EXISTENCE RESULTS FOR MINIMIZERS OF PARAMETRIC ELLIPTIC FUNCTIONALS

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Abstract. We prove a compactness principle for the anisotropic formulation of the Plateau problem in any codimension, in the same spirit of the previous works of the authors [DGM14, DPDRG15, DLDRG16]. In particular, we perform a new strategy for the proof of the rectifiability of the minimal set, based on the new anisotropic counterpart of the Allard rectifiability theorem proved by the authors in [DPDRG16]. As a consequence we provide a new proof of Reifenberg existence theorem.

1. Introduction and Main Result

This paper concludes a series of works by the authors on the Plateau problem: we here provide a general and flexible existence result for sets that minimize an anisotropic energy, which can be applied to several notions of boundary conditions. In the spirit of the previous works [DGM14],[DPDRG15] and [DLDRG16], we use the direct methods of the calculus of variations to find a generalised minimisers (namely a Radon measure) via standard compactness arguments, and then we aim at proving that it is actually a fairly regular surface. To do this, we employed several techniques to first establish the rectifiability of the limit measure: in the case of the area integrand this property was initially deduced from a powerful result due to Preiss [Pre87, De 08], as well as, in codimension one, from the theory of sets of finite perimeter. These two techniques are no longer available in the case of anisotropic problems in higher codimension (in particular due to the lack of a monotonicity formula for anisotropic problems). A new rectifiability criterion was found in [DPDRG16, Theorem 1.2], for varifolds having positive lower density and a bounded anisotropic first variation, extending the celebrated result by Allard [All72] (see also [De 16]).

The proof of the existence theorem can be applied to the minimization of the energy in several classes of sets (also treated in the works [HP15, HP16]), corresponding to several notions of boundary conditions: in particular we discuss the existence theorem for minimizers with a homological notion of boundary, originally considered by Reifenberg in the isotropic case [Rei60], see Section 3. Our techniques can as well be extended to prove existence for the anisotropic Plateau problem under co-homological boundary conditions, first considered in [HP16] where however more general assumptions on the integrand are assumed, see Remark 1.7 and 3.5 below. Recently a related existence theorem has been proved also in [FK17], following the strategy of [Alm68, Alm76].

In order to precisely state our main result, we introduce some notations and definitions. We will always work in $\mathbb{R}^n$ and $1 \leq d \leq n$ will always be an integer number, we recall that a set K is said to be d-rectifiable if it can be covered, up to an $\mathcal{H}^d$ negligible set, by countably many C^1 manifolds where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure. We also denote by $G = G(n,d)$ the Grassmannian of unoriented d-dimensional hyperplanes in $\mathbb{R}^n$ and, for every $U \subset \mathbb{R}^n$, we define $G(U) := U \times G$. Given a d-rectifiable set K, we denote by $T_K(x)$ the approximate tangent space of K at x, which exists for $\mathcal{H}^d$-almost every point $x \in K$ [Sim83, Chapter 3]. We also let $\mathsf{Lip}(\mathbb{R}^n)$ be the space of Lipschitz maps in $\mathbb{R}^n$.

The anisotropic Lagrangians considered in the rest of the note will be C^1 maps

$$F : \mathbb{R}^n \times G \ni (x, T) \mapsto F(x, T) \in \mathbb{R}^+,$$

verifying the lower and upper bounds

$$0 < \lambda \le F(x, T) \le \Lambda < \infty. \tag{1.1}$$

Given a d-rectifiable set K and an open subset $U \subset \mathbb{R}^n$, we define:

$$\mathbf{F}(K, U) := \int_{K \cap U} F(x, T_K(x))\, d\mathcal{H}^d(x) \quad \text{and} \quad \mathbf{F}(K) := \mathbf{F}(K, \mathbb{R}^n). \tag{1.2}$$

It will be also convenient to look at the frozen Lagrangian: for $y \in \mathbb{R}^n$, we let

$$\mathbf{F}^y(K, U) := \int_{K \cap U} F(y, T_K(x))\, d\mathcal{H}^d(x).$$

We note that given a d-dimensional varifold V (i.e. a positive Radon measure on the Grassmannian $G(U)$) we can define its anisotropic energy as

$$\mathbf{F}(V, U) := \int F(x, T)\, dV(x, T),$$

which is coherent with (1.2), since to any rectifiable set K we will naturally associate the varifold $\mathcal{H}^d \llcorner K \otimes \delta_{T_K(x)}$. In this setting we define the *anisotropic first variation* of a varifold V as the order one distribution whose action on $g \in C^1_c(U, \mathbb{R}^n)$ is given by

$$\begin{aligned} \delta_F V(g) &:= \frac{d}{dt} \mathbf{F}\big(\varphi_t^\# V\big)\Big|_{t=0} \\ &= \int_{\Omega \times G(n,d)} \Big[\langle d_x F(x, T), g(x)\rangle + B_F(x, T) : Dg(x)\Big] dV(x, T), \end{aligned}$$

where $\varphi_t(x) = x + tg(x)$, $\varphi_t^\# V$ is the image varifold of V through φ_t see [Sim83, Chapter 8], $B_F(x, T) \in \mathbb{R}^n \otimes \mathbb{R}^n$ is an explicitly computable $n \times n$ matrix and $\langle A, B\rangle := \operatorname{tr} A^* B$ for $A, B \in \mathbb{R}^n \otimes \mathbb{R}^n$, see for instance [DPDRG16] for the relevant computations. A varifold V is said to be F-stationary in an open set U if $\delta_F V = 0$ as a distribution in U.

Throughout all the paper, $H \subset \mathbb{R}^n$ will denote a closed subset of $\mathbb{R}^n$. Assume to have a class $\mathcal{P}(H)$ of relatively closed d-rectifiable subsets K of $\mathbb{R}^n \setminus H$: one can then formulate the anisotropic Plateau problem by asking whether the infimum

$$m_0 := \inf \big\{\mathbf{F}(K) : K \in \mathcal{P}(H)\big\} \tag{1.3}$$

is achieved by some set (which should be a suitable limit of a minimizing sequence), if it belongs to the chosen class $\mathcal{P}(H)$ and which additional regularity properties it satisfies. We will say that a sequence $(K_j) \subset \mathcal{P}(H)$ is a *minimizing sequence* if $\mathbf{F}(K_j) \downarrow m_0$.

