A variational proof of partial regularity for optimal transportation maps

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Abstract

We provide a new proof of the known partial regularity result for the optimal transportation map (Brenier map) between two Hölder continuous densities. Contrary to the existing regularity theory for the Monge-Ampère equation, which is based on the maximum principle, our approach is purely variational. By constructing a competitor on the level of the Eulerian (Benamou-Brenier) formulation, we show that locally, the velocity is close to the gradient of a harmonic function provided the transportation cost is small. We then translate back to the Lagrangian description and perform a Campanato iteration to obtain an ε-regularity result.

1 Introduction

For $\alpha \in (0,1)$, let ρ_0 and ρ_1 be two probability densities with bounded support which are $C^{0,\alpha}$ continuous, bounded and bounded away from zero on their support and let T be the solution of the optimal transportation problem

$$\min_{T\sharp\rho_0=\rho_1} \int_{\mathbb{R}^d} |T(x)-x|^2 \rho_0(x)dx, \tag{1.1}$$

where with a slight abuse of notation $T\sharp\rho_0$ denotes the push-forward by T of the measure $\rho_0 dx$ (existence and characterization of T as the gradient of a convex function ψ are given by Brenier's Theorem, see [21, Th. 2.12]). Our main result is a partial regularity theorem for T:

Theorem 1.1. *There exist open sets $E \subseteq \operatorname{spt}\rho_0$ and $F \subseteq \operatorname{spt}\rho_1$ of full measure such that T is a $C^{1,\alpha}$-diffeomorphism between E and F.*

This theorem is a consequence of Alexandrov Theorem [22, Th. 14.25] and the following $\varepsilon-$regularity theorem:

Theorem 1.2. *Let T be the minimizer of (1.1) and assume that $\rho_0(0) = \rho_1(0) = 1$. There exists $\varepsilon(\alpha, d)$ such that if*[i]

$$\frac{1}{(2R)^{d+2}} \int_{B_{2R}} |T - x|^2 \rho_0 dx + R^{2\alpha}[\rho_0]^2_{\alpha,2R} + R^{2\alpha}[\rho_1]^2_{\alpha,2R} \leq \varepsilon,$$

then, T is $C^{1,\alpha}$ inside B_R.

Theorem 1.1 was already obtained by Figalli [11] in the case of planar transportation between sets and is a slightly weaker version of a result obtained by Figalli and Kim [12] (see also [9, 15] for a far-reaching generalization), but our proof departs from the usual scheme for proving regularity for the Monge-Ampère equation. Indeed, while most proofs use some variants of the maximum principle, our proof is variational. The classical approach operates on the level of the convex potential ψ and the ground-breaking paper in that respect is Caffarelli's [5]: By comparison with simple barriers it is shown that an Alexandrov (and thus viscosity) solution ψ of the Monge-Ampère equation is C^1, provided its convexity does not degenerate along a line crossing the entire domain of definition. The same author shows in [6] by similar arguments that the potential ψ of the Brenier map is a strictly convex Alexandrov solution, and thus regular, provided the target domain $\operatorname{spt} \rho_1$ is convex. The challenge in [12] is to follow the above line of arguments while avoiding the notion of Alexandrov solution, more precisely, without having access to the Upper Alexandrov estimate. The $\varepsilon-$regularity theorem in [12] in turn is used by Figalli and De Philippis as the core for a generalization to general cost functions by means of a Campanato iteration. On the contrary to these papers, we work directly at the level of the optimal transportation map T, and besides the L^∞ bound (see (3.11)) given by McCann's displacement convexity, we only use variational arguments.

Let us now give an outline of the proof. As in many $\varepsilon-$regularity results, it goes through a Campanato iteration (see Proposition 3.7). This scheme relies on two ingredients. The first is an 'improvement of flatness by tilting", see Proposition 3.6. This means that if the energy in a given ball is small then, up to a change of coordinates, the energy has a geometric decay on a smaller scale. The second ingredient is the invariance of the variational problem under affine transformations.
The main idea behind the proof of the one-step improvement Proposition 3.6 is the well-known fact that the linearization of the Monge-Ampère equation gives rise to the Laplace equation [21, Sec. 7.6]. In Proposition 3.5, we make this statement more quantitative and prove that if the energy in a given ball is small enough, then in the half-sized ball and up to an error which is super-linear in the energy, T is equal to the gradient of a harmonic function. Once we have this approximation result, using that by classical elliptic regularity, harmonic functions are close to their second-order Taylor expansion, we obtain Proposition 3.6.
The core of our proof is thus Proposition 3.5, which is actually established at the Eulerian

[i] here $[\rho]_{\alpha,R} := \sup_{x,y\in B_R} \frac{|\rho(x)-\rho(y)|}{|x-y|^\alpha}$ denotes the $C^{0,\alpha}-$semi-norm.

level (i.e. for the solutions of the Benamou-Brenier formulation of optimal transportation, see [21, Th. 8.1] or [3, Chap. 8]), see Proposition 3.3. It is for this result that we need the outcome of McCann's displacement convexity, cf. Lemma 3.2, since it is required for the quasi-orthogonality property (see Step 3 of the proof of Proposition 3.3). Our argument is variational and proceeds by defining a competitor based on the solution of a Poisson equation with suitable flux boundary conditions, and a boundary-layer construction. The boundary-layer construction is carried out in Lemma 2.4; by a duality argument it reduces to a trace estimate (see Lemma 2.3). This part of the proof is reminiscent of arguments from [1].

This entire approach to ε-regularity is guided by De Giorgi's strategy for minimal surfaces (see [16] for instance). Let us notice that because of the natural scaling of the problem, our Campanato iteration operates directly at the $C^{1,\alpha}$-level for T, as opposed to [12, 9], where $C^{0,\alpha}$-regularity is obtained first.

Motivated by applications to the optimal matching problem, we extended in [14] together with M. Huesmann, Proposition 3.5 to arbitrary target measures.

The plan of the paper is the following. In Section 2, we recall some well-known facts about the Poisson equation and then prove Lemma 2.4, the proof of which is based on the trace estimate given by Lemma 2.3. In the final section, we prove Theorem 1.2 and then Theorem 1.1.

Since it simplifies some of the technicalities, we suggest at first reading to consider the simpler case of transportation between sets i.e. $\rho_0 = \chi_E$ and $\rho_1 = \chi_F$ for some sets E and F. A previous version of this paper treating that case is available on our webpages.

Notation

In the paper we will use the following notation. The symbols $\sim$, $\gtrsim$, $\lesssim$ indicate estimates that hold up to a global constant C, which only depends on the dimension d and the Hölder exponent α (if applicable). For instance, $f \lesssim g$ means that there exists such a constant with $f \le Cg$, $f \sim g$ means $f \lesssim g$ and $g \lesssim f$. An assumption of the form $f \ll 1$ means that there exists $\varepsilon > 0$, only depending on the dimension and the Hölder exponent, such that if $f \le \varepsilon$, then the conclusion holds. We write $|E|$ for the Lebesgue measure of a set E. Inclusions will always be understood as holding up to a set of Lebesgue measure zero, that is for two sets E and F, $E \subseteq F$ means that $|E\backslash F| = 0$. When no confusion is possible, we will drop the integration measures in the integrals. For $R > 0$ and $x_0 \in \mathbb{R}^d$, $B_R(x_0)$ denotes the ball of radius R centered in x_0. When $x_0 = 0$, we will simply write B_R for $B_R(0)$. We will also use the notation

$$\fint_{B_R} f := \frac{1}{|B_R|} \int_{B_R} f.$$

For a function ρ defined on a ball B_R we introduce the Hölder semi-norm of exponent $\alpha \in (0,1)$

$$[\rho]_{\alpha,R} := \sup_{x \neq y \in B_R} \frac{|\rho(x) - \rho(y)|}{|x-y|^\alpha}.$$

2 Preliminaries

In this section, we first recall some well-known estimates for harmonic functions.

Lemma 2.1. *Given $f \in L^2(\partial B_1)$ with average zero, we consider a solution φ of*

$$\begin{cases} -\Delta\varphi = 0 & \text{in } B_1 \\ \frac{\partial\varphi}{\partial\nu} = f & \text{on } \partial B_1, \end{cases} \tag{2.1}$$

where ν denotes the outer normal to ∂B_1. We have

$$\int_{B_1} |\nabla\varphi|^2 \lesssim \int_{\partial B_1} f^2, \tag{2.2}$$

$$\sup_{B_{1/2}} \left(|\nabla^3\varphi|^2 + |\nabla^2\varphi|^2 + |\nabla\varphi|^2\right) \lesssim \int_{B_1} |\nabla\varphi|^2, \tag{2.3}$$

and for every $0 < r \leq 1$, letting $A_r := B_1 \backslash B_{1-r}$,

$$\int_{A_r} |\nabla\varphi|^2 \lesssim r \int_{\partial B_1} f^2. \tag{2.4}$$

Proof. We start with (2.2). Changing φ by an additive constant, we may assume that $\int_{B_1} \varphi = 0$. Testing (2.1) with φ, we obtain

$$\begin{aligned} \int_{B_1} |\nabla\varphi|^2 &= \int_{\partial B_1} f\varphi \\ &\leq \left(\int_{\partial B_1} f^2\right)^{1/2} \left(\int_{\partial B_1} \varphi^2\right)^{1/2} \\ &\lesssim \left(\int_{\partial B_1} f^2\right)^{1/2} \left(\int_{B_1} |\nabla\varphi|^2\right)^{1/2}, \end{aligned}$$

where we used the trace estimate in conjunction with Poincaré's estimate for mean-value zero. This yields (2.2).

Estimate (2.3) follows from the mean-value property of harmonic functions applied to $\nabla\varphi$ and its derivatives.

We finally turn to (2.4). By sub-harmonicity of $|\nabla\varphi|^2$ (which can for instance be inferred from the Bochner formula), we have the mean-value property in the form

$$\int_{\partial B_r} |\nabla\varphi|^2 \leq \int_{\partial B_1} |\nabla\varphi|^2 \quad \text{for } r \leq 1.$$

Integrating this inequality between r and 1, using Pohozaev identity (see [10, Lem. 8.3.2]), that is,

$$(d-2)\int_{B_1}|\nabla\varphi|^2=\int_{\partial B_1}|\nabla_\tau\varphi|^2-\int_{\partial B_1}\left(\frac{\partial\varphi}{\partial\nu}\right)^2, \tag{2.5}$$

where ∇_τ is the tangential part of the gradient of φ, and (2.2), we obtain (2.4). □

We also need similar estimates for solutions of Poisson equation.

