - z_k^\tau}{\tau}\right) + \frac{\epsilon}{\tau}\left\|\frac{z - z_k^\tau}{\tau}\right\|_{L^2(\Omega)}^2 \ : \ z \in \mathcal{Z} \Big\}, \qquad k \in \{1, \ldots, N-1\}. \tag{3.29}$$

The analogue of Prop. 3.1 holds. In particular, the (left- and right-continuous) piecewise constant and linear interpolants $\overline{z}_{\tau,\nu}$, $\underline{z}_{\tau,\nu}$ and $\widehat{z}_{\tau,\nu}$ of the elements $(z_k^{\tau,\nu})_{k=0}^N$ satisfy the following approximate version of (3.8)

$$\mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\widehat{z}'_{\tau,\nu}(t)) + \epsilon\widehat{z}'_{\tau,\nu}(t) + A_q\overline{z}_{\tau,\nu}(t) + \mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_\tau(t), \overline{z}_{\tau,\nu}(t)) = 0 \quad \text{in } L^2(\Omega) \quad \text{for a.a. } t \in (0, T), \tag{3.30}$$

where we have in fact used that $\mathrm{D}_{\mathcal{Z},\mathcal{Z}^*}\overline{\mathcal{R}}_{1,\nu}(\widehat{z}'_{\tau,\nu}) = \mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\widehat{z}'_{\tau,\nu})$. In particular, observe that, by comparison in (3.30), there holds

$$A_q\overline{z}_{\tau,\nu}(t) \in L^2(\Omega) \qquad \text{for almost all } t \in (0, T). \tag{3.31}$$

For the functions $(\overline{z}_{\tau,\nu}, \widehat{z}_{\tau,\nu}, \overline{u}_{\tau,\nu}, \widehat{u}_{\tau,\nu})_{\tau,\nu}$ (with $\overline{u}_{\tau,\nu}$, $\widehat{u}_{\tau,\nu}$ the interpolants of the elements $u_{\min}(t_\tau^k, z_k^{\tau,\nu})$), we are now able to derive *rigorously* estimates (3.9), in fact *uniformly* w.r.t. both parameters τ and ν.

Lemma 3.6. *Under Assumptions 2.1, 2.2, and 2.8, and under condition* (3.6) *on the initial datum* z_0*, estimates* (3.9) *hold for the functions* $(\overline{z}_{\tau,\nu}, \widehat{z}_{\tau,\nu}, \overline{u}_{\tau,\nu}, \widehat{u}_{\tau,\nu})_{\tau,\nu}$ *(in particular,* (3.9g) *for* $\overline{\omega}_{\tau,\nu} := \mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\widehat{z}'_{\tau,\nu})$*), with constants* C, $C(\epsilon)$, $C(\sigma) > 0$ uniform *w.r.t.* τ *and* ν*.*

Proof. Estimates (3.9a)–(3.9b) (and, consequently, (3.9h) for $\overline{u}_{\tau,\nu}$) can be derived by the very same arguments as in the proof of Lemma 3.4. Let us point out that we may suppose that $\sup_{\tau,\nu}\|\overline{z}_{\tau,\nu}\|_{L^\infty(0,T;W^{1,q}(\Omega))} \leq M$, with M the same constant as in (3.14).

Instead, the calculations for (3.9d)–(3.9e) have to be slightly modified, as the ones developed in the proof of Lemma 3.4 rely on the 1-homogeneity of $\mathcal{R}_1$, whereas $\overline{\mathcal{R}}_{1,\nu}$ no longer has this property. Therefore, we argue in this way: keeping the short-hand notation $\widehat{h}_{\tau,\nu}(t) := -(\epsilon\widehat{z}'_{\tau,\nu}(t) + A_q\overline{z}_{\tau,\nu}(t) + \mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_\tau(t), \overline{z}_{\tau,\nu}(t)))$, and writing $\overline{\omega}_{\tau,\nu}(t)$ in place of $\mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\widehat{z}'_{\tau,\nu}(t))$, (3.30) rephrases as $\overline{\omega}_{\tau,\nu}(t) = \widehat{h}_{\tau,\nu}(t)$. We subtract (3.30) at time $r \in (t_{k-1}^\tau, t_k^\tau)$ from (3.30) at time $t \in (t_k^\tau, t_{k+1}^\tau)$ and test the resulting relation by $\widehat{z}'_{\tau,\nu}(t)$. Therefore we obtain

$$\overline{\mathcal{R}}^*_{1,\nu}(\overline{\omega}_{\tau,\nu}(t)) - \overline{\mathcal{R}}^*_{1,\nu}(\overline{\omega}_{\tau,\nu}(r)) \leq \int_\Omega (\overline{\omega}_{\tau,\nu}(t) - \overline{\omega}_{\tau,\nu}(r))\,\widehat{z}'_{\tau,\nu}(t)\,\mathrm{d}x = \int_\Omega \Big(\widehat{h}_{\tau,\nu}(t) - \widehat{h}_{\tau,\nu}(r)\Big)\,\widehat{z}'_{\tau,\nu}(t)\,\mathrm{d}x, \tag{3.32}$$

where $\overline{\mathcal{R}}^*_{1,\nu}$ denotes the Fenchel-Moreau convex conjugate of $\overline{\mathcal{R}}_{1,\nu}$, and we have used that

$$\widehat{z}'_{\tau,\nu}(t) \in \partial\overline{\mathcal{R}}^*_{1,\nu}(\overline{\omega}_{\tau,\nu}(t)) \quad \text{for all } t \in (t_k^\tau, t_{k+1}^\tau) \text{ and for all } k = 0, \ldots, N-1. \tag{3.33}$$

From (3.32) we then obtain the analogue of (3.15), namely

$$\begin{aligned} &\frac{1}{\tau}\overline{\mathcal{R}}^*_{1,\nu}(\overline{\omega}_{\tau,\nu}(t)) + \frac{\epsilon}{\tau}\int_\Omega (\widehat{z}'_{\tau,\nu}(t) - \widehat{z}'_{\tau,\nu}(r))\widehat{z}'_{\tau,\nu}(t)\,\mathrm{d}x + \frac{1}{\tau}\int_\Omega (A_q\overline{z}_{\tau,\nu}(t) - A_q\overline{z}_{\tau,\nu}(r))\widehat{z}'_{\tau,\nu}(t)\,\mathrm{d}x \\ &\qquad \leq \frac{1}{\tau}\overline{\mathcal{R}}^*_{1,\nu}(\overline{\omega}_{\tau,\nu}(r)) - \frac{1}{\tau}\int_\Omega (\mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_\tau(t), \overline{z}_{\tau,\nu}(t)) - \mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_\tau(r), \overline{z}_{\tau,\nu}(r)))\widehat{z}'_{\tau,\nu}(t)\,\mathrm{d}x. \end{aligned} \tag{3.34}$$

Observe that (3.34) contains the same terms as in (3.15), but with the additional contribution coming from $\overline{\mathcal{R}}^*_{1,\nu}$. Following the lines of the proof of Lemma 3.4 (see also [KRZ15, Lemma 5.3]) we "integrate" over the time interval (t_0, t) with $t_0 \in (0, t_1^\tau)$ and $t \in (t_k^\tau, t_{k+1}^\tau)$ and get

$$\begin{aligned} &\overline{\mathcal{R}}^*_{1,\nu}(\overline{\omega}_{\tau,\nu}(t)) + \frac{\epsilon}{2}\|\widehat{z}'_{\tau,\nu}(t)\|^2_{L^2(\Omega)} + C_q \int_{t_1^\tau}^{\bar{t}_\tau(t)} \int_\Omega \left(1 + |\nabla \widehat{z}_{\tau,\nu}(\rho)|^2\right)^{(q-2)/2} |\nabla \widehat{z}'_{\tau,\nu}(\rho)|^2 \,\mathrm{d}x\,\mathrm{d}\rho \\ &\leq \overline{\mathcal{R}}^*_{1,\nu}(\overline{\omega}_{\tau,\nu}(t_0)) + \frac{\epsilon}{2}\|\widehat{z}'_{\tau,\nu}(t_0)\|^2_{L^2(\Omega)} + C \int_{t_1^\tau}^{\bar{t}_\tau(t)} (1 + \|\widehat{z}'_{\tau,\nu}(\rho)\|_{L^6(\Omega)}) \|\widehat{z}'_{\tau,\nu}(\rho)\|_{L^2(\Omega)} \,\mathrm{d}\rho, \end{aligned} \tag{3.35}$$

with C_q from (3.16). We observe that $\overline{\mathcal{R}}^*_{1,\nu}(\overline{\omega}_{\tau,\nu}(t)) \geq 0$, and therefore on the left-hand side we get the exact analogue of the left-hand side of (3.19). For the right-hand side, we have to deal with the "extra"-term $\overline{\mathcal{R}}^*_{1,\nu}(\overline{\omega}_{\tau,\nu}(t_0))$. For this, we observe that

$$\begin{aligned} \overline{\mathcal{R}}^*_{1,\nu}(\overline{\omega}_{\tau,\nu}(t_0)) &= \overline{\mathcal{R}}^*_{1,\nu}(\overline{\omega}_{\tau,\nu}(t_0)) - \overline{\mathcal{R}}^*_{1,\nu}(0) \leq \int_\Omega \left(\frac{z_1^{\tau,\nu} - z_0}{\tau}\right) \overline{\omega}_{\tau,\nu}(t_0)\,\mathrm{d}x \\ &= \int_\Omega \frac{(z_1^{\tau,\nu} - z_0)}{\tau} \left(-\epsilon \frac{z_1^{\tau,\nu} - z_0}{\tau} - \mathrm{D}_z \mathcal{I}(t_1^\tau, z_1^{\tau,\nu})\right) \mathrm{d}x \\ &= -\epsilon \|\widehat{z}'_{\tau,\nu}(t_0)\|^2_{L^2(\Omega)} - \int_\Omega \mathrm{D}_z \mathcal{I}(t_1^\tau, z_1^{\tau,\nu}) \widehat{z}'_{\tau,\nu}(t_0)\,\mathrm{d}x \end{aligned} \tag{3.36}$$

and therefore, the right-hand side of (3.35) can be bounded as follows

$$\text{R.H.S.} \leq -\int_\Omega \mathrm{D}_z \mathcal{I}(t_1^\tau, z_1^{\tau,\nu}) \widehat{z}'_{\tau,\nu}(t_0)\,\mathrm{d}x - \frac{\epsilon}{2}\|\widehat{z}'_{\tau,\nu}(t_0)\|^2_{L^2(\Omega)} + C \int_{t_1^\tau}^{\bar{t}_\tau(t)} (1 + \|\widehat{z}'_{\tau,\nu}(\rho)\|_{L^6(\Omega)}) \|\widehat{z}'_{\tau,\nu}(\rho)\|_{L^2(\Omega)} \,\mathrm{d}\rho. \tag{3.37}$$

Writing $\mathrm{D}_z \mathcal{I}(t_1^\tau, z_1^{\tau,\nu}) = A_q(z_1^{\tau,\nu}) - A_q(z_0) + \mathrm{D}_z \widetilde{\mathcal{I}}(t_1^\tau, z_1^{\tau,\nu}) - \mathrm{D}_z \widetilde{\mathcal{I}}(0, z_0) + \mathrm{D}_z \mathcal{I}(0, z_0)$ and performing calculations analogous to those developed in the proof of Lemma 3.4, we obtain

$$\begin{aligned} -\int_\Omega \mathrm{D}_z \mathcal{I}(t_1^\tau, z_1^{\tau,\nu}) \widehat{z}'_{\tau,\nu}(t_0)\,\mathrm{d}x \leq -C_q \tau \int_\Omega (1 + |\nabla \widehat{z}_{\tau,\nu}(t_0)|^2)^{(q-2)/2} |\nabla \widehat{z}'_{\tau,\nu}(t_0)|^2 \,\mathrm{d}x + \frac{\epsilon}{2}\|\widehat{z}'_{\tau,\nu}(t_0)\|^2_{L^2(\Omega)} \\ + \epsilon^{-1} \|\mathrm{D}_z \mathcal{I}(0, z_0)\|^2_{L^2(\Omega)} + c\tau (1 + \|\widehat{z}'_{\tau,\nu}(t_0)\|^2_{L^6(\Omega)}) \|\widehat{z}'_{\tau,\nu}(t_0)\|^2_{L^2(\Omega)}. \end{aligned}$$

Combining this with (3.37), summing the resulting inequality with (3.35), and adding $C_q \int_0^{\bar{t}_\tau(t)} \|\widehat{z}'_{\tau,\nu}(\rho)\|^2_{L^2(\Omega)} \,\mathrm{d}\rho$ to both terms of the resulting estimate, we obtain

$$\begin{aligned} &\frac{\epsilon}{2}\|\widehat{z}'_{\tau,\nu}(t)\|^2_{L^2(\Omega)} + C_q \int_0^{\bar{t}_\tau(t)} \|\widehat{z}'_{\tau,\nu}(\rho)\|^2_{L^2(\Omega)} \,\mathrm{d}\rho + C_q \int_0^{\bar{t}_\tau(t)} \int_\Omega \left(1 + |\nabla \widehat{z}_{\tau,\nu}(\rho)|^2\right)^{(q-2)/2} |\nabla \widehat{z}'_{\tau,\nu}(\rho)|^2 \,\mathrm{d}x\,\mathrm{d}\rho \\ &\leq \epsilon^{-1} \|\mathrm{D}_z \mathcal{I}(0, z_0)\|^2_{L^2(\Omega)} + C_q \int_0^{\bar{t}_\tau(t)} \|\widehat{z}'_{\tau,\nu}(\rho)\|^2_{L^2(\Omega)} \,\mathrm{d}\rho + C \int_0^{\bar{t}_\tau(t)} (1 + \|\widehat{z}'_{\tau,\nu}(\rho)\|_{L^6(\Omega)}) \|\widehat{z}'_{\tau,\nu}(\rho)\|_{L^2(\Omega)} \,\mathrm{d}\rho \\ &\leq C + \epsilon^{-1} \|\mathrm{D}_z \mathcal{I}(0, z_0)\|^2_{L^2(\Omega)} + C \int_0^{\bar{t}_\tau(t)} \|\widehat{z}'_{\tau,.\nu}(\rho)\|^2_{L^2(\Omega)} \,\mathrm{d}\rho + \frac{C_q}{4} \int_0^{\bar{t}_\tau(t)} \|\widehat{z}'_{\tau,\nu}(\rho)\|^2_{H^1(\Omega)} \,\mathrm{d}\rho, \end{aligned} \tag{3.38}$$

where in the last inequality we have used Young's inequality, and the continuous embedding $H^1(\Omega) \subset L^6(\Omega)$, for the last term in the r.h.s. of (3.37) exactly as in the proof of Lemma 3.4. Absorbing $\int_0^{\bar{t}_\tau(t)} \|\widehat{z}'_\tau(\rho)\|^2_{H^1(\Omega)} \,\mathrm{d}\rho$, into the left-hand side, we conclude estimate (3.9d) for $\widehat{z}'_{\tau,\nu}$, uniformly with respect to τ and ν.

Combining the arguments in the proof of Lemma 3.4 with the above arguments related to $\overline{\mathcal{R}}^*_{1,\nu}$ we also obtain estimate (3.9e) for $\widehat{z}'_{\tau,\nu}$ uniformly with respect to ϵ, τ and ν, and therefore also the bounds (3.9i)–(3.9j) for $\widehat{u}'_{\tau,\nu}$.

