FREQUENCY OF SOBOLEV AND QUASICONFORMAL DIMENSION DISTORTION

ZOLTÁN M. BALOGH, ROBERTO MONTI, AND JEREMY T. TYSON

Dedicated to Robert Kaufman

Abstract. We study Hausdorff and Minkowski dimension distortion for images of generic affine subspaces of Euclidean space under Sobolev and quasiconformal maps. For a supercritical Sobolev mapping f defined on a domain in $\mathbb{R}^n$, we estimate from above the Hausdorff dimension of the set of affine subspaces parallel to a fixed m-dimensional linear subspace, whose image under f has positive $\mathcal{H}^\alpha$ measure for some fixed $\alpha > m$. As a consequence, we obtain new dimension distortion and absolute continuity statements valid for almost every affine subspace. Our results hold for mappings taking values in arbitrary metric spaces, yet are new even for quasiconformal maps of the plane. Our theory extends to cover mappings in Sobolev–Lorentz spaces as well as pseudomonotone mappings in the critical Sobolev class. In particular, we obtain new absolute continuity statements for quasisymmetric maps from Euclidean domains into metric spaces. We illustrate our results with numerous examples.

Contents

1. Introduction 2
2. Sobolev mappings valued in metric spaces 6
3. Exceptional sets for Sobolev mappings 9
3.1. Exceptional sets for supercritical Sobolev mappings 9
3.2. Exceptional sets for Sobolev–Lorentz and critical Sobolev mappings 12
3.3. Remarks on quasiconformal mappings 17
4. Sobolev maps which increase the dimension of many affine subspaces 18
4.1. A probabilistic lemma 22
5. Examples 23
5.1. Quasiconformal maps which increase the Minkowski dimension of many lines 23
5.2. Space-filling mappings in subcritical Sobolev classes 28
6. Open problems and questions 32
References 33

2010 Mathematics Subject Classification. Primary: 46E35, 30C65; Secondary: 28A78, 46E40, 30L99, 31B15.
Keywords and Phrases. Hausdorff dimension, Sobolev mapping, potential theory, quasiconformal mapping, space-filling mapping.

Date: August 10, 2010.
Z.M.B. supported by the Swiss National Science Foundation, European Research Council Project GALA and European Science Foundation Project HCAA. R.M. supported by Mathematisches Institut, Universität Bern. J.T.T. supported by US National Science Foundation Grant DMS-0901620.

1. Introduction

Every continuous Sobolev mapping is ACL, i.e., its components are absolutely continuous when restricted on almost every line. In particular, almost every line parallel to any fixed vector is mapped onto a locally rectifiable curve, and hence onto a curve of Hausdorff dimension one. Moreover, every supercritical Sobolev mapping satisfies Lusin's condition N, i.e., sets of Lebesgue measure zero are mapped to sets of measure zero. Condition N persists for critical Sobolev mappings under extra topological or analytic assumptions.

It is natural to investigate regularity properties of Sobolev mappings on subspaces of intermediate dimension. For a fixed set this was done by Kaufman [32] and earlier by Astala [2] and Gehring–Väisälä [19] for quasiconformal maps. In this paper, we study absolute continuity and dimension distortion properties for the restriction of Sobolev and quasiconformal mappings to *generic* affine subspaces. Our main results are Theorems 1.3, 1.4 and 1.6.

The literature on generic dimension estimates is extensive. We highlight papers by Mattila, Kaufman and Falconer. This line of inquiry initiates in a paper of Kaufman [31] on dimensions of generic projections of Euclidean sets. Dimensions of generic projections have been thoroughly investigated by Mattila [39], [42], [44], Kaufman and Mattila [33], Falconer [12], Falconer and Howroyd [16], [17], and others. Mattila [40], [41] proved an important series of results on dimensions of generic intersections of translates or rigid motions of Euclidean sets. These results gave signficant impetus and visibility to the subject of generic dimension estimates. More recently, Falconer [13], [15] investigated the dimensions of invariant sets for generic elements in parameterized families of self-affine iterated function systems. See also Solomyak [53] and Falconer–Miao [11] for further work on this subject. Ideas from these papers were taken up by the first and third authors in [4] and [5] for the study of dimensions of generic invariant sets associated to sub-Riemannian iterated function systems.

Our goal in this paper is to apply the techniques of geometric measure theory commonly used for this type of theorem to understand the generic dimension distortion behavior of Sobolev maps on affine subspaces. Our main results suggest many extensions and generalizations. Section 6 contains open problems and questions motivated by this study.

We consider the foliation of $\mathbb{R}^n$ by m-dimensional affine subspaces

$$V_a := V + a,$$

where V is an m-dimensional linear subspace of $\mathbb{R}^n$, i.e., an element of the Grassmannian $G(n,m)$, and a ranges over the orthogonal complement $V^\perp$ of V. We will assume throughout this paper that m and n are fixed integers satisfying

$$1 \le m \le n-1. \tag{1.1}$$

The notion of genericity is measured by suitable Hausdorff measures on $V^\perp$. For instance, the ACL property of a Sobolev map $f : \Omega \to \mathbb{R}^m$ asserts that, for a given $V \in G(n,1)$,

$$\begin{aligned} &f|_{V_a \cap \Omega} : (V_a \cap \Omega, \mathcal{H}^1) \to (f(V_a \cap \Omega), \mathcal{H}^1) \text{ is absolutely continuous} \\ &\qquad \text{for } \mathcal{H}^{n-1} \text{ almost every point } a \text{ in } V^\perp \in G(n,n-1). \end{aligned} \tag{1.2}$$

Since $f(V_a \cap \Omega)$ has locally finite Hausdorff 1-measure at such points a, we also conclude

$$\dim f(V_a \cap \Omega) \le 1 \text{ for } \mathcal{H}^{n-1} \text{ almost every } a \in V^\perp. \tag{1.3}$$

Throughout this paper, we denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure and by dim the Hausdorff dimension.

In this paper, we shall prove a sweeping generalization of (1.2) and (1.3) for families of affine subspaces of arbitrary dimension.

We take advantage of recent developments in analysis in metric spaces to formulate our results for Sobolev mappings taking values in arbitrary metric spaces. The notion of a metric space-valued Sobolev mapping has been introduced by Ambrosio [1] and Reshetnyak [48]. It was used in [57] and [29] to provide an analytic characterization of quasisymmetric maps in metric spaces, and in [3] to investigate the mapping properties of quasiconformal maps with Sobolev boundary values from the perspective of conformal densities.

Despite this general framework, we stress that many of our results are already new for Sobolev and quasiconformal maps between Euclidean domains, even domains in the plane.

Definition 1.1. Let Ω be a domain in some Euclidean space and let B be a Banach space. A map $f : \Omega \to B$ is said to lie in $W^{1,p}(\Omega, B)$ if $\langle b^*, f\rangle \in W^{1,p}(\Omega)$ for every b^* in the dual space B^*, and if the weak gradients of the functions $\langle b^*, f\rangle$, $b^* \in B^*$, $||b^*|| \le 1$, are uniformly bounded in $L^p(\Omega)$.

Let Y be a separable metric space. A map $f : \Omega \to Y$ is said to lie in $W^{1,p}(\Omega, Y)$ if $\iota \circ f$ lies in the Sobolev space $W^{1,p}(\Omega, \ell^\infty)$, where $\iota : Y \to \ell^\infty$ denotes an isometric embedding.

Fix n and m as in (1.1). Let Ω and Y be as in Definition 1.1, and let f be an element of $W^{1,p}(\Omega, Y)$. For the moment we restrict our attention to the case of *supercritical* mappings, i.e., the case $p > n$. The Sobolev embedding theorem in this case implies that f is Hölder continuous. The following proposition gives an a priori estimate for the distortion of dimension of an m-dimensional affine subspace under a supercritical Sobolev map. Kaufman [32] proved a more general statement covering subsets of arbitrary Hausdorff dimension. See Proposition 2.5. Although Kaufman's paper is the first place where we have seen this explicit result in print, the underlying principle (increased Sobolev regularity implies improved dimension distortion bounds), had apparently already been recognized for some time. In the category of quasiconformal maps, it was used by both Astala [2] and Gehring–Väisälä [19].

Proposition 1.2 (Kaufman)**.** *Let $f \in W^{1,p}(\Omega, Y)$ for $p > n$ and let $V \in G(n, m)$. Then $f(V_a \cap \Omega)$ has zero $\mathcal{H}^{pm/(p-n+m)}$ measure for each $a \in V^\perp$. In particular,*

$$\dim f(V_a \cap \Omega) \le \frac{pm}{p-n+m}. \tag{1.4}$$

Note that a naive application of the $(1-n/p)$-Hölder regularity of f would yield the weaker estimate

$$\dim f(V_a \cap \Omega) \le \frac{pm}{p-n}.$$

Proposition 1.2 provides an upper bound, strictly smaller than n, for the dimension of the image of an arbitrary m-dimensional subspace under a supercritical $W^{1,p}$ mapping f. How frequently can the intermediate values

$$m < \alpha < \frac{pm}{p-n+m}$$

be exceeded? Our first main theorem provides a quantitative measurement of this frequency.

Fix n and m satisfying (1.1). For $p \ge 1$ and $m \le \alpha \le \frac{pm}{p-n+m}$, set

$$\beta(p, \alpha) := (n-m) - \left(1 - \frac{m}{\alpha}\right) p. \tag{1.5}$$

The following theorem, which is the primary result of this paper, asserts an $\mathcal{H}^\beta$-almost everywhere upper bound on the dimensions of images of affine subspaces parallel to a fixed m-dimensional linear subspace of $\mathbb{R}^n$ under a supercritical Sobolev map.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $f \in W^{1,p}(\Omega, Y)$, $p > n$, $V \in G(n,m)$, and*

$$m < \alpha \le \frac{pm}{p-n+m}. \tag{1.6}$$

Then $f(V_a \cap \Omega)$ has zero $\mathcal{H}^\alpha$ measure for $\mathcal{H}^\beta$-almost every $a \in V^\perp$, where $\beta = \beta(p,\alpha)$.

Note that $\beta(p,\alpha) = 0$ if and only if $\alpha = \frac{pm}{p-n+m}$, which shows that Theorem 1.3 includes Proposition 1.2 as a special case. Theorem 1.3 implies both the dimension estimate

$$\dim f(V_a \cap \Omega) \le \alpha \tag{1.7}$$

as well as the absolute continuity of

$$f|_{V_a\cap\Omega} : (V_a \cap \Omega, \mathcal{H}^m) \to (f(V_a \cap \Omega), \mathcal{H}^\alpha) \tag{1.8}$$

for $\mathcal{H}^\beta$-a.e. $a \in V^\perp$.

Theorem 1.3 is sharp. In the following theorem, we construct a $W^{1,p}$ map which increases from m to α the dimension of *each* element in a $\beta(p,\alpha)$-dimensional set of parallel affine m-dimensional subspaces of $\mathbb{R}^n$. In order to describe precisely the class of sets to which the theorem applies, we fix some useful notation. For a bounded set $E \subset \mathbb{R}^n$ and for $r > 0$, we denote by $\mathbf{N}(E,r)$ the smallest number of balls of radius r needed to cover E.

Theorem 1.4. *Let $p \ge 1$, let α satisfy $m < \alpha \le \frac{pm}{p-n+m}$, and define $\beta(p,\alpha)$ by the formula (1.5). Let $E \subset \mathbb{R}^{n-m}$ be any bounded Borel set for which*

$$\limsup_{r\to 0} r^\beta \mathbf{N}(E,r) < \infty, \tag{1.9}$$

where $\beta = \beta(p,\alpha)$. Then, for any integer $N > \alpha$, there exists a map $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$ so that $f(\{a\} \times \mathbb{R}^m)$ has Hausdorff dimension at least α, for $\mathcal{H}^\beta$-almost every $a \in E$.

Note that we only assume $p \ge 1$ in the statement of Theorem 1.4. Choosing $p > n$ and a set $E \subset \mathbb{R}^{n-m}$ with positive and finite Hausdorff $\mathcal{H}^\beta$ measure which satisfies the assumptions of the theorem shows that Theorem 1.3 is sharp. Sets of this type exist in abundance. For instance, we may take any compact subset $E \subset \mathbb{R}^{n-m}$ which is Ahlfors regular of dimension $\beta(p,\alpha)$, e.g., self-similar Cantor sets.

The map in Theorem 1.4 is obtained by a random construction. We exhibit a large family of $W^{1,p}$ maps and show that almost every map in this family has the desired property.

Theorem 1.3 holds in particular for Euclidean quasiconformal maps. We obtain almost sure dimension estimates for the size of the exceptional set of points a in $V^\perp$ for which some component of the quasiconformal m-manifold $f(V_a \cap \Omega)$ has positive $\mathcal{H}^\alpha$ measure. By Gehring's higher integrability theorem [18], quasiconformal maps in $\mathbb{R}^n$ lie in $W^{1,p}$ for some $p > n$. Since

$$\beta(p,\alpha) < \beta(n,\alpha) = m\left(\frac{n}{\alpha} - 1\right)$$

for all $p > n$, we obtain the following

Corollary 1.5. *Let $f : \Omega \to \Omega'$ be a quasiconformal map between domains in $\mathbb{R}^n$, let $V \in G(n,m)$, and let $m < \alpha < n$. Then $\mathcal{H}^\alpha(f(V_a \cap \Omega)) = 0$ for $\mathcal{H}^{m(\frac{n}{\alpha}-1)}$-a.e. $a \in V^\perp$.*

In particular,

$$\dim f(V_a \cap \Omega) \le \alpha \tag{1.10}$$

for $\mathcal{H}^{m(\frac{n}{\alpha}-1)}$-a.e. $a \in V^\perp$.

Estimates for quasiconformal dimension distortion are often obtained via conformal modulus techniques. Our proof makes no explicit use of modulus, although it is motivated by modulus arguments used in estimates of conformal dimension (Remark 3.4). Quasiconformal and quasisymmetric dimension distortion is a classical subject ([19], [56], [2]), but we are unaware of prior theorems yielding simultaneous dimension estimates for the images of a large family of parallel subspaces. See Remark 5.8 for more details.

Remarkably, even Corollary 1.5 is sharp, provided we replace Hausdorff dimension by upper Minkowski dimension in (1.10). To simplify the exposition here in the introduction, we only state the following theorem in the case $m = 1$, i.e., for parameterized families of lines and their images. A similar result holds for higher dimensional subspaces, but only for a restricted choice of image dimensions α. See Remark 5.6.

Theorem 1.6. *Let $n \geq 2$. For each $\alpha \in (1, n)$ and each $\epsilon > 0$, there exists a Borel set $E \subset \mathbb{R}^{n-1}$ of Hausdorff dimension at least*

$$\left(\frac{n}{\alpha} - 1\right) - \epsilon$$

and a quasiconformal map $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\{a\} \times \mathbb{R})$ has upper Minkowski dimension at least α, for all $a \in E$.

We recall that the upper Minkowski dimension of E is the infimum of those values $\beta > 0$ for which (1.9) is satisfied.

Theorem 1.6 provides the first example of which we are aware of a quasiconformal map which simultaneously increases the (Minkowski) dimension of a family of parallel subspaces of optimal size. We do not know any example of a quasiconformal map which simultaneously increases the Hausdorff dimension of such a large family of subspaces, although previous examples of Bishop [6], David–Toro [9] and Kovalev–Onninen [37] should be noted. We review the examples of Bishop, David–Toro and Kovalev–Onninen in Remark 5.8.

Our main theorem extends to other classes of Sobolev mappings. The key tool which we employ is the regularity of supercritical Sobolev mappings arising from the Morrey–Sobolev inequality. Our result sharpens Lusin's condition N for such mappings. It is now well understood ([35], [47], [50], [59]) that the correct borderline integrability condition for continuity and condition N is membership in the Sobolev–Lorentz space $W^{1,n,1}$. In that context, the role of the Morrey–Sobolev inequality is taken over by the Rado–Reichelderfer condition (3.6). Our results extend to the Ambrosio–Reshetnyak–Sobolev–Lorentz class $W^{1,n,1}(\Omega, Y)$, where we show that the dimension of the exceptional set is bounded above by $\beta(n, \alpha)$, which is still strictly less than $n - m$. The same dimension estimate holds for continuous pseudomonotone mappings in the critical Sobolev class $W^{1,n}$. We discuss Sobolev–Lorentz spaces and pseudomonotone mappings in section 3.2.

The situation for weaker integrability criteria is more intriguing. In Example 5.10 we construct mappings in

$$W^{1,m}([0, 1]^n, \ell^2),$$

for any $2 \leq m < n$, with the property that **every** image $f(V_a \cap [0, 1]^n)$, $a \in V^\perp$, is infinite-dimensional. In fact, every such image coincides with a fixed infinite-dimensional cube. The construction makes use of space-filling Sobolev mappings with metric space targets, as constucted by Hajłasz–Tyson [27] and Wildrick–Zürcher [59], [58]. The methods can be adapted to construct a mapping in $W^{1,p}$ for

$$m < p < n \tag{1.11}$$

with similar properties, but at present, a complete understanding of the generic dimension distortion behavior of m-dimensional affine subspaces by $W^{1,p}$ maps from $\mathbb{R}^n$, for p satisfying (1.11), remains a challenging open problem.