Lemma 2.2. *Given $g\in C^{0,\alpha}(\overline{B}_1)$ such that $g(0)=0$, we consider a solution φ of*

$$\begin{cases}-\Delta\varphi=g & \text{in } B_1\\ \frac{\partial\varphi}{\partial\nu}=-\frac{1}{\mathcal{H}^{d-1}(\partial B_1)}\int_{B_1}g & \text{on } \partial B_1,\end{cases} \tag{2.6}$$

where ν denotes the outer normal to ∂B_1. We have

$$\sup_{B_1}\left(|\nabla^2\varphi|^2+|\nabla\varphi|^2\right)\lesssim[g]_{\alpha,1}^2. \tag{2.7}$$

In particular,

$$\int_{B_1}|\nabla\varphi|^2\lesssim[g]_{\alpha,1}^2, \tag{2.8}$$

and letting for $r\le 1$, $A_r:=B_1\backslash B_{1-r}$, there holds

$$\int_{A_r}|\nabla\varphi|^2\lesssim r[g]_{\alpha,1}^2. \tag{2.9}$$

Proof. Estimate (2.7) follows from global Schauder estimates [19] and the fact that since $g(0)=0$, $\|g\|_{L^\infty(B_1)}\lesssim[g]_{\alpha,1}$. □

We will need a trace estimate in the spirit of [1, Lem. 3.2].

Lemma 2.3. *For $0<r\le 1$, letting $A_r:=B_1\backslash B_{1-r}$, there holds for every suitable function ψ,*

$$\left(\int_0^1\int_{\partial B_1}(\psi-\overline{\psi})^2\right)^{1/2}\lesssim r^{1/2}\left(\int_0^1\int_{A_r}|\nabla\psi|^2\right)^{1/2}+\frac{1}{r^{(d+1)/2}}\int_0^1\int_{A_r}|\partial_t\psi|, \tag{2.10}$$

where $\overline{\psi}(x):=\int_0^1\psi(t,x)dt$. Here suitable means that ψ is integrable on $A_r\times(0,1)$, so that $\nabla\psi$, $\partial_t\psi$ are defined as distributions and $\overline{\psi}(x)$ is well-defined, with the understanding that the right-hand side is finite when these distributions admit Lebesgue densities that are square integrable and integrable, respectively (this amounts to a Sobolev space of mixed integrability).

Proof. Without loss of generality, we may assume that the right-hand side of (2.10) is finite. Since by (even) reflection and subsequent convolution, we learn that $C^1(\overline{A_r}\times[0,1])$ is dense with respect to the topology defined by the right-hand side of (2.10), we may assume that $\psi\in C^1(\overline{A_r}\times[0,1])$.
Because of $\int_0^1|\nabla(\psi-\overline{\psi})|^2\le\int_0^1|\nabla\psi|^2$, we may rewrite (2.10) in terms of $v:=\psi-\overline{\psi}$ as

$$\left(\int_0^1\int_{\partial B_1}v^2\right)^{1/2}\lesssim r^{1/2}\left(\int_0^1\int_{A_r}|\nabla v|^2\right)^{1/2}+\frac{1}{r^{(d+1)/2}}\int_0^1\int_{A_r}|\partial_t v|.$$

Since for every $x\in B_1$, $\int_0^1 v=0$, we have $\left(\int_0^1 v^2\right)^{1/2}\le\int_0^1|\partial_t v|$, so that it is enough to prove

$$\left(\int_{\partial B_1}\int_0^1 v^2\right)^{1/2}\lesssim r^{1/2}\left(\int_{A_r}\int_0^1|\nabla v|^2\right)^{1/2}+\frac{1}{r^{(d+1)/2}}\int_{A_r}\left(\int_0^1 v^2\right)^{1/2}.$$

Introducing $V:=\left(\int_0^1 v^2\right)^{1/2}$ and noting that $|\nabla V|^2\le\int_0^1|\nabla v|^2$, we see that it is sufficient to establish

$$\left(\int_{\partial B_1}V^2\right)^{1/2}\lesssim r^{1/2}\left(\int_{A_r}|\nabla V|^2\right)^{1/2}+\frac{1}{r^{(d+1)/2}}\int_{A_r}|V|. \tag{2.11}$$

We now cover the sphere ∂B_1 with (geodesic) cubes Q of side-length $\sim r$ in such a way that there is only a locally finite overlap. Then the annulus A_r is covered with the corresponding conical sets Q_r. By summation over Q and the super-additivity of the square function, for (2.11) it is enough to prove for every Q

$$\left(\int_Q V^2\right)^{1/2}\lesssim r^{1/2}\left(\int_{Q_r}|\nabla V|^2\right)^{1/2}+\frac{1}{r^{(d+1)/2}}\int_{Q_r}|V|.$$

Since Q_r is the bi-Lipschitz image of the Euclidean cube $(0,r)^d$, it is enough to establish

$$\int_{\{0\}\times(0,r)^{d-1}}V^2\lesssim r\int_{(0,r)^d}|\nabla V|^2+\frac{1}{r^{d+1}}\left(\int_{(0,r)^d}|V|\right)^2. \tag{2.12}$$

By rescaling, for (2.12) it is sufficient to consider $r=1$. By the fundamental theorem of calculus we have for every $x'\in(0,1)^{d-1}$

$$|V(0,x')|\lesssim\int_0^1|\partial_1 V(x_1,x')|dx_1+\int_0^1|V(x_1,x')|dx_1.$$

Taking squares, integrating and using Jensen's inequality, we get

$$\int_{\{0\}\times(0,1)^{d-1}}V^2\lesssim\int_{(0,1)^d}|\partial_1 V|^2+\int_{(0,1)^d}V^2.$$

Using Poincaré inequality in the form $\int_{(0,1)^d}V^2\lesssim\int_{(0,1)^d}|\nabla V|^2+\left(\int_{(0,1)^d}|V|\right)^2$, we obtain (2.12). □

This trace estimate is used in a similar spirit as in [1, Lem. 3.3] to obtain

Lemma 2.4. *Let $f \in L^2(\partial B_1 \times (0,1))$ be such that for a.e. $x \in \partial B_1$, $\int_0^1 f(x,t)dt = 0$. For $r > 0$ we introduce $A_r := B_1 \backslash B_{1-r}$ and define Λ as the set of measurable pairs $(s,q)\colon A_r \times (0,1) \to \mathbb{R} \times \mathbb{R}^d$ with $|s| \le 1/2$, q square-integrable, and such that for $\psi \in C^1(\overline{B_1} \times [0,1])$*[ii]*,*

$$\int_0^1 \int_{A_r} s\partial_t \psi + q \cdot \nabla \psi = \int_0^1 \int_{\partial B_1} f\psi. \tag{2.13}$$

Provided $r \gg \left(\int_0^1 \int_{\partial B_1} f^2\right)^{1/(d+1)}$, there exists $(s,q) \in \Lambda$ with

$$\int_0^1 \int_{A_r} \frac{1}{2}|q|^2 \lesssim r \int_0^1 \int_{\partial B_1} f^2. \tag{2.14}$$

Proof. We first note that the class Λ is not empty: For $t \in (0,1)$, let u_t be defined as the (mean-free) solution of the Neumann problem

$$\begin{cases} -\Delta u_t = -\frac{1}{|A_r|}\int_{\partial B_1} f & \text{in } A_r \times (0,1) \\ \frac{\partial u_t}{\partial \nu} = f & \text{on } \partial B_1 \times (0,1) \\ \frac{\partial u_t}{\partial \nu} = 0 & \text{on } \partial B_{1-r} \times (0,1), \end{cases}$$

and set $q(x,t) := \nabla u_t(x)$. The definition $s(x,t) := -\int_0^t \operatorname{div} q(x,z)dz = -\frac{1}{|A_r|}\int_0^t \int_{\partial B_1} f$ then ensures that (2.13) is satisfied, and $r \gg \left(\int_0^1 \int_{\partial B_1} f^2\right)^{\frac{1}{2}}$ yields $|s| \le 1/2$.

As in [1, Lem. 3.3], we now prove (2.14) with help of duality:

$$\begin{aligned} \inf_{(s,q)\in\Lambda} \int_0^1 \int_{A_r} \frac{1}{2}|q|^2 &= \inf_{(s,q),|s|\le 1/2} \sup_{\psi} \left\{ \int_0^1 \int_{A_r} \frac{1}{2}|q|^2 - \int_0^1 \int_{A_r} s\partial_t\psi + q\cdot\nabla\psi \right. \\ &\qquad \left. + \int_0^1 \int_{\partial B_1} f\psi \right\} \\ &= \sup_{\psi} \inf_{(s,q),|s|\le 1/2} \left\{ \int_0^1 \int_{A_r} \frac{1}{2}|q|^2 - \int_0^1 \int_{A_r} s\partial_t\psi + q\cdot\nabla\psi \right. \\ &\qquad \left. + \int_0^1 \int_{\partial B_1} f\psi \right\}, \end{aligned}$$

where the swapping of the sup and inf is allowed since the functional is convex in (s,q) and linear in ψ (see for instance [4, Prop. 1.1]). Minimizing in (s,q), and using $\int_0^1 f = 0$

[ii] For (s,q) regular, (2.13) just means $\partial_t s + \operatorname{div} q = 0$ in A_r, $s(\cdot,0) = s(\cdot,1) = 0$, $q\cdot\nu = 0$ on $\partial B_{1-r} \times (0,1)$ and $q \cdot \nu = f$ on $\partial B_1 \times (0,1)$

which allows us to smuggle in $\overline{\psi} := \int_0^1 \psi$, we obtain

$$
\begin{aligned}
\inf_{(s,q)\in\Lambda} \int_0^1 \int_{A_r} \frac{1}{2}|q|^2 &= \sup_{\psi} \left\{ -\int_0^1 \int_{A_r} \frac{1}{2}(|\nabla\psi|^2 + |\partial_t\psi|) + \int_0^1 \int_{\partial B_1} f\psi \right\} \\
&= \sup_{\psi} \left\{ -\int_0^1 \int_{A_r} \frac{1}{2}(|\nabla\psi|^2 + |\partial_t\psi|) + \int_0^1 \int_{\partial B_1} f(\psi - \bar{\psi}) \right\} \\
&\le \sup_{\psi} \left\{ -\int_0^1 \int_{A_r} \frac{1}{2}(|\nabla\psi|^2 + |\partial_t\psi|) \right. \\
&\qquad \left. + \left(\int_0^1 \int_{\partial B_1} f^2\right)^{1/2} \left(\int_0^1 \int_{\partial B_1} (\psi - \bar{\psi})^2\right)^{1/2} \right\}.
\end{aligned}
$$

With the abbreviation $F := \left(\int_0^1 \int_{\partial B_1} f^2\right)^{1/2}$ we have just established the inequality

$$
\inf_{(s,q)\in\Lambda} \int_0^1 \int_{A_r} \frac{1}{2}|q|^2 \le \sup_{\psi} \left\{ F\left(\int_0^1 \int_{\partial B_1} (\psi - \overline{\psi})^2\right)^{1/2} - \frac{1}{2}\int_0^1 \int_{A_r} |\nabla\psi|^2 + |\partial_t\psi| \right\}.
$$

Using now (2.10), where we denote the constant by C_0, and Young's inequality, we find that provided $r \ge (2C_0F)^{2/(d+1)}$ (in line with our assumption $r \gg \left(\int_0^1 \int_{\partial B_1} f^2\right)^{1/(d+1)}$),

$$
\begin{aligned}
\inf_{(s,q)\in\Lambda} \int_0^1 \int_{A_r} \frac{1}{2}|q|^2 &\le \sup_{\psi} \left\{ \frac{1}{2}C_0^2F^2r + C_0\frac{F}{r^{(d+1)/2}} \int_0^1 \int_{A_r} |\partial_t\psi| - \frac{1}{2}\int_0^1 \int_{A_r} |\partial_t\psi| \right\} \\
&\lesssim F^2 r = r\int_0^1 \int_{\partial B_1} f^2.
\end{aligned}
$$

This concludes the proof of (2.14). □

3 Proofs of the main results

3.1 The optimal transport problem: Lagrangian and Eulerian formulations

In order to set up notation, we start by recalling some well-known facts about optimal transportation. Let ρ_0 and ρ_1 be two densities with compact support in $\mathbb{R}^d$ and equal mass and let T be the minimizer of

$$
\min_{T\sharp\rho_0=\rho_1} \int_{\mathbb{R}^d} |T(x) - x|^2 \rho_0(x)dx, \tag{3.1}
$$

where by a slight abuse of notation $T\sharp\rho_0$ denotes the push-forward by T of the measure $\rho_0 dx$. If T' is the optimal transportation map between ρ_1 and ρ_0, then (see for instance [3, Rem. 6.2.11])

$$
T'(T(x)) = x, \qquad \text{and} \qquad T(T'(y)) = y \qquad \text{for a.e. } (x,y) \in \operatorname{spt}\rho_0 \times \operatorname{spt}\rho_1. \tag{3.2}
$$

By another abuse of notation, we will denote $T^{-1} := T'$.