We next outline a set of flexible and rather weak requirements for $\mathcal{P}(H)$: the key property for K' to be a competitor of K is that K' is close in energy to sets obtained from K via deformation maps as in Definition 1.1. This allows a larger flexibility on the choice of the admissible sets, since a priori K' might not belong to the competition class.

Definition 1.1 (Lipschitz deformations)**.** Given a ball $B(x, r)$, we let $\mathfrak{D}(x, r)$ be the set of functions $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi(z) = z$ in $\mathbb{R}^n \setminus B(x, r)$ and which are smoothly isotopic to the identity inside $B(x, r)$, namely those for which there exists an isotopy $\lambda \in C^\infty([0, 1] \times \mathbb{R}^n; \mathbb{R}^n)$ such that

$$\lambda(0, \cdot) = \mathrm{Id}, \quad \lambda(1, \cdot) = \varphi, \quad \lambda(t, y) = y \quad \forall\, (t, y) \in [0, 1] \times (\mathbb{R}^n \setminus B(x, r)) \quad \text{and}$$
$$\lambda(t, \cdot) \text{ is a diffeomorphism of } \mathbb{R}^n\ \forall t \in [0, 1].$$

We finally set $\mathsf{D}(x,r) := \overline{\mathfrak{D}(x,r)}^{w^*-W^{1,\infty}}$, the sequential closure of $\mathfrak{D}(x,r)$ with respect to the uniform convergence with equibounded differentials.

Observe that in the definition of $\mathsf{D}(x,r)$ it is equivalent to require any C^k regularity on the isotopy λ, for $k \geq 1$, as C^k isotopies "supported" in $B(x,r)$ can be approximated in C^k by smooth ones also supported in the same set.

Definition 1.2 (Deformed competitors and good class)**.** Let $H \subset \mathbb{R}^n$ be closed, $K \subset \mathbb{R}^n \setminus H$ be a relatively closed countably $\mathcal{H}^d$-rectifiable and $B(x,r) \subset \mathbb{R}^n \setminus H$. A *deformed competitor* for K in $B(x,r)$ is any set of the form

$$\varphi(K) \qquad \text{where} \qquad \varphi \in \mathsf{D}(x,r).$$

Given a family $\mathcal{P}(H)$ of relatively closed d-rectifiable subsets $K \subset \mathbb{R}^n \setminus H$, we say that $\mathcal{P}(H)$ is a *good class* if for every $K \in \mathcal{P}(H)$, for every $x \in K$ and for a.e. $r \in (0, \operatorname{dist}(x,H))$

$$\inf\left\{\mathbf{F}(J) : J \in \mathcal{P}(H)\,, J \setminus \overline{B(x,r)} = K \setminus \overline{B(x,r)}\right\} \leq \mathbf{F}(L) \tag{1.4}$$

whenever L is any deformed competitor for K in $B(x,r)$.

We will assume the following ellipticity condition on the energy $\mathbf{F}$, introduced in [Alm68], which is a geometric version of quasiconvexity, cf. [Mor66]:

Definition 1.3 (Elliptic integrand, [Alm68, 1.2])**.** The anisotropic Lagrangian F is said to be *elliptic* if there exists $\Gamma \geq 0$ such that, whenever $x \in \mathbb{R}^n$ and D is a d-disk centered in x and with radius r, then the inequality

$$\mathbf{F}^x(K, B(x,r)) - \mathbf{F}^x(D, B(x,r)) \geq \Gamma(\mathcal{H}^d(K \cap B(x,r)) - \mathcal{H}^d(D)) \tag{1.5}$$

holds for every d-rectifiable set K such that $K \cap \overline{B(x,r)}$ is closed, $K \cap \partial B(x,r) = \partial D \times \{0\}$ and K cannot be deformed into $\partial D \times \{0\}$ via a map $\varphi \in \mathsf{D}(x,r)$.

Remark 1.4. Given a d-rectifiable set K and a deformation $\varphi \in \mathsf{D}(x,r)$, using property (1.1), we deduce the quasiminimality property

$$\mathbf{F}(\varphi(K)) \leq \Lambda \mathcal{H}^d(\varphi(K)) \leq \Lambda (\mathrm{Lip}\,(\varphi))^d \mathcal{H}^d(K) \leq \frac{\Lambda}{\lambda} (\mathrm{Lip}\,(\varphi))^d \mathbf{F}(K). \tag{1.6}$$

Moreover, whenever $U \subset\subset \mathbb{R}^n$, the following holds

$$\sup_{x,y \in U\ S,T \in G} |F(x,T) - F(y,S)| \leq \omega_U(|x-y| + \|T-S\|), \tag{1.7}$$

for some modulus of continuity ω_U for F in $G(U)$.

In [DPDRG16] the authors obtained an extension of Allard's rectifiability Theorem for stationary varifolds to anisotropic integrands. In order to obtain the validity of this theorem in the anisotropic setting, a necessary and sufficient condition on the Lagrangian has been identified in [DPDRG16].

Definition 1.5. For a given integrand $F \in C^1(\Omega \times G(n,d))$, $x \in \Omega$ and a Borel probability measure $\mu \in \mathcal{P}(G(n,d))$, let us define

$$A_x(\mu) := \int_{G(n,d)} B_F(x,T)\, d\mu(T) \in \mathbb{R}^n \otimes \mathbb{R}^n.$$

We say that F verifies the *atomic condition* (AC) at x if the following two conditions are satisfied:

(i) $\dim \ker A_x(\mu) \leq n-d$ for all $\mu \in \mathcal{P}(G(n,d))$,
(ii) if $\dim \ker A_x(\mu) = n-d$, then $\mu = \delta_{T_0}$ for some $T_0 \in G(n,d)$.

An immediate consequence of the main result in [DPDRG16] is the following theorem:

Theorem 1.6. *Let $F \in C^1(G(\mathbb{R}^n), \mathbb{R}^+)$ be a positive integrand satisfying the (AC) condition, and let us suppose that V is a d-dimensional varifold such that:*

- *V has bounded anisotropic first variation: $\delta_F V$ is a Radon measure.*
- *V has lower density bound: there exists $\theta_0 > 0$ such that*

$$\frac{\|V\|(B_r(x))}{r^d} = \frac{V(B_r(x) \times G(n,d))}{r^d} \geq \theta_0 \qquad \text{for all } x \in K \text{ and } r < dist(x, H).$$

Then V is d-rectifiable.

