On a class of conserved phase field systems with a maximal monotone perturbation

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Abstract

We prove existence and regularity for the solutions to a Cahn–Hilliard system describing the phenomenon of phase separation for a material contained in a bounded and regular domain. Since the first equation of the system is perturbed by the presence of an additional maximal monotone operator, we show our results using suitable regularization of the nonlinearities of the problem and performing some a priori estimates which allow us to pass to the limit thanks to compactness and monotonicity arguments. Next, under further assumptions, we deduce a continuous dependence estimate whence the uniqueness property is also achieved. Then, we consider the related sliding mode control (SMC) problem and show that the chosen SMC law forces a suitable linear combination of the temperature and the phase to reach a given (space-dependent) value within finite time.

Key words: Cahn–Hilliard system, phase separation, initial-boundary value problem, existence, continuous dependence, sliding mode control.

AMS (MOS) subject classification: 35K61, 35K25, 35D30, 34H05, 80A22.

1 Introduction

The Cahn–Hilliard equation, originally introduced in [6] and first studied mathematically in the seminal paper [22], yields a description of the evolution phenomenon of the solid–solid phase separation. In general, an evolution process goes on diffusively. However, the phenomenon of the solid–solid phase separation does not seem to follow this structure: e.g., when a binary alloy is cooled down sufficiently, each phase concentrates and the material quickly becomes inhomogeneous, forming a fine-grained structure in which each of the two components appears more or less alternatively (see, e.g., [32]). The Cahn–Hilliard equation is a celebrated model which describes this process (usually known as spinodal decomposition) by the simple framework of partial differential equations. The mathematical literature concerning this problem is rather vast. Let us quote [7, 11, 15, 24, 26, 33, 36, 37, 42] and also refer to [10] in which a forced mass constraint on the boundary is considered.

In the present contribution, we consider the following Cahn–Hilliard system perturbed by the presence of an additional maximal monotone nonlinearity:

$$\partial_t(\vartheta + \ell\varphi) - \Delta\vartheta + \zeta = f \quad \text{a.e. in } Q := \Omega \times (0,T), \tag{1.1}$$

$$\partial_t\varphi - \Delta\mu = 0 \quad \text{a.e. in } Q, \tag{1.2}$$

$$\mu = -\nu\Delta\varphi + \xi + \pi(\varphi) - \gamma\vartheta \quad \text{a.e. in } Q, \tag{1.3}$$

$$\zeta(t) \in A(a\vartheta(t) + b\varphi(t) - \eta^*) \quad \text{for a.e. } t \in (0,T), \tag{1.4}$$

$$\xi \in \beta(\varphi) \quad \text{a.e. in } Q, \tag{1.5}$$

where $\Omega \subseteq \mathbb{R}^3$ is an open, bounded, connected subset of class C^1, T is some final time, ϑ, φ and μ denote the temperature, the order parameter and the chemical potential, respectively. We point out that here ϑ does not represent the absolute temperature, but it is related to it by

$$\vartheta = \Theta - \Theta_c, \tag{1.6}$$

where Θ_c denotes a critical temperature. Moreover, η^* is a function in $H^2(\Omega)$ with null outward normal derivative on the boundary of Ω, f is a source term and a, b, ℓ, γ are constants. In particular, let ℓ and γ be positive. The above system is complemented by homogeneous Neumann boundary conditions for both ϑ and φ, that is,

$$\partial_{\mathbf{n}}\vartheta = \partial_{\mathbf{n}}\varphi = \partial_{\mathbf{n}}\mu = 0 \quad \text{on } \Sigma := \Gamma \times (0,T), \tag{1.7}$$

where Γ is the boundary of Ω and $\partial_{\mathbf{n}}$ is the outward normal derivative, and by the initial conditions

$$\vartheta(0) = \vartheta_0, \qquad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{1.8}$$

The term $\xi + \pi(\varphi)$, appearing in (1.3), represents the derivative of the potential $\mathcal{W}$ associated with the phase configuration. In the literature (see, e.g., [14, 23]), $\mathcal{W}$ is frequently assumed to be a double-well potential. More generally, $\mathcal{W}$ can be defined as the sum $\mathcal{W} = \widehat{\beta} + \widehat{\pi}$, where $\widehat{\beta} : \mathbb{R} \to [0, +\infty]$ is a proper, l.s.c. and convex function and $\widehat{\pi} : \mathbb{R} \to \mathbb{R}$ is a function in $C^1(\mathbb{R})$ such that $\pi := \widehat{\pi}'$ is Lipschitz continuous. Due to the properties of $\widehat{\beta}$, the subdifferential $\partial\widehat{\beta} =: \beta$ is well defined and is a maximal monotone graph. In our problem a maximal monotone operator $A : \ H := L^2(\Omega) \to 2^H$ also appears. We assume that $0 \in A(0)$ and $\|v\|_H \le C(1 + \|x\|_H)$ for all $x \in H$, $v \in Ax$, for some constant $C > 0$. For a comprehensive presentation of the theory of maximal monotone operators, we refer, e.g., to [1, 3, 38].

Let us spend some words about the thermodynamic derivation of the system (1.1)–(1.5). We move from the following expression for the local free energy:

$$\Psi(\Theta, \varphi) = c_0\Theta(1 - \ln\Theta) - \gamma_0(\Theta - \Theta_c)\varphi + \Theta\Big(- \mu\varphi + \mathcal{W}(\varphi) + \frac{\nu}{2}|\nabla\varphi|^2\Big), \tag{1.9}$$

where Θ denotes the absolute temperature, as in (1.6). The quantity

$$e = \vartheta + \ell\varphi \tag{1.10}$$

appearing (under time derivative) in (1.1) represents a rescaled internal energy of the system, since a standard thermodynamic relation postulates that

$$\mathcal{U} = \Psi + \Theta\mathcal{S}, \tag{1.11}$$

where $\mathcal{U}$ is the internal energy and $\mathcal{S} := -\frac{\partial\Psi}{\partial\Theta}$ denotes the entropy. Then, by an easy computation, we find out that

$$\begin{aligned}\mathcal{U} = \; & c_0\Theta(1-\ln\Theta) - \gamma_0(\Theta-\Theta_c)\varphi + \Theta\Big(-\mu\varphi + \mathcal{W}(\varphi) + \frac{\nu}{2}|\nabla\varphi|^2\Big) \\ & - \Theta\Big(c_0(1-\ln\Theta) - c_0 - \gamma_0\varphi - \mu\varphi + \mathcal{W}(\varphi) + \frac{\nu}{2}|\nabla\varphi|^2\Big) = c_0\Theta + \gamma_0\Theta_c\varphi, \end{aligned} \tag{1.12}$$

so that, by adding the constant $-c_0\Theta_c$ and dividing by c_0, from (1.6) we exactly have that

$$e = \vartheta + \ell\varphi, \qquad \text{with} \qquad \ell = \frac{\gamma_0\Theta_c}{c_0}. \tag{1.13}$$

On the other hand, by considering the total free energy

$$\mathcal{F}(\Theta,\varphi) = \int_\Omega \Psi(\Theta,\varphi) \tag{1.14}$$

and taking the variational derivative $\frac{\delta\mathcal{F}}{\delta\varphi}$, we actually recover the phase equation (1.3). Indeed, by a standard computation, we infer that

$$-\gamma_0(\Theta-\Theta_c)\varphi + \Theta\big(-\mu + \mathcal{W}'(\varphi) - \nu\Delta\varphi\big) - \nu\nabla\Theta\cdot\nabla\varphi = 0. \tag{1.15}$$

Dividing by Θ, observing that (cf. (1.6))

$$-\gamma_0\Big(1-\frac{\Theta_c}{\Theta}\Big) \cong \frac{\gamma_0}{\Theta_c}\vartheta, \tag{1.16}$$

thanks to a first order linearization, and neglecting the higher order term $-\frac{\nu}{\Theta}\nabla\Theta\cdot\nabla\varphi$, we finally obtain (1.3) with $\gamma \cong \frac{\gamma_0}{\Theta_c}$.

As usual for Cahn–Hilliard system, in the Problem (P) stated by (1.1)–(1.8) the integral mean value of $\varphi(t)$ remains constant during the whole evolution. Indeed, fixing an arbitrary $t\in(0,T)$ and integrating (1.2) over Ω, we infer that

$$\frac{d}{dt}\int_\Omega \varphi(t) = 0, \tag{1.17}$$

whence it immediately follows that

$$m(\varphi(t)) := \frac{1}{|\Omega|}\int_\Omega \varphi(t) = \frac{1}{|\Omega|}\int_\Omega \varphi_0 \qquad \text{for every } t\in(0,T). \tag{1.18}$$

We also observe that system (P) is a fourth-order problem constructed as the conserved version of the following phase-field system:

$$\partial_t(\vartheta+\ell\varphi) - k\Delta\vartheta + \zeta = f \quad \text{a.e. in } Q, \tag{1.19}$$

$$\partial_t\varphi - \nu\Delta\varphi + \xi + \pi(\varphi) = \gamma\vartheta \quad \text{a.e. in } Q, \tag{1.20}$$

$$\zeta(t) \in A(\vartheta(t) + \alpha\varphi(t) - \eta^*) \text{ for a.e. } t\in(0,T), \tag{1.21}$$

$$\xi \in \beta(\varphi) \text{ a.e. in } Q, \tag{1.22}$$

$$\partial_{\mathbf{n}}\vartheta = 0, \qquad \partial_{\mathbf{n}}\varphi = 0 \quad \text{on } \Sigma, \tag{1.23}$$

$$\vartheta(0) = \vartheta_0, \qquad \varphi(0) = \varphi_0 \quad \text{in } \Omega, \tag{1.24}$$

where k and α are positive coefficients. Phase-field systems have been widely studied in the literature. We refer, without any sake of completeness, e.g., to [4,5,19,23,25,31,32,34,40] and references therein for the well-posedness and long-time behavior results. In particular, the above system (1.19)–(1.24) has been thoroughly discussed in [21], where existence and regularity of the solutions is proved and, under further assumptions, uniqueness and continuous dependence on the initial data are deduced.

In the paper, we first show the existence of solutions for Problem (P) (see (1.1)–(1.8)). In order to carry out this purpose, we consider the approximating problem (P_ε), obtained from (P) by approximating A and β by their Yosida regularizations. In performing our uniform estimates we often refer to [12], where the authors propose the study of a nonlinear diffusion problem as an asymptotic limit of a particular Cahn–Hilliard system. Then, we pass to the limit as $\varepsilon \searrow 0$ and show that some limit of a subsequence of solutions for (P_ε) yields a solution of (P). Next, we let $a\ell = b$ which is, in some sense, a physical restriction since the argument of the variable in the operator A is thus proportional to the internal energy of the system. We also write Problem (P) for two different sets of data f_i, η_i^*, ϑ_{0_i} and φ_{0_i}, $i = 1, 2$. By suitably performing contracting estimates for the difference of the corresponding solutions, we deduce the continuous dependence result whence the uniqueness property is also achieved.

The second part of the paper is devoted to the sliding mode control (SMC) problem. Hence, the main idea behind this scheme is first to identify a manifold of lower dimension (called the sliding manifold) where the control goal is fulfilled and such that the original system restricted to this sliding manifold has a desired behavior, and then to act on the system through the control in order to constrain the evolution on it, that is, to design a SMC-law that forces the trajectories of the system to reach the sliding surface and maintains them on it (see, e.g., [30, 35]). The main advantage of sliding mode control is that it allows the separation of the motion of the overall system in independent partial components of lower dimensions, and consequently it reduces the complexity of the control problem. In particular, we prove the existence of sliding modes for the solutions of our system (P) for a suitable choice of the operator A and the coefficients a and b. We take $a = 1$, $b = \ell$ and $A = \rho\,\mathrm{Sign}$, where ρ is a positive coefficient and $\mathrm{Sign} : H \longrightarrow 2^H$ is a maximal monotone operator defined as $\mathrm{Sign}(v) = \frac{v}{\|v\|}$, if $v \neq 0$ and $\mathrm{Sign}(0) = B_1(0)$, if $v = 0$ (here, $B_1(0)$ denotes the closed unit ball of H). Thus we prescribe a state-feedback control law acting on the rescaled internal energy $(\vartheta + \ell\varphi)$ of the system in order that the dynamics of the system modified in this way forces the value $(\vartheta(t) + \ell\varphi(t))$ to reach a manifold of the phase space in a finite time and then lie there with a sliding mode (cf. [2, 18]).

Concerning the study of optimal control problems for phase-field systems, we quote [13, 14, 20, 29]. Recent investigations have been also addressed to the optimal control problem for Cahn-Hilliard systems: let us mention [8,9,15–17,27]. We also refer to [43,44] which deals with the convective Cahn–Hilliard equation, and to [28, 41], where some discretized versions of the general Cahn–Hilliard systems are studied.

In the present contribution, assuming $a = 1$, $b = \ell$ and $A = \rho\,\mathrm{Sign}$ in (1.1)–(1.8), we prove the existence of sliding modes for Problem (P) by identifying $\rho^* > 0$ such that the following property is fulfilled: for every $\rho > \rho^*$, there exists a solution $(\vartheta, \varphi, \mu)$ to Problem (P) and a time T^* such that, for every $t \in [T^*, T]$

$$\vartheta(t) + \ell\varphi(t) = \eta^* \qquad \text{a.e. in } \Omega. \tag{1.25}$$

It is curious and interesting that we are able to handle a feedback law and prove the mentioned property just for the internal energy of the system, which is a special linear combination of the variables ϑ and φ. However, for a discussion of the SMC laws, linear and nonlinear, that can be considered for phase field systems, we refer to the Introduction of [2].

The paper is organized as follows. In Section 1, we list our assumptions, state the problem in a precise form and present our results. The next sections are devoted to the corresponding proofs: Section 3–6 deal with existence and regularity, while uniqueness and continuous dependence are proved in Section 7. In Section 8, we show the existence of sliding modes.

2 Main results

In this section, we state the main results.

2.1 Preliminary assumptions

We assume $\Omega \subseteq \mathbb{R}^3$ to be open, bounded, connected, of class C^1 and we write $|\Omega|$ for its Lebesgue measure. Moreover, Γ and $\partial_{\mathbf{n}}$ still stand for the boundary of Ω and the outward normal derivative, respectively. Given a finite final time $T > 0$, for every $t \in (0, T]$ we set

$$Q_t = (0, t) \times \Omega, \quad Q = Q_T, \tag{2.1}$$

$$\Sigma_t = (0, t) \times \Gamma, \quad \Sigma = \Sigma_T. \tag{2.2}$$

In the following, we set for brevity:

$$H = L^2(\Omega), \quad V = H^1(\Omega), \quad V_0 = H^1_0(\Omega), \quad W = \{u \in H^2(\Omega) : \ \partial_{\mathbf{n}} u = 0 \text{ on } \partial\Omega\}, \tag{2.3}$$

with usual norms $\|\cdot\|_H$, $\|\cdot\|_V$ and inner products $(\cdot,\cdot)_H$, $(\cdot,\cdot)_V$, respectively. The symbol V^* denotes the dual space of V while the pair $\langle\cdot,\cdot\rangle_{V^*,V}$ represents the duality pairing between V^* and V. Moreover, we identify H with its dual space.

2.2 Operators

In this subsection we describe the operators appearing in the problem under study.

The operator m. We consider the operator $m : V^* \to \mathbb{R}$ defined by

$$m(z^*) := \frac{1}{|\Omega|}\langle z^*, 1\rangle_{V^*,V} \quad \text{for all } z^* \in V^*. \tag{2.4}$$

We observe that, if $z^* \in H$, then

$$m(z^*) = \frac{1}{|\Omega|} \int_\Omega z^* dx. \tag{2.5}$$

The double-well potential $\mathcal{W}$. We introduce the double-well potential $\mathcal{W}$ as the sum

$$\mathcal{W} = \widehat{\beta} + \widehat{\pi}, \tag{2.6}$$

where

$$\widehat{\beta} : \mathbb{R} \longrightarrow [0, +\infty] \text{ is proper, l.s.c. and convex with } \widehat{\beta}(0) = 0, \tag{2.7}$$

$$\widehat{\pi} : \mathbb{R} \to \mathbb{R},\ \widehat{\pi} \in C^1(\mathbb{R}) \text{ with } \pi := \widehat{\pi}' \text{ Lipschitz continuous.} \tag{2.8}$$

Since $\widehat{\beta}$ is proper, l.s.c. and convex, the subdifferential $\beta := \partial\widehat{\beta}$ is well defined. We denote by $D(\beta)$ and $D(\widehat{\beta})$ the effective domains of β and $\widehat{\beta}$, respectively, and also assume that $\operatorname{int}(D(\beta)) \neq \emptyset$. Thanks to these assumptions, β is a maximal monotone graph. Moreover, as $\widehat{\beta}$ takes its minimum in 0, we have that $0 \in \beta(0)$.