Now for $t \in [0,1]$ and $x \in \mathbb{R}^d$ we set $T_t(x) := tT(x) + (1-t)x$ and consider the non-negative and $\mathbb{R}^d$-valued measures respectively defined through

$$\rho(\cdot,t) := T_t \sharp \rho_0 \qquad \text{and} \qquad j(\cdot,t) := T_t \sharp \left[(T - Id)\rho_0 \right]. \tag{3.3}$$

It is easy to check that $j(\cdot,t)$ is absolutely continuous with respect to $\rho(\cdot,t)$. The couple (ρ, j) solves the Eulerian (or Benamou-Brenier) formulation of optimal transportation (see [21, Th. 8.1] or [3, Chap. 8], see also [20, Prop. 5.32] for the uniqueness), i.e. it is the minimizer of

$$\min \left\{ \int_0^1 \int_{\mathbb{R}^d} \frac{1}{\rho} |j|^2 \ : \ \partial_t \rho + \operatorname{div} j = 0, \quad \rho(\cdot,0) = \rho_0, \ \rho(\cdot,1) = \rho_1 \right\}, \tag{3.4}$$

where the continuity equation including its boundary conditions are imposed in a distributional sense and where the functional is defined through (see [2, Th. 2.34]),

$$\int_0^1 \int_{\mathbb{R}^d} \frac{1}{\rho} |j|^2 := \begin{cases} \displaystyle\int_0^1 \int_{\mathbb{R}^d} \left| \frac{dj}{d\rho} \right|^2 d\rho & \text{if } j \text{ is absolutely continuous with respect to } \rho, \\ +\infty & \text{otherwise.} \end{cases}$$

Since T is the gradient of a convex function, by Alexandrov Theorem [22, Th. 14.25], T is differentiable a.e., that is for a.e. x_0, there exists a symmetric matrix A such that

$$T(x) = T(x_0) + A(x - x_0) + o(|x - x_0|).$$

Moreover, A coincide a.e. with the absolutely continuous part of the distributional derivative DT of the map T. We will from now on denote $\nabla T(x_0) := A$. For $t \in [0,1]$, by [21, Prop. 5.9], $\rho(\cdot,t)$ (and thus also j) is absolutely continuous with respect to the Lebesgue measure. The functional can be therefore rewritten as

$$\int_0^1 \int_{\mathbb{R}^d} \frac{1}{\rho} |j|^2 = \int_0^1 \int_{\mathbb{R}^d} \frac{1}{\rho} |j|^2 (x,t) dx dt,$$

where

$$\frac{1}{\rho} |j|^2 (x,t) := \begin{cases} \frac{1}{\rho(x,t)} |j(x,t)|^2 & \text{if } \rho(x,t) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the Jacobian equation

$$\rho(t, T_t(x)) \det \nabla T_t(x) = \rho_0(x), \tag{3.5}$$

holds a.e. (see [22, Ex. 11.2] or [21, Th. 4.8]) and in particular, $\rho_1(T(x)) \det \nabla T(x) = \rho_0(x)$.

The proof of Theorem 1.2 is based on the decay properties of the excess energy

$$\mathcal{E}(\rho_0, \rho_1, T, R) := R^{-2} \fint_{B_R} |T - x|^2 \rho_0. \tag{3.6}$$

When clear from the context we will use the short-hand notation

$$\mathcal{E} := \mathcal{E}(\rho_0, \rho_1, T, 1).$$

As will be shown in the proofs of Proposition 3.7 and Theorem 1.1, up to a change of variables it is not restrictive to assume that $\rho_0(0) = \rho_1(0) = 1$.

3.2 An L^∞ bound on the transport

Let us first prove an L^∞ control on $T - x$ by the energy. This result is important in order to localize the transport problem.

Lemma 3.1. *Let T be a minimizer of* (3.1) *and assume that $\frac{1}{2} \le \rho_0 \le 2$ in B_1 and that $\mathcal{E} \ll 1$. Then,*

$$\sup_{B_{3/4}} |T - x| + |T^{-1} - x| \lesssim \mathcal{E}^{1/(d+2)}. \tag{3.7}$$

As a consequence, for every $t \in [0, 1]$

$$T_t(B_{1/8}) \subseteq B_{3/16}. \tag{3.8}$$

Moreover, for $t \in [0, 1]$, we have for the pre-image

$$T_t^{-1}(B_{1/2}) \subseteq B_{3/4}. \tag{3.9}$$

Proof. We begin with the proof of (3.7). Since we assume that $\frac{1}{2} \le \rho_0 \le 2$, it is enough to prove that

$$\sup_{B_{3/4}} |T - x| + |T^{-1} - x| \lesssim \left(\int_{B_1} |T - x|^2 \right)^{1/(d+2)}.$$

We first prove the estimate on T. Let $u(x) := T(x) - x$. By monotonicity of T, for a.e. $x, y \in B_1$,

$$(u(x) - u(y)) \cdot (x - y) \ge -|x - y|^2. \tag{3.10}$$

Let $y \in B_{3/4}$ be such that (3.10) holds for a.e. $x \in B_1$. By translation we may assume that $y = 0$. By rotation, it is enough to prove for the first coordinate of u that

$$u_1(0) \lesssim \left(\int_{B_{1/4}} |u|^2 \right)^{1/(d+2)}.$$

Taking $y = 0$ in (3.10), we find for a.e. $x \in B_{1/4}$

$$u(0) \cdot x \le u(x) \cdot x + |x|^2 \lesssim |u(x)|^2 + |x|^2.$$

Integrating the previous inequality over the ball $B_r(re_1)$, we obtain

$$u(0)\cdot re_1 \lesssim \fint_{B_r(re_1)} |u|^2 + r^2,$$

so that

$$u_1(0) \lesssim \frac{1}{r^{d+1}} \int_{B_1} |u|^2 + r.$$

Optimizing in r yields the first part of (3.7). We now prove the estimate on T^{-1}. By the above argument for T in the ball $B_{4/5}$ instead of $B_{3/4}$, it is enough to show that $T^{-1}(B_{3/4}) \subseteq B_{4/5}$. Assume that there exists $y \in B_{3/4}$ and $x \in \mathbb{R}^d$ with $T(x) = y$ but $|x| \geq 4/5$. Let then $z \in \partial B_{\frac{1}{2}(\frac{3}{4}+\frac{4}{5})} \cap [x, y]$. By monotonicity of T,

$$\begin{aligned}
0 &\leq (T(x) - T(z))\cdot(x - z)\\
&= (y - z)\cdot(x - z) + (z - T(z))\cdot(x - z)\\
&\leq -\frac{1}{40}|x - z| + |x - z||T(z) - z|\\
&\leq |x - z|\left(-\frac{1}{40} + \sup_{B_{\frac{1}{2}(\frac{3}{4}+\frac{4}{5})}} |T - x|\right),
\end{aligned}$$

which is absurd if $\mathcal{E} \ll 1$ by the L^∞ bound on T on the ball $B_{\frac{1}{2}(\frac{3}{4}+\frac{4}{5})}$.

Since (3.8) is a direct consequence of (3.7), we are left with the proof of (3.9). If $x \in \mathbb{R}^d$ is such that $T_t(x) \in B_{1/2}$, then by (3.7) in the form of $|T_t(0)| = o(1)$, where $o(1)$ denotes a function that goes to zero as $\mathcal{E}$ goes to zero,

$$\begin{aligned}
\frac{1}{4}(1 + o(1)) &\geq |T_t(x) - T_t(0)|^2\\
&= t^2|T(x) - T(0)|^2 + 2t(1 - t)(T(x) - T(0))\cdot x + (1 - t)^2|x|^2\\
&\overset{(3.10)}{\geq} t^2|T(x) - T(0)|^2 + (1 - t)^2|x|^2\\
&\geq \frac{1}{2}\min\left\{|T(x) - T(0)|^2, |x|^2\right\}.
\end{aligned}$$

From this we see that x or $T(x)$ is in $B_{\frac{1}{\sqrt{2}}+o(1)} \subseteq B_{3/4}$. In the first case, (3.9) is proven while in the second, we have thanks to (3.7) that $x \in T^{-1}(T(x)) \subseteq T^{-1}(B_{\frac{1}{\sqrt{2}}+o(1)}) \subseteq B_{3/4}$ from which we get (3.9) as well. □

3.3 McCann's displacement convexity

Our second lemma is a localized version of McCann's displacement convexity (see [17, Cor. 4.4]), wich gives an upper bound for the density.