Remark 1.7. As pointed out in [DPDRG16] the AC condition is essentially necessary in order to obtain the validity of the above theorem. In [DPDRG16], it is shown that if $d = n-1$ (or if $d = 1$) this condition is equivalent to the the strict convexity of F and thus to (1.5).

In the general case $2 \leq d \leq n-2$, no implication between AC and (1.5) is currently known. Since our strategy of the proof heavily relies on Theorem 1.6, in our result we are forced to assume both (1.5) and the AC condition on F.

The following theorem is our main result and establishes the behaviour of minimizing sequences.

Theorem 1.8. *Let $F \in C^1(G(\mathbb{R}^n))$ be an integrand satisfying* (1.1), (1.5) *and the AC condition.*

Let $H \subset \mathbb{R}^n$ be closed and $\mathcal{P}(H)$ be a good class. Assume the infimum in Plateau problem (1.3) *is finite and let $(K_j) \subset \mathcal{P}(H)$ be a minimizing sequence. Then, up to subsequences, the measures $\mu_j := F(\cdot, T_{K_j}(\cdot))\mathcal{H}^d \llcorner K_j$ converge weakly* in $\mathbb{R}^n \setminus H$ to the measure $\mu = F(\cdot, T_K(\cdot))\mathcal{H}^d \llcorner K$, where $K = \operatorname{spt}\mu \setminus H$ is a d-rectifiable set. Furthermore, the integral varifold naturally associated to μ is F-stationary in $\mathbb{R}^n \setminus H$. In particular, $\liminf_j \mathbf{F}(K_j) \geq \mathbf{F}(K)$ and if $K \in \mathcal{P}(H)$, then K is a minimum for* (1.3).

Remark 1.9. We observe that in case the set K provided by the Theorem 1.8 belongs to $\mathcal{P}(H)$, it has minimal $\mathbf{F}$ energy with respect to deformations in the classes $\mathsf{D}(x,r)$ of Definition 1.1, with $x \in K$ and $H \cap B_r(x) = \emptyset$.

While the union of these classes is strictly contained in the class of all Lipschitz deformations, however such union is rich enough to generate the comparison sets in [Alm76] which are needed to prove the almost everywhere regularity of K, under the assumption of strict ellipticity in Definition 1.3, see [Alm76, III.1 and III.3].

We remark that, as in the previous works of the authors [DGM14, DPDRG15, DLDRG16], Theorem 1.8 can be applied to the two definitions of boundary conditions considered in [HP13] and in [Dav14, Dav13]. In this paper we extend this approach to the case of homological and cohomological boundary conditions in the spirit of the original paper by Reifenberg, see Section 3.

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2. Proof of Theorem 1.8

The proof of Theorem 1.8 has several points in common with the proofs in [DPDRG15] to which we will often refer, we assume the reader to be familiar with it.

Proof of Theorem 1.8. Since the infimum in Plateau problem (1.3) is finite, there exists a minimizing sequence $(K_j) \subset \mathcal{P}(H)$ and a Radon measure μ on $\mathbb{R}^n \setminus H$ such that

$$\mu_j \stackrel{*}{\rightharpoonup} \mu\,, \qquad \text{as Radon measures on } \mathbb{R}^n \setminus H\,, \tag{2.1}$$

where $\mu_j = F(\cdot, T_{K_j}(\cdot))\mathcal{H}^d \llcorner K_j$. We set $K = \operatorname{spt}\mu \setminus H$ and consider also the canonical density one rectifiable varifolds V^j associated to K_j:

$$V^j := \mathcal{H}^d \llcorner K_j \otimes \delta_{T_x K_j}.$$

Since K_j is a minimizing sequence in (1.3) and $F \geq \lambda$, we can assume to have the bound[1] $\|V^j\|(\mathbb{R}^n) \leq \frac{2m_0}{\lambda}$ and therefore we can assume that V^j converges to V in the sense of varifolds.

Step 1: V is F-stationary in $\mathbb{R}^n \setminus H$. Assume indeed the existence of $g \in C^1_c(\mathbb{R}^n \setminus H, \mathbb{R}^n)$ such that $\delta_F V(g) < 0$. By standard partition of unity argument for the compact set $\operatorname{supp}(g)$ in the open set $\mathbb{R}^n \setminus H$, we get the existence of a ball $B_{x,r} \subset\subset \mathbb{R}^n \setminus H$ and a vector field (not relabeled) $g \in C^1_c(B_{x,r}, \mathbb{R}^n)$ such that $\delta_F V(g) = -2c < 0$. For an arbitrarily small time $s > 0$, we have $(\mathrm{Id} + sg) \in \mathsf{D}(x, r)$. Moreover, there exists an open set $B_{x,r} \subset A \subset \mathbb{R}^n$, satisfying $\|(\mathrm{Id} + sg)^{\#}V\|(\partial A) = 0$. We consequently have

$$\mathbf{F}((\mathrm{Id} + sg)^{\#}V, A) \leq -cs + \mathbf{F}(V, A).$$

By lower semicontinuity and by the hypothesis on ∂A, for j large enough it holds true:

$$\mathbf{F}((\mathrm{Id} + sg)^{\#}V^j, A) - \frac{1}{j} \leq -cs + \mathbf{F}(V^j, A) + \frac{1}{j}.$$

Note that $\mathbf{F}((\mathrm{Id} + sg)^{\#}V^j, A) = \mathbf{F}((\mathrm{Id} + sg)(K_j), A)$ as well as $\mathbf{F}(V^j, A) = \mathbf{F}(K_j, A)$: adding to both members $\mathbf{F}(K_j, \mathbb{R}^n \setminus A)$ and noting that $(\mathrm{Id} + sg)(K_j) \setminus A = K_j \setminus A$, we obtain

$$\mathbf{F}((\mathrm{Id} + sg)(K_j), \mathbb{R}^n) \leq \frac{2}{j} - cs + \mathbf{F}(K_j, \mathbb{R}^n).$$

Since $(\mathrm{Id} + sg) \in \mathsf{D}(x, r)$ and $B_{x,r} \subset\subset \mathbb{R}^n \setminus H$, this is a contradiction with the minimizing property of the sequence (K_j) in $\mathcal{P}(H)$.