The operator $\mathcal{B}$. We introduce the operator $\mathcal{B}$ induced by β on $L^2(Q)$ in the following way:

$$\mathcal{B} : L^2(Q) \longrightarrow L^2(Q) \tag{2.9}$$

$$\xi \in \mathcal{B}(\varphi) \Longleftrightarrow \xi(x,t) \in \beta(\varphi(x,t)) \quad \text{for a.e. } (x,t) \in Q. \tag{2.10}$$

We notice that

$$\beta = \partial\widehat{\beta}, \qquad \mathcal{B} = \partial\Phi, \tag{2.11}$$

where

$$\Phi : \ L^2(Q) \longrightarrow (-\infty, +\infty] \tag{2.12}$$

$$\Phi(u) = \begin{cases} \int_Q \widehat{\beta}(u) & \text{if } u \in L^2(Q) \text{ and } \widehat{\beta}(u) \in L^1(Q), \\ +\infty & \text{elsewhere, with } u \in L^2(Q). \end{cases} \tag{2.13}$$

The operator A. We consider the maximal monotone operator

$$A : H \longrightarrow H. \tag{2.14}$$

We assume that

$$0 \in A(0) \tag{2.15}$$

and that there exists a constant $C_A > 0$ such that

$$\|v\|_H \leq C_A(1 + \|\eta\|_H) \quad \text{for every } \eta \in H,\ v \in A\eta. \tag{2.16}$$

The operator $\mathcal{A}$. We introduce the operator $\mathcal{A}$ induced by A on $L^2(0,T;H)$ in the following way

$$\mathcal{A} : L^2(0,T;H) \longrightarrow L^2(0,T;H) \tag{2.17}$$

$$\zeta \in \mathcal{A}(\eta) \Longleftrightarrow \zeta(t) \in A(\eta(t)) \quad \text{for a.e. } t \in (0,T). \tag{2.18}$$

We notice that also $\mathcal{A}$ is a maximal monotone operator.

The operator Sign. An example of maximal monotone operator A which satisfies (2.14)–(2.16) is the operator

$$\mathrm{Sign}: \qquad H \longrightarrow 2^H \tag{2.19}$$

$$\mathrm{Sign}(v) = \begin{cases} \frac{v}{\|v\|} & \text{if } v \neq 0, \\ B_1(0) & \text{if } v = 0, \end{cases} \tag{2.20}$$

where $B_1(0)$ is the closed unit ball of H. Sign is the subdifferential of the map $\|\cdot\| : H \to \mathbb{R}$ and is a maximal monotone operator on H which satisfies (2.15)–(2.16).

The operator $\mathcal{N}$. We also consider the operator

$$\mathcal{N} : D(\mathcal{N}) \subseteq V^* \to V, \tag{2.21}$$

defined on its domain

$$D(\mathcal{N}) := \{w \in V^* : m(w^*) = 0\}. \tag{2.22}$$

For every $w^* \in D(\mathcal{N})$, we define $w = \mathcal{N}w^*$ if $w \in V$, $m(w) = 0$ and w is a solution of the following variational equation

$$\int_\Omega \nabla w \cdot \nabla z dx = \langle w^*, z\rangle_{V^*,V} \quad \text{for all } z \in V. \tag{2.23}$$

If $w^* \in D(\mathcal{N}) \cap H$, then w is the unique solution to the elliptic problem

$$\begin{cases} -\Delta w = w^* & \text{a.e. in } \Omega, \\ \partial_\nu w = 0 & \text{a.e. in } \Gamma, \\ m(w) = 0. \end{cases} \tag{2.24}$$

We observe that, due to elliptic regularity, $w \in W$. Moreover, for every $v^*, w^* \in D(\mathcal{N})$, $v = \mathcal{N}v^*$ and $w = \mathcal{N}w^*$ we have that

$$\begin{aligned} \langle w^*, \mathcal{N}v^*\rangle_{V^*,V} = \langle w^*, v\rangle_{V^*,V} &= \int_\Omega \nabla w \cdot \nabla v dx \\ &= \langle v^*, w\rangle_{V^*,V} = \langle v^*, \mathcal{N}w^*\rangle_{V^*,V}. \end{aligned}$$

Consequently, by defining

$$\|w^*\|^2_{V^*} := \big\|\nabla\mathcal{N}\big(w^* - m(w^*)\big)\big\|^2_{H^3} + \big|m(w^*)\big|^2 \quad \text{for all } w^* \in V^*, \tag{2.25}$$

it turns out that $\|\cdot\|_{V^*}$ is a norm in V^*.

2.3 Setting of the problem and results

Now, we describe the state system. We assume

$$\ell,\ \nu,\ \gamma \in (0, +\infty), \qquad a,\ b \in \mathbb{R}, \tag{2.26}$$

$$f \in L^2(0, T, H), \tag{2.27}$$

$$\eta^* \in W, \quad \vartheta_0 \in H, \quad \varphi_0 \in V, \quad \widehat{\beta}(\varphi_0) \in L^1(\Omega), \quad m(\varphi_0) =: m_0 \in \operatorname{int}(D(\beta)). \tag{2.28}$$

We look for a triplet $(\vartheta, \varphi, \mu)$ satisfying at least the regularity requirements

$$\vartheta \in H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V), \tag{2.29}$$

$$\varphi \in H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{2.30}$$

$$\mu \in L^2(0,T;V), \tag{2.31}$$

and solving the Problem (P), that is,

$$\partial_t(\vartheta + \ell\varphi) - \Delta\vartheta + \zeta = f \quad \text{a.e. in } Q, \tag{2.32}$$

$$\partial_t\varphi - \Delta\mu = 0 \quad \text{a.e. in } Q, \tag{2.33}$$

$$\mu = -\nu\Delta\varphi + \xi + \pi(\varphi) - \gamma\vartheta \quad \text{a.e. in } Q, \tag{2.34}$$

$$\zeta(t) \in A(a\vartheta(t) + b\varphi(t) - \eta^*) \text{ for a.e. } t \in (0,T), \tag{2.35}$$

$$\xi \in \beta(\varphi) \text{ a.e. in } Q, \tag{2.36}$$

$$\partial_{\mathbf{n}}\vartheta = \partial_{\mathbf{n}}\varphi = \partial_{\mathbf{n}}\mu = 0 \quad \text{on } \Sigma, \tag{2.37}$$

$$\vartheta(0) = \vartheta_0, \qquad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{2.38}$$

Theorem 2.1 (Existence) *Assume* (2.7)–(2.8), (2.14)–(2.16) *and* (2.26)–(2.28). *Then Problem* (P) *(see* (2.32)–(2.38)*) has at least one solution* $(\vartheta, \varphi, \mu)$ *satisfying* (2.29)–(2.31).

Theorem 2.2 (Regularity) *Assume* (2.7)–(2.8), (2.14)–(2.16), (2.26)–(2.27),

$$\eta^* \in W, \quad \vartheta_0 \in V, \quad \varphi_0 \in W, \quad \beta^0(\varphi_0) \in H, \quad m_0 \in \operatorname{int}(D(\beta)) \tag{2.39}$$

and that there exists $\varepsilon_0 \in (0,1]$ *such that*

$$\| - \nu\Delta\varphi_0 + \beta_\varepsilon(\varphi_0) + \pi(\varphi_0) - \gamma\vartheta_0\|_V \le c \qquad \textit{for every } \varepsilon \in (0, \varepsilon_0], \tag{2.40}$$

for some positive constant c*, where* β_ε *is the Yosida regularization of* β *(see* (3.9)*). Then Problem* (P) *(see* (2.32)–(2.38)*) has at least one solution* $(\vartheta, \varphi, \mu)$ *satisfying*

$$\vartheta \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{2.41}$$

$$\varphi \in W^{1,\infty}(0,T;V^*) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{2.42}$$

$$\mu \in L^\infty(0,T;V) \cap L^2(0,T;W). \tag{2.43}$$

Remark. We fix $t \in (0,T)$ and integrate (2.33) over Ω. We infer that

$$\int_\Omega \partial_t\varphi(t) - \int_\Omega \Delta\mu(t) = 0. \tag{2.44}$$

Integrating by parts the second term of the left-hand side of (2.44), we obtain that

$$\frac{d}{dt}\int_\Omega \varphi(t) = 0. \tag{2.45}$$

Consequently we conclude that

$$m(\varphi(t)) = \frac{1}{|\Omega|}\int_\Omega \varphi(t) = \frac{1}{|\Omega|}\int_\Omega \varphi_0 = m(\varphi_0) =: m_0 \quad \text{for every } t \in (0,T). \tag{2.46}$$

Change of variables. In the following it we will be useful to consider the equivalent modified form of the initial Problem (P) (see (2.32)–(2.38)). We make a change of variables and set

$$\eta = a\vartheta + b\varphi - \eta^*, \qquad \eta_0 = a\vartheta_0 + b\varphi_0 - \eta^*. \tag{2.47}$$

Due to (2.47), from (2.32)–(2.38) we obtain the modified problem $(\widetilde{P})$:

$$\partial_t(\eta + (a\ell - b)\varphi) - \Delta\eta + b\Delta\varphi - \Delta\eta^* + a\zeta = af \ \text{ a.e. in } Q, \tag{2.48}$$

$$\partial_t\varphi - \Delta\mu = 0 \ \text{ a.e. in } Q, \tag{2.49}$$

$$\mu = -\nu\Delta\varphi + \xi + \pi(\varphi) - \frac{\gamma}{a}(\eta - b\varphi + \eta^*) \ \text{ a.e. in } Q, \tag{2.50}$$

$$\zeta(t) \in A(\eta(t)) \text{ for a.e. } t \in (0,T), \tag{2.51}$$

$$\xi \in \beta(\varphi) \text{ a.e. in } Q, \tag{2.52}$$

$$\partial_{\mathbf{n}}\eta = \partial_{\mathbf{n}}\varphi = \partial_{\mathbf{n}}\mu = 0 \ \text{ on } \Sigma, \tag{2.53}$$

$$\eta(0) = \eta_0, \qquad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{2.54}$$

Theorem 2.3 (Uniqueness and continuous dependence) *Assume* (2.7)–(2.8), (2.14)–(2.16) *and* (2.26)–(2.28). *If* a, $b > 0$ *and* $a\ell = b$, *then the solution* (η, φ, μ) *of problem* $(\widetilde{P})$ *(see* (2.48)–(2.54)*) is unique. Moreover, we assume that* f_i, η_i^*, η_{0_i}, φ_{0_i}, $i = 1, 2$, *are given as in* (2.27)–(2.28) *and* $(\eta_i, \varphi_i, \mu_i)$, $i = 1, 2$, *are the corresponding solutions. If*

$$m(\varphi_{0_1}) = m(\varphi_{0_2}), \tag{2.55}$$

then the estimate

$$\|\eta_1 - \eta_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\varphi_1 - \varphi_2\|_{L^\infty(0,T;V^*)\cap L^2(0,T;V)}$$

$$\leq c\big(\|\varphi_{0_1} - \varphi_{0_2}\|_{V^*} + \|\eta_{0_1} - \eta_{0_2}\|_H + \|f_1 - f_2\|_{L^2(0,T;H)} + \|\eta_1^* - \eta_2^*\|_W\big) \tag{2.56}$$

holds true for some constant c *that depends only on* Ω, T *and the structure* (2.7)–(2.8), (2.14)–(2.16) *and* (2.26)–(2.28) *of the system.*

Theorem 2.4 (Sliding mode control) *Assume* (2.7)–(2.8), (2.14)–(2.16), (2.26), $a = 1$, $b = \ell$ *and*

$$f \in L^\infty(0,T,H), \tag{2.57}$$

$$\eta^* \in W, \quad \vartheta_0 \in V, \quad \varphi_0 \in W, \quad \beta^0(\varphi_0) \in H, \quad m_0 \in \operatorname{int}(D(\beta)). \tag{2.58}$$

We consider $A = \rho\,\mathrm{Sign}$, *where* ρ *is a positive coefficient,* Sign *is defined as in* (2.19) *and* σ *is an element of the range of* Sign, *i.e.,*

$$\sigma(t) \in \mathrm{Sign}(\vartheta(t) + \ell\varphi(t) - \eta^*) \textit{ for a.e. } t \in (0,T), \tag{2.59}$$

Then, for some $\rho^* > 0$ *and for every* $\rho > \rho^*$, *there exists a solution* $(\vartheta, \varphi, \mu)$ *to Problem* (P) *(see* (2.32)–(2.38)*) and a time* T^* *such that, for every* $t \in [T^*, T]$

$$\vartheta(t) + \ell\varphi(t) = \eta^* \qquad \textit{a.e. in } \Omega. \tag{2.60}$$

3 Existence - The approximating problem (P_ε)

The following three sections are devoted to the proof of the existence Theorem 2.1.

Let us stress that, from now on, the symbol c stands for different positive constants which depend only on $|\Omega|$, on the final time T, the shape of the nonlinearities and on the constants and the norms of the functions involved in the assumptions of our statements.

Yosida regularization of A. We consider the Yosida regularization of A. For $\varepsilon > 0$ we define

$$A_\varepsilon : H \longrightarrow H, \qquad A_\varepsilon = \frac{I-(I+\varepsilon A)^{-1}}{\varepsilon}, \tag{3.1}$$

where I denotes the identity operator. Note that A_ε is Lipschitz continuous and maximal monotone, with Lipschitz constant $1/\varepsilon$, and satisfies the following properties. Denoting by $J_\varepsilon = (I+\varepsilon A)^{-1}$ the resolvent operator, for all $\delta > 0$ we have that

$$A_\varepsilon \eta \in A(J_\varepsilon \eta), \tag{3.2}$$

$$(A_\varepsilon)_\delta = A_{\varepsilon+\delta}, \tag{3.3}$$

$$\|A_\varepsilon \eta\|_H \le \|A^0 \eta\|_H, \tag{3.4}$$

$$\lim_{\varepsilon\to 0} \|A_\varepsilon \eta\|_H = \|A^0\eta\|_H, \tag{3.5}$$

where $A^0\eta$ is the element of the range of $A\eta$ having minimum norm.

