Variational Approximation of Vectorial Free-Discontinuity Problems: the Discrete and Continuous Case

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Chapter 1

Introduction

Many mathematical problems arising from Computer Vision Theory, Fracture Mechanics, Liquid Crystals Theory, Minimal Partitions Problems (see for instance [15],[37],[57],[86], the book [20] and the references therein for a more accurate list) are characterized by a competition between volume and surface energies. The variational formulation of those problems leads to the minimization (under boundary or confinement conditions) of functionals represented by

$$F(u,K) := \int_{\Omega\setminus K} f(x,u,\nabla u)\;dx + \int_K \varphi\left(x,u^-,u^+,\nu\right)\;d\mathcal{H}^{n-1}, \tag{1.0.1}$$

where $\Omega \subset \mathbf{R}^n$ is a fixed open set, K is a (sufficiently regular) closed subset of Ω and $u:\Omega\setminus K\to\mathbf{R}^N$ belongs to a (sufficiently regular) class of functions with traces $u^{\pm}$ defined on K. We remark that K is not assigned a priori and is not a boundary in general. Therefore, these problems are not free-boundary problems, and new ideas and techniques have to be developed to solve them.

Since no known topology on the closed subsets of Ω ensures at the same time compactness for minimizing sequences and lower semicontinuity of the surface energy of (1.0.1) even if $\varphi\equiv 1$ (although some recent results in this direction have been obtained in the paper [80]), the direct methods do not apply and a weak formulation of the problem is needed. To do this, De Giorgi [56] proposed to interpret K as the set of discontinuity points of u. This idea motivates the terminology *free-discontinuity problem* for the minimization of (1.0.1), to underline the fact that one looks for a function the discontinuities of which are not assigned a priori.

As a first attempt one may try to set the relaxed problem in the space $BV\left(\Omega;\mathbf{R}^N\right)$ of *function of bounded variation on* Ω, i.e., functions u which are summable and with the first order distributional derivative representable by a finite Radon measure Du on Ω. Actually, since free-discontinuity problems as in (1.0.1) involve energies with volume and surface terms, it is natural to allow in these problems only BV functions whose distributional derivative has a similar structure. Indeed, De Giorgi and Ambrosio [57] relaxed the problem in the space $SBV\left(\Omega;\mathbf{R}^N\right)$ of *special functions of bounded variation*, i.e., functions u in $BV\left(\Omega;\mathbf{R}^N\right)$ such that the singular part of Du with respect to the Lebesgue measure is supported in S_u, the complement of the set of Lebesgue's points for u. Naively, setting $K=S_u$ in (1.0.1), and defining $\mathcal{F}(u) = E(u, S_u)$, the free-discontinuity problem reduces to

$$\min_{u \in SBV(\Omega;\mathbf{R}^N)} \mathcal{F}(u) . \tag{1.0.2}$$

The abstract theory for such problems developed during the last few years: Ambrosio (see [10],[11],[14],[20]) established the existence theory, and many authors studied the regularity of weak solutions u (see [58],[19],[20],[23],[47]), proving in some cases that the jump set S_u differs from its closure by a set of zero $\mathcal{H}^{n-1}$ measure, and that u is smooth outside $\overline{S_u}$. As a consequence, the pair $(u, \overline{S_u})$ is a strong solution, i.e., a minimizer of the original functional in (1.0.1).

Hence, from a theoretical point of view, the minimization of (1.0.1) has been solved under very general assumptions. Nevertheless, explicit solutions can be computed only in very few cases, also for simple choices of f and φ.

Thus, the numerical approximation of the problem (1.0.2) is not only interesting by itself, but also crucial in order to get more pieces of information on the solutions of (1.0.1). The problem revealed to be a hard task because of the use of spaces of discontinuous functions. One method to overcome this difficulty is to perform a preliminary variational approximation of the functional $\mathcal{F}$, in the sense of De Giorgi's Γ-convergence [59], via simpler functionals easier to be handled numerically, and then to discretize each of the approximating functionals.

Many approaches have been proposed for the approximation problem in the scalar case (see [5],[24],[25],[40],[48],[49],[75] and the book [37]), while the vectorial case had not been treated, yet.

In this Thesis we have collected some recent results dealing with the vectorial case, which also generalize some of the scalar ones quoted above.

To begin with, in Chapter 2, we extend the so called Ambrosio-Tortorelli's construction [24],[25]. The model was originally built to obtain in the limit (the weak formulation of) the Mumford-Shah's functional of image reconstruction [86], defined on $SBV(\Omega)$ by

$$MS(u) = \int_\Omega |\nabla u|^2\, dx + \alpha \mathcal{H}^{n-1}(S_u) + \beta \int_\Omega |u - g|^2\, dx \tag{1.0.3}$$

where g is a given function in $L^2(\Omega)$, $\alpha > 0$ and $\beta \geq 0$.

For our purposes the last term in (1.0.3) is irrelevant, since it does not affect Γ-convergence. Hence, we will drop it in the discussion in the sequel, i.e., we set $\beta = 0$.

The main issue of the model is the introduction of an auxiliary variable which asymptotically approaches $1 - \mathcal{X}_{S_u}$, thus detecting the discontinuity set of u. In [25] (see [31],[81] for numerical simulations) it was proved that the family of elliptic functionals

$$AT_\varepsilon(u, v) = \int_\Omega v^2 \left|\nabla u\right|^2\, dx + \int_\Omega \left(\frac{1}{\varepsilon}(1-v)^2 + \varepsilon \left|\nabla v\right|^2\right) dx, \tag{1.0.4}$$

defined for $u, v \in W^{1,2}(\Omega)$, Γ-converges with respect to the $L^1(\Omega;\mathbf{R}^2)$ convergence to the functional defined on $SBV(\Omega)$ by

$$AT(u,v) = \begin{cases} MS(u) & \text{if } v = 1 \text{ a.e. in } \Omega \\ \\ +\infty & \text{otherwise,} \end{cases}$$

equivalent to MS as far as minimum problems are concerned. The analysis is restricted to the scalar isotropic case where the use of an integral-geometric argument, called *slicing procedure*, allows to reduce the n-dimensional problem to the one-dimensional case.

Let us remark that the penalization term in (1.0.4) is strongly related to the Modica-Mortola-type singular perturbation problems for phase transformations (see [83],[30],[34] and [70]).

Pushing forward the construction of [25], and with obvious substitutions in the definition of the approximating functionals (1.0.4) (see (1.1.2)), we extend the approximation to energies defined on $SBV\left(\Omega;\mathbf{R}^N\right)$, $N \geq 1$, by

$$\int_\Omega f\left(x,u,\nabla u\right)\,dx + \int_{S_u} \varphi\left(\nu_u\right)\,d\mathcal{H}^{n-1}, \tag{1.0.5}$$

where f is a positive Carathéodory integrand with superlinear growth and quasiconvex in the gradient variable and φ is a norm on $\mathbf{R}^n$.

This extension, apart from its own mathematical interest, is motivated by some applications to Griffith's theory of brittle fracture [15], for which the vectorial setting is more natural than the scalar one.

Due to the general form of the volume term in (1.0.5), the slicing methods no longer applies, and other techniques have to be exploited to deal with the anisotropic vectorial case (see Compendium of Chapter 3 for a detailed discussion).

To approximate more complex surface energies depending also on the traces $u^\pm$, which arise for instance in fracture models of Barenblatt's type (see [15],[28]), a variant of the Ambrosio-Tortorelli's construction is studied in [6], obtained by replacing in (1.0.4) $|\nabla u|^2$ with $f\left(|\nabla u|\right)$, where f is convex and with linear growth. Indeed, this weaker penalization of ∇u enables a stronger interaction between the two competing terms in (1.0.4).

An obvious consequence of the linear growth assumption is the presence of a term accounting for the Cantor part $D^c u$ of Du in the limit energy, which has the form on $BV(\Omega)$

$$\int_\Omega f\left(|\nabla u|\right)\,dx + \|D^c u\|\left(\Omega\right) + \int_{S_u} g\left(|u^+ - u^-|\right)\,d\mathcal{H}^{n-1},$$

where g is defined by a suitable minimization formula highlighting the contribute of the two terms of (1.0.4). Again, the analysis in [6] is restricted to the scalar isotropic case in order to exploit the usual one-dimensional reduction argument.

In Chapter 4 we consider the full vectorial problem by studying the asymptotic behaviour of the family of functionals defined for $u \in W^{1,1}\left(\Omega;\mathbf{R}^N\right)$, $N \geq 1$, $v \in W^{1,2}\left(\Omega\right)$ by

$$F_\varepsilon(u,v) = \int_\Omega v^2 f\left(x,u,\nabla u\right)\,dx + \int_\Omega \left(\frac{1}{\varepsilon}(1-v)^2 + \varepsilon\left|\nabla v\right|^2\right)\,dx, \tag{1.0.6}$$

where f is a quasiconvex function in the gradient variable satisfying linear growth conditions.

The slicing procedure is not useful to deal with this problem since the general form of the integrand f. In order to get more pieces of information on the interaction between the two terms in (1.0.6) we couple the two variables u, v as being a single vector-valued one. This choice highlights the vectorial nature of the model, which is studied by exploiting techniques typical in vectorial problems of Calculus of Variations. Indeed, by using the blow-up methods of Fonseca-Müller [67],[68] we are able to prove the Γ-convergence of the functionals in (1.0.6), with respect to the $L^1\left(\Omega; \mathbf{R}^{N+1}\right)$ convergence, to the functional defined for u in $BV\left(\Omega; \mathbf{R}^N\right)$ and $v \equiv 1$ a.e. on Ω by

$$\int_\Omega f\left(x, u, \nabla u\right)\, dx + \int_\Omega f^\infty\left(x, u, dD^c u\right) + \int_{S_u} K\left(x, u^+, u^-, \nu_u\right)\, d\mathcal{H}^{n-1},$$

where f^∞ is the *recession function* of f, and the surface energy density K is calculated as an asymptotic limit of Dirichlet's boundary values problems in the spirit of the Global Method for Relaxation introduced in [35] (see Compendium of Chapter 4).

A different approach to approximate the Mumford-Shah's functional in (1.0.3) can be followed by considering non-local functionals where the gradient is replaced by finite differences. It was conjectured by De Giorgi, and proved by Gobbino [75] (see [76] for extensions and [49] for numerical implementation), the Γ-convergence (up to some multiplicative constants) to the Mumford-Shah's functional with respect to the $L^1(\Omega)$ topology of the family

$$DG_\varepsilon(u) = \frac{1}{\varepsilon}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f\left(\varepsilon|D_{\varepsilon,\xi}u(x)|^2\right)\rho(\xi)\, dx\, d\xi, \tag{1.0.7}$$

where $f : [0, +\infty) \to [0, +\infty)$ is any increasing function such that $f(0) = 0$, $f'(0^+) > 0$ and $f(+\infty) < +\infty$; ρ is a symmetric convolution kernel and $D_{\varepsilon,\xi}u(x) = \frac{1}{\varepsilon}(u(x + \varepsilon\xi) - u(x))$.

In Chapter 5, dealing with the two- and three-dimensional cases, we pursue this approach to provide an approximation of functionals defined on (sufficiently regular) vector fields $u : \Omega \setminus K \to \mathbf{R}^n$ by

$$\mu\int_{\Omega\setminus K}|\mathcal{E}u(x)|^2\, dx + \frac{\lambda}{2}\int_{\Omega\setminus K}|\operatorname{div} u(x)|^2\, dx + \gamma\mathcal{H}^{n-1}(K), \tag{1.0.8}$$

that is by taking in (1.0.1) $N = n$ and $f(\nabla u) = \mu|\mathcal{E}u|^2 + \frac{\lambda}{2}|\operatorname{div} u|^2$, $\mathcal{E}u$ being the symmetric part of ∇u. In this case, f is a linear elasticity density degenerate as a quadratic form with respect to ∇u. Incidentally, the functional setting of the weak formulation of this problem changes from $SBV\left(\Omega; \mathbf{R}^n\right)$ functions to $SBD(\Omega)$ ones, i.e., summable vector fields $u : \Omega \to \mathbf{R}^n$ with symmetrized distributional derivative Eu represented by a finite Radon measure on Ω, the singular part of which, with respect to Lebesgue's measure, is concentrated on an $(n-1)$-dimensional set J_u.

In order to approximate functionals of the type (1.0.8) for any choice of the parameters μ and λ, we have to introduce in the model a suitable difference quotient representing the divergence, call it $\mathrm{Div}_{\varepsilon,\xi}$, and consider functionals of the form

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{1}{\varepsilon}f\left(\varepsilon\left(|\langle D_{\varepsilon,\xi}u(x), \xi\rangle|^2 + \theta|\mathrm{Div}_{\varepsilon,\xi}u(x)|^2\right)\right)\rho(\xi)\, dx\, d\xi,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbf{R}^n$. The presence of the divergence term does not allow us to reduce to the one-dimensional case by the standard slicing procedures. The convergence result is then recovered by a discretization argument which leads to the study of discrete functionals of the form

$$\sum_{\alpha\in\varepsilon\mathbf{Z}^n}\varepsilon^{n-1}f\left(\varepsilon\left(|D_{\varepsilon,\xi}u(\alpha)|^2+\theta|\mathrm{Div}_{\varepsilon,\xi}u(\alpha)|^2\right)\right), \tag{1.0.9}$$

with θ a strictly positive parameter.

The study of the asymptotic limits of discrete systems is a recent trend of research, trying to give a theoretical justification of the behaviour at a macroscopic level of homogeneous media by means of a microscopic analysis (see [43],[44],[45],[92]).