Step 2: V satisfies density lower bound. We claim that there exists $\theta_0 = \theta_0(, n, d, \lambda, \Lambda) > 0$ such that

$$\|V\|(B(x, r)) \geq \theta_0\, \omega_d r^d\,, \qquad x \in \operatorname{spt}\|V\| \text{ and } r < d_x := \operatorname{dist}(x, H). \tag{2.2}$$

This can be achieved by the same techniques of [DPDRG15, Theorem 1.3, Step 1]. Indeed by (1.1) the integrand F is comparable to the area and this is the only property needed in the proof in [DPDRG15], see also [De 16].

Step 3: V is rectifiable. Combining the lower bound (2.2) with the F-stationarity in $\mathbb{R}^n \setminus H$ and applying Theorem 1.6 we conclude that V is a d-rectifiable varifold and in turn, that $\mu = \mathbf{F}(V, \cdot) = \theta\mathcal{H}^d \llcorner \tilde{K}$ for some countably $\mathcal{H}^d$-rectifiable set $\tilde{K}$ and some positive Borel function θ. Since K is the support of μ, then $\mathcal{H}^d(\tilde{K} \setminus K) = 0$. On the other hand, by differentiation of Hausdorff measures, (2.2) yields $\mathcal{H}^d(K \setminus \tilde{K}) = 0$. Hence K is d-rectifiable and

$$\mu = \theta\mathcal{H}^d \llcorner K. \tag{2.3}$$

We now proceed to compute the exact value of the density θ: to this end we need the following elementary Lemma whose proof can be obtained as in [DLDRG16, Lemma 3.2].

Lemma 2.1. *Let K be the d-rectifiable set obtained in the previous section. For every x where K has an approximate tangent plane $T_K(x)$, let O_x be the special orthogonal transformation of $\mathbb{R}^n$ mapping $\{x_{d+1} = \cdots = x_n = 0\}$ onto $T_K(x)$ and set $\bar{Q}_{x,r} = O_x(Q_{x,r})$ and $\bar{R}_{x,r,\varepsilon r} =$*

[1] Here $\|V\|$ is the projection of the measure V on the first factor, i.e. $\|V\|(A) = V(A \times \mathbb{G})$ for every Borel set $A \subset \mathbb{R}^d$

$O_x(R_{x,r\varepsilon r})$. *At almost every* $x \in K$ *the following holds: for every* $\varepsilon > 0$ *there exist* $r_0 = r_0(x) \leq \frac{1}{\sqrt{n+1}} \operatorname{dist}(x, H)$ *such that, for* $r \leq r_0/2$,

$$(\theta_0\omega_d - \varepsilon)r^d \leq \mu(B(x,r)) \leq (\theta(x)\omega_d + \varepsilon)r^d, \qquad (\theta(x) - \varepsilon)r^d < \mu(\bar{Q}_{x,r}) < (\theta(x) + \varepsilon)r^d, \tag{2.4}$$

$$\sup_{y \in B_{x,r_0}, S \in G} |F(y,S) - F(x,S)| \leq \varepsilon, \tag{2.5}$$

where $\theta_0 = \theta_0(n,d)$ *is the universal lower bound obtained in* (2.2). *Moreover, for almost every such* r, *there exists* $j_0(r) \in \mathbb{N}$ *such that for every* $j \geq j_0$:

$$(\theta(x)\omega_d - \varepsilon)r^d \leq \mathbf{F}(K_j, B(x,r)) \leq (\theta(x)\omega_d + \varepsilon)r^d, \tag{2.6}$$

$$(\theta(x) - \varepsilon)r^d \leq \mathbf{F}(K_j, \bar{Q}_{x,r}) \leq (\theta(x) + \varepsilon)r^d, \qquad \mathbf{F}(K_j, \bar{Q}_{x,r} \setminus \bar{R}_{x,r,\varepsilon r}) < \varepsilon r^d. \tag{2.7}$$

We are now ready to complete the proof of Theorem 1.8, namely to show $\liminf_j \mathbf{F}(K_j) \geq \mathbf{F}(K)$ and moreover $\mu = F(x, T_K(x))\mathcal{H}^d \llcorner K$.

Step 4: $\theta(x) \geq F(x, T_K)$ *for almost every* $x \in K$. Let $x \in K$ satisfy the properties of Lemma 2.1. Assume w.l.o.g. $x = 0$ and let T_K be the tangent plane of K at 0. Let us fix $\varepsilon < r_0/2$ and choose a radius r such that both r and $(1-\sqrt{\varepsilon})r$ satisfy properties (2.4)-(2.7): in order to apply the ellipticity assumption in Definition 1.3 of F, we need to compare our set with $T_K \cap \partial B_r$. We reach this comparison with the help of a map $P \in \mathsf{D}(0,r)$ that squeezes a large portion of B_r onto T_K. Before doing this, we need to preliminary deform our competing sequence into another one, of approximately the same energy, whose associated measures are concentrated near T_K. In turn this, with the help of the density lower bound (2.2), can be achieved by applying a polyhedral deformation in B_{2r} outside the slab $R_{0,2r,\varepsilon r}$: for this construction we refer the reader to [DPDRG15, Theorem 1.3, Step 4]. Once we have ensured that, up to a deformation $\phi \in \mathsf{D}(0,2r)$,

$$\mathcal{H}^d(\phi(K_j) \cap B_{2r} \setminus R_{0,2r,\varepsilon r}) = 0 \tag{2.8}$$

(see equation (3.12) there) we can proceed to the squeezing deformation. With abuse of notation we will rename this new sequence $(\phi(K_j))$ with (K_j). Consider now a map S satisfying

- $S = Id$ in $\overline{B}_{1-\sqrt{\varepsilon}} \cup (\mathbb{R}^n \setminus \overline{B}_{1+\sqrt{\varepsilon}})$,
- $S(\partial B \cap U_\varepsilon(T_K)) = \partial B \cap T_K$,
- S stretches $\partial B \setminus U_\varepsilon(T_K)$ onto $\partial B \setminus T_K$

where $U_\varepsilon(T_K) = \{x : \operatorname{dist}(x, T_K) < \varepsilon\}$. It is not hard to construct an extension $S \in \mathsf{D}(0,1)$ fulfilling the previous requirements and such that $S|_{B_{1+\sqrt{\varepsilon}} \setminus B}$ and $S|_{B \setminus B_{1-\sqrt{\varepsilon}}}$ are interpolations between the values of S on the three spheres $S|_{\partial B_{1+\sqrt{\varepsilon}}}$, $S|_{\partial B}$ and $S|_{B \setminus B_{1-\sqrt{\varepsilon}}}$. Furthermore , we can also ensure that $\|S - Id\|_\infty + \operatorname{Lip}(S - Id) \leq C\sqrt{\varepsilon}$ and then obtain the desired map by rescaling $P(\cdot) = rS(\frac{\cdot}{r})$.