Remark. We point out a key property of A_ε, which is a consequence of (2.16):

$$\|v\|_H \le C_A(1+\|\eta\|_H) \qquad \text{for all } \eta\in H,\ v\in A_\varepsilon\eta. \tag{3.6}$$

Indeed notice that $0\in A(0)$ and $0\in I(0)$: consequently, for every $\varepsilon>0$, $0\in(I+\varepsilon A)(0)$. This fact implies that $J_\varepsilon(0)=0$. We also recall that A is a maximal monotone operator, hence J_ε is a contraction. Then, from (2.16) and (3.2), it follows that

$$\begin{aligned}\|A_\varepsilon\eta\|_H &\le C_A(\|J_\varepsilon\eta\|_H+1)\\ &\le C_A(\|J_\varepsilon\eta - J_\varepsilon 0\|_H + \|J_\varepsilon 0\|_H + 1)\\ &\le C_A(\|\eta\|_H+1).\end{aligned}$$

Yosida regularization of Sign. Let us introduce the operator $\mathrm{Sign}_\varepsilon : H\to H$ as the Yosida regularization at level $\varepsilon>0$ of the operator Sign. We observe that $\mathrm{Sign}_\varepsilon(v)$ is the gradient at v of the C^1 functional $\|\cdot\|_{H,\varepsilon}$ defined as

$$\|v\|_{H,\varepsilon} := \min_{w\in H}\{\frac{1}{2\varepsilon}\|w-v\|_H^2 + \|w\|_H\} = \int_0^{\|v\|_H} \min\{s/\varepsilon,1\}\,ds \quad \text{for every } v\in H. \tag{3.7}$$

We also recall that

$$\mathrm{Sign}_\varepsilon(v) = \frac{v}{\max\{\varepsilon,\|v\|_H\}} \quad \text{for every } v\in H. \tag{3.8}$$

Moreau-Yosida regularization of β and $\widehat{\beta}$. We introduce the Yosida regularization of β. For every $\varepsilon > 0$ we define

$$\beta_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}, \qquad \beta_\varepsilon = \frac{I - (I + \varepsilon\beta)^{-1}}{\varepsilon}. \tag{3.9}$$

We remark that β_ε is Lipschitz continuous with Lipschitz constant $1/\varepsilon$ and satisfies the following properties. Denoting by $R_\varepsilon = (I + \varepsilon\beta)^{-1}$ the resolvent operator, for all $\delta > 0$ and for every $\varphi \in D(\beta)$ we have that

$$\beta_\varepsilon(\varphi) \in \beta(R_\varepsilon\varphi), \tag{3.10}$$

$$(\beta_\varepsilon)_\delta = \beta_{\varepsilon+\delta}, \tag{3.11}$$

$$|\beta_\varepsilon(\varphi)| \leq |\beta^0(\varphi)|, \tag{3.12}$$

$$\lim_{\varepsilon\to 0} \beta_\varepsilon(\varphi) = \beta^0(\varphi), \tag{3.13}$$

where $\beta^0(\varphi)$ is the element of the range of β having minimum modulus. For $\varepsilon > 0$, we also introduce $\widehat{\beta}_\varepsilon : \mathbb{R} \to [0, +\infty]$ as the standard Moreau-Yosida regularization of $\widehat{\beta}$

$$\widehat{\beta}_\varepsilon := \min_{y\in\mathbb{R}} \left\{ \widehat{\beta}(x) + \frac{1}{2\varepsilon}|x - y| \right\} \tag{3.14}$$

and we recall that, for every $\varphi \in D(\widehat{\beta})$,

$$\widehat{\beta}_\varepsilon(\varphi) \leq \widehat{\beta}(\varphi). \tag{3.15}$$

Moreover, β_ε is the Frechet derivative of $\widehat{\beta}_\varepsilon$. Then, for every $\varphi_1, \varphi_2 \in D(\widehat{\beta})$, we have that

$$\widehat{\beta}_\varepsilon(\varphi_2) = \widehat{\beta}_\varepsilon(\varphi_1) + \int_{\varphi_1}^{\varphi_2} \beta_\varepsilon(s)\ ds. \tag{3.16}$$

Regularization of the initial data. We denote by $\vartheta_{0\varepsilon}$ and $\varphi_{0\varepsilon}$ the regularization of the initial data ϑ_0 and φ_0, respectively, obtained solving the following elliptic problems:

$$\begin{cases} \vartheta_{0\varepsilon} - \varepsilon\Delta\vartheta_{0\varepsilon} = \vartheta_0 & \text{in } \Omega, \\ \partial_{\mathbf{n}}\vartheta_{0\varepsilon} = 0 & \text{on } \Gamma. \end{cases} \tag{3.17}$$

$$\begin{cases} \varphi_{0\varepsilon} - \varepsilon\Delta\varphi_{0\varepsilon} = \varphi_0 & \text{in } \Omega, \\ \partial_{\mathbf{n}}\varphi_{0\varepsilon} = 0 & \text{on } \Gamma. \end{cases} \tag{3.18}$$

Since $\vartheta_0 \in H$ and $\varphi_0 \in V$, by elliptic regularity we infer that $\vartheta_{0\varepsilon} \in W$ and $\varphi_{0\varepsilon} \in W \cap H^3(\Omega)$. Moreover, integrating over Ω the first equation of (3.18), we obtain that

$$m_0 = \frac{1}{|\Omega|} \int_\Omega \varphi_0 = \frac{1}{|\Omega|} \int_\Omega \varphi_{0\varepsilon} =: m_{0\varepsilon}. \tag{3.19}$$

From (2.28) and (2.46) it immediately follows that $m_{0\varepsilon} \in \operatorname{int}(D(\beta))$. Since β is maximal monotone, testing the first equation of (3.18) by $\beta_\varepsilon(\varphi_{0\varepsilon})$ and integrating over Ω, we have that

$$\int_\Omega (\varphi_{0\varepsilon} - \varphi_0)\beta_\varepsilon(\varphi_{0\varepsilon}) = -\varepsilon \int_\Omega |\nabla\varphi_{0\varepsilon}|^2 \beta_\varepsilon'(\varphi_{0\varepsilon}) \leq 0. \tag{3.20}$$

Recalling that β_ε is the subdifferential of $\widehat{\beta}_\varepsilon$, from (3.20) we infer that

$$\int_\Omega \widehat{\beta}_\varepsilon(\varphi_{0\varepsilon}) - \int_\Omega \widehat{\beta}_\varepsilon(\varphi_0) \le \int_\Omega (\varphi_{0\varepsilon} - \varphi_0)\beta_\varepsilon(\varphi_{0\varepsilon}) \le 0. \tag{3.21}$$

Consequently, due to (2.28), (3.15), (3.21) and the definition of $\widehat{\beta}_\varepsilon$, we conclude that

$$0 \le \int_\Omega \widehat{\beta}_\varepsilon(\varphi_{0\varepsilon}) \le \int_\Omega \widehat{\beta}_\varepsilon(\varphi_0) \le \int_\Omega \widehat{\beta}(\varphi_0) < +\infty, \tag{3.22}$$

whence there exists a positive constant c, independent of ε, such that $\|\widehat{\beta}(\varphi_{0\varepsilon})\|_{L^1(\Omega)} \le c$. Now, we test (3.17) by $\vartheta_{0\varepsilon}$ and integrate over Ω. We obtain that

$$\int_\Omega |\vartheta_{0\varepsilon}|^2 + \varepsilon \int_\Omega |\nabla \vartheta_{0\varepsilon}|^2 = \int_\Omega \vartheta_0 \vartheta_{0\varepsilon} \le \frac{1}{2}\int_\Omega |\vartheta_0|^2 + \frac{1}{2}\int_\Omega |\vartheta_{0\varepsilon}|^2. \tag{3.23}$$

Since $\vartheta_0 \in H$, from (3.23) it immediately follows that $\varepsilon \vartheta_{0\varepsilon} \longrightarrow 0$ in V as $\varepsilon \searrow 0$. Besides, there exists a positive constant c, independent of ε, such that $\|\vartheta_{0\varepsilon}\|_H \le c$. Then, testing the first equation of the system (3.17) by an arbitrary function $v \in V$ and passing to the limit as $\varepsilon \searrow 0$, we obtain that

$$\lim_{\varepsilon \searrow 0} \left(\int_\Omega \vartheta_{0\varepsilon} v + \varepsilon \int_\Omega \nabla \vartheta_{0\varepsilon} \cdot \nabla v - \int_\Omega \vartheta_0 v \right) = 0 \qquad \text{for all } v \in V, \tag{3.24}$$

whence $\vartheta_{0\varepsilon} \rightharpoonup \vartheta_0$ in H. Moreover, from (3.23) and (3.24) we infer that

$$\int_\Omega |\vartheta_0|^2 \le \liminf_{\varepsilon \searrow 0} \int_\Omega |\vartheta_{0\varepsilon}|^2 \le \limsup_{\varepsilon \searrow 0} \int_\Omega |\vartheta_{0\varepsilon}|^2 \le \int_\Omega |\vartheta_0|^2. \tag{3.25}$$

Thanks to (3.25), $\|\vartheta_{0\varepsilon}\|_H \longrightarrow \|\vartheta_0\|_H$ and this ensures, due to the weak convergence already proved, that $\vartheta_{0\varepsilon} \longrightarrow \vartheta_0$ in H.

With a similar technique, testing (3.18) by $\varphi_{0\varepsilon}$ and integrating over Ω, we obtain that $\varphi_{0\varepsilon} \longrightarrow \varphi_0$ in H. Now, we test (3.18) by $-\Delta \varphi_{0\varepsilon}$ and integrate over Ω. We obtain that

$$\int_\Omega |\nabla \varphi_{0\varepsilon}|^2 + \varepsilon \int_\Omega |\Delta \varphi_{0\varepsilon}|^2 = \int_\Omega \nabla \varphi_0 \cdot \nabla \varphi_{0\varepsilon} \le \frac{1}{2}\int_\Omega |\nabla \varphi_0|^2 + \frac{1}{2}\int_\Omega |\nabla \varphi_{0\varepsilon}|^2. \tag{3.26}$$

Since $\varphi_0 \in V$, from (3.26) it immediately follows that $\varepsilon \varphi_{0\varepsilon} \longrightarrow 0$ in W as $\varepsilon \searrow 0$. Furthermore, there exists a positive constant c, independent of ε, such that $\|\nabla \varphi_{0\varepsilon}\|_H \le c$. Recalling that $\|\varphi_{0\varepsilon}\|_H \le c$, we conclude that $\|\varphi_{0\varepsilon}\|_V \le c$. Then, testing the the first equation of the system (3.18) by $-\Delta w$, where w is an arbitrary function in W, and passing to the limit as $\varepsilon \searrow 0$, we obtain

$$\lim_{\varepsilon \searrow 0} \left(\int_\Omega \nabla \varphi_{0\varepsilon} \cdot \nabla w + \varepsilon \int_\Omega \Delta \varphi_{0\varepsilon} \cdot \Delta w - \int_\Omega \nabla \varphi_0 \cdot \nabla w \right) = 0 \qquad \text{for all } w \in W, \tag{3.27}$$

whence $\varphi_{0\varepsilon} \rightharpoonup \varphi_0$ in V. Moreover, from (3.26)–(3.27) we infer that

$$\int_\Omega |\nabla \varphi_0|^2 \le \liminf_{\varepsilon \searrow 0} \int_\Omega |\nabla \varphi_{0\varepsilon}|^2 \le \limsup_{\varepsilon \searrow 0} \int_\Omega |\nabla \varphi_{0\varepsilon}|^2 \le \int_\Omega |\nabla \varphi_0|^2. \tag{3.28}$$

Thanks to (3.28), $\|\nabla\varphi_{0\varepsilon}\|_H \longrightarrow \|\nabla\varphi_0\|_H$ and this ensures, due to the weak convergence already proved, that $\varphi_{0\varepsilon} \longrightarrow \varphi_0$ in V. Now, let us summarize the main properties of $\vartheta_{0\varepsilon}$ and $\varphi_{0\varepsilon}$. For every $\varepsilon \in (0,1)$ we have that

$$\vartheta_{0\varepsilon} \in W, \quad \varphi_{0\varepsilon} \in W \cap H^3(\Omega), \quad m_{0\varepsilon} \in \operatorname{int}(D(\beta)), \quad \|\widehat{\beta}(\varphi_{0\varepsilon})\|_{L^1(\Omega)} \le c, \tag{3.29}$$

$$\lim_{\varepsilon\searrow 0} \|\vartheta_0 - \vartheta_{0\varepsilon}\|_H = 0, \qquad \lim_{\varepsilon\searrow 0} \|\varphi_0 - \varphi_{0\varepsilon}\|_V = 0, \tag{3.30}$$

$$-\nu\Delta\varphi_{0\varepsilon} + \beta_\varepsilon(\varphi_{0\varepsilon}) + \pi(\varphi_{0\varepsilon}) - \gamma\vartheta_{0\varepsilon} \in V. \tag{3.31}$$

Regularization of f. We denote by f_ε the regularization of f, constructed in such a way that

$$f_\varepsilon \in C^1([0,T];H) \text{ for all } \varepsilon > 0, \qquad \lim_{\varepsilon\searrow 0} \|f_\varepsilon - f\|_{L^2(0,T;H)} = 0. \tag{3.32}$$

For example, we can consider f_ε as the solution the following system:

$$\begin{cases} -\varepsilon f_\varepsilon''(t) + f_\varepsilon(t) = f(t), \quad t \in (0,T), \\ f_\varepsilon(0) = f_\varepsilon(T) = 0. \end{cases} \tag{3.33}$$

Approximating problem (P_ε). We look for a triplet $(\vartheta_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon)$ satisfying at least the regularity requirements

$$\vartheta_\varepsilon \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{3.34}$$

$$\varphi_\varepsilon \in W^{1,\infty}(0,T;V^*) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{3.35}$$

$$\mu_\varepsilon \in L^\infty(0,T;V) \cap L^2(0,T;W), \tag{3.36}$$

and solving the approximating problem (P_ε):

$$\partial_t(\vartheta_\varepsilon + \ell\varphi_\varepsilon) - \Delta\vartheta_\varepsilon + \zeta_\varepsilon = f_\varepsilon \ \text{ a.e. in } Q, \tag{3.37}$$

$$\partial_t\varphi_\varepsilon - \Delta\mu_\varepsilon = 0 \ \text{ a.e. in } Q, \tag{3.38}$$

$$\mu_\varepsilon = -\nu\Delta\varphi_\varepsilon + \xi_\varepsilon + \pi(\varphi_\varepsilon) - \gamma\vartheta_\varepsilon \ \text{ a.e. in } Q, \tag{3.39}$$

$$\zeta_\varepsilon(t) \in A_\varepsilon(a\vartheta_\varepsilon(t) + b\varphi_\varepsilon(t) - \eta^*) \text{ for a.e. } t \in (0,T), \tag{3.40}$$

$$\xi_\varepsilon \in \beta_\varepsilon(\varphi_\varepsilon) \text{ a.e. in } Q, \tag{3.41}$$

$$\partial_{\mathbf{n}}\vartheta_\varepsilon = \partial_{\mathbf{n}}\varphi_\varepsilon = \partial_{\mathbf{n}}\mu_\varepsilon = 0 \ \text{ on } \Sigma, \tag{3.42}$$

$$\vartheta_\varepsilon(0) = \vartheta_{0\varepsilon}, \qquad \varphi_\varepsilon(0) = \varphi_{0\varepsilon} \ \text{ in } \Omega, \tag{3.43}$$

where β_ε and A_ε are the Yosida regularizations of β and A defined in (3.1) and (3.9). We notice that the homogeneous Neumann boundary conditions are already contained in the conditions (3.34)–(3.36) due to the definition of W (see (2.3)).

We observe that, for almost every $t \in (0,T)$, we can re-write the approximating problem (P_ε) in the following way:

$$\langle \partial_t(\vartheta_\varepsilon + \ell\varphi_\varepsilon)(t), z\rangle_{V^*,V} + \int_\Omega \nabla\vartheta_\varepsilon(t)\cdot\nabla z + \langle \zeta_\varepsilon(t), z\rangle_{V^*,V} = \langle f_\varepsilon(t), z\rangle_{V^*,V} \ \text{ for all } z \in V, \tag{3.44}$$

$$\langle \partial_t \varphi_\varepsilon(t), z\rangle_{V^*,V} + \int_\Omega \nabla \mu_\varepsilon(t) \cdot \nabla z = 0 \ \text{ for all } z \in V, \tag{3.45}$$

$$\mu_\varepsilon(t) = -\nu \Delta \varphi_\varepsilon(t) + \xi_\varepsilon(t) + \pi(\varphi_\varepsilon(t)) - \gamma \vartheta_\varepsilon(t) \ \text{ in } H, \tag{3.46}$$

$$\zeta_\varepsilon(t) \in A_\varepsilon(a\vartheta_\varepsilon(t) + b\varphi_\varepsilon(t) - \eta^*), \tag{3.47}$$

$$\xi \in \beta_\varepsilon(\varphi_\varepsilon) \text{ a.e. in } Q, \tag{3.48}$$

$$\partial_{\mathbf{n}} \varphi_\varepsilon = 0 \quad \text{a.e. on } \Sigma, \tag{3.49}$$

$$\vartheta_\varepsilon(0) = \vartheta_{0\varepsilon}, \qquad \varphi_\varepsilon(0) = \varphi_{0\varepsilon} \quad \text{in } \Omega. \tag{3.50}$$

Since $m_{0\varepsilon} = m_0$, recalling the definition of $\mathcal{N}$ (see (2.21)–(2.24)), we have that $\partial_t \varphi_\varepsilon(t) \in D(\mathcal{N})$. Hence, (3.45) can be written as

$$\mathcal{N} \partial_t \varphi_\varepsilon(t) = m(\mu_\varepsilon(t)) - \mu_\varepsilon(t) \ \text{ in } V, \tag{3.51}$$

and this and (3.45) entail

$$m(\mu_\varepsilon(t)) - \mathcal{N} \partial_t \varphi_\varepsilon(t) = -\nu \Delta \varphi_\varepsilon(t) + \xi_\varepsilon(t) + \pi(\varphi_\varepsilon(t)) - \gamma \vartheta_\varepsilon(t) \ \text{ in } H. \tag{3.52}$$

4 Existence - Global a priori estimates

In this section, we will deduce some a priori estimates inferred from (3.44)–(3.52).