We are now in a position to carry out the time-discrete analogue of the *Third regularity estimate*. We multiply (3.30), written at time $\rho \in (t_k^\tau, t_{k+1}^\tau)$, by the difference $(A_q \overline{z}_{\tau,\nu}(\rho) - A_q \overline{z}_{\tau,\nu}(r))$, with $r \in (t_{k-1}^\tau, t_k^\tau)$, and integrate in space. Observe that this is now a legal test, in view of (3.31). We thus obtain

$$\underbrace{\int_\Omega \mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\widehat{z}'_{\tau,\nu}(\rho))(A_q\overline{z}_{\tau,\nu}(\rho)-A_q\overline{z}_{\tau,\nu}(r))\,\mathrm{d}x}_{I_1} \quad +\epsilon \quad \underbrace{\int_\Omega \widehat{z}'_{\tau,\nu}(\rho)(A_q\overline{z}_{\tau,\nu}(\rho)-A_q\overline{z}_{\tau,\nu}(r))\,\mathrm{d}x}_{I_2}$$
$$+ \quad \underbrace{\int_\Omega A_q\overline{z}_{\tau,\nu}(\rho)(A_q\overline{z}_{\tau,\nu}(\rho)-A_q\overline{z}_{\tau,\nu}(r))\,\mathrm{d}x}_{I_3} \quad = - \quad \underbrace{\int_\Omega \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(\rho),\overline{z}_{\tau,\nu}(\rho))(A_q\overline{z}_{\tau,\nu}(\rho)-A_q\overline{z}_{\tau,\nu}(r))\,\mathrm{d}x}_{I_4} \quad . \tag{3.39}$$

Now, we have that

$$\begin{aligned}
I_1 &= \int_\Omega \nabla\left(\overline{\mathcal{R}}'_{1,\nu}(\widehat{z}'_{\tau,\nu}(\rho))\right)\cdot\Big((1+|\nabla\overline{z}_{\tau,\nu}(\rho)|^2)^{q/2-1}\nabla\overline{z}_{\tau,\nu}(\rho)-(1+|\nabla\overline{z}_{\tau,\nu}(r)|^2)^{q/2-1}\nabla\overline{z}_{\tau,\nu}(r)\Big)\,\mathrm{d}x \\
&= \int_\Omega \overline{\mathcal{R}}''_{1,\nu}(\widehat{z}'_{\tau,\nu}(\rho))\nabla\widehat{z}'_{\tau,\nu}(\rho)\cdot\Big((1+|\nabla\overline{z}_{\tau,\nu}(\rho)|^2)^{q/2-1}\nabla\overline{z}_{\tau,\nu}(\rho)-(1+|\nabla\overline{z}_{\tau,\nu}(r)|^2)^{q/2-1}\nabla\overline{z}_{\tau,\nu}(r)\Big)\,\mathrm{d}x \overset{(1)}{\geq} 0,
\end{aligned}$$

where for the first equality we have used that $\mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\widehat{z}'_{\tau,\nu}(\rho)) = \overline{\mathcal{R}}'_{1,\nu}(\widehat{z}'_{\tau,\nu}(\rho))$ is an element in $W^{1,q}(\Omega)$: indeed, $\widehat{z}'_{\tau,\nu}(\rho)\in W^{1,q}(\Omega)\subset \mathrm{C}^0(\overline{\Omega})$, so that there exists a constant $M>0$ with $|\widehat{z}'_{\tau,\nu}(\rho)|\leq M$ a.e. in Ω; on the other hand $\overline{\mathcal{R}}'_{1,\nu}\in \mathrm{C}^\infty(\mathbb{R})$, hence its restriction to the ball $\overline{B}_M(0)$ is Lipschitz, and the composition of a Lipschitz function with an element in $W^{1,q}(\Omega)$ belongs to $W^{1,q}(\Omega)$. Estimate (1) follows from the fact that $\overline{\mathcal{R}}''_{1,\nu}\geq 0$ on $\mathbb{R}$, and from the convexity inequality

$$(A-B)\cdot\Big((1+|A|^2)^{q/2-1}A-(1+|B)|^2)^{q/2-1}B\Big)\geq 0 \quad \text{for all } A,\, B\in\mathbb{R}^3,$$

applied with $A=\nabla\overline{z}_{\tau,\nu}(\rho)$ and $B=\nabla\overline{z}_{\tau,\nu}(r)$. Analogously, we have

$$I_2 = \int_\Omega \nabla\widehat{z}'_{\tau,\nu}(\rho)\cdot\Big((1+|\nabla\overline{z}_{\tau,\nu}(\rho)|^2)^{q/2-1}\nabla\overline{z}_{\tau,\nu}(\rho)-(1+|\nabla\overline{z}_{\tau,\nu}(r)|^2)^{q/2-1}\nabla\overline{z}_{\tau,\nu}(r)\Big)\,\mathrm{d}x\geq 0.$$

We have

$$I_3 \geq \frac{1}{2}\|A_q\overline{z}_{\tau,\nu}(\rho)\|^2_{L^2(\Omega)} - \frac{1}{2}\|A_q\overline{z}_{\tau,\nu}(r)\|^2_{L^2(\Omega)}\,.$$

Finally,

$$\begin{aligned}
I_4 = &\int_\Omega \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(\rho),\overline{z}_{\tau,\nu}(\rho))A_q\overline{z}_{\tau,\nu}(\rho)\,\mathrm{d}x - \int_\Omega \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(r),\overline{z}_{\tau,\nu}(r))A_q\overline{z}_{\tau,\nu}(r)\,\mathrm{d}x \\
&- \int_\Omega \Big(\mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(\rho),\overline{z}_{\tau,\nu}(\rho))-\mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(r),\overline{z}_{\tau,\nu}(r))\Big)\,A_q\overline{z}_{\tau,\nu}(r)\,\mathrm{d}x\,.
\end{aligned}$$

Summing with respect to the index k, we thus obtain for any $t\in(t_1^\tau,T)$ and for $\sigma\in(0,t_1^\tau)$ (remember that $\overline{z}_{\tau,\nu}(r)=\underline{z}_{\tau,\nu}(\rho)$ and $\overline{t}_\tau(r)=\underline{t}_\tau(\rho)$ for $r\in(t_{k-1}^\tau,t_k^\tau]$ and $\rho\in[t_k^\tau,t_{k+1}^\tau)$)

$$\begin{aligned}
\frac{1}{2}\|A_q\overline{z}_{\tau,\nu}(t)\|^2_{L^2(\Omega)} \leq &\frac{1}{2}\|A_q\overline{z}_{\tau,\nu}(\sigma)\|^2_{L^2(\Omega)} + \int_\Omega \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(\sigma),\overline{z}_{\tau,\nu}(\sigma))A_q\overline{z}_{\tau,\nu}(\sigma)\,\mathrm{d}x - \int_\Omega \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(t),\overline{z}_{\tau,\nu}(t))A_q\overline{z}_{\tau,\nu}(t)\,\mathrm{d}x \\
&+\int_{t_1^\tau}^{\overline{t}_\tau(t)}\int_\Omega \frac{1}{\tau}\Big(\mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(\rho),\overline{z}_{\tau,\nu}(\rho))-\mathrm{D}\widetilde{\mathcal{I}}(\underline{t}_\tau(\rho),\underline{z}_{\tau,\nu}(\rho))\Big)\,A_q\underline{z}_{\tau,\nu}(\rho)\,\mathrm{d}x\,\mathrm{d}\rho \doteq I_5+I_6+I_7+I_8.
\end{aligned}$$

We estimate via Hölder's and Young's inequalities

$$
\begin{aligned}
|I_6| &\leq \|\mathrm{D}\widetilde{\mathfrak{I}}(\overline{t}_\tau(\sigma), \overline{z}_{\tau,\nu}(\sigma))\|_{L^2(\Omega)}^2 + \frac{1}{4}\|A_q \overline{z}_{\tau,\nu}(\sigma)\|_{L^2(\Omega)}^2 \overset{(2)}{\leq} C + \frac{1}{4}\|A_q \overline{z}_{\tau,\nu}(\sigma)\|_{L^2(\Omega)}^2, \\
|I_7| &\leq \|\mathrm{D}\widetilde{\mathfrak{I}}(\overline{t}_\tau(t), \overline{z}_{\tau,\nu}(t))\|_{L^2(\Omega)}^2 + \frac{1}{4}\|A_q \overline{z}_{\tau,\nu}(t)\|_{L^2(\Omega)}^2 \overset{(1)}{\leq} C + \frac{1}{4}\|A_q \overline{z}_{\tau,\nu}(t)\|_{L^2(\Omega)}^2, \\
|I_8| &\leq \int_0^{\overline{t}_\tau(t)} \frac{1}{\tau}\|\mathrm{D}\widetilde{\mathfrak{I}}(\overline{t}_\tau(\rho), \overline{z}_{\tau,\nu}(\rho)) - \mathrm{D}\widetilde{\mathfrak{I}}(\underline{t}_\tau(\rho), \underline{z}_{\tau,\nu}(\rho))\|_{L^2(\Omega)}\|A_q \underline{z}_{\tau,\nu}(\rho)\|_{L^2(\Omega)}\,\mathrm{d}\rho \\
&\overset{(3)}{\leq} C\int_0^{\overline{t}_\tau(t)} \frac{1}{\tau}\|\overline{z}_{\tau,\nu}(\rho) - \underline{z}_{\tau,\nu}(\rho)\|_{L^6(\Omega)}\|A_q \underline{z}_{\tau,\nu}(\rho)\|_{L^2(\Omega)}\,\mathrm{d}\rho.
\end{aligned}
$$

where (1) and (2) follow from (2.34) and from the bound $\|f'(\overline{z}_{\tau,\nu}(t)\|_{L^\infty(\Omega)} + P(\overline{z}_{\tau,\nu}(t), 0) \leq C$, for a constant uniform w.r.t. $t \in [0,T]$, thanks to estimate (3.9b) for $(\overline{z}_{\tau,\nu})_{\tau,\nu}$; instead, (3) is due to (2.32), again taking into account that $\sup_{\rho\in[0,T]}(C_{f''}(\overline{z}_{\tau,\nu}(\rho), \underline{z}_{\tau,\nu}(\rho)) + P(\overline{z}_{\tau,\nu}(\rho), \underline{z}_{\tau,\nu}(\rho))^3) \leq C$ due to the bound (3.9b). All in all, we conclude

$$
\frac{1}{4}\|A_q \overline{z}_{\tau,\nu}(t)\|_{L^2(\Omega)}^2 \leq \frac{3}{4}\|A_q \overline{z}_{\tau,\nu}(\sigma)\|_{L^2(\Omega)}^2 + C\left(1 + \int_0^{\overline{t}_\tau(t)} \|\widehat{z}'_{\tau,\nu}(\rho)\|_{L^6(\Omega)}\|A_q \underline{z}_{\tau,\nu}(\rho)\|_{L^2(\Omega)}\,\mathrm{d}\rho\right),
$$

and, with a version of Gronwall's Lemma (cf. e.g. [Bré73, Lemme A.5]), we conclude that

$$
\|A_q \overline{z}_{\tau,\nu}(t)\|_{L^2(\Omega)} \leq C\left(1 + \|A_q \overline{z}_{\tau,\nu}(\sigma)\|_{L^2(\Omega)} + \int_0^{\overline{t}_\tau(t)} \|\widehat{z}'_{\tau,\nu}(\rho)\|_{L^6(\Omega)}\,\mathrm{d}\rho\right). \tag{3.40}
$$

It now remains to estimate $\|A_q \overline{z}_{\tau,\nu}(\sigma)\|_{L^2(\Omega)} = \|A_q z_1^{\tau,\nu}\|_{L^2(\Omega)}$. For this, we use the Euler-Lagrange equation

$$
\mathrm{D}\overline{\mathcal{R}}_{1,\nu}\left(\frac{z_1^{\tau,\nu} - z_0}{\tau}\right) + \epsilon\frac{z_1^{\tau,\nu} - z_0}{\tau} + A_q z_1^{\tau,\nu} + \mathrm{D}_z\widetilde{\mathfrak{I}}(t_1^{\tau,\nu}, z_1^{\tau,\nu}) = 0
$$

and test it by $A_q z_1^{\tau,\nu} - A_q z_0$. We repeat the same calculations as above and arrive at

$$
\begin{aligned}
\frac{1}{2}\|A_q z_1^{\tau,\nu}\|_{L^2(\Omega)}^2 \leq &\frac{1}{2}\|A_q z_0\|_{L^2(\Omega)}^2 + \int_\Omega \mathrm{D}\widetilde{\mathfrak{I}}(0, z_0)A_q z_0\,\mathrm{d}x - \int_\Omega \mathrm{D}\widetilde{\mathfrak{I}}(t_1^{\tau,\nu}, z_1^{\tau,\nu})A_q z_1^{\tau,\nu}\,\mathrm{d}x \\
&+ \int_\Omega \left(\mathrm{D}\widetilde{\mathfrak{I}}(t_1^{\tau,\nu}, z_1^{\tau,\nu}) - \mathrm{D}\widetilde{\mathfrak{I}}(0, z_0)\right)A_q z_0\,\mathrm{d}x,
\end{aligned}
$$

whence

$$
\|A_q z_1^{\tau,\nu}\|_{L^2(\Omega)}^2 \leq C\left(1 + \|A_q z_0\|_{L^2(\Omega)}^2 + \|z_1^{\tau,\nu} - z_0\|_{L^6(\Omega)}^2\right) \leq C,
$$

the last inequality due to (3.6) and bound (3.9b). Combining the above estimate with (3.40), we conclude estimate (3.9f) in view of the previously proved bound (3.9e) for $\widehat{z}'_{\tau,\nu}$.

Finally, estimate (3.9g) for $\overline{\omega}_{\tau,\nu} = \mathrm{D}\overline{\mathcal{R}}_{1,\nu}(\widehat{z}'_{\tau,\nu})$ follows from a comparison argument in (3.30), in view of estimate (3.9d), and the previously used bound for $\mathrm{D}\widetilde{\mathfrak{I}}(\overline{t}_\tau(\cdot), \overline{z}_{\tau,\nu}(\cdot))$ in $L^\infty(0,T;L^2(\Omega))$ due to (2.34) and (3.9b). □

We are now in a position to conclude the **proof of Lemma 3.5**: For *fixed* positive τ and ϵ, let $(\overline{z}_{\tau,\nu}, \widehat{z}_{\tau,\nu})_\nu$ a family of solutions to (3.30). It follows from estimates (3.9) proved in Lemma 3.6 and from Proposition 2.7, that the sequence $(\overline{z}_{\tau,\nu})_\nu$ is also uniformly bounded in $L^\infty(0,T;W^{1+\sigma,q}(\Omega))$ for all $0 < \sigma < \frac{1}{q}$, whence estimate (3.9c). Hence, also the sequence $(\widehat{z}_{\tau,\nu})_\nu$ is bounded in that space. Therefore, applying the Aubin-Lions type compactness results from [Sim87] to $(\widehat{z}_{\tau,\nu})_\nu$, we infer that there exists a function $\widehat{z}$ such that, along a (not relabeled) subsequence, as $\nu \downarrow 0$ the following convergences hold

$$
\begin{aligned}
&\widehat{z}_{\tau,\nu} \rightharpoonup^* \widehat{z} \quad \text{in } L^\infty(0,T;W^{1+\sigma,q}(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega)) \quad \text{for all } 0 < \sigma < \frac{1}{q}, \\
&\widehat{z}_{\tau,\nu} \to \widehat{z} \quad \text{in } \mathrm{C}^0([0,T];\mathcal{Z}),
\end{aligned} \tag{3.41a}
$$

where the last convergence follows from the compact embedding $W^{1+\sigma,q}(\Omega) \Subset \mathcal{Z}$ for all $\sigma \in (0, \frac{1}{q})$. From the estimate for $(\widehat{z}'_{\tau,\nu})_\nu$ in $L^1(0,T;H^1(\Omega))$ we gather that

$$\|\overline{z}_{\tau,\nu}\|_{\mathrm{BV}([0,T];H^1(\Omega))} \leq C$$

for a constant independent of ν (and τ). Therefore, thanks to an infinite-dimensional version of Helly's Theorem, see e.g. [MT04, Thm. 6.1], we conclude that there exists $\overline{z} \in \mathrm{BV}([0,T];H^1(\Omega))$ such that, up to the further extraction of a subsequence, $\overline{z}_{\tau,\nu}(t) \rightharpoonup \overline{z}(t)$ in $H^1(\Omega)$, as $\nu \downarrow 0$ for every $t \in [0,T]$. Since $(\overline{z}_{\tau,\nu})_\nu$ is bounded in $L^\infty(0,T;W^{1+\sigma,q}(\Omega))$, we ultimately conclude that $\overline{z}_{\tau,\nu}(t) \rightharpoonup \overline{z}(t)$ in $W^{1+\sigma,q}(\Omega)$ for every $t \in [0,T]$. Thus, we infer

$$\overline{z}_{\tau,\nu}(t) \to \overline{z}(t) \qquad \text{in } \mathcal{Z} \quad \text{for every } t \in [0,T]. \tag{3.41b}$$