We set $K_j' := P(K_j)$ and since $K_j \cap B_{(1-\sqrt{\varepsilon})r} = K_j' \cap B_{(1-\sqrt{\varepsilon})r}$, using Remark 1.4 and property (2.6), we estimate:

$$\begin{aligned}
\mathbf{F}(K_j, B_r) &\geq \mathbf{F}\left(K_j', B_{(1-\sqrt{\varepsilon})r}\right) \geq \mathbf{F}\left(K_j', B_r\right) - \mathbf{F}\left(K_j', B_r \setminus B_{(1-\sqrt{\varepsilon})r}\right) \\
&\geq \mathbf{F}(K_j', B_r) - \frac{\Lambda}{\lambda}\operatorname{Lip}(P)^d \mathbf{F}\left(K_j, B_r \setminus B_{(1-\sqrt{\varepsilon})r}\right) \\
&= \mathbf{F}(K_j', B_r) - \frac{\Lambda}{\lambda}\operatorname{Lip}(P)^d \left(\mathbf{F}(K_j, B_r) - \mathbf{F}\left(K_j, B_{(1-\sqrt{\varepsilon})r}\right)\right) \\
&\geq \mathbf{F}(K_j', B_r) - \frac{\Lambda}{\lambda}\operatorname{Lip}(P)^d(\theta(0)\omega_d + \varepsilon)r^d + \frac{\Lambda}{\lambda}\operatorname{Lip}(P)^d(\theta(0)\omega_d - \varepsilon)(1-\sqrt{\varepsilon})^d r^d \\
&\geq \mathbf{F}(K_j', B_r) - C\sqrt{\varepsilon}r^d.
\end{aligned} \tag{2.9}$$

We furthermore observe that $K_j' \cap B_r$ cannot be deformed via any map $Q \in \mathsf{D}(0,r)$ onto $\partial B_r \cap T_K$. Otherwise, being $\mathcal{P}(H)$ a deformation class, there would exist a competitor $J_j \in \mathcal{P}(H)$, εr^d-close in energy to $K_j'' := Q(P(K_j))$, with $K_j'' \cap B_r = \emptyset$. Since $K_j'' \cap (\mathbb{R}^n \setminus B_{(1+\sqrt{\varepsilon})r}) = K_j \cap (\mathbb{R}^n \setminus B_{(1+\sqrt{\varepsilon})r})$, using (2.6) and equation (1.6), we would get:

$$\begin{aligned}
\mathbf{F}(K_j) - \mathbf{F}(J_j) &\geq \mathbf{F}\left(K_j, B_{(1+\sqrt{\varepsilon})r}\right) - \mathbf{F}\left(K_j'', B_{(1+\sqrt{\varepsilon})r}\right) - \varepsilon r^d \\
&\geq \mathbf{F}(K_j, B_r) + \mathbf{F}\left(K_j, B_{(1+\sqrt{\varepsilon})r} \setminus B_r\right) - \mathbf{F}\left(K_j', B_{(1+\sqrt{\varepsilon})r} \setminus B_r\right) - \varepsilon r^d \\
&\geq (\theta_0 \omega_d - \varepsilon) r^d + (\theta(0)\omega_d + \varepsilon)((1+\sqrt{\varepsilon})^d - 1)\left(1 - \frac{\Lambda}{\lambda} \mathrm{Lip}\,(P)^d\right) r^d - \varepsilon r^d \\
&\geq (\theta_0 \omega_d - C\sqrt{\varepsilon}) r^d > 0,
\end{aligned}$$

which contradicts the minimizing property of the sequence $\{K_j\}$ if ε is small enough.

In order to apply the ellipticity condition in Definition 1.3, we want to costruct another closed set $\tilde{K}_j$ such that

(i) $\tilde{K}_j \cap \partial B_r = \partial B_r \cap T_K$,
(ii) $\tilde{K}_j \subset \overline{B_r}$ cannot be deformed via any map $Q \in \mathsf{D}(0,r)$ onto $\partial B_r \cap T_K$,
(iii) $\mathbf{F}^0(\tilde{K}_j, B_r) = \mathbf{F}^0(K_j', B_r)$.

We can achieve this set in the following way

$$\tilde{K}_j := (\partial B_r \cap T_K) \cup (K_j' \cap \overline{B_r} \cap R_{0,r,\varepsilon r} \setminus \{|x_{||}| > r - |x_\perp|\}),$$

where $x_{||}$ and $x_\perp$ denote respectively the projections of x on T_K and its orthogonal. Indeed condition (i) is by construction, condition (ii) is a straightforward consequence of the fact that $K_j' \cap B_r$ cannot be deformed via any map $Q \in \mathsf{D}(0,r)$ onto $\partial B_r \cap T_K$. Condition (iii) follows by (2.8) and the properties of the map S, for ε small enough.

Therefore, the ellipticity of F, (2.5), (2.6), (1.6) and (2.2) imply that

$$\mathbf{F}^0(T_K, B_r) \leq \mathbf{F}^0(\tilde{K}_j, B_r) = \mathbf{F}^0(K_j', B_r) \leq \mathbf{F}(K_j', B_r) + C\varepsilon r^d. \tag{2.10}$$

We can now sum up as follows

$$\begin{aligned}
\theta(0)\omega_d r^d &\overset{(2.6)}{\geq} \mathbf{F}(K_j, B_r) - \varepsilon r^d \overset{(2.9)}{\geq} \mathbf{F}(K_j', B_r) - C\sqrt{\varepsilon} r^d \\
&\overset{(2.10)}{\geq} \mathbf{F}^0(T_K, B_r) - C\sqrt{\varepsilon} r^d = F(0, T_K)\omega_d r^d - C\sqrt{\varepsilon} r^d
\end{aligned}$$

which easily implies the desired inequality $\theta(0) \geq F(0, T_K)$.