In the remainder of the paper we often owe to the Hölder inequality and to the elementary Young inequalities in performing our a priori estimates. For every $x, y > 0$, $\alpha \in (0, 1)$ and $\delta > 0$ there hold

$$xy \le \alpha x^{\frac{1}{\alpha}} + (1 - \alpha) y^{\frac{1}{1-\alpha}}, \tag{4.1}$$

$$xy \le \delta x^2 + \frac{1}{4\delta} y^2. \tag{4.2}$$

Moreover, we also use the inequality deduced from the compactness of the embedding $V \subset H \subset V^*$ (see [39, Lemma 8, p. 84]): for all $\delta > 0$ there exists a constant $K > 0$ such that

$$\|z\|_H \le \delta \|z\|_V + K \|z\|_{V^*} \qquad \text{for all } z \in H. \tag{4.3}$$

In the following, the symbol c stands for different positive constants which depend only on $|\Omega|$, on the final time T, on the shape of the nonlinearities and on the constants and the norms of the functions involved in the assumptions of our statements.

First a priori estimate. According to (3.19), $m(\partial_t \varphi_\varepsilon) = 0$. Consequently, $\partial_t \varphi_\varepsilon \in D(\mathcal{N})$ and we can test (3.45) by $\mathcal{N} \partial_t \varphi_\varepsilon$. Integrating over $(0, t)$, $t \in (0, T]$, we obtain that

$$\int_0^t \|\partial_t \varphi_\varepsilon(s)\|_{V^*}^2 \, ds + \int_{Q_t} \nabla \mu_\varepsilon \cdot \nabla \mathcal{N} \partial_t \varphi_\varepsilon = \int_0^t \|\partial_t \varphi_\varepsilon\|_{V^*}^2 + \int_{Q_t} \mu_\varepsilon \partial_t \varphi_\varepsilon = 0. \tag{4.4}$$

Recalling that

$$\nu \int_{Q_t} \varphi_\varepsilon \partial_t \varphi_\varepsilon = \frac{\nu}{2} \int_\Omega |\varphi_\varepsilon(t)|^2 - \frac{\nu}{2} \int_\Omega |\varphi_{0\varepsilon}|^2, \tag{4.5}$$

we combine (3.44) tested by $\frac{\gamma}{\ell}\vartheta_\varepsilon$, (4.4) and (4.5). Then we subtract (3.46) tested by $\partial_t\varphi_\varepsilon$ and integrate over $(0,t)$. We have that

$$\frac{\gamma}{2\ell}\int_\Omega |\vartheta_\varepsilon(t)|^2 + \frac{\gamma}{\ell}\int_{Q_t}|\nabla\vartheta_\varepsilon|^2 + \int_0^t \|\partial_t\varphi_\varepsilon(s)\|^2_{V^*}\,ds + \frac{\nu}{2}\|\varphi_\varepsilon(t)\|^2_V + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_\varepsilon(t))$$

$$= \frac{\gamma}{2\ell}\|\vartheta_{0\varepsilon}\|^2_H + \frac{\nu}{2}\|\varphi_{0\varepsilon}\|^2_V + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_{0\varepsilon}) + \frac{\gamma}{\ell}\int_{Q_t} f_\varepsilon\vartheta_\varepsilon - \frac{\gamma}{\ell}\int_{Q_t}\zeta_\varepsilon\vartheta_\varepsilon + \int_{Q_t}(\nu\varphi_\varepsilon - \pi(\varphi_\varepsilon))\partial_t\varphi_\varepsilon. \tag{4.6}$$

As π is a Lipschitz continuous function with Lipschitz constant $C_\pi = \|\pi'\|_\infty$, we obtain that

$$|\pi(\varphi_\varepsilon(s))| \le |\pi(\varphi_\varepsilon(s)) - \pi(0)| + |\pi(0)| \le C_\pi|\varphi_\varepsilon(s)| + |\pi(0)|. \tag{4.7}$$

Consequently, thanks to (4.7), we infer that

$$\|\nu\varphi_\varepsilon(s) - \pi(\varphi_\varepsilon(s))\|^2_V = \int_\Omega |\nu\varphi_\varepsilon(s) - \pi(\varphi_\varepsilon(s))|^2 + \int_\Omega |\nu\nabla\varphi_\varepsilon(s) - \pi'(\varphi_\varepsilon(s))\nabla\varphi_\varepsilon(s)|^2$$

$$\le 2\int_\Omega \Big(\nu^2|\varphi_\varepsilon(s)|^2 + |\pi(\varphi_\varepsilon(s))|^2\Big) + 2\int_\Omega \Big(\nu^2|\nabla\varphi_\varepsilon(s)|^2 + \|\pi'\|^2_\infty|\nabla\varphi_\varepsilon(s)|^2\Big)$$

$$\le 2\nu^2\int_\Omega |\varphi_\varepsilon(s)|^2 + 4C_\pi^2\int_\Omega|\varphi_\varepsilon(s)|^2 + 4|\Omega||\pi(0)|^2 + 2\nu^2\int_\Omega|\nabla\varphi_\varepsilon(s)|^2 + 2C_\pi^2\int_\Omega|\nabla\varphi_\varepsilon(s)|^2$$

$$= (2\nu^2 + 4C_\pi^2)\int_\Omega|\varphi_\varepsilon(s)|^2 + (2\nu^2+2C_\pi^2)\int_\Omega|\nabla\varphi_\varepsilon(s)|^2 + 4|\pi(0)|^2|\Omega| \le c(\|\varphi_\varepsilon(s)\|^2_V + 1). \tag{4.8}$$

Due to (4.8), we obtain that the last term on the right-hand side of (4.6) is estimated as follows

$$\begin{aligned}\int_{Q_t}(\nu\varphi_\varepsilon - \pi(\varphi_\varepsilon))\partial_t\varphi_\varepsilon \ &\le\ \frac12\int_0^t \|\partial_t\varphi_\varepsilon(s)\|^2_{V^*}\,ds + \frac12\int_0^t \|\nu\varphi_\varepsilon(s) - \pi(\varphi_\varepsilon(s))\|^2_V\,ds \\ &\le\ \frac12\int_0^t\|\partial_t\varphi_\varepsilon(s)\|^2_{V^*}\,ds + c\int_0^t(\|\varphi_\varepsilon(s)\|^2_V + 1)\,ds. \end{aligned}\tag{4.9}$$

Due to the liear growth of A_ε stated by (3.6), we have that

$$-\frac{\gamma}{\ell}\int_{Q_t}\zeta_\varepsilon\vartheta_\varepsilon \le \frac{\gamma}{\ell}\int_{Q_t}|\zeta_\varepsilon(s)||\vartheta_\varepsilon(s)|\,ds \le \frac{\gamma}{\ell}\int_0^t\|\zeta_\varepsilon(s)\|^2_H\,ds + \frac{\gamma}{\ell}\int_0^t\|\vartheta_\varepsilon(s)\|^2_H\,ds$$

$$\le \frac{\gamma}{\ell}\int_0^t C_A^2(1 + \|a\vartheta_\varepsilon(s) + b\varphi_\varepsilon(s) - \eta^*\|_H)^2\,ds + \frac{\gamma}{\ell}\int_0^t\|\vartheta_\varepsilon(s)\|^2_H\,ds$$

$$\le \frac{\gamma}{\ell}\int_0^t 4C_A^2(1 + |a|^2\|\vartheta_\varepsilon(s)\|^2_H + |b|^2\|\varphi_\varepsilon(s)\|^2_H + \|\eta^*\|^2_H)\,ds + \frac{\gamma}{\ell}\int_0^t\|\vartheta_\varepsilon(s)\|^2_H\,ds$$

$$\le \frac{\gamma}{\ell}4C_A^2T + \frac{\gamma}{\ell}4C_A^2|a|^2\int_0^t\|\vartheta_\varepsilon(s)\|^2_H\,ds + \frac{\gamma}{\ell}4C_A^2|b|^2\int_0^t\|\varphi_\varepsilon(s)\|^2_H\,ds$$

$$+\frac{\gamma}{\ell}4C_A^2T\|\eta^*\|^2_H + \frac{\gamma}{\ell}\int_0^t\|\vartheta_\varepsilon(s)\|^2_H\,ds$$

$$\le c\Bigg(\int_0^t\|\vartheta_\varepsilon(s)\|^2_H\,ds + \int_0^t\|\varphi_\varepsilon(s)\|^2_H\,ds + 1\Bigg). \tag{4.10}$$

Moreover, by applying (4.2) to the fourth term on the right-hand side of (4.6), we have that

$$\frac{\gamma}{\ell}\int_{Q_t} f_\varepsilon\vartheta_\varepsilon \le \frac{\gamma}{\ell}\int_{Q_t}|f_\varepsilon|^2 + \frac{\gamma}{4\ell}\int_{Q_t}|\vartheta_\varepsilon|^2 = \frac{\gamma}{\ell}\int_{Q_t}|f_\varepsilon|^2 + \frac{\gamma}{4\ell}\int_0^t \|\vartheta_\varepsilon(s)\|_H^2\, ds. \tag{4.11}$$

We rearrange the right-hand side of (4.6) using (4.9)–(4.11) and obtain that

$$\begin{aligned}\frac{\gamma}{2\ell}\int_\Omega |\vartheta_\varepsilon(t)|^2 + \frac{\gamma}{\ell}\int_{Q_t}|\nabla\vartheta_\varepsilon|^2 + \frac{1}{2}\int_0^t \|\partial_t\varphi_\varepsilon(s)\|_{V^*}^2\, ds + \frac{\nu}{2}\|\varphi_\varepsilon(t)\|_V^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_\varepsilon(t))\\
\le \frac{\gamma}{2\ell}\|\vartheta_{0\varepsilon}\|_H^2 + \frac{\nu}{2}\|\varphi_{0\varepsilon}\|_V^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_{0\varepsilon}) + \frac{\gamma}{\ell}\int_0^t \|f_\varepsilon(s)\|_H^2\, ds\\
+c\left(\int_0^t \|\varphi_\varepsilon(s)\|_V^2\, ds + \int_0^t \|\vartheta_\varepsilon(s)\|_H^2\, ds + 1\right) + \frac{\gamma}{4\ell}\int_0^t \|\vartheta_\varepsilon(s)\|_H^2\, ds.\end{aligned} \tag{4.12}$$

Due to (3.29)–(3.30), the first three terms of the right-hand side of (4.12) are bounded and similarly the fourth term, thanks to (3.32). Then, applying the Gronwall lemma, we conclude that there exists a positive constant c, independent of ε, such that

$$\frac{\gamma}{2\ell}\int_\Omega |\vartheta_\varepsilon(t)|^2 + \int_{Q_t}|\nabla\vartheta_\varepsilon|^2 + \frac{1}{2}\int_0^t \|\partial_t\varphi_\varepsilon(s)\|_{V^*}^2\, ds + \frac{\nu}{2}\|\varphi_\varepsilon(t)\|_V^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_\varepsilon(t)) \le c, \tag{4.13}$$

whence it immediately follows that

$$\|\vartheta_\varepsilon\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c, \tag{4.14}$$
$$\|\varphi_\varepsilon\|_{H^1(0,T;V^*)\cap L^\infty(0,T;V)} \le c, \tag{4.15}$$
$$\|\widehat{\beta}_\varepsilon(\varphi_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \le c. \tag{4.16}$$

Due to (4.14)–(4.16), by (3.6) we have that

$$\|\zeta_\varepsilon\|_{L^\infty(0,T;H)} \le c, \tag{4.17}$$

and, consequently, by comparison in (3.44) we infer that

$$\|\partial_t\vartheta_\varepsilon\|_{L^2(0,T;V^*)} \le c. \tag{4.18}$$

Second a priori estimate. Recalling that $m_{0\varepsilon} = m_0$ due to (3.45), we have that $\varphi_\varepsilon(s) - m_0 \in D(\mathcal{N})$ for every $s \in (0,T)$. We test (3.52) at time s by $(\varphi_\varepsilon(s) - m_0) \in D(\mathcal{N})$ and we infer that

$$\begin{aligned}(\xi_\varepsilon(s), \varphi_\varepsilon(s) - m_0)_H = -(\mathcal{N}\partial_t\varphi_\varepsilon(s), \varphi_\varepsilon(s) - m_0)_H + (m(\mu_\varepsilon(s)), \varphi_\varepsilon(s) - m_0)_H\\
+\nu(\Delta\varphi_\varepsilon(s), \varphi_\varepsilon(s) - m_0)_H - (\pi(\varphi_\varepsilon(s)), \varphi_\varepsilon(s) - m_0)_H + \gamma(\vartheta_\varepsilon(s), \varphi_\varepsilon(s) - m_0)_H.\end{aligned} \tag{4.19}$$

We recall that there exists a positive constant c such that $\|z\|_{V^*} \le c\|z\|_H$ for all $z \in H$. Consequently the first term of the right-hand side of (4.19) is estimated as follows:

$$\begin{aligned}-(\mathcal{N}\partial_t\varphi_\varepsilon(s), \varphi_\varepsilon(s) - m_0)_H &= -(\partial_t\varphi_\varepsilon(s), \varphi_\varepsilon(s) - m_0)_{V^*}\\
&\le \|\partial_t\varphi_\varepsilon(s)\|_{V^*}(\|\varphi_\varepsilon(s)\|_{V^*} + |m_0||\Omega|)\\
&= c\|\partial_t\varphi_\varepsilon(s)\|_{V^*}(\|\varphi_\varepsilon(s)\|_H + 1).\end{aligned} \tag{4.20}$$

Recalling (3.19), we have that

$$(m(\mu_\varepsilon(s)), \varphi_\varepsilon(s) - m_0)_H = m(\mu_\varepsilon(s))\Bigg(\int_\Omega \varphi_\varepsilon(s) - |\Omega| m_0\Bigg) = 0. \tag{4.21}$$

Due to the Neumann homogeneous boundary conditions for φ_ε, we have that

$$\int_\Omega \Delta\varphi_\varepsilon(s) = 0. \tag{4.22}$$

Thanks to (4.22), we infer that

$$\nu(\Delta\varphi_\varepsilon(s), \varphi_\varepsilon(s) - m_0)_H = -\nu\|\nabla\varphi_\varepsilon(s)\|_H^2 - m_0\int_\Omega \Delta\varphi_\varepsilon(s) = -\nu\|\nabla\varphi_\varepsilon(s)\|_H^2 \le 0. \tag{4.23}$$

As π is a Lipschitz continuous function with Lipschitz constant C_π, we obtain that

$$\begin{aligned}
-(\pi(\varphi_\varepsilon(s)), \varphi_\varepsilon(s) - m_0)_H &\le \int_\Omega |\pi(\varphi_\varepsilon(s))||\varphi_\varepsilon(s) - m_0| \\
&\le \int_\Omega \Big(|\pi(\varphi_\varepsilon(s)) - \pi(0)| + |\pi(0)|\Big)\Big(|\varphi_\varepsilon(s)| + |m_0|\Big) \\
&\le \int_\Omega \Big(C_\pi|\varphi_\varepsilon(s)| + |\pi(0)|\Big)\Big(|\varphi_\varepsilon(s)| + |m_0|\Big) \\
&\le C_\pi\|\varphi_\varepsilon(s)\|_H^2 + \Big(C_\pi|m_0| + |\pi(0)|\Big)\|\varphi_\varepsilon(s)\|_H^2 + c \\
&\le c(\|\varphi_\varepsilon(s)\|_H^2 + 1).
\end{aligned} \tag{4.24}$$