Then, a fortiori one has that

$$\overline{z}_{\tau,\nu} \rightharpoonup^* \overline{z} \text{ in } L^\infty(0,T;\mathcal{Z}), \qquad \overline{z}_{\tau,\nu} \to \overline{z} \text{ in } L^p(0,T;\mathcal{Z}) \text{ for every } 1 \leq p < \infty. \tag{3.41c}$$

Finally, there exists $\overline{\omega} \in L^\infty(0,T;L^2(\Omega))$ such that, up to a further extraction,

$$\overline{\omega}_{\tau,\nu} \rightharpoonup^* \overline{\omega} \qquad \text{in } L^\infty(0,T;L^2(\Omega)). \tag{3.41d}$$

It follows from (3.41b), combined with the bound (3.9e), that

$$A_q\overline{z}_{\tau,\nu}(t) \rightharpoonup A_q\overline{z}(t) \qquad \text{in } L^2(\Omega) \quad \text{for every } t \in [0,T].$$

Also in view of (3.41c) it is not difficult to deduce that

$$A_q\overline{z}_{\tau,\nu} \rightharpoonup^* A_q\overline{z} \qquad \text{in } L^\infty(0,T;L^2(\Omega)).$$

Furthermore, combining estimate (2.32) with (3.9b) and convergence (3.41b) we find that for every $t \in [0,T]$

$$\begin{aligned}\|\mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(t),\overline{z}_{\tau,\nu}(t)) - \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(t),\overline{z}(t))\|_{L^2(\Omega)} &\leq C\left(C'_f(\overline{z}_{\tau,\nu}(t),\overline{z}(t)) + P(\overline{z}_{\tau,\nu}(t),\overline{z}(t))^3\right)\|\overline{z}_{\tau,\nu}(t)-\overline{z}(t)\|_{L^6(\Omega)}\\ &\leq C\|\overline{z}_{\tau,\nu}(t)-\overline{z}(t)\|_{L^6(\Omega)} \to 0\end{aligned}$$

as $\nu \downarrow 0$. Since $(\mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau,\overline{z}_{\tau,\nu}))_\nu$ is bounded in $L^\infty(0,T;L^2(\Omega))$ by (2.34) and (3.9b), we also have

$$\begin{aligned}&\mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau,\overline{z}_{\tau,\nu}) \rightharpoonup^* \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau,\overline{z}) \text{ in } L^\infty(0,T;L^2(\Omega)),\\ &\mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau,\overline{z}_{\tau,\nu}) \to \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau,\overline{z}) \text{ in } L^p(0,T;L^2(\Omega)) \quad \text{for every } 1 \leq p < \infty.\end{aligned}$$

Therefore, also on account of convergences (3.41a) and (3.41d) we can pass to the limit as $\nu \downarrow 0$ in (3.30) and conclude that the triple $(\overline{z},\widehat{z},\overline{\omega})$ satisfies

$$\overline{\omega}(t) + \epsilon\widehat{z}'(t) + A_q\overline{z}(t) + \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(t),\overline{z}(t)) = 0 \quad \text{in } L^2(\Omega) \quad \text{for a.a.}\, t \in (t_k^\tau, t_{k+1}^\tau)$$

and for every $k \in \{0,\ldots,N-1\}$. We can also prove that

$$\limsup_{\nu\downarrow 0} \int_{t_k^\tau}^{t_{k+1}^\tau}\int_\Omega \overline{\omega}_{\tau,\nu}\widehat{z}'_{\tau,\nu}\,\mathrm{d}x\,\mathrm{d}t \leq \int_{t_k^\tau}^{t_{k+1}^\tau}\int_\Omega \overline{\omega}\widehat{z}'\,\mathrm{d}x\,\mathrm{d}t\,.$$

This follows from multiplying (3.30) by $\widehat{z}'_{\tau,\nu}$ and taking the limit in each of the terms, on account of the convergences so far proved.

Therefore, thanks to (3.28d), we infer that $\overline{\omega}(t) \in \partial\mathcal{R}_1(\widehat{z}'(t))$ for almost all $t \in (t_k^\tau, t_{k+1}^\tau)$. All in all, the pair $(\overline{z},\widehat{z})$ fulfills the differential inclusion

$$\partial\mathcal{R}_1(\widehat{z}'(t)) + \epsilon\widehat{z}'(t) + A_q\overline{z}(t) + \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(t),\overline{z}(t)) \ni 0 \text{ in } L^2(\Omega) \quad \text{for a.a.}\, t \in (t_k^\tau,t_{k+1}^\tau)\ \forall\, k \in \{0,\ldots,N-1\}\,. \tag{3.42}$$

A fortiori, since $\partial\mathcal{R}_1(\widehat{z}'(t)) \subset \partial_{\mathcal{Z},\mathcal{Z}^*}\mathcal{R}_1(\widehat{z}'(t))$, we conclude that $(\overline{z},\widehat{z})$ fulfill

$$\partial_{\mathcal{Z},\mathcal{Z}^*}\mathcal{R}_1(\widehat{z}'(t)) + \epsilon\widehat{z}'(t) + A_q\overline{z}(t) + \mathrm{D}\widetilde{\mathcal{I}}(\overline{t}_\tau(t),\overline{z}(t)) \ni 0 \quad \text{in } \mathcal{Z}^* \quad \text{for a.a.}\, t \in (t_k^\tau,t_{k+1}^\tau) \quad \forall\, k \in \{0,\ldots,N-1\}\,.$$

Since the latter has a unique solution in the closed ball $\overline{B}_M(0)$ of $\mathcal{Z}$ for $\tau < \overline{\tau}_\epsilon$ (cf. Prop. 3.1), and since $\overline{z}$ and $\overline{z}_\tau$ take value in that ball, we get that

$$\overline{z}(t) = \overline{z}_\tau(t), \qquad \widehat{z}'(t) = \widehat{z}'_\tau(t) \qquad \text{for a.a.}\, t \in (t_k^\tau,t_{k+1}^\tau) \quad \forall\, k \in \{0,\ldots,N-1\}\,,$$

and, therefore, a.e. in $(0,T)$. In particular, we find that $A_q\overline{z}_\tau \in L^\infty(0,T;L^2(\Omega))$. Furthermore, since estimates (3.9f) and (3.9g) are uniform both w.r.t. $\nu > 0$ and w.r.t. $\tau > 0$, they are inherited in the limit as $\nu \downarrow 0$. Therefore,

$$\|A_q\overline{z}_\tau\|_{L^\infty(0,T;L^2(\Omega))} + \|\overline{\omega}\|_{L^\infty(0,T;L^2(\Omega))} \leq C$$

for a constant independent of $\tau < \bar{\tau}_\epsilon$. We set $\overline{\omega}_\tau := \overline{\omega}$ and ultimately conclude (3.7) as well as (3.9f) and (3.9g). Finally, from (3.42) we gather the validity of (3.8). This concludes the proof of Lemma 3.5. ∎

4. Existence of viscous solutions

In this section, we briefly comment on the existence of solutions to the viscous system (1.2). By passing to the limit with $\epsilon > 0$ fixed in the time discrete scheme (3.5), we are able to prove the existence of a solution to (1.2), formulated as a subdifferential inclusion in $L^2(\Omega)$, namely

$$\omega(t) + \epsilon z'(t) + A_q(z(t)) + \mathrm{D}_z\widetilde{\mathcal{I}}(t,z(t)) \ni 0 \quad \text{in } L^2(\Omega) \text{ for a.a. } t \in (0,T), \tag{4.1}$$

with $\omega(\cdot)$ a selection in the subdifferential $\partial\mathcal{R}_1(z'(\cdot)) \subset L^2(\Omega)$. Furthermore, along the footsteps of [MRS13] we obtain an energy-dissipation balance featuring the conjugate $\mathcal{R}_\epsilon^*$ of $\mathcal{R}_\epsilon$, cf. (3.10).

Theorem 4.1. *Let $\epsilon > 0$ be fixed. Under Assumptions 2.1, 2.2, and 2.8, and under condition* (3.6) *on the initial datum z_0, there exist*

$$\begin{gathered} z \in L^\infty(0,T;W^{1+\sigma,q}(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega)) \text{ for every } \sigma \in (0,\tfrac{1}{q}), \text{ with} \\ A_q z \in L^\infty(0,T;L^2(\Omega)) \end{gathered} \tag{4.2}$$

and $\omega \in L^\infty(0,T;L^2(\Omega))$ fulfilling the subdifferential inclusion (4.1) *and the Cauchy condition $z(0) = z_0$.*

Furthermore, z complies with the energy-dissipation balance

$$\int_s^t \mathcal{R}_\epsilon(z'(r))\,\mathrm{d}r + \int_s^t \mathcal{R}_\epsilon^*(-A_q(z(r)) - \mathrm{D}_z\widetilde{\mathcal{I}}(r,z(r)))\,\mathrm{d}r + \mathcal{I}(t,z(t)) = \mathcal{I}(s,z(s)) + \int_s^t \partial_t\mathcal{I}(r,z(r))\,\mathrm{d}r \tag{4.3}$$

for every $0 \leq s \leq t \leq T$.

Proof. Let $(\tau_j)_j$ be a null sequence of time steps, and let $(\overline{z}_{\tau_j})_j$, $(\widehat{z}_{\tau_j})_j$ be the approximate solutions to the viscous subdifferential inclusion (1.2) constructed in Section 3. For them, estimates (3.9) hold with a constant uniform w.r.t. $j \in \mathbb{N}$ (recall that $\epsilon > 0$ is fixed).

Adapting the arguments from the proof of [KRZ15, Prop. 6.2], combining (3.9) with Aubin-Lions type compactness results (cf., e.g., [Sim87, Thm. 5, Cor. 4]) and arguing in the same way as in the proof of Lemma 3.5, cf. also Lemma 6.2 ahead, we may show that there exist a (not relabeled) subsequence and a curve z as in (4.2) such that the following convergences hold

$$\begin{aligned}
&\overline{z}_{\tau_j},\ \widehat{z}_{\tau_j} \to z && \text{in } L^\infty(0,T;\mathcal{Z}),\\
&\widehat{z}_{\tau_j} \rightharpoonup^* z && \text{in } H^1(0,T;H^1(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega)),\\
&\mathcal{I}(\bar{t}_{\tau_j}(t),\overline{z}_{\tau_j}(t)),\ \mathcal{I}(t,\widehat{z}_{\tau_j}(t)) \to \mathcal{I}(t,z(t)) && \text{for all } t \in [0,T],\\
&\mathrm{D}_z\mathcal{I}(\bar{t}_{\tau_j}(t),\overline{z}_{\tau_j}(t)) \rightharpoonup^* \mathrm{D}_z\mathcal{I}(t,z(t)) && \text{in } L^\infty(0,T;L^2(\Omega)),\\
&\mathrm{D}_z\mathcal{I}(\bar{t}_{\tau_j}(t),\overline{z}_{\tau_j}(t)) \to \mathrm{D}_z\mathcal{I}(t,z(t)) && \text{in } L^\infty(0,T;\mathcal{Z}^*).
\end{aligned}$$

With the limit passage arguments from [KRZ15, Thm. 3.5] we deduce that z complies with the variational inequality

$$\begin{aligned}
\mathcal{R}_\epsilon(w) - \mathcal{R}_\epsilon(z'(t)) \geq \langle -A_q z(t), w\rangle_{\mathcal{Z}} &+ \int_\Omega (1+|\nabla z(t)|^2)^{\frac{q-2}{2}}\nabla z(t)\cdot\nabla z'(t)\,\mathrm{d}x \\
&- \int_\Omega \mathrm{D}_z\widetilde{\mathcal{I}}(t,z(t))(w - z'(t))\,\mathrm{d}x \quad \text{for all } w \in \mathcal{Z} \qquad \text{for a.a. } t \in (0,T)\,,
\end{aligned} \tag{4.4}$$

which in fact defined the concept of *weak solution* to the viscous system considered in [KRZ15].

We now enhance (4.4) by relying on the information that $A_q z \in L^\infty(0,T;L^2(\Omega))$. Due to this, $\int_\Omega (1+|\nabla z(t)|^2)^{\frac{q-2}{2}} \nabla z(t)\cdot\nabla z'(t)\,\mathrm{d}x = \int_\Omega A_q(z(t))z'(t)\,\mathrm{d}x$, so that (4.4) reads for almost all $t\in(0,T)$

$$\mathcal{R}_\epsilon(w)-\mathcal{R}_\epsilon(z'(t)) \geq -\int_\Omega A_q z(t)(w-z'(t))\,\mathrm{d}x - \int_\Omega \mathrm{D}_z\widetilde{\mathcal{I}}(t,z(t))(w-z'(t))\,\mathrm{d}x \quad \text{for all } w\in\mathcal{Z}\,.$$

This extends to all $w\in L^2(\Omega)$ by a density argument, and therefore we conclude that

$$-A_q z(t) - \mathrm{D}_z\widetilde{\mathcal{I}}(t,z(t)) \in \partial\mathcal{R}_\epsilon(z'(t)) \qquad \text{in } L^2(\Omega) \tag{4.5}$$

for almost all $t\in(0,T)$, namely the validity of (4.1).

The energy-dissipation balance (4.3) ensues from integrating on the generic interval $(s,t)\subset(0,T)$ the following chain of identities

$$\begin{aligned}\mathcal{R}_\epsilon(z'(r)) + \mathcal{R}_\epsilon^*(-A_q z(r) - \mathrm{D}_z\widetilde{\mathcal{I}}(r,z(r))) &\overset{(1)}{=} \int_\Omega \Big(-A_q z(r) - \mathrm{D}_z\widetilde{\mathcal{I}}(r,z(r))\Big) z'(t)\,\mathrm{d}x \\ &\overset{(2)}{=} -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(r,z(r)) + \partial_t\mathcal{I}(r,z(r)) \quad \text{for a.a. } r\in(0,T),\end{aligned}$$

where (1) is a reformulation of (4.5), while (2) follows from the chain rule (2.43). □

5. Balanced Viscosity solutions to the rate-independent damage system

The main result of this section, Theorem 5.7 ahead, states the convergence of the sequences

$$(\overline{z}_{\tau,\epsilon})_{\tau,\epsilon},\ (\widehat{z}_{\tau,\epsilon})_{\tau,\epsilon} \tag{5.1}$$

of discrete solutions constructed in Section 3 to a Balanced Viscosity solution of the rate-independent damage system (1.1), as ϵ and τ *simultaneously* tend to zero (that is why, we stress the dependence on the parameter ϵ in the notation (5.1)). The proof of Thm. 5.7 will be carried out in Section 6.

In Section 5.1 we provide a precise definition of this solution concept, after revisiting, and suitably modifying, all the preliminary definitions and notions given in [MRS16, Sec. 3.1]. Indeed, the latter paper addressed the case of a *nonsmooth* energy functional driving the (abstract) gradient system under consideration, and developed the vanishing-viscosity analysis under the sole *basic energy* estimates for viscous solutions. In the present context, on the one hand we will work with simpler definitions, tailored to the smoothness properties of $\mathcal{I}$, and to the enhanced estimates holding for our own damage system. On the other hand, our definitions shall reflect the fact that the dissipation potential $\mathcal{R}_1$ takes the value $+\infty$, whereas the analysis in [MRS16] is confined to the case of a *continuous* potential $\mathcal{R}_1$.

In Sec. 5.2 we gain further insight into to the properties of Balanced Viscosity solutions and again revisit and adapt a series of results given in [MRS16, Secs. 3.2, 3.3, 3.4].