Outline of the paper. In Section 2 we review the Ambrosio/Reshetnyak framework for metric space-valued Sobolev maps, emphasizing dimension distortion and absolute continuity properties. Section 3 contains the proof of Theorem 1.3. In subsection 3.2, we discuss how the method can be adapted to verify the analogous statements for Sobolev–Lorentz functions or continuous pseudomonotone mappings in the critical Sobolev class. Throughout this section, we use the technique of energy integrals to obtain generic lower bounds on dimension.

In section 4 we prove Theorem 1.4. The desired Sobolev map is obtained via a random method, as a generic representative in a parameterized family of mappings. The idea goes back to Kaufman [32].

Section 5 is devoted to examples. Here we prove Theorem 1.6. The quasiconformal map in Theorem 1.6 is constructed in a piecewise fashion on a Whitney decomposition of the complement of a codimension one subspace. The construction is a refined version of an earlier one by Heinonen and Rohde [30], who constructed a quasiconformal map of the unit ball in $\mathbb{R}^n$ sending $\mathcal{H}^{n-1}$-a.e. radial segment onto a curve of infinite length.

In Section 5 we also discuss subcritical Sobolev mappings. The space-filling constructions of Hajłasz–Tyson [27] yield an example of a $W^{1,m}$ mapping f from $\mathbb{R}^n$ to the Hilbert space ℓ^2 for which $f(V_a)$ is infinite-dimensional for every $a \in V^\perp \in G(n, n-m)$. In subsection 5.2 we generalize the constructions from [27] to build similar maps in $W^{1,p}$ for $m < p < n$.

The final Section 6 is reserved for open problems and questions arising out of this study.

Convention. Throughout the paper we denote unspecified positive constants by C or c. We write $C = C(a, b, \ldots)$ to mean that C depends on the data $a, b, \ldots$. We employ the following convention: we write C if we wish to emphasize that a certain constant is *finite*, and we write c if we wish to emphasize that it is *positive*.

We denote by $\#S$ the cardinality of a finite set S. The Lebesgue measure in $\mathbb{R}^n$ will be written $\mathcal{L}^n$.

Acknowledgements. Research for this paper was completed while the second and third authors were visitors in the Mathematics Institute of the University of Bern. The hospitality of the institute is acknowledged. We would like to thank Kari Astala, István Prause and Tadeusz Iwaniec for helpful comments related to Problem 6.2. We are also grateful to Pekka Koskela for numerous discussions on the topic of this paper.

2. Sobolev mappings valued in metric spaces

Our results are naturally phrased in the modern language of metric space-valued Sobolev mappings (see Definition 1.1). This notion was introduced by Ambrosio [1] in 1990 and later studied by Reshetnyak [48]. For the reader's convience, we repeat the definition.

Let B be a Banach space, let $1 \le p < \infty$, and let Ω be a domain in $\mathbb{R}^n$, $n \ge 2$. The Bochner–Lebesgue space $L^p(\Omega, B)$ consists of all weakly measurable, essentially separably valued maps $f : \Omega \to B$ satisfying $\int_\Omega ||f(x)||^p \, dx < \infty$.

Definition 2.1. A map $f : \Omega \to B$ in the Bochner–Lebesgue space $L^p(\Omega, B)$ belongs to the *Ambrosio–Reshetnyak–Sobolev space* $W^{1,p}(\Omega, B)$ if there exists $g \in L^p(\Omega)$ so that for every $b^* \in B^*$ with $||b^*|| \le 1$, we have $\langle b^*, f\rangle \in W^{1,p}(\Omega)$ and $|\nabla\langle b^*, f\rangle| \le g$ a.e.

A function g as in the definition will be called an *upper gradient* for f. Thus $W^{1,p}(\Omega, B)$ consists of those functions in $L^p(\Omega, B)$ which admit an L^p upper gradient.

We may equip $W^{1,p}(\Omega, B)$ with the norm

$$||f||_{1,p} := ||f||_{L^p(\Omega,B)} + \inf_g ||g||_{L^p(\Omega)}. \tag{2.1}$$

Here the infimum is taken over all upper gradients $g \in L^p(\Omega)$ for f. Endowed with this norm, $W^{1,p}(\Omega, B)$ is a Banach space. See, for example Theorem 3.13 in [29].

Furthermore, when $1 < p < \infty$ there exists an upper gradient $g_f \in L^p(\Omega)$ so that

$$||f||_{1,p} = ||f||_{L^p(\Omega,B)} + ||g_f||_{L^p(\Omega)}.$$

Moreover, g_f is unique up to modification on a set of measure zero. The existence of such a *minimal upper gradient* g_f follows by a standard convexity argument.

The space $W^{1,p}(\Omega, B)$ admits the following weak characterization.

Proposition 2.2. *Let B be the dual of a separable Banach space. Then $W^{1,p}(\Omega, B)$ coincides with the space of all functions $f \in L^p(\Omega, B)$ which have weak partial derivatives in $L^p(\Omega, B)$.*

As usual, we say that $f : \Omega \to B$ has $g_i : \Omega \to B$ as a *weak i-th partial derivative* if

$$\int_\Omega (\partial_i \varphi) f \, dx = - \int_\Omega \varphi g_i \, dx \tag{2.2}$$

for all C^∞ functions φ which are compactly supported in Ω. Here $i \in \{1, \ldots, n\}$ and the identity (2.2) is understood in the sense of the Bochner integral, as an equality between elements of B. For a proof of Proposition 2.2, see for example [27].

Now suppose that (Y, d) is a separable metric space. Fix an isometric embedding ι of Y into ℓ^∞. In this case, we say that $f : \Omega \to Y$ is in the *Ambrosio–Reshetnyak–Sobolev space* $W^{1,p}(\Omega, Y)$ if $\iota \circ f \in W^{1,p}(\Omega, \ell^\infty)$. Since ℓ^∞ is the dual of the separable Banach space ℓ^1, the membership of $\iota \circ f$ in $W^{1,p}(\Omega, \ell^\infty)$ can be understood in the weak sense via Proposition 2.2. When $1 < p < \infty$ we write $g_f = g_{\iota \circ f}$ and call this the *minimal upper gradient* of f.

The existence of isometric embeddings of separable metric spaces in ℓ^∞ is well known. For instance, we may use the Kuratowski embedding [28, Chapter 12].

The space $W^{1,p}(\Omega, Y)$ is naturally equipped with a metric by the rule

$$d(f_1, f_2) = ||\iota \circ f_1 - \iota \circ f_2||_{1,p},$$

where $|| \cdot ||_{1,p}$ denotes the norm in (2.1). We emphasize that this metric depends on the choice of the isometric embedding ι. While membership in the class $W^{1,p}(\Omega, Y)$ turns out to be independent of the choice of ι, the metric structure of the space is highly dependent on that choice. This fact has been explored in detail by Hajłasz [23], [24], [21] who has shown, for example, the surprising result that the question of density of Lipschitz mappings in the Sobolev space can admit a different answer depending on the choice of ι.

For additional information on this notion of metric space-valued Sobolev space, we recommend the clear and readable survey [25] by Hajłasz.

Sobolev maps from Ω to Y are absolutely continuous along almost every line, and restrict to Sobolev maps on almost every affine subspace of dimension at least two. We record this fact in the following proposition. It is easily deduced from Proposition 2.2 by standard arguments. See Theorem 2.1.4 and Remark 2.1.5 in [60].

Proposition 2.3. *Let $f \in W^{1,p}(\Omega, Y)$, $p \geq 1$. Then f has an ACL representative $\overline{f}$. In particular, for any $V \in G(n, 1)$, the set of $a \in V^\perp$ for which $\overline{f}|_{V_a \cap \Omega}$ is not absolutely continuous as a map from $(V_a \cap \Omega, \mathcal{H}^1)$ to $(\overline{f}(V_a \cap \Omega), \mathcal{H}^1)$ has zero $\mathcal{H}^{n-1}$-measure. Moreover,* *for any $V \in G(n,m)$, $m \ge 2$, the set of $a \in V^\perp$ for which $\overline{f}|_{V_a\cap\Omega} \notin W^{1,p}(V_a \cap \Omega, Y)$ has zero $\mathcal{H}^{n-m}$-measure.*

By the Morrey–Sobolev embedding theorem, each supercritical mapping $f \in W^{1,p}(\Omega, Y)$, $p > n$, has a representative which is locally $(1 - n/p)$-Hölder continuous. In the remainder of the paper we always work with this representative. In the following proposition, we summarize several basic properties of supercritical Sobolev mappings.

Proposition 2.4. *Let Y be a separable metric space, let $\Omega \subset \mathbb{R}^n$, and let $f \in W^{1,p}(\Omega, Y)$, $p > n$, be represented as above. Let g_f denote the minimal upper gradient for f. Then*

(i) *for all cubes Q compactly contained in Ω, we have*

$$\operatorname{diam} f(Q) \le C(n,p)(\operatorname{diam} Q)^{1-n/p} \left(\int_Q g_f^p \, dx \right)^{1/p}, \tag{2.3}$$

(ii) *the map f satisfies the following quantitative version of the Lusin condition N:*

$$\mathcal{H}^n(f(E)) \le C(n,p)\mathcal{L}^n(E)^{1-n/p} ||g_f||^n_{L^p(\Omega)} \tag{2.4}$$

for all measurable sets $E \subset \Omega$.

The local Hölder continuity and the estimate in (2.3) are established by standard arguments as in the Euclidean case, beginning from the Sobolev–Poincaré inequality for supercritical Sobolev functions. For details, we refer to Ziemer [60, Theorem 2.4.4] or Hajłasz–Koskela [26]. We prove the quantitative Lusin property (2.4). While this argument is also standard, it serves as a model for other proofs which occur in this paper.

We make repeated use of the fact that Hausdorff dimension can be computed using coverings by dyadic cubes. By a *dyadic cube* of size 2^{-j}, $j \in \mathbb{Z}$, we mean a closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, with side length 2^{-j} and with vertices in the set $2^{-j} \cdot \mathbb{Z}^n$. The *$s$-dimensional dyadic Hausdorff measure* $\mathcal{H}^s_{dyadic}$ is defined by the usual Carathéodory procedure to be

$$\mathcal{H}^s_{dyadic}(E) = \lim_{\delta \to 0} \mathcal{H}^s_{dyadic,\delta}(E)$$

where $\mathcal{H}^s_{dyadic,\delta}(E)$ is the infimum of the expressions $\sum_j (\operatorname{diam} Q_j)^s$ over all coverings $\{Q_j\}$ of E by dyadic cubes of diameter no more than δ. The inequalities

$$\mathcal{H}^s_\delta(E) \le \mathcal{H}^s_{dyadic,\delta}(E) \le (4\sqrt{n})^s \mathcal{H}^s_\delta(E), \qquad E \subset \mathbb{R}^n, 0 \le \delta \le \infty, \tag{2.5}$$

show that the dyadic Hausdorff measures generate the same dimension value as do the standard Hausdorff measures. See Mattila [43, §5.2] for details. We recall that the dyadic cubes of a fixed size form an essentially disjoint decomposition of $\mathbb{R}^n$ (that is, they have disjoint interiors).

To prove (2.4), let $\epsilon > 0$, choose $\delta > 0$ sufficiently small relative to ϵ, and consider an arbitrary covering $\{Q_i\}$ of E by essentially disjoint dyadic cubes with side length $r_i < \delta$. Then $f(E)$ is covered by the sets $\{f(Q_i)\}$, and

$$\operatorname{diam} f(Q_i) \le C(n,p) r_i^{1-n/p} \left(\int_{Q_i} g_f^p \, dx \right)^{1/p} \le C(n,p) ||g_f||_{L^p(\Omega)} \delta^{1-n/p} < \epsilon \tag{2.6}$$

by (2.3), provided δ is chosen appropriately. Summing the n-th powers of (2.6) over i, applying Hölder's inequality together with the essential disjointedness of the family $\{Q_i\}$, and taking the infimum over all such coverings $\{Q_i\}$ yields

$$\mathcal{H}^n_\epsilon(f(E)) \le C(n,p) ||g_f||^n_{L^p(\Omega)} \mathcal{H}^n_{dyadic,\delta}(E)^{1-n/p}. \tag{2.7}$$

Letting δ and ϵ tend to zero and recalling the equivalence of $\mathcal{H}^s$ and $\mathcal{H}^s_{dyadic}$ completes the proof of (ii).

Kaufman [32] generalized Proposition 2.4(ii) to cover the full range of Hausdorff measures $\mathcal{H}^s$, $0 < s < n$. Proposition 1.2 is a special case of the following theorem.

Proposition 2.5 (Kaufman)**.** *Let $E \subset \Omega$ be a set of σ-finite $\mathcal{H}^\alpha$ measure for some $0 < \alpha < n$. Let $f \in W^{1,p}(\Omega, Y)$ for some $p > n$. Then $f(E)$ has zero $\mathcal{H}^{p\alpha/(p-n+\alpha)}$ measure.*

The proof of Proposition 2.5 proceeds along exactly the same lines as that of Proposition 2.4(ii) with one additional modification. Since $\alpha < n$, we have that E is a null set for the Lebesgue measure in Ω. Instead of (2.7) we obtain

$$\mathcal{H}^{p\alpha/(p-n+\alpha)}_\epsilon(f(E)) \le C(n,p,\alpha)||g_f||_{L^p(U)}^{\frac{p\alpha}{p-n+\alpha}} \mathcal{H}^\alpha_{dyadic,\delta}(E)^{\frac{p-n}{p-n+\alpha}}$$

for each open set U containing E. Taking the infimum over all such open sets and using the outer regularity of the Lebesgue measure yields the desired conclusion.

3. Exceptional sets for Sobolev mappings

3.1. **Exceptional sets for supercritical Sobolev mappings.** In this subsection, we prove Theorem 1.3.

For $\delta > 0$ we denote by $\mathcal{H}^\alpha_\delta$ the α-dimensional Hausdorff premeasure at scale δ. In particular, $\mathcal{H}^\alpha_\infty$ denotes the α-dimensional Hausdorff content. See [43, Chapter 4] for definitions.

Using countable stability of Hausdorff measure and the invariance of Hausdorff measure under rigid motions of $\mathbb{R}^n$, it suffices to assume that Ω is bounded and $V = \{0\} \times \mathbb{R}^m$. Since the null sets for $\mathcal{H}^\alpha$ and $\mathcal{H}^\alpha_\infty$ coincide [43, Lemma 4.6], the exceptional set of points from the statement of Theorem 1.3 consists of those points $a \in V^\perp$ for which

$$\mathcal{H}^\alpha_\infty(f(V_a \cap \Omega)) > 0.$$

Let us denote this set by $\mathrm{Exc}_f(\alpha)$.

Our first task is to show that $\mathrm{Exc}_f(\alpha)$ is a Borel set. This will permit us to use Frostman's lemma in later proofs.

Lemma 3.1. *For each $\alpha \in [m, n)$, $\mathrm{Exc}_f(\alpha)$ is a Borel set.*

For a linear subspace $W \subset \mathbb{R}^n$, let $P_W : \mathbb{R}^n \to W$ denote the orthogonal projection onto W.

Proof. As described above, we may assume that Ω is bounded. Exhaust Ω with an increasing sequence of compact sets $\{K_i\}$. For $\delta > 0$, let $E(\alpha, i, \delta)$ be the set of points $a \in V^\perp$ with the following property: *whenever $f(V_a \cap K_i)$ is covered by a countable family of open sets, $\{A_k\}$, then $\sum_k (\operatorname{diam} A_k)^\alpha > \delta$.* Then

$$\mathrm{Exc}_f(\alpha) = \bigcup_i \bigcup_{\delta > 0} E(\alpha, i, \delta).$$

We will prove that $E(\alpha, i, \delta)$ is a closed set.

Let (a_j) be a sequence of points in $E(\alpha, i, \delta)$ with $\lim_{j\to\infty} a_j = a$. Let $\{A_k\}$ be a countable family of open sets covering $f(V_a \cap K_i)$. For each k, let $B_k = f^{-1}(A_k)$. Since f is continuous and $V_a \cap K_i$ is compact, it follows from the Tube Lemma [45, Lemma 5.8] that there exists a neighborhood U of a in $V^\perp$ so that $P^{-1}_{V^\perp}(U) \cap K_i \subset \cup_k B_k$. For sufficiently large j, $a_j \in U$ and hence $f(V_{a_j} \cap K_i) \subset \bigcup_k A_k$. Since $\sum_k (\operatorname{diam} A_k)^\alpha > \delta$ we conclude that $a \in E(\alpha, i, \delta)$. This completes the proof. $\square$

Denote by $B_{V^\perp}(a,r)$ the ball in $V^\perp$ with center a and radius $r>0$. We will deduce Theorem 1.3 from the following proposition.