Lemma 3.2. *Assume that* $\rho_0(0) = \rho_1(0) = 1$ *and that* $\mathcal{E} + [\rho_0]_{\alpha,1} + [\rho_1]_{\alpha,1} \ll 1$*. Then for* $t \in [0,1]$*, there holds*

$$\sup_{B_{1/2}} \rho(t,\cdot) \leq 1 + [\rho_0]_{\alpha,1} + [\rho_1]_{\alpha,1}. \tag{3.11}$$

Proof. We start by pointing out that since $\rho_0(0) = \rho_1(0) = 1$ and $[\rho_0]_{\alpha,1} + [\rho_1]_{\alpha,1} \ll 1$ we have for $i = 0, 1$,

$$\sup_{B_1} |1 - \rho_i| \leq [\rho_i]_{\alpha,1} \ll 1. \tag{3.12}$$

For every $t \in (0,1)$, the map T_t has a well-defined inverse $\rho(t,\cdot)$−a.e. (see the proof of [21, Th. 8.1]) so that for $x \in B_{1/2}$, (3.5) can be written as

$$\rho(t,x) = \frac{\rho_0(T_t^{-1}(x))}{\det \nabla T_t(T_t^{-1}(x))}.$$

By concavity of $\det(\cdot)^{1/d}$ on non-negative symmetric matrices, we have

$$\det \nabla T_t(T_t^{-1}(x)) \geq \left(\det \nabla T(T_t^{-1}(x))\right)^t.$$

By (3.5), $\det \nabla T(T_t^{-1}(x)) = \frac{\rho_0(T_t^{-1}(x))}{\rho_1(T(T_t^{-1}(x)))}$, so that

$$\rho(t,x) \leq \left(\rho_0(T_t^{-1}(x))\right)^{1-t} \left(\rho_1(T(T_t^{-1}(x)))\right)^t.$$

Since $\mathcal{E} \ll 1$ and (3.12) holds, by (3.9) and (3.7), we have $T_t^{-1}(B_{1/2}) \subseteq B_1$ and $T(T_t^{-1}(B_{1/2})) \subseteq B_1$. By (3.12), we then have

$$\rho(t,x) \leq (1 + [\rho_0]_{\alpha,1})^{1-t} (1 + [\rho_1]_{\alpha,1})^t,$$

which by Young's inequality concludes the proof of (3.11). □

3.4 The harmonic approximation lemma: Eulerian version

We now can turn to the proof of Theorem 1.2. We first prove that the deviation of the velocity field $v := \frac{dj}{d\rho}$ from being the gradient of a harmonic function is locally controlled by the Eulerian energy. The construction we use is somewhat reminiscent of the Dacorogna-Moser construction (see [20]). As stated in the introduction, the crucial point in (3.15) below is that the right-hand side is strictly super-linear in $\int_0^1 \int_{B_1} \frac{1}{\rho}|j|^2$, and at least quadratic in $[\rho_0]_{\alpha,1} + [\rho_1]_{\alpha,1}$.

Proposition 3.3. *Let* (ρ, j) *be the minimizer of* (3.4)*. Assume that* $\rho_0(0) = \rho_1(0) = 1$ *and that*

$$\mathcal{E} + [\rho_0]_{\alpha,1} + [\rho_1]_{\alpha,1} \ll 1. \tag{3.13}$$

Then, there exists φ *harmonic in* $B_{1/2}$ *and such that*

$$\int_{B_{1/2}} |\nabla\varphi|^2 \lesssim \int_0^1 \int_{B_1} \frac{1}{\rho}|j|^2 + [\rho_0]_{\alpha,1}^2 + [\rho_1]_{\alpha,1}^2, \tag{3.14}$$

and

$$\int_0^1 \int_{B_{1/2}} \frac{1}{\rho}|j-\rho\nabla\varphi|^2 \lesssim \left(\int_0^1 \int_{B_1} \frac{1}{\rho}|j|^2\right)^{\frac{d+2}{d+1}} + [\rho_0]^2_{\alpha,1} + [\rho_1]^2_{\alpha,1}. \tag{3.15}$$

Proof. Without loss of generality, we may assume that $\int_0^1 \int_{B_1} \frac{1}{\rho}|j|^2 \ll 1$ since otherwise we can take $\varphi = 0$. Notice that since $\rho_0(0) = \rho_1(0) = 1$, thanks to (3.13), if we let

$$\gamma := [\rho_0]_{\alpha,1} + [\rho_1]_{\alpha,1} \qquad \text{and} \qquad \delta\rho := \rho_1 - \rho_0,$$

we have by (3.11)

$$\rho \leq 1 + \gamma \tag{3.16}$$

and since $\rho_0(0) = \rho_1(0) = 1$,

$$\sup_{B_1} |\delta\rho| \lesssim [\rho_0]_{\alpha,1} + [\rho_1]_{\alpha,1} = \gamma. \tag{3.17}$$

The proof is divided into four steps. In the first one we choose a good radius R for which the flux of j through ∂B_R is well controlled in L^2. This leads to the definition of the function φ in the second step. In the third step we prove a quasi-orthogonality property which is used in the last step to take advantage of the minimality property of (ρ, j). By constructing a competitor with small energy we then conclude the proof of (3.15).

Step 1 [Choice of a good radius] Using (3.16), and Fubini, we have

$$\int_{1/2}^1 \int_{\partial B_R} \int_0^1 |j|^2 = \int_{B_1\setminus B_{1/2}} \int_0^1 |j|^2 \lesssim \int_0^1 \int_{B_1} \frac{1}{\rho}|j|^2. \tag{3.18}$$

We can thus find $R \in (1/2, 1)$ such that

$$\int_{\partial B_R} \int_0^1 |j|^2 \lesssim \int_0^1 \int_{B_1} \frac{1}{\rho}|j|^2. \tag{3.19}$$

We then have for every function $\zeta \in H^1(B_R \times (0,1))$ [iii],

$$\int_0^1 \int_{B_R} \rho\partial_t\zeta + j\cdot\nabla\zeta = \int_0^1 \int_{\partial B_R} \zeta f + \int_{B_R} \zeta(\cdot,1)\rho_1 - \zeta(\cdot,0)\rho_0, \tag{3.20}$$

where $f := j\cdot\nu$ denotes the normal component of j.

For the convenience of the reader, we give a short proof of (3.20). From (3.18), $j \in L^2(B_1 \times (0,1))$ and the map $J(R) := j(R\,\cdot,\cdot) \in L^2((1/2,1), L^2(\partial B_1 \times (0,1)))$. We can thus

[iii] we consider here are larger class of test functions than $C^1(\overline{B_R} \times [0,1])$ since we want to apply (3.20) to the function $\widetilde{\varphi}$ defined in (3.23).

find a Lebesgue point $R \in (1/2, 1)$ of J such that (3.19) holds. This means in particular that

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{R-\varepsilon}^{R+\varepsilon} \int_{\partial B_1} \int_0^1 |j(rx,t) - j(Rx,t)|^2 \, dt \, d\mathcal{H}^{d-1}(x) \, dr = 0 \tag{3.21}$$

For $0 < \varepsilon \ll 1$ we now introduce the cut-off function

$$\eta_\varepsilon(x) := \begin{cases} 1 & \text{if } |x| \le R - \varepsilon \\ \frac{R-|x|}{\varepsilon} & \text{if } R - \varepsilon \le |x| \le R \\ 0 & \text{otherwise} \end{cases}$$

and obtain by admissibility of (ρ, j)

$$\begin{aligned} \int_{\mathbb{R}^2} \eta_\varepsilon(\zeta(\cdot,1)\rho_1 - \zeta(\cdot,0)\rho_0) &= \int_0^1 \int_{\mathbb{R}^2} \partial_t(\zeta\eta_\varepsilon)\rho + \nabla(\zeta\eta_\varepsilon) \cdot j \\ &= \int_0^1 \int_{\mathbb{R}^2} \eta_\varepsilon \partial_t \zeta \rho + \eta_\varepsilon \nabla\zeta \cdot j - \frac{1}{\varepsilon} \int_0^1 \int_{B_R \setminus B_{R-\varepsilon}} \zeta j \cdot \nu. \end{aligned}$$

Letting ε go to zero and using (3.21), we obtain (3.20).

Step 2 [Definition of φ] Let φ be a solution of

$$\begin{cases} -\Delta\varphi = 0 & \text{in } B_R \\ \frac{\partial\varphi}{\partial\nu} = \overline{f} + \frac{1}{\mathcal{H}^{d-1}(\partial B_R)} \int_{B_R} \delta\rho & \text{on } \partial B_R, \end{cases} \tag{3.22}$$

with $\overline{f} := \int_0^1 f dt$. Notice that by (3.20) applied to $\zeta = 1$, we get

$$\int_{B_R} \delta\rho = -\int_{\partial B_R} \bar{f},$$

so that (3.22) is indeed solvable. We then define

$$\begin{cases} -\Delta\widetilde{\varphi} = \delta\rho & \text{in } B_R \\ \frac{\partial\widetilde{\varphi}}{\partial\nu} = \overline{f} & \text{on } \partial B_R, \end{cases} \qquad \begin{cases} -\Delta\hat{\varphi} = \delta\rho & \text{in } B_R \\ \frac{\partial\hat{\varphi}}{\partial\nu} = -\frac{1}{\mathcal{H}^{d-1}(\partial B_R)} \int_{B_R} \delta\rho & \text{on } \partial B_R, \end{cases} \tag{3.23}$$

so that $\widetilde{\varphi} = \varphi + \hat{\varphi}$. We will argue that it is enough to establish

$$\int_0^1 \int_{B_R} \frac{1}{\rho} |j - \rho\nabla\widetilde{\varphi}|^2 \lesssim \left(\int_0^1 \int_{B_1} \frac{1}{\rho} |j|^2 \right)^{\frac{d+2}{d+1}} + \gamma^2, \tag{3.24}$$

and

$$\int_{B_R} |\nabla\widetilde{\varphi}|^2 \lesssim \int_0^1 \int_{B_1} \frac{1}{\rho} |j|^2 + \gamma^2. \tag{3.25}$$

Moreover, defining for $1 \gg r > 0$, $A_r := B_R \backslash B_{R(1-r)}$, we will show that

$$\int_{A_r} |\nabla \widetilde{\varphi}|^2 \lesssim r \left(\int_{\partial B_R} |\overline{f}|^2 + \gamma^2 \right). \tag{3.26}$$

Applying (2.2) from Lemma 2.1 (with the radius 1 replaced by $R \sim 1$) we have,

$$\int_{B_{1/2}} |\nabla \varphi|^2 \leq \int_{B_R} |\nabla \varphi|^2 \lesssim \int_{\partial B_R} |\overline{f}|^2 + \sup_{B_R} |\delta \rho|^2 \overset{(3.19)\&(3.17)}{\lesssim} \int_0^1 \int_{B_1} \frac{1}{\rho} |j|^2 + \gamma^2,$$

and thus (3.14) holds. Since by (2.8) from Lemma 2.2 (with the radius 1 replaced by $R \sim 1$),

$$\int_{B_R} |\nabla \hat{\varphi}|^2 \lesssim \gamma^2, \tag{3.27}$$

estimate (3.25) is obtained by

$$\int_{B_R} |\nabla \widetilde{\varphi}|^2 \lesssim \int_{B_R} |\nabla \varphi|^2 + \int_{B_R} |\nabla \hat{\varphi}|^2 \lesssim \int_0^1 \int_{B_1} \frac{1}{\rho} |j|^2 + \gamma^2.$$

Similarly, (3.26) follows from (2.4) and (2.9).
Assume now that (3.24) is established. We then get (3.15):

$$\begin{aligned} \int_0^1 \int_{B_R} \frac{1}{\rho} |j - \rho \nabla \varphi|^2 &= \int_0^1 \int_{B_R} \frac{1}{\rho} |j - \rho \nabla \widetilde{\varphi} + \rho \nabla \hat{\varphi}|^2 \\ &\lesssim \int_0^1 \int_{B_R} \frac{1}{\rho} |j - \rho \nabla \widetilde{\varphi}|^2 + \int_0^1 \int_{B_R} \rho |\nabla \hat{\varphi}|^2 \\ &\overset{(3.24)\&(3.16)\&(3.27)}{\lesssim} \left(\int_0^1 \int_{B_1} \frac{1}{\rho} |j|^2 \right)^{\frac{d+2}{d+1}} + \gamma^2. \end{aligned}$$