So far, only the scalar setting has been investigated, and very general results are available in such a case. In Chapter 5 and Chapter 6 we begin the analysis of functionals defined on vector-valued discrete deformations.

In particular, in Chapter 6 we provide an approximation result, in the two- and three-dimensional cases, for autonomous energies defined on $SBV\left(\Omega;\mathbf{R}^N\right)$ by

$$\int_\Omega\psi\left(\nabla u\right)\,dx+\int_{S_u}g(u^+-u^-,\nu_u)\,d\mathcal{H}^{n-1} \tag{1.0.10}$$

for any quasiconvex function ψ with superlinear growth and g suitable.

The model adopted is analogous to that of (1.0.9). Our aim is to obtain in the limit autonomous energies with any quasiconvex function ψ as bulk energy density . Then, in order to recover the global behaviour of the gradient matrix ∇u, we introduce a finite-differences matrix $D_\varepsilon u$ which plays the role of the difference quotient $D_{\varepsilon,\xi}u$ in (1.0.9). Hence, the approximating functionals have the form

$$\sum_{\alpha\in\varepsilon\mathbf{Z}^n}\varepsilon^{n-1}\psi_\varepsilon\left(\varepsilon D_\varepsilon u(\alpha)\right), \tag{1.0.11}$$

where ψ_ε is an interaction potential, underlying a separation of scales, obtained by rescaling and truncating the function ψ suitably. The behaviour of ψ_ε at infinity influences the formula defining the surface energy density g, as shown in the two-dimensional case by discussing different models.

What has been exposed in a descriptive way above, will be now discussed in a more detailed and rigorous framework in the following four compendia. Each compendium summarizes the contents of a corresponding paper, annexed as a chapter of this Thesis. The contents of Chapters 3,4,5 and 6 are published in the papers [64],[7],[8] and [65] respectively; and are the result of a research activity carried on by the Author at Scuola Normale Superiore di Pisa and at Università degli Studi di Firenze in collaboration with R. Alicandro and M.S. Gelli.

1.1 Compendium of Chapter 3

In this Chapter we provide a variational approximation for functionals defined on the space $GSBV\left(\Omega;\mathbf{R}^N\right)$ of the form

$$\int_\Omega f\left(x,u,\nabla u\right)\,dx+\int_{S_u}\varphi\left(\nu_u\right)\,d\mathcal{H}^{n-1}, \tag{1.1.1}$$

where $\varphi : \mathbf{R}^n \to [0, +\infty)$ is a norm and $f : \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times n} \to [0, +\infty)$ is a Carathéodory integrand, quasiconvex in z, satisfying for every $(x, u, z) \in \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times n}$

$$c_1 |z|^p + b(u) - a(x) \leq f(x, u, z) \leq c_2(|z|^p + b(u) + a(x)),$$

with $p \in (1, +\infty)$, $c_1, c_2 > 0$, $a \in L^1(\Omega)$ and $b \in C^0\left(\mathbf{R}^N\right)$ a non negative function.

The model we use is the one of Ambrosio-Tortorelli [25], that is by taking into account the family of functionals defined on the space of Borel functions by

$$F_\varepsilon(u, v) = \begin{cases} \displaystyle\int_\Omega (\psi(v) + \eta_\varepsilon)\, f(x, u, \nabla u)\, dx + \int_\Omega \left(\frac{1}{\varepsilon p'} W(v) + \frac{\varepsilon^{p-1}}{p} \varphi^p(\nabla v) \right) dx \\ \qquad\qquad\qquad\qquad (u, v) \in W^{1,p}\left(\Omega; \mathbf{R}^{N+1}\right),\ 0 \leq v \leq 1 \text{ a.e. in } \Omega \\ +\infty \qquad\qquad\qquad \text{otherwise,} \end{cases} \tag{1.1.2}$$

where $\psi : [0, 1] \to [0, 1]$ is any increasing lower semicontinuous function such that $\psi(0) = 0$, $\psi(1) = 1$, and $\psi(t) > 0$ if $t > 0$; $p' = \frac{p}{p-1}$; η_ε is any positive infinitesimal faster than ε^{p-1} for $\varepsilon \to 0^+$; $W(t) = c_W(1 - t)^p$ with $c_W = \left(2 \int_0^1 (1 - s)^{\frac{p}{p'}} ds\right)^{-p'}$.

Let us briefly comment the heuristical idea of the Γ-convergence for this model choosing for simplicity $f(\nabla u) = |\nabla u|^p$, φ to be the euclidean norm and $\eta_\varepsilon \equiv 0$ (it turns out from the discussion below that $\eta_\varepsilon > 0$ is essential only for the coercivity of F_ε but it does not affect the Γ-convergence of the family (F_ε), see Theorem 3.4.2).

Assume that we are given a family $(u_\varepsilon, v_\varepsilon)$ converging to $(u, 1)$ in measure on Ω and such that

$$\liminf_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty.$$

On one hand v_ε is forced to stay very close to 1 since the potential W vanishes only for $t = 1$ and it is strictly positive otherwise; on the other hand u_ε approximates a (possibly) discontinuous function u, hence the term $\psi(v_\varepsilon)$ in front of $|\nabla u_\varepsilon|^p$ (which prevents coercivity in Sobolev spaces) must go to 0 to keep the Dirichlet's integral bounded. Hence, v_ε is forced to make transitions (which are sharper and sharper as $\varepsilon \to 0^+$) between 0 and 1 near to discontinuities of u. The balance between the two terms in the second integral of (1.1.2) shows that the cheapest energetical transition cost is proportional to $\mathcal{H}^{n-1}(S_u)$.

Actually, this idea can be made formal, it is indeed exploited in [25] for the proof of the lower bound inequality in the one-dimensional case, to which the authors reduce the general one by the usual integral-geometric argument.

Due to the general form of the volume term in (1.1.1), the slicing procedure no longer applies to obtain a lower bound for it in the Γ-limit. The idea, then, is to deduce lower estimates on the bulk term and on the surface term separately by using a global technique proposed by Ambrosio in [12], from which one also identify the domain of the limit functional.

Let us point out that in the vectorial setting $N > 1$, the lower semicontinuity inequality cannot be obtained by means of the slicing techniques even for $f(\nabla u) = |\nabla u|^p$. Indeed, a one-dimensional reduction argument yields the operator norm of the gradient matrix in the Γ-limit, instead of the euclidean one.

Denote by $I_\varepsilon(u,v)$ and $G_\varepsilon(v)$ the first and second integral in (1.1.2), respectively. In Lemma 3.2.1 we prove that for any family $(u_\varepsilon, v_\varepsilon)$ converging to $(u,1)$ in measure on Ω the following hold true

$$\liminf_{\varepsilon\to 0^+} I_\varepsilon(u_\varepsilon, v_\varepsilon) \geq \int_\Omega f\left(x, u, \nabla u\right)\, dx, \tag{1.1.3}$$

and

$$\liminf_{\varepsilon\to 0^+} G_\varepsilon\left(v_\varepsilon\right) \geq \int_{S_u} \varphi\left(\nu_u\right)\, d\mathcal{H}^{n-1}. \tag{1.1.4}$$

The main idea to prove such inequalities is to gain coercivity in $GSBV$ by 'cutting around' the discontinuity set S_u of the limit function u.

According to the heuristic interpretation given above, S_u is detected by means of the superlevel sets $U_{\varepsilon,t} = \{x \in \Omega : v_\varepsilon(x) > t\}$ of v_ε. Consider the $GSBV$ functions $w_{\varepsilon,t} = u_\varepsilon \mathcal{X}_{U_{\varepsilon,t}}$, notice that since v_ε converges to 1 in measure on Ω, then $\mathcal{L}^n(\Omega \backslash U_{\varepsilon,t})$ is infinitesimal, which in turn implies the convergence of $w_{\varepsilon,t}$ to u in measure on Ω. Moreover, with fixed $0 < \lambda < \lambda' < 1$, we may choose $\lambda < t_\varepsilon < \lambda'$ in such a way that $(w_{\varepsilon,t_\varepsilon})$ is pre-compact in $GSBV$ according to the $GSBV$ Compactness Theorem 2.7.11. Hence, the $GSBV$ Closure Theorem 2.7.10 implies that u is a $GSBV\left(\Omega;\mathbf{R}^N\right)$ function. It is easy to check that

$$I_\varepsilon(u_\varepsilon, v_\varepsilon) \geq \psi(\lambda) \int_\Omega f\left(x, w_{\varepsilon,t_\varepsilon}, \nabla w_{\varepsilon,t_\varepsilon}\right)\, dx + o(1), \tag{1.1.5}$$

then by using the $GSBV$ Lower Semicontinuity Theorem 2.7.17 we may pass to the limit in (1.1.5) on $\varepsilon \to 0^+$, and eventually get the desired estimate (1.1.3) by letting $\lambda \to 1^-$.

For what the surface inequality (1.1.4) is concerned, here we give a proof different from the one originally appeared in [64], which was based on the slicing methods. The new proof exploits the global procedure used for the bulk term. To explain the idea assume for simplicity φ to be the euclidean norm. Then, by following Modica and Mortola [83], Young's inequality yields the pointwise estimate

$$G_\varepsilon(v) \geq \int_\Omega |\nabla(\Phi(v))|\, dx \tag{1.1.6}$$

for every $v \in W^{1,p}(\Omega)$ and $\varepsilon > 0$, where $\Phi : [0,1] \to [0,+\infty)$ is the auxiliary function defined by

$$\Phi(t) := \int_0^t \left(W(s)\right)^{\frac{1}{p'}}\, ds. \tag{1.1.7}$$

Notice that c_W is such that $\Phi(1) = \frac{1}{2}$.

By taking into account the BV Coarea Formula (see Theorem 2.6.6) we have

$$\int_\Omega |\nabla\left(\Phi(v_\varepsilon)\right)|\, dx = \int_0^{\Phi(1)} \mathcal{H}^{n-1}\left(J_{\mathcal{X}_{U_{\varepsilon,\Phi^{-1}(s)}}}\right)\, ds.$$

Then, since $U_{\varepsilon,t} \supseteq S_{w_{\varepsilon,t}}$, to conclude it suffices to estimate the perimeter of the discontinuity set of $w_{\varepsilon,t}$ and to prove that it is asymptotically greater than $2\mathcal{H}^{n-1}(S_u)$, as one would heuristically expects.

Let us point out that the whole information carried in F_ε is needed to prove the estimates (1.1.3) and (1.1.4), even though each of them involves only one term between I_ε and G_ε.

Indeed, assuming the convergence of v_ε to 1 in measure on Ω without any additional information, one can only infer the trivial inequality

$$\liminf_{\varepsilon\to 0^+} G_\varepsilon(v_\varepsilon) \geq 0.$$

It is easy to show families (v_ε) for which the limit of $G_\varepsilon(v_\varepsilon)$ exists and equals 0, for instance $v_\varepsilon \equiv 1$.

Moreover, take $\eta_\varepsilon \equiv 0$, then I_ε coincide on $W^{1,1}\left(\Omega;\mathbf{R}^{N+1}\right)$ with

$$I(u,v) = \int_\Omega \psi(v) f(\nabla u)\, dx.$$

The functional I, in general, is not lower semicontinuous in the $L^1\left(\Omega;\mathbf{R}^{N+1}\right)$ topology, as pointed out by Eisen in his famous counterexample [61]. Indeed, for $n = N = 1$, $\Omega = (0,1)$ and the simple choices $f(\dot{u}) = |\dot{u}|^2$ and $\psi(v) = v^2$, Eisen constructs a sequence $(u_j, v_j) \to (x,1)$ in $L^1\left((0,1);\mathbf{R}^2\right)$ for which $I(u_j,v_j) = 0$ for every $j \in \mathbf{N}$, so that

$$I(x,1) = 1 > \liminf_j I(u_j,v_j) = 0,$$

and the lower semicontinuity of I is then violated. Hence, one cannot prove (1.1.3) by using directly a lower semicontinuity argument on I_ε.

The loss of semicontinuity for I is due to the lack of coercivity in the v variable of the integrand above, as one can deduce by the results in [66], where the deep relation between coercivity (or the lack of it) and lower semicontinuity has been investigated. In our case it is the term G_ε which recovers coercivity in the v variable for F_ε. Indeed, by (1.1.6), for every (u,v) and $\varepsilon > 0$ there holds

$$F_\varepsilon(u,v) \geq \int_\Omega \tilde{\psi}(\Phi(v)) f\left(x,u,\nabla u\right)\, dx + \int_\Omega |\nabla(\Phi(v))|\, dx,$$

where $\tilde{\psi}(t) = \psi(\Phi^{-1}(t))$. Hence, for target functions in SBV, one could also prove the lower bound inequality for the bulk term in (1.1.1), by using directly a lower semicontinuity argument similar to those of [66],[71] (see also Subsection 4.6.2).

In Lemma 3.3.1 we prove the constructive inequality for functions with a polyhedral discontinuity set S_u and smooth outside. This is not restrictive, since the limit functional in (1.1.1) is continuous on a class of functions, dense in its domain, which have a polyhedral discontinuity set and are smooth outside (see Theorem 2.7.14).

For those regular functions the construction of the recovery sequence (v_ε) is quite easy since ν_u has a finite range and thus $\varphi(\nu_u)$ is piecewise constant. Then, one may use a one-dimensional construction in a ε-neighbourhood of S_u, in such a way that v_ε makes a transition from 0 to a value approaching 1 as $\varepsilon \to 0^+$ according to an underlying minimality criterion analogous to that of [24],[25].