Moreover, we have that

$$\begin{aligned}
\gamma(\vartheta_\varepsilon(s), \varphi_\varepsilon(s) - m_0)_H &\le \gamma\int_\Omega |\vartheta_\varepsilon(s)||\varphi_\varepsilon(s)| + \gamma|m_0|\int_\Omega |\vartheta_\varepsilon(s)| \\
&\le \gamma\|\vartheta_\varepsilon(s)\|_H^2 + \gamma\|\varphi_\varepsilon(s)\|_H^2 + \gamma|m_0|\|\vartheta_\varepsilon(s)\|_H^2 + \gamma|m_0||\Omega| \\
&\le c(\|\vartheta_\varepsilon(s)\|_H^2 + \|\varphi_\varepsilon(s)\|_H^2 + 1).
\end{aligned} \tag{4.25}$$

Consequently, rearranging the right-hand side of (4.19) using (4.20)–(4.21) and (4.23)–(4.25), we obtain that

$$(\xi_\varepsilon(s), \varphi_\varepsilon(s) - m_0)_H \le c\Bigg(\|\partial_t\varphi_\varepsilon(s)\|_{V^*} + \|\varphi_\varepsilon(s)\|_H^2 + \|\vartheta_\varepsilon(s)\|_H^2 + 1\Bigg). \tag{4.26}$$

Due to a useful inequality stated in [24, Section 5], it turns out that

$$|\xi_\varepsilon(s)| \le c[\xi_\varepsilon(s)(\varphi_\varepsilon(s) - m_0) + 1]. \tag{4.27}$$

We integrate (4.27) over Ω and, due to (4.26), we infer that

$$\begin{aligned}
\|\xi_\varepsilon(s)\|_{L^1(\Omega)} &\le c\Big[(\xi_\varepsilon(s), \varphi_\varepsilon(s) - m_0)_H + 1\Big] \\
&\le c\Bigg(\|\partial_t\varphi_\varepsilon(s)\|_{V^*} + \|\varphi_\varepsilon(s)\|_H^2 + \|\vartheta_\varepsilon(s)\|_H^2 + 1\Bigg).
\end{aligned} \tag{4.28}$$

Due to (4.14)–(4.15), from (4.28) we conclude that there exists a positive constant c, independent of ε, such that

$$\|\xi_\varepsilon\|_{L^2(0,T;L^1(\Omega))} \le c. \tag{4.29}$$

Third a priori estimate. As π is a Lipschitz continuous function with Lipschitz constant C_π, for every $s \in (0,T)$ we have that

$$\begin{aligned}
|\pi(\varphi_\varepsilon(s))|^2 &\leq \big(|\pi(\varphi_\varepsilon(s)) - \pi(0)| + |\pi(0)|\big)^2 \\
&\leq \big(C_\pi|\varphi_\varepsilon(s)| + |\pi(0)|\big)^2 \\
&\leq c\big(|\varphi_\varepsilon(s)|^2 + 1\big). \qquad (4.30)
\end{aligned}$$

Now, integrating (3.52) over Ω, squaring the resultant and using (4.14)–(4.18) and (4.30), we obtain that

$$\begin{aligned}
|m(\mu_\varepsilon(s))|^2 &\leq \frac{3}{|\Omega|^2}\Bigg(\|\xi_\varepsilon(s)\|^2_{L^1(\Omega)} + |\Omega|\|\pi(\varphi_\varepsilon(s))\|^2_H + \gamma\|\vartheta_\varepsilon(s)\|^2_H\Bigg) \\
&\leq c\Bigg(\|\xi_\varepsilon(s)\|^2_{L^1(\Omega)} + \|\varphi_\varepsilon(s)\|^2_H + \|\vartheta_\varepsilon(s)\|^2_H + 1\Bigg). \qquad (4.31)
\end{aligned}$$

Consequently, integrating (4.31) over $(0,T)$ and recalling the previous a priori estimates (4.14)–(4.15) and (4.29), we conclude that there exists a positive constant c, independent of ε, such that

$$\|m(\mu_\varepsilon)\|_{L^2(0,T)} \leq c. \qquad (4.32)$$

Fourth a priori estimate. We recall that the Poincaré inequality states that there exists a positive constant c_p such that

$$\|z\|^2_V \leq c_p\|\nabla z\|^2_H \quad \text{for all } z \in V \text{ with } m(z) = 0. \qquad (4.33)$$

We integrate over $(0,T)$ the square of the norms in V of each term of (3.51). Then, applying (4.32) and (4.33), we obtain that

$$\begin{aligned}
\int_0^T \|\mu_\varepsilon(s)\|^2_V\, ds &\leq 2\int_0^T \|m(\mu_\varepsilon(s))\|^2_V\, ds + 2\int_0^T \|\mathcal{N}\partial_t\varphi_\varepsilon(s)\|^2_V\, ds \\
&\leq 2\int_0^T |m(\mu_\varepsilon(s))|^2\, ds + 2c_p\int_0^T \|\nabla\mathcal{N}\partial_t\varphi_\varepsilon(s)\|^2_H\, ds \\
&\leq c + 2c_p\int_0^T \|\partial_t\varphi_\varepsilon(s)\|^2_{V^*}\, ds. \qquad (4.34)
\end{aligned}$$

Due to (4.15), we conclude that there exists a positive constant c, independent of ε, such that

$$\|\mu_\varepsilon\|_{L^2(0,T;V)} \leq c. \qquad (4.35)$$

Fifth a priori estimate. We test (3.46) at time $s \in (0,T)$ by $\xi_\varepsilon(s) \in V$ and integrate the resultant over Ω. We obtain that

$$\|\xi_\varepsilon(s)\|^2_H = \big(\mu_\varepsilon(s) + \nu\Delta\varphi_\varepsilon(s) - \pi(\varphi_\varepsilon(s)) + \gamma\vartheta_\varepsilon(s), \xi_\varepsilon(s)\big)_H. \qquad (4.36)$$

Due to the monotonicity of β_ε, we have that

$$\begin{aligned} (\nu\Delta\varphi_\varepsilon(s),\xi_\varepsilon(s))_H &= \nu\int_\Omega \Delta\varphi_\varepsilon(s)\xi_\varepsilon(s) \\ &= -\nu\int_\Omega \nabla\varphi_\varepsilon(s)\cdot\nabla\xi_\varepsilon(s) \\ &= -\nu\int_\Omega |\nabla\varphi_\varepsilon(s)|^2\beta'_\varepsilon(\varphi_\varepsilon(s)) \le 0. \end{aligned} \tag{4.37}$$

Using (4.37) and the Young inequality, we can estimate (4.36) as follows

$$\begin{aligned} \|\xi_\varepsilon(s)\|_H^2 &\le \big(\mu_\varepsilon(s)-\pi(\varphi_\varepsilon(s))+\gamma\vartheta_\varepsilon(s),\xi_\varepsilon(s)\big)_H \\ &\le \|\mu_\varepsilon(s)-\pi(\varphi_\varepsilon(s))+\gamma\vartheta_\varepsilon(s)\|_H\|\xi_\varepsilon(s)\|_H \\ &\le \frac{1}{2}\|\xi_\varepsilon(s)\|_H^2+2\big(\|\mu_\varepsilon(s)\|_H^2+\|\pi(\varphi_\varepsilon(s))\|_H^2+\gamma^2\|\vartheta_\varepsilon(s)\|_H^2\big). \end{aligned} \tag{4.38}$$

Due to (4.30), from (4.38) we infer that

$$\|\xi_\varepsilon(s)\|_H^2 \le c\big(\|\mu_\varepsilon(s)\|_H^2+\|\varphi_\varepsilon(s)\|_H^2+\|\vartheta_\varepsilon(s)\|_H^2+1\big). \tag{4.39}$$

Then, integrating (4.39) over $(0,T)$ with respect to s and using (4.14)–(4.15) and (4.35), we have that

$$\|\xi_\varepsilon\|_{L^2(0,T;H)} \le c, \tag{4.40}$$

for some positive constant c, independent of ε.

Sixth a priori estimate. We integrate over $(0,T)$ the square of the norms in H of each term of (3.46). Then, using (4.30), (4.35) and (4.40), we obtain that

$$\begin{aligned} &\nu^2\int_0^T \|\Delta\varphi_\varepsilon(s)\|_H^2\,ds \\ &\le 4\int_0^T \|\mu_\varepsilon(s)\|_H^2\,ds+4\int_0^T \|\xi_\varepsilon(s)\|_H^2\,ds+4\int_0^T \|\pi(\varphi_\varepsilon(s))\|_H^2\,ds+4\gamma^2\int_0^T \|\vartheta_\varepsilon(s)\|_H^2\,ds \\ &\le c\left(\int_0^T \|\varphi_\varepsilon(s)\|_H^2\,ds+\int_0^T \|\vartheta_\varepsilon(s)\|_H^2\,ds+1\right). \end{aligned} \tag{4.41}$$

Thanks to (4.14)–(4.15), we conclude that there exists a positive constant c, independent of ε, such that

$$\|\varphi_\varepsilon\|_{L^2(0,T;W)} \le c. \tag{4.42}$$

Summary of the a priori estimates. Let us summarize the a priori estimates. From (4.14)–(4.18), (4.35), (4.40) and (4.42) we conclude that there exists a constant $c>0$, independent of ε, such that

$$\|\vartheta_\varepsilon\|_{H^1(0,T;V^*)\cap L^\infty(0,T;H)\cap L^2(0,T;V)} \le c, \tag{4.43}$$
$$\|\varphi_\varepsilon\|_{H^1(0,T;V^*)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le c, \tag{4.44}$$
$$\|\zeta_\varepsilon\|_{L^\infty(0,T;H)} \le c, \tag{4.45}$$
$$\|\xi_\varepsilon\|_{L^2(0,T;H)} \le c, \tag{4.46}$$
$$\|\mu_\varepsilon\|_{L^2(0,T;V)} \le c. \tag{4.47}$$

5 Existence - Passage to the limit as $\varepsilon \searrow 0$

Based on available results (cf., e.g., [10]), it turns out that there exists a solution $(\vartheta_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon)$ of (P_ε) satisfying the regularity requirements (3.34)–(3.36) and solving (3.37)-(3.43). In this section we pass to the limit as $\varepsilon \searrow 0$ and prove that the limit of subsequences of solutions $(\vartheta_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon)$ for (P_ε) (see (3.37)–(3.43)) yields a solution $(\vartheta, \varphi, \mu)$ of (P) (see (2.32)–(2.38)).

Thanks to the uniform estimates (4.43)–(4.47), there exists a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ with $\varepsilon_k \searrow 0$ as $k \to +\infty$ and some limit functions $\vartheta \in H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V)$, $\varphi \in H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;W)$, $\mu \in L^2(0,T;V)$, $\xi \in L^2(0,T;H)$ and $\zeta \in L^\infty(0,T;H)$ such that

$$
\begin{aligned}
\vartheta_{\varepsilon_k} &\rightharpoonup^* \vartheta \quad \text{in} \quad H^1(0,T;V^*) \cap L^\infty(0,T;H), && (5.1)\\
\vartheta_{\varepsilon_k} &\rightharpoonup \vartheta \quad \text{in} \quad L^2(0,T;V), && (5.2)\\
\varphi_{\varepsilon_k} &\rightharpoonup^* \varphi \quad \text{in} \quad H^1(0,T;V^*) \cap L^\infty(0,T;V), && (5.3)\\
\varphi_{\varepsilon_k} &\rightharpoonup \varphi \quad \text{in} \quad L^2(0,T;W), && (5.4)\\
\mu_{\varepsilon_k} &\rightharpoonup \mu \quad \text{in} \quad L^2(0,T;V), && (5.5)\\
\xi_{\varepsilon_k} &\rightharpoonup \xi \quad \text{in} \quad L^2(0,T;H), && (5.6)\\
\zeta_{\varepsilon_k} &\rightharpoonup^* \zeta \quad \text{in} \quad L^\infty(0,T;H), && (5.7)
\end{aligned}
$$

as $k \to +\infty$. From (5.1)–(5.4) and the well-known Ascoli–Arzelá theorem (see, e.g., [39, Sect. 8, Cor. 4]), we infer that

$$
\begin{aligned}
\vartheta_{\varepsilon_k} &\longrightarrow \vartheta \quad \text{in} \quad C^0([0,T];V^*) \cap L^2(0,T;H), && (5.8)\\
\varphi_{\varepsilon_k} &\longrightarrow \varphi \quad \text{in} \quad C^0([0,T];H) \cap L^2(0,T;V), && (5.9)
\end{aligned}
$$

as $k \to +\infty$. As π is a Lipschitz continuous function, for a.e. $s \in [0,T]$ we have that

$$
|\pi(\varphi_{\varepsilon_k}(s)) - \pi(\varphi(s))| \le C_\pi |\varphi_{\varepsilon_k}(s) - \varphi(s)|.
$$

Thanks to (5.9), we conclude that

$$
\pi(\varphi_{\varepsilon_k}(s)) \longrightarrow \pi(\varphi(s)) \quad \text{in } L^2(0,T;H), \tag{5.10}
$$

as $k \to +\infty$.

Passage to the limit on ξ_ε. In this paragraph we check that $\xi \in \beta(\varphi)$ a.e. in Q. To this aim, we recall that

$$
\varphi_{\varepsilon_k} \to \varphi \qquad \text{in } L^2(0,T;H) \equiv L^2(Q), \tag{5.11}
$$

$$
\xi_{\varepsilon_k} \rightharpoonup \xi \qquad \text{in } L^2(0,T;H), \tag{5.12}
$$

as $k \to +\infty$. Now, we introduce the operator $\mathcal{B}_\varepsilon$ induced by β_ε on $L^2(Q)$ in the following way

$$
\mathcal{B}_\varepsilon : L^2(Q) \longrightarrow L^2(Q)
$$

$$
\xi_\varepsilon \in \mathcal{B}_\varepsilon(\varphi_\varepsilon) \Longleftrightarrow \xi_\varepsilon(x,t) \in \beta_\varepsilon(\varphi_\varepsilon(x,t)) \quad \text{for a.e. } (x,t) \in Q. \tag{5.13}
$$

Due to (5.11)–(5.12), as $k \to +\infty$, we have that

$$\begin{cases} \mathcal{B}_{\varepsilon_k}(\varphi_{\varepsilon_k}) \rightharpoonup \xi & \text{in } L^2(Q), \\ \varphi_{\varepsilon_k} \to \varphi & \text{in } L^2(Q), \end{cases} \tag{5.14}$$

$$\limsup_{k\to+\infty} \int_Q \xi_{\varepsilon_k}\varphi_{\varepsilon_k} = \int_Q \xi\varphi. \tag{5.15}$$

Thanks to (5.14)–(5.15) and to the general result [1, Proposition 2.2, p. 38], we conclude that

$$\xi \in \mathcal{B}(\varphi) \quad \text{in } L^2(Q), \tag{5.16}$$

with analogous definition for $\mathcal{B}$ (see (2.9)–(2.10)). This is equivalent to saying that

$$\xi \in \beta(\varphi) \quad \text{a.e. in } Q. \tag{5.17}$$

Passage to the limit on ζ_ε. In this paragraph we check that $\zeta(t) \in A(a\vartheta(t)+b\varphi(t)-\eta^*)$ for a.e. $t \in [0,T]$. Let us recall that

$$\vartheta_{\varepsilon_k} \to \vartheta \quad \text{in} \quad L^2(0,T;H), \tag{5.18}$$

$$\varphi_{\varepsilon_k} \to \varphi \quad \text{in} \quad L^2(0,T;H), \tag{5.19}$$

$$\zeta_{\varepsilon_k} \rightharpoonup \zeta \quad \text{in} \quad L^2(0,T;H), \tag{5.20}$$

as $k \to +\infty$. Setting

$$\eta_{\varepsilon_k} := a\vartheta_{\varepsilon_k} + b\varphi_{\varepsilon_k} - \eta^*, \qquad \eta := a\vartheta + b\varphi - \eta^*,$$

thanks to (5.18)–(5.19), we have that

$$\eta_{\varepsilon_k} \longrightarrow \eta \quad \text{in } L^2(0,T;H), \tag{5.21}$$

as $k \to +\infty$. Now, we introduce the operator $\mathcal{A}_\varepsilon$ induced by A_ε on $L^2(0,T;H)$ in the following way

$$\mathcal{A}_\varepsilon : L^2(0,T;H) \longrightarrow L^2(0,T;H)$$

$$\zeta_\varepsilon \in \mathcal{A}_\varepsilon(\eta_\varepsilon) \Longleftrightarrow \zeta_\varepsilon(t) \in A_\varepsilon(\eta_\varepsilon(t)) \quad \text{for a.e. } t \in [0,T]. \tag{5.22}$$

Due to (5.18)–(5.20), we have that

$$\begin{cases} \mathcal{A}_{\varepsilon_k}(\eta_{\varepsilon_k}) \rightharpoonup \zeta & \text{in } L^2(0,T;H), \\ \eta_{\varepsilon_k} \to \eta & \text{in } L^2(0,T;H), \end{cases} \tag{5.23}$$

$$\limsup_{k\to+\infty} \int_Q \zeta_{\varepsilon_k}\eta_{\varepsilon_k} = \int_Q \zeta\eta. \tag{5.24}$$

Thanks to (5.23)–(5.24) and the convergence result [1, Proposition 2.2, p. 38], we conclude that

$$\zeta \in \mathcal{A}(\eta) \quad \text{in } L^2(0,T;H), \tag{5.25}$$

with obvious definition for $\mathcal{A}$ (see (2.17)–(2.18)). This is equivalent to saying that

$$\zeta(t) \in A(a\vartheta(t) + b\varphi(t) - \eta^*) \quad \text{for a.e. } t \in [0,T]. \tag{5.26}$$

Conclusion of the proof Using (5.1)–(5.10), (5.17) and (5.26), we can pass to the limit as $\varepsilon \searrow 0$ in (3.37)–(3.43) obtaining (2.32)–(2.38) for the limiting functions ϑ, φ and μ.