Step 3 [Quasi-orthogonality] We start the proof of (3.24). To simplify notation, we will assume from now on that $R = 1/2$. Here we prove that

$$\int_0^1 \int_{B_{1/2}} \frac{1}{\rho} |j - \rho \nabla \widetilde{\varphi}|^2 \leq \int_0^1 \int_{B_{1/2}} \frac{1}{\rho} |j|^2 - (1 - \gamma) \int_{B_{1/2}} |\nabla \widetilde{\varphi}|^2. \tag{3.28}$$

Notice that if $\rho = 0$ then $j = 0$ and thus also $j - \rho \nabla \widetilde{\varphi} = 0$, so that the left-hand side of (3.28) is well defined (see the discussion below (3.4)). Based on this we compute

$$\begin{aligned} \frac{1}{2} \int_0^1 \int_{B_{1/2}} \frac{1}{\rho} |j - \rho \nabla \widetilde{\varphi}|^2 &= \frac{1}{2} \int_0^1 \int_{B_{1/2}} \frac{1}{\rho} |j|^2 - \int_0^1 \int_{B_{1/2}} j \cdot \nabla \widetilde{\varphi} + \frac{1}{2} \int_0^1 \int_{B_1} \rho |\nabla \widetilde{\varphi}|^2 \\ &= \frac{1}{2} \int_0^1 \int_{B_{1/2}} \frac{1}{\rho} |j|^2 - \int_0^1 \int_{B_{1/2}} \left(1 - \frac{\rho}{2}\right) |\nabla \widetilde{\varphi}|^2 - \int_0^1 \int_{B_{1/2}} (j - \nabla \widetilde{\varphi}) \cdot \nabla \widetilde{\varphi} \\ &\overset{(3.16)}{\leq} \frac{1}{2} \int_0^1 \int_{B_{1/2}} \frac{1}{\rho} |j|^2 - \frac{1 - \gamma}{2} \int_0^1 \int_{B_{1/2}} |\nabla \widetilde{\varphi}|^2 - \int_0^1 \int_{B_{1/2}} (j - \nabla \widetilde{\varphi}) \cdot \nabla \widetilde{\varphi}. \end{aligned}$$

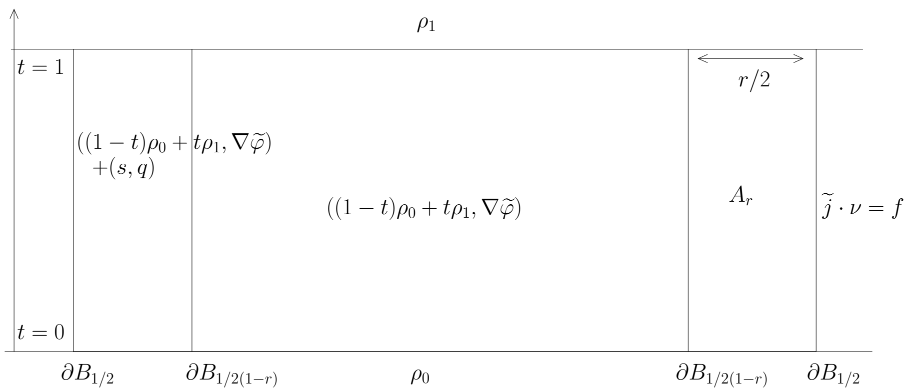

Figure 1: The definition of $(\widetilde{\rho}, \widetilde{j})$.

Using (3.20) with $\zeta = \widetilde{\varphi}$ and testing (3.23) with $\widetilde{\varphi}$, we have

$$\int_0^1 \int_{B_{1/2}} (j - \nabla\widetilde{\varphi}) \cdot \nabla\widetilde{\varphi} = \int_{\partial B_{1/2}} \widetilde{\varphi} \left(\int_0^1 f - \overline{f} \right) = 0,$$

where we recall that $\overline{f} = \int_0^1 f$. This proves (3.28).

Step 4 [The main estimate] In this last step, we establish that

$$\int_0^1 \int_{B_{1/2}} \frac{1}{\rho}|j|^2 - \int_{B_{1/2}} |\nabla\widetilde{\varphi}|^2 \lesssim \left(\int_0^1 \int_{B_1} \frac{1}{\rho}|j|^2 \right)^{\frac{d+2}{d+1}} + \gamma^2. \tag{3.29}$$

Thanks to (3.28) and (3.25), this would yield (3.24). By minimality of (ρ, j), it is enough to construct a competitor $(\widetilde{\rho}, \widetilde{j})$ that agrees with (ρ, j) outside of $B_{1/2} \times (0,1)$ and that satisfies the upper bound given through (3.29). We now make the following ansatz (see Figure 1)

$$(\widetilde{\rho}, \widetilde{j}) := \begin{cases} (t\rho_1 + (1-t)\rho_0, \nabla\widetilde{\varphi}) & \text{in } B_{1/2(1-r)} \times (0,1), \\ (t\rho_1 + (1-t)\rho_0 + s, \nabla\widetilde{\varphi} + q) & \text{in } A_r \times (0,1), \end{cases}$$

with $(s,q) \in \Lambda$, where Λ is the set defined in Lemma 2.4 with f replaced by $f - \overline{f}$ and the radius 1 replaced by 1/2. Notice that if $|s| \leq 1/2$, by (3.13) and $\rho_0(0) = \rho_1(0) = 1$,

$$\frac{1}{4} \leq \widetilde{\rho}. \tag{3.30}$$

Thanks to (3.23) for $\widetilde{\varphi}$, (2.13) for (s,q) and (3.20) for (ρ, j), $(\widetilde{\rho}, \widetilde{j})$ extended by (ρ, j) outside of $B_{1/2} \times (0,1)$ is indeed admissible for (3.4).
By Lemma 2.4, if $r \gg \left(\int_0^1 \int_{\partial B_{1/2}} (f - \overline{f})^2 \right)^{1/(d+1)}$, we may choose $(s,q) \in \Lambda$ such that

$$\int_0^1 \int_{A_r} |q|^2 \lesssim r \int_0^1 \int_{\partial B_{1/2}} (f - \overline{f})^2. \tag{3.31}$$

By definition of $(\widetilde{\rho},\widetilde{j})$,

$$\int_0^1\int_{B_{1/2}}\frac{1}{\widetilde{\rho}}|\widetilde{j}|^2-\int_{B_{1/2}}|\nabla\widetilde{\varphi}|^2\le\int_0^1\int_{B_{1/2(1-r)}}\frac{1}{t\rho_1+(1-t)\rho_0}|\nabla\widetilde{\varphi}|^2-\int_{B_{1/2(1-r)}}|\nabla\widetilde{\varphi}|^2 + \int_0^1\int_{A_r}\frac{1}{\widetilde{\rho}}|\nabla\widetilde{\varphi}+q|^2. \quad (3.32)$$

The first two terms on the right-hand side can be estimated as

$$\begin{aligned}\int_0^1\int_{B_{1/2(1-r)}}\frac{1}{t\rho_1+(1-t)\rho_0}|\nabla\widetilde{\varphi}|^2-|\nabla\widetilde{\varphi}|^2 &= \int_0^1\int_{B_{1/2(1-r)}}\frac{t(1-\rho_0)+(1-t)(1-\rho_1)}{t\rho_1+(1-t)\rho_0}|\nabla\widetilde{\varphi}|^2\\ &\overset{(3.13)}{\lesssim}\gamma\int_{B_{1/2}(1-r)}|\nabla\widetilde{\varphi}|^2\\ &\overset{(3.25)}{\lesssim}\gamma\left(\int_0^1\int_{B_1}\frac{1}{\rho}|j|^2+\gamma^2\right). \end{aligned}\quad (3.33)$$

We now estimate the last term of (3.32):

$$\begin{aligned}\int_0^1\int_{A_r}\frac{1}{\widetilde{\rho}}|\nabla\widetilde{\varphi}+q|^2 &\overset{(3.30)}{\lesssim}\int_0^1\int_{A_r}|\nabla\widetilde{\varphi}|^2+|q|^2\\ &\overset{(3.31)}{\lesssim}\int_{A_r}|\nabla\widetilde{\varphi}|^2+r\int_{\partial B_{1/2}}(f-\overline{f})^2\\ &\overset{(3.26)}{\lesssim} r\left(\int_{\partial B_{1/2}}\overline{f}^2+\gamma^2\right).\end{aligned}$$

Taking r to be a large but order-one multiple of

$$\left(\int_0^1\int_{\partial B_{1/2}}(f-\overline{f})^2\right)^{1/(d+1)}\le\left(\int_0^1\int_{\partial B_{1/2}}f^2\right)^{1/(d+1)}\overset{(3.19)}{\lesssim}\left(\int_0^1\int_{B_1}\frac{1}{\rho}|j|^2\right)^{1/(d+1)}$$

yields

$$\int_0^1\int_{A_r}\frac{1}{\widetilde{\rho}}|\nabla\widetilde{\varphi}+q|^2\lesssim\left(\int_0^1\int_{B_1}\frac{1}{\rho}|j|^2\right)^{1/(d+1)}\left(\int_0^1\int_{B_1}\frac{1}{\rho}|j|^2+\gamma^2\right).$$

Plugging this and (3.33) into (3.32),

$$\begin{aligned}\int_0^1\int_{B_{1/2}}\frac{1}{\widetilde{\rho}}|\widetilde{j}|^2-\int_{B_{1/2}}|\nabla\widetilde{\varphi}|^2 &\lesssim\left(\left(\int_0^1\int_{B_1}\frac{1}{\rho}|j|^2\right)^{1/(d+1)}+\gamma\right)\left(\int_0^1\int_{B_1}\frac{1}{\rho}|j|^2+\gamma^2\right)\\ &\lesssim\left(\int_0^1\int_{B_1}\frac{1}{\rho}|j|^2\right)^{\frac{d+2}{d+1}}+\gamma^2,\end{aligned}$$

where we have used Young's inequality and the fact that $2>\frac{d+2}{d+1}$. This proves (3.29). □

Remark 3.4. *The quasi-orthogonality property* (3.28) *is a generalization of the following classical fact: If φ is a harmonic function with $\frac{\partial\varphi}{\partial\nu} = f$ on ∂B_1, then for every divergence-free vector-field b with $b \cdot \nu = f$ on ∂B_1*

$$\int_{B_1} |b - \nabla\varphi|^2 = \int_{B_1} |b|^2 - \int_{B_1} |\nabla\varphi|^2,$$

so that the minimizers b of the left-hand side coincide with the minimizers of the right-hand side. See for instance [18, Lem. 2.2] for an application of this idea in a different context.

3.5 The harmonic approximation lemma: Lagrangian version

We now prove that (3.15) together with the L^∞ bounds of Lemma 3.1 and elliptic regularity imply a similar statement in the Lagrangian setting, namely that the distance of the displacement $T - x$ to the set of gradients of harmonic functions is (locally) controlled by a super-linear power of the energy. This is reminiscent of the harmonic approximation property for minimal surfaces (see [16, Sec. III.5]).