5.1. The notion of Balanced Viscosity solution.

In order to define the notion of Balanced Viscosity solution for the damage system (1.1), we start by introducing the *vanishing-viscosity contact potential* $\mathfrak{p}$ induced by the viscous dissipation potentials $\mathcal{R}_\epsilon$ from (1.3). Such functional will enter into the Finsler cost describing the energy dissipated at jumps. We define $\mathfrak{p}: L^2(\Omega)\times L^2(\Omega)\to[0,+\infty]$ via

$$\begin{aligned}\mathfrak{p}(v,\xi) &:= \inf_{\epsilon>0}\left(\mathcal{R}_\epsilon(v)+\mathcal{R}_\epsilon^*(\xi)\right) \\ &= \mathcal{R}_1(v) + \|v\|_{L^2(\Omega)} \inf_{z\in\partial\mathcal{R}_1(0)} \|\xi - z\|_{L^2(\Omega)}\,.\end{aligned}$$

From this, one defines the *dissipation functional* $\mathfrak{f}:[0,T]\times\mathcal{Z}\times L^2(\Omega)\to[0,+\infty]$ via

$$\mathfrak{f}_t(z,v) := \mathfrak{p}(v,-\mathrm{D}_z\mathcal{I}(t,z)) = \mathcal{R}_1(v) + \|v\|_{L^2(\Omega)} \min_{\zeta\in\partial\mathcal{R}_1(0)} \|-\mathrm{D}_z\mathcal{I}(t,z)-\zeta\|_{L^2(\Omega)}\,,$$

where v plays the role of z'. Observe that for all $z\in\mathcal{Z}, v\in L^2(\Omega)$ we have

$$\mathfrak{f}_t(z,v) \geq \langle -\mathrm{D}_z\mathcal{I}(t,z), v\rangle_{L^2(\Omega)}$$

provided that $\mathrm{D}_z\mathcal{I}(t,z) \in L^2(\Omega)$. We are now in a position to define the Finsler cost associated with $\mathfrak{f}$, obtained by minimizing suitable integral quantities along *admissible curves*. Let us mention in advance that our definition of the class of admissible curves reflects the enhanced estimates available in the present setting for the discrete viscous solutions, cf. Remark 5.2 below for more details.

Definition 5.1. Let $t \in [0,T]$ and $z_0,\, z_1 \in \mathcal{Z}$ be fixed.

(1) We call a curve $\vartheta : [r_0, r_1] \to \mathcal{Z}$, for some $r_0 < r_1$, an *admissible transition curve* between z_0 and z_1, at the time $t \in [0,T]$, if
   (a) $\vartheta \in L^\infty(r_0, r_1; \mathcal{Z}) \cap \mathrm{AC}([r_0, r_1]; L^2(\Omega))$;
   (b) $\mathrm{D}_z\mathcal{I}(t, \vartheta(\cdot)) \in L^\infty(r_0, r_1; L^2(\Omega))$.

   We denote by $\mathcal{T}_t(z_0, z_1)$ the set of admissible curves connecting z_0 and z_1.

(2) The (possibly asymmetric) Finsler cost induced by $\mathfrak{f}_t$ at the time t is given by

$$\Delta_{\mathfrak{f}}(t; z_0, z_1) := \inf_{\vartheta \in \mathcal{T}_t(z_0,z_1)} \int_{r_0}^{r_1} \mathfrak{f}_t(\vartheta(r), \vartheta'(r))\,\mathrm{d}r \tag{5.2}$$

   with the usual convention of setting $\Delta_{\mathfrak{f}}(t; u_0, u_1) = +\infty$ if the set $\mathcal{T}_t(z_0, z_1)$ of admissible curves connecting z_0 and z_1 is empty.

Along the footsteps of Remark 2.17, we observe that, since $\vartheta \in L^\infty(r_0, r_1; \mathcal{Z})$, requiring $\mathrm{D}_z\mathcal{I}(t, \vartheta(\cdot)) \in L^\infty(r_0, r_1; L^2(\Omega))$ is equivalent to asking for $A_q(\vartheta(\cdot)) \in L^\infty(r_0, r_1; L^2(\Omega))$.

We trivially have

$$\Delta_{\mathfrak{f}}(t; z_0, z_1) \geq \mathcal{R}_1(z_1 - z_0) \quad \text{for every } t \in [0,T] \text{ and } z_0,\, z_1 \in \mathcal{Z}. \tag{5.3}$$

Up to a reparameterization, due to the positive homogeneity of the Finsler metric $\mathfrak{f}_t(z, \cdot)$, we can suppose that the admissible transition curves are defined on $[0,1]$. For later use we also introduce, for a fixed $\varrho > 0$, the set of admissible transition curves lying in a suitable ball of radius ϱ, i.e.

$$\mathcal{T}_t^\varrho(z_0, z_1) := \{\vartheta \in \mathcal{T}_t(z_0, z_1) \,:\, \|\vartheta\|_{L^\infty(0,1;\mathcal{Z})} + \|\vartheta'\|_{L^1(0,1;L^2(\Omega))} + \|\mathrm{D}_z\mathcal{I}(t, \vartheta(\cdot))\|_{L^\infty(0,1;L^2(\Omega))} \leq \varrho\} \tag{5.4a}$$

and, accordingly,

$$\Delta_{\mathfrak{f}}^\varrho(t; z_0, z_1) := \inf_{\vartheta \in \mathcal{T}_t^\varrho(z_0,z_1)} \int_{r_0}^{r_1} \mathfrak{f}_t(\vartheta(r), \vartheta'(r))\,\mathrm{d}r\,. \tag{5.4b}$$

Since for every $\varrho > 0$ there holds $\mathcal{T}_t^\varrho(z_0, z_1) \subset \mathcal{T}_t(z_0, z_1)$, one has $\Delta_{\mathfrak{f}}(t; z_0, z_1) \leq \Delta_{\mathfrak{f}}^\varrho(t; z_0, z_1)$. Indeed,

$$\Delta_{\mathfrak{f}}(t; z_0, z_1) = \inf_{\varrho>0} \Delta_{\mathfrak{f}}^\varrho(t; z_0, z_1) \qquad \text{for every } t \in [0,T] \text{ and } z_0,\, z_1 \in \mathcal{Z}. \tag{5.5}$$

For later use, we also record the following monotonicity property

$$\Delta_{\mathfrak{f}}^{\bar\varrho}(t; z_0, z_1) = \inf_{0<\varrho<\bar\varrho} \Delta_{\mathfrak{f}}^\varrho(t; z_0, z_1) = \sup_{\varrho>\bar\varrho} \Delta_{\mathfrak{f}}^\varrho(t; z_0, z_1) \text{ for every } t \in [0,T],\ z_0,\, z_1 \in \mathcal{Z} \text{ and } \bar\varrho > 0, \tag{5.6}$$

since $\mathcal{T}_t^\varrho(z_0, z_1) \subset \mathcal{T}_t^{\bar\varrho}(z_0, z_1)$ for every $0 < \varrho < \bar\varrho$. Observe that, for every fixed $\varrho > 0$, the inf in definition (5.4b) is attained, cf. Proposition 6.1 ahead, whereas it need not be attained in the definition of $\Delta_{\mathfrak{f}}$. In fact, the dissipation functional $\mathfrak{f}$ does not control the norms of the spaces where we look for admissible transition curves.

Remark 5.2. The most striking difference between the present definition of admissible curve and the one given in [MRS16, Def. 3.4] resides in the fact that, in contrast with conditions (a) & (b) from Definition 5.1, in [MRS16] it was only required

$$\begin{gathered} \vartheta|_{G_t[\vartheta]} \in \mathrm{AC}(G_t[\vartheta]; L^2(\Omega)) \quad \text{with the open set} \\ G_t[\vartheta] := \{r \in [r_0, r_1] \,:\, \min_{\zeta \in \partial\mathcal{R}_1(0)} \|-\mathrm{D}_z\mathcal{I}(t,z) - \zeta\|_{L^2(\Omega)} > 0\}\,. \end{gathered} \tag{5.7}$$

The stronger condition $\vartheta \in \mathrm{AC}([r_0.r_1]; L^2(\Omega))$ reflects the fact that the discrete viscous solutions $(\overline{z}_\tau)_\tau$ enjoy a (uniform, w.r.t. both parameters ϵ and τ) estimate in $\mathrm{BV}([0,T]; L^2(\Omega))$ (even in $\mathrm{BV}([0,T]; H^1(\Omega))$, cf. (3.9e)). Instead, in the general framework considered in [MRS16] only the *basic* energy estimate

$$\int_0^T \mathfrak{p}(\widehat{z}_\tau'(t), -\mathrm{D}_z\mathcal{I}(\overline{t}_\tau(t), \overline{z}_\tau(t)))\,\mathrm{d}t \leq \int_0^T \big(\mathcal{R}_\epsilon(\widehat{z}_\tau'(t)){+}\mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\overline{t}_\tau(t), \overline{z}_\tau(t)))\big)\,\mathrm{d}t \leq C$$

was available. In accordance with that, only (5.7) was required on admissible curves.

Condition (b) in Def. 5.1 reflects the enhanced estimate (3.9k). It is also peculiar of the present framework, and in particular it is motivated by the fact that we impose unidirectionality of damage evolution, thus allowing $\mathcal{R}_1$ to take the value $+\infty$. In order to explain this, let us observe that, in the setting considered in [MRS16], it was not necessary to specify the summability properties of $\mathrm{D}_z\mathcal{I}(t, \vartheta(\cdot))$ within the definition of admissible curve. Indeed, outside the set $G_t[\vartheta]$ one had $\mathrm{D}_z\mathcal{I}(t, \vartheta(\cdot)) \in \partial\mathcal{R}_1(0)$, a bounded subset of $L^2(\Omega)$ since the dissipation potential $\mathcal{R}_1$ was everywhere continuous. Instead, on the set $G_t[\vartheta]$ an estimate for the quantity $\min_{\zeta \in \partial\mathcal{R}_1(0)} \| - \mathrm{D}_z\mathcal{I}(t,z) - \zeta\|_{L^2(\Omega)}$ would morally provide a bound for $-\mathrm{D}_z\mathcal{I}(t,z)$, as well, by comparison arguments, again thanks to the boundedness $\partial\mathcal{R}_1(0)$. Instead, in the present setting, since the set $\partial\mathcal{R}_1(0)$ is unbounded, it is necessary to encompass a suitable summability condition on $\mathrm{D}_z\mathcal{I}(t, \vartheta(\cdot))$ in the definition of admissible curve.

We are now ready to introduce the jump variation induced by $\mathfrak{f}$, accounting for the energy dissipated at the jumps of a given curve $z \in \mathrm{BV}([0,T]; L^1(\Omega))$, with (countable) jump set

$$\mathrm{J}_z := \{t \in [0,T] \ : \ z(t_-) \neq z(t) \text{ or } z(t_+) \neq z(t)\}$$

and $z(t_\pm)$ the right/left limits of z at $t \in [0,T]$. Based on the jump variation associated with $\mathfrak{f}$ in (5.10) ahead, we introduce a novel notion of total variation for the curve z, alternative to the total variation induced by the dissipation potential $\mathcal{R}_1$. We recall that, for a given curve $z \in \mathrm{BV}([0,T]; L^1(\Omega))$ and $[a,b] \subset [0,T]$, the latter is given by

$$\mathrm{Var}_{\mathcal{R}_1}(z; [a,b]) := \sup\{\sum_{m=1}^M \mathcal{R}_1(z(t_m){-}z(t_{m-1})) \ : \ a = t_0 < t_1 < \ldots < t_{M-1} < t_M = b\}. \tag{5.8}$$

In particular, the contribution at the jumps induced by $\mathcal{R}_1$ is

$$\mathrm{Jump}_{\mathcal{R}_1}(z; [a,b]) := \mathcal{R}_1(z(a_+){-}z(a)) + \mathcal{R}_1(z(b_-){-}z(b)) + \sum_{t \in \mathrm{J}_z \cap (a,b)} \mathcal{R}_1(z(t_+){-}z(t)) + \mathcal{R}_1(z(t_-){-}z(t)).$$

For later convenience, we also introduce the scalar function

$$V(t) := \begin{cases} 0 & \text{if } t \leq 0, \\ \mathrm{Jump}_{\mathcal{R}_1}(z; [0,t]) & \text{if } t \in (0,T), \\ \mathrm{Jump}_{\mathcal{R}_1}(z; [0,T]) & \text{if } t \geq T \end{cases} \qquad \text{with distributional derivative } \mu = \frac{\mathrm{d}}{\mathrm{d}t} V\,. \tag{5.9}$$

Recall that μ is a finite Borel measure supported on $[0,T]$, and it can be decomposed as $\mu = \mu_\mathrm{d} + \mu_\mathrm{J}$, with μ_J the jump part, concentrated on the (countable) jump set J_z, and μ_d the diffuse part, given by the sum of the absolutely continuous and of the Cantor parts, so that $\mu_\mathrm{d}(\{t\}) = 0$ for every $t \in \mathbb{R}$.

We are now in a position to give the notion of total variation induced by $\mathfrak{f}$. Let us mention in advance that it is obtained by replacing the $\mathrm{Jump}_{\mathcal{R}_1}$-contribution to the total variation $\mathrm{Var}_{\mathcal{R}_1}$, with the $\mathfrak{f}$-jump variation, cf. (5.11) below.

Definition 5.3 (Jump and total variation induced by $\mathfrak{f}$)**.** Let z in $\mathrm{BV}([0,T]; L^1(\Omega))$, with $z(t) \in \mathcal{Z}$ for all $t \in [0,T]$, be a given curve with jump set J_z. Let $[a,b] \subset [0,T]$:

(1) The *jump variation* of z on $[a,b]$ induced by $\mathfrak{f}$ is

$$\begin{aligned}\mathrm{Jump}_{\mathfrak{f}}(z;[a,b]) := {} & \Delta_{\mathfrak{f}}(a;z(a),z(a_+)) + \Delta_{\mathfrak{f}}(b;z(b_-),z(b)) \\ & + \sum_{t\in \mathrm{J}_z\cap(a,b)} \big(\Delta_{\mathfrak{f}}(t;z(t_-),z(t)) + \Delta_{\mathfrak{f}}(t;z(t),z(t_+))\big).\end{aligned} \tag{5.10}$$

(2) The total variation of z on $[a,b]$ induced by $\mathfrak{f}$ is

$$\begin{aligned}\mathit{Var}_{\mathfrak{f}}(z;[a,b]) & := \mathrm{Var}_{\mathcal{R}_1}(z;[a,b]) - \mathrm{Jump}_{\mathcal{R}_1}(z;[a,b]) + \mathrm{Jump}_{\mathfrak{f}}(z;[a,b]) & (5.11)\\ & = \mu_{\mathrm{d}}([a,b]) + \mathrm{Jump}_{\mathfrak{f}}(z;[a,b])\,. & (5.12)\end{aligned}$$

For a given $\varrho > 0$, we use the symbols $\mathrm{Jump}_{\mathfrak{f}}^{\varrho}(z;[a,b])$ and $\mathit{Var}_{\mathfrak{f}}^{\varrho}$ for the total variation induced by the cost $\Delta_{\mathfrak{f}}^{\varrho}$.

As already pointed out in [MRS12b, Rmk. 3.5], $\mathit{Var}_{\mathfrak{f}}$ is not a *standard* total variational functional: it is neither induced by any distance on $L^1(\Omega)$, nor is it lower semicontinuous w.r.t. pointwise convergence in $L^1(\Omega)$. Yet, it enjoys the additivity property.