Proposition 3.2. *Let α satisfy (1.6), let $p>n$, and define $\beta=\beta(p,\alpha)$ by the formula (1.5). Let $E\subset V^\perp$ be a set of finite $\mathcal{H}^\beta$ measure and assume that μ is a positive Borel measure supported on E and satisfying the growth condition*

$$\mu(B_{V^\perp}(a,r))\le r^\beta \qquad \text{for all } a\in V^\perp \text{ and } r>0. \tag{3.1}$$

Finally, let $f\in W^{1,p}(\Omega,Y)$. Then $\mathcal{H}^\alpha(f(V_a\cap\Omega))=0$ for μ-a.e. $a\in E$.

Proof. We may assume without loss of generality that $\Omega=(0,1)^n$ and that $E\subset P_{V^\perp}(\Omega)$. Fix $\delta>0$. Since $\beta<n-m$, E can be included in an open set $U_\delta\subset\mathbb{R}^{n-m}$ of $\mathcal{H}^{n-m}$ measure at most δ. Since $g_f\in L^p(\Omega)$,

$$\lim_{\delta\to 0}\int_{U_\delta\times(0,1)^m} g_f^p\,dx=0. \tag{3.2}$$

Consider an essentially disjoint collection of dyadic cubes, $\{R_i\}$, contained in U_δ and covering E, for which

$$\sum_i r_i^\beta<\mathcal{H}^\beta_{dyadic,\delta}(E)+\delta.$$

Here r_i denotes the side length of R_i; we assume without loss of generality that $r_i<\delta$ for all i. For each i, let $\{Q_{ij}\}_{j=1}^{N_i}$ be a family of essentially disjoint dyadic cubes in $\mathbb{R}^n$, each of which has side length r_i, with the property that $\bigcup_j Q_{ij}=R_i\times(0,1)^m$. For fixed i, the number N_i of cubes Q_{ij} is on the order of r_i^{-m}.

By Proposition 2.4(i),

$$\operatorname{diam} f(Q_{ij})\le Cr_i^{1-n/p}\left(\int_{Q_{ij}} g_f^p\,dx\right)^{1/p}\le C||g_f||_{L^p(Q_{ij})}\delta^{1-n/p}=:\epsilon. \tag{3.3}$$

Here g_f denotes the minimal L^p upper gradient for f.

For each $a\in E$, we have

$$\mathcal{H}^\alpha_\epsilon(f(V_a\cap\Omega))\le\sum_{j=1}^{N_i}(\operatorname{diam} f(Q_{ij}))^\alpha$$

for each i so that $a\in R_i$. For fixed i and $a\in E$, let

$$\chi(i,a)=\begin{cases}1, & \text{if } a\in R_i,\\ 0, & \text{else.}\end{cases}$$

Then $\mathcal{H}^\alpha_\epsilon(f(V_a\cap\Omega))\le\sum_i\chi(i,a)\sum_{j=1}^{N_i}(\operatorname{diam} f(Q_{ij}))^\alpha$ and so

$$\begin{aligned}\int^*_{V^\perp}\mathcal{H}^\alpha_\epsilon(f(V_a\cap\Omega))\,d\mu(a)&\le\int^*_{V^\perp}\sum_i\chi(i,a)\sum_{j=1}^{N_i}(\operatorname{diam} f(Q_{ij}))^\alpha\,d\mu(a)\\&=\sum_i\mu(R_i)\sum_j(\operatorname{diam} f(Q_{ij}))^\alpha\\&\le C(n,p)\sum_i r_i^\beta r_i^{\alpha(1-n/p)}\sum_j\left(\int_{Q_{ij}} g_f^p\,dx\right)^{\alpha/p},\end{aligned}$$

where we used (2.3) and (3.1). (Here we employed the upper integral $\int^*$ to avoid the difficult issue of measurability of the integrand $a \mapsto \mathcal{H}^\alpha_\epsilon(f(V_a \cap \Omega))$.)

Applying Hölder's inequality to the inner sum, we obtain

$$\begin{aligned}\int^*_{V^\perp} \mathcal{H}^\alpha_\epsilon(f(V_a \cap \Omega))\, d\mu(a) &\le C(n,p) \sum_i r_i^{\beta+\alpha(1-n/p)} (N_i)^{1-\alpha/p} \left(\sum_{j=1}^{N_i} \int_{Q_{ij}} g_f^p\, dx \right)^{\alpha/p} \\ &\le C(n,p) \sum_i r_i^{\beta+\alpha(1-n/p)-m(1-\alpha/p)} \left(\int_{R_i \times (0,1)^m} g_f^p\, dx \right)^{\alpha/p}.\end{aligned}$$

Applying Hölder's inequality again yields

$$\int^*_{V^\perp} \mathcal{H}^\alpha_\epsilon(f(V_a \cap \Omega))\, d\mu(a) \le C(n,p) \left(\sum_i \int_{R_i \times (0,1)^m} g_f^p\, dx \right)^{\frac{\alpha}{p}} \left(\sum_i r_i^{(\beta+\alpha(1-\frac{n}{p})-m(1-\frac{\alpha}{p}))\frac{p}{p-\alpha}} \right)^{1-\frac{\alpha}{p}}.$$

Since $\beta = \beta(p,\alpha)$,

$$\left(\beta + \alpha(1-\tfrac{n}{p}) - m(1-\tfrac{\alpha}{p})\right) \left(\frac{p}{p-\alpha} \right) = \beta.$$

Thus

$$\begin{aligned}\int^*_{V^\perp} \mathcal{H}^\alpha_\epsilon(f(V_a \cap \Omega))\, d\mu(a) &\le C(n,p) \left(\int_{U_\delta \times (0,1)^m} g_f^p\, dx \right)^{\alpha/p} \left(\sum_i r_i^\beta \right)^{1-\alpha/p} \\ &\le C(n,p) ||g_f||^\alpha_{L^p(U_\delta \times (0,1)^m)} \left(\mathcal{H}^\beta_{dyadic,\delta}(E) + \delta \right)^{1-\alpha/p}.\end{aligned} \tag{3.4}$$

Letting $\delta \to 0$ and using the Monotone Convergence Theorem, the equivalence of $\mathcal{H}^s$ and $\mathcal{H}^s_{dyadic}$, and (3.2), we conclude that $\int^*_{V^\perp} \mathcal{H}^\alpha(f(V_a \cap \Omega))\, d\mu(a) = 0$. This completes the proof of the proposition. □

Remark 3.3. The reader may have noticed that we only used the condition $\alpha < p$ in the preceding proof, while the hypotheses include the stronger restriction

$$\alpha < \frac{pm}{p-n+m}. \tag{3.5}$$

The reason for (3.5) is implicit in the proof: recall that (3.5) holds if and only if $\beta > 0$. In practice, the desired measure μ will be obtained by an application of Frostman's lemma, which requires the growth exponent β to be positive.

Remark 3.4. Some aspects of the preceding proof are modelled on a lemma of Bourdon [8] (see also Pansu [46]) which provides lower estimates for the conformal dimension of a metric space. This formal similarity is not surprising. Lower bounds on the conformal dimension of a metric space indicate that a large family of (quasisymmetrically equivalent) spaces have uniformly large dimension, while Theorem 1.3 indicates restrictions on the set of parameters a for which the dimensions of the fiber images $f(V_a \cap \Omega)$ are all uniformly large.

Proof of Theorem 1.3. Let $\beta = \beta(p,\alpha)$. Suppose $\mathrm{Exc}_f(\alpha)$ has positive $\mathcal{H}^\beta$ measure. By Lemma 3.1 and Theorem 8.13 in [43], there exists a compact set $E \subset \mathrm{Exc}_f(\alpha)$ so that $0 < \mathcal{H}^\beta(E) < \infty$. By Frostman's lemma ([43, Theorem 8.9]), there exists a positive Borel measure $\mu \neq 0$ supported on E such that $\mu(B_{V^\perp}(a,r)) \le r^\beta$ for all $a \in E$ and $r > 0$. Then μ is absolutely continuous with respect to $\mathcal{H}^\beta \llcorner E$, so $\mu(E) < \infty$. By Proposition 3.2, $\mathcal{H}^\alpha(f(V_a \cap \Omega)) = 0$ for μ-a.e. $a \in E$. This contradicts the definition of $\mathrm{Exc}_f(\alpha)$. □

3.2. **Exceptional sets for Sobolev–Lorentz and critical Sobolev mappings.** Theorem 1.3 generalizes in several directions. In this subsection, we indicate such generalizations and describe what changes to the previous argument are necessary for their proofs.

We begin with the definition of the Ambrosio–Reshetnyak–Sobolev–Lorentz spaces $W^{1,n,q}$. Sobolev–Lorentz spaces measure a finer scale of integrability criteria. In the metric space-valued case, Ambrosio–Reshetnyak–Sobolev–Lorentz spaces have been considered by Wildrick and Zürcher [59].

We first recall the Lorentz spaces. As before, let Ω be a domain in $\mathbb{R}^n$. The *non-increasing rearrangement* $f^* : [0,\infty) \to [0,\infty]$ of a measurable function $f : \Omega \to \mathbb{R}$ is

$$f^*(t) = \inf\{\alpha \ge 0 : \omega_f(\alpha) \le t\},$$

where $\omega_f(\alpha) = |\{x \in \Omega : |f(x)| > \alpha\}|$ denotes the distribution function of f. For $1 \le p < \infty$ and $1 \le q < \infty$, the Lorentz space $L^{p,q}(\Omega)$ consists of those measurable functions $f : \Omega \to \mathbb{R}$ for which

$$||f||_{p,q} := \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \, \frac{dt}{t} \right)^{1/q}$$

is finite. Equipped with the norm $||\cdot||_{p,q}$, $L^{p,q}(\Omega)$ is a Banach space. Note that $L^{p,p}(\Omega) = L^p(\Omega)$ isometrically as Banach spaces. For the following result of Calderón, see Ziemer [60, Lemma 1.8.13].

Proposition 3.5. *If $p > 1$ and $1 \le q < r < \infty$, then $||f||_{p,r} \le C(p,q,r)||f||_{p,q}$ for all measurable functions f. In particular, $L^{p,q}(\Omega)$ admits a bounded embedding into $L^{p,r}(\Omega)$.*

The *Sobolev–Lorentz space* $W^{1,p,q}(\Omega)$ consists of those functions $f \in L^p(\Omega)$ whose distributional first-order partial derivatives exist as functions in $L^{p,q}(\Omega)$.

Definition 3.6. Let B be a Banach space. A map $f : \Omega \to B$ in $L^p(\Omega, B)$ belongs to the *Ambrosio–Reshetnyak–Sobolev–Lorentz space* $W^{1,p,q}(\Omega, B)$ if $\langle b^*, f\rangle \in W^{1,p,q}(\Omega)$ for every $b^* \in B^*$ with $||b^*|| \le 1$ and f admits an upper gradient in $L^{p,q}(\Omega)$.

For a separable metric space (Y, d) equipped with a fixed isometric embedding ι into ℓ^∞, we say that $f : \Omega \to Y$ is in the *Ambrosio–Reshetnyak–Sobolev–Lorentz space* $W^{1,p,q}(\Omega, Y)$ if $\iota \circ f \in W^{1,p,q}(\Omega, \ell^\infty)$.

Maps in $W^{1,n,1}$ retain many analytic and geometric properties enjoyed by supercritical Sobolev maps. The following result is the analog of Proposition 2.4 for the space $W^{1,n,1}$. For a proof of part (i), see Kauhanen–Koskela–Malý [35] for the case of real-valued functions and Wildrick–Zürcher [59] for the general case. Part (ii) follows from part (i) as in Proposition 2.4.

Proposition 3.7. *Let Y be a separable metric space, $\Omega \subset \mathbb{R}^n$, and $f \in W^{1,n,1}(\Omega, Y)$. Then*

(i) *f satisfies the Rado–Reichelderfer condition: there exists $\Theta_f \in L^1(\Omega)$ so that*

$$(\operatorname{diam} f(Q))^n \le \int_Q \Theta_f \, dx \tag{3.6}$$

for all cubes Q compactly contained in Ω. In particular, f is continuous.

(ii) *f satisfies the following quantitative version of the Lusin condition N: there exists a constant $C(n)$ depending only on the dimension n so that*

$$\mathcal{H}^n(f(E)) \le C(n) \int_E \Theta_f \, dx \tag{3.7}$$

for all measurable sets $E \subset \Omega$.

For maps in the Sobolev–Lorentz class $W^{1,n,1}$, we have the following result, which generalizes and extends our main Theorem 1.3. The dimension estimate for the exceptional sets which we obtain matches that in Corollary 1.5.

Theorem 3.8. *Let $f : \Omega \to Y$ be a map in the class $W^{1,n,1}(\Omega, Y)$, let $V \in G(n,m)$ and let $m < \alpha < n$. Then $\mathcal{H}^\alpha(f(V_a \cap \Omega)) = 0$ for $\mathcal{H}^{m\left(\frac{n}{\alpha}-1\right)}$-a.e. $a \in V^\perp$.*

To prove Theorem 3.8, we set $\beta = m(\frac{n}{\alpha} - 1)$ and replace (3.4) with

$$\int_{V^\perp}^* \mathcal{H}^\alpha_\epsilon(f(V_a \cap \Omega))\, d\mu(a) \le C \left(\mathcal{H}^\beta(E)\right)^{1-\alpha/n} ||\Theta_f||^{\alpha/n}_{L^1(U_\delta \times (0,1)^m)}, \tag{3.8}$$

where Θ_f denotes the Rado–Reichelderfer control function from Proposition 3.7. To prove (3.8), we replace (3.3) with (3.6) and use the arguments from the proof of Proposition 3.2.

Mappings in the critical Sobolev class $W^{1,n}$ need not enjoy the good analytic properties such as continuity and the validity of condition N. We require additional conditions. To this end, we recall the notion of pseudomonotonicity.

Definition 3.9. A mapping $f : \Omega \to Y$ from a domain $\Omega \subset \mathbb{R}^n$ to a metric space Y is called *K-pseudomonotone*, $K \ge 1$, if $\operatorname{diam} f(B(x,r)) \le K \operatorname{diam} f(\partial B(x,r))$ for all $x \in \Omega$ and all $r < \operatorname{dist}(x, \partial\Omega)$. If f is K-pseudomonotone for some K, we say that f is *pseudomonotone*.

For example, all homeomorphisms between Euclidean domains are 1-pseudomonotone. Also, all η-quasisymmetric embeddings of Euclidean domains into metric spaces are K-pseudomonotone for some K depending only on η, see Lemma 5.5 in [57]. A theorem of Malý and Martio [38] (see also [57] and [29]) asserts that continuous pseudomonotone mappings in the class $W^{1,n}$ verify Lusin's condition N.

Theorem 3.10. *Let $f : \Omega \to Y$ be a continuous, pseudomonotone map in the class $W^{1,n}(\Omega, Y)$. Let $V \in G(n,m)$ and let $m < \alpha < n$. Then $\mathcal{H}^\alpha(f(V_a \cap \Omega)) = 0$ for $\mathcal{H}^{m\left(\frac{n}{\alpha}-1\right)}$-a.e. $a \in V^\perp$.*

Theorems 3.8 and 3.10 extend to the case of subspaces of intermediate dimension the results of Malý and Martio [38] and Kauhanen–Koskela–Malý [35] on the validity of condition N. In turn, Theorem 3.10 yields corresponding conclusions for quasisymmetric maps taking values in metric spaces, thereby generalizing Corollary 1.5.

Corollary 3.11. *The conclusion of Theorem 3.8 holds for an arbitrary quasisymmetric map f from a domain $\Omega \subset \mathbb{R}^n$ onto a metric space Y of locally finite Hausdorff $\mathcal{H}^n$ measure.*

In the remainder of this section, we prove Theorem 3.10. While the overall structure of the proof generally follows that of our main Theorem 1.4, we must make several subtle modifications to the argument to deal with the lack of universal, scale-invariant estimates such as the Morrey–Sobolev inequality, and the concomitant use of covering theorems, in the present situation.

The key point of the argument is to prove the following analog of (3.4) for continuous pseudomonotone maps $f \in W^{1,n}$:

$$\int_{V^\perp}^* \mathcal{H}^\alpha_\epsilon(f(V_a \cap \Omega))\, d\mu(a) \le C \left(\mathcal{H}^\beta(E)\right)^{1-\alpha/n} \left(||g_f||^\alpha_{L^n(U_\delta \times (0,1)^m)} + |U_\delta \times (0,1)^m|^{\alpha/n}\right). \tag{3.9}$$

The proof of (3.9) is more difficult than that of its predecessor (3.4).