6 Regularity

This section is devoted to the proof of Theorem 2.2. In order to obtain additional regularity for the solutions, we need further a priori estimates obtained from the approximating problem (P_ε) (see (3.37)–(3.43)) in which we take $\vartheta_{0\varepsilon} = \vartheta_0$ and $\varphi_{0\varepsilon} = \varphi_0$.

Seventh a priori estimate. We test (3.37) by $\partial_t \vartheta_\varepsilon$ and integrate over Q_t, $t \in (0, T]$. We have that

$$\int_{Q_t} |\partial_t \vartheta_\varepsilon|^2 + \ell \int_{Q_t} \partial_t \varphi_\varepsilon \partial_t \vartheta_\varepsilon + \frac{1}{2} \int_\Omega |\nabla \vartheta_\varepsilon(t)| + \int_{Q_t} \zeta_\varepsilon \partial_t \vartheta_\varepsilon = \int_{Q_t} f_\varepsilon \partial_t \vartheta_\varepsilon + \frac{1}{2} \int_\Omega |\nabla \vartheta_0|. \quad (6.1)$$

We now proceed with a formal estimate since we have to differentiate (3.38) and (3.39) with respect to time. For a rigorous approach, one can argue, e.g., as in [11, Subsection 4.4]. By time differentiation of (3.38) and (3.39) we have

$$\partial_{tt} \varphi_\varepsilon - \Delta \partial_t \mu_\varepsilon = 0, \quad (6.2)$$

$$\partial_t \mu_\varepsilon = -\nu \Delta \partial_t \varphi_\varepsilon + \beta_\varepsilon'(\varphi_\varepsilon) \partial_t \varphi_\varepsilon + \pi'(\varphi_\varepsilon) \partial_t \varphi_\varepsilon - \gamma \partial_t \vartheta_\varepsilon. \quad (6.3)$$

According to (3.19), $m(\partial_t \varphi_\varepsilon) = 0$. Consequently, $\partial_t \varphi_\varepsilon \in D(\mathcal{N})$ and we can test (6.2) by $\frac{\ell}{\gamma} \mathcal{N}(\partial_t \varphi_\varepsilon)$. Integrating the resultant over Q_t, we obtain that

$$-\frac{\ell}{\gamma} \int_{Q_t} \partial_t \mu_\varepsilon \partial_t \varphi_\varepsilon = \frac{\ell}{2\gamma} \|\partial_t \varphi_\varepsilon(t)\|_{V^*}^2 - \frac{\ell}{2\gamma} \|\partial_t \varphi_\varepsilon(0)\|_{V^*}^2. \quad (6.4)$$

We test (6.3) by $\frac{\ell}{\gamma} \partial_t \varphi_\varepsilon$ and integrate over Q_t. We have that

$$\frac{\ell}{\gamma} \int_{Q_t} \partial_t \mu_\varepsilon \partial_t \varphi_\varepsilon$$

$$= \frac{\nu \ell}{\gamma} \int_{Q_t} |\nabla \partial_t \varphi_\varepsilon|^2 + \frac{\ell}{\gamma} \int_{Q_t} \beta_\varepsilon'(\varphi_\varepsilon) |\partial_t \varphi_\varepsilon|^2 + \frac{\ell}{\gamma} \int_{Q_t} \pi'(\varphi_\varepsilon) |\partial_t \varphi_\varepsilon|^2 - \ell \int_{Q_t} \partial_t \varphi_\varepsilon \partial_t \vartheta_\varepsilon. \quad (6.5)$$

By combining (6.1), (6.4) and (6.5), we infer that

$$\int_{Q_t} |\partial_t \vartheta_\varepsilon|^2 + \frac{1}{2} \int_\Omega |\nabla \vartheta_\varepsilon(t)| + \frac{\nu \ell}{\gamma} \int_{Q_t} |\nabla \partial_t \varphi_\varepsilon|^2 + \frac{\ell}{2\gamma} \|\partial_t \varphi_\varepsilon(t)\|_{V^*}^2 = \frac{1}{2} \int_\Omega |\nabla \vartheta_0|$$

$$+ \int_{Q_t} f_\varepsilon \partial_t \vartheta_\varepsilon + \frac{\ell}{2\gamma} \|\partial_t \varphi_\varepsilon(0)\|_{V^*}^2 - \int_{Q_t} \zeta_\varepsilon \partial_t \vartheta_\varepsilon - \frac{\ell}{\gamma} \int_{Q_t} \beta_\varepsilon'(\varphi_\varepsilon) |\partial_t \varphi_\varepsilon|^2 - \frac{\ell}{\gamma} \int_{Q_t} \pi'(\varphi_\varepsilon) |\partial_t \varphi_\varepsilon|^2. \quad (6.6)$$

By applying inequality (4.2) to the second term on the right-hand side of (6.6), we infer that

$$\int_{Q_t} f_\varepsilon \partial_t \vartheta_\varepsilon \le \|f_\varepsilon\|_{L^2(0,T;H)}^2 + \frac{1}{4} \int_{Q_t} |\partial_t \vartheta_\varepsilon|^2. \quad (6.7)$$

Moreover, as β_ε is a maximal monotone operator, we have that $\beta_\varepsilon' > 0$ and consequently

$$-\frac{\ell}{\gamma}\int_{Q_t}\beta_\varepsilon'(\varphi_\varepsilon)|\partial_t\varphi_\varepsilon|^2 \le 0. \tag{6.8}$$

Due to (4.17), we have that

$$-\int_{Q_t}\zeta_\varepsilon\partial_t\vartheta_\varepsilon \le \int_{Q_t}|\zeta_\varepsilon|^2 + \frac{1}{4}\int_{Q_t}|\partial_t\vartheta_\varepsilon|^2 \le c + \frac{1}{4}\int_{Q_t}|\partial_t\vartheta_\varepsilon|^2. \tag{6.9}$$

As π is a Lipschitz continuous function with Lipschitz constant C_π, we have that

$$-\frac{\ell}{\gamma}\int_{Q_t}\pi'(\varphi_\varepsilon)|\partial_t\varphi_\varepsilon|^2 \le \frac{\ell}{\gamma}\int_{Q_t}|\pi'(\varphi_\varepsilon)||\partial_t\varphi_\varepsilon|^2 \le \frac{C_\pi\ell}{\gamma}\int_{Q_t}|\partial_t\varphi_\varepsilon|^2. \tag{6.10}$$

Adding $\frac{\nu\ell}{\gamma}\int_{Q_t}|\partial_t\varphi_\varepsilon|^2$ to both side of (6.6) and rearranging the right-hand side of (6.6) using (6.7)–(6.10), we obtain that

$$\begin{aligned}&\frac{1}{2}\int_{Q_t}|\partial_t\vartheta_\varepsilon|^2 + \frac{1}{2}\int_\Omega|\nabla\vartheta_\varepsilon(t)| + \frac{\nu\ell}{\gamma}\int_0^t\|\partial_t\varphi_\varepsilon(s)\|_V^2\,ds + \frac{\ell}{2\gamma}\|\partial_t\varphi_\varepsilon(t)\|_{V^*}^2\\ &\le \frac{1}{2}\|\vartheta_0\|_V^2 + \frac{\ell}{2\gamma}\|\partial_t\varphi_\varepsilon(0)\|_{V^*}^2 + \|f_\varepsilon\|_{L^2(0,T;H)}^2 + \left(\frac{C_\pi\ell}{\gamma} + \frac{\nu\ell}{\gamma}\right)\int_0^t\|\partial_t\varphi_\varepsilon(s)\|_{V^*}^2\,ds + c.\end{aligned} \tag{6.11}$$

Thanks to the compactness of the embedding $V \subset H \subset V^*$, the inequality stated by [39, Lemma 8, p. 84] ensures that, choosing

$$\delta = \left(\frac{\nu\ell}{4\gamma}\left(\frac{C_\pi\ell}{\gamma} + \frac{\nu\ell}{\gamma}\right)^{-1}\right)^{\frac{1}{2}},$$

we can estimate the fourth term on the right-hand side of (6.11) as follows

$$\left(\frac{C_\pi\ell}{\gamma} + \frac{\nu\ell}{\gamma}\right)\int_0^t\|\partial_t\varphi_\varepsilon(s)\|_H^2\,ds \le \frac{\nu\ell}{2\gamma}\int_0^t\|\partial_t\varphi_\varepsilon(s)\|_V^2\,ds + c\int_0^t\|\partial_t\varphi_\varepsilon(s)\|_{V^*}^2\,ds. \tag{6.12}$$

Due to (6.12), from (6.11) we have that

$$\begin{aligned}&\frac{1}{2}\int_{Q_t}|\partial_t\vartheta_\varepsilon|^2 + \frac{1}{2}\int_\Omega|\nabla\vartheta_\varepsilon(t)| + \frac{\nu\ell}{2\gamma}\int_0^t\|\partial_t\varphi_\varepsilon(s)\|_V^2\,ds + \frac{\ell}{2\gamma}\|\partial_t\varphi_\varepsilon(t)\|_{V^*}^2\\ &\le \frac{1}{2}\|\vartheta_0\|_V^2 + \frac{\ell}{2\gamma}\|\partial_t\varphi_\varepsilon(0)\|_{V^*}^2 + \|f_\varepsilon\|_{L^2(0,T;H)}^2 + c\int_0^t\frac{\ell}{2\gamma}\|\partial_t\varphi_\varepsilon(s)\|_{V^*}^2\,ds\\ &\le \frac{1}{2}\|\vartheta_0\|_V^2 + \frac{\ell}{2\gamma}\|\partial_t\varphi_\varepsilon(0)\|_{V^*}^2 + \|f_\varepsilon\|_{L^2(0,T;H)}^2 + c\|\varphi_\varepsilon\|_{H^1(0,T;V^*)}^2 + c.\end{aligned} \tag{6.13}$$

Since $(-\nu\Delta\varphi_0 + \beta_\varepsilon(\varphi_0) + \pi(\varphi_0) - \gamma\vartheta_0)$ is bounded in V uniformly with respect to ε according to (2.40), we deduce, by comparison in (3.38)–(3.39), that the second term on the right-hand side of (6.13) is estimated by a positive constant. Hence, due to (2.39), (3.30)–(3.36) and (4.44), the right-hand side of (6.13) is bounded and we conclude that there exists a positive constant c, independent of ε, such that

$$\|\vartheta_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\varphi_\varepsilon\|_{W^{1,\infty}(0,T;V^*)\cap H^1(0,T;V)} \le c. \tag{6.14}$$

Eighth a priori estimate. From (3.37), we have that

$$\Delta\vartheta_\varepsilon = \partial_t(\vartheta_\varepsilon + \ell\varphi_\varepsilon) + \zeta_\varepsilon - f_\varepsilon =: h_\varepsilon. \tag{6.15}$$

We observe that (6.14) ensures that h_ε is bounded in $L^2(0,T;H)$ uniformly with respect to ε. Then we infer that there exists a constant $c > 0$, independent of ε, such that

$$\|\vartheta_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le c. \tag{6.16}$$

Ninth a priori estimate. Due to (6.14)–(6.16), from (4.28) we deduce that

$$\|\xi_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \le c. \tag{6.17}$$

Now, using (4.31), we infer that $\|m(\mu_\varepsilon)\|_{L^\infty(0,T)} \le c$. By comparison in (3.35) and (3.51), it follows that

$$\|\mu_\varepsilon\|_{L^\infty(0,T;V)} \le c. \tag{6.18}$$

Moreover, from (4.39), we obtain that $\|\xi_\varepsilon\|_{L^\infty(0,T;H)} \le c$. Then, by comparison in (3.39), we conclude that

$$\|\Delta\varphi_\varepsilon\|_{L^\infty(0,T;H)\cap L^2(0,T;W)} \le c. \tag{6.19}$$

Conclusion of the proof. As (6.14), (6.16) and (6.17)–(6.19) follow uniformly with respect to ε, the same estimates hold true for the limiting functions ϑ, φ and μ. Hence, (2.41)–(2.43) are fulfilled and

$$\|\vartheta\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le c, \tag{6.20}$$

$$\|\varphi\|_{W^{1,\infty}(0,T;V^*)\cap H^1(0,T;V)\cap L^\infty(0,T;W)} \le c, \tag{6.21}$$

$$\|\mu\|_{L^\infty(0,T;V)} \le c. \tag{6.22}$$

7 Uniqueness and continuous dependence

This section is devoted to the proof of Theorem 2.3.

Assume $a\ell = b$. If f_i, η_i^*, η_{0_i}, φ_{0_i}, $i = 1,2$, are given as in (2.27)–(2.28) and (η_i, φ_i), $i = 1,2$, are the corresponding solutions of problem $(\widetilde{P})$ (see (2.48)–(2.54)), then we can write problem $(\widetilde{P})$ for both (η_i, φ_i), $i = 1,2$ and take the difference between the respective equations. Setting $\eta := \eta_1 - \eta_2$, $\varphi := \varphi_1 - \varphi_2$, $\mu := \mu_1 - \mu_2$, $f := f_1 - f_2$, $\eta^* := \eta_1^* - \eta_2^*$, $\eta_0 := \eta_{0_1} - \eta_{0_2}$, $\varphi_0 := \varphi_{0_1} - \varphi_{0_2}$, we obtain that

$$\partial_t\eta - \Delta\eta + b\Delta\varphi - \Delta\eta^* + a(\zeta_1 - \zeta_2) = af, \tag{7.1}$$

$$\partial_t\varphi - \Delta\mu = 0, \tag{7.2}$$

$$\mu = -\nu\Delta\varphi + \xi_1 - \xi_2 + \pi(\varphi_1) - \pi(\varphi_2) - \frac{\gamma}{a}(\eta - b\varphi + \eta^*). \tag{7.3}$$

We observe that, due to (2.55), $m(\varphi_0) = 0$. Consequently, thanks to (2.46), $m(\varphi) = 0$ and $\varphi \in D(\mathcal{N})$ a.e. in $(0,T)$ (see (2.22)). Now, we test (7.1) by η. Integrating over Q_t, $t \in (0,T]$, we have that

$$\frac{1}{2}\int_\Omega |\eta(t)|^2 + \int_{Q_t} |\nabla\eta|^2 - b\int_{Q_t} \nabla\varphi\cdot\nabla\eta + a\int_{Q_t}(\zeta_1-\zeta_2)(\eta_1-\eta_2)$$
$$= \frac{1}{2}\int_\Omega |\eta_0|^2 + \int_{Q_t}(af+\Delta\eta^*)\eta. \tag{7.4}$$

We test (7.2) by $\frac{b^2}{\nu}\mathcal{N}\varphi$. Integrating over $(0,t)$, we obtain that

$$\frac{b^2}{\nu}\int_0^t \langle\partial_t\varphi(s),\mathcal{N}\varphi(s)\rangle_{V^*,V}\,ds + \frac{b^2}{\nu}\int_{Q_t}\nabla\mu\cdot\nabla\mathcal{N}\varphi = 0,$$
$$\frac{b^2}{2\nu}\|\varphi(t)\|_{V^*}^2 + \frac{b^2}{\nu}\int_{Q_t}\mu\varphi = \frac{b^2}{2\nu}\|\varphi_0\|_{V^*}^2. \tag{7.5}$$