Proposition 3.5. *Let T be the minimizer of* (3.1) *and assume that $\rho_0(0) = \rho_1(0) = 1$. Then there exists a function φ harmonic in $B_{1/8}$, such that*

$$\int_{B_{1/8}} |T - (x + \nabla\varphi)|^2 \rho_0 \lesssim \mathcal{E}^{\frac{d+2}{d+1}} + [\rho_0]_{\alpha,1}^2 + [\rho_1]_{\alpha,1}^2 \tag{3.34}$$

and

$$\int_{B_{1/8}} |\nabla\varphi|^2 \lesssim \mathcal{E} + [\rho_0]_{\alpha,1}^2 + [\rho_1]_{\alpha,1}^2. \tag{3.35}$$

Proof. Notice first that we may assume that $\mathcal{E} + [\rho_0]_{\alpha,1}^2 + [\rho_1]_{\alpha,1}^2 \ll 1$ since otherwise we can take $\varphi = 0$.

We recall the definitions of the measures

$$\rho(\cdot, t) := T_t \sharp \rho_0 \qquad \text{and} \qquad j(\cdot, t) := T_t \sharp \left[(T - Id)\rho_0\right].$$

We note that the velocity field $v = \frac{dj}{d\rho}$ satisfies $v(T_t(x), t) = T(x) - x$ for a.e. $x \in \operatorname{spt} \rho_0$ (this can be seen arguing for instance as in the proof of [21, Th. 8.1]). Hence, by definition of the expression $\frac{1}{\rho}|j|^2$ and that of ρ,

$$\int_0^1 \int_{B_{1/2}} \frac{1}{\rho}|j|^2 = \int_0^1 \int_{B_{1/2}} |v|^2 d\rho = \int_0^1 \int_{T_t^{-1}(B_{1/2})} |T - x|^2 \rho_0 \overset{(3.9)}{\lesssim} \int_{B_1} |T - x|^2 \rho_0 = \mathcal{E}.$$

By Proposition 3.3, we infer that there exists a function φ harmonic in $B_{1/4}$ such that

$$\int_0^1 \int_{B_{1/4}} \frac{1}{\rho}|j - \rho\nabla\varphi|^2 \lesssim \mathcal{E}^{\frac{d+2}{d+1}} + [\rho_0]_{\alpha,1}^2 + [\rho_1]_{\alpha,1}^2 \qquad \text{and} \qquad \int_{B_{1/4}} |\nabla\varphi|^2 \lesssim \mathcal{E} + [\rho_0]_{\alpha,1}^2 + [\rho_1]_{\alpha,1}^2. \tag{3.36}$$

We now prove (3.34). By the triangle inequality we have

$$\int_{B_{1/8}} |T-(x+\nabla\varphi)|^2\rho_0 \lesssim \int_0^1\int_{B_{1/8}} |T-(x+\nabla\varphi\circ T_t)|^2\rho_0 + \int_0^1\int_{B_{1/8}} |\nabla\varphi-\nabla\varphi\circ T_t|^2\rho_0.$$

Using that for $t\in[0,1]$, $|T_t(x)-x|\le|T(x)-x|$, the second term on the right-hand side is estimated as above:

$$\begin{aligned}\int_0^1\int_{B_{1/8}} |\nabla\varphi-\nabla\varphi\circ T_t|^2\rho_0 &\overset{(3.8)}{\lesssim} \sup_{B_{3/16}}|\nabla^2\varphi|^2\int_0^1\int_{B_{1/8}}|T_t-x|^2\rho_0\\ &\overset{(2.3)}{\lesssim} \mathcal{E}\int_{B_{1/4}}|\nabla\varphi|^2\\ &\overset{(3.36)}{\lesssim} \mathcal{E}\left(\mathcal{E}+[\rho_0]_{\alpha,1}^2+[\rho_1]_{\alpha,1}^2\right).\end{aligned}$$

We thus just need to estimate the first term. Recall that $v=\frac{dj}{d\rho}$ satisfies $v(T_t(x),t)=T(x)-x$, so that we obtain for the integrand $T(x)-(x+\nabla\varphi(T_t(x))=(v(t,\cdot)-\nabla\varphi)(T_t(x))$ for a.e. $x\in\operatorname{spt}\rho_0$. Hence, by definition of ρ and by our convention on how to interpret $\frac{1}{\rho}|j-\rho\nabla\varphi|^2$ when ρ vanishes,

$$\begin{aligned}\int_0^1\int_{B_{1/8}} |T-(x+\nabla\varphi\circ T_t)|^2\rho_0 &= \int_0^1\int_{T_t(B_{1/8})}|v-\nabla\varphi|^2 d\rho\\ &= \int_0^1\int_{T_t(B_{1/8})}\frac{1}{\rho}|j-\rho\nabla\varphi|^2\\ &\overset{(3.8)}{\le} \int_0^1\int_{B_{1/4}}\frac{1}{\rho}|j-\rho\nabla\varphi|^2 \overset{(3.36)}{\lesssim} \mathcal{E}^{\frac{d+2}{d+1}}+[\rho_0]_{\alpha,1}^2+[\rho_1]_{\alpha,1}^2.\end{aligned}$$

□

3.6 The one-step improvement result

Analogously to De Giorgi's proof of regularity for minimal surfaces (see for instance [16, Chap. 25.2]), we are going to prove an "excess improvement by tilting"-estimate. By this we mean that if at a certain scale R, the map T is close to the identity, i.e. if $\mathcal{E}(\rho_0,\rho_1,T,R)+R^{2\alpha}([\rho_0]_{\alpha,1}^2+[\rho_1]_{\alpha,1}^2)\ll 1$, then on a scale θR, after an affine change of coordinates, it is even closer to be the identity by a geometric factor. Together with (3.34) from Proposition 3.5, the main ingredient of the proof are the regularity estimates (2.3) from Lemma 2.1 for harmonic functions.

Proposition 3.6. *For every $\alpha'\in(0,1)$ there exist $\theta=\theta(d,\alpha,\alpha')>0$, $\varepsilon(d,\alpha,\alpha')>0$ and $C_\theta(d,\alpha,\alpha')>0$ with the property that for every $R>0$ such that $\rho_0(0)=\rho_1(0)=1$ and*

$$\mathcal{E}(\rho_0,\rho_1,T,R)+R^{2\alpha}([\rho_0]_{\alpha,R}^2+[\rho_1]_{\alpha,R}^2)\le\varepsilon, \tag{3.37}$$

there exist a symmetric matrix B with $\det B = 1$ and a vector b such that, letting $\lambda := \rho_1(b)^{\frac{1}{d}}$, $\hat{x} := B^{-1}x$, $\hat{y} := \lambda B(y-b)$ and then

$$\hat{T}(\hat{x}) := \lambda B(T(x)-b), \quad \hat{\rho}_0(\hat{x}) := \rho_0(x) \quad \text{and} \quad \hat{\rho}_1(\hat{y}) := \lambda^{-d}\rho_1(y), \tag{3.38}$$

there holds

$$\mathcal{E}(\hat{\rho}_0, \hat{\rho}_1, \hat{T}, \theta R) \le \theta^{2\alpha'}\mathcal{E}(\rho_0,\rho_1,T,R) + C_\theta R^{2\alpha}\left([\rho_0]^2_{\alpha,R} + [\rho_1]^2_{\alpha,R}\right). \tag{3.39}$$

Moreover, $\hat{\rho}_0(0) = \hat{\rho}_1(0) = 1$ and

$$|B - Id|^2 + \frac{1}{R^2}|b|^2 + |\lambda - 1|^2 \lesssim \mathcal{E}(\rho_0,\rho_1,T,R) + R^{2\alpha}([\rho_0]^2_{\alpha,R} + [\rho_1]^2_{\alpha,R}). \tag{3.40}$$

Proof. By a rescaling $\widetilde{x} = R^{-1}x$, which amounts to the re-definition $\widetilde{T}(\widetilde{x}) := R^{-1}T(R\widetilde{x})$ (which preserves optimality) and $\widetilde{b} := R^{-1}b$, we may assume that $R = 1$.
As usual, we let $\mathcal{E} := \mathcal{E}(\rho_0,\rho_1,T,1)$. Let φ be the harmonic function given by Proposition 3.5 and then define $b := \nabla\varphi(0)$, $A := \nabla^2\varphi(0)$ and set $B := e^{-A/2}$, so that $\det B = 1$. Using (2.3) from Lemma 2.1 and (3.35) from Proposition 3.5, we see that the first part of (3.40) is satisfied. By definition of λ and since $\rho_1(0) = 1$, and $[\rho_0]_{\alpha,1} + [\rho_1]_{\alpha,1} \lesssim 1$,

$$|\lambda - 1|^2 = |\rho_1(b) - 1|^2 \le |b|^{2\alpha}[\rho_1]^2_{1,\alpha} \overset{(3.40)}{\lesssim} \left(\mathcal{E}^\alpha + 1\right)[\rho_1]^2_{\alpha,1},$$

which gives the last part of (3.40). Using Young's inequality with $p = \alpha^{-1}$ and $q = (1-\alpha)^{-1}$ we obtain for $\delta > 0$,

$$|\lambda - 1|^2 \le \delta\mathcal{E} + \frac{C_\alpha}{\delta}[\rho_1]^2_{\alpha,1}, \tag{3.41}$$

where C_α is a constant which depends only on α.

Defining $\hat{\rho}_i$ and $\hat{T}$ as in (3.38) we have by (3.40) and (3.37)

$$\begin{aligned}
\fint_{B_\theta} |\hat{T} - \hat{x}|^2\hat{\rho}_0 &= \fint_{BB_\theta} |\lambda B(T-b) - B^{-1}x|^2\rho_0 \\
&\lesssim \lambda^2 \fint_{B_{2\theta}} |T - (B^{-2}x + b)|^2\rho_0 + |1-\lambda|^2 \fint_{B_{2\theta}} |B^{-1}x|^2\rho_0 \\
&\lesssim \fint_{B_{2\theta}} |T - (B^{-2}x+b)|^2\rho_0 + \theta^2\left(\theta^2\mathcal{E} + \theta^{-2}[\rho_1]^2_{\alpha,1}\right),
\end{aligned}$$

where in the last line we used (3.41) with $\delta = \theta^2$ and the fact that $\rho_0 \lesssim 1$ on B_1. We split the first term on the right-hand side into three terms

$$\begin{aligned}
&\fint_{B_{2\theta}} |T - (B^{-2}x + b)|^2\rho_0 \\
&\lesssim \fint_{B_{2\theta}} |T - (x + \nabla\varphi)|^2\rho_0 + \fint_{B_{2\theta}} |(B^{-2} - Id - A)x|^2\rho_0 + \fint_{B_{2\theta}} |\nabla\varphi - b - Ax|^2\rho_0 \\
&\lesssim \fint_{B_{2\theta}} |T - (x+\nabla\varphi)|^2\rho_0 + \theta^2|B^{-2} - Id - A|^2 + \sup_{B_{2\theta}}|\nabla\varphi - b - Ax|^2.
\end{aligned}$$