We denote by $Q(x,r)$ a cube of side length r centered at a point x. Let us observe that if the estimate

$$(\operatorname{diam} f(Q(x,r)))^n \le C \int_{Q(x,r)} (1+g_f^n)\,dx \tag{3.10}$$

were valid for all cubes $Q(x,r)$ (compactly contained in Ω), and for some fixed finite constant C, then (3.9) would follow by the previous arguments. However, it turns out that this estimate is not necessarily true for all locations and scales.

For a fixed $C<\infty$, let us say that a cube $Q(x,r)\subset\Omega$ is a *C-good cube* if the estimate (3.10) holds.

Definition 3.12. A collection $\mathcal{Q}$ of cubes in Ω is called a *frequent cover* of a set $E\subset\Omega$ if for each point $x\in E$ and for each $0<r_0<\operatorname{dist}(x,\partial\Omega)$, there exists a cube $Q(x,r)\in\mathcal{Q}$ with $r_0/2\le r\le r_0$.

In other words, $\mathcal{Q}$ contains cubes centered at every point x of E, whose side lengths differ by at most a factor of two from the side length of any cube centered at x and contained in Ω.

Proposition 3.13. *Let f be a continuous, K-pseudomonotone map in the class $W^{1,n}(\Omega,Y)$. There exists a finite constant $C=C(n,K)$ so that the collection $\mathcal{Q}$ of C-good cubes $Q(x,r)$ in Ω is a frequent cover of Ω.*

Proposition 3.13 follows from two known auxiliary results, stated in the following two lemmas. Lemma 3.14 is a standard fact of real analysis. For a proof of the analogous result with cubes replaced by balls, see pp. 22–23 of [38]. There are no complications involved in replacing balls by cubes in this proof. Lemma 3.15 follows from the Sobolev embedding theorem applied on the codimension one sets $\partial Q(x,r)$. See, for instance, Proposition 4.9 in [57].

Lemma 3.14. *Let $h\in L^1(\Omega)$ be nonnegative. There exists a finite constant C so that the following holds: for each $x\in\Omega$ and for any $r_0>0$ so that $Q(x,r_0)$ is compactly contained in Ω, the set of values $r\in(r_0/2,r_0)$ for which*

$$\int_{\partial Q(x,r)} h(y)\,d\sigma(y) \le \frac{C}{r}\int_{Q(x,r)} (1+h(y))\,dy \tag{3.11}$$

has positive Lebesgue 1-measure.

Lemma 3.15. *Let Ω be a domain in $\mathbb{R}^n$, let Y be separable metric, and let $f\in W^{1,n}(\Omega,Y)$ be continuous. For each $x\in\Omega$ and a.e. $r>0$ such that $Q(x,r)$ is compactly contained in Ω, we have*

$$\operatorname{diam} f(\partial Q(x,r)) \le C r^{1/n}\left(\int_{\partial Q(x,r)} g_f(y)^n\,d\sigma(y)\right)^{1/n}. \tag{3.12}$$

In (3.11) and (3.12) the integral over $\partial Q(x,r)$ is taken with respect to the surface area measure $d\sigma$. Note that although the integrands are merely Lebesgue functions, the value of such integrals is well-defined for almost every $r>0$ by Fubini's theorem (after choosing suitable representatives for the integrands).

Proof of Proposition 3.13. By the definition of pseudomonotonicity, the estimate

$$\operatorname{diam} f(Q(x,r)) \le K \operatorname{diam} f(\partial Q(x,r))$$

holds for all $x \in \Omega$ and $r > 0$ so that $Q(x,r)$ is compactly contained in Ω. By Lemma 3.15, the estimate

$$(\operatorname{diam} f(\partial Q(x,r)))^n \le Cr \int_{\partial Q(x,r)} g_f^n \, d\sigma$$

holds for all $x \in \Omega$ and a.e. $r > 0$ so that $Q(x,r)$ is compactly contained in Ω. Finally, by Lemma 3.14, the estimate

$$\int_{\partial Q(x,r)} g_f^n \, d\sigma \le \frac{C}{r} \int_{Q(x,r)} (1 + g_f^n) \, dx$$

holds for a frequent cover $\{Q(x,r)\}$ of Ω by cubes. The proof is complete. □

For the remainder of this section, we assume that a constant C has been fixed so that the conclusion of Proposition 3.13 is satisfied. Henceforth, we say that a cube is a *good cube* if it is a C-good cube for this value of C.

Proof of Theorem 3.10. First, we remark that since the conclusion involves only the Hausdorff measures of subsets of Ω, we may utilize the countable stability of Hausdorff measure to restrict our attention to subdomains Ω' which are compactly contained in Ω. For points x in such a subdomain Ω', the choice of r_0 in Definition 3.12 can be made independently of x. Indeed, we may choose $r_0 = \operatorname{dist}(\Omega', \partial\Omega)$.

We begin the proof of (3.9) in exactly the same fashion as before. Assuming for the sake of contradiction that the exceptional set E has positive $\mathcal{H}^\beta$ measure, we select a Frostman measure μ supported on E for the exponent β. We claim that $\mathcal{H}^\alpha(f(V_a \cap \Omega)) = 0$ for μ-a.e. $a \in E$.

Fix $\delta > 0$; without loss of generality we may assume that $\delta \le \operatorname{dist}(\Omega', \partial\Omega)$. Since

$$\beta = m\left(\frac{n}{\alpha} - 1\right) < n - m,$$

E can be included in an open set $U_\delta \subset \mathbb{R}^{n-m}$ of $\mathcal{H}^{n-m}$ measure at most δ. Since $g_f \in L^n(\Omega)$, $\int_{U_\delta \times (0,1)^m} g_f^n \, dx$ tends to zero as δ tends to zero. We consider an essentially disjoint collection of dyadic cubes, $\{R_i\}$, contained in U_δ and covering E, for which

$$\sum_i r_i^\beta < \mathcal{H}^\beta_{dyadic,\delta}(E) + \delta,$$

where $r_i < \delta$ denotes the side length of R_i.

At this stage of the proof, we remark that we can no longer guarantee that the cubes Q_{ij} considered in the previous version of the proof are good cubes. Since the collection of good cubes is a frequent cover, we know that any cube is contained in a good cube with comparable side length. However, we must deal with the possible resulting overlap. We do this using the Besicovitch Covering Theorem.

Consider the set $R_i \times [0,1]^m$. For each point (a,x) in this set, choose a side length r_{ax} satisfying

$$\frac{r_i}{2} \le r_{ax} \le r_i$$

so that the cube

$$Q_{ax} := Q((a,x), r_{ax})$$

is a good cube. The collection $\{Q_{ax}\}$ of such cubes, as a varies over $\cup_i R_i$ and x varies over $[0,1]^m$, is a Besicovitch cover of $E \times [0,1]^m$. By the Besicovitch Covering Theorem (see Theorem 1.1 in [10] or Theorem 2.7 in [43]), we can extract a countable collection of cubes,

$\{Q_j = Q_{a_j x_j}\} \subset \{Q_{ax}\}$, which continues to cover $E \times [0,1]^m$ and which has bounded overlap, i.e., no point of Ω lies in more than M of the cubes Q_j, where M depends only on n.

Let us observe that the number N_i of these cubes Q_j for which $a_j \in R_i$ is still bounded by a multiple of r_i^{-m}. Indeed, any such cube Q_j is contained within $3R_i \times [0,1]^m$ (where $3R_i$ denotes the cube in $V^\perp$ concentric with R_i, whose side length is equal to $3r_i$). The conclusion then follows by volume considerations:

$$r_i^{n-m} = \mathcal{L}^n(R_i \times [0,1]^m) \ge \frac{1}{3^{n-m}M} \sum_j \mathcal{L}^n(Q_j) \ge \frac{1}{3^{n-m}M} \left(\frac{r_i}{2}\right)^n N_i,$$

whence

$$N_i \le M \cdot 2^n \cdot 3^{n-m} \cdot r_i^{-m}. \tag{3.13}$$

Since each of the cubes Q_j is good, the estimate

$$(\operatorname{diam} f(Q_j))^n \le C(n,K) \int_{Q_j} (1 + g_f^n)\, dx \le C(n,K)(\delta^n + \omega_{g_f^n}(\delta^n)) =: \epsilon$$

holds, where $\omega_{g_f^n}(b) := \sup_{E:|E|\le b} \int_E g_f^n\, dx$ denotes a modulus of continuity for the set function $E \mapsto \int_E g_f^n\, dx$.

For each $a \in E$, we have

$$\mathcal{H}^\alpha_\epsilon(f(V_a \cap \Omega)) \le \sum_{\substack{j \\ V_a \cap Q_j \ne \emptyset}} (\operatorname{diam} f(Q_j))^\alpha.$$

Again we introduce, for fixed j and a point $a \in E$, the characteristic function

$$\chi(j,a) = \begin{cases} 1, & \text{if } V_a \cap Q_j \ne \emptyset, \\ 0, & \text{else.} \end{cases}$$

Then $\mathcal{H}^\alpha_\epsilon(f(V_a \cap \Omega)) \le \sum_j \chi(j,a)(\operatorname{diam} f(Q_j))^\alpha$ and so

$$\begin{aligned} \int^*_{V^\perp} \mathcal{H}^\alpha_\epsilon(f(V_a \cap \Omega))\, d\mu(a) &\le \int^*_{V^\perp} \sum_j \chi(j,a)(\operatorname{diam} f(Q_j))^\alpha\, d\mu(a) \\ &= \sum_j \mu(P_{V^\perp}(Q_j))(\operatorname{diam} f(Q_j))^\alpha \\ &\le \sum_j (\operatorname{diam} Q_j)^\beta (\operatorname{diam} f(Q_j))^\alpha \\ &\le C(n,K) \sum_i \sum_{\substack{j \\ a_j \in R_i}} r_i^\beta \left(\int_{Q_j} (1 + g_f^n)\, dx \right)^{\alpha/n} \end{aligned}$$

Applying Hölder's inequality and (3.13) we obtain

$$\int^*_{V^\perp} \mathcal{H}^\alpha_\epsilon(f(V_a \cap \Omega))\, d\mu(a) \le C(n,K) \sum_i r_i^{\beta - m(1-\alpha/n)} \left(\sum_{\substack{j \\ a_j \in R_i}} \int_{Q_j} (1 + g_f^n)\, dx \right)^{\alpha/n}.$$

Applying Hölder's inequality again yields

$$\int_{V^\perp}^{*} \mathcal{H}^\alpha_\epsilon(f(V_a \cap \Omega))\,d\mu(a) \le C(n,K)\left(\sum_{\substack{i,j\\ a_j\in R_i}} \int_{Q_j}(1+g_f^n)\,dx\right)^{\frac{\alpha}{n}} \left(\sum_i r_i^{(\beta-m(1-\frac{\alpha}{n}))\frac{n}{n-\alpha}}\right)^{1-\frac{\alpha}{n}}.$$

Observe that $(\beta - m(1-\frac{\alpha}{n}))(\frac{n}{n-\alpha}) = \beta$. Since the collection $\{Q_j\}$ is a Besicovitch cover, we obtain

$$\int_{V^\perp}^{*} \mathcal{H}^\alpha_\epsilon(f(V_a \cap \Omega))\,d\mu(a) \le C(n,K,M,\alpha)\left(\int_{U_\delta\times(0,1)^m}(1+g_f^n)\,dx\right)^{\alpha/n}\left(\sum_i r_i^\beta\right)^{1-\alpha/n}.$$

From here, we derive (3.9) and complete the proof of the theorem exactly as before. We omit the remaining details. □

3.3. **Remarks on quasiconformal mappings.** Quasiconformal self-maps of $\mathbb{R}^n$, $n \ge 2$, lie in $W^{1,p}$ for some $p > n$. This is Gehring's higher integrability theorem [18]. Corollary 1.5 follows from this fact and Theorem 1.3. More precisely, if f is K-quasiconformal then

$$\dim \operatorname{Exc}_f(\alpha) \le (n-m) - \left(1 - \frac{m}{\alpha}\right) p(n,K) \tag{3.14}$$

where $p(n,K) > n$ denotes the sharp exponent of higher integrability for the partial derivatives of a K-quasiconformal mapping. We say that f is K-*quasiconformal* if $H_f(x) \le K$ for all $x \in \Omega$, where

$$H_f(x) = \limsup_{r\to 0} \frac{\sup\{|f(x)-f(y)| : |x-y| = r\}}{\inf\{|f(x)-f(z)| : |x-z| = r\}}$$

denotes the *metric dilatation* of a homeomorphism $f : \Omega \to \Omega'$ between domains in $\mathbb{R}^n$.

A celebrated theorem of Astala [2] asserts that

$$p(2,K) = \frac{2K}{K-1}; \tag{3.15}$$

the corresponding value $p(n,K) = \frac{nK}{K-1}$ remains a conjecture when $n \ge 3$.

Astala's theorem yields sharp bounds on dimension distortion by planar quasiconformal maps. If f is a K-quasiconformal map between planar domains Ω, Ω' and $E \subset \Omega$, then

$$\frac{1}{K}\left(\frac{1}{\dim E} - \frac{1}{2}\right) \le \frac{1}{\dim f(E)} - \frac{1}{2} \le K\left(\frac{1}{\dim E} - \frac{1}{2}\right). \tag{3.16}$$

We deduce from (3.14) and (3.15) that

$$\dim \operatorname{Exc}_f(\alpha) \le \frac{2K - (K+1)\alpha}{\alpha(K-1)} \tag{3.17}$$

whenever f is a K-quasiconformal map between planar domains, $V \in G(2,1)$, and $\alpha \in [1,2)$. Note that the right hand side of (3.17) is equal to zero precisely when

$$\alpha = \frac{2K}{K+1} = 1 + \left(\frac{K-1}{K+1}\right).$$

This agrees with the upper bound in (3.16) for the dimension of the image of any 1-dimensional set under a planar K-quasiconformal map. In fact, the proof of (3.16) given in [2] uses only the higher Sobolev integrability of f.

We discuss the case of quasiconformal mappings further in Problem 6.2.

4. Sobolev maps which increase the dimension of many affine subspaces

In this section we prove Theorem 1.4. The proof which we give is modelled closely on that of an analogous result of Kaufman [32, Theorem 3], which exhibits Sobolev mappings which increase maximally the dimension of a fixed subset. Our situation is complicated by the fact that we work with the orthogonal splitting of $\mathbb{R}^n$ into $V = \{0\} \times \mathbb{R}^m$ and $V^\perp = \mathbb{R}^{n-m} \times \{0\}$ and look for a mapping which simultaneously increases the dimension of many fibers.

Recall that our goal is to construct a $W^{1,p}$ map of $\mathbb{R}^n$ which increases the dimensions of all of the fibers V_a over the points a in a certain set $E \subset V^\perp$ from m to α. To achieve this, we will use a random construction. We will define a family of maps (f_ξ) parameterized by sequences ξ of independent and identically distributed random variables. All of these maps will lie in the Sobolev class $W^{1,p}$, and we will show that, almost surely with respect to ξ, such maps have the desired property. We do not know whether a deterministic construction can be given.

Recall also that in the statement of Theorem 1.4 we assume that the set E satisfies the growth condition

$$\mathbf{N}(E, r) \le C r^{-\beta} \tag{4.1}$$

for all $r < r_0$, for some constants C and $r_0 > 0$. Here $\beta = \beta(p, \alpha)$ is the value given in (1.5). In particular, $\mathcal{H}^\beta(E) < \infty$ and so

$$\dim E \le \beta. \tag{4.2}$$

Recall that in the statement of Theorem 1.4 we merely assume that $p \ge 1$. For (4.2) to hold, we necessarily must have $\beta \ge 0$. When $p > n - m$, this imposes the usual restriction (1.6) on α. When $p \le n - m$, no upper bound on the value of α is required. We fix an integer $N > \alpha$; this value will be the dimension of the target space for our mapping. When $p > n$, we may set $N = n$.

We are ready to begin the proof of the theorem.

Proof of Theorem 1.4. Let E be a bounded subset of $\mathbb{R}^{n-m}$ satisfying (4.1) for all $0 < r < r_0$, for suitable constants C and r_0. By applying a preliminary homothety, we may assume that $E \subset [0,1]^{n-m}$. The maps $f_\xi \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$ which we will construct (see (4.5)) will satisfy

$$\mathcal{H}^{\alpha'}(f_\xi(V_a \cap [0,1]^n)) = \infty \tag{4.3}$$

for $\mathcal{H}^\beta$-almost every $a \in E$ and almost surely in ξ, for each $\alpha' < \alpha$. This clearly suffices to obtain the desired conclusion $\dim f_\xi(V_a) \ge \alpha$ for $\mathcal{H}^\beta$-a.e. $a \in E$ and almost surely in ξ.

Before continuing with the proof, we pause to review terminology from symbolic dynamics. Let $W = \{1, \ldots, 2^n\}$, let W^j be the set of (ordered) j-tuples of elements of W, and let

$$W^* = \bigcup_{j \ge 0} W^j$$

be the set of all finite sequences of elements of W (including the empty sequence). We call the elements of W^* *words* comprised of the *letters* in W. If $v = (v_1, \ldots, v_j)$ and $w = (w_1, \ldots, w_k)$ are words with $j \ge k$, we say that w is a *subword* of v if $v_i = w_i$ for all $i = 1, \ldots, k$. The *length* of a word $w \in W^j$ is equal to j.