Step 5: $\theta(x) \leq F(x, T_K(x))$ for almost every $x \in K$: Again we assume that $x = 0$ is a point satisfying the conclusion of Lemma 2.1 and we argue as in [DPDRG15, Theorem 1.3, Step 5].

Arguing by contradiction, we assume that $\theta(0) = F(0, T_K(0)) + \sigma$ for some $\sigma > 0$ and let $\varepsilon < \min\{\frac{\sigma}{2}, \frac{\lambda\sigma}{4\Lambda}\}$. As a consequence of (2.7), there exist r and $j_0 = j_0(r)$ such that

$$\mathbf{F}(K_j, Q_r) > \left(F(0,T) + \frac{\sigma}{2}\right) r^d, \qquad \mathbf{F}(K_j, Q_r \setminus R_{r,\varepsilon r}) < \frac{\lambda\sigma}{4\Lambda} r^d, \qquad \forall j \geq j_0\,. \tag{2.11}$$

Consider the map $P \in \mathsf{D}(0,r)$ defined in [DPDRG15, Equation 3.14] which collapses $R_{r(1-\sqrt{\varepsilon}),\varepsilon r}$ onto the tangent plane T_K and satisfies $\|P - Id\|_\infty + \mathrm{Lip}\,(P - Id) \leq C\sqrt{\varepsilon}$. Exploiting the fact that $\mathcal{P}(H)$ is a deformation class and by almost minimality of K_j, we find that

$$\begin{aligned}
\mathbf{F}(K_j, Q_r) - o_j(1) \leq{}& \underbrace{\mathbf{F}(P(K_j), P(R_{(1-\sqrt{\varepsilon})r,\varepsilon r}))}_{I_1} + \underbrace{\mathbf{F}(P(K_j), P(R_{r,\varepsilon r} \setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r}))}_{I_2} \\
&+ \underbrace{\mathbf{F}(P(K_j), P(Q_r \setminus R_{r,\varepsilon r}))}_{I_3}\,.
\end{aligned}$$

By the properties of P and (2.5), we get $I_1 \le (F(0,T_K)+\varepsilon)r^d$, while, by (2.11) and equation (1.6)

$$I_3 \le \frac{\Lambda}{\lambda}\,(\mathrm{Lip}\,P)^d\,\mathbf{F}(K_j, Q_r \setminus R_{r,\varepsilon r}) < (1+C\,\sqrt{\varepsilon})^d\,\frac{\sigma}{4}\,r^d\,.$$

Since $\mathbf{F}(P(K_j), P(R_{r,\varepsilon r}\setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r})) \le \frac{\Lambda}{\lambda}(1+C\sqrt{\varepsilon})^d\mathbf{F}(K_j, R_{r,\varepsilon r}\setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r})$ and $R_{r,\varepsilon r}\setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r} \subset Q_{(1-\sqrt{\varepsilon})r}\setminus Q_r$, by (2.7) we can also bound

$$\begin{aligned} I_2 = \mathbf{F}(P(K_j), P(R_{r,\varepsilon r}\setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r})) &\le \frac{\Lambda}{\lambda}(1+C\sqrt{\varepsilon})^d\mathbf{F}(K_j, Q_r\setminus Q_{(1-\sqrt{\varepsilon})r}) \\ &\le C(1+C\sqrt{\varepsilon})^d\Big((F(0,T_K)+\sigma+\varepsilon)-(F(0,T_K)+\sigma-\varepsilon)(1-\sqrt{\varepsilon})^d\Big)\,r^d \le C\sqrt{\varepsilon}r^d. \end{aligned}$$

Hence, as $j\to\infty$, by (2.4)

$$\Big(F(0,T_K)+\frac{\sigma}{2}\Big)r^d \le (F(0,T_K)+\varepsilon)r^d + C\sqrt{\varepsilon}r^d + (1+C\,\sqrt{\varepsilon})^d\,\frac{\sigma}{4}\,r^d\ :$$

dividing by r^d and letting $\varepsilon\downarrow 0$ provides the desired contradiction.

□

3. Solution to Reifenberg's formulation of Plateau problem

The original formulation of Plateau problem given by Reifenberg in [Rei60] involves an algebraic notion of boundary described in terms of Čech homology groups. The particular choice of an homology theory defined on compact spaces and with coefficient groups that are abelian and compact has three motivations: with these assumptions this homology is well behaved under Hausdorff convergence of compact sets, it satisfies the classical axioms of Eilenberg and Steenrod, enabling the use of the Mayer-Vietoris exact sequence [ES52, Chapter X] and, crucially, ensures that $\check{H}_\ell(K)=0$ if $\mathcal{H}^\ell(K)=0$ (a fact that is false for other homology theory, see [BM62]). In this section we follow Reifenberg's approach and show that we can obtain a minimizing set in the chosen homology class.

Let $\mathbb{G}$ be a compact Abelian group and let K be a closed set in $\mathbb{R}^n$. For every $m\ge 0$ we denote with $\check{H}_m(K;\mathbb{G})$ (often omitting the explicit mention of the group $\mathbb{G}$) the m^{th}-Čech homology group of K with coefficients in $\mathbb{G}$, [ES52, Chapter IX].

Recall that, if $H\subset K$ is a compact set, the inclusion map $i_{H,K}: H\to K$ induces a graded homomorphism between the homology groups (of every grade m, again often omitted)

$$i_{*\,H,K}: \check{H}_m(H,\mathbb{G}) \to \check{H}_m(K,\mathbb{G}).$$

(For any given continuous maps of compact spaces $f: X\to Y$, the induced homomorphisms f_* between homology groups are functorial). Note, in the next definition, the role of the dimension d, inherent of our variational problem.

Definition 3.1 (Boundary in the sense of Reifenberg)**.** Let $\mathbb{G}, H, K$ be as above and let $L\subset \check{H}_{d-1}(H,\mathbb{G})$ be a subgroup. We say that K has boundary L if

$$\mathrm{Ker}(i_{*\,H,K}) \supset L. \tag{3.1}$$

Definition 3.2 (Reifenberg class)**.** Given $\mathbb{G}$ a compact Abelian group and H a compact set, we let $\mathcal{R}(H)$ be the class of closed d-rectifiable subsets K of $\mathbb{R}^n\setminus H$ uniformly contained in a ball $B\Supset H$ and such that $K\cup H$ has boundary L in the sense of Definition 3.1.