We are finally in a position to give our definition of Balanced Viscosity solution to the rate-independent damage system. Again, we will consider a slightly stronger version than that given in [MRS16, Def. 3.10], where $z \in \mathrm{BV}([0,T];L^1(\Omega))$ was only required. Instead, here we will consider curves z in $\mathrm{BV}([0,T];L^2(\Omega))$ and, for technical reasons that will be apparent in the proof of the BV-chain rule from Proposition 5.8 ahead, we will also restrict to curves z such that $\mathrm{D}_z\mathcal{I}(\cdot,z(\cdot)) \in L^\infty(0,T;L^2(\Omega))$. Furthermore, unlike what was done in [MRS16], we will claim an energy balance involving a total variation $\mathit{Var}_{\mathfrak{f}}^{\varrho}(z;[0,t])$ with a threshold $\varrho > 0$ such that

$$\varrho \geq \|z\|_{L^\infty(0,T;\mathcal{Z})\cap \mathrm{BV}([0,T];L^2(\Omega))} + \|\mathrm{D}_z\mathcal{I}(\cdot,z(\cdot))\|_{L^\infty(0,T;L^2(\Omega))}\,. \tag{5.13}$$

Definition 5.4. A curve z in $L^\infty(0,T;\mathcal{Z}) \cap \mathrm{BV}([0,T];L^2(\Omega))$, with

$$z(t) \in \mathcal{Z} \ \text{ and } \ \mathrm{D}_z\mathcal{I}(t,z(t)) \in L^2(\Omega) \quad \text{ for all } t \in [0,T] \tag{5.14}$$

and $\mathrm{D}_z\mathcal{I}(\cdot,z(\cdot)) \in L^\infty(0,T;L^2(\Omega))$, is a *Balanced Viscosity* solution of the rate-independent damage system (1.1) if the *local stability* $(\mathrm{S}_{\mathrm{loc}})$ and the $(\mathrm{E}_{\mathfrak{f}})$-*energy balance* hold:

$$-\mathrm{D}_z\mathcal{I}(t,z(t)) \in \partial\mathcal{R}_1(0) \quad \text{for all} \quad t \in [0,T]\setminus \mathrm{J}_z, \tag{$\mathrm{S}_{\mathrm{loc}}$}$$

$$\mathit{Var}_{\mathfrak{f}}^{\varrho}(z;[0,t]) + \mathcal{I}(t,z(t)) = \mathcal{I}(0,z(0)) + \int_0^t \partial_t\mathcal{I}(s,z(s))\,\mathrm{d}s \quad \text{ for all } t \in (0,T]. \tag{$\mathrm{E}_{\mathfrak{f}}$}$$

with $\varrho > 0$ fulfilling (5.13).

Remark 5.5. The requirement $z \in L^\infty(0,T;\mathcal{Z})$ in Def. 5.4 is redundant and has been added only for the sake of clarity. Indeed, since $\mathcal{I}(0,z(0)) \leq C$ as $z(0) \in \mathcal{Z}$ (cf. (2.15)), and taking into account that $t \mapsto \partial_t\mathcal{I}(t,z(t))$ is in $L^\infty(0,T)$ thanks to (2.23), from $(\mathrm{E}_{\mathfrak{f}})$ we deduce that $|\mathcal{I}(t,z(t))| \leq C$ (recall that $\mathcal{I}$ is bounded from below thanks to (2.18)). In turn, this gives $z \in L^\infty(0,T;\mathcal{Z})$.

On the other hand, combining the information $z \in L^\infty(0,T;\mathcal{Z})$ with estimate (2.34) for $\mathrm{D}_z\widetilde{\mathcal{I}}$, we conclude that $\mathrm{D}_z\widetilde{\mathcal{I}}(\cdot,z(\cdot)) \in L^\infty(0,T;L^2(\Omega))$. Therefore, what we are really requiring in Def. 5.4 is that $A_q z \in L^\infty(0,T;L^2(\Omega))$, which enhances the regularity of z to the space $L^\infty(0,T;W^{1+\sigma,q}(\Omega))$ for every $0 < \sigma < \frac{1}{q}$ by Proposition 2.7.

Prior to stating the **main result of the paper**, Theorem 5.7 below, we need to give the following definition, where z_- and z_+ are place-holders for the left and right limits of a curve z at a jump point.

Definition 5.6. Let $\varrho > 0$, $t \in [0,T]$, and z_-, $z_+ \in \mathcal{Z}$ be such that

$$-\mathrm{D}_z\mathcal{I}(t,z_-) \in \partial\mathcal{R}_1(0) \text{ and } -\mathrm{D}_z\mathcal{I}(t,z_+) \in \partial\mathcal{R}_1(0)\,. \tag{5.15}$$

We say that an admissible transition curve $\vartheta \in \mathcal{T}_t^{\varrho}(z_-,z_+)$ is an *optimal transition* between z_- and z_+ if

$$\mathcal{I}(t,z_-) - \mathcal{I}(t,z_+) = \Delta_{\mathfrak{f}}^{\varrho}(t;z_-,z_+) = \int_0^1 \mathfrak{f}_t(\vartheta(r),\vartheta'(r))\,\mathrm{d}r = \mathfrak{f}_t(\vartheta(r),\vartheta'(r)) \qquad \text{for a.a. } r \in (0,1). \tag{5.16}$$

We will denote by $\mathcal{O}_t^\varrho(z_-, z_+)$ the collection of such transitions.

A few comments are in order. First of all, with (5.15) we are imposing that the points z_- and z_+ to be connected fulfill the local stability condition. It is not difficult to check that this is verified whenever z_- and z_+ are the left and right limits at a jump point of a Balanced Viscosity solution. Secondly, let us gain further insight into (5.16): with the second equality, we are asking that ϑ (which we may always suppose to be defined on $[0,1]$) is a minimizer in the definition of $\Delta_{\mathfrak{f}}^\varrho(t; z_-, z_+)$; with the third one, that ϑ has constant "$\mathfrak{f}_t$-velocity", which can be obtained by a rescaling argument. The first equality relates to the jump conditions verified along any Balanced Viscosity solution, cf. (5.26) ahead.

We are now in a position to give Thm. 5.7 , stating the convergence of the discrete solutions of the viscous damage system to a Balanced Viscosity solution of the rate-independent damage system, as the parameters ϵ and τ tend to zero *simultaneously*, with $\frac{\varepsilon}{\tau} \uparrow \infty$. In fact, we will retrieve a Balanced Viscosity solution z with enhanced properties:

(i) we have that $z \in \mathrm{BV}([0,T]; H^1(\Omega))$, which reflects the enhanced discrete BV-estimate (3.9e);
(ii) at all jump points t of z, the left and right limits $z(t_-)$ and $z(t_+)$ can be connected by an optimal jump transition in the sense of Definition 5.6, so that the set $\mathcal{O}_t^{\bar\varrho}(z(t_-), z(t_+))$ is non-empty. Additionally, such transition has finite $H^1(\Omega)$-length. Furthermore, the total $H^1(\Omega)$-length of the connecting paths is finite.

Observe that property (ii) is not encoded in Definition 5.4, which gives $\mathit{Var}_{\mathfrak{f}}(z; [0,T]) < \infty$, since $\mathit{Var}_{\mathfrak{f}}(z; [0,T])$ only controls the "$\mathfrak{f}$-length" of the optimal jump paths.

This enhanced concept of Balanced Viscosity solution was already introduced in the general setting of [MRS16], cf. Section 3.4 therein. Along the footsteps of [MRS16], we will refer to these solutions as $H^1(\Omega)$-*parameterizable Balanced Viscosity solutions.*

Theorem 5.7. *Under Assumptions 2.1, 2.2, and 2.8, let $z_0 \in \mathcal{Z}$, fulfilling* (3.6), *be approximated by discrete initial data $(z_{\tau,\epsilon}^0)_{\tau,\epsilon}$ such that*

$$z_{\tau,\epsilon}^0 \to z_0 \quad \text{in } \mathcal{Z}, \qquad \mathcal{I}(0, z_{\tau,\epsilon}^0) \to \mathcal{I}(0, z_0), \quad \mathrm{D}_z\mathcal{I}(0, z_{\tau,\epsilon}^0) \rightharpoonup \mathrm{D}_z\mathcal{I}(0, z_0) \quad \text{in } L^2(\Omega), \tag{5.17}$$

and let $(\overline{z}_{\tau,\epsilon})_{\tau,\epsilon}$, $(\widehat{z}_{\tau,\epsilon})_{\tau,\epsilon}$ be the discrete solutions to the viscous damage system (1.2) *starting from the data $(z_{\tau,\epsilon}^0)_{\tau,\epsilon}$.*

Then, there exists $\bar\varrho > 0$, only depending on the problem data (cf. (6.2) *below) and fulfilling* (5.13), *such that for all sequences $(\tau_k, \epsilon_k)_k$ satisfying*

$$\lim_{k\to\infty} \epsilon_k = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{\tau_k}{\epsilon_k} = 0, \tag{5.18}$$

there exist a (not relabeled) subsequence, and a Balanced Viscosity solution z to the rate-independent damage system (1.1), *fulfilling $z(0) = z_0$, the energy balance* ($\mathrm{E}_{\mathfrak{f}}$) *with*

$$\mathrm{Var}_{\mathfrak{f}}^{\bar\varrho}(z; [0,t]) = \sup_{\varrho \ge \bar\varrho} \mathrm{Var}_{\mathfrak{f}}^{\varrho}(z; [0,t]) = \inf_{\varrho \ge \bar\varrho} \mathrm{Var}_{\mathfrak{f}}^{\varrho}(z; [0,t]) \quad \text{for every } t \in [0,T] \tag{5.19}$$

and such that the following convergences hold as $k \to \infty$, at every $t \in [0,T]$:

$$\overline{z}_{\tau_k,\epsilon_k}(t), \, \widehat{z}_{\tau_k,\epsilon_k}(t) \to z(t) \quad \text{in } \mathcal{Z}, \tag{5.20a}$$

$$\mathcal{I}(t, \overline{z}_{\tau_k,\epsilon_k}(t)), \, \mathcal{I}(t, \widehat{z}_{\tau_k,\epsilon_k}(t)) \to \mathcal{I}(t, z(t)), \tag{5.20b}$$

$$\int_0^{\bar t_\tau(t)} \left(\mathcal{R}_\epsilon(\widehat{z}_\tau'(r)) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\bar t_\tau(r), \overline{z}_\tau(r))) \right) \mathrm{d}r \to \mathrm{Var}_{\mathfrak{f}}^{\bar\varrho}(z; [0,t]) \,. \tag{5.20c}$$

Furthermore, z is a $H^1(\Omega)$-parameterizable Balanced Viscosity solution, namely $z \in \mathrm{BV}([0,T]; H^1(\Omega))$, and

$$\forall\, t \in \mathrm{J}_z \ \exists\, \vartheta_t \in \mathcal{O}_t^{\bar\varrho}(z(t_-), z(t_+)) \ \text{s.t.}\ \vartheta_t \in \mathrm{AC}([0,1]; H^1(\Omega)); \tag{5.21a}$$

(1)

$$\sum_{t \in \mathrm{J}_z} \int_0^1 \|\vartheta_t'(r)\|_{H^1(\Omega)}\, \mathrm{d}r < \infty. \tag{5.21b}$$

(2)

Observe that (5.19) is an additional property, cf. (5.6). The constant $\bar\varrho$ will be specified along the proof of Theorem 5.7, postponed to Section 6. Instead, in the forthcoming Sec. 5.2 we gain further insight into the notion of Balanced Viscosity solution for our damage system, in particular focusing on the description of the behavior of the system at jumps.

5.2. **Properties of Balanced Viscosity solutions.** One of the cornerstones of the proof of Thm. 5.7 is a characterization of Balanced Viscosity solutions in terms of the local stability condition ($\mathrm{S}_{\mathrm{loc}}$), combined with the *upper energy estimate* in ($\mathrm{E}_{\mathfrak{f}}$). The proof of this characterization relies on *chain-rule inequality* for $\mathcal{E}$, evaluated along a *locally stable* curve with the regularity and summability properties specified in Definition 5.4. This inequality involves the non-standard total variation functional $\mathit{Var}_{\mathfrak{f}}$.

Proposition 5.8 (BV-chain rule inequality)**.** *Under Assumptions 2.1, 2.2, and 2.8, let $z \in L^\infty(0,T;\mathcal{Z}) \cap \mathrm{BV}([0,T];L^2(\Omega))$, with $\mathrm{D}_z\mathcal{I}(\cdot, z(\cdot)) \in L^\infty(0,T;L^2(\Omega))$, also fulfill* (5.14)*. Let ϱ fulfill* (5.13)*. Suppose that z satisfies the local stability condition* ($\mathrm{S}_{\mathrm{loc}}$)*, with $\mathrm{Var}_{\mathfrak{f}}^{\varrho}(z;[0,T]) < \infty$. Then, the map $t \mapsto \mathcal{I}(t,u(t))$ belongs to $\mathrm{BV}([0,T])$ and satisfies the chain rule inequality*

$$\left|\mathcal{I}(t_1,u(t_1)) - \mathcal{I}(t_0,u(t_0)) - \int_{t_0}^{t_1} \partial_t \mathcal{I}(t,z(t))\,\mathrm{d}t\right| \leq \mathit{Var}_{\mathfrak{f}}^{\varrho}(z;[t_0,t_1]) \qquad \text{for all } 0 \leq t_0 \leq t_1 \leq T\,. \tag{5.22}$$

We postpone its *proof* to Section 6. We now characterize Balanced Viscosity solutions in terms of the local stability ($\mathrm{S}_{\mathrm{loc}}$), joint with the upper energy estimate in ($\mathrm{E}_{\mathfrak{f}}$), which it is sufficient to give on the whole time interval $[0,T]$. Namely we have

Corollary 5.9. *Under Assumptions 2.1, 2.2, and 2.8, a curve $z \in \mathrm{BV}([0,T];L^2(\Omega))$ is a Balanced Viscosity solution of the rate-independent damage system* (1.1) *(in the sense of Definition 5.4) if and only if it satisfies* ($\mathrm{S}_{\mathrm{loc}}$) *and*

$$\mathit{Var}_{\mathfrak{f}}^{\varrho}(z;[0,T]) + \mathcal{I}(T,z(T)) \leq \mathcal{I}(0,z(0)) + \int_0^T \partial_t\mathcal{I}(s,z(s))\,\mathrm{d}s \tag{5.23}$$

for some ϱ fulfilling (5.13)*.*

For the *proof*, we refer the reader to the argument for [MRS16, Cor. 3.14]. Corollary 5.9 will play a crucial role in the proof of Theorem 5.7, for it will allow us to focus on the proof of ($\mathrm{S}_{\mathrm{loc}}$) and of the energy inequality (5.23), only, in place of the balance ($\mathrm{E}_{\mathfrak{f}}$). In turn, (5.23) will be achieved by means of careful lower semicontinuity arguments. The second outcome of the characterization provided by Cor. 5.9 is the following Proposition 5.10, which was proved in the abstract setting in [MRS16, Thm. 3.15]. It shows that a locally stable curve is a Balanced Viscosity solution of the rate-independent system if and only if it fulfills

- (i) an energy-dissipation inequality only featuring the $\mathcal{R}_1$-total variation functional from (5.8), cf. (5.25) below, and
- (ii) at each jump point, the jump conditions (5.26) featuring the Finsler cost $\Delta_{\mathfrak{f}}$ induced by $\mathfrak{f}$.

Concerning (i), let us also mention that it is possible to show (cf. [MRS16, Thm. 3.16]) that any Balanced Viscosity solution also satisfies the subdifferential inclusion

$$\partial\mathcal{R}_1(z'(t)) + \mathrm{D}_z\mathcal{I}(t,z(t)) \ni 0 \quad \text{in } L^2(\Omega) \tag{5.24}$$

at every $t \in (0,T)$ that is not a jump point, hence for almost all $t \in (0,T)$. The system behavior at jump points is instead described by the jump conditions (5.26) below. This further characterization of the Balanced Viscosity concept in terms of (i) and (ii) highlights how it differs in comparison to the standard Global Energetic notion. The latter can be characterized in terms of the *global stability* condition, the energy-dissipation inequality (5.25), and the analogues of the jump conditions (5.26), with the cost $\Delta_{\mathfrak{f}}(t;\cdot,\cdot)$ replaced by $\mathcal{R}_1$. Conditions (5.26) highlight that the viscous approximation from which Balanced Viscosity solutions originate enters into play in the description of the energetic behavior of the system at jumps.

Proposition 5.10. *A curve $z \in \mathrm{BV}([0,T];L^2(\Omega))$ is a Balanced Viscosity solution of the rate-independent damage system* (1.1) *if and only if it satisfies* ($\mathrm{S_{loc}}$)*, the* ($\mathcal{R}_1$)*-energy dissipation inequality*

$$\mathrm{Var}_{\mathcal{R}_1}(z;[s,t]) + \mathcal{I}(t,z(t)) \leq \mathcal{I}(s,z(s)) + \int_s^t \partial_t \mathcal{I}(s,z(s))\,\mathrm{d}s \quad \text{for all } 0 \leq s \leq t \leq T, \tag{5.25}$$

and the jump conditions

$$\begin{aligned} \mathcal{I}(t,z(t)) - \mathcal{I}(t,z(t_-)) &= -\Delta_{\mathfrak{f}}^{\varrho}(t;z(t_-),z(t)), \\ \mathcal{I}(t,z(t_+)) - \mathcal{I}(t,z(t)) &= -\Delta_{\mathfrak{f}}^{\varrho}(t;z(t),z(t_+)), \\ \mathcal{I}(t,z(t_+)) - \mathcal{I}(t,z(t_-)) &= -\Delta_{\mathfrak{f}}^{\varrho}(t;z(t_-),z(t_+)) \\ &= -\Big(\Delta_{\mathfrak{f}}^{\varrho}(t;z(t_-),z(t)) + \Delta_{\mathfrak{f}}^{\varrho}(t;z(t),z(t_+))\Big) \end{aligned} \tag{5.26}$$

at every $t \in \mathrm{J}_z$.