We use the set W^* to index the cubes in the standard dyadic decomposition

$$\mathcal{D} = \{Q_w\}_{w \in W^*}$$

of $Q = [0,1]^n$. We choose this indexing in such a way that the side length $s(Q_w)$ of Q_w is equal to 2^{-j} if w has length j, and also that $Q_w \subset Q_v$ if v is a subword of w. For each j, the cubes $\{Q_w\}_{w\in W^j}$ form an essentially disjoint decomposition of Q.

We also introduce a second collection of cubes, obtained by dilating the elements of $\mathcal{D}$. For each $w \in W_*$, let $Q'_w = 100Q_w$. It is important to note that, for fixed j, the collection $\{Q'_w\}_{w\in W^j}$ has bounded overlap: no points of $\mathbb{R}^n$ lies in more than C of the cubes in this collection, where C is a constant depending only on the dimension n.

We project these cubes into the subspaces V and $V^\perp$. In order to maintain a consistent notation we write

$$Q_w^{V^\perp} = P_{V^\perp}(Q_w) \qquad \text{and} \qquad Q_w^V = P_V(Q_w)$$

for such projections. We view these as cubes in $\mathbb{R}^{n-m}$ and $\mathbb{R}^m$ respectively. Similarly, we define $(Q_w^{V^\perp})'$ and $(Q_w^V)'$ to be the corresponding dilated cubes. Note that Q_w, $Q_w^{V^\perp}$ and Q_w^V all have the same side length $2^{-|w|}$. Similarly, Q'_w, $(Q_w^{V^\perp})'$ and $(Q_w^V)'$ all have the same side length $100\cdot 2^{-|w|}$. In particular, we denote by $Q^V = P_V(Q)$ the unit cube $[0,1]^m$ and by $Q^{V^\perp} = P_{V^\perp}(Q)$ the unit cube $[0,1]^{n-m}$.

For each $w \in W^*$, let ψ_w be a function in $C_0^\infty(\mathbb{R}^n)$ satisfying the following conditions:

(i) $0 \le \psi_w \le 1$,
(ii) $\psi_w \equiv 1$ on Q_w,
(iii) $\psi_w \equiv 0$ on the complement of $\frac{5}{4}Q_w$,
(iv) $|\nabla\psi_w| \le \frac{C}{s(Q_w)} = C2^{|w|}$.

Let $\xi = (\xi_w)$ be a countable sequence of elements, indexed by the words w in W^*, each lying in the unit ball $B \subset \mathbb{R}^N$. We define the mappings f_ξ. For each $j \ge 0$, we first define mappings $f_{\xi,j} : \mathbb{R}^n \to \mathbb{R}^N$ by the formula

$$f_{\xi,j}(a,x) = \sum_{\substack{w\in W^j \\ Q_w^{V^\perp}\cap E\neq\emptyset}} \mathcal{H}^m((Q_w^V)')^{1/\alpha}\psi_w(a,x)\xi_w, \qquad x \in V, a \in V^\perp.$$

Note that

$$\mathcal{H}^m((Q_w^V)') = C(m)2^{-jm} \tag{4.4}$$

whenever $w \in W^j$, for some fixed constant $C(m)$.

Lemma 4.1. *For all ξ as above and all $j \ge 0$, the map $f_{\xi,j}$ is in $W^{1,p}(\mathbb{R}^n,\mathbb{R}^N)$, with $||f_{\xi,j}||_{1,p}$ bounded above by a finite constant which is independent of ξ and j.*

We now define $f_\xi : \mathbb{R}^n \to \mathbb{R}^N$ by the formula

$$f_\xi(a,x) = \sum_{j\ge 0}(1+j)^{-2} f_{\xi,j}(a,x). \tag{4.5}$$

Corollary 4.2. *For all ξ as above, f_ξ is in $W^{1,p}(\mathbb{R}^n,\mathbb{R}^N)$, with $||f_\xi||_{1,p}$ bounded above by a finite constant which is independent of ξ.*

To simplify the notation, we henceforth write

$$W^j(E) := \{w \in W^j : Q_w^{V^\perp} \cap E \neq \emptyset\}$$

and $W^*(E) = \bigcup_{j\ge 0} W^j(E)$.

Proof of Lemma 4.1. It is easy to see that the functions $f_{\xi,j}$ are uniformly bounded, so it suffices to check the integrability condition on the partial derivatives

$$\partial_i f_{\xi,j}(a,x) = \sum_{w \in W^j(E)} \mathcal{H}^m((Q_w^V)')^{1/\alpha} \partial_i \psi_w(a,x) \xi_w,$$

where $\partial_i = \partial_{a_i}$ if $i = 1, \ldots, n-m$ and $\partial_i = \partial_{x_{i-n+m}}$ if $i = n-m+1, \ldots, n$. Since the cubes $\{\frac{5}{4}Q_w\}$ have bounded overlap, we obtain

$$\begin{aligned} \int_Q |\partial_i f_{\xi,j}|^p &\le C \int_Q \sum_{w \in W^j(E)} \mathcal{H}^m((Q_w^V)')^{p/\alpha} |\nabla \psi_w(a,x)|^p \, da \, dx \\ &\le C \sum_{w \in W^j(E)} \mathcal{H}^m((Q_w^V)') \cdot 2^{-jm(p/\alpha - 1)} 2^{jp} 2^{-jn}. \end{aligned} \tag{4.6}$$

Here we wrote

$$\mathcal{H}^m((Q_w^V)')^{p/\alpha} = C(m,p,\alpha) 2^{-jm(p/\alpha-1)} \mathcal{H}^m((Q_w^V)'). \tag{4.7}$$

Note that the expression $\frac{p}{\alpha} - 1$ may be negative, however, the identity (4.7) still holds since $(Q_w^V)'$ is a cube in $\mathbb{R}^m$ of side length exactly equal to $100 \cdot 2^{-j}$.

To estimate the term $\sum_{w \in W^j(E)} \mathcal{H}^m((Q_w^V)')$, we rewrite it as

$$\sum_{\substack{t \in T^j \\ R_t \cap E \neq \emptyset}} \left(\sum_{\substack{w \in W^j \\ Q_w^{V^\perp} = R_t}} \mathcal{H}^m((Q_w^V)') \right), \tag{4.8}$$

where $T = \{1, \ldots, 2^{n-m}\}$, $T^* = \bigcup_{j \ge 0} T^j$, and $\{R_t\}_{t \in T^*}$ denotes the usual dyadic decomposition in $Q^{V^\perp}$. The term in parentheses in (4.8) is bounded by a constant independent of t, so we obtain

$$\sum_{w \in W^j(E)} \mathcal{H}^m((Q_w^V)') \le C \# \{ t \in T^j : R_t \cap E \neq \emptyset \} \le C 2^{j\beta}$$

by (4.1). Returning to (4.6), we find

$$\int_Q |\partial_i f_{\xi,j}|^p \le C 2^{j(\beta + p - n + m - \frac{mp}{\alpha})} = C$$

for all i, independent of ξ and j. This completes the proof of the lemma. □

In the second part of the proof, we show that a generic choice of ξ yields a map f_ξ with the desired property. To this end, we now view $\xi = (\xi_w)$ as a sequence of independent random variables, identically distributed according to the uniform probability distribution on B.

For $\alpha > 0$, denote by

$$I_\alpha(\mu) := \iint |x-y|^{-\alpha} \, d\mu(x) \, d\mu(y)$$

the *α-energy* of a finite Borel measure μ in $\mathbb{R}^N$.

For each $a \in E$, consider the measure $(f_\xi)_\#(\mathcal{H}^m \llcorner V_a)$, i.e., the pushforward of the Hausdorff m-measure on the affine subspace V_a via the map f_ξ. We claim that the expectation

$$\mathbb{E}_\xi \left(\int_E I_{\alpha'}((f_\xi)_\#(\mathcal{H}^m \llcorner V_a)) \, d\mathcal{H}^\beta(a) \right) \tag{4.9}$$

is finite for each $\alpha' < \alpha$. If we can prove this claim, then almost surely with respect to ξ, we have

$$\int_E I_{\alpha'}((f_\xi)_\#(\mathcal{H}^m \llcorner V_a))\, d\mathcal{H}^\beta(a) < \infty$$

and hence $I_{\alpha'}((f_\xi)_\#(\mathcal{H}^m \llcorner V_a))$ is finite for $\mathcal{H}^\beta$-a.e. $a \in E$. By considering a sequence $\alpha'_n \nearrow \alpha$ and using the countable stability of the Hausdorff measures and Frostman's lemma [43, Theorem 8.9(1)], we reach our desired conclusion (4.3).

It remains to verify the finiteness of the value in (4.9). By Tonelli's theorem, (4.9) equals

$$\int_{[0,1]^m} \int_{[0,1]^m} \int_E \mathbb{E}_\xi(|f_\xi(a,x) - f_\xi(a,y)|^{-\alpha'})\, d\mathcal{H}^\beta(a)\, d\mathcal{H}^m(x)\, d\mathcal{H}^m(y).$$

To estimate the integrand, we write

$$f_\xi(a,x) - f_\xi(a,y) = \sum_{w \in W^*(E)} c_w(a,x,y)\xi_w$$

where the coefficients are given by

$$c_w(a,x,y) := (1+j)^{-2}\mathcal{H}^m((Q_w^V)')^{1/\alpha}\Big(\psi_w(a,x) - \psi_w(a,y)\Big), \qquad w \in W^j. \tag{4.10}$$

For the sake of clarity we emphasize that the coefficient $(1+j)^{-2}\mathcal{H}^m((Q_w^V)')^{1/\alpha}$ in (4.10) is a constant depending only on m, α and j.

For a bounded sequence $\mathbf{c} = (c_i)$, we consider the supremum

$$||\mathbf{c}||_\infty := \sup_i |c_i| \tag{4.11}$$

and the *second largest value*

$$\rho(\mathbf{c}) := \begin{cases} ||\mathbf{c}||_\infty, & \text{if the supremum in (4.11) is not attained,} \\ \sup_{i \neq i_0} |c_i|, & \text{if } \sup_i |c_i| \text{ is attained at } i = i_0. \end{cases} \tag{4.12}$$

For $a \in V^\perp$ and $x, y \in V$, we let $\mathbf{c}(a,x,y) = (c_w(a,x,y))$ be the sequence of coefficients defined in (4.10). Clearly, $\mathbf{c}(a,x,y) \in \ell^\infty$. We denote by $||\mathbf{c}(a,x,y)||_\infty$ the supremum and by $\rho(\mathbf{c}(a,x,y))$ the corresponding second largest value.

We require an elementary lemma from probability theory. As we were unable to locate a precise reference in the literature, we opt to give a short proof at the end of this section. See subsection 4.1. The proof uses an upper bound for the Fourier transform of the density function for the random variables X_i. In order to obtain a sufficiently good bound on large scales in the frequency domain, we must use at least two of the random variables. This results in the appearance of the second largest value $\rho(\mathbf{c})$ in (4.13).

Lemma 4.3. *Let $\{X_i\}$ be a countable sequence of independent random variables, identically distributed according to the uniform distribution on the unit ball B in $\mathbb{R}^N$. Let $\mathbf{c} = (c_i) \in \ell^\infty$. Finally, let $0 < \alpha < N+1$. Then there exists a constant C which depends only on N and α so that*

$$\mathbb{E}\left(\left|\sum_i c_i X_i\right|^{-\alpha}\right) \le C\rho(\mathbf{c})^{-\alpha}. \tag{4.13}$$

Using this lemma, we finish the proof of Theorem 1.4. Applying Lemma 4.3 to the sequences ξ and $\mathbf{c}(a,x,y)$, and noting that $\mathcal{H}^m(Q^V)$ and $\mathcal{H}^\beta(E)$ are finite, we observe by another application of Tonelli's theorem that it suffices to prove the estimate

$$\int_{Q^V} \rho(\mathbf{c}(a,x,y))^{-\alpha'}\, d\mathcal{H}^m(y) \le C < \infty,$$

where C denotes a constant which is independent of $a \in E$ and $x \in Q^V$.

Fix $a \in E$ and $x \in Q^V$. For $y \in Q^V$, let $j(y)$ be the largest integer $j \ge 0$ with the property that x and y lie in identical or adjacent dyadic cubes Q_w^V of level j. It follows from the construction that there exists a word w_0 in $W^{j(y)+1}$ so that $x \in Q_{w_0}^V$ and $y \in (Q_{w_0}^V)'$, but $y \notin \frac{5}{4}Q_{w_0}^V$. Furthermore, we may choose the word w_0 so that $Q_{w_0}^{V^\perp} \cap E \neq \emptyset$, i.e., $w_0 \in W^*(E)$. Observe that

$$|c_{w_0}(a,x,y)| = (2+j(y))^{-2}\mathcal{H}^m((Q_{w_0}^V)')^{1/\alpha}.$$

Moreover, w_0 is a subword of another word $w_1 \in W^*(E)$ with $|w_1| = |w_0|+1$ and

$$(3+j(y))^{-2}\mathcal{H}^m((Q_{w_1}^V)')^{1/\alpha} = |c_{w_1}(a,x,y)| \le |c_{w_0}(a,x,y)|.$$

From the previous discussion, we deduce that

$$\rho(\mathbf{c}(a,x,y)) \ge |c_{w_1}(a,x,y)| = (3+j(y))^{-2}\mathcal{H}^m((Q_{w_1}^V)')^{1/\alpha}.$$

Let F_j denote the set of points $y \in Q^V$ for which $j(y) = j$. Note that $F_j \subset (Q_{w_0}^V)'$. We have

$$\int_{Q^V} \rho(\mathbf{c}(a,x,y))^{-\alpha'}\, d\mathcal{H}^m(y) = \sum_{j\ge 0}\int_{F_j} \rho(\mathbf{c}(a,x,y))^{-\alpha'}\, d\mathcal{H}^m(y) \le C\sum_{j\ge 0}(3+j)^{2\alpha'}2^{-jm(1-\alpha'/\alpha)}$$

by (4.4). Since $\alpha' < \alpha$, the series converges. The proof of Theorem 1.4 is complete. □

4.1. **A probabilistic lemma.** Here we give a short proof of Lemma 4.3. Such a result is implicitly used by Kaufman [32, p. 429]. As we have been unable to find a proof for this result in the literature, we have chosen to provide the details here.

The Fourier transform of an integrable radial function $g(x) = \varphi(|x|)$ on $\mathbb{R}^n$ is given by a Hankel transform (see, e.g., [43, (12.8)]):

$$\widehat{g}(x) = c(n)|x|^{1-n/2}\int_0^\infty \varphi(s)J_{n/2-1}(s|x|)s^{n/2}\, ds. \tag{4.14}$$

Here $J_\nu(t)$ denotes the Bessel function of the first kind of order ν.

Let X be a random variable which is uniformly distributed on the unit ball $B \subset \mathbb{R}^N$. The density function for X is $\mathbf{f} = \mathbf{f}_X = |B|^{-1}\chi_B$. Using (4.14) we compute its Fourier transform:

$$\widehat{\mathbf{f}}(x) = c(N)|x|^{1-N/2}\int_0^1 s^{N/2}J_{N/2-1}(s|x|)\, ds. \tag{4.15}$$

Using the differentiation relation $\frac{d}{ds}(s^\nu J_\nu(s)) = s^\nu J_{\nu-1}(s)$ we simplify (4.15) to

$$\widehat{\mathbf{f}}(x) = c(N)|x|^{-N/2}J_{N/2}(|x|). \tag{4.16}$$

We are now prepared to prove Lemma 4.3. By an elementary limiting argument, it suffices to assume that the sequences (X_i) and $\mathbf{c}$ have finitely many terms. Without loss of generality, assume that $|c_1| = ||\mathbf{c}||_\infty$ and $|c_2| = \rho(\mathbf{c})$. The density function $\mathbf{f}_{\sum_i c_iX_i}$ for the random variable $\sum_i c_iX_i$ is given by the iterated convolution of the dilated functions $c_i^{-n}\mathbf{f}_{X_i}(c_i^{-1}\cdot)$.

Expressing the expectation in (4.13) as the L^2 inner product of $|x|^{-\alpha}$ and the density $\mathbf{f}_{\sum_i c_i X_i}$ and using Plancherel's theorem, we obtain that

$$\mathbb{E}\left(\left|\sum_i c_i X_i\right|^{-\alpha}\right) = c(N,\alpha)\int_{\mathbb{R}^N} |x|^{\alpha-N}\prod_i \widehat{\mathbf{f}}_{X_i}(c_i x)\,dx. \tag{4.17}$$

Here we used the well known formula $(|\cdot|^{-\alpha})\widehat{\ } = c(N,\alpha)|\cdot|^{\alpha-N}$ for the Fourier transform of the Riesz kernel; see e.g. [54, III.3.3] for a proof.