Testing (7.3) by $-\frac{b^2}{\nu}\varphi$ and integrating over Q_t, we have that

$$-\frac{b^2}{\nu}\int_{Q_t}\mu\varphi = -b^2\int_{Q_t}|\nabla\varphi|^2 - \frac{b^2}{\nu}\int_{Q_t}(\xi_1-\xi_2)(\varphi_1-\varphi_2)$$
$$-\frac{b^2}{\nu}\int_{Q_t}[\pi(\varphi_1)-\pi(\varphi_2)](\varphi_1-\varphi_2) + \frac{\gamma b^2}{a\nu}\int_{Q_t}(\eta-b\varphi+\eta^*)\varphi. \tag{7.6}$$

Then, we combine (7.4)–(7.6) and infer that

$$\frac{1}{2}\|\eta(t)\|_H^2 + \int_{Q_t}(|\nabla\eta|^2 - b\nabla\varphi\cdot\nabla\eta + b^2|\nabla\varphi|^2) + \frac{b^2}{2\nu}\|\varphi(t)\|_{V^*}^2$$
$$+a\int_{Q_t}(\zeta_1-\zeta_2)(\eta_1-\eta_2) + \frac{b^2}{\nu}\int_{Q_t}(\xi_1-\xi_2)(\varphi_1-\varphi_2)$$
$$= -\frac{b^2}{\nu}\int_{Q_t}[\pi(\varphi_1)-\pi(\varphi_2)](\varphi_1-\varphi_2) + \frac{\gamma b^2}{a\nu}\int_{Q_t}(\eta-b\varphi+\eta^*)\varphi$$
$$+\frac{b^2}{2\nu}\|\varphi_0\|_{V^*}^2 + \frac{1}{2}\|\eta_0\|_H^2 + \int_{Q_t}(af+\Delta\eta^*)\eta. \tag{7.7}$$

Since A and β are maximal monotone, we have that

$$a\int_{Q_t}(\zeta_1-\zeta_2)(\eta_1-\eta_2) \geq 0, \tag{7.8}$$

$$\frac{b^2}{\nu}\int_{Q_t}(\xi_1-\xi_2)(\varphi_1-\varphi_2) \geq 0. \tag{7.9}$$

Moreover, thanks to the Lipschitz continuity of π, we infer that

$$\begin{aligned} -\frac{b^2}{\nu}\int_{Q_t}[\pi(\varphi_1)-\pi(\varphi_2)](\varphi_1-\varphi_2) &\leq \frac{b^2}{\nu}\int_{Q_t}|\pi(\varphi_1)-\pi(\varphi_2)||\varphi_1-\varphi_2| \\ &\leq \frac{C_\pi b^2}{\nu}\int_{Q_t}|\varphi|^2. \end{aligned} \tag{7.10}$$

We also notice that the integral involving the gradients is estimated from below in this way:

$$\int_{Q_t} (|\nabla\eta|^2 - b\nabla\varphi\cdot\nabla\eta + b^2|\nabla\varphi|^2) \geq \frac{1}{2}\int_{Q_t} (|\nabla\eta|^2 + b^2|\nabla\varphi|^2). \tag{7.11}$$

Recalling that

$$-\frac{\gamma b^3}{a\nu}\int_{Q_t} |\varphi|^2 \leq 0, \tag{7.12}$$

applying inequality (4.2) to the second and fifth term on the right-hand side of (7.7), using (7.8)–(7.11) and adding to both sides $b^2\int_0^t \|\varphi(s)\|_H^2\, ds$, we infer that

$$\frac{1}{2}\|\eta(t)\|_H^2 + \int_{Q_t} |\nabla\eta|^2 + b^2\int_0^t \|\varphi(s)\|_V^2\, ds + \frac{b^2}{2\nu}\|\varphi(t)\|_{V^*}^2$$

$$\leq (K+b^2)\int_0^t \|\varphi(s)\|_H^2\, ds + \frac{1}{2}\int_{Q_t} |\eta|^2 + \frac{b^2}{2\nu}\|\varphi_0\|_{V^*}^2 + \frac{1}{2}\|\eta_0\|_H^2 + 2a^2\|f\|_{L^2(0,T;H)}^2 + 3T\|\eta^*\|_W^2, \tag{7.13}$$

where

$$K = \left[\frac{C_\pi b^2}{\nu} + 2\Big(\frac{\gamma b^2}{a\nu}\Big)^2\right].$$

We observe that, for every $\delta > 0$,

$$\|\varphi(t)\|_H^2 = \langle\varphi(t), \varphi(t)\rangle_{V^*,V} \leq \|\varphi(t)\|_{V^*}\|\varphi(t)\|_V \leq \frac{\delta}{2}\|\varphi(t)\|_V^2 + \frac{1}{2\delta}\|\varphi(t)\|_{V^*}^2. \tag{7.14}$$

Choosing $\delta = \frac{b^2}{K+b^2}$ in (7.14), we can estimate the first term of the right-hand side of (7.13) as follows:

$$(K+b^2)\int_0^t \|\varphi(s)\|_H^2\, ds \leq \frac{b^2}{2}\int_0^t \|\varphi(s)\|_V^2\, ds + \frac{(K+b^2)^2\nu}{b^4}\int_0^t \frac{b^2}{2\nu}\|\varphi(s)\|_{V^*}^2\, ds. \tag{7.15}$$

Then, due to (7.15), from (7.13) we obtain that

$$\frac{1}{2}\|\eta(t)\|_H^2 + \int_{Q_t} |\nabla\eta|^2 + \frac{b^2}{2}\int_0^t \|\varphi(s)\|_V^2\, ds + \frac{b^2}{2\nu}\|\varphi(t)\|_{V^*}^2$$

$$\leq c\int_0^t \left(\frac{1}{2}\|\eta(s)\|_H^2 + \frac{b^2}{2\nu}\|\varphi(s)\|_{V^*}^2\right) ds + \frac{b^2}{2\nu}\|\varphi_0\|_{V^*}^2 + \frac{1}{2}\|\eta_0\|_H^2 + 2a^2\|f\|_{L^2(0,T;H)}^2 + 3T\|\eta^*\|_W^2. \tag{7.16}$$

Due to (2.27)–(2.31), the last four terms on the right-hand side of (7.16) are bounded uniformly with respect to ε . Then, by applying the Gronwall lemma, we conclude that

$$\|\eta(t)\|_H + \|\nabla\eta\|_{L^2(0,T;H)} + \|\varphi\|_{L^2(0,T;V)} + \|\varphi(t)\|_{V^*}$$

$$\leq c\Big(\|\varphi_0\|_{V^*} + \|\eta_0\|_H + \|f\|_{L^2(0,T;H)} + \|\eta^*\|_W\Big) \tag{7.17}$$

for some positive constant c which depends only on Ω, T and the structure (2.7)–(2.8), (2.14)–(2.16) and (2.26)–(2.28) of the system. Now, we recall that (7.17) is equivalent to

$$\|\eta_1 - \eta_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\varphi_1 - \varphi_2\|_{L^\infty(0,T;V^*)\cap L^2(0,T;V)}$$

$$\le c\big(\|\varphi_{0_1}-\varphi_{0_2}\|_{V^*}+\|\eta_{0_1}-\eta_{0_2}\|_H+\|f_1-f_2\|_{L^2(0,T;H)}+\|\eta_1^*-\eta_2^*\|_W\big). \tag{7.18}$$

If $f_1 = f_2$, $\eta_1^* = \eta_2^*$, $\eta_{0_1} = \eta_{0_2}$ and $\varphi_{0_1} = \varphi_{0_2}$, from (7.18) we conclude that $\eta_1 = \eta_2$ and $\varphi_1 = \varphi_2$, i.e., the solution of problem $(\widetilde{P})$ (see (2.48)–(2.54)) is unique. From this fact, we immediately infer the uniqueness of the solution for our initial Problem (P) (see (2.32)–(2.38)).

8 Sliding mode control

This section is devoted to the proof of Theorem 2.4. The argument we use in the proof relies in the following Lemma (see [2, Lemma 4.1, p. 20]).

Lemma 8.1 *Let a_0, b_0, ψ_0, $\rho\in\mathbb{R}$ be such that*

$$a_0,\ b_0,\ \psi_0\ge 0 \quad \textit{and} \quad \rho > a_0^2+2b_0+2\frac{\psi_0}{T} \tag{8.1}$$

and let $\psi:\ [0,T]\to[0,+\infty)$ be an absolutely continuous function satisfying $\psi(0)=\psi_0$ and

$$\psi'+\rho\le a_0\rho^{1/2}+b_0 \quad \textit{a.e. in the set } P:=\{t\in(0,T):\psi(t)>0\}. \tag{8.2}$$

Then, the following conditions hold true:

1. *If $\psi_0=0$, then ψ vanishes identically.*
2. *If $\psi_0>0$, then there exists $T^*\in(0,T)$ satisfying $T^*\le 2\psi_0/(\rho-a_0^2-2b_0)$ such that ψ is strictly decreasing in $(0,T^*)$ and ψ vanishes in $[T^*,T]$.*

We assume $a=1$, $b=\ell$ and $A=\rho\,\mathrm{Sign}$ and consider the approximating problem $(\widetilde{P}_\varepsilon)$ obtained from (P_ε) (see (3.37)–(3.43)) with the usual change of variables

$$\eta_\varepsilon=\vartheta_\varepsilon+\ell\varphi_\varepsilon-\eta^*, \qquad \eta_{0\varepsilon}=\vartheta_{0\varepsilon}+\ell\varphi_{0\varepsilon}-\eta^*. \tag{8.3}$$

We have that

$$\partial_t\eta_\varepsilon-\Delta\eta_\varepsilon+\ell\Delta\varphi_\varepsilon-\Delta\eta^*+\rho\sigma_\varepsilon=f_\varepsilon \ \text{ a.e. in } Q, \tag{8.4}$$

$$\partial_t\varphi_\varepsilon-\Delta\mu_\varepsilon=0 \ \text{ a.e. in } Q, \tag{8.5}$$

$$\mu_\varepsilon=-\nu\Delta\varphi_\varepsilon+\xi_\varepsilon+\pi(\varphi_\varepsilon)-\gamma(\eta_\varepsilon-\ell\varphi_\varepsilon+\eta^*) \ \text{ a.e. in } Q, \tag{8.6}$$

$$\sigma_\varepsilon(t)\in\mathrm{Sign}_\varepsilon(\eta_\varepsilon(t)) \text{ for a.e. } t\in(0,T), \tag{8.7}$$

$$\xi_\varepsilon\in\beta_\varepsilon(\varphi_\varepsilon) \text{ a.e. in } Q, \tag{8.8}$$

$$\partial_{\mathbf{n}}\eta_\varepsilon=\partial_{\mathbf{n}}\varphi_\varepsilon=\partial_{\mathbf{n}}\mu_\varepsilon=0 \ \text{ on } \Sigma, \tag{8.9}$$

$$\eta_\varepsilon(0)=\eta_{0\varepsilon}, \qquad \varphi_\varepsilon(0)=\varphi_{0\varepsilon} \quad \text{in } \Omega. \tag{8.10}$$

Further a priori uniform estimates. We test (8.4) by $\partial_t\eta_\varepsilon$ and integrate over Q_t. Recalling that

$$\int_{Q_t} \rho\sigma_\varepsilon\partial_t\eta_\varepsilon = \rho\|\eta_\varepsilon(t)\|_{H,\varepsilon} - \rho\|\eta_0\|_{H,\varepsilon}, \tag{8.11}$$

we have that

$$\int_{Q_t} |\partial_t\eta_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\nabla\eta_\varepsilon(t)|^2 + \rho\|\eta_\varepsilon(t)\|_{H,\varepsilon} = \frac{1}{2}\int_\Omega |\nabla\eta_0|^2$$

$$+\rho\|\eta_0\|_{H,\varepsilon} + \int_{Q_t} \Delta\eta^*\partial_t\eta_\varepsilon + \int_{Q_t} f_\varepsilon\partial_t\eta_\varepsilon - \int_{Q_t} \ell\Delta\varphi_\varepsilon\partial_t\eta_\varepsilon. \tag{8.12}$$

We observe that $\|\eta_0\|_{H,\varepsilon} \le \|\eta_0\|_H$ (cf. (3.7)). Then, thanks to (2.28) and (2.40), the first two terms on the right-hand side of (8.12) are estimated as follows:

$$\frac{1}{2}\int_\Omega |\nabla\eta_0|^2 + \rho\|\eta_0\|_{H,\varepsilon} \le c(1+\rho). \tag{8.13}$$

Due to (2.28) and (3.32), applying (4.2) to the third and fourth term on the right-hand side of (8.12), we have that

$$\int_{Q_t} \Delta\eta^*\partial_t\eta_\varepsilon \le \frac{1}{4}\int_{Q_t} |\partial_t\eta_\varepsilon|^2 + \int_{Q_t} |\Delta\eta^*|^2 = \frac{1}{4}\int_{Q_t} |\partial_t\eta_\varepsilon|^2 + c, \tag{8.14}$$

$$\int_{Q_t} f_\varepsilon\partial_t\eta_\varepsilon \le \frac{1}{4}\int_{Q_t} |\partial_t\eta_\varepsilon|^2 + \int_{Q_t} |f_\varepsilon|^2 \le \frac{1}{4}\int_{Q_t} |\partial_t\eta_\varepsilon|^2 + c. \tag{8.15}$$

Moreover, integrating by parts the last term of (8.12), we formally have that

$$-\int_{Q_t} \ell\Delta\varphi_\varepsilon\partial_t\eta_\varepsilon = \ell\int_{Q_t} \nabla\varphi_\varepsilon\cdot\nabla(\partial_t\eta_\varepsilon)$$

$$= \ell\int_\Omega \nabla\varphi_\varepsilon(t)\cdot\nabla\eta_\varepsilon(t) - \ell\int_\Omega \nabla\varphi_0\cdot\nabla\eta_0 - \ell\int_{Q_t} \nabla(\partial_t\varphi_\varepsilon)\cdot\nabla\eta_\varepsilon. \tag{8.16}$$

Using (4.2) and the Hölder inequality, the first term on the right-hand side of (8.16) is estimated as follows:

$$\left|\ell\int_\Omega \nabla\varphi_\varepsilon(t)\cdot\nabla\eta_\varepsilon(t)\right| \le \frac{1}{4}\int_\Omega |\nabla\eta_\varepsilon(t)|^2 + \ell^2\int_\Omega |\nabla\varphi_\varepsilon(t)|^2$$

$$= \frac{1}{4}\int_\Omega |\nabla\eta_\varepsilon(t)|^2 + \ell^2\int_\Omega \left|\nabla\left(\varphi_0 + \int_0^t \partial_t\varphi_\varepsilon(s)\,ds\right)\right|^2$$

$$\le \frac{1}{4}\int_\Omega |\nabla\eta_\varepsilon(t)|^2 + 2\ell^2\int_\Omega |\nabla\varphi_0|^2 + 2\ell^2\int_\Omega \left|\int_0^t \nabla(\partial_t\varphi_\varepsilon(s))\,ds\right|^2$$

$$\le \frac{1}{4}\int_\Omega |\nabla\eta_\varepsilon(t)|^2 + 2\ell^2\int_\Omega |\nabla\varphi_0|^2 + 2T\ell^2\int_{Q_t} |\nabla(\partial_t\varphi_\varepsilon)|^2. \tag{8.17}$$

Due to (2.28), the second term on the right-hand side of (8.16) and similarly the second term on the right-hand side of (8.17) are estimated by a positive constant c independent of ρ and ε. Indeed

$$-\ell \int_\Omega \nabla\varphi_0 \cdot \nabla\eta_0 \le \ell^2 \int_\Omega |\nabla\varphi_0|^2 + \frac{1}{4}\int_\Omega |\nabla\eta_0|^2 \le c. \tag{8.18}$$