The *proof* follows the very same lines as the argument for [MRS16, Thm. 3.15].

We conclude this section by shedding further light into the the fine properties of optimal jump transitions. Following [MRS16, Sec. 3.4], we say that an optimal transition $\vartheta \in \mathcal{O}_t^{\varrho}(z_-,z_+)$ is of

- *sliding* type if $-\mathrm{D}_z\mathcal{I}(t,\vartheta(r)) \in \mathcal{R}_1(0)$ for every $r \in [0,1]$;
- *viscous* type if $-\mathrm{D}_z\mathcal{I}(t,\vartheta(r)) \notin \mathcal{R}_1(0)$ for every $r \in [0,1]$.

The forthcoming result on sliding and viscous optimal transitions follows from the very same argument as in the proof of [MRS16, Prop. 3.19].

Proposition 5.11. *Let $\varrho > 0$, $t \in [0,T]$, and z_-, $z_+ \in \mathcal{Z}$ fulfilling* (5.15) *be given. Let $\vartheta \in \mathcal{O}_t^{\varrho}(z_-,z_+)$. Then,*

(1) ϑ is of sliding type if and only if it satisfies

$$\partial\mathcal{R}_1(\vartheta'(r)) + \mathrm{D}_z\mathcal{I}(t,\vartheta(r)) \ni 0 \quad \text{in } L^2(\Omega) \text{ for a.a. } r \in (0,1);$$

(2) ϑ is of viscous type if and only if there exists a map $\epsilon : (0,1) \to (0,+\infty)$ such that ϑ and ϵ satisfy

$$\partial\mathcal{R}_1(\vartheta'(r)) + \epsilon(r)\vartheta'(r) + \mathrm{D}_z\mathcal{I}(t,\vartheta(r)) \ni 0 \quad \text{in } L^2(\Omega) \text{ for a.a. } r \in (0,1);$$

(3) Every optimal transition ϑ can be decomposed in a canonical way into an (at most) countable collection of optimal sliding *and* viscous *transitions.*

6. Proofs

We start by giving Proposition 6.1, which is the counterpart to [MRS16, Thm. 3.7]. A comparison between the latter result and Proposition 6.1 below reflects the major differences between the present context and that of [MRS16]: The transition curves by means of which the Finsler cost $\Delta_{\mathfrak{f}}$ from (5.2) is defined have better properties than their analogues in [MRS16], cf. also Remark 5.2. This is also apparent from item (3) of the ensuing statement, yielding the existence of a transition path ϑ in the space $W^{1,\infty}(0,1;H^1(\Omega))$, even, in accordance with the uniform bound (3.9e) for the discrete solutions.

Proposition 6.1. *Let $t \in [0,T]$ and z_0, $z_1 \in \mathcal{Z}$ be fixed. Then:*

(1) For every $\varrho > 0$ such that $\max_{i=0,1}(\|z_i\|_{\mathcal{Z}} + \|\mathrm{D}_z\mathcal{I}(t,z_i)\|_{L^2(\Omega)}) \leq \varrho$ and $\Delta_{\mathfrak{f}}^{\varrho}(t;z_0,z_1) < +\infty$, there exists an optimal transition path $\vartheta \in \mathcal{T}_t^{\varrho}(z_0,z_1)$ attaining the inf *in the definition of $\Delta_{\mathfrak{f}}^{\varrho}(t;z_0,z_1)$, cf.* (5.4);

(2) *Let* $(z_0^n)_n$, $(z_1^n)_n \subset \mathcal{Z}$ *fulfill*

$$z_0^n \to z_0, \quad z_1^n \to z_1 \quad \text{in } \mathcal{Z}.$$

Then,

$$\liminf_{n\to\infty} \Delta_{\mathfrak{f}}^{\varrho}(t; z_0^n, z_1^n) \geq \Delta_{\mathfrak{f}}^{\varrho}(t; z_0, z_1) \tag{6.1}$$

for every $\varrho \geq \sup_{i=1,2,n\in\mathbb{N}}(\|z_i\|_{\mathcal{Z}} + \|\mathrm{D}_z\mathcal{I}(t,z_i)\|_{L^2(\Omega)})$.

(3) *Let the sequences* $(\alpha_k)_k$, $(\beta_k)_k \subset [0,T]$, $(\widehat{z}_k)_k \subset L^\infty(\alpha_k,\beta_k;\mathcal{Z})\cap\mathrm{AC}([\alpha_k,\beta_k];H^1(\Omega))$, $(\overline{z}_k)_k \subset L^\infty(\alpha_k,\beta_k;\mathcal{Z})$, *fulfill*

$$\begin{aligned}
&\lim_{k\to\infty}\alpha_k = t = \lim_{k\to\infty}\beta_k, \quad \overline{z}_k(\alpha_k)\to z_0 \text{ in } \mathcal{Z}, \quad \overline{z}_k(\beta_k)\to z_1 \text{ in } \mathcal{Z},\\
&\lim_{k\to\infty}\sup_{r\in[\alpha_k,\beta_k]}\|\overline{z}_k(r)-\widehat{z}_k(r)\|_{H^1(\Omega)} = 0,\\
&\exists\,\bar\varrho>0 \quad \forall\,k\in\mathbb{N}:\\
&\qquad \|\widehat{z}_k\|_{L^\infty(\alpha_k,\beta_k;\mathcal{Z})\cap W^{1,1}(\alpha_k,\beta_k;H^1(\Omega))} + \|\overline{z}_k\|_{L^\infty(\alpha_k,\beta_k;\mathcal{Z})} + \|\mathrm{D}_z\mathcal{I}(\bar{t}_{\tau_k},\overline{z}_k)\|_{L^\infty(\alpha_k,\beta_k;L^2(\Omega))} \leq \bar\varrho\,.
\end{aligned} \tag{6.2}$$

Then, there exists a (not relabeled) increasing subsequence of (k), *increasing and surjective time rescalings* $\mathsf{t}_k \subset \mathrm{AC}([0,1];[\alpha_k,\beta_k])$ *and an admissible transition* $\vartheta \in \mathcal{T}_t^{\bar\varrho}(z_0,z_1)$ *such that*

$$\lim_{k\to\infty}\sup_{s\in[0,1]}\|\overline{z}_k\circ\mathsf{t}_k(s)-\vartheta(s)\|_{H^1(\Omega)} = \lim_{k\to\infty}\sup_{s\in[0,1]}\|\widehat{z}_k\circ\mathsf{t}_k(s)-\vartheta(s)\|_{H^1(\Omega)} = 0, \tag{6.3a}$$

in addition, ϑ *is in* $W^{1,\infty}(0,1;H^1(\Omega))$, *and* (6.3b)

$$\Delta_{\mathfrak{f}}^{\bar\varrho}(t;z_0,z_1) \leq \int_0^1 \mathfrak{f}_t[\vartheta(s),\vartheta'(s)]\,\mathrm{d}s \leq \liminf_{k\to\infty}\int_{\alpha_k}^{\beta_k}\left(\mathcal{R}_{\epsilon_k}(\widehat{z}_k'(r))+\mathcal{R}_{\epsilon_k}^*(-\mathrm{D}_z\mathcal{I}(\bar{t}_{\tau_k}(r),\overline{z}_k(r)))\right)\mathrm{d}r\,. \tag{6.3c}$$

Proof. We start by addressing the proof of **(2)**: Along the footsteps of the proof of [MRS16, Thm. 3.7], we consider a sequence of admissible transitions $\vartheta_n \in \mathcal{T}_t^{\varrho}(z_0^n,z_1^n)$ such that

$$\int_0^1 \mathfrak{f}_t(\vartheta_n(r),\vartheta_n'(r))\,\mathrm{d}r \leq \Delta_{\mathfrak{f}}^{\varrho}(t;z_0^n,z_1^n)+\eta_n \qquad \text{with } \eta_n\geq 0 \text{ and } \lim_{n\to\infty}\eta_n=\eta\geq 0\,.$$

We perform the change of variable

$$\mathsf{s}_n(r) := c_n\left(r+\int_0^r\|\vartheta_n'(\sigma)\|_{L^2(\Omega)}\,\mathrm{d}\sigma\right), \qquad \mathsf{r}_n := \mathsf{s}_n^{-1}:[0,\mathsf{S}]\to[0,1], \qquad \theta_n := \vartheta_n\circ\mathsf{r}_n:[0,\mathsf{S}]\to\mathcal{Z}, \tag{6.4}$$

with c_n a normalization constant such that $\mathsf{S} = \mathsf{s}_n(1)$ is independent of $n\in\mathbb{N}$. In view of the estimate $\|\vartheta_n'\|_{L^1(0,1;L^2(\Omega))} \leq \varrho$ encoded in the definition of $\Delta_{\mathfrak{f}}^{\varrho}$, we have that $c_n \geq \bar{c} > 0$ for all $n\in\mathbb{N}$. The curves $(\mathsf{r}_n,\theta_n)_n$ fulfill the normalization condition

$$\mathsf{r}_n'(s)+\|\theta_n'(s)\|_{L^2(\Omega)} = \frac{1}{c_n}\leq\frac{1}{\bar{c}} \qquad \text{for a.a. } s\in(0,\mathsf{S}) \tag{6.5a}$$

and, moreover,

$$\|\theta_n\|_{L^\infty(0,\mathsf{S};\mathcal{Z})}+\|\theta_n'\|_{L^1(0,\mathsf{S};L^2(\Omega))}+\|\mathrm{D}_z\mathcal{I}(t,\theta_n(\cdot))\|_{L^\infty(0,\mathsf{S};L^2(\Omega))}\leq\varrho. \tag{6.5b}$$

It follows from the first bound in (6.5b) and from (2.34) that $\|\mathrm{D}_z\widetilde{\mathcal{I}}(t,\theta_n(\cdot))\|_{L^\infty(0,\mathsf{S};L^2(\Omega))}\leq C$. Therefore we deduce that $\|A_q(\theta_n)\|_{L^\infty(0,\mathsf{S};L^2(\Omega))}\leq C$, which yields, in view of the aforementioned regularity results from Proposition 2.7, a bound for $(\theta_n)_n$ in $L^\infty(0,\mathsf{S};W^{1+\sigma,q}(\Omega)))$ for all $0<\sigma<\frac{1}{q}$. In view of (6.5a), there exists $\mathsf{r}\in W^{1,\infty}(0,\mathsf{S})$ such that, up to a not relabeled subsequence, $\mathsf{r}_n\to r$ uniformly in $[0,\mathsf{S}]$ and weakly* in $W^{1,\infty}(0,\mathsf{S})$. Furthermore, by Aubin-Lions type compactness results (cf., e.g. [Sim87, Thm. 5, Cor. 4]), there exists a curve $\theta\in L^\infty(0,\mathsf{S};W^{1+\sigma,q}(\Omega))\cap\mathrm{C}^0([0,\mathsf{S}];\mathcal{Z})\cap W^{1,\infty}(0,\mathsf{S};L^2(\Omega))$ for all $0<\sigma<\frac{1}{q}$, with $\mathrm{D}_z\mathcal{I}(t,\theta(\cdot))\in L^\infty(0,\mathsf{S};L^2(\Omega))$, such that

$$\begin{aligned}
&\theta_n \rightharpoonup^* \theta && \text{in } L^\infty(0,\mathsf{S};W^{1+\sigma,q}(\Omega))\cap W^{1,\infty}(0,\mathsf{S};L^2(\Omega)) \quad \text{for all } 0<\sigma<\frac{1}{q},\\
&\theta_n\to\theta && \text{in } \mathrm{C}^0([0,\mathsf{S}];\mathcal{Z})\,,\\
&\mathrm{D}_z\mathcal{I}(t,\theta_n)\rightharpoonup^*\mathrm{D}_z\mathcal{I}(t,\theta) && \text{in } L^\infty(0,\mathsf{S};L^2(\Omega))
\end{aligned} \tag{6.6}$$

(the latter convergence property following from the fact that $\mathrm{D}_z\mathcal{I}(t,\theta_n) = A_q(\theta_n) + \mathrm{D}_z\widetilde{\mathcal{I}}(t,\theta_n)$ converges strongly to $\mathrm{D}_z\mathcal{I}(t,\theta)$ in $L^\infty(0,\mathsf{S};\mathcal{Z}^*)$ in view of the second of (6.6), combined with (2.36)). Therefore,

$$\|\theta\|_{L^\infty(0,\mathsf{S};\mathcal{Z})} + \|\theta'\|_{L^1(0,\mathsf{S};L^2(\Omega))} + \|\mathrm{D}_z\mathcal{I}(t,\theta(\cdot))\|_{L^\infty(0,\mathsf{S};L^2(\Omega))} \le \varrho.$$

We thus conclude that $\theta \in \mathcal{T}_t^\varrho(z_0,z_1)$; up to a reparameterization, we may suppose θ to be defined on $[0,1]$. Arguing in the very same way as in the proof of [KRZ13, Thm. 5.1], [KRZ15, Thm. 7.4], we see that

$$\begin{aligned}\eta + \liminf_{n\to\infty} \Delta_{\mathfrak{f}}^\varrho(t;z_0^n,z_1^n) &\ge \liminf_{n\to\infty} \int_0^1 \mathfrak{f}_t(\vartheta_n(r),\vartheta_n'(r))\,\mathrm{d}r = \liminf_{n\to\infty}\int_0^{\mathsf{S}} \mathfrak{f}_t(\theta_n(s),\theta_n'(s))\,\mathrm{d}s \\ &\ge \int_0^{\mathsf{S}} \mathfrak{f}_t(\theta(s),\theta'(s))\,\mathrm{d}s \ge \Delta_{\mathfrak{f}}^\varrho(t;z_0,z_1)\,.\end{aligned}$$

Observe that the last inequality follows from the fact that θ is an admissible curve between z_0 and z_1. Since $\eta \ge 0$ is arbitrary, this concludes the proof of **(2)**; a slight modification of this argument yields **(1)**, as well.