We fix $R = \rho(\mathbf{c})^{-1}$ and decompose the integral in (4.17) into two terms: the integral over the ball $B = B(0,R)$ and the integral over its complement B^c. Since

$$||\widehat{\mathbf{f}}||_\infty \le ||\mathbf{f}||_1 = 1,$$

we conclude that the desired expectation is bounded above by a constant multiple of

$$\int_B |x|^{\alpha-N}\,|\widehat{\mathbf{f}}_{X_1}(c_1x)|\,dx + \int_{B^c}|x|^{\alpha-N}\,|\widehat{\mathbf{f}}_{X_1}(c_1x)|\,|\widehat{\mathbf{f}}_{X_2}(c_2x)|\,dx.$$

We insert the formula from (4.16), write the resulting integrals in polar coordinates, and use the elementary estimates $|J_\nu(t)| \le Ct^\nu$ and $|J_\nu(t)| \le Ct^{-1/2}$ to obtain

$$\mathbb{E}\left(\left|\sum_i c_i X_i\right|^{-\alpha}\right) \le C\left(\int_0^R r^{\alpha-1}\,dr + (|c_1|\,|c_2|)^{-(N+1)/2}\int_R^\infty r^{\alpha-N-2}\,dr\right).$$

Since $R^{-1} = \rho(\mathbf{c}) = |c_2| \le |c_1| = ||\mathbf{c}||_\infty$, we obtain the desired conclusion (4.13). This completes the proof of the lemma.

5. Examples

5.1. **Quasiconformal maps which increase the Minkowski dimension of many lines.** Theorem 1.3 applies in particular to quasiconformal maps. It is natural to ask how sharp the theorem is in that category.

In this section, we prove Theorem 1.6. We construct a quasiconformal mapping for which the exceptional set associated to upper Minkowski dimension distortion has close-to-optimal dimension. We do not have a corresponding example asociated to Hausdorff dimension distortion.

Let us recall the definition of the Minkowski dimension.

Definition 5.1. Let S be a bounded subset of $\mathbb{R}^n$. The *upper Minkowski dimension* of S is

$$\overline{\dim}_M S := \limsup_{r\to 0}\frac{\log \mathbf{N}(S,r)}{\log 1/r}.$$

The *lower Minkowski dimension* of S, denoted $\underline{\dim}_M S$, is defined similarly, with $\liminf$ replacing $\limsup$. In case the limit exists, the corresponding value is called the *Minkowski dimension*

Theorem 1.6 corresponds to the case $m = 1$ in the following more general theorem. As we will see in the proof, we may choose

$$\delta_{n,1} = 1 - \frac{1}{n}$$

and so the full range $1 < \alpha < n$ is allowed. Note that Minkowski dimension is only defined for bounded sets, which explains the reason why we only consider the compact set $f(\{a\}\times[0,1]^m)$ in the conclusion of the theorem.

Theorem 5.2. *Let $n \geq 2$ and $1 \leq m \leq n-1$ be integers. Then there exists a positive constant $\delta_{n,m}$ so that for each α satisfying*

$$m < \alpha < \frac{m}{1-\delta_{n,m}}$$

and for each $\epsilon > 0$, there exists a compact set $E \subset \mathbb{R}^{n-m}$ of Hausdorff dimension at least

$$m\left(\frac{n}{\alpha}-1\right)-\epsilon$$

and a quasiconformal map $f : \mathbb{R}^n \to \mathbb{R}^n$ so that $\overline{\dim}_M f(\{a\} \times [0,1]^m) \geq \alpha$ for all $a \in E$.

To simplify the exposition, we will only prove the case $n = 2$, $m = 1$ in what follows. In Remarks 5.5 and 5.6 we comment on the changes required to cover the general situation.

Recall that

$$\dim E \leq \underline{\dim}_M E \leq \overline{\dim}_M E$$

for all bounded sets E, with equality throughout if E is sufficiently nice (for instance, if E is Ahlfors regular). While Hausdorff dimension is countably stable (the dimension of any countable union is the supremum of the dimensions of the pieces), Minkowski dimension is only finitely stable (the dimension of any finite union is the maximum of the dimensions of the pieces).

We begin with a lemma of Heinonen and Rohde. The quasiconformal map g_T in the following lemma maps an interior segment of the unit square in the xy-plane onto a nonrectifiable arc of von Koch snowflake type. The image of this segment under g_T has an increased (Minkowski or Hausdorff) dimension. Nearby segments are mapped onto smooth arcs, hence we realize no increase in their Hausdorff dimension. However, such nearby segments are stretched significantly by the mapping (due to local quasisymmetry), which increases their contribution to the covering number $\mathbf{N}(g_T(\{a\} \times \mathbb{R}), \epsilon)$. To complete the proof of Theorem 5.2, we sum these contributions over all squares in a Whitney-style decomposition of the x-axis.

In the following lemma, we use the notation $A \simeq B$ to indicate that two quantities A and B are comparable up to an absolute multiplicative constant.

For an arbitrary square $T \subset \mathbb{R}^2$ with sides parallel to the coordinate axes, we use the following notation: $\varphi_T : Q \to T$ denotes the unique orientation-preserving homothety from the unit square $Q = [0,1]^2$ onto T which maps vertical sides to vertical sides, s_T denotes the side length of T, and $M_T = \varphi_T(\{\frac{1}{2}\} \times [\frac{1}{4}, \frac{3}{4}])$ denotes a vertical segment in the middle of T of length $\frac{1}{2}s_T$. For $a \in \mathbb{R}$, we denote by γ_a the set $\{a\} \times \mathbb{R}$.

Lemma 5.3 (Heinonen–Rohde)**.** *Fix a real number D, $1 < D < 2$. Let T be any square in the plane. Then there exists a homeomorphism $g_T : T \to T$ with the following properties:*

(i) *g_T is quasiconformal on the interior of T,*
(ii) *$g_T|_{\partial T}$ is the identity,*
(iii) *if $p, q \in T$ are within distance $\frac{1}{8}s_T$ from M_T and $|p-q| \geq \max\{\mathrm{dist}(p, M_T), \mathrm{dist}(q, M_T)\}$, then*

$$|g_T(p) - g_T(q)| \simeq |p-q|^{1/D} s_T^{1-1/D}. \tag{5.1}$$

(iv) *if $a \in \mathbb{R}$ satisfies $d := \mathrm{dist}(\gamma_a, M_T) \leq \frac{1}{8}s_T$, then*

$$\mathbf{N}(g_T(\gamma_a \cap T), c d^{1/D} s_T^{1-1/D}) \geq \frac{s_T}{d} \tag{5.2}$$

for some positive constant c.

We remark that the quantities $\mathbf{N}(g_T(\gamma_a \cap T), cd^{1/D}s_T^{1-1/D})$ and s_T/d from (5.2) are in fact comparable, in view of the local quasisymmetry of g_T. However, we only need the stated lower bound in what follows.

Proof. Parts (i), (ii) and (iii) of this lemma coincide with Lemma 3.2 on page 401 in [30]; see also the discussion on page 402. Briefly, the map g_T is constructed as follows. Choose a quasiconformal map h of $\mathbb{R}^2$ which sends M_T onto a D-dimensional snowflake curve of von Koch type contained in the interior of T. Such a map can be chosen so that the estimate in (5.1) holds for all $p, q \in M_T$. For a construction of such a map h, see for instance [55, p. 151]. Next, by a standard technique from quasiconformal function theory, we may choose a map $g_T : \mathbb{R}^2 \to \mathbb{R}^2$ which is equal to the identity on the complement of T, and which agrees with h on a neighborhood of M_T. This is the desired map.

To complete the proof, we need only verify part (iv). Let a be a point satisfying the stated conditions, choose an integer N satisfying

$$\frac{s_T}{N} \ge d > \frac{s_T}{N+1},$$

and choose $N+1$ points $p_0, \dots, p_N$ on $\gamma_a \cap T$ so that $|p_i - p_{i-1}| = s_T/N$ for all $i = 1, \dots, N$. If $i \ne j$, then $|p_i - p_j| \ge \frac{s_T}{N} \ge d$ and hence (by part (iii)),

$$|g_T(p_i) - g_T(p_j)| \ge \frac{1}{C}|p_i - p_j|^{1/D} s_T^{1-1/D} \ge \frac{1}{C}\left(\frac{s_T}{N}\right)^{1/D} s_T^{1-1/D}$$

for some constant C. Hence we require at least $N+1$ balls of radius $c\left(\frac{s_T}{N}\right)^{1/D} s_T^{1-1/D}$ to cover $g_T(\gamma_a \cap T)$, where $c = \frac{1}{3C}$. A fortiori, we require at least $N+1$ balls of radius $cd^{1/D}s_T^{1-1/D}$ to cover $g_T(\gamma_a \cap T)$. We conclude the proof by observing that $N + 1 > \frac{s_T}{d}$. $\square$

In the proof of Theorem 5.2, and also in the example in the following subsection, we will use the following calculation of the Hausdorff dimension of certain Cantor-type sets in the real line. See, for instance, Example 4.6 in [14].

Proposition 5.4. *Let $W_1, W_2, \dots$ be finite sets with $M_j := \#W_j \ge 2$ for each j, let $W^* = \bigcup_{j\ge 0}(W_1 \times \cdots \times W_j)$, and let $\{I_w\}_{w\in W^*}$ be a family of closed intervals satisfying the following conditions:*

(i) *$I_w \subset I_v$ whenever v is a subword of w,*
(ii) *$\max\{|I_w| : w \in W_1 \times \cdots \times W_j\} \to 0$ as $j \to \infty$, and*
(iii) *there exists a decreasing sequence (ϵ_j) of positive real numbers so that $\operatorname{dist}(I_v, I_w) \ge \epsilon_j$ whenever $v, w \in W_1 \times \cdots \times W_j$ are distinct.*

Let

$$E = \bigcap_{j\ge 1} \bigcup_{w\in W_1\times\cdots\times W_j} I_w.$$

Then

$$\dim E \ge \liminf_{j\to\infty} \frac{\sum_{i=1}^{j} \log M_i}{-\log(\epsilon_{j+1} M_{j+1})}. \tag{5.3}$$

Proof of Theorem 5.2. Let $\alpha \in (1, 2)$ and $\epsilon > 0$ be fixed. Without loss of generality, we may assume that $\epsilon < \frac{2}{\alpha} - 1$. Choose a rational number $b > 1$ satisfying

$$\frac{\alpha}{2-\alpha} < b < \frac{\alpha}{2-(1+\epsilon)\alpha}$$

and define

$$D := \alpha\left(\frac{b-1}{b-\alpha}\right).$$

Observe that

$$\frac{1}{b} > \left(\frac{2}{\alpha}-1\right)-\epsilon$$

and also that $\alpha < D < 2$.

Let $(n_j)_{j\geq 1}$ be any increasing sequence of positive integers with the following properties:

(i) $n_{j+1} - bn_j$ is an integer for each $j \geq 1$, and

(ii) the limit of $\frac{\sum_{i=1}^{j} n_i}{n_{j+1}}$ as $j \to \infty$ is equal to zero.

For instance, if $b = \frac{P}{Q}$ in lowest terms, we may choose $n_j = QP^j 2^{2^j}$.

We associate to the sequence (n_j) a *sub-Whitney decomposition* $\mathcal{W}$ of the upper half plane, or more precisely, of the domain $\Omega = (0,1)\times(-2,2)$ relative to the x-axis. This means that we begin with the standard Whitney decomposition of Ω relative to the x-axis, and subdivide all squares in this decomposition with size between 2^{-n_j} and $2^{-n_{j+1}}$ into subsquares of size $2^{-n_{j+1}}$. Note that the resulting squares T have the property that $\operatorname{diam} T$ is bounded above by a constant multiple of the distance d from T to the x-axis, however, $\operatorname{diam} T$ may be significantly smaller than d.

Define a map $f:\Omega\to\Omega$ by setting $f|_T = g_T$ for each $T \in \mathcal{W}$. Since g_T is the identity on the boundary of T, this map is well-defined and continuous. Extend it to a map f of $\mathbb{R}^2$ to itself by the identity. Then f is quasiconformal.

We now define a Cantor set on the x-axis by an iterative procedure. For each $j \geq 1$ and each square $T \in \mathcal{W}$ with $s_T = 2^{-n_j}$ and $T \cap \{(x,y) : y = 2^{-n_j}\} \neq \emptyset$, the projection P of the set $T \cap \{(x,y) : y = 2^{-n_j}\}$ onto the x-axis consists of $2^{n_{j+1}-n_j}$ essentially disjoint closed intervals, each of length $2^{-n_{j+1}}$. Note that the total length of all of these intervals is equal to 2^{-n_j}, which is the side length of P. Select the subcollection of these intervals, centered around the middle of P, of total length 2^{-bn_j}. Observe that this subcollection consists of $2^{n_{j+1}-bn_j}$ intervals each of length $2^{-n_{j+1}}$. In the inductive step, we consider only squares in some vertical column corresponding to one of these intervals and repeat the construction.

For each j, let

$$W_j = \{1,\ldots,2^{n_j-bn_{j-1}}\}$$

and denote by I_w, $w \in W_1\times\cdots\times W_j$, the intervals at the jth level in the construction in the previous paragraph. The Cantor set in question is

$$E = \bigcap_{j\geq 1}\ \bigcup_{w\in W_1\times\cdots\times W_j} I_w.$$

Using Proposition 5.4 with $M_j = 2^{n_j-bn_{j-1}}$ and $\epsilon_j \simeq 2^{-n_j}$ we find

$$\dim E \geq \lim_{j\to\infty}\frac{n_j-(b-1)\sum_{i=1}^{j-1} n_i}{bn_j} = \frac{1}{b} > (\frac{2}{\alpha}-1)-\epsilon.$$

Now suppose that $a \in E$ and fix an integer $j \geq 1$. Then a is contained in a unique interval I_w with $w \in W_1\times\cdots\times W_{j+1}$ which in turn is contained in a unique interval $I_{\hat{w}}$ with $\hat{w} \in W_1\times\cdots\times W_j$. Let T be any square from $\mathcal{W}$ lying above the interval $I_{\hat{w}}$. Then the distance from γ_a to M_T is bounded above by $\frac{1}{2}2^{-bn_j}$ which is smaller than $\frac{1}{8}s_T = \frac{1}{8}2^{-n_j}$ provided that j is chosen sufficiently large. Note that there are

$$2^{n_j-n_{j-1}}-1$$

such squares T. We define a sequence of scales (ϵ_j) depending on the point a; the desired estimate for the upper Minkowski dimension of $f(\gamma_a)$ will come from analyzing the covering number on this sequence of scales by an application of Lemma 5.3.

Let

$$\epsilon_j = c \operatorname{dist}(\gamma_a, M_T)^{1/D} s_T^{1-1/D} = c|a - m_j|^{1/D} 2^{-n_j(1-1/D)},$$

where m_j denotes the x-coordinate of the midline M_T. By Lemma 5.3(iv), we have

$$\mathbf{N}(g_T(\gamma_a \cap T), \epsilon_j) \geq \frac{s_T}{\operatorname{dist}(\gamma_a, M_T)} = \frac{2^{-n_j}}{|a - m_j|}.$$

Summing this over all of the relevant squares gives

$$\mathbf{N}(f(\gamma_a \cap Q), \epsilon_j) \geq (2^{n_j - n_{j-1}} - 1)\frac{2^{-n_j}}{|a - m_j|} \geq \frac{2^{-n_{j-1}}}{2|a - m_j|}.$$

We conclude that

$$\overline{\dim}_M f(\gamma_a \cap Q) \geq \limsup_{j\to\infty} \frac{-\log_2|a - m_j| - n_{j-1} - 1}{-\frac{1}{D}\log_2|a - m_j| + (1 - \frac{1}{D})n_j + C}. \tag{5.4}$$

Observing that $|a - m_j| \leq 2^{-bn_j - 1}$ and that the expression inside the limit on the right hand side of (5.4) is nondecreasing in the variable $-\log_2|a - m_j|$, we conclude that

$$\overline{\dim}_M f(\gamma_a \cap Q) \geq D \cdot \limsup_{j\to\infty} \frac{bn_j - n_{j-1}}{(b + D - 1)n_j + DC + D} = \frac{bD}{b + D - 1} = \alpha$$

by the choice of D. This completes the proof. □

Remark 5.5. For general n (still assuming $m = 1$) the proof is similar. We require the existence of D-dimensional von Koch snowflake curves in $\mathbb{R}^n$ for each $1 < D < n$. More precisely, we require a curve $\Gamma \subset \mathbb{R}^n$ such that $\Gamma = g(\mathbb{R})$, where $g : \mathbb{R}^n \to \mathbb{R}^n$ is a quasiconformal map so that $|g(x) - g(y)| \simeq |x - y|^{1/D}$ for all $x, y \in \mathbb{R}$ with $|x - y| \leq 1$. For a construction of such curves in $\mathbb{R}^3$, see Bonk and Heinonen [7]. A similar construction has been given by Ghamsari and Herron [20]. Using this construction, the proof of Theorem 5.2 for $m = 1$ and general n proceeds in a similar fashion.