Remark 3.3. We remark that $\mathcal{R}(H)$ is a good class in the sense of Definition 1.2. Indeed for every $K\in\mathcal{R}(H)$, every $x\in K$, $r\in(0,\mathrm{dist}(x,H))$ and $\varphi\in\mathsf{D}(x,r)$ (which is in particular continuous),

$$\varphi(K\cup H) = \varphi(K)\cup H,$$

and moreover by functoriality $\mathrm{Ker}(\varphi \circ i_{H,K\cup H})_* \supset \mathrm{Ker}(i_{*\,H,K\cup H}) \supset L$, which implies that $\varphi(K) \in \mathcal{R}(H)$. We can therefore apply Theorem 1.8 to the class $\mathcal{R}(H)$: we immediately obtain the existence of a relatively closed subset K of $\mathbb{R}^n \setminus H$ satisfying

$$\mathbf{F}(K) = \inf_{S\in\mathcal{R}(H)} \{\mathbf{F}(S)\}.$$

We address now the question whether K belongs to the Reifenberg class $\mathcal{R}(H)$. Recall the definition of Hausdorff distance between two compact sets C_1, C_2 of a metric space X:

$$d_{\mathcal{H}}(C_1, C_2) := \inf\{r > 0 : C_1 \subset U_r(C_2) \text{ and } C_2 \subset U_r(C_1)\}.$$

Theorem 3.4. *For every minimizing sequence $(K_j) \subset \mathcal{R}(H)$ the associated limit set given by Theorem 1.8 satisfies $K \in \mathcal{R}(H)$.*

The proof of the above Theorem will be obtained by constructing another minimizing sequence, $(K_j^1) \subset \mathcal{R}(H)$, yielding the same set K but with the further property that

$$d_{\mathcal{H}}(K_j^1 \cup H, K \cup H) \to 0. \tag{3.2}$$

The new sequence is obtained by the original one by first deforming the part of the sequence K_j outside a given ε neighbourhood $U_\varepsilon(K)$ onto a $(d-1)$ skeleton, and then by considering the intersection of the deformed sequence with $U_\varepsilon(K)$. By a simple applications of the Mayer-Vietoris sequence and since the deformed sequence has vanishing $\mathcal{H}^{d-1}$ measure in $\partial U_\varepsilon(K)$, we can show that the new sequence still satisfies the homological boundary conditions (here we crucially use that we are working with the Čech homology, see the discussion at the beginning of the Section).

Proof. Step 1: Construction of the new sequence. From Theorem 1.8 we know that $\mu_j := F(\cdot, T_{(\cdot)}K_j)\mathcal{H}^d \llcorner K_j$ converge weakly* in $\mathbb{R}^n \setminus H$ to the measure $\mu = F(\cdot, T_{(\cdot)}K)\mathcal{H}^d \llcorner K$. Then, for every $\varepsilon > 0$, there exists $j(\varepsilon)$ big enough so that for every $j \geq j(\varepsilon)$ we get

$$\mu_j(\mathbb{R}^n \setminus U_\varepsilon(K \cup H)) < \frac{\lambda\varepsilon^d}{k_1(4n)^d}, \tag{3.3}$$

where we denoted with $U_\varepsilon(K \cup H)$ the ε-tubular neighborhood of $K \cup H$, with Λ the constant in equation (1.1) and k_1 is the constant of the deformation [DPDRG15, Theorem 2.4].

We cover $U_{5\varepsilon}(K\cup H) \setminus U_{2\varepsilon}(K\cup H)$ with a complex Δ of closed cubes with side length equal to $\frac{\varepsilon}{4n}$ contained in $U_{6\varepsilon}(K \cup H) \setminus U_\varepsilon(K \cup H)$. We can apply an adaptation of the Deformation Theorem [DPDRG15, Theorem 2.4] relative to the set K_j and obtain a Lipschitz deformation $\varphi_j := \varphi_{\frac{\varepsilon}{4n},K_j}$. Observe that $\varphi(K_j) \cap (U_{4\varepsilon}(K \cup H) \setminus U_{3\varepsilon}(K \cup H)) \subset \Delta_d$ (the d-skeleton of the complex): we claim that

$$\varphi_j(K_j) \cap (U_{4\varepsilon}(K \cup H) \setminus U_{3\varepsilon}(K \cup H)) \subset \Delta_{d-1}. \tag{3.4}$$

Otherwise by [DPDRG15, Theorem 2.4] point (5), $\varphi_j(K_j) \cap (U_{4\varepsilon}(K\cup H) \setminus U_{3\varepsilon}(K\cup H))$ should contain an entire d-face of edge length $\frac{\varepsilon}{4n}$, leading together with (3.3) to a contradiction:

$$\begin{aligned}\frac{\varepsilon^d}{(4n)^d} &\leq \mathcal{H}^d(\varphi(K_j) \cap (U_{4\varepsilon}(K \cup H) \setminus U_{3\varepsilon}(K \cup H))) \leq k_1\mathcal{H}^d(K_j \setminus U_\varepsilon(K \cup H))\\ &\leq \frac{k_1}{\lambda}\mathbf{F}(K_j \setminus U_\varepsilon(K \cup H)) \leq \frac{k_1}{\lambda}\mu_j(\mathbb{R}^n \setminus U_\varepsilon(K \cup H)) < \frac{\varepsilon^d}{(4n)^d}.\end{aligned}$$

Set $\widetilde{K_j} := \varphi_j(K_j)$: by (3.4) and the coarea formula [Fed69, 3.2.22(3)], there exists $\alpha \in (3, 4)$ such that

$$\mathcal{H}^{d-1}(\widetilde{K_j} \cap \partial U_{\alpha\varepsilon}(K \cup H)) = 0. \tag{3.5}$$

We let

$$K_j^1 := \widetilde{K_j} \cap \overline{U_{\alpha\varepsilon}(K \cup H)} \quad \text{and} \quad K_j^2 := \widetilde{K_j} \setminus U_{\alpha\varepsilon}(K \cup H). \tag{3.6}$$

Step 2: proof of the property (3.2). Recall that by construction,

$$\forall \varepsilon > 0 \quad K_j^1 \cup H \subset U_{4\varepsilon}(K \cup H). \tag{3.7}$$

If on the other hand there were $x \in K \cup H \setminus U_\varepsilon(K^1_{j(h)} \cup H)$ for some subsequence $j(h)$, then necessarily $d(x, H) \geq \varepsilon$ as well as $d(x, K^1_{j(h)}) \geq \varepsilon$: the weak convergence $\mu_{j(h)} \overset{*}{\rightharpoonup} \mu$ would then fail the uniform density lower bounds (2.2) on $B(x, \varepsilon/2)$. This implies (3.2).