As usual, the recovery sequence (u_ε) for u is obtained by considering $u_\varepsilon = \varphi_\varepsilon u$, where φ_ε is a smooth cut-off function such that u_ε does not charge the set where v_ε makes the transition.

Finally, we provide a coercivity result for the approximating functionals F_ε perturbed by a term of the type

$$\int_\Omega |u-g|^q\,dx$$

for $q \in (0,+\infty)$ and $g \in L^q\left(\Omega;\mathbf{R}^N\right)$, which does not affect Γ-convergence.

1.2 Compendium of Chapter 4

Chapter 4 is devoted to study the asymptotic behaviour of the family of functionals defined for $u \in W^{1,1}\left(\Omega;\mathbf{R}^N\right)$, $v \in W^{1,2}(\Omega)$ by

$$F_\varepsilon(u,v) = \int_\Omega \psi(v) f(x,u,\nabla u)\ dx + \int_\Omega \left(\frac{1}{\varepsilon}W(v) + \varepsilon\,|\nabla v|^2\right)\,dx, \tag{1.2.1}$$

where f is a quasiconvex function in the gradient variable and satisfies linear growth conditions (for the set of assumptions on ψ, f and W see Section 4.1).

This model is a variant of the Ambrosio-Tortorelli's construction discussed in Compendium of Chapter 3, obtained by taking a linearly growing potential in (1.1.2).

We prove that (F_ε) Γ-converges with respect to the $L^1\left(\Omega;\mathbf{R}^{N+1}\right)$ topology to the functional defined on $(GBV(\Omega))^N \times \{1\}$ by

$$\begin{aligned}\mathcal{F}(u) = &\int_\Omega f(x,u,\nabla u)\ dx \\ &+ \int_\Omega f^\infty(x,u,dD^c u) + \int_{S_u} K\left(x,u^+,u^-,\nu_u\right)\,d\mathcal{H}^{n-1},\end{aligned} \tag{1.2.2}$$

where f^∞ is the recession function of f and K is suitably defined (see (1.2.4) below).

Let us point out that the functional (1.2.2) is well defined also for GBV functions, since in Lemma 2.7.4 we show that one can define a vector measure which can be regarded as the Cantor part of the generalized distributional derivative.

The model provides a simplified variational formulation for problems in fracture mechanics involving crack initiation energies of Barenblatt's type, i.e., depending on the size of the crack opening (see [15],[28],[79]).

The convex isotropic scalar case was studied in the paper [6]. More precisely, for $N=1$ and assuming $f = f(|\nabla u|)$ to be convex and such that $\frac{f(t)}{t} \to 1$ for $t \to +\infty$, it is proved the Γ-convergence of the family in (1.2.1) to a functional which takes the form on $BV(\Omega) \times \{1\}$

$$\int_\Omega f(|\nabla u|)\ dx + \|D^c u\|(\Omega) + \int_{S_u} g\left(|u^+-u^-|\right)\,d\mathcal{H}^{n-1},$$

where $g:[0,+\infty) \to [0,+\infty)$ is the concave function defined by

$$g(t) = \min_{r\in[0,1]}\left\{\psi(r)t + 4\int_r^1 \sqrt{W(s)}\,ds\right\}. \tag{1.2.3}$$

Let us briefly comment formula (1.2.3). To draw a parallel with the Ambrosio-Tortorelli's model take $W(v) = (1-v)^2$ and $f(\nabla u) = |\nabla u|$ in (1.2.1). Going back to the heuristical explanation of Compendium of Chapter 3, in this case the linear growth assumption on ∇u doesn't force any longer the term $\psi(v)$ to go to 0 near to discontinuities of u. Hence, v makes a transition from 1 to a quota $r \in [0,1]$ which is selected according to the minimality criterion in (1.2.3).

In [6] the use of the slicing techniques allows to recover the n-dimensional problem from the one-dimensional case; while, due to the generality of the functionals in (1.2.1), the mentioned integral-geometric approach does not longer apply and different arguments have to be exploited. The main tool of our analysis is the blow-up technique of Fonseca-Müller [67],[68] which has been intensively used for the study of the relaxation and lower semicontinuity properties of functional with linear growth (see [66],[67],[68]) and for the study of anisotropic singular perturbations of non-convex functionals in the vector-valued case (see [30]).

The proofs of the lower estimates on the diffuse and jump part of the limit functional rely on different arguments.

The analysis of the diffuse part is reduced to the identification of the relaxation of functionals with linear growth in the vectorial case, as considered in [66]. In fact, by arguing as in (1.1.6) of Compendium of Chapter 3, one can note that for every (u,v) and $\varepsilon > 0$ we have

$$F_\varepsilon(u,v) \geq \int_\Omega \psi(v) f\left(x,u,\nabla u\right)\, dx + 2\int_\Omega |\nabla(\Phi(v))|\, dx,$$

where Φ is defined by

$$\Phi(t) = \int_0^t (W(s))^{\frac{1}{2}}\, ds.$$

The diffuse part of the relaxation of the functional on the right-hand side above turns out to be the corresponding part of the limit functional.

For what concerns the surface part, a non trivial use of blow-up techniques and De Giorgi's type averaging-slicing lemma (see Subsection 4.2.2) is needed to show that the surface energy density K can be written in terms of Dirichlet's boundary value problems, in the spirit of [30] and [35], that is

$$\begin{aligned} K\left(x_o,a,b,\nu\right) := \inf \Bigg\{ \int_{Q_\nu} \left(\psi(v) f^\infty\left(x_o,u,\nabla u\right) + L\, W(v) + \frac{1}{L}|\nabla v|^2\right) dx :\\ L>0,\ (u,v) \in W^{1,1}\left(Q_\nu;\mathbf{R}^{N+1}\right), \qquad (1.2.4)\\ (u,v) = (a,1) \text{ on } \left(\partial Q_\nu\right)^-,\ (u,v) = (b,1) \text{ on } \left(\partial Q_\nu\right)^+ \Bigg\}, \end{aligned}$$

where Q_ν is an open unit cube with two faces orthogonal to the direction ν and $(\partial Q_\nu)^\pm = \partial Q_\nu \cap \{\pm\langle x,\nu\rangle > 0\}$.

We point out that, even for scalar valued functions u, the minimization problems above are of vectorial type. This fact places some difficulty in order to give an explicit expression to K in the general case, while this can be done under isotropy assumptions on f^∞, as we show in Subsection 4.1.1. In such a case we prove that K can be calculated by restricting the infimum to functions (u,v) with one-dimensional profile. By virtue of this characterization, we provide an extension to the isotropic vector-valued case of the result of [6] (see Remarks 4.1.6, 4.1.11).

The upper bound inequality for functions u in $BV\left(\Omega; \mathbf{R}^N\right)$ is obtained by exploiting an abstract approach via integral representation methods. Indeed, it turns out that for every subsequence of (F_ε) $\overline{\Gamma}$-converging, the limit, as a set function, is a Borel measure absolutely continuous with respect to the total variation measure $\|Du\|$ and that it does not depend on the extracted subsequence. To show this we compute the Radon-Nikodým's derivatives of any $\overline{\Gamma}$-limit with respect to each mutually singular part of $\|Du\|$, and then we prove that they are exactly the densities of $\mathcal{F}(u)$ in (1.2.2) $\|Du\|$ almost everywhere. Hence, Uryshon's property implies the Γ-convergence of the whole family (F_ε).

The representation of the diffuse part then follows by a relaxation argument noting that for every $\varepsilon > 0$ and u in $W^{1,1}\left(\Omega; \mathbf{R}^N\right)$

$$F_\varepsilon(u, 1) \equiv \int_\Omega f\left(x, u, \nabla u\right)\, dx.$$

The representation of the surface part, thanks to a very general integral representation result in [35] (see Theorem 2.6.15), can be proved only for piecewise constant functions jumping along an hyperplane, for which the upper inequality follows by a standard homogenization technique.

To recover the full Γ-convergence result in GBV we use De Giorgi's type averaging-slicing techniques on the range and the continuity of the limit functional (1.2.2) with respect to truncations.

Eventually, by suitably diagonalizing the family of functionals in (1.2.1), we provide an approximation result for energies with superlinear growth as considered in (1.1.1) of Compendium of Chapter 3 (see Theorem 4.6.3). A similar approach, but relying on a double Γ-limit procedure, had already been used in [6] to obtain in the limit functionals with superlinear bulk energy density and with surface energy depending on the one-sided traces, i.e., of the type

$$\int_{S_u} \theta\left(|u^+ - u^-|\right)\, d\mathcal{H}^{n-1},$$

for any positive concave function θ such that $\lim_{t\to 0^+} \frac{\theta(t)}{t} = +\infty$.

1.3 Compendium of Chapter 5

In this Chapter we provide a variational approximation by discrete energies of functionals of the type

$$\mu \int_{\Omega\setminus K} |\mathcal{E}u(x)|^2\, dx + \frac{\lambda}{2} \int_{\Omega\setminus K} |\operatorname{div} u(x)|^2\, dx + \int_K \Phi(u^+ - u^-, \nu)\, d\mathcal{H}^{n-1} \tag{1.3.1}$$

defined for every closed hypersurface $K \subseteq \Omega$ with normal ν and $u \in \mathcal{C}^1(\Omega \setminus K; \mathbf{R}^n)$, where $\Omega \subseteq \mathbf{R}^n$ is a bounded domain of $\mathbf{R}^n$. Here $\mathcal{E}u = \frac{1}{2}(\nabla u + \nabla^t u)$ denotes the symmetric part of the gradient of u, $u^\pm$ are the one-sided traces of u on K and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

These functionals are related to variational models in fracture mechanics for linearly elastic materials in the framework of Griffith's theory of brittle fracture (see [77]). In this context u represents the displacement field of the body, with Ω as a reference configuration. The volume term in (1.3.1) represents the bulk energy of the body in the "solid region", where linear elasticity is supposed to hold, μ, λ being the Lamé constants of the material. The surface term is the energy necessary to produce the fracture, proportional to the crack surface K in the isotropic case and, in general, depending on the normal ν to K and on the jump $u^+ - u^-$.

The weak formulation of the problem leads to functionals of the type

$$\mu \int_\Omega |\mathcal{E}u(x)|^2\, dx + \frac{\lambda}{2}\int_\Omega |\operatorname{div} u(x)|^2\, dx + \int_{J_u} \Phi(u^+ - u^-, \nu_u)\, d\mathcal{H}^{n-1} \tag{1.3.2}$$

defined on the space $SBD(\Omega)$ of special functions of bounded deformation on Ω.

The description of continuum models in Fracture Mechanics as variational limits of discrete systems has been the object of recent research (see [38],[43],[44],[45],[84] and [92]). In particular, in [44] an asymptotic analysis has been performed for discrete energies of the form

$$\mathcal{H}_\varepsilon(u) = \sum_{\alpha,\beta\in R_\varepsilon,\ \alpha\neq\beta} \Psi_\varepsilon(u(\alpha) - u\,(\beta)\,, \alpha-\beta), \tag{1.3.3}$$

where R_ε is the portion of the lattice $\varepsilon\mathbf{Z}^n$ of step size $\varepsilon > 0$ contained in Ω and $u : R_\varepsilon \to \mathbf{R}^n$ may be interpreted as the displacement of a particle parameterized by $\alpha \in R_\varepsilon$. In this model the energy of the system is obtained by superposition of energies which take into account pairwise interactions, according to the classical theory of crystalline structures. Upon identifying u in (1.3.3) with the function in L^1 constant on each cell of the lattice $\varepsilon\mathbf{Z}^n$, the asymptotic behaviour of functionals $\mathcal{H}_\varepsilon$ can be studied in the framework of Γ-convergence of energies defined on L^1. Actually, as shown in Proposition 5.1.14, we may as well identify u with any piecewise-affine function which is obtained on each cell of the lattice as a convex interpolation of the values of u on the nodes of the cell itself.

A complete study of the asymptotic behaviour of energies in (1.3.3) has been developed when u is scalar-valued; in this setting the proper space where the limit energies are defined is the one of SBV functions. An important model case is when $\Psi_\varepsilon(z,w) = \rho(\frac{w}{\varepsilon})\varepsilon^{n-1} f\left(\frac{|z|^2}{\varepsilon}\right)$, so that we may rewrite $\mathcal{H}_\varepsilon$ as

$$\sum_{\xi\in\mathbf{Z}^n} \rho(\xi) \sum_{\alpha\in R_\varepsilon^\xi} \varepsilon^{n-1} f\left(\varepsilon|D_{\varepsilon,\xi}u(\alpha)|^2\right),$$

where R_ε^ξ is a suitable portion of R_ε and the symbol $D_{\varepsilon,\xi}u(\alpha)$ denotes the difference quotient $\frac{1}{\varepsilon}\left(u(\alpha+\varepsilon\xi) - u(\alpha)\right)$. Functionals of this type have been studied also in [49] in the framework of Computer Vision. In [49] and, in a general framework, in [44] it has been proved that, if f is any increasing function with $f(0) = 0$, $f'(0^+) > 0$ and $f(+\infty) < +\infty$, ρ is a positive function with suitable summability and symmetry properties, then $\mathcal{H}_\varepsilon$ approximates functionals of the type

$$c\int_\Omega |\nabla u(x)|^2\, dx + \int_{J_u} \Phi(u^+ - u^-, \nu_u)\, d\mathcal{H}^{n-1} \tag{1.3.4}$$

defined for $u \in SBV(\Omega)$, which are formally very similar to that in (1.3.2) (see also [50],[87] for a finite element approximation of the Mumford-Shah functional).