In order to prove **(3)**, we can confine the discussion to the case $z_0 \neq z_1$, so that

$$\lim_{k\to\infty}\int_{\alpha_k}^{\beta_k} \left(\mathcal{R}_{\epsilon_k}(\widehat{z}_k'(r)) + \mathcal{R}_{\epsilon_k}^*(-\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau_k}(r),\overline{z}_k(r)))\right)\,\mathrm{d}r =: L \ge \mathcal{R}_1(z_1 - z_0) > 0\,.$$

In analogy with (6.4), but taking now into account that $(\widehat{z}_k)_k$ is bounded in $W^{1,1}(\alpha_k,\beta_k;H^1(\Omega))$ by (6.2), we define

$$\mathsf{s}_k(r) := c_k\left(r + \int_0^r \|\widehat{z}_k'(\sigma)\|_{H^1(\Omega)}\,\mathrm{d}\sigma\right) \quad \text{for all } r \in [0,\beta_k-\alpha_k]$$

where the normalization constant c_k is now chosen in such a way as to have $\mathsf{s}_k(\beta_k-\alpha_k) = 1$. Thus, we set

$$\mathsf{t}_k := \mathsf{s}_k^{-1} : [0,1] \to [\alpha_k,\beta_k], \qquad \overline{\mathsf{z}}_k := \overline{z}_k \circ \mathsf{t}_k,\ \widehat{\mathsf{z}}_k := \widehat{z}_k \circ \mathsf{t}_k : [0,1] \to \mathcal{Z},$$

and observe that the following estimates hold

$$\|\mathsf{t}_k\|_{W^{1,\infty}(0,1)} + \|\widehat{\mathsf{z}}_k\|_{W^{1,\infty}(0,1;H^1(\Omega))} \le C, \tag{6.7a}$$

$$\|\overline{\mathsf{z}}_k\|_{L^\infty(0,1;\mathcal{Z})} + \|\widehat{\mathsf{z}}_k\|_{L^\infty(0,1;\mathcal{Z})} + \|\widehat{\mathsf{z}}_k'\|_{L^1(0,1;H^1(\Omega))} + \|\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau_k}\circ\mathsf{t}_k,\overline{\mathsf{z}}_k)\|_{L^\infty(0,1;L^2(\Omega))} \le \bar{\varrho}\,, \tag{6.7b}$$

where (6.7a) is due to the analogue of the normalization condition (6.5a), while (6.7b) derives from (6.2). From the bound for $\|\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau_k}\circ\mathsf{t}_k,\overline{\mathsf{z}}_k)\|_{L^\infty(0,1;L^2(\Omega))}$, taking into account that $\|\mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_{\tau_k}\circ\mathsf{t}_k,\overline{\mathsf{z}}_k)\|_{L^\infty(0,1;L^2(\Omega))} \le C$ in view of (2.34) and the estimate $\|\overline{\mathsf{z}}_k\|_{L^\infty(0,1;\mathcal{Z})} \le C$, we also deduce

$$\|A_q(\overline{\mathsf{z}}_k)\|_{L^\infty(0,1;L^2(\Omega))} \le C\,. \tag{6.7c}$$

Combining estimates (6.7) with, again, the compactness results [Sim87, Thm. 5, Cor. 4], and taking into account that $(\overline{\mathsf{z}}_k)$ and $(\widehat{\mathsf{z}}_k)_k$ converge to the same limit in view of the second of (6.2), with the very same arguments as in the proof of **(2)** we conclude that there exists ϑ such that

$$\widehat{\mathsf{z}}_k \rightharpoonup^* \vartheta \qquad \text{in } L^\infty(0,1;\mathcal{Z}) \cap W^{1,\infty}(0,1;H^1(\Omega)), \tag{6.8a}$$

$$\overline{\mathsf{z}}_k \rightharpoonup^* \vartheta \qquad \text{in } L^\infty(0,1;W^{1+\sigma,q}(\Omega)) \quad \text{for all } 0<\sigma<\frac{1}{q}, \tag{6.8b}$$

$$\overline{\mathsf{z}}_k \to \vartheta \qquad \text{in } L^\infty(0,1;\mathcal{Z}), \tag{6.8c}$$

$$\widehat{\mathsf{z}}_k \to \vartheta \qquad \text{in } \mathrm{C}^0([0,1],H^1(\Omega))\,, \tag{6.8d}$$

whence (6.3a) and (6.3b). Furthermore, observe that $A_q(\overline{\mathsf{z}}_k) \rightharpoonup^* A_q(\vartheta)$ in $L^\infty(0,1;L^2(\Omega))$ and that, as $k\to\infty$,

$$\|\mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_{\tau_k}\circ\mathsf{t}_k,\overline{\mathsf{z}}_k) - \mathrm{D}_z\widetilde{\mathcal{I}}(t,\vartheta)\|_{L^\infty(0,1;L^2(\Omega))} \overset{(1)}{\le} C\sup_{s\in[0,1]}\left(|\overline{t}_{\tau_k}(\mathsf{t}_k(s)) - t| + \|\overline{\mathsf{z}}_k(s)-\vartheta(s)\|_{L^6(\Omega)}\right) \overset{(2)}{\to} 0 \tag{6.8e}$$

with (1) due to (2.32), and convergence (2) due to (6.8c), joint with the fact that $\sup_{s\in[0,1]}|\mathsf{t}_k(s)) - t| \to 0$ as t_k takes values in the interval $[\alpha_k,\beta_k]$ which shrinks to $\{t\}$. All in all, $\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau_k}\circ\mathsf{t}_k,\overline{\mathsf{z}}_k) \rightharpoonup^* \mathrm{D}_z\mathcal{I}(t,\vartheta)$ in $L^\infty(0,1;L^2(\Omega))$. It follows from estimates (6.7b) and convergences (6.8) that $\vartheta \in \mathcal{T}_t^{\bar{\varrho}}(z_0,z_1)$. It remains to conclude (6.3c). For this limit passage, we rely on convergences (6.8) and refer the reader to the proof of [MRS16, Prop. 7.1], cf. also [KRZ13, Thm. 5.1], [KRZ15, Thm. 7.4].

This finishes the proof of Proposition 6.1. □

We continue this section by carrying out the **proof of Proposition 5.8**, by suitably adapting the argument for the chain-rule result [MRS16, Thm. 3.13]. From now on, we will suppose that $t_0 = 0$ and $t_1 = T$ for the sake of simplicity. Let $\varrho > 0$ fulfill (5.13).

First of all, for any $z \in \mathrm{BV}([0,T]; L^2(\Omega))$ fulfilling the conditions of the statement we construct a *parameterized* curve $(\mathsf{t},\mathsf{z}) : [0,\mathsf{S}] \to [0,T] \times \mathcal{Z}$ with the following properties:

$$z(t) \in \{\mathsf{z}(s) \ : \mathsf{t}(s) = t\}$$

and

- t is non-decreasing, surjective, Lipschitz,
- $\mathsf{z} \in L^\infty(0,\mathsf{S};\mathcal{Z}) \cap \mathrm{AC}([0,\mathsf{S}]; L^2(\Omega))$ and $\mathrm{D}_z\mathcal{I}(\cdot,\mathsf{z}(\cdot)) \in L^\infty(0,\mathsf{S}; L^2(\Omega))$.

The integrability and regularity requirements on z coincide with those on admissible transition curves, cf. Definition 5.1. Hence, we will call (t,z) *admissible parameterized curve*. We borrow the construction of (t,z), starting from the BV-curve z, from the proof of [MRS16, Prop. 4.7]: first, we introduce the parameterization

$$\mathsf{s}(t) := t + \mathrm{Var}_{L^2(\Omega)}(z;[0,t]), \quad \mathsf{S} := \mathsf{s}(T).$$

We define

$$\mathsf{t} := \mathsf{s}^{-1} : [0,\mathsf{S}] \setminus I \to [0,T], \qquad \mathsf{z} := z \circ \mathsf{t},$$

where the set I is given by $I = \cup_n I_n$, with $I_n = (\mathsf{s}(t_{n-}), \mathsf{s}(t_{n+}))$ and the points $(t_n)_n$ constitute the countable jump set of z, which in fact coincides with the jump set of s. We extend t and z to I by setting

$$\mathsf{t}(s) := t_n, \quad \mathsf{z}(s) := \vartheta_n(\mathsf{r}_n(s)) \quad \text{if } s \in I_n,$$

with $\mathsf{r}_n : \overline{I_n} \to [0,1]$ the unique affine and strictly increasing function from I_n to $[0,1]$ and $\vartheta_n \in \mathcal{T}^{\varrho}_{t_n}(z(t_{n-}), z(t_{n+}))$ an admissible transition curve satisfying $\vartheta_n(\mathsf{r}_n(\mathsf{s}(t_n))) = z(t_n)$ and the optimality condition

$$\int_0^1 \mathfrak{f}_{t_n}(\vartheta_n(r), \vartheta_n'(r))\,\mathrm{d}r = \Delta^{\varrho}_{\mathfrak{f}}(t_n; z(t_{n-}), z(t_n)) + \Delta^{\varrho}_{\mathfrak{f}}(t_n; z(t_n), z(t_{n+}))\,.$$

The existence of such an optimal transition follows from Proposition 6.1(1). Indeed, let $t_* \in \mathrm{J}_z$. Observe that in $(t_*, z(t_{*\pm}))$ the assumptions of the proposition are satisfied, which can be seen as follows. First of all, $\Delta^{\varrho}_{\mathfrak{f}}(t_n; z(t_{n-}), z(t_n)) < \infty$ and $\Delta^{\varrho}_{\mathfrak{f}}(t_n; z(t_n), z(t_{n+})) < \infty$ since $\mathrm{Var}^{\varrho}_{\mathfrak{f}}(z;[0,T]) < +\infty$. Moreover, choose a sequence $s_k \to t_{*-}$ for $k \to \infty$ such that the assumptions of Prop. 6.1(1) are satisfied along this sequence and such that $z(s_k) \rightharpoonup z(t_{*-})$ in $\mathcal{Z}$. Consequently, by Corollary 2.13, $\mathrm{D}_z\widetilde{\mathcal{I}}(s_k, z(s_k)) \to \mathrm{D}_z\widetilde{\mathcal{I}}(t_*, z(t_{*-}))$ and $\|A_q(z(s_k))\|_{L^2(\Omega)} \le C$, which translates into a uniform bound of the sequence $(z(s_k))_k$ in $W^{1+\sigma,q}(\Omega)$ for $0 < \sigma < \frac{1}{q}$, cf. Proposition 2.7. Thus, we finally conclude that $\mathrm{D}_z\mathcal{I}(t_*, z(t_{*-})) \in L^2(\Omega)$ and that $\|z(t_{*-})\|_{\mathcal{Z}} + \|\mathrm{D}_z\mathcal{I}(t_*, z(t_{*-}))\|_{L^2(\Omega)} \le \varrho$. A similar argument applies to t_{*+}.

By construction, $\mathsf{z} \in W^{1,\infty}(0,\mathrm{S}; L^2(\Omega))$. Indeed, let $s_1 < s_2 \in [0,\mathsf{S}]$ and $\sigma_i := \mathsf{t}(s_i)$. Hence, $s_i = \sigma_i + \mathrm{Var}_{L^2(\Omega)}(z;[0,\sigma_i])$. This implies that

$$\|\mathsf{z}(s_1) - \mathsf{z}(s_2)\|_{L^2(\Omega)} \le |\sigma_2 + \mathrm{Var}_{L^2(\Omega)}(z;[0,\sigma_2]) - (\sigma_1 + \mathrm{Var}_{L^2(\Omega)}(z;[0,\sigma_1]))| = |s_2 - s_1|\,.$$

Hence, altogether (t,z) is an admissible parameterized curve.

By repeating the very same calculations as in the proof of [MRS16, Prop. 4.7], we may show that

$$\mathrm{Var}^{\varrho}_{\mathfrak{f}}(z;[0,T]) = \int_0^{\mathsf{S}} \mathfrak{f}_{\mathsf{t}(s)}(\mathsf{z}(s), \mathsf{z}'(s))\,\mathrm{d}s\,. \tag{6.9}$$

Secondly, we observe that the chain rule from Lemma 2.16 (cf. also Remark 2.17) extends to the admissible parameterized curve (t,z), yielding

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{I}(\mathsf{t}(s),\mathsf{z}(s)) - \partial_t\mathcal{I}(\mathsf{t}(s),\mathsf{z}(s))\mathsf{t}'(s) = \int_\Omega \mathrm{D}_z\mathcal{I}(\mathsf{t}(s),\mathsf{z}(s))\mathsf{z}'(s)\,\mathrm{d}x \qquad \text{for a.a. } s\in(0,\mathsf{S})\,.$$

Therefore, with a simple calculation (cf. also the proof of [MRS16, Thm. 4.4]) we infer that

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{I}(\mathsf{t}(s),\mathsf{z}(s)) - \partial_t\mathcal{I}(\mathsf{t}(s),\mathsf{z}(s))\mathsf{t}'(s)\right| \leq \mathfrak{f}_{\mathsf{t}(s)}(\mathsf{z}(s),\mathsf{z}'(s)) \qquad \text{for a.a. } s\in(0,\mathsf{S})\,. \tag{6.10}$$

Combining (6.9) & (6.10) we obtain the desired chain-rule inequality (5.22). ∎

We are now in a position to give the **proof of Theorem 5.7**. We will split the proof in several steps and give some intermediate results. Let us mention in advance that, in their statements, we will always tacitly suppose that Assumptions 2.1, 2.2, and 2.8, as well as condition (5.17), from Theorem 5.7 hold. More precisely,

- we start by fixing the compactness properties of the sequences $(\overline{z}_{\tau_k,\epsilon_k})_k$, $(\widehat{z}_{\tau_k,\epsilon_k})_k$ in Lemma 6.2 below.
- Throughout Steps 1–3 we show that any limit curve z of $(\overline{z}_{\tau_k,\epsilon_k})_k$, $(\widehat{z}_{\tau_k,\epsilon_k})_k$ complies with the local stability ($\mathrm{S}_{\mathrm{loc}}$) and with the energy-dissipation inequality (5.23), obtained by passing to the limit in its discrete counterpart (3.11). By virtue of Corollary 5.9 we thus conclude that z is a Balanced Viscosity solution to the rate-independent system (1.1).
- Steps 4 & 5 are devoted to finalizing the proof of convergences (5.20), and to showing that z is a $H^1(\Omega)$-parameterizable solution, cf. (5.21).

Step 0**: Compactness.** We prove the following

Lemma 6.2. *Let $(\tau_k,\epsilon_k)_k$ be null sequences. There holds*

$$\exists\, C>0\ \forall\, k\in\mathbb{N}:\ \sup_{t\in[0,T]}\|\overline{z}_{\tau_k,\epsilon_k}(t)-\widehat{z}_{\tau_k,\epsilon_k}(t)\|_{H^1(\Omega)} \leq C\left(\frac{\tau_k}{\epsilon_k}\right)^{1/2}\,. \tag{6.11}$$

Suppose in addition (5.18)*. Then, there exists a curve $z\in L^\infty(0,T;\mathcal{Z})\cap \mathrm{BV}([0,T];H^1(\Omega))$ such that, up to a (not relabeled) subsequence, the following convergences hold:*

$$\overline{z}_{\tau_k,\epsilon_k},\ \widehat{z}_{\tau_k,\epsilon_k} \rightharpoonup^* z \qquad \text{in } L^\infty(0,T;\mathcal{Z}), \tag{6.12a}$$

$$\overline{z}_{\tau_k,\epsilon_k}(t),\ \widehat{z}_{\tau_k,\epsilon_k}(t) \to z(t) \qquad \text{in } \mathcal{Z} \quad \text{for all } t\in[0,T], \tag{6.12b}$$

$$\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau_k}(t),\overline{z}_{\tau_k,\epsilon_k}(t)) \rightharpoonup \mathrm{D}_z\mathcal{I}(t,z(t)) \qquad \text{in } L^2(\Omega) \quad \text{for all } t\in[0,T]. \tag{6.12c}$$

Proof. The first estimate follows from observing that for every $t\in(0,T)$

$$\|\overline{z}_{\tau_k,\epsilon_k}(t)-\widehat{z}_{\tau_k,\epsilon_k}(t)\|_{H^1(\Omega)} \leq \int_{\underline{t}_\tau(t)}^{\overline{t}_\tau(t)} \|\widehat{z}'_{\tau_k,\epsilon_k}(r)\|_{H^1(\Omega)}\,\mathrm{d}r \leq \tau_k^{1/2}\|\widehat{z}'_{\tau_k,\epsilon_k}\|_{L^2(\underline{t}_\tau(t),\overline{t}_\tau(t);H^1(\Omega))},$$

and then (6.11) is a consequence of the a priori estimate (3.9d).

Convergences (6.12a) follow from estimate (3.9b): observe that the sequences $(\overline{z}_{\tau_k,\epsilon_k})_k$, $(\widehat{z}_{\tau_k,\epsilon_k})_k$ converge to the same limit, weakly star in $L^\infty(0,T;\mathcal{Z})$, in view of the fact that

$$\|\overline{z}_{\tau_k,\epsilon_k}-\widehat{z}_{\tau_k,\epsilon_k}\|_{L^\infty(0,T;H^1(\Omega))} \to 0 \tag{6.13}$$

as $k\to\infty$ by (6.11) combined with condition (5.18) on the sequences $(\tau_k,\epsilon_k)_k$.