Recalling $B = e^{-A/2}$, $A = \nabla^2\varphi(0)$, and $b = \nabla\varphi(0)$, we obtain

$$\begin{aligned}\theta^{-2}\fint_{B_\theta} |\hat{T}-x|^2\hat{\rho}_0 &\overset{(3.34)}{\lesssim} \theta^{-(d+2)}\left(\mathcal{E}^{\frac{d+2}{d+1}} + [\rho_0]_{\alpha,1}^2 + [\rho_1]_{\alpha,1}^2\right) + |\nabla^2\varphi(0)|^4 + \theta^2 \sup_{B_{2\theta}} |\nabla^3\varphi|^2 \\ &\quad + \theta^2\mathcal{E} + \theta^{-2}[\rho_1]_{\alpha,1}^2 \\ &\overset{(2.3)\&(3.35)}{\lesssim} \theta^{-(d+2)}\mathcal{E}^{\frac{d+2}{d+1}} + \left(\mathcal{E} + [\rho_0]_{\alpha,1}^2 + [\rho_1]_{\alpha,1}^2\right)^2 + \theta^2\left(\mathcal{E} + [\rho_0]_{\alpha,1}^2 + [\rho_1]_{\alpha,1}^2\right) \\ &\quad + \theta^2\mathcal{E} + \theta^{-(d+2)}\left([\rho_0]_{\alpha,1}^2 + [\rho_1]_{\alpha,1}^2\right).\end{aligned}$$

Since $\frac{d+2}{d+1} < 2$ and $\mathcal{E} + [\rho_0]_{\alpha,1}^2 + [\rho_1]_{\alpha,1}^2 \ll \theta^2$ (recall that θ has not been fixed yet), this simplifies to

$$\theta^{-2}\fint_{B_\theta} |\hat{T}-x|^2\hat{\rho}_0 \lesssim \theta^{-(d+2)}\mathcal{E}^{\frac{d+2}{d+1}} + \theta^2\mathcal{E} + \theta^{-(d+2)}\left([\rho_0]_{\alpha,1}^2 + [\rho_1]_{\alpha,1}^2\right). \tag{3.42}$$

We may thus find a constant $C(d,\alpha) > 0$ such that

$$\theta^{-2}\fint_{B_\theta} |\hat{T}-x|^2\hat{\rho}_0 \le C\left(\theta^{-(d+2)}\mathcal{E}^{\frac{d+2}{d+1}} + \theta^2\mathcal{E}\right) + \theta^{-(d+2)}\left([\rho_0]_{\alpha,1}^2 + [\rho_1]_{\alpha,1}^2\right).$$

We now fix $\theta(d,\alpha,\alpha')$ such that $C\theta^2 \le \frac{1}{2}\theta^{2\alpha'}$, which is possible because $\alpha' < 1$. If $\mathcal{E}$ is small enough, $C\theta^{-(d+2)}\mathcal{E}^{\frac{d+2}{d+1}} \le \frac{1}{2}\theta^{2\alpha'}\mathcal{E}$ and thus

$$\theta^{-2}\fint_{B_\theta} |\hat{T}-x|^2\hat{\rho}_0 \le \theta^{2\alpha'}\mathcal{E} + \theta^{-(d+2)}\left([\rho_0]_{\alpha,1}^2 + [\rho_1]_{\alpha,1}^2\right).$$

□

3.7 The iteration

Equipped with the one-step-improvement of Proposition 3.6, the next proposition is the outcome of a Campanato iteration (see for instance [13, Chap. 5] for an application of Campanato iteration to obtain Schauder theory for linear elliptic systems). It is a Campanato iteration on the $C^{1,\alpha}$ level for the transportation map T and thus proceeds by comparing T to affine maps. The main ingredient is the *affine* invariance of transportation. Proposition 3.7, which contains Theorem 1.2, amounts to an ε-regularity result.

Proposition 3.7. *Assume that $\rho_0(0) = \rho_1(0) = 1$ and that*

$$\mathcal{E}(\rho_0,\rho_1,T,2R) + R^{2\alpha}([\rho_0]_{\alpha,2R}^2 + [\rho_1]_{\alpha,2R}^2) \ll 1, \tag{3.43}$$

then T is of class $C^{1,\alpha}$ in B_R, with

$$[\nabla T]_{\alpha,R} \lesssim R^{-\alpha}\mathcal{E}(\rho_0,\rho_1,T,2R)^{1/2} + [\rho_0]_{\alpha,2R} + [\rho_1]_{\alpha,2R}.$$

Proof. By Campanato's theory, see [8, Th. 5.I], we have to prove that (3.43) implies

$$\sup_{x_0\in B_R}\sup_{r\le\frac{1}{2}R}\min_{A,b}\frac{1}{r^{2(1+\alpha)}}\fint_{B_r(x_0)}|T-(Ax+b)|^2\lesssim R^{-2\alpha}\mathcal{E}(\rho_0,\rho_1,T,2R)+[\rho_0]^2_{\alpha,2R}+[\rho_1]^2_{\alpha,2R}. \tag{3.44}$$

Let us first notice that (3.43) implies that for every $x_0\in B_R$

$$\mathcal{E}:=R^{-2}\fint_{B_R(x_0)}|T-x|^2\rho_0\ll 1\qquad\text{and}\qquad R^\alpha\left([\rho_0]_{\alpha,2R}+[\rho_1]_{\alpha,2R}\right)\ll 1. \tag{3.45}$$

Therefore, in order to prove (3.44), it is enough to prove that (3.45) implies that for $r\le\frac{1}{2}R$,

$$\min_{A,b}\frac{1}{r^2}\fint_{B_r(x_0)}|T-(Ax+b)|^2\lesssim r^{2\alpha}\left(R^{-2\alpha}\mathcal{E}+[\rho_0]^2_{\alpha,2R}+[\rho_1]^2_{\alpha,2R}\right). \tag{3.46}$$

Replacing ρ_0 by $\rho_0(x_0)^{-1}\rho_0$ and ρ_1 by $\rho_1(x_0)^{-1}\rho_1(x_0+\left(\frac{\rho_0(x_0)}{\rho_1(x_0)}\right)^{\frac{1}{d}}(\cdot-x_0))$ and thus T by $x_0+\left(\frac{\rho_1(x_0)}{\rho_0(x_0)}\right)^{\frac{1}{d}}(T-x_0)$ which still satisfies (3.45) thanks to $\rho_0(0)=\rho_1(0)=1$ and (3.43), we may assume that $\rho_0(x_0)=\rho_1(x_0)=1$. Without loss of generality we may thus assume that $x_0=0$.

Fix from now on an $\alpha'\in(\alpha,1)$. Thanks to (3.45), Proposition 3.6 applies and there exist a (symmetric) matrix B_1 of unit determinant, a vector b_1 and a positive number λ_1 such that $T_1(x):=\lambda_1B_1(T(B_1x)-b_1)$, $\rho_0^1(x):=\rho_0(B_1x)$ and $\rho_1^1(x):=\lambda_1^{-d}\rho_1(\lambda_1^{-1}B_1^{-1}x+b_1)$ satisfy

$$\mathcal{E}_1:=\mathcal{E}(\rho_0^1,\rho_1^1,T_1,\theta R)\le\theta^{2\alpha'}\mathcal{E}+C_\theta R^{2\alpha}\left([\rho_0]^2_{\alpha,R}+[\rho_1]^2_{\alpha,R}\right). \tag{3.47}$$

If T is a minimizer of (3.1), then so is T_1 with (ρ_0,ρ_1) replaced by (ρ_0^1,ρ_1^1). Indeed, because $\det B_1=1$, T_1 sends ρ_0^1 on ρ_1^1 and if T is the gradient of a convex function ψ then $T_1=\nabla\psi_1$ where $\psi_1(x):=\lambda_1(\psi(B_1x)-b_1\cdot B_1x)$ is also a convex function, which characterizes optimality [21, Th. 2.12]. Moreover, for $i=0,1$

$$[\rho_i^1]_{\alpha,\theta R}\le\left(1+C(\mathcal{E}^{1/2}+R^\alpha[\rho_0]_{\alpha,R}+R^\alpha[\rho_0]_{\alpha,R})\right)[\rho_i]_{\alpha,R}. \tag{3.48}$$

Indeed, (we argue only for ρ_1^1 since the proof for ρ_0^1 is simpler), using that $\lambda_1^{-1}B_1^{-1}B_{\theta R}+b_1\subseteq B_R$ by (3.40),

$$\begin{aligned}
[\rho_1^1]_{\alpha,\theta R}&=\lambda_1^{-d}\sup_{x,y\in B_{\theta R}}\frac{|\rho_1(\lambda_1^{-1}B_1^{-1}x+b_1)-\rho_1(\lambda_1^{-1}B_1^{-1}y+b_1)|}{|x-y|^\alpha}\\
&=\lambda_1^{-d}\sup_{x,y\in B_{\theta R}}\frac{|\rho_1(\lambda_1^{-1}B_1^{-1}x+b_1)-\rho_1(\lambda_1^{-1}B_1^{-1}y+b_1)|}{|\lambda_1B_1[(\lambda_1^{-1}B_1^{-1}x+b_1)-(\lambda_1^{-1}B_1^{-1}y+b_1)]|^\alpha}\\
&\le\lambda_1^{-d}|\lambda_1^{-1}B_1^{-1}|^\alpha\sup_{x,y\in B_{\theta R}}\frac{|\rho_1(\lambda_1^{-1}B_1^{-1}x+b_1)-\rho_1(\lambda_1^{-1}B_1^{-1}y+b_1)|}{|(\lambda_1^{-1}B_1^{-1}x+b_1)-(\lambda_1^{-1}B_1^{-1}y+b_1)|^\alpha}\\
&\le\lambda_1^{-(d+\alpha)}|B_1^{-1}|^\alpha\sup_{x',y'\in B_R}\frac{|\rho_1(x')-\rho_1(y')|}{|x'-y'|^\alpha}\\
&=\lambda_1^{-(d+\alpha)}|B_1^{-1}|^\alpha[\rho_1]_{\alpha,R}.
\end{aligned}$$

By (3.40), we get (3.48).
Therefore, we may iterate Proposition 3.6 $K > 1$ times to find a sequence of (symmetric) matrices B_k with $\det B_k = 1$, a sequence of vectors b_k, a sequence of real numbers λ_k and a sequence of maps T_k such that setting for $1 \le k \le K$,

$$T_k(x) := \lambda_k B_k(T_{k-1}(B_k x) - b_k), \quad \rho_0^k(x) := \rho_0^{k-1}(B_k x) \quad \text{and} \quad \rho_1^k(x) := \lambda_k^{-d} \rho_1^{k-1}(\lambda_k^{-1} B_k^{-1} x + b_k),$$

there holds $\rho_0^k(0) = \rho_1^k(0) = 1$,

$$\mathcal{E}_k := \mathcal{E}(\rho_0^k, \rho_1^k, T_k, \theta^k R) \le \theta^{2\alpha'} \mathcal{E}_{k-1} + C_\theta \theta^{2(k-1)\alpha} R^{2\alpha} \left([\rho_0^{k-1}]^2_{\alpha, \theta^{k-1} R} + [\rho_1^{k-1}]^2_{\alpha, \theta^{k-1} R} \right) \quad (3.49)$$

and

$$|B_k - Id|^2 + \frac{1}{(\theta^{k-1} R)^2} |b_k|^2 + |\lambda_k - 1|^2 \lesssim \mathcal{E}_{k-1} + \theta^{2k\alpha} R^{2\alpha} \left([\rho_0^{k-1}]^2_{\alpha, \theta^{k-1} R} + [\rho_1^{k-1}]^2_{\alpha, \theta^{k-1} R} \right). \quad (3.50)$$

Arguing as for (3.48), we obtain that there exists $C_1(d, \alpha) > 0$ such that

$$[\rho_i^k]_{\alpha, \theta^k R} \le \left(1 + C_1 (\mathcal{E}_{k-1}^{1/2} + R^\alpha \theta^{k\alpha} \left([\rho_0^{k-1}]_{\alpha, \theta^{k-1} R} + [\rho_0^{k-1}]_{\alpha, \theta^{k-1} R} \right) \right) [\rho_i^{k-1}]_{\alpha, \theta^{k-1} R}. \quad (3.51)$$

Let us prove by induction that the above together with (3.45) implies that there exists $C_2(d, \alpha, \alpha') > 0$ such that for every $1 \le k \le K$,

$$[\rho_i^k]_{\alpha, \theta^k R} \le (1 + \theta^{k\alpha}) [\rho_i^{k-1}]_{\alpha, \theta^{k-1} R}, \quad \theta^{-2k\alpha} \mathcal{E}_k \le C_2 \left(\mathcal{E} + R^{2\alpha} [\rho_0]^2_{\alpha, R} + R^{2\alpha} [\rho_1]^2_{\alpha, R} \right). \quad (3.52)$$

This will show that we can keep on iterating Proposition 3.6.