Applying inequality (4.2) to the last term on the right-hand side of (8.16) we obtain that

$$-\ell \int_{Q_t} \nabla(\partial_t\varphi_\varepsilon) \cdot \nabla\eta_\varepsilon \le \frac{1}{4}\int_{Q_t} |\nabla\eta_\varepsilon|^2 + \ell^2 \int_{Q_t} |\nabla(\partial_t\varphi_\varepsilon)|^2. \tag{8.19}$$

Then, thanks to (8.13)–(8.19), from (8.12) we infer that

$$\begin{aligned}
&\frac{1}{2}\int_{Q_t} |\partial_t\eta_\varepsilon|^2 + \frac{1}{4}\int_\Omega |\nabla\eta_\varepsilon(t)|^2 + \rho\|\eta_\varepsilon(t)\|_{H,\varepsilon} \\
&\le c(1+\rho) + \ell^2(1+2T)\int_{Q_t} |\nabla(\partial_t\varphi_\varepsilon)|^2 + \frac{1}{4}\int_{Q_t} |\nabla\eta_\varepsilon|^2.
\end{aligned} \tag{8.20}$$

Now, we formally differentiate (8.5) and (8.6) with respect to time and obtain that

$$\partial_{tt}\varphi_\varepsilon - \Delta\partial_t\mu_\varepsilon = 0, \tag{8.21}$$

$$\partial_t\mu_\varepsilon = -\nu\Delta\partial_t\varphi_\varepsilon + \beta_\varepsilon'(\varphi_\varepsilon)\partial_t\varphi_\varepsilon + \pi'(\varphi_\varepsilon)\partial_t\varphi_\varepsilon - \gamma(\partial_t\eta_\varepsilon - \ell\partial_t\varphi_\varepsilon). \tag{8.22}$$

According to (3.19), $m(\partial_t\varphi_\varepsilon) = 0$. Consequently, $\partial_t\varphi_\varepsilon \in D(\mathcal{N})$ and we can test (8.21) by $\mathcal{N}(\partial_t\varphi_\varepsilon)$ and (8.22) by $\partial_t\varphi_\varepsilon$, respectively. Integrating over Q_t, we have that

$$-\int_{Q_t} \partial_t\mu_\varepsilon\partial_t\varphi_\varepsilon = \frac{1}{2}\|\partial_t\varphi_\varepsilon(t)\|_{V^*}^2 - \frac{1}{2}\|\partial_t\varphi_\varepsilon(0)\|_{V^*}^2, \tag{8.23}$$

$$\begin{aligned}
\int_{Q_t} \partial_t\mu_\varepsilon\partial_t\varphi_\varepsilon &= \nu\int_{Q_t} |\nabla\partial_t\varphi_\varepsilon|^2 + \int_{Q_t} \beta_\varepsilon'(\varphi_\varepsilon)|\partial_t\varphi_\varepsilon|^2 \\
&+ \int_{Q_t} \pi'(\varphi_\varepsilon)|\partial_t\varphi_\varepsilon|^2 - \gamma\int_{Q_t} \partial_t\varphi_\varepsilon\partial_t\eta_\varepsilon + \ell\gamma\int_{Q_t} |\partial_t\varphi_\varepsilon|^2.
\end{aligned} \tag{8.24}$$

Combining (8.23) and (8.24) we obtain that

$$\begin{aligned}
&\frac{1}{2}\|\partial_t\varphi_\varepsilon(t)\|_{V^*}^2 + \nu\int_{Q_t} |\nabla\partial_t\varphi_\varepsilon|^2 + \ell\gamma\int_{Q_t} |\partial_t\varphi_\varepsilon|^2 = \frac{1}{2}\|\partial_t\varphi_\varepsilon(0)\|_{V^*}^2 \\
&- \int_{Q_t} \beta_\varepsilon'(\varphi_\varepsilon)|\partial_t\varphi_\varepsilon|^2 - \int_{Q_t} \pi'(\varphi_\varepsilon)|\partial_t\varphi_\varepsilon|^2 + \gamma\int_{Q_t} \partial_t\eta_\varepsilon\partial_t\varphi_\varepsilon.
\end{aligned} \tag{8.25}$$

Thanks to (2.28) and (2.40), the first term on the right-hand side of (8.25) is bounded by a positive constant c independent of ρ and ε (cf. the analogous bound discussed below (6.13)). Since β_ε is maximal monotone, the second term on the right-hand side of (8.25) is non-positive. As π is a Lipschitz continuous function with Lipschitz constant C_π, we have that

$$-\int_{Q_t} \pi'(\varphi_\varepsilon)|\partial_t\varphi_\varepsilon|^2 \le C_\pi \int_{Q_t} |\partial_t\varphi_\varepsilon|^2. \tag{8.26}$$

Finally, using (4.2), the last term on the right-hand side of (8.25) is estimated as follows:

$$\gamma \int_{Q_t} \partial_t \eta_\varepsilon \partial_t \varphi_\varepsilon \leq \frac{1}{4}\left(\frac{\nu}{\ell^2(1+2T)+1}\right) \int_{Q_t} |\partial_t \eta_\varepsilon|^2 + \gamma^2 \frac{\ell^2(1+2T)+1}{\nu} \int_{Q_t} |\partial_t \varphi_\varepsilon|^2, \quad (8.27)$$

where the reason of such involved constants will be clear in a moment. Due to (8.26)–(8.27) and the previous observations, from (8.25) we infer that

$$\frac{1}{2}\|\partial_t \varphi_\varepsilon(t)\|_{V^*}^2 + \nu \int_{Q_t} |\nabla \partial_t \varphi_\varepsilon|^2 + \ell\gamma \int_{Q_t} |\partial_t \varphi_\varepsilon|^2$$

$$\leq c + \frac{1}{4}\left(\frac{\nu}{\ell^2(1+2T)+1}\right) \int_{Q_t} |\partial_t \eta_\varepsilon|^2 + \left(\gamma^2 \frac{\ell^2(1+2T)+1}{\nu} + C_\pi\right) \int_{Q_t} |\partial_t \varphi_\varepsilon|^2. \quad (8.28)$$

Multiplying (8.28) by $(\ell^2(1+2T)+1)/\nu$ and adding it to (8.20), we infer that

$$\frac{1}{4}\int_{Q_t} |\partial_t \eta_\varepsilon|^2 + \frac{1}{4}\int_\Omega |\nabla \eta_\varepsilon(t)|^2 + \rho\|\eta_\varepsilon(t)\|_{H,\varepsilon} + \int_{Q_t} |\nabla \partial_t \varphi_\varepsilon|^2 + C_1 \int_{Q_t} |\partial_t \varphi_\varepsilon|^2$$

$$+ C_2\|\partial_t \varphi_\varepsilon(t)\|_{V^*}^2 \leq c(1+\rho) + \frac{1}{4}\int_{Q_t} |\nabla \eta_\varepsilon|^2 + C_3 \int_{Q_t} |\partial_t \varphi_\varepsilon|^2, \quad (8.29)$$

where

$$C_1 = \frac{\ell^3\gamma(1+2T)+\ell\gamma}{\nu}, \qquad C_2 = \frac{\ell^2(1+2T)+1}{2\nu},$$

$$C_3 = \gamma^2\left(\frac{\ell^2(1+2T)+1}{\nu}\right)^2 + C_\pi \frac{\ell^2(1+2T)+1}{\nu} + \ell^2(1+2T).$$

Denoting by C_4 the minimum between 1 and C_1, and applying the inequality (4.3) with $\delta = \sqrt{C_4}/\sqrt{2C_3}$ to the last term on the right-hand side of (8.29), we obtain that

$$C_3 \int_{Q_t} |\partial_t \varphi_\varepsilon|^2 \leq \frac{C_4}{2}\int_0^t \|\partial_t \varphi_\varepsilon(s)\|_V^2 \, ds + 2K^2 C_3 \int_0^t \|\partial_t \varphi_\varepsilon(s)\|_{V^*}^2 \, ds. \quad (8.30)$$

Thanks to (8.30), from (8.29) we infer that

$$\frac{1}{4}\int_{Q_t} |\partial_t \eta_\varepsilon|^2 + \frac{1}{4}\int_\Omega |\nabla \eta_\varepsilon(t)|^2 + \rho\|\eta_\varepsilon(t)\|_{H,\varepsilon} + \frac{C_4}{2}\int_0^t \|\partial_t \varphi_\varepsilon(s)\|_V^2 + C_2\|\partial_t \varphi_\varepsilon(t)\|_{V^*}^2$$

$$\leq c(1+\rho) + \frac{1}{4}\int_{Q_t} |\nabla \eta_\varepsilon|^2 + 2K^2 C_3 \int_0^t \|\partial_t \varphi_\varepsilon(s)\|_{V^*}^2. \quad (8.31)$$

From (8.31), by applying the Gronwall lemma, we conclude that

$$\|\partial_t \eta_\varepsilon\|_{L^2(0,T;H)} + \|\eta_\varepsilon\|_{L^\infty(0,T;V)} + \|\partial_t \varphi_\varepsilon\|_{L^\infty(0,T;V^*)} + \|\partial_t \varphi_\varepsilon\|_{L^2(0,T;V)} \leq c(1+\rho^{1/2}), \quad (8.32)$$

whence

$$\|\eta_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq c(1+\rho^{1/2}), \quad (8.33)$$

$$\|\varphi_\varepsilon\|_{W^{1,\infty}(0,T;V^*)\cap H^1(0,T;V)} \leq c(1+\rho^{1/2}). \quad (8.34)$$

Due to (8.33)–(8.34) and the change of variables stated by (8.3), we have that

$$\|\vartheta_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq c(1+\rho^{1/2}). \quad (8.35)$$

Proceeding as in the second a priori estimate (cf. (4.19)–(4.27)) and recalling (8.34)–(8.35), from (4.28) we infer that

$$\|\xi_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq c(1+\rho^{1/2}). \tag{8.36}$$

Now, with the analogous technique applied in the third a priori estimate, thanks to (8.34)–(8.36), from (4.31) we obtain that

$$\|m(\mu_\varepsilon)\|_{L^\infty(0,T)} \leq c(1+\rho^{1/2}). \tag{8.37}$$

Then, due to (8.37) and the Poincaré inequality, by comparison in (8.5) we deduce that

$$\|\mu_\varepsilon\|_{L^\infty(0,T;V)} \leq c(1+\rho^{1/2}). \tag{8.38}$$

Finally, with the same computations as explained in the fifth a priori estimate (cf. (4.36)–(4.38)), thanks to (8.34)–(8.35) and (8.38), from (4.39) we infer that

$$\|\xi_\varepsilon\|_{L^\infty(0,T;H)} \leq c(1+\rho^{1/2}), \tag{8.39}$$

whence, by comparison of every term in (8.6), we conclude that

$$\|\Delta\varphi_\varepsilon\|_{L^\infty(0,T;H)} \leq c(1+\rho^{1/2}). \tag{8.40}$$

Existence of sliding mode. Due to (2.28), (3.32) and (8.40), we can rewrite (8.4) in the form

$$\partial_t\eta_\varepsilon - \Delta\eta_\varepsilon + \rho\sigma_\varepsilon = g_\varepsilon := f_\varepsilon - \ell\Delta\varphi_\varepsilon + \Delta\eta^*, \tag{8.41}$$

with

$$\|g_\varepsilon\|_{L^\infty(0,T;H)} \leq c(1+\rho^{1/2}), \tag{8.42}$$

where c depends only on the structure and the data involved in the statement. In order to prove the existence of sliding mode, we fix the constant c appearing in (8.42) and set

$$\rho^* := c^2 + 2c + \frac{2}{T}\|\vartheta_0 + \ell\varphi_0 - \eta^*\|_H \tag{8.43}$$

and assume $\rho > \rho^*$. We also set

$$\psi_\varepsilon(t) := \|\eta_\varepsilon(t)\|_H \quad \text{for } t \in [0,T]. \tag{8.44}$$

By assuming $h \in (0,T)$ and $t \in (0, T-h)$, we multiply (8.41) by $\sigma_\varepsilon = \mathrm{Sign}_\varepsilon(\eta_\varepsilon)$ and integrate over $(t,t+h)\times\Omega$. We have that

$$\begin{aligned}\int_t^{t+h} (\partial_t\eta_\varepsilon(s), \sigma_\varepsilon(s))_H \, ds + \int_t^{t+h}\int_\Omega \nabla\eta_\varepsilon\cdot\nabla\sigma_\varepsilon + \rho\int_t^{t+h}\|\sigma_\varepsilon(s)\|_H^2\, ds \\ = \int_t^{t+h} (g_\varepsilon(s), \sigma_\varepsilon(s))_H\, ds.\end{aligned} \tag{8.45}$$

Recalling that $\mathrm{Sign}_\varepsilon(v)$ is the gradient at v of the C^1 functional $\|\cdot\|_{H,\varepsilon}$, from (3.7)–(3.8) we deduce that

$$(\partial_t\eta_\varepsilon(s), \sigma_\varepsilon(s))_H = \frac{d}{dt}\int_0^{\psi_\varepsilon(t)} \min\{s/\varepsilon,\ 1\}\, ds \quad \text{for a.a. } t \in (0,T).$$

Then, for the first term on the right-hand side of (8.45) we have that

$$\int_t^{t+h} (\partial_t \eta_\varepsilon(s), \sigma_\varepsilon(s))_H \, ds = \int_{\psi_\varepsilon(t)}^{\psi_\varepsilon(t+h)} \min\{s/\varepsilon, 1\} \, ds.$$

We also notice that (3.8) implies that

$$\nabla \eta_\varepsilon(t) \cdot \nabla \sigma_\varepsilon(t) = \frac{|\nabla \eta_\varepsilon(t)|^2}{\max\{\varepsilon, \|\eta_\varepsilon(t)\|_H\}} \geq 0 \quad \text{a.e. in } \Omega, \text{ for a.e. } t \in (0,T),$$

whence the second integral on the left-hand side of (8.45) is nonnegative. Moreover, as $\|\sigma_\varepsilon(s)\|_H \leq 1$ for every s (see (2.20)) and (8.42) holds, we infer from (8.45) that

$$\int_{\psi_\varepsilon(t)}^{\psi_\varepsilon(t+h)} \min\{s/\varepsilon, 1\} \, ds + \rho \int_t^{t+h} \|\sigma_\varepsilon(s)\|_H^2 \, ds \leq hc(\rho^{1/2} + 1). \tag{8.46}$$

At this point, we let $\varepsilon \searrow 0$. Due to (5.8)–(5.9), (8.3) and the uniqueness of the solution of the limit Problem (2.48)–(2.54) (cf. Theorem 2.3) we have that

$$\eta_\varepsilon \to \eta \qquad \text{in } C^0(0,T;H). \tag{8.47}$$

Besides, using standard weak, weakstar and compactness results, from (8.46) we infer that

$$\sigma_\varepsilon \rightharpoonup^* \sigma \qquad \text{in } L^\infty(0,T;H). \tag{8.48}$$

Then, taking the limit as $\varepsilon \searrow 0$ in (8.46) and denoting by

$$\psi(t) := \|\eta(t)\|_H \qquad \text{for } t \in [0,T], \tag{8.49}$$

we obtain that

$$\begin{aligned} &\psi(t+h) - \psi(t) + \rho \int_t^{t+h} \|\sigma(s)\|_H^2 \, ds \\ &\leq \lim_{\varepsilon \searrow 0} \int_{\psi_\varepsilon(t)}^{\psi_\varepsilon(t+h)} \min\{s/\varepsilon, 1\} \, ds + \rho \liminf_{\varepsilon \searrow 0} \int_t^{t+h} \|\sigma_\varepsilon(s)\|_H^2 \, ds \leq hc(\rho^{1/2}+1) \end{aligned} \tag{8.50}$$

for every $h \in (0,T)$ and $t \in (0, T-h)$. Finally, we multiply (8.50) by $1/h$ and let h tend to zero. We conclude that

$$\psi'(t) + \rho \|\sigma(t)\|_H^2 \leq c(\rho^{1/2} + 1) \quad \text{for a.a. } t \in (0,T). \tag{8.51}$$

As $\|\sigma(t)\|_H = 1$ if $\|\eta(t)\|_H > 0$ (see (2.20)), we can apply Lemma 8.1 with $a_0 = b_0 = c$ and we observe that our condition $\rho > \rho^*$ completely fits the assumptions by (8.43). Thus, we find $T^* \in [0,T)$ such that $\eta(t) = 0$ for every $t \in [T^*, T]$, i.e., (2.60).

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