It follows from estimate (3.9e) that the sequences $(\overline{z}_{\tau_k,\epsilon_k})_k$, $(\widehat{z}_{\tau_k,\epsilon_k})_k$ are bounded in $\mathrm{BV}([0,T];H^1(\Omega))$. Due to the previously mentioned [MT04, Thm. 6.1], up to a subsequence they pointwise converge on $[0,T]$, w.r.t. the weak $H^1(\Omega)$-topology, to (the same, by (6.13)) function $\tilde{z}$. Now, by the additional estimate (3.9f), $(\overline{z}_{\tau_k,\epsilon_k})_k$ is bounded in $L^\infty(0,T;W^{1+\sigma,q}(\Omega))$ for every $0<\sigma<\frac{1}{q}$, cf. Proposition 2.7, and so is $(\widehat{z}_{\tau_k,\epsilon_k})_k$. Therefore, by compactness the above pointwise convergence to $\tilde{z}$ improves to a strong convergence in $\mathcal{Z}$. But then, $\overline{z}_{\tau_k,\epsilon_k}$, $\widehat{z}_{\tau_k,\epsilon_k}\to\tilde{z}$ in $L^p(0,T;\mathcal{Z})$ for every $1\leq p<\infty$, which allows us to conclude that $\tilde{z}=z$. All in all, we have obtained convergence (6.12b).

Finally, we address (6.12c): Observe that $A_q(\overline{z}_{\tau_k,\epsilon_k}(t)) \to A_q(z(t))$ in $\mathcal{Z}^*$ as a consequence of the strong convergence (6.12b). A fortiori, by the $L^\infty(0,T;L^2(\Omega))$-bound on $(A_q(\overline{z}_{\tau_k,\epsilon_k}))_k$, we find that $A_q(\overline{z}_{\tau_k,\epsilon_k}(t)) \rightharpoonup A_q(z(t))$ in $L^2(\Omega)$. We combine this with (2.36), giving that $\mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_{\tau_k}(t),\overline{z}_{\tau_k,\epsilon_k}(t)) \rightharpoonup \mathrm{D}_z\widetilde{\mathcal{I}}(t,z(t))$ in $L^2(\Omega)$, and arrive at (6.12c). □

Step 1: ad the local stability ($\mathrm{S}_{\mathrm{loc}}$)**.** On the one hand, the very same argument leading to the proof of estimate (3.12a) in Corollary 3.3 also shows that

$$\sup_k \int_0^T \mathcal{R}^*_{\epsilon_k}(-\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau_k}(r),\overline{z}_{\tau_k,\epsilon_k}(r)))\,\mathrm{d}r \le C\,. \tag{6.14}$$

On the other hand, $\mathcal{R}^*_\epsilon$ Mosco-converges, w.r.t. the $L^2(\Omega)$-topology, to the indicator functional

$$\mathrm{I}_{\partial\mathcal{R}(0)} : L^2(\Omega) \to [0,+\infty] \quad \text{defined by} \quad \mathrm{I}_{\partial\mathcal{R}(0)}(v) := \begin{cases} 0 & \text{if } v \in \partial\mathcal{R}_1(0), \\ +\infty & \text{else.} \end{cases}$$

Hence we have in view of (6.12c) that

$$\liminf_{k\to\infty} \mathcal{R}^*_{\epsilon_k}(-\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau_k}(t),\overline{z}_{\tau_k,\epsilon_k}(t))) \ge \mathrm{I}_{\partial\mathcal{R}(0)}(-\mathrm{D}_z\mathcal{I}(t,z(t))) \qquad \text{for every } t \in [0,T]. \tag{6.15}$$

Therefore, from (6.14) and (6.15) via the Fatou Lemma we infer that

$$\int_0^T \mathrm{I}_{\partial\mathcal{R}(0)}(-\mathrm{D}_z\mathcal{I}(t,z(t)))\,\mathrm{d}t < +\infty \qquad \text{whence} \qquad \mathrm{I}_{\partial\mathcal{R}(0)}(-\mathrm{D}_z\mathcal{I}(t,z(t))) = 0 \quad \text{for a.a. } t \in (0,T).$$

From this we conclude with an approximation argument $-\mathrm{D}_z\mathcal{I}(t,z(t)) \in \partial\mathcal{R}_1(0)$ for every $t \in [0,T]\setminus \mathrm{J}_z$, and that $-\mathrm{D}_z\mathcal{I}(t,z(t_\pm)) \in \partial\mathcal{R}_1(0)$ for every $t \in \mathrm{J}_z$, i.e. ($\mathrm{S}_{\mathrm{loc}}$).

Step 2: the key lower semicontinuity inequality. We aim to prove the following

Lemma 6.3. *For every $0 \le s \le t \le T$ there holds*

$$\liminf_{k\to\infty} \int_{\underline{t}_{\tau_k}(s)}^{\overline{t}_{\tau_k}(t)} \mathcal{R}_{\epsilon_k}(\widehat{z}'_{\tau_k,\epsilon_k}(r))\,\mathrm{d}r + \mathcal{R}^*_{\epsilon_k}(-\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau_k}(r),\overline{z}_{\tau_k,\epsilon_k}(r)))\,\mathrm{d}r \ge \mathrm{Var}_{\mathfrak{f}}^{\bar\varrho}(z;[s,t]) \tag{6.16}$$

with $\bar\varrho$ given by

$$\begin{aligned} \bar\varrho := \sup_k \Big(\int_0^T \big(\mathcal{R}_{\epsilon_k}(\widehat{z}'_k(r)) + \mathcal{R}^*_{\epsilon_k}(-\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau_k}(r),\overline{z}_k(r)))\big)\,\mathrm{d}r + \|\widehat{z}_k\|_{L^\infty(0,T;\mathcal{Z})\cap W^{1,1}(0,T;H^1(\Omega))} \\ + \|\overline{z}_k\|_{L^\infty(0,T;\mathcal{Z})} + \|\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau_k},\overline{z}_k))\|_{L^\infty(0,T;L^2(\Omega))}\Big) \end{aligned}$$

Proof. Along the footsteps of the [MRS16, proof of Thm. 7.3], we introduce the non-negative Borel measures on $[0,T]$

$$\nu_k := \big(\mathcal{R}_{\epsilon_k}(\widehat{z}'_{\tau_k,\epsilon_k}) + \mathcal{R}^*_{\epsilon_k}(-\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau_k},\overline{z}_{\tau_k,\epsilon_k}))\big)\mathscr{L}^1,$$

with $\mathscr{L}^1$ the Lebesgue measure. It follows from estimate (3.12b) that the sequence $(\nu_k)_k$ is bounded in the space of Radon measures, hence there exists a positive measure ν such that $\nu_k \rightharpoonup^* \nu$ as $k\to\infty$. Like in the proof of [MRS16, Thm. 7.3], we observe that for every interval $[a,b] \subset [0,T]$

$$\begin{aligned} \nu([a,b]) \ge \limsup_{k\to\infty} \nu_k([a,b]) &\ge \limsup_{k\to\infty} \int_a^b \big(\mathcal{R}_{\epsilon_k}(\widehat{z}'_{\tau_k,\epsilon_k}(r)) + \mathcal{R}^*_{\epsilon_k}(-\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau_k}(r),\overline{z}_{\tau_k,\epsilon_k}(r)))\big)\,\mathrm{d}r \\ &\ge \liminf_{k\to\infty} \int_a^b \mathcal{R}_{\epsilon_k}(\widehat{z}'_{\tau_k,\epsilon_k}(r))\,\mathrm{d}r \\ &\ge \liminf_{k\to\infty} \mathrm{Var}_{\mathcal{R}_1}(\overline{z}_{\tau_k,\epsilon_k};[a,b]) \overset{(1)}{\ge} \mathrm{Var}_{\mathcal{R}_1}(z;[a,b]) \overset{(2)}{\ge} \mu_{\mathrm{d}}([a,b]), \end{aligned}$$

where (1) follows from the pointwise convergence (6.12b) and the lower semicontinuity of the variation functional $\mathrm{Var}_{\mathcal{R}_1}$, and (2) from the definition (5.9) of the measure μ. We thus conclude that

$$\nu \geq \mu_{\mathrm{d}}. \tag{6.17}$$

We now check

$$\nu(\{t\}) \geq \Delta_{\mathfrak{f}}^{\bar{\varrho}}(t; z(t_-), z(t)) + \Delta_{\mathfrak{f}}^{\bar{\varrho}}(t; z(t), z(t_+)) \geq \mu_{\mathrm{J}}(\{t\}) \qquad \text{for every } t \in \mathrm{J}_z. \tag{6.18}$$

With this aim, for fixed $t \in \mathrm{J}_z$ let us fix two sequences $\alpha_k \uparrow t$ and $\beta_k \downarrow t$ such that

$$\begin{cases} \overline{z}_{\tau_k,\epsilon_k}(\alpha_k) \to z(t_-), \\ \overline{z}_{\tau_k,\epsilon_k}(\beta_k) \to z(t_+) \end{cases} \qquad \text{in } \mathcal{Z} \text{ as } k \to \infty.$$

Thus we have

$$\limsup_{k\to\infty} \nu_k([\alpha_k,\beta_k]) \geq \liminf_{k\to\infty} \int_{\alpha_k}^{\beta_k} \left(\mathcal{R}_{\epsilon_k}(\widehat{z}'_{\tau_k,\epsilon_k}(r)) + \mathcal{R}^*_{\epsilon_k}(-\mathrm{D}_z \mathcal{I}(\bar{t}_{\tau_k}(r), \overline{z}_{\tau_k,\epsilon_k}(r))) \right) \mathrm{d}r \overset{(1)}{\geq} \Delta_{\mathfrak{f}}^{\bar{\varrho}}(t; z(t_-), z(t_+)) \,,$$

where (1) ensues from Proposition 6.1, applying (6.3) with the choices $\overline{z}_k := \overline{z}_{\tau_k,\epsilon_k}$, $\widehat{z}_k := \widehat{z}_{\tau_k,\epsilon_k}$. With analogous arguments check we check that

$$\liminf_{k\to\infty} \nu_k([\alpha_k, t]) \geq \Delta_{\mathfrak{f}}^{\bar{\varrho}}(t; z(t_-), z(t)), \qquad \liminf_{k\to\infty} \nu_k([t, \beta_k]) \geq \Delta_{\mathfrak{f}}^{\bar{\varrho}}(t; z(t), z(t_+)) \,. \tag{6.19}$$

All in all, we have

$$\begin{aligned} \nu(\{t\}) &\overset{(1)}{\geq} \limsup_{k\to\infty} \nu_k([\alpha_k,\beta_k]) \geq \liminf_{k\to\infty} \nu_k([\alpha_k,t]) + \liminf_{k\to\infty} \nu_k([t,\beta_k]) \geq \Delta_{\mathfrak{f}}^{\bar{\varrho}}(t; z(t_-), z(t)) + \Delta_{\mathfrak{f}}^{\bar{\varrho}}(t; z(t), z(t_+)) \\ &\overset{(2)}{\geq} \mu_{\mathrm{J}}(\{t\}), \end{aligned}$$

where (1) is a property of the weak*-convergence of measures and (2) ensues from (5.3). Hence inequality (6.18) is proved.

Combining (6.17), (6.18), and (6.19) and repeating the very same calculations as in the proof of [MRS16, Thm. 7.3], we ultimately conclude (6.16). □

Step 3: ad the energy-dissipation inequality (5.23)**.** We now pass to the limit in the discrete energy-dissipation inequality (3.11), written for $s = 0$ and $t = T$. For the first term on the left-hand side, we resort to the lower semicontinuity inequality (6.16) from Step 2. It follows from the pointwise convergence (6.12b) and the lower semicontinuity (2.36) of $\mathcal{I}$ that

$$\liminf_{k\to\infty} \mathcal{I}(T, \widehat{z}_{\tau_k,\epsilon_k}(T)) \geq \mathcal{I}(T, z(T)),$$

whereas by hypothesis we have that $\mathcal{I}(0, \widehat{z}_{\tau_k,\epsilon_k}(0)) \to \mathcal{I}(0, z_0)$. Furthermore, it follows from (2.23), (2.24), and the Lebesgue Theorem that

$$\lim_{k\to\infty} \int_0^T \partial_t \mathcal{I}(t, \widehat{z}_{\tau_k,\epsilon_k}(t)) \,\mathrm{d}t = \int_0^T \partial_t \mathcal{I}(t, z(t)) \,\mathrm{d}t \,.$$

Finally, observe that the very last term on the right-hand side of (3.11) converges to zero by virtue of estimates (3.9) and convergence (6.13).

Thus, (5.23) is proven with $\mathit{Var}_{\mathfrak{f}}^{\bar{\varrho}}(z; [0,T])$ and, by virtue of Corollary 5.9, we deduce that z is a Balanced Viscosity solution to the rate-independent damage system (1.1).

Finally, (5.19) follows from the following chain of inequalities (which in fact holds for every $t \in [0,T]$)

$$\sup_{\varrho \geq \bar{\varrho}} \mathit{Var}_{\mathfrak{f}}^{\varrho}(z; [0,T]) \overset{(1)}{=} \mathit{Var}_{\mathfrak{f}}^{\bar{\varrho}}(z; [0,T]) \overset{(2)}{=} \mathcal{I}(0, z(0)) - \mathcal{I}(T, z(T)) + \int_0^T \partial_t \mathcal{I}(s, z(s)) \,\mathrm{d}s \overset{(3)}{\leq} \inf_{\varrho \geq \bar{\varrho}} \mathit{Var}_{\mathfrak{f}}^{\varrho}(z; [0,T]),$$

with (1) due to (5.6), (2) to ($\mathrm{E}_{\mathfrak{f}}$) involving the total variation functional $\mathit{Var}_{\mathfrak{f}}^{\bar{\varrho}}(z; [0,T])$, and (3) from the chain-rule inequality (5.22) (observe that ϱ therein is arbitrary, provided it fulfills (5.13)).

Step 4: ad convergences (5.20)**.** The convergences of the energies $(\mathcal{I}(t, \overline{z}_{\tau_k,\epsilon_k}(t)))_k$ follows from the pointwise convergence (6.12a) of $(\overline{z}_{\tau_k,\epsilon_k}(t))_k$. In order to prove the convergence of $(\mathcal{I}(t, \widehat{z}_{\tau_k,\epsilon_k}(t)))_k$ and of the dissipation integrals in (5.20c), we repeat the very same arguments as in the proof of [MRS16, Thm. 3.11].

Step 5: ad (5.21)**.** We may repeat the proof of [MRS16, Thm. 3.22], to which we refer the reader, relying on Proposition 6.1(3).

This concludes the proof of Theorem 5.7. ■

Appendix A. Some references on elliptic regularity

For $d \geq 2$ let $\Omega \subset \mathbb{R}^d$ be a bounded $\mathrm{C}^{1,1}$-domain with Dirichlet boundary $\partial\Omega$. Let further $\mathbb{C}$ satisfy (2.5). Reference [Val78, Theorem 3], see also [MR03, Theorem 7.1], yields

Theorem A.1. *For every* $p \in (1,\infty)$ *the operator* $L_{\mathbb{C}} : W^{1,p}_0(\Omega) \to W^{-1,p}(\Omega)$ *is a continuous isomorphism.*

Moreover, Theorem 10.5 from [ADN64] (there it is assumed that the domain has a C^2-boundary, but the coefficients need to be continuous, only, instead of Lipschitz continuous) provides the following a priori estimate:

Theorem A.2. *For every* $p \in (1,\infty)$ *there exist constants* $c_p, \widetilde{c}_p > 0$ *such that for every* $u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ *it holds*

$$\|u\|_{W^{2,p}(\Omega)} \leq c_p \left(\|L_{\mathbb{C}} u\|_{L^p(\Omega)} + \widetilde{c}_p \|u\|_{L^p(\Omega)} \right) . \tag{A.1}$$

Thanks to Theorem A.1, for every $p \in (1,\infty)$ the operator

$$L_{\mathbb{C}} : W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \to L^p(\Omega), \quad u \mapsto -\mathrm{div}(\mathbb{C}\varepsilon(u)) \tag{A.2}$$

is injective, which implies that estimate (A.1) is valid with $\widetilde{c}_p = 0$ and that $L_{\mathbb{C}}$ has a closed range. By [Kat84, Chapter 3.5.5], one finally concludes that the operator $L_{\mathbb{C}}$ from (A.2) is surjective for every $p \in (1,\infty)$. This finally results in

Theorem A.3. *For every* $p \in (1,\infty)$ *the operator in* (A.2) *is a continuous isomorphism.*

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