Remark 5.6. The case $m \geq 2$ in Theorem 5.2 is more challenging. We require the existence of D-dimensional quasiconformal submanifolds of $\mathbb{R}^n$ of von Koch type. More precisely, we require a topological m-manifold $\Sigma \subset \mathbb{R}^n$ so that $\Sigma = g(\mathbb{R}^m)$, where $g : \mathbb{R}^n \to \mathbb{R}^n$ is a quasiconformal map so that

$$|g(x) - g(y)| \simeq |x - y|^{m/D}, \qquad \forall x, y \in \mathbb{R}^m, |x - y| \leq 1. \tag{5.5}$$

Such *snowflaked quasiconformal submanifolds* were constructed by David and Toro [9] for a small range of values $D \in [m, m + \epsilon_{n,m})$ for some $\epsilon_{n,m} > 0$. Using such submanifolds, one can establish an analog for Lemma 5.3 and thereby establish Theorem 5.2 for general m satisfying (1.1). The value of $\delta_{n,m}$ in Theorem 5.2 depends on the size of the interval $[m, m + \epsilon_{n,m})$ of dimensions of such snowflaked quasiconformal submanifolds. We leave to the interested reader the computation of a precise relationship between $\delta_{n,m}$ and $\epsilon_{n,m}$.

Snowflaked quasiconformal submanifolds were previously used in [7] and [34] to study the effect of smoothness on branching phenomena for quasiregular mappings.

Remark 5.7. Bishop [6] previously constructed a quasiconformal map g of $\mathbb{R}^3$ so that $g(W)$ contains no rectifiable curves, where $W \in G(3,2)$ is a fixed plane. In particular, choosing $V \in G(3,1)$ with $V \subset W$ and expressing $\mathbb{R}^3$ as an orthogonal sum

$$V \oplus (V^\perp \cap W) \oplus W^\perp \tag{5.6}$$

exhibits a one-dimensional family of parallel lines V_a, $a \in V^\perp \cap W$, all of whose images under g have no nontrivial rectifiable subcurves. The construction in [6], however, did not guarantee any dimension increase for the sets $g(V_a)$.

Using the aforementioned result of David and Toro and expressing $\mathbb{R}^n$ as an orthogonal sum of the form (5.6) for some $V \in G(n,k)$, $k < m$, $V \subset W$, we can exhibit an $(m-k)$-dimensional family of parallel lines V_a, $a \in V^\perp \cap W$, all of whose images under g have Hausdorff dimension at least a fixed value $D > m$.

Remark 5.8. Kovalev and Onninen [37, Corollary 1.6] have recently shown that, to every countable family of parallel lines $\{V_a\}$ in the plane, there corresponds a *reduced* quasiconformal map f of $\mathbb{R}^2$ with the property that each curve $f(V_a)$ has no nontrivial rectifiable subcurve. (See Definition 1.4 in [37] for the definition of reduced planar quasiconformal map.) It is not clear how to extend their construction to higher dimensions. Reduced quasiconformality implies that the image curves $f(V_a)$ necessarily have Hausdorff dimension equal to one [37, Theorem 1.7]. In Theorem 1.6, the curves $f(V_a)$ are nonrectifiable but locally rectifiable and also have Hausdorff dimension equal to one. However, the size of the family of lines allowed in Theorem 1.6 is substantially larger than that in [37].

5.2. **Space-filling mappings in subcritical Sobolev classes.** We continue with a discussion of the nonsupercritical case

$$p \le n.$$

We are interested in understanding the frequency of Hausdorff dimension distortion by a map f in $W^{1,p}(\Omega, Y)$. The first point to emphasize is that the problem is not precisely defined in this setting. Indeed, Sobolev maps in the critical class $W^{1,n}$ need not have continuous representatives. Varying the representative of f can affect the dimension distortion properties.

It is a standard fact of Sobolev space theory ([60, Corollary 3.3.4]) that maps in $W^{1,p}$ admit p-quasicontinuous representatives, i.e., representatives which are continuously defined on the complement of a set of zero Bessel capacity $B_{1,p}$. This observation continues to hold true for metric space-valued maps. We omit the definition of the Bessel capacity $B_{1,p}$ but recall that the null sets for $B_{1,p}$ correspond roughly to the sets of Hausdorff dimension $n-p$. More precisely, $B_{1,p}(E) = 0$ whenever $\mathcal{H}^{n-p}(E) < \infty$, and $B_{1,p}(E) = 0$ implies that $\mathcal{H}^{n-p+\epsilon}(E) = 0$ for any $\epsilon > 0$. See, for instance [60, Theorem 2.6.16]. It is natural to restrict our attention to such p-quasicontinuous representatives. For such a representative f we have no information whatsoever about the behavior of f on the exceptional set of Hausdorff dimension $n-p$. The following example, which illustrates this remark in the case $p = n$, is a special case of [27, Theorem 1.3].

Example 5.9 (Hajłasz–Tyson)**.** Let $n \ge 2$. There exists a continuous map $g \in W^{1,n}(\mathbb{R}^n, \ell^2)$ which is constant on the complement of $[0,1]^n$ and a set $F \subset [0,1]^n$ of Hausdorff dimension zero so that $\dim g(F) = \infty$. In particular, $\dim g([0,1]^n) = \infty$.

Next, we use Example 5.9 to illustrate what type of dimension distortion behavior can occur for maps in $W^{1,m}$. Note that here, in contrast with the rest of this paper, we require $m \ge 2$, since we appeal to Example 5.9. It is easy to see that Example 5.9 cannot extend to the case $n = 1$. Indeed, every $W^{1,1}$ map from $\mathbb{R}$ is absolutely continuous and the target has dimension at most one.

Example 5.10. Let $n \geq 3$ and $2 \leq m \leq n-1$ be integers. Then there exists a continuous map $f \in W^{1,m}(\mathbb{R}^n, \ell^2)$ which is constant on the complement of $[0,1]^n$ with the property that $\dim f(\{a\} \times [0,1]^m) = \infty$ for all $a \in [0,1]^{n-m}$.

Proof. Let $g : [0,1]^m \to \ell^2$ be a continuous map in the class $W^{1,m}$ which is constant on the boundary of $[0,1]^m$ and for which $\dim g([0,1]^m) = \infty$. Define $f : [0,1]^n \to \ell^2$ by

$$f(a,x) = g(x), \qquad a \in \mathbb{R}^{n-m}, x \in \mathbb{R}^m.$$

Extend f to be constant on the complement of $[0,1]^n$. Then $f \in W^{1,m}(\mathbb{R}^n, \ell^2)$ and f is continuous. Moreover, for each $a \in [0,1]^{n-m}$, the set $f(\{a\} \times [0,1]^m) = g([0,1]^m)$ is infinite-dimensional. $\square$

We next modify the preceding example to illustrate what can happen for maps in $W^{1,p}$, $m < p < n$, with regard to almost sure dimension distortion of parallel subspaces. To accomplish this, we will need to modify the details of the construction of Example 5.9.

Example 5.11. Fix integers $1 \leq m < n$ and let $m < p < n$. Then there exists a map $f \in W^{1,p}(\mathbb{R}^n, \ell^2)$ which is constant on the complement of $[0,1]^n$ and there exist compact sets $F \subset [0,1]^m$ and $E \subset [0,1]^{n-m}$ so that

(1) the Hausdorff dimension of F is strictly less than $\frac{m}{p+1}$,
(2) the Hausdorff dimension of E is in the interval $\left(n-p-\frac{m}{p+1}, n-p\right]$,
(3) $\dim E \times F = \dim E + \dim F = n-p$, and
(4) $\dim f(\{a\} \times F) = \infty$ for all $a \in E$.

The proof will show that when p is an integer, we may choose $\dim F = 0$ and $\dim E = n-p$.

We begin with some remarks.

The construction in Example 5.9 uses the fact that the n-capacity of a point in $\mathbb{R}^n$ is equal to zero. This allows us to build a $W^{1,n}$ map from a domain in $\mathbb{R}^n$ whose image is large with very small n-energy. In fact, the map is constructed first on the zero dimensional Cantor set F and then is extended to all of $[0,1]^n$ while preserving the finiteness of the n-energy.

The corresponding construction in Example 5.11 will use the p-capacity. The details are more technical, however, since we must work explicitly with the precise value of this capacity and relate it to the cardinality of various prefractals associated to the Cantor set F.

Let us recall the definition of capacity.

Definition 5.12. Let $E \subset F$ be compact sets in $\mathbb{R}^n$. Let $p \geq 1$. The *p-capacity* of the pair (E,F) is the value

$$\mathrm{Cap}_p(E,F) = \inf \int_{\mathbb{R}^n} |\nabla \varphi|^p,$$

where the infimum is taken over all functions $\varphi \in C_c^\infty(\mathbb{R}^n)$ so that $\varphi|_E = 1$ and $\varphi|_{\mathbb{R}^n \setminus F} = 0$.

We require knowledge of the behavior of the p-capacity of a ring domain. The following lemma is standard. Denote by $Q^n(r) = \{x \in \mathbb{R}^n : |x_i| \leq r \ \forall i = 1, \dots, n\}$ the cube of side length $2r$ centered at the origin.

Lemma 5.13. *Let $0 < r < R < \infty$ and $1 < p < \infty$. Then*

$$\mathrm{Cap}_p(Q^n(r), Q^n(R)) = \begin{cases} c(n,p) \left| R^{\frac{p-n}{p-1}} - r^{\frac{p-n}{p-1}} \right|^{1-p}, & \text{if } p \neq n, \\ c(n)(\log R/r)^{1-n}, & \text{if } p = n. \end{cases}$$

In particular, if $1 < p < n$ and $2r < R$, then

$$C^{-1} r^{n-p} \le \mathrm{Cap}_p(Q^n(r), Q^n(R)) \le C r^{n-p} \tag{5.7}$$

for some constant $C = C(n,p)$.

Let $\varphi_{r,R;n,p} \in C_0^\infty(\mathbb{R}^n)$ be quasiextremal for the p-capacity of the ring domain $(Q^n(r), Q^n(R))$, i.e.,

$$\varphi_{r,R;n,p}|_{Q^n(r)} = 1, \tag{5.8}$$

$$\varphi_{r,R;n,p}|_{\mathbb{R}^n \setminus Q^n(R)} = 0, \tag{5.9}$$

and

$$\int_{\mathbb{R}^n} |\nabla \varphi_{r,R;n,p}|^p \le C r^{n-p} \tag{5.10}$$

for some constant $C = C(n,p)$.

We now begin the construction of the mapping described in Example 5.11. The target will be the (compact) Hilbert cube

$$Y = \{y = (y_i) \in \ell^2 : 0 \le |y_i| \le \frac{1}{i}\}.$$

In fact, any compact infinite-dimensional subset of ℓ^2 would work for our purposes.

There exists an increasing sequence of positive integers $N_1 < N_2 < N_3 < \cdots$ and an increasing sequence of finite subsets $Y_1 \subset Y_2 \subset Y_3 \subset \cdots \subset Y$ with the following properties:

- Y_j is 2^{-j}-dense in Y, i.e., every point of Y lies within distance 2^{-j} from a point of Y_j, and
- we can assign to each element y of Y_j a *parent* in Y_{j-1} which lies at distance 2^{-j} from y, so that each point in Y_{j-1} has at most 2^{N_j} children.

From the second condition, it follows that the cardinality of Y_j is at most $2^{\tilde N_j}$, where

$$\tilde N_j = N_1 + N_2 + \cdots + N_j.$$

We may assume that each of the integers N_j is a multiple of m.

For each point $y \in Y_j$, denote by γ_y the line segment in ℓ^2 joining y to its parent $\hat y$. The length of γ_y is at most 2^{-j}. We parameterize γ_y at constant speed by the interval $[0, 2^{-j}]$, in such a way that $\gamma_y(0) = y$ and $\gamma_y(2^{-j}) = \hat y$. As a map from $[0, 2^{-j}]$ to ℓ^2, γ_y is 1-Lipschitz.

We now return to the source space. Let k be the smallest integer greater than or equal to $p - m$ and write

$$\mathbb{R}^n = \mathbb{R}^{n-m-k} \times \mathbb{R}^k \times \mathbb{R}^m.$$

We will write points of $\mathbb{R}^n$ according to this splitting in the form $(a_1, a_2, x) = (a, x)$, where $a \in \mathbb{R}^{n-m}$ and $x \in \mathbb{R}^m$.

First, we construct a Cantor set in $\mathbb{R}^{k+m}$. Let $Q = [0,1]^{k+m}$ be the unit cube in $\mathbb{R}^{k+m}$. We partition Q into

$$2^{(\frac{k}{m}+1)N_1}$$

essentially disjoint subcubes of side length $2^{-N_1/m}$. We index these subcubes by a parameter w_1 ranging over

$$W_1 = \{1, \dots, 2^{kN_1/m}\} \times \{1, \dots, 2^{N_1}\}.$$

Next, fix $\gamma < 1$. Inside each of the above subcubes, consider two further subcubes $Q_w \subset Q'_w$ so that

(1) Q'_w has side length $R_1 = \beta_1 = \gamma \cdot 2^{-\frac{N_1}{m}}$,

(2) Q_w has side length $r_1 = \alpha_1 = \gamma \cdot 2^{-\frac{m+k}{m+k-p}\cdot\frac{N_1}{m}}$, and
(3) the distance between any two distinct cubes in $\{Q'_w\}_{w\in W_1}$ is comparable to $2^{-N_1/m}$.

For instance, we may choose Q_w and Q'_w to be concentric with each other and with the original cube with index w.

We now describe the inductive step. Assume that we are given a collection of disjoint cubes $\{Q_w\}$ indexed by the elements w in $W_1 \times \cdots \times W_j$, where

$$W_i = \{1, \dots, 2^{kN_i/m}\} \times \{1, \dots, 2^{N_i}\}.$$

We further assume that each of the cubes Q_w has side length $r_j = \alpha_1 \cdots \alpha_j$ where

$$\alpha_i = \gamma \cdot 2^{-\frac{m+k}{m+k-p}\cdot\frac{N_i}{m}}.$$

Let $R_j = \alpha_1 \cdots \alpha_{j-1} \cdot \beta_j$, where

$$\beta_i = \gamma \cdot 2^{-\frac{N_i}{m}}.$$

We partition each of the cubes Q_w into

$$2^{(\frac{k}{m}+1)N_{j+1}}$$

essentially disjoint subcubes of side length $2^{-N_{j+1}/m}$, which we index by a parameter w_{j+1} ranging over W_{j+1}.

Inside each of these subcubes, consider two further subcubes $Q_{ww_{j+1}} \subset Q'_{ww_{j+1}}$ so that

(1) $Q'_{ww_{j+1}}$ has side length $R_{j+1} = r_j\beta_{j+1}$,
(2) $Q_{ww_{j+1}}$ has side length $r_{j+1} = r_j\alpha_{j+1}$, and
(3) the distance between any two distinct cubes in $\{Q'_w\}_{w\in W_1\times\cdots\times W_{j+1}}$ is comparable to $2^{-N_{j+1}/m} r_j$.

The Cantor set in question is

$$C = \bigcap_{j\geq 1} \bigcup_{w\in W_1\times\cdots\times W_j} Q_w.$$

For each j, map W_j to the set $V_j := \{1, \dots, 2^{N_j}\}$ by projecting to the second factor. This induces a map from $W_1 \times \cdots \times W_j$ to $V_1 \times \cdots \times V_j$.

By the choice of the sets Y_j, we can choose a surjective map from $V_1 \times \cdots \times V_j$ to Y_j for all j so that the following diagram commutes:

$$\begin{array}{ccccc} W_1 \times \cdots \times W_{j+1} & \to & V_1 \times \cdots \times V_{j+1} & \to & Y_{j+1} \\ \downarrow & & \downarrow & & \downarrow \\ W_1 \times \cdots \times W_j & \to & V_1 \times \cdots \times V_j & \to & Y_j \end{array}.$$

Here the left hand and central vertical maps are the natural projections, while the right hand map is the one which assigns to each point $y \in Y_{j+1}$ its parent $\hat{y} \in Y_j$. We denote by y_w the point in Y_j which corresponds to a given $w \in W_1 \times \cdots \times W_j$.