Following this approach, in order to approximate (1.3.2), one may think to "symmetrize" the effect of the difference quotient by considering the family of functionals

$$\sum_{\xi \in \mathbf{Z}^n} \rho(\xi) \sum_{\alpha \in R^\xi_\varepsilon} \varepsilon^{n-1} f\left(\varepsilon |\langle D_{\varepsilon,\xi} u(\alpha), \xi\rangle|^2\right)$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbf{R}^n$.

By letting ε tend to 0, we obtain as limit a proper subclass of functionals (1.3.2). Indeed, the two coefficients μ and λ of the limit functionals are related by a fixed ratio. This limitation corresponds to the well-known fact that pairwise interactions produce only particular choices of the Lamé constants (see [33]).

To overcome this difficulty we are forced to take into account in the model non-central interactions, in such a way that the energy contribution due to each pair of interacting points depends only on the projection of their relative displacement onto their difference vector in the reference configuration. The idea underlying our approach is to introduce a suitable discretization of the divergence, call it $\mathrm{Div}_{\varepsilon,\xi} u$, that takes into account also interactions in directions orthogonal to ξ, and to consider functionals of the form

$$\sum_{\xi \in \mathbf{Z}^n} \rho(\xi) \sum_{\alpha \in R^\xi_\varepsilon} \varepsilon^{n-1} f\left(\varepsilon \left(|D_{\varepsilon,\xi} u(\alpha)|^2 + \theta |\mathrm{Div}_{\varepsilon,\xi} u(\alpha)|^2\right)\right), \tag{1.3.5}$$

with θ a strictly positive parameter (for more precise definitions see Sections 5.1 and 5.6). In Theorem 5.1.1 we prove that with suitable choices of f, ρ and θ we can approximate functionals of type (1.3.2) in dimension 2 and 3 with arbitrary μ, λ and Φ satisfying some symmetry properties due to the geometry of the lattice. Actually, the general form of the limit functional is the following

$$\int_\Omega W(\mathcal{E}u(x))\,dx + c\int_\Omega |\mathrm{div}\, u(x)|^2\,dx + \int_{J_u} \Phi(u^+ - u^-, \nu_u)\,d\mathcal{H}^{n-1} \tag{1.3.6}$$

with W explicitly given; in particular we may choose $W(\mathcal{E}u(x)) = \mu|\mathcal{E}u(x)|^2$ and $c = \frac{\lambda}{2}$. We underline that the energy density of the limit surface term is always anisotropic due to the symmetries of the lattices $\varepsilon\mathbf{Z}^n$. The dependence on $u^+ - u^-$ and ν_u arises in a natural way from the discretizations chosen and the vectorial framework of the problem.

To drop the anisotropy of the limit surface energy we consider as well a continuous version of the approximating functionals (1.3.5) given by

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \frac{1}{\varepsilon} f\left(\varepsilon\left(|\langle D_{\varepsilon,\xi} u(x), \xi\rangle|^2 + \theta|\mathrm{Div}_{\varepsilon,\xi} u(x)|^2\right)\right) \rho(\xi)\,dx\,d\xi,$$

where in this case ρ is a symmetric convolution kernel which corresponds to a polycrystalline approach. By varying f, ρ and θ, as stated in Theorem 5.1.8, we obtain as limit functionals of the form

$$\mu\int_\Omega |\mathcal{E}u(x)|^2\,dx + \frac{\lambda}{2}\int_\Omega |\mathrm{div}\, u(x)|^2\,dx + \gamma\mathcal{H}^{n-1}(J_u) \tag{1.3.7}$$

for any choice of positive constants μ, λ and γ. This continuous model generalizes the one proposed by E. De Giorgi and studied by M. Gobbino in [75], to approximate the Mumford-Shah functional. Let us also mention that a finite element approximation of energies as in (1.3.7) has been proposed in [88] in case $n = 2$.

The main technical issue of the Chapter is that, in the proof of the discrete approximation from which we recover the continuous one by a discretization procedure, we cannot reduce to the one-dimensional case by an integral-geometric approach as in [44],[49],[75], due to the presence of the divergence term. Hence, a direct n-dimensional approach must be followed, by comparing each functional

$$\sum_{\alpha \in R_\varepsilon^\xi} \varepsilon^{n-1} f\left(\varepsilon\left(|D_{\varepsilon,\xi}u(\alpha)|^2 + \theta|\mathrm{Div}_{\varepsilon,\xi}u(\alpha)|^2\right)\right),$$

with a suitable n-dimensional energy defined on $SBD(\Omega)$.

For a deeper insight of the techniques used we refer to Sections 5.3 and 5.4, we just underline that the proofs of the two approximations (discrete and continuous) are strictly related.

Analogously to [44], in Section 5.5 we treat boundary value problems in the discrete scheme for the two-dimensional case and a convergence result for such problems is derived (see Proposition 5.5.3 and Theorem 5.5.4).

1.4 Compendium of Chapter 6

In this Chapter we provide a variational approximation for energies defined on functions u in $SBV\left(\Omega; \mathbf{R}^N\right)$ by

$$\int_\Omega \psi\left(\nabla u\right)\, dx + \int_{S_u} g(u^+ - u^-)\phi(\nu_u)\, d\mathcal{H}^2, \tag{1.4.8}$$

where Ω is an open bounded set of $\mathbf{R}^3$, $\psi : \mathbf{R}^{N\times 3} \to [0, +\infty)$, $g : \mathbf{R}^N \setminus \{0\} \to [0, +\infty)$ and $\phi : \mathbf{S}^2 \to [0, +\infty)$ are assigned.

These models derive from the theory of brittle fracture for hyperelastic materials. For such materials the elastic deformation outside the fracture can be modeled by an elastic energy density independent of the crack. The assumptions required on ψ are quasiconvexity and superlinear growth; while, for what the surface term is concerned, g is a subadditive and continuous function superlinear at 0 and $\phi(\nu) = \sum_{\ell=1}^{3} |\langle \nu, \mathrm{e}_\ell\rangle|$ (for more details see Chapter 6).

Our approximation relies on finite-differences discretization schemes, following the approach proposed by De Giorgi to treat the Mumford-Shah's problem in Computer Vision (see [75]), and applied to Fracture Mechanics firstly by Braides, Dal Maso and Garroni [43] and then by Braides and Gelli [44],[45], in order to deduce continuum theories starting from an atomistic description of the media (see also [8],[92]),

While the previous results mainly study the scalar case, here we deal with the vectorial one. In this Chapter we prove the Γ-convergence, with respect to both the convergence in measure on Ω and $L^1\left(\Omega;\mathbf{R}^N\right)$, to energies of type (1.4.8) of the family of approximating functionals defined as

$$\int_{\mathcal{T}_\varepsilon\cap\Omega}\psi_\varepsilon\left(\nabla u(x)\right)\,dx, \tag{1.4.9}$$

where $\mathcal{T}_\varepsilon$ is a regular triangulation of $\mathbf{R}^3$, ψ_ε is a suitable non-convex interaction potential and $u:\mathbf{R}^3\to\mathbf{R}^N$ is continuous and affine on each element of $\mathcal{T}_\varepsilon$.

The main problem in the vectorial case is to give a definition of discrete schemes that is consistent with the 'discrete method', i.e., find a suitable 'ε-discretization of the gradient', $D_\varepsilon u$, by finite-differences, and find proper potentials ψ_ε, in order to obtain, by means of a separation of scales, the assigned bulk density ψ, and the corresponding surface one.

Since we are interested in non-isotropic bulk energy densities, a quite natural choice for $D_\varepsilon u$, in order to recover the global behaviour of the gradient matrix, is the finite-differences matrix below

$$D_\varepsilon u=\frac{1}{\varepsilon}(\langle u(\alpha+\varepsilon\mathrm{e}_\ell)-u(\alpha),e_k\rangle)_{\substack{\ell=1,2,3\\ k=1,\dots,N}}. \tag{1.4.10}$$

Let us remark that the gradient ∇u in (1.4.9) coincides exactly with the matrix $D_\varepsilon u$ defined above, since we choose to identify a 'discrete function' u, i.e., defined on the nodes of the simplices of the triangulation $\mathcal{T}_\varepsilon$, with its continuous piecewise affine interpolation, still denoted by u. This is done only for simplicity of notation (see Proposition 5.1.14 and also Compendium of Chapter 5).

The scalar models considered so far are based on discretizations, $D^\ell_\varepsilon u$, accounting for increments only along given integer directions, i.e., $D_\varepsilon u$ in (1.4.10) has to be replaced by

$$\frac{1}{\varepsilon}(u(\alpha+\varepsilon\mathrm{e}_\ell)-u(\alpha)).$$

In addition, in the case of linear elasticity (see Chapter 5), $D_\varepsilon u$ is chosen to be the projection along a fixed direction of the increment of u in the same direction, i.e.,

$$\frac{1}{\varepsilon}\langle u(\alpha+\varepsilon\mathrm{e}_\ell)-u(\alpha),\mathrm{e}_\ell\rangle.$$

Both these approaches allow to get a complete characterization of the limit by studying the asymptotic behaviour of one-dimensional functionals. On the other hand, the only possible bulk energy densities obtained as limits are those determined by summing up all the contribution on fixed directions.

We overcome this drawback by defining $\psi_\varepsilon:\mathbf{R}^{N\times 3}\to[0,+\infty)$ as

$$\psi_\varepsilon\left(X\right):=\begin{cases}\psi\left(X\right) & \text{if } |X|\le\lambda_\varepsilon\\[2mm] \dfrac{1}{\varepsilon}\displaystyle\sum_{\ell=1}^{3}g\left(\varepsilon X\mathrm{e}_\ell\right) & \text{otherwise,}\end{cases} \tag{1.4.11}$$

where $(\lambda_\varepsilon)\subset[0,+\infty)$ is such that $\lambda_\varepsilon\to+\infty$ as $\varepsilon\to 0^+$ and $\sup_{\varepsilon>0}\left(\varepsilon\lambda^p_\varepsilon\right)<+\infty$.

The lower bound inequality is obtained by proving separate estimates on the bulk and surface terms. With given u_ε converging to u in measure on Ω, consider the sets of simplices

$$\mathcal{N}_\varepsilon:=\{T\in\mathcal{T}_\varepsilon:\ |\nabla u_\varepsilon||_T>\lambda_\varepsilon\},$$

we show that the family $(\mathcal{N}_\varepsilon)$ detects the discontinuity set of the limiting function u, and that there holds

$$\liminf_{\varepsilon\to 0^+} \int_{(\mathcal{T}_\varepsilon\setminus\mathcal{N}_\varepsilon)\cap\Omega} \psi_\varepsilon\left(\nabla u_\varepsilon\right)\, dx \geq \int_\Omega \psi(\nabla u)\, dx, \tag{1.4.12}$$

and

$$\liminf_{\varepsilon\to 0^+} \int_{\mathcal{N}_\varepsilon\cap\Omega} \psi_\varepsilon\left(\nabla u_\varepsilon\right)\, dx \geq \int_{S_u} g\left(u^+ - u^-\right) |\langle \nu_u, \mathrm{e}_\ell\rangle|\, d\mathcal{H}^2. \tag{1.4.13}$$

To this aim we construct a sequence $(v_\varepsilon) \subset SBV\left(\Omega;\mathbf{R}^N\right)$ such that v_ε converges to u in measure on Ω, (v_ε) satisfies locally all the assumptions of the GSBV Closure Theorem 2.7.10 and

$$\liminf_{\varepsilon\to 0^+} \int_{(\mathcal{T}_\varepsilon\setminus\mathcal{N}_\varepsilon)\cap\Omega} \psi_\varepsilon\left(\nabla u_\varepsilon\right)\, dx \geq \liminf_{\varepsilon\to 0^+} \int_\Omega \psi(\nabla v_\varepsilon)\, dx.$$

Hence, we infer $u \in (GSBV(\Omega))^N$ and the bulk inequality (1.4.12) thanks to the GSBV lower semicontinuity Theorem 2.7.17. The function v_ε coincides with u_ε on $\mathcal{T}_\varepsilon \setminus \mathcal{N}_\varepsilon$ and is constant on $\mathcal{N}_\varepsilon$ in such a way that the measures of the jump sets S_{u_ε} are uniformly bounded.

A similar technique is used to prove the surface energy inequality (1.4.13), but, with fixed $\ell \in \{1,2,3\}$, comparing the energy of (u_ε) on $\mathcal{N}_\varepsilon$ with the corresponding one of a sequence with one-dimensional profile along e_ℓ, which is locally pre-compact in SBV in this given direction, but in general not globally in $GSBV$ (see Proposition 6.1.3).

As usual the upper bound inequality is first proven for regular functions and then the conclusion follows by a density argument.

Eventually, in Section 6.2 we consider the two-dimensional setting for which we exhibit two approximation results related to different definitions of ψ_ε.

The first model is the formulation in the two-dimensional case of the result discussed previously. The same techniques can be used to prove a slightly more general result. Indeed, one can work out that the Γ-convergence does not depend on the triangulation of $\mathbf{R}^2$ chosen.