Step 3: boundary constraint of the new sequence. To conclude the proof of Theorem 3.4, we need to check that $(K_j^1) \subset \mathcal{R}(H)$. By (3.6) we get

$$\widetilde{K_j} = K_j^1 \cup K_j^2, \qquad \text{and} \qquad K_j^1 \cap K_j^2 = \widetilde{K_j} \cap \partial U_{\alpha\varepsilon}(K)$$

and (3.5),(3.6) yield

$$\mathcal{H}^{d-1}((K_j^1 \cup H) \cap K_j^2) = \mathcal{H}^{d-1}(K_j^1 \cap K_j^2) = 0.$$

Therefore by [HW41, Theorem VIII 3']:

$$\check{H}_{d-1}((K_j^1 \cup H) \cap K_j^2) = (0). \tag{3.8}$$

We furthermore observe that the sets $\widetilde{K_j}$ are obtained as deformations via Lipschitz maps strongly approximable via isotopies, and therefore belong to $\mathcal{R}(H)$. Since the map φ_j coincides with the identity on H, we have

$$i_{H, \widetilde{K_j} \cup H} = \varphi_j \circ i_{H, K_j \cup H};$$

moreover, trivially $i_{H, \widetilde{K_j} \cup H} = i_{K_j^1 \cup H, \widetilde{K_j} \cup H} \circ i_{H, K_j^1 \cup H}$. Hence by functoriality

$$\mathrm{Ker}(i_{*\, K_j^1 \cup H, \widetilde{K_j} \cup H} \circ i_{*\, H, K_j^1 \cup H}) = \mathrm{Ker}(i_{*\, H, \widetilde{K_j} \cup H}) = \mathrm{Ker}((\varphi_j)_* \circ i_{*\, H, K_j \cup H}) \supset L.$$

We claim that $i_{*\, K_j^1 \cup H, \widetilde{K_j} \cup H}$ is injective: this implies that

$$\mathrm{Ker}(i_{*\, H, K_j^1 \cup H}) \supset L, \tag{3.9}$$

namely $(K_j^1) \subset \mathcal{R}(H)$.

Step 4: the map $i_{*\, K_j^1 \cup H, \widetilde{K_j} \cup H}$ is injective. We can write the Mayer-Vietoris sequence (which for the Čech homology holds true for compact spaces and with coefficients in a compact group, due to the necessity of having the excision axiom, [ES52, Theorem 7.6 p.248]) and use (3.8):

$$(0) \overset{(3.8)}{=} \check{H}_{d-1}((K_j^1 \cup H) \cap K_j^2) \xrightarrow{f} \check{H}_{d-1}(K_j^1 \cup H) \oplus \check{H}_{d-1}(K_j^2) \xrightarrow{g} \check{H}_{d-1}(\widetilde{K_j} \cup H) \longrightarrow ...$$

where $f = (i_{*\, (K_j^1 \cup H) \cap K_j^2, K_j^1 \cup H}, i_{*\, (K_j^1 \cup H) \cap K_j^2, K_j^2})$ and $g(\sigma, \tau) = \sigma - \tau$. The exactness of the sequence implies that g is injective: in particular the map g is injective when restricted to the subgroup $\check{H}_{d-1}(K_j^1 \cup H) \oplus (0)$, where it coincides with $i_{*\, K_j^1 \cup H, \widetilde{K_j} \cup H}$. This concludes the proof of Step 4.

Step 5: boundary constraint for the limit set. Setting

$$Y_n := \overline{\bigcup_{j \geq n} K_j^1 \cup H},$$

by (3.2) we get

$$d_{\mathcal{H}}(Y_n, K \cup H) \to 0. \tag{3.10}$$

Therefore $K \cup H$ is the inverse limit of the sequence Y_n. Since the sets $(K_j^1 \cup H)$ are in the Reifenberg class $\mathcal{R}(H)$, namely the inclusion (3.9) holds, by composing the two injections $i_{*\, K_n^1 \cup H, Y_n}$ and $i_{*\, H, K_n^1 \cup H}$ we obtain that

$$L \subset \mathrm{Ker}(i_{*\, H, Y_n}).$$

Since the Čech homology with coefficients in compact groups is continuous [ES52, Definition 2.3], the latter inclusion is stable under Hausdorff convergence, see [ES52, Theorem 3.1] (see also [Rei60, Lemma 21A]): therefore, by (3.10), we conclude

$$L \subset \mathrm{Ker}(i_{*\,H,K\cup H}),$$

and eventually $K \in \mathcal{R}(H)$. □

Remark 3.5. Using the contravariance of cohomology theory, the same results can be obtained when considering a cohomological definition of boundary, again in the Čech theory, as introduced in [HP16]. In particular a new proof the the theorem there can be obtained with our assumption on the Lagrangian.

Note that in the cohomological definition of boundary all the Eilenberg-Steenrod axioms are satisfied even with a non-compact group $\mathbb{G}$. This allows us to consider as coefficients set the natural group $\mathbb{Z}$.

Remark 3.6. We observe that any minimizer K as in Theorem 3.4 is also an $(\mathbf{F}, 0, \infty)$ minimal set in the sense of [Alm76, Definition III.1]. Indeed the boundary condition introduced in Definition 3.1 is preserved under Lipschitz maps (not necessarily in $\mathsf{D}(x,r)$). In particular, by [Alm76, Theorem III.3(7)], if F is smooth and strictly elliptic (Γ in Definition 1.3 is strictly positive), then K is smooth away from the boundary, outside of a relative closed set of $\mathcal{H}^d$-measure zero (the theorem gives actually $C^{1,\alpha}$ almost everywhere regularity for all $\alpha < 1/2$ if $F \in C^3$ and elliptic).

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