We now define a map $g : \mathbb{R}^{k+m} \to \ell^2$. If $w \in W_1 \times \cdots \times W_j$ and $(a_2, x) \in Q'_w \setminus Q_w$, then

$$g(a_2, x) = \gamma_{y_w}\left(2^{-j}\varphi_{r_j,R_j;m+k,p}((a_2, x) - c_w)\right),$$

where c_w denotes the center of the square Q_w. Observe that $g|_{\partial Q'_w} = \gamma_{y_w}(2^{-j}) = \widehat{y_w}$ and $g|_{\partial Q_w} = \gamma_{y_w}(0) = y_w$ by (5.8) and (5.9), respectively. Thus we may extend g to the sets $Q_w \setminus \bigcup_{w_{j+1}} Q'_{ww_{j+1}}$ for each w, and also to the set $\mathbb{R}^{k+m} \setminus Q$ in a continuous fashion, by setting g to an appropriate constant value in each of those sets. This defines g on the complement of C; we extend g by continuity to all of $\mathbb{R}^{k+m}$. Observe that for each $a_2 \in P_{\mathbb{R}^k}(C)$, the closed set $g(\{a_2\} \times P_{\mathbb{R}^m}(C))$ contains each of the sets Y_j, and hence contains all of Y.

We now define a map $f : \mathbb{R}^n \to \ell^2$ by setting $f(a, x) = f(a_1, a_2, x) = g(a_2, x)$ for all $a_1 \in [0,1]^{n-m-k}$ and extending by a suitable constant value for other values of a_1.

We claim that f is in the Sobolev space $W^{1,p}$. Since f is bounded, it suffices to verify that it has an upper gradient in L^p. For any w and for all (a_1, a_2, x) in the set $[0,1]^{n-m-k} \times (Q'_w \setminus Q_w)$,

$$|\nabla f(a_1, a_2, x)| = |\nabla g(a_2, x)| \le 2^{-j} |\nabla \varphi_{r_j, R_j; m+k, p}((a_2, x) - c_w)|.$$

At other points, ∇f vanishes. Thus we can estimate

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla f|^p &= \sum_w \int_{[0,1]^{n-m-k} \times (Q'_w \setminus Q_w)} |\nabla f|^p \\ &\le \sum_{j=1}^{\infty} 2^{-jp} \sum_{w \in W_1 \times \cdots \times W_j} ||\nabla \varphi_{r_j, R_j; m+k, p}||^p_{L^p(\mathbb{R}^{k+m})} \\ &\le C \sum_{j=1}^{\infty} 2^{-jp} r_j^{m+k-p} \#(W_1 \times \cdots \times W_j)\end{aligned}$$

by (5.10)

$$\le C \sum_{j=1}^{\infty} 2^{-jp} r_j^{m+k-p} 2^{(\frac{k}{m}+1)\tilde{N}_j} = C \sum_{j=1}^{\infty} 2^{-jp} \prod_{i=1}^{j} \left(\alpha_i^{m+k-p} 2^{(\frac{k}{m}+1)N_i} \right).$$

By the choice of α_i, we easily see that

$$\alpha_i^{m+k-p} 2^{(\frac{k}{m}+1)N_i} = \gamma^{m+k-p} \le 1,$$

so the above product is bounded above by one and the sum converges. This shows that f is an element of the Sobolev space $W^{1,p}$.

Let F be the projection of C into the $\mathbb{R}^m$ factor, let E_2 be the projection of C into the $\mathbb{R}^k$ factor, and let $E = [0,1]^{n-m-k} \times E_2$. Using again the estimate in [14, Example 4.6], we find

$$\dim F = \lim_{j \to \infty} \frac{\log 2^{\tilde{N}_j}}{\log(1/r_j)} = m - \frac{pm}{m+k} < \frac{m}{p+1}$$

and

$$\dim E = n - m - k + \lim_{j \to \infty} \frac{\log 2^{\frac{k}{m}\tilde{N}_j}}{\log(1/r_j)} = n - m - \frac{pk}{m+k}.$$

Recalling that k is the smallest integer greater than or equal to $p - m$, we leave the details of the remaining claims to the reader. Note that $f(\{a\} \times F) \supset Y$ whenever $a \in E$, since for such a, the closed set $f(\{a\} \times F)$ contains each of the sets Y_j.

6. Open problems and questions

Problem 6.1. Our main theorem estimates the size of the collection of parallel affine subspaces whose image under a fixed supercritical Sobolev mapping f exhibits a prespecified dimension jump. Do similar results hold for other parameterized families of subspaces?

As a sample of the type of problems which could be posed, we present the following variation on our main theme.

The Grassmanian manifold $G(n,m)$ is a smooth manifold of dimension $m(n-m)$. How many subspaces $V \in G(n,m)$ can have the property that their image under f exhibits a prespecified dimension jump? To be more precise, fix $p > n$ and α satisfying $m < \alpha < \frac{pm}{p-n+m}$. We ask for an estimate from above for the dimension of the set of subspaces $V \in G(n,m)$ for which $\dim f(V) \geq \alpha$. In fact, we seek an estimate of the form

$$\dim\{V \in G(n,m) : \dim f(V) \geq \alpha\} \leq m(n-m) - \delta,$$

where $\delta = \delta(n,m,\alpha,p) > 0$.

The Grassmanian $G(n,1)$ coincides with the real projective space $P^{n-1}_{\mathbb{R}}$, which has dimension $n-1$. Using local triviality of the tautological line bundle over $G(n,1)$, one can recast the above problem into the framework of the product decomposition considered in our main theorem. The eventual conclusion matches that from Theorem 1.3, in the case $m = 1$. We omit the details, reserving discussion of this question for a later paper.

Problem 6.2. We anticipate that (3.17) is not sharp. Indeed, the dimension bounds in (3.16) can be improved in the case when E is a line. Smirnov [52] has shown that

$$\dim f(E) \leq 1 + \left(\frac{K-1}{K+1}\right)^2 \tag{6.1}$$

whenever $E \subset \mathbb{R}^2$ is a line segment and $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a K-quasiconformal map. We expect that (3.17) can be improved in the planar case to an estimate which recovers (6.1) at the borderline, when the exceptional set has zero dimension. Such a result could be obtained, for instance, by introducing the thermodynamic formalism as in [2] in the context of the proof of our main theorem, to estimate the Hausdorff dimension of the exceptional set.

Problem 6.3. Does Theorem 1.6 hold with Minkowski dimension replaced by Hausdorff dimension?

Problem 6.4. In subsection 5.2 we gave examples demonstrating the limitations of any potential generic dimension distortion results for nonsupercritical Sobolev mappings. Give any positive result concerning the frequency of dimension distortion by such mappings. In view of the remarks at the beginning of subsection 5.2, one may wish to restrict attention to the p-quasicontinuous representative of such a mapping.

Problem 6.5. What can be said for other source spaces? The notion of Sobolev space defined on a metric measure space is by now well understood, see for instance [51], [22], [29], [25]. Even in the potentially simplest non-Euclidean setting, when the source is the sub-Riemannian Heisenberg group, it is unclear whether results analogous to those of this paper hold. We make substantial use of several purely Euclidean features, such as the Besicovitch covering theorem and the fact that the projection mappings $P_V : \mathbb{R}^n \to V$ are Lipschitz. In the Heisenberg group, the Besicovitch covering theorem is false and retractions along the fibers of a horizontal foliation are never Lipschitz. See [36] or [49] for details. At present, it appears that these complications preclude the development of a theory similar to that presented in this paper, in more general, non-Riemannian, contexts.

References

[1] Ambrosio, L. Metric space valued functions of bounded variation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 17*, 3 (1990), 439–478.

[2] Astala, K. Area distortion of quasiconformal mappings. *Acta Math. 173*, 1 (1994), 37–60.

[3] Astala, K., Bonk, M., and Heinonen, J. Quasiconformal mappings with Sobolev boundary values. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 1*, 3 (2002), 687–731.

[4] Balogh, Z. M., and Tyson, J. T. Hausdorff dimensions of self-similar and self-affine fractals in the Heisenberg group. *Proc. London Math. Soc. (3) 91*, 1 (2005), 153–183.

[5] Balogh, Z. M., Tyson, J. T., and Warhurst, B. Sub-Riemannian vs. Euclidean dimension comparison and fractal geometry on Carnot groups. *Adv. Math. 220*, 2 (2009), 560–619.

[6] Bishop, C. J. A quasisymmetric surface with no rectifiable curves. *Proc. Amer. Math. Soc. 127*, 7 (1999), 2035–2040.

[7] Bonk, M., and Heinonen, J. Smooth quasiregular mappings with branching. *Publ. Math. Inst. Hautes Études Sci. 100* (2004), 153–170.

[8] Bourdon, M. Au bord de certains polyèdres hyperboliques. *Ann. Inst. Fourier (Grenoble) 45* (1995), 119–141.

[9] David, G., and Toro, T. Reifenberg flat metric spaces, snowballs, and embeddings. *Math. Ann. 315*, 4 (1999), 641–710.

[10] de Guzmán, M. *Differentiation of integrals in R^n*. Lecture Notes in Mathematics, Vol. 481. Springer-Verlag, Berlin, 1975.

[11] Falconer, K., and Miao, J. Exceptional sets for self-affine fractals. *Math. Proc. Cambridge Philos. Soc. 145*, 3 (2008), 669–684.

[12] Falconer, K. J. Hausdorff dimension and the exceptional set of projections. *Mathematika 29*, 1 (1982), 109–115.

[13] Falconer, K. J. The Hausdorff dimension of self-affine fractals. *Math. Proc. Cambridge Philos. Soc. 103* (1988), 339–350.

[14] Falconer, K. J. *Fractal geometry.* Mathematical Foundations and Applications. John Wiley and Sons Ltd., Chichester, 1990.

[15] Falconer, K. J. The dimension of self-affine fractals. II. *Math. Proc. Cambridge Philos. Soc. 111* (1992), 169–179.

[16] Falconer, K. J., and Howroyd, J. D. Projection theorems for box and packing dimensions. *Math. Proc. Cambridge Philos. Soc. 119*, 2 (1996), 287–295.

[17] Falconer, K. J., and Howroyd, J. D. Packing dimensions of projections and dimension profiles. *Math. Proc. Cambridge Philos. Soc. 121*, 2 (1997), 269–286.

[18] Gehring, F. W. The L^p-integrability of the partial derivatives of a quasiconformal mapping. *Acta Math. 130* (1973), 265–277.

[19] Gehring, F. W., and Väisälä, J. Hausdorff dimension and quasiconformal mappings. *J. London Math. Soc. (2) 6* (1973), 504–512.

[20] Ghamsari, M., and Herron, D. A. Higher dimensional Ahlfors regular sets and chordarc curves in $\mathbf{R}^n$. *Rocky Mountain J. Math. 28*, 1 (1998), 191–222.

[21] Hajłasz, P. Sobolev mappings: Lipschitz density is not an isometric invariant of the target. Preprint, 2010.

[22] Hajłasz, P. Sobolev spaces on an arbitrary metric space. *Potential Anal. 5* (1996), 403–415.

[23] Hajłasz, P. Sobolev mappings: Lipschitz density is not a bi-Lipschitz invariant of the target. *Geom. Funct. Anal. 17*, 2 (2007), 435–467.

[24] Hajłasz, P. Density of Lipschitz mappings in the class of Sobolev mappings between metric spaces. *Math. Ann. 343*, 4 (2009), 801–823.

[25] Hajłasz, P. Sobolev mappings between manifolds and metric spaces. In *Sobolev spaces in mathematics. I*, vol. 8 of *Int. Math. Ser. (N.Y.)*. Springer, New York, 2009, pp. 185–222.

[26] Hajłasz, P., and Koskela, P. Sobolev met Poincaré. *Memoirs Amer. Math. Soc. 145*, 688 (2000).

[27] Hajłasz, P., and Tyson, J. T. Sobolev Peano cubes. *Michigan Math. J. 56*, 3 (2008), 687–702.

[28] Heinonen, J. *Lectures on analysis on metric spaces.* Springer-Verlag, New York, 2001.

[29] Heinonen, J., Koskela, P., Shanmugalingam, N., and Tyson, J. T. Sobolev classes of Banach space-valued functions and quasiconformal mappings. *J. Anal. Math. 85* (2001), 87–139.

[30] Heinonen, J., and Rohde, S. The Gehring-Hayman inequality for quasihyperbolic geodesics. *Math. Proc. Cambridge Philos. Soc. 114*, 3 (1993), 393–405.

[31] Kaufman, R. On Hausdorff dimension of projections. *Mathematika 15* (1968), 153–155.

[32] Kaufman, R. Sobolev spaces, dimension, and random series. *Proc. Amer. Math. Soc. 128*, 2 (2000), 427–431.

[33] Kaufman, R., and Mattila, P. Hausdorff dimension and exceptional sets of linear transformations. *Ann. Acad. Sci. Fenn. Ser. A I Math. 1*, 2 (1975), 387–392.

[34] Kaufman, R., Tyson, J. T., and Wu, J.-M. Smooth quasiregular mappings with branching in $\mathbf{R}^n$. *Publ. Math. Inst. Hautes Études Sci.*, 101 (2005), 209–241.

[35] Kauhanen, J., Koskela, P., and Malý, J. On functions with derivatives in a Lorentz space. *Manuscripta Math. 100*, 1 (1999), 87–101.

[36] Korányi, A., and Reimann, H. M. Foundations for the theory of quasiconformal mappings on the Heisenberg group. *Adv. Math. 111* (1995), 1–87.

[37] Kovalev, L. V., and Onninen, J. Variation of quasiconformal mappings on lines. *Studia Math. 195*, 3 (2009), 257–274.

[38] Malý, J., and Martio, O. Lusin's condition (N) and mappings of the class $W^{1,n}$. *J. Reine Angew. Math. 458* (1995), 19–36.

[39] Mattila, P. Hausdorff dimension, orthogonal projections and intersections with planes. *Ann. Acad. Sci. Fenn. Ser. A I Math. 1*, 2 (1975), 227–244.

[40] Mattila, P. Hausdorff dimension and capacities of intersections of sets in n-space. *Acta Math. 152*, 1-2 (1984), 77–105.

[41] Mattila, P. On the Hausdorff dimension and capacities of intersections. *Mathematika 32*, 2 (1985), 213–217 (1986).

[42] Mattila, P. Orthogonal projections, Riesz capacities, and Minkowski content. *Indiana Univ. Math. J. 39*, 1 (1990), 185–198.

[43] Mattila, P. *Geometry of sets and measures in Euclidean spaces*, vol. 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.

[44] Mattila, P. Hausdorff dimension, projections, and the Fourier transform. *Publ. Mat. 48* (2004), 3–48.

[45] Munkres, J. R. *Topology: a first course*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.

[46] Pansu, P. Dimension conforme et sphère à l'infini des variétés à courbure négative. *Ann. Acad. Sci. Fenn. Ser. A I Math. 14* (1989), 177–212.

[47] Ranjbar-Motlagh, A. An embedding theorem for Sobolev type functions with gradients in a Lorentz space. *Studia Math. 191*, 1 (2009), 1–9.

[48] Reshetnyak, Y. G. Sobolev classes of functions with values in a metric space. *Sibirsk. Mat. Zh. 38* (1997), 657–675.

[49] Rigot, S. Counter example to the Besicovitch covering property for some Carnot groups equipped with their Carnot-Carathéodory metric. *Math. Z. 248*, 4 (2004), 827–848.

[50] Romanov, A. S. On the absolute continuity of Sobolev-type functions on metric spaces. *Sibirsk. Mat. Zh. 49*, 5 (2008), 1147–1156.

[51] Shanmugalingam, N. Newtonian spaces: an extension of Sobolev spaces to metric measure spaces. *Rev. Mat. Iberoamericana 16*, 2 (2000), 243–279.

[52] Smirnov, S. Dimension of quasicircles. *Acta Math.* to appear.

[53] Solomyak, B. Measure and dimension for some fractal families. *Math. Proc. Cambridge Philos. Soc. 124*, 3 (1998), 531–546.

[54] Stein, E. M. *Singular integrals and differentiability properties of functions*. Princeton University Press, Princeton, N.J., 1970. Princeton Mathematical Series, No. 30.

[55] Tukia, P. A quasiconformal group not isomorphic to a Möbius group. *Ann. Acad. Sci. Fenn. Ser. A I Math. 6* (1981), 149–160.

[56] Tukia, P. Hausdorff dimension and quasisymmetric mappings. *Math. Scand. 65* (1989), 152–160.

[57] Tyson, J. T. Analytic properties of locally quasisymmetric mappings from Euclidean domains. *Indiana Univ. Math. J. 49*, 3 (2000), 995–1016.

[58] Wildrick, K., and Zürcher, T. Space filling with metric measure spaces. Preprint, 2009.

[59] Wildrick, K., and Zürcher, T. Peano cubes with derivatives in a Lorentz space. *Illinois J. Math. 53*, 2 (2009), 365–378.

[60] Ziemer, W. P. *Weakly Differentiable Functions*, vol. 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989.

Mathematisches Institut, Universität Bern, Sidlerstrasse 5, 3012 Bern, Switzerland.
E-mail address: zoltan.balogh@math.unibe.ch

Dipartimento di Matematica Pura e Applicata, Università di Padova, Via Trieste, 63 35121 Padova, Italy
E-mail address: monti@math.unipd.it

Department of Mathematics, University of Illinois, 1409 West Green St., Urbana, IL 61801, USA.
E-mail address: tyson@math.uiuc.edu