The second model is related to functionals defined on $SBV\left(\Omega;\mathbf{R}^N\right)$ of the form

$$\int_\Omega \psi(\nabla u) dx + \beta \int_{S_u} \varphi(\nu_u)\, d\mathcal{H}^1,$$

where $\beta > 0$ and $\varphi : \mathbf{S}^1 \to [0,+\infty)$ is given by

$$\varphi(\nu) := \begin{cases} |\langle \nu, \mathrm{e}_1\rangle| \vee |\langle \nu, \mathrm{e}_2\rangle| & \text{if } \langle \nu, \mathrm{e}_1\rangle\langle \nu, \mathrm{e}_2\rangle \geq 0 \\ \\ |\langle \nu, \mathrm{e}_1\rangle| + |\langle \nu, \mathrm{e}_2\rangle| & \text{if } \langle \nu, \mathrm{e}_1\rangle\langle \nu, \mathrm{e}_2\rangle < 0. \end{cases}$$

We point out that the surface term is anisotropic and penalizes crack sites in different ways, according to their orientation with respect to the basis $\{\mathrm{e}_1, \mathrm{e}_2\}$.

The family of approximating functionals is defined as in (1.4.9), with the function $\psi_\varepsilon : \mathbf{R}^{N\times 2} \to [0,+\infty)$ now given by

$$\psi_\varepsilon\left(X\right) := \begin{cases} \psi\left(X\right) & \text{if } |X| \leq \lambda_\varepsilon \\ \\ \dfrac{1}{\varepsilon}\beta & \text{otherwise.} \end{cases}$$

Such a model requires more sophisticated tools (see Lemma 2.2.4) and a new construction must be performed in order to have an estimate along direction $e_2 - e_1$, that is, a direction in which difference quotients are not involved. This difficulty can be bypassed by considering the lattice generated by the vectors $e_1, e_2 - e_1$ and by constructing a one-dimensional profiled function affine on the slanted unitary cell P_ε of such lattice (see Proposition 6.2.3 and compare it with Proposition 6.1.3).

Chapter 2

Preliminaries

2.1 Basic Notation

For every $t \in \mathbf{R}$, $[t]$ denotes its integer part. If $a, b \in \mathbf{R}$ we write $a \wedge b$ and $a \vee b$ for the minimum and maximum between a and b, respectively.

For $x, y \in \mathbf{R}^n$, $[x, y]$ denotes the segment between x and y.

Given $a \in \mathbf{R}^N$ and $b \in \mathbf{R}^n$, $a \otimes b$ is the matrix with entries equal to $a_i b^j$, $1 \leq i \leq N$ and $1 \leq j \leq n$.

We denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbf{R}^n$ and with $|\cdot|$ the usual euclidean norm, without specifying the dimension n when there is no risk of confusion. Given two sets A, $B \subseteq \mathbf{R}^n$ the distance of A from B is defined as usual, i.e., $d(A, B) := \inf\{|a - b| : a \in A, b \in B\}$.

With fixed $\rho > 0$ and $x \in \mathbf{R}^n$ set $B_\rho(x) := \{y \in \mathbf{R}^n : |y - x| < \rho\}$; and we denote by $\mathbf{S}^{n-1}$ the boundary of the unit ball centered in the origin. Moreover, given $\nu \in \mathbf{S}^{n-1}$, Q_ν is the unitary cube with two faces parallel to ν.

Let n, $k \in \mathbf{N}$, then $\mathcal{L}^n$ denotes the Lebesgue measure and $\mathcal{H}^k$ denotes the k dimensional Hausdorff measure in $\mathbf{R}^n$. The notation *a.e.* stands for almost everywhere with respect to the Lebesgue measure, unless otherwise specified.

In the sequel Ω will always be an open set of $\mathbf{R}^n$. Denote by $\mathcal{A}(\Omega)$, $\mathcal{B}(\Omega)$ the families of open and Borel subsets of Ω, respectively; and by $\mathcal{X}_B$ the characteristic function of the set $B \in \mathcal{B}(\Omega)$.

Define

$$\mathcal{B}\left(\Omega; \mathbf{R}^N\right) := \left\{u : \Omega \to \mathbf{R}^N : u \text{ is a Borel function}\right\},$$

and recall the following notion:

Definition 2.1.1 *We say that a sequence* $(u_j) \subset \mathcal{B}\left(\Omega; \mathbf{R}^N\right)$ converges to $u \in \mathcal{B}\left(\Omega; \mathbf{R}^N\right)$ in measure on Ω *if for every* $\eta > 0$ *we have*

$$\lim_j \mathcal{L}^n\left(\{x \in \Omega : |u_j(x) - u(x)| > \eta\}\right) = 0.$$

In case Ω is a set of finite $\mathcal{L}^n$ measure, such a convergence is induced by the metric defined as

$$d(u, v) := \int_\Omega \frac{|u - v|}{1 + |u - v|}\, dx \tag{2.1.1}$$

for every $u, v \in \mathcal{B}\left(\Omega;\mathbf{R}^N\right)$.

We use standard notation for Lebesgue and Sobolev spaces.

2.2 Overview of measure theory

The set of all Borel measures $\mu : \mathcal{B}(\Omega) \to \mathbf{R}^N$ is denoted by $\mathcal{M}(\Omega;\mathbf{R}^N)$. If $N = 1$ we simply write $\mathcal{M}(\Omega)$ instead of $\mathcal{M}(\Omega;\mathbf{R})$. In the sequel, we will always understand measures as a Borel measures, unless otherwise specified.

If $B \in \mathcal{B}(\Omega)$, then the measure $\mu \llcorner B$ is defined as $\mu \llcorner B(A) = \mu(A \cap B)$. For any $\mu \in \mathcal{M}(\Omega;\mathbf{R}^N)$ we will indicate by $\|\mu\|$ the total variation of μ, that is the positive finite measure defined by

$$\|\mu\|(B) := \sup\Big\{\sum_{i\in\mathbf{N}} |\mu(B_i)| : \ B_i \text{ disjoint}, \ B = \bigcup_i B_i\Big\}$$

for any $B \in \mathcal{B}(\Omega)$. $\mathcal{M}(\Omega;\mathbf{R}^N)$ is a Banach space when equipped with the norm $\|\mu\| := \|\mu\|(\Omega)$ and it is the dual of $C_0\left(\Omega;\mathbf{R}^N\right)$, closure with respect to the uniform convergence of the space $C_c(\Omega;\mathbf{R}^N)$ of continuous functions with compact support in Ω. By virtue of the duality above, a notion of weak $*$ convergence on $\mathcal{M}(\Omega;\mathbf{R}^N)$ can be introduced:

Definition 2.2.1 *We say that a sequence $(\mu_j) \subset \mathcal{M}(\Omega;\mathbf{R}^N)$* converges weakly $*$ to μ (in the sense of measures) *if for any $\varphi \in C_0(\Omega;\mathbf{R}^N)$*

$$\lim_{j\to+\infty} \int_\Omega \varphi\, d\mu_j = \int_\Omega \varphi\, d\mu.$$

Note that by the lower semicontinuity of the dual norm with respect to weak $*$ convergence we have that $\mu \mapsto \|\mu\|(\Omega)$ is weakly lower semicontinuous, i.e., $\|\mu\|(\Omega) \le \liminf_j \|\mu_j\|(\Omega)$ if (μ_j) converges weakly $*$ to μ.

In the following proposition we collect some results concerning weak convergence in the sense of measures.

Proposition 2.2.2 *(1) Let $(\mu_j) \subset \mathcal{M}(\Omega;\mathbf{R}^N)$ be a sequence of Radon measures converging weakly $*$ to μ. If $\|\mu_j\|$ converges weakly $*$ to λ, then $\lambda \ge \|\mu\|$.*

Moreover, for every $B \in \mathcal{B}(\Omega)$ such that $\lambda(\partial B) = 0$, then $\lim_j \mu_j(B) = \mu(B)$.

(2) Let $(\mu_j) \subset \mathcal{M}(\Omega)$ be a sequence of positive Radon measures converging weakly $*$ to μ. Then, for every upper semicontinuous function $v : \Omega \to [0,+\infty)$ with compact support

$$\limsup_j \int_\Omega v\, d\mu_j \le \int_\Omega v\, d\mu.$$

(3) Let $(\mu_j) \subset \mathcal{M}(\Omega)$ be a sequence of positive Radon measures, and assume the existence of a positive finite Radon measure μ on Ω such that

$$\lim_j \mu_j(\Omega) = \mu(\Omega); \quad \lim_j \mu_j(A) \ge \mu(A)$$

for every $A \in \mathcal{A}(\Omega)$. Then (μ_j) converges weakly $*$ to μ.

The following celebrated result of De Giorgi and Letta gives a criterion to establish when an increasing set function defined on $\mathcal{A}(\Omega)$ is the trace of a positive measure (see Theorem 1.53 [20]).

Lemma 2.2.3 *Let $\lambda : \mathcal{A}(\Omega) \to [0, +\infty]$ be an increasing set function such that $\lambda(\emptyset) = 0$. Then λ is the trace on $\mathcal{A}(\Omega)$ of a Borel measure if and only if*

(i) Superadditivity: *for every A, $A' \in \mathcal{A}(\Omega)$ such that $A \cap A' = \emptyset$*

$$\lambda(A \cup A') \geq \lambda(A) + \lambda(A');$$

(ii) Subadditivity: *for every A, $A' \in \mathcal{A}(\Omega)$*

$$\lambda(A \cup A') \leq \lambda(A) + \lambda(A');$$

(iii) Inner regularity: *for every $A \in \mathcal{A}(\Omega)$*

$$\lambda(A) = \sup\{\lambda(A') : A' \in \mathcal{A}(\Omega),\ A' \subset\subset A\}.$$

Eventually, we include the following proposition on the supremum of a family of measures which will be useful in the sequel and that can be easily deduced from the regularity properties of positive measures (see Proposition 1.16 [37]).

Lemma 2.2.4 *Let $\lambda : \mathcal{A}(\Omega) \to [0, +\infty)$ be a superadditive function on disjoint open sets, let μ be a positive measure on Ω and let $\psi_j : \Omega \to [0, +\infty]$ be a countable family of Borel functions such that $\lambda(A) \geq \int_A \psi_j \, d\mu$ for every $A \in \mathcal{A}(\Omega)$.*

Set $\psi = \sup_{j \in \mathbf{N}} \psi_j$, then

$$\lambda(A) \geq \int_A \psi \, d\mu$$

for every $A \in \mathcal{A}(\Omega)$.

2.3 Γ-convergence

In this section we introduce the notions of Γ-convergence and state its main properties. For a detailed introduction to this subject we refer to Dal Maso [53] (see also [26],[37],[39],[41]).

In what follows $X = (X, d)$ denotes a metric space. In 1975 De Giorgi and Franzoni [59] introduced a very general notion of variational convergence which turned out to be a useful tool to study many "limit problems" in the Calculus of Variations.

Definition 2.3.1 *We say that a sequence $F_j : X \to [-\infty, +\infty]$* Γ-converges *to $F : X \to [-\infty, +\infty]$, and we write $F(u) = \Gamma\text{-}\lim_j F_j(u)$, if for all $u \in X$ the following two conditions hold:*

(LB) Lower Bound inequality*: for every sequence $u_j \xrightarrow{d} u$ there holds*

$$F(u) \leq \liminf_j F_j(u_j); \tag{2.3.1}$$

(UB) Upper Bound inequality*: there exists a sequence $u_j \xrightarrow{d} u$ such that*

$$F(u) \geq \limsup_j F_j(u_j). \tag{2.3.2}$$

The function F is uniquely determined by conditions (LB) and (UB) and it is called the Γ-limit *of (F_j).*

Moreover, given a family of functions (F_ε) labelled by a real parameter $\varepsilon > 0$, we say that F_ε Γ-converges *to F if F is the Γ-limit of $\left(F_{\varepsilon_j}\right)$ for every sequence $\varepsilon_j \to 0^+$.*

We call *recovery sequence* any sequence satisfying (2.3.2); for such a sequence, combining (2.3.1) and (2.3.2), there holds

$$F(u) = \lim_j F_j\left(u_j\right).$$

The main properties of Γ-convergence are listed in the following theorem. In particular, statement *(iii)* below explains why the notion of Γ-convergence is convenient in the study of the asymptotic analysis of variational problems.

Theorem 2.3.2 *Let F_ε, $F : X \to [-\infty, +\infty]$ be such that $\Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon = F$, then*

(i) Lower semicontinuity*: F is d-lower semicontinuous on X;*

(ii) Stability under continuous perturbations: *if $G : X \to \mathbf{R}$ is continuous, then*

$$\Gamma\text{-}\lim_{\varepsilon\to 0^+} (F_\varepsilon + G) = F + G;$$

(iii) Stability of minimizing sequences: *if (u_ε) is asymptotically minimizing, i.e.,*

$$\lim_{\varepsilon\to 0^+} \left(F_\varepsilon\left(u_\varepsilon\right) - \inf_X F_\varepsilon\right) = 0,$$

then every cluster point u of (u_ε) minimizes F over X, and

$$\lim_{\varepsilon\to 0^+} \inf_X F_\varepsilon = F(u) \left(= \min_X F\right). \tag{2.3.3}$$

More generally, we introduce the notions of lower and upper Γ-limits.

Definition 2.3.3 *Given a family (F_ε) of functions and $u \in X$, we define the* lower and upper Γ-limits *by*

$$F'(u) = \Gamma\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon(u) = \inf\left\{\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) :\ u_\varepsilon \xrightarrow{d} u\right\};$$
$$F''(u) = \Gamma\text{-}\limsup_{\varepsilon\to 0^+} F_\varepsilon(u) = \inf\left\{\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) :\ u_\varepsilon \xrightarrow{d} u\right\},$$

respectively.

It can be easily proved that the functions F' and F'' are d-lower semicontinuous. Notice that, conditions (LB) and (UB) are equivalent to $F'(u) = F''(u) = F(u)$ for all $u \in X$.

In the sequel we will sometimes write $\Gamma(d)$-$\liminf$, $\Gamma(d)$-$\limsup$ and $\Gamma(d)$-$\lim$ to emphasize the dependence on the metric d with respect to which the convergence is taken. In particular, we will write $\Gamma(meas)$ in case the metric d is the one in (2.1.1) inducing the convergence in measure.

The so called Uryshon's property holds for Γ-convergence.

Proposition 2.3.4 *A family (F_ε) Γ-converges to F if and only if for every $\varepsilon_j \to 0^+$ the subsequence (F_{ε_j}) contains a further subsequence which Γ-converges to F.*

Let us now recall the notion of relaxed functional.

Definition 2.3.5 *Let $F : X \to [-\infty, +\infty]$. Then the* relaxed functional $\overline{F} : X \to [-\infty, +\infty]$ *of F, or* relaxation *of F, is the greatest d-lower semicontinuous functional less than or equal to F, i.e., for every $u \in X$*

$$\overline{F}(u) = \sup\{G(u) : G \text{ } d\text{-lower semicontinuous}, G \leq F\}.$$

We remark that relaxation theory can be studied as a particular case of Γ-convergence. Indeed, consider the constant sequence $F_j \equiv F$, then the relaxation $\overline{F}$ of F can be characterized as follows

$$\overline{F} = \Gamma\text{-}\lim_j F_j,$$

that is $\overline{F}(u) = \inf\{\liminf_j F(u_j) : \ u_j \to u\}$ for every $u \in X$.

2.3.1 $\overline{\Gamma}$-convergence

In this subsection we recall the notion of $\overline{\Gamma}$-convergence, which is useful when dealing with the integral representation of the Γ-limit of a family of integral functionals (see Chapter 16 [53]).

Definition 2.3.6 *Let $F_\varepsilon : X \times \mathcal{A}(\Omega) \to [0, +\infty]$ be such that for every $u \in X$ the set function $F_\varepsilon(u, \cdot)$ is increasing on $\mathcal{A}(\Omega)$ and define*

$$F'(\cdot, A) := \Gamma\text{-}\liminf_{\varepsilon \to 0^+} F_\varepsilon(\cdot, A), \quad F''(\cdot, A) := \Gamma\text{-}\limsup_{\varepsilon \to 0^+} F_\varepsilon(\cdot, A)$$

for every $A \in \mathcal{A}(\Omega)$. We say that (F_ε) $\overline{\Gamma}$-converges *to $F : X \times \mathcal{A}(\Omega) \to [0, +\infty]$, if F is the inner regular envelope of both functionals F' and F'', i.e.,*

$$\begin{aligned} F(u, A) &= \sup\{F'(u, A') : A' \in \mathcal{A}(\Omega), A' \subset\subset A\} \\ &= \sup\{F''(u, A') : A' \in \mathcal{A}(\Omega), A' \subset\subset A\}, \end{aligned}$$

for every $(u, A) \in X \times \mathcal{A}(\Omega)$.

The following theorem shows that $\overline{\Gamma}$-convergence enjoys useful compactness properties.

Theorem 2.3.7 *Every sequence $F_j : X \times \mathcal{A}(\Omega) \to [0, +\infty]$ has a $\overline{\Gamma}$-convergent subsequence.*

The following results give us a criterion to establish when the $\overline{\Gamma}$-limit, as a set function, is a Borel measure. We recall that, according to the De Giorgi-Letta's Lemma 2.2.3, an increasing set function $\lambda : \mathcal{A}(\Omega) \to [0, +\infty]$ is a measure if and only if it is superadditive, subadditive and inner regular.

Proposition 2.3.8 *Let $F_j : X \times \mathcal{A}(\Omega) \to [0, +\infty]$ be such that $F_j(u, \cdot)$ is increasing and superadditive. Then both $F'(u, \cdot)$ and its inner regular envelope are superadditive.*

In particular, if (F_j) $\overline{\Gamma}$-converges to F, then $F(u, \cdot)$ is superadditive.

Proposition 2.3.9 *Let $F_j : X \times \mathcal{A}(\Omega) \to [0, +\infty]$ be such that*

$$F''(u, A' \cup B) \leq F''(u, A) + F''(u, B)$$

for every $u \in X$ and for every A', A, $B \in \mathcal{A}(\Omega)$ with $A' \subset\subset A$. Then the inner regular envelope of $F''(u, \cdot)$ is subadditive.

In particular, if (F_j) $\overline{\Gamma}$-converge to F, then $F(u, \cdot)$ is subadditive.

If, in addition, there exists $G : X \times \mathcal{A}(\Omega) \to [0, +\infty]$ such that $G(u, \cdot)$ is a measure and $F'' \leq G$, then $F''(u, \cdot)$ coincides with its inner regular envelope for every $A \in \mathcal{A}(\Omega)$ for which $G(u, A) < +\infty$.

In particular if (F_j) $\overline{\Gamma}$-converges to F, then

$$F(u, A) = \Gamma\text{-}\lim_j F_j(u, A)$$

for every $A \in \mathcal{A}(\Omega)$ such that $G(u, A) < +\infty$.

2.4 Rectifiable sets

Let us recall an important measure theoretic property of sets and some results concerning it.

Definition 2.4.1 *Given a Borel set $J \subset \mathbf{R}^n$, we say that J is* countably $\mathcal{H}^{n-1}$ rectifiable *if*

$$J = N \cup \bigcup_{i \geq 1} K_i$$

where $\mathcal{H}^{n-1}(N) = 0$ and each K_i is a compact subset of a C^1 $(n-1)$-dimensional manifold.

For rectifiable sets a generalization of the coarea formula holds true (see [63]). Let us fix some notation. Let $\xi \in \mathbf{S}^{n-1}$ be a fixed direction, denote by Π^ξ be the orthogonal space to ξ, i.e., $\Pi^\xi = \{y \in \mathbf{R}^n : \langle \xi, y \rangle = 0\}$, and by $\pi_\xi : \mathbf{R}^n \to \Pi^\xi$ the orthogonal projection. If $y \in \Pi^\xi$ and $E \subset \mathbf{R}^n$ define $E^\xi_y = \{t \in \mathbf{R} : y + t\xi \in E\}$ and $E_\xi = \{y \in \Pi^\xi : E^\xi_y \neq \emptyset\}$. Moreover, given $g : E \to \mathbf{R}^N$ define, for $y \in E_\xi$, $g_{\xi,y} : E^\xi_y \to \mathbf{R}^N$ by

$$g_{\xi,y}(t) := g(y + t\xi).$$

Lemma 2.4.2 *For every countably $\mathcal{H}^{n-1}$ rectifiable set $J \subset \mathbf{R}^n$ there exists a Borel function $\nu_J : J \to \mathbf{S}^{n-1}$ such that for every $\xi \in \mathbf{S}^{n-1}$, $A \in \mathcal{A}(\Omega)$ and $g \in L^1\left(J; \mathcal{H}^{n-1}\right)$ there holds*

$$\int_{J\cap A} g(x)|\langle \nu_J(x), \xi\rangle| \, d\mathcal{H}^{n-1}(x) = \int_{A_\xi} \int_{J_y^\xi \cap A} g_{\xi,y}(t) \, d\mathcal{H}^0(t) \, d\mathcal{H}^{n-1}(y). \tag{2.4.1}$$

An interesting property of countably $\mathcal{H}^{n-1}$ rectifiable sets is that their $\mathcal{H}^{n-1}$ measure can be recovered from the $\mathcal{L}^{n-1}$ measure of their projections onto hyperplanes (see Proposition 2.66 [20]).

Proposition 2.4.3 *For any countably $\mathcal{H}^{n-1}$ rectifiable set $J \subset \mathbf{R}^n$, $\mathcal{H}^{n-1}(J)$ is equal to*

$$\sup \left\{ \sum_{i=1}^{N} \mathcal{L}^{n-1}\left(\pi_{\xi_i}(K_i)\right) : K_i \subseteq J \text{ compact pairwise disjoint, } \xi_i \in \mathbf{S}^{n-1} \right\}.$$

2.5 Approximate limits and approximate differentials

Let $\mathbf{S} = \mathbf{R}^N \cup \{\infty\}$ be the one point compactification of $\mathbf{R}^N$.

Definition 2.5.1 *Let $B \in \mathcal{B}(\Omega)$ such that $\mathcal{L}^n\left(B_\rho(x) \cap B\right) > 0$ for every $\rho > 0$. We say that $z \in \mathbf{S}$ is the* approximate limit *in $x \in \Omega$ of $u \in \mathcal{B}\left(\Omega; \mathbf{R}^N\right)$* in the domain B, *and we write $z = \mathrm{ap} - \lim_{\substack{y\to x \\ y\in B}} u(y)$, if for every neighbourhood U of z in $\mathbf{S}$ there holds*

$$\lim_{\rho \to 0^+} \frac{\mathcal{L}^n\left(\{y \in B_\rho(x) \cap B : u(y) \notin U\}\right)}{\mathcal{L}^n\left(B_\rho(x) \cap B\right)} = 0.$$

Denote by S_u the set of points where the approximate limit of u in Ω doesn't exists; it is well known that $\mathcal{L}^n\left(S_u\right) = 0$. Define the function $\tilde{u} : \Omega \setminus S_u \to S$ by

$$\tilde{u}(x) = \mathrm{ap} - \lim_{\substack{y\to x \\ y\in\Omega}} u(y),$$

thus u is equal a.e. on Ω to $\tilde{u}$. Notice that $\tilde{u}$ is allowed to take the value ∞ but $\mathcal{L}^n\left(\{\tilde{u} = \infty\}\right) = 0$.

Definition 2.5.2 *We say that $x \in \Omega$ is a* jump point *of u, and we write $x \in J_u$, if there exist a, $b \in \mathbf{S}$, and a vector $\nu \in \mathbf{S}^{n-1}$ such that $a \neq b$ and*

$$a = \mathrm{ap} - \lim_{\substack{y\to x \\ y\in\Pi_-^\nu(x)}} u(y), \quad b = \mathrm{ap} - \lim_{\substack{y\to x \\ y\in\Pi_+^\nu(x)}} u(y), \tag{2.5.1}$$

where $\Pi_\pm^\nu(x) = \{y \in \Omega : \pm\langle y - x, \nu\rangle > 0\}$.

The triplet (a, b, ν), uniquely determined by (2.5.1) up to a permutation of (a, b) and a change of sign of ν, will be denoted by $(u^+(x), u^-(x), \nu_u(x))$.

Moreover, if $x \in J_u$, the quantity $[u](x) := u^+(x) - u^-(x)$ is called the jump *of u at x.*

The definitions above, given in terms of approximate limits, take into account only the geometry of the level sets of u and don't need any local summability assumption. Let us recall that in case $u \in L^1_{loc}(\Omega; \mathbf{R}^N)$ we can define similar concepts by means of integral averages.

Indeed, if $u \in L^1_{loc}(\Omega; \mathbf{R}^N)$ the complement of the Lebesgue' set of u is denoted by $S^\star_u$, i.e., $x \notin S^\star_u$ if and only if

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{B_\rho(x)} |u(y) - z|\, dy = 0 \tag{2.5.2}$$

for some $z \in \mathbf{R}^N$. If z exists then it is unique, and we denote it by $\bar{u}(x)$. The set $S^\star_u$ is $\mathcal{L}^n$ negligible and $\bar{u}$ is a Borel function equal to u a.e. in Ω.

Moreover, we denote by $J^\star_u$ the set of all points $x \in S^\star_u$, for which there exist $a, b \in \mathbf{R}^N$ and $\nu \in \mathbf{S}^{n-1}$ such that $a \neq b$ and

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{B^+_\rho(x,\nu)} |u(y) - a|\, dy = 0, \qquad \lim_{\rho \to 0^+} \rho^{-n} \int_{B^-_\rho(x,\nu)} |u(y) - b|\, dy = 0, \tag{2.5.3}$$

where $B^\pm_\rho(x,\nu) := B_\rho(x) \cap \Pi^\nu_\pm(x)$. The triplet (a, b, ν), uniquely determined by (2.5.3) up to a permutation of (a, b) and a change of sign of ν, will be denoted by $(u^+_\star(x), u^-_\star(x), \nu^\star_u(x))$.

Notice that $J^\star_u$ is a Borel subset of $S^\star_u$, and the following inclusions hold true

$$J^\star_u \subseteq J_u \subseteq S_u \subseteq S^\star_u. \tag{2.5.4}$$

Hence, $\tilde{u} \equiv \bar{u}$ on $\Omega \setminus S^\star_u$.

It is easy both to prove that if $u \in L^\infty_{loc}\left(\Omega; \mathbf{R}^N\right)$ the notions introduced in (2.5.2) and (2.5.3) coincide with the ones of Definitions 2.5.1, 2.5.2, respectively; and to show counter-examples without that additional assumption.

We refer to Remark 2.6.10 for a deeper analisys in a more specific case.

We can also introduce a notion of approximate differentiability.

Definition 2.5.3 *We say that u is* approximately differentiable *at a point $x \in \Omega \setminus S_u$ such that $\tilde{u}(x) \neq \infty$, if there exists a matrix $L \in \mathbf{R}^{N \times n}$ such that*

$$\mathrm{ap} - \lim_{\substack{y \to x \\ y \in \Omega}} \frac{|u(y) - \tilde{u}(x) - L(y - x)|}{|y - x|} = 0. \tag{2.5.5}$$

If u is approximately differentiable at a point x, the matrix L uniquely determined by (2.5.5), will be denoted by $\nabla u(x)$ and will be called the approximate gradient *of u at x.*

Even for approximate differentials a stronger definition is available in terms of integral averages, i.e.,

$$\lim_{\rho \to 0^+} \rho^{-n-1} \int_{B_\rho(x)} |u(y) - \bar{u}(x) - L(y - x)|\, dy = 0 \tag{2.5.6}$$

with $x \in \Omega \setminus S^\star_u$ and for some matrix $L \in \mathbf{R}^{N \times n}$.

It is easy to check that if for a point x (2.5.6) holds, then also (2.5.5) does, and the matrix L, uniquely determined by (2.5.6), equals $\nabla u(x)$.

2.6 Functions of Bounded Variation

We recall some definitions and basic results on functions with bounded variation which we will use in the sequel. Our main reference is the book [20] (see also [62],[73],[93],[94]).

Definition 2.6.1 *Let $u \in L^1\left(\Omega;\mathbf{R}^N\right)$. We say that u is a* function of Bounded Variation *in Ω, we write $u \in BV\left(\Omega;\mathbf{R}^N\right)$, if the distributional derivative Du of u is representable by a $N \times n$ matrix valued Radon measure on Ω whose entries are denoted by $D_i u^\alpha$, i.e., if $\varphi \in C_c^1\left(\Omega;\mathbf{R}^N\right)$ then*

$$\sum_{\alpha=1}^{N}\int_\Omega u^\alpha div\varphi^\alpha\, dx = -\sum_{\alpha=1}^{N}\sum_{i=1}^{n}\int_\Omega \varphi_i^\alpha dD_i u^\alpha.$$

Moreover, if $u \in L^1_{loc}\left(\Omega;\mathbf{R}^N\right)$ then we say that u is a function of locally Bounded Variation, *we write $u \in BV_{loc}\left(\Omega;\mathbf{R}^N\right)$, if $u \in BV\left(A;\mathbf{R}^N\right)$ for every $A \in \mathcal{A}(\Omega)$ with $A \subset\subset \Omega$.*

If we define

$$\|u\|_{BV(\Omega;\mathbf{R}^N)} := \|u\|_{L^1(\Omega;\mathbf{R}^N)} + \|Du\|(\Omega),$$

then $BV\left(\Omega;\mathbf{R}^N\right)$ turns out to be a Banach space. The next theorem shows that its embedding in $L^1\left(\Omega;\mathbf{R}^N\right)$ is compact.

Theorem 2.6.2 *If $(u_j) \subset BV\left(\Omega;\mathbf{R}^N\right)$ is such that $\sup_j \|u_j\|_{BV(\Omega;\mathbf{R}^N)} < +\infty$, then there exists a subsequence (u_{j_k}) and a function $u \in BV\left(\Omega;\mathbf{R}^N\right)$ such that $u_{j_k} \to u$ in $L^1\left(\Omega;\mathbf{R}^N\right)$ and $Du_{j_k} \to Du$ weakly $*$ in the sense of measures.*

The following result extends the theory of traces to BV functions.

Theorem 2.6.3 *Let $\Omega \subset \mathbf{R}^n$ be an open set with bounded Lipschitz boundary, and let $u \in BV\left(\Omega;\mathbf{R}^N\right)$. Then, for $\mathcal{H}^{n-1}$ a.e. $x \in \partial\Omega$ there exists $u^\Omega(x) \in \mathbf{R}^N$ such that*

$$\lim_{\rho\to 0^+} \rho^{-n}\int_{\Omega\cap B_\rho(x)} \left|u(y) - u^\Omega(x)\right| dy = 0.$$

Moreover, $\|u^\Omega\|_{L^1(\partial\Omega;\mathcal{H}^{n-1})} \leq C\|u\|_{BV(\Omega;\mathbf{R}^N)}$ for some constant $C > 0$ depending only on Ω.

From now on, the equality of BV functions on the boundary of a regular set is to be intended in the sense of traces.

Let us recall a density result in BV of Sobolev's functions with prescribed boundary conditions (see Lemma 2.5 [35]).

Lemma 2.6.4 *Let $\Omega \subseteq \mathbf{R}^n$ be an open set with Lipschitz boundary. Given $u \in BV\left(\Omega;\mathbf{R}^N\right)$ we may find $(u_j) \subset W^{1,1}\left(\Omega;\mathbf{R}^N\right)$ such that $u_j \to u$ in $L^1\left(\Omega;\mathbf{R}^N\right)$ and*

$$\|Du_j\|(\Omega) \to \|Du\|(\Omega),\ u_j = u \text{ on } \partial\Omega.$$

The class of characteristic functions in BV is particularly interesting.

Definition 2.6.5 *We say that a set $E \subset \mathbf{R}^n$ is a* set of finite perimeter in Ω *if $\mathcal{X}_E \in BV(\Omega)$. The quantity $\|D\mathcal{X}_E\|(\Omega)$ is called the* perimeter *of E in Ω.*

The following result is a generalized version of the Fleming-Rishel's coarea formula (see Lemma 2.4 [54]).

Theorem 2.6.6 *Let $u \in BV(\Omega)$. Then, for $\mathcal{L}^1$ a.e. $t \in \mathbf{R}$ the set $\{u > t\}$ has finite perimeter in Ω and for any $B \in \mathcal{B}(\Omega)$ there holds*

$$Du(B) = \int_{\mathbf{R}} D\mathcal{X}_{\{u>t\}}(B)\, dt, \quad \|Du\|(B) = \int_{\mathbf{R}} \|D\mathcal{X}_{\{u>t\}}\|(B)\, dt.$$

Moreover, for every Borel function $f : \Omega \times \mathbf{R} \times \mathbf{R}^n \to [0, +\infty)$ such that $f(x, u, \cdot)$ is convex and positively one-homogeneous for each $(x, u) \in \Omega \times \mathbf{R}$ there holds

$$\int_\Omega f(x, u, dDu) = \int_{\mathbf{R}} dt \int_\Omega f\left(x, t, dD\mathcal{X}_{\{u>t\}}\right). \tag{2.6.1}$$

Remark 2.6.7 *In (2.6.1) we have used the notation commonly adopted in literature for functionals defined on measures: with fixed a Borel function $g : \Omega \times \mathbf{R}^n \to [0, +\infty]$ convex and positively one-homogeneous in the second variable and $\mu \in \mathcal{M}(\Omega; \mathbf{R}^n)$, then*

$$\int_\Omega g(x, d\mu) := \int_\Omega g\left(x, \frac{d\mu}{d\|\mu\|}(x)\right) d\|\mu\|(x).$$

Let's now describe the structure of the distributional derivative of a BV function u. By the Radon-Nikodým's Theorem we have

$$Du = D^a u + D^s u,$$

where $D^a u$, $D^s u$ are the absolutely continuous and singular part with respect to $\mathcal{L}^n$, respectively.

We may further decompose the singular part $D^s u$. Define the *jump part* of Du, $D^j u$, to be the restriction of $D^s u$ to S_u and the *Cantor part*, $D^c u$, to be the restriction of $D^s u$ to $\Omega \setminus S_u$. Thus, we have

$$Du = D^a u + D^j u + D^c u.$$

We will denote by C_u the support of the measure $D^c u$.

In the following theorem we collect some properties of $D^a u$, $D^c u$, $D^j u$.

Theorem 2.6.8 *Let $u \in BV\left(\Omega; \mathbf{R}^N\right)$, then*

(i) u is approximately differentiable at a.e. $x \in \Omega$ and for every $B \in \mathcal{B}(\Omega)$

$$D^a u(B) = \int_B \nabla u\, dx.$$

Moreover, if $E \subset \mathbf{R}^N$ is $\mathcal{H}^{n-1}$ negligible, then ∇u vanishes a.e. on $u^{-1}(E)$;

(ii) $\|D^c u\|$ *vanishes on sets* $B \in \mathcal{B}(\Omega)$ *such that* $\mathcal{H}^{n-1}(B) < +\infty$, *and on sets of the form* $\tilde{u}^{-1}(E)$ *with* $E \subset \mathbf{R}^N$ $\mathcal{H}^1$ *negligible;*

(iii) S_u *is countably* $\mathcal{H}^{n-1}$ *rectifiable. In addition,* $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$, *and for every* $B \in \mathcal{B}(\Omega)$

$$D^j u(B) = \int_{B \cap J_u} (u^+ - u^-) \otimes \nu_u \, d\mathcal{H}^{n-1}.$$

Remark 2.6.9 *A useful locality property of* Du *can be deduced by Theorem 2.6.8. Let* u_1, $u_2 \in BV\left(\Omega; \mathbf{R}^N\right)$ *and define*

$$L = \{x \in \Omega \setminus (S_{u_1} \cup S_{u_2}) : \tilde{u}_1(x) = \tilde{u}_2(x)\}.$$

By applying Theorem 2.6.8 with $E = \{0\}$ *to* $u = u_1 - u_2$ *we obtain* $Du_1 \llcorner L = Du_2 \llcorner L$.

Remark 2.6.10 *Let us point out that, in case* $u \in BV\left(\Omega; \mathbf{R}^N\right)$, *the definitions of* S_u, J_u *and* $u^{\pm}$ *given in Definitions 2.5.1-2.5.3 are essentially equivalent to those of* $S_u^\star$, $J_u^\star$, $u_\star^{\pm}$.

Indeed, one can refine statement (iii) of Theorem 2.6.8 and prove that

$$\mathcal{H}^{n-1}(S_u^\star \setminus J_u^\star) = 0$$

(see Theorem 3.78 [20]). Hence, by (2.5.4) and the result above, the sets $J_u^\star$, J_u, S_u *and* $S_u^\star$ *differ up to a* $\mathcal{H}^{n-1}$ *negligible set. Thus, for* $x \in J_u^\star$ *either* $(u_\star^+(x), u_\star^-(x), \nu_u^\star(x)) = (u^+(x), u^-(x), \nu_u(x))$ *or* $(u_\star^+(x), u_\star^-(x), \nu_u^\star(x)) = (u^-(x), u^+(x), -\nu_u(x))$.

Moreover, by a classical result of Calderón and Zygmund, BV functions are approximately differentiable in the stronger sense (2.5.6) (see Theorem 3.83 [20]).

We need the measure theoretic definitions given in Definitions 2.5.1-2.5.3 since they make sense also in the more general framework of GBV functions as we will see in Section 2.7.

The $(n-1)$-dimensional density of the measure $\|D^j u\|$ is identified in the following lemma (see Lemma 2.6 [68]).

Lemma 2.6.11 *For* $\mathcal{H}^{n-1}$ *a.e.* $x_o \in J_u$

$$\lim_{\delta \to 0^+} \frac{1}{\delta^{n-1}} \int_{J_u \cap \left(x_o + \delta Q_{\nu_u(x_o)}\right)} \left|u^+(x) - u^-(x)\right| \, d\mathcal{H}^{n-1}(x) = \left|u^+(x_o) - u^-(x_o)\right|.$$

Let us recall a chain rule for BV functions.

Theorem 2.6.12 *Let* $u \in BV\left(\Omega; \mathbf{R}^N\right)$ *and* $f \in C^1\left(\mathbf{R}^N; \mathbf{R}^p\right)$ *be a Lipschitz function satisfying* $f(0) = 0$ *if* $\mathcal{L}^n(\Omega) = +\infty$. *Then,* $w = f(u) \in BV(\Omega; \mathbf{R}^p)$ *and*

$$\begin{cases} D^a w = \nabla f(u) \nabla u \, \mathcal{L}^n; \; D^c w = \nabla f(\tilde{u}) \, D^c u; \\ D^j w = (f(u^+) - f(u^-)) \otimes \nu_u \, \mathcal{H}^{n-1} \llcorner J_u \end{cases}.$$

Eventually, we introduce a special subspace of BV functions, which appeared in [57] as the relaxed domain of free-discontinuity energies taking into account only volume and surface terms and with an imposed confinement condition.

Definition 2.6.13 *Let $u \in BV\left(\Omega; \mathbf{R}^N\right)$, we say that u is a* Special function of Bounded Variation *in Ω, we write $u \in SBV\left(\Omega; \mathbf{R}^N\right)$, if $D^c u = 0$.*

Equivalently, $u \in SBV\left(\Omega; \mathbf{R}^N\right)$ if and only if $D^s u$ is concentrated on S_u.

Many properties and results about SBV functions will be stated in the more general framework of $GSBV$ functions (see Subsection 2.7.1). Here, we only recall an extension result.

Theorem 2.6.14 *Let $\Omega \subset \mathbf{R}^n$ be a bounded and open set with Lipschitz boundary. Let $u \in SBV \cap L^\infty\left(\Omega; \mathbf{R}^N\right)$ be such that*

$$\int_\Omega |\nabla u|^p \, dx + \mathcal{H}^{n-1}\left(S_u\right) < +\infty$$

for some $p \in (1, +\infty)$.

Let $\Omega' \subset \mathbf{R}^n$ be a bounded and open set such that $\Omega \subset\subset \Omega'$. Then there exists a function $\hat{u} \in SBV \cap L^\infty\left(\Omega'; \mathbf{R}^N\right)$ such that $\hat{u}|_\Omega \equiv u$, $\mathcal{H}^{n-1}\left(S_{\hat{u}} \cap \partial\Omega\right) = 0$, $\|\hat{u}\|_{L^\infty(\Omega'; \mathbf{R}^N)} = \|u\|_{L^\infty(\Omega; \mathbf{R}^N)}$ and

$$\int_{\Omega'} |\nabla \hat{u}|^p \, dx + \mathcal{H}^{n-1}\left(S_{\hat{u}}\right) < +\infty.$$

2.6.1 Lower semicontinuity and integral representation in BV

Given a Borel function $f : \Omega \times \mathbf{R}^N \times \mathbf{R}^{N \times n} \to [0, +\infty)$ consider the integral functional $F : L^1\left(\Omega; \mathbf{R}^N\right) \to [0, +\infty]$ defined by

$$F(u; \Omega) := \begin{cases} \displaystyle\int_\Omega f\left(x, u, \nabla u\right) \, dx & \text{if } u \in W^{1,1}\left(\Omega; \mathbf{R}^N\right) \\ +\infty & \text{if } u \in L^1\left(\Omega; \mathbf{R}^N\right) \setminus W^{1,1}\left(\Omega; \mathbf{R}^N\right), \end{cases}$$

and denote by $\overline{F}$ its relaxation in the strong $L^1\left(\Omega; \mathbf{R}^N\right)$ topology. Actually, it is convenient to localize the functional F, and thus $\overline{F}$, by considering a set dependence of F on the domain of integration, which we assume to vary among sets in $\mathcal{A}(\Omega)$.

Recently, there has been a great effort to find an explicit representation for $\overline{F}$ under very mild assumptions on f, at least for target functions u in $BV\left(\Omega; \mathbf{R}^N\right)$ (see Chapter 5 of [20],[18],[21],[35],[36],[66]).

As a first step observe that if f has linear growth in z then (the localized version of) the relaxed functional $\overline{F} : BV\left(\Omega; \mathbf{R}^N\right) \times \mathcal{A}(\Omega) \to [0, +\infty]$ is a *variational functional* with respect to the strong $L^1\left(\Omega; \mathbf{R}^N\right)$ topology in the sense of [55], that is: $\overline{F}(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$

of a Borel measure for every $u \in BV\left(\Omega;\mathbf{R}^N\right)$; $\overline{F}(\cdot,\Omega)$ is $L^1\left(\Omega;\mathbf{R}^N\right)$ lower semicontinuous; and $\overline{F}$ is local, i.e., $\overline{F}(u,A)=\overline{F}(v,A)$ whenever $u,v \in BV\left(A,\mathbf{R}^N\right)$, $A \in \mathcal{A}(\Omega)$ and $u=v$ a.e. in A (see Proposition 4.2 [21]).

Hence, we may address the more general problem of finding an explicit integral representation formula for variational functionals.

In the sequel we recall some results concerning these kind of problems, in a form which is useful for our purposes. The first theorem is an integral representation result which summarizes Lemma 3.5 and Theorem 3.7 of [35].

Theorem 2.6.15 *Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set, and let $\mathcal{F} : BV\left(\Omega;\mathbf{R}^N\right) \times \mathcal{A}(\Omega) \to [0,+\infty]$ be a variational functional such that for every $u \in BV\left(\Omega;\mathbf{R}^N\right)$ and $A \in \mathcal{A}(\Omega)$*

$$0 \leq \mathcal{F}(u;A) \leq c\left(\mathcal{L}^n(A)+\|Du\|(A)\right).$$

Then, for every $u \in BV\left(\Omega;\mathbf{R}^N\right)$

(1) for a.e. $x_o \in \Omega$

$$\frac{d\mathcal{F}(u,\cdot)}{d\mathcal{L}^n}(x_o)=\limsup_{\delta\to 0^+}\frac{1}{\delta^n}\inf\Big\{\mathcal{F}(w,x_o+\delta Q) : w \in BV\left(\Omega;\mathbf{R}^N\right), \\ w=u(x_o)+\langle\nabla u(x_o),(\cdot-x_o)\rangle \text{ on } x_o+\delta\partial Q\Big\};$$

(2) for $\|D^c u\|$ a.e. $x_o \in C_u$

$$\frac{d\mathcal{F}(u,\cdot)}{d\|D^c u\|}(x_o)=\limsup_{\delta\to 0^+}\frac{1}{\|D^c u\|(x_o+\delta C)}\inf\Big\{\mathcal{F}(w,x_o+\delta C) : \\ w \in BV\left(\Omega;\mathbf{R}^N\right), w=u \text{ on } x_o+\delta\partial C\Big\},$$

where C is any convex bounded open set containing the origin;

(3) for $\mathcal{H}^{n-1}$ a.e. $x_o \in J_u$

$$\frac{d\mathcal{F}(u,\cdot)}{d\left(\mathcal{H}^{n-1}\llcorner J_u\right)}(x_o)=\limsup_{\delta\to 0^+}\frac{1}{\delta^{n-1}}\inf\Big\{\mathcal{F}\left(w,x_o+\delta Q_{\nu_u(x_o)}\right) : \\ w \in BV\left(\Omega;\mathbf{R}^N\right), w=u_o \text{ on } x_o+\delta\partial Q_{\nu_u(x_o)}\Big\}, \tag{2.6.2}$$

where

$$u_o(x):=\begin{cases} u^+(x_o) & \text{if } \langle x,\nu_u(x_o)\rangle \geq 0 \\ u^-(x_o) & \text{if } \langle x,\nu_u(x_o)\rangle < 0 \end{cases}.$$

In the case of relaxation one can make more specific the statements given above and find lower and upper bounds on the densities of $\overline{F}$ in terms of the original integrand f. Different sets of hypotheses are needed to deal with the diffuse and jump part, and since in the sequel we are interested only in the former, we state partial results.

We need to recall a few concepts. The first condition is the well known Morrey's *quasiconvexity* [85], which turned out to be crucial in vectorial Calculus of Variations (see [52],[74]).

Definition 2.6.16 *Let Ω be a bounded open set of $\mathbf{R}^n$ and $f : \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times n} \to [0, +\infty)$ be a Borel function. We say that f is* quasiconvex *in z if for a.e. $x \in \Omega$ and for every $u \in \mathbf{R}^N$*

$$f(x, u, z)\, \mathcal{L}^n(\Omega) \leq \int_\Omega f(x, u, z + D\varphi(y))\, dy$$

for every $\varphi \in C^1_c\left(\Omega; \mathbf{R}^N\right)$.

Definition 2.6.17 *Let $f : \Omega\times\mathbf{R}^N\times\mathbf{R}^{N\times n} \to [0, +\infty)$ be a Borel function. For any $(x, u, z) \in \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times n}$ define the* recession function *of f by*

$$f^\infty(x, u, z) = \limsup_{t\to+\infty} \frac{f(x, u, tz)}{t}.$$

The following result is due to Fonseca-Leoni (Theorem 1.8 [66]).

Theorem 2.6.18 *Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and let $f : \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times n} \to [0, +\infty)$ be a Borel integrand. Assume that*

(i) $f(x, u, \cdot)$ is quasiconvex for every $(x, u) \in \Omega \times \mathbf{R}^N$ and there exists $c > 0$ such that

$$0 \leq f(x, u, z) \leq c\,(|z| + 1) \tag{2.6.3}$$

for every $(x, u, z) \in \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times n}$;

(ii) for all $(x_o, u_o) \in \Omega \times \mathbf{R}^N$ either $f(x_o, u_o, z) \equiv 0$ for all $z \in \mathbf{R}^{N\times n}$, or for every $\eta > 0$ there exist $c_0, c_1, \delta > 0$ such that

$$f(x_o, u_o, z) - f(x, u, z) \leq \eta\,(1 + f(x, u, z))\,, \tag{2.6.4}$$

$$f(x, u, z) \geq c_1|z| - c_0$$

for all $(x, u) \in \Omega \times \mathbf{R}^N$ with $|x - x_o| + |u - u_o| \leq \delta$ and for all $z \in \mathbf{R}^{N\times n}$.

Let $u \in BV\left(\Omega; \mathbf{R}^N\right)$, then $\overline{F}(u, \cdot)$ is the trace of a finite measure on $\mathcal{A}(\Omega)$, and for every $A \in \mathcal{A}(\Omega)$

$$\overline{F}(u, A) \geq \int_A f(x, u, \nabla u))\, dx + \int_A f^\infty(x, \tilde{u}, dD^c u)\,.$$

The statements of Theorem 2.6.18 are complemented by the ones in the following, which are contained in Theorem 1.9 [66].

Definition 2.6.19 *We say that* $f : \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times n} \to [0, +\infty)$ *is a* Carathéodory integrand *if* $f(\cdot, u, z)$ *is Borel measurable for every* $(u, z) \in \mathbf{R}^N \times \mathbf{R}^{N\times n}$ *and* $f(x, \cdot, \cdot)$ *is continuous for a.e.* $x \in \Omega$.

Theorem 2.6.20 *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded open set, and assume that* f *satisfies condition (i) of Theorem 2.6.18.*

Let $u \in BV\left(\Omega; \mathbf{R}^N\right)$, *then* $\overline{F}(u, \cdot)$ *is the trace of a finite measure on* $\mathcal{A}(\Omega)$, *and for every* $A \in \mathcal{A}(\Omega)$

(1) if f *is Carathéodory or* $f(\cdot, \cdot, z)$ *is upper semicontinuous then*

$$\overline{F}(u, A \setminus (J_u \cup C_u)) \leq \int_A f(x, u, \nabla u)\, dx;$$

(2) if $f^\infty(\cdot, \cdot, z)$ *is upper semicontinuous then*

$$\overline{F}(u, A \cap C_u) \leq \int_A f^\infty(x, \tilde{u}, dD^c u).$$

Remark 2.6.21 *Let us point out that the notation adopted in Theorem 2.6.18, 2.6.20 for the term accounting for the Cantor part is consistent with the discussion of Remark 2.6.7. Indeed, it is easy to check that, in case (2.6.3) holds,* $f^\infty(x, u, \cdot)$ *inherits the quasiconvexity property from* $f(x, u, \cdot)$; *so that* $f^\infty(x, u, \cdot)$ *is convex on rank-one matrices (see [52],[74]). Eventually, a result of Alberti [1] ensures that* $\frac{dD^c u}{d\|D^c u\|}(x)$ *has rank-one for* $\|D^c u\|$ *a.e.* $x \in \Omega$.

2.7 Generalized functions of Bounded Variation

Functionals involved in free-discontinuity problems are not coercive in the space BV if no L^∞ bound on the norms is imposed. Then, it is useful to consider the following larger class (see [57], Chapter 4 of [20]).

Definition 2.7.1 *Given* $u \in \mathcal{B}\left(\Omega; \mathbf{R}^N\right)$, *we say that* u *is a* Generalized function of Bounded Variation *in* Ω, *we write* $u \in GBV\left(\Omega; \mathbf{R}^N\right)$, *if* $g(u) \in BV_{loc}(\Omega)$ *for every* $g \in C^1\left(\mathbf{R}^N\right)$ *such that* ∇g *has compact support.*

The above generalization is based on a double localization, with respect to both the dependent and independent variables. Moreover, GBV functions are not even locally summable in general. Nevertheless, GBV functions have generalized derivatives which keep the same structure as those of BV functions.

Notice that by the very definition we have $BV\left(\Omega; \mathbf{R}^N\right) \subset GBV\left(\Omega; \mathbf{R}^N\right)$, and also $GBV \cap L^\infty\left(\Omega; \mathbf{R}^N\right) = BV \cap L^\infty\left(\Omega; \mathbf{R}^N\right)$.

Remark 2.7.2 *In case* $N = 1$, *it can be easily checked that* $u \in GBV(\Omega)$ *if and only if* $((-T) \vee u \wedge T) \in BV(\Omega)$ *for every* $T > 0$.

While, for $N > 1$ *the product space* $(GBV(\Omega))^N$ *is strictly contained in* $GBV\left(\Omega; \mathbf{R}^N\right)$ *even if* $\Omega \subseteq \mathbf{R}$ *(see Remark 4.27 of [20]).*

The space GBV inherits some of the main properties of BV (see Proposition 1.3 [11]).

Theorem 2.7.3 *Let $u \in GBV\left(\Omega;\mathbf{R}^N\right)$, then*

(i) u is approximately differentiable a.e. in Ω;

(ii) S_u is countably $\mathcal{H}^{n-1}$ rectifiable and $\mathcal{H}^{n-1}\left(S_u \setminus J_u\right) = 0$.

To give a rigorous mathematical sense to variational functionals involved in our problems, we need to associate to a particular class of GBV functions a vector measure which can be regarded as the Cantor part of the generalized distributional derivative.

Let us first recall that if $u \in GBV\left(\Omega;\mathbf{R}^N\right)$ then a positive measure $\|D^c u\|$ is associated to u. Indeed, for every $i \in \mathbf{N}$, let Ψ_i be defined as

$$\Psi_i(u) := \begin{cases} u & \text{if } |u| \le a_i \\ \\ 0 & \text{if } |u| \ge a_{i+1} \end{cases}, \tag{2.7.1}$$

where $(a_i) \subset (0,+\infty)$ is a strictly increasing and diverging sequence, $\Psi_i \in C^1\left(\mathbf{R}^N;\mathbf{R}^N\right)$ and $\|\nabla\Psi_i\|_{L^\infty(\mathbf{R}^N;\mathbf{R}^{N\times N})} \le 1$.

Then, set $u^i := \Psi_i(u) \in BV\left(\Omega;\mathbf{R}^N\right)$, and define for every $B \in \mathcal{B}(\Omega)$ the positive measure $\|D^c u\|$ by

$$\|D^c u\|(B) := \sup_i \|D^c u^i\|(B).$$

Actually, the sup above is independent of the truncation performed on u and it is also the pointwise limit and the least upper bound measure of the family $\left(\|D^c u^i\|\right)$.

For a GBV function u for which $\|D^c u\|$ is a finite measure, we define a