By (3.47) and (3.48), (3.52) holds for $K = 1$ provided $C_2 \gtrsim C_\theta \theta^{-2\alpha}$. Assume that this holds for $K - 1$. Let us start by the first part of (3.52). Notice that the induction hypothesis implies that

$$[\rho_i^{K-1}]_{\alpha, \theta^{K-1} R} \le \prod_{k=1}^{K-2} (1 + \theta^{k\alpha}) [\rho_i]_{\alpha, R} \le C_3 [\rho_i]_{\alpha, R}, \quad (3.53)$$

where $C_3 := \prod_{k=1}^\infty (1 + \theta^{k\alpha}) < \infty$. From (3.52) and (3.53) for $k = K - 1$ we learn that we may choose the implicit small constant in (3.45) such that we have

$$C_1 \left(\theta^{-\alpha} \left(\sup_{1 \le k \le K-1} \theta^{-2k\alpha} \mathcal{E}_k \right)^{1/2} + R^\alpha \sup_{1 \le k \le K-1} [\rho_0^k]_{\alpha, \theta^k R} + [\rho_1^k]_{\alpha, \theta^k R} \right) \le 1.$$

Plugging this into (3.51), we obtain the first part of (3.52) for $k = K$.
Let us now turn to the second part of (3.52). Dividing (3.49) by $\theta^{2k\alpha}$ and taking the sup over $k \in [1, K]$, we obtain by (3.53),

$$\sup_{1 \le k \le K} \theta^{-2k\alpha} \mathcal{E}_k \le \theta^{2(\alpha' - \alpha)} (\mathcal{E} + \sup_{1 \le k \le K-1} \theta^{-2k\alpha} \mathcal{E}_k) + C_\theta C_3^2 R^{2\alpha} \left([\rho_0]^2_{\alpha, R} + [\rho_1]^2_{\alpha, R} \right).$$

Since $\alpha' > \alpha$, $\theta^{2(\alpha'-\alpha)} < 1$ and thus

$$\sup_{1\le k\le K} \theta^{-2k\alpha}\mathcal{E}_k \le (1-\theta^{2(\alpha'-\alpha)})^{-1}\left[\mathcal{E} + C_\theta C_3^2 R^{2\alpha}\left([\rho_0]_{\alpha,R}^2 + [\rho_1]_{\alpha,R}^2\right)\right].$$

Choosing $C_2 := (1-\theta^{2(\alpha'-\alpha)})^{-1}\max\{1, C_\theta C_3^2\}$ we see that also the second part of (3.52) holds for $k = K$.

Letting $\Lambda_k := \prod_{i=1}^k \lambda_i$, $A_k := B_k B_{k-1}\cdots B_1$ and $d_k := \sum_{i=1}^k (\lambda_k B_k)(\lambda_{k-1}B_{k-1})\cdots(\lambda_i B_i)b_i$, we see that $T_k(x) = \Lambda_k A_k T(A_k^* x) - d_k$. By (3.50), (3.52) and (3.53),

$$|A_k - Id|^2 \lesssim \mathcal{E} + R^{2\alpha}[\rho_0]_{\alpha,R}^2 + R^{2\alpha}[\rho_1]_{\alpha,R}^2 \ll 1, \tag{3.54}$$

so that $B_{\frac{1}{2}\theta^k R} \subseteq A_k^*(B_{\theta^k R})$. By the same reasoning, we obtain from (3.50),

$$|\Lambda_k - 1| \ll 1. \tag{3.55}$$

We then conclude by definition of T_k that

$$\begin{aligned}
\min_{A,b}\frac{1}{(\frac{1}{2}\theta^k R)^2}\fint_{B_{\frac{1}{2}\theta^k R}} |T-(Ax+b)|^2 &\lesssim \frac{1}{(\theta^k R)^2}\fint_{A_k^*(B_{\theta^k R})} |T - \Lambda_k^{-1}A_k^{-1}A_k^{-*}x + \Lambda_k^{-1}A_k^{-1}d_k)|^2\\
&= \frac{1}{(\theta^k R)^2}\fint_{B_{\theta^k R}} |A_k^{-1}\Lambda_k^{-1}(T_k - x)|^2\\
&\overset{(3.54)\&(3.55)}{\lesssim} \frac{1}{(\theta^k R)^2}\fint_{B_{\theta^k R}} |T_k - x|^2\\
&\overset{(3.52)}{\lesssim} \theta^{2k\alpha}\left(\mathcal{E} + R^{2\alpha}[\rho_0]_{\alpha,R}^2 + R^{2\alpha}[\rho_1]_{\alpha,R}^2\right).
\end{aligned}$$

From this (3.46) follows, which concludes the proof of (3.44).

□

With this ε-regularity result at hand, we now may prove Theorem 1.1 i.e. that T is a $C^{1,\alpha}$ diffeomorphism outside of a set of measure zero.

Theorem. *For E and F two bounded open sets, let $\rho_0 : E \to \mathbb{R}^+$ and $\rho_1 : F \to \mathbb{R}^+$ be two $C^{0,\alpha}$ densities with equal masses, both bounded and bounded away from zero and let T be the minimizer of (3.1). There exist open sets $E' \subseteq E$ and $F' \subseteq F$ with $|E\backslash E'| = |F\backslash F'| = 0$ and such that T is a $C^{1,\alpha}$ diffeomorphism between E' and F'.*

Proof. By the Alexandrov Theorem [22, Th. 14.25], there exist two sets of full measure $E_1 \subseteq E$ and $F_1 \subseteq F$ such that for all $(x_0, y_0) \in E_1 \times F_1$, T and T^{-1} are differentiable at x_0 and y_0, respectively, in the sense that there exist A, B symmetric such that for a.e. $(x, y) \in E \times F$,

$$T(x) = T(x_0) + A(x-x_0) + o(|x-x_0|) \quad\text{and}\quad T^{-1}(y) = T^{-1}(y_0) + B(y-y_0) + o(|y-y_0|). \tag{3.56}$$

Moreover, we may assume that (3.2) holds for every $(x_0, y_0) \in E_1 \times F_1$. Using (3.2), it is not hard to show that if $T(x_0) = y_0$, then $A = B^{-1}$ and then by (3.5)

$$\rho_1(y_0) \det A = \rho_0(x_0). \tag{3.57}$$

We finally let $E' := E_1 \cap T^{-1}(F_1)$ and $F' := T(E') = F_1 \cap T(E_1)$. Notice that since T sends sets of measure zero to sets of measure zero, $|E\backslash E'| = |F\backslash F'| = 0$. We are going to prove that E' and F' are open sets and that T is a $C^{1,\alpha}$ diffeomorphism from E' to F'. Let $x_0 \in E'$, and thus automatically $y_0 := T(x_0) \in F'$, be given; we shall prove that T is of class $C^{1,\alpha}$ in a neighborhood of x_0. By (3.56) and the fact that ρ_0 and ρ_1 are bounded we have in particular

$$\lim_{R\to 0} \frac{1}{R^2} \fint_{B_R(x_0)} |T - y_0 - A(x - x_0)|^2 \rho_0 = 0. \tag{3.58}$$

We make the change of variables $x = A^{-1/2}\hat{x} + x_0$, $y = A^{1/2}\hat{y} + y_0$, which leads to $\hat{T}(\hat{x}) := A^{-1/2}(T(x) - y_0)$, and then define $\hat{\rho}_0(\hat{x}) := \rho_0(x_0)^{-1}\rho_0(x)$ and $\hat{\rho}_1(\hat{y}) := \rho_0(x_0)^{-1} \det^{-2} A \rho_1(y)$. Note that $\hat{T}$ is the optimal transportation map between $\hat{\rho}_0$ and $\hat{\rho}_1$ (indeed, if $T = \nabla\psi$ for a convex function ψ, then $\hat{T} = \hat{\nabla}\hat{\psi}$, where $\hat{\psi}(\hat{x}) = \psi(x) - y_0 \cdot \hat{x}$) and that by (3.57), $\hat{\rho}_0(0) = \hat{\rho}_1(0) = 1$. Moreover, since ρ_0 and ρ_1 are bounded and bounded away from zero, $\hat{\rho}_0$ and $\hat{\rho}_1$ are $C^{0,\alpha}$ continuous with Hölder semi-norms controlled by the ones of ρ_0 and ρ_1, so that

$$\lim_{R\to 0} R^\alpha \left([\hat{\rho}_0]_{\alpha, B_R} + [\hat{\rho}_1]_{\alpha, R} \right) = 0.$$

Finally, the change of variables is made such that (3.58) turns into

$$\lim_{R\to 0} \frac{1}{R^2} \fint_{B_R} |\hat{T} - \hat{x}|^2 \hat{\rho}_0 = 0.$$

Hence, we may apply Proposition 3.7 to $\hat{T}$ to obtain that $\hat{T}$ is of class $C^{1,\alpha}$ in a neighborhood of zero. Similarly, we obtain that $\hat{T}^{-1}$ is $C^{1,\alpha}$ in a neighborhood of zero. Going back to the original map, this means that T is a $C^{1,\alpha}$ diffeomorphism of a neighborhood U of x_0 on the neighborhood $T(U)$ of $T(x_0)$. In particular, $U \times T(U) \subseteq E' \times F'$ so that E' and F' are both open and thanks to (3.2), T is a global $C^{1,\alpha}$ diffeomorphism from E' to F'. □

Remark 3.8. *If ψ is a convex function such that $\nabla\psi = T$, Theorem 1.1 shows that $\psi \in C^{2,\alpha}(E')$ and it solves (in the classical sense) the Monge-Ampère equation which is now a uniformly elliptic equation. If the densities are more regular then by the Evans-Krylov Theorem (see [7]) and Schauder estimates we may obtain higher regularity of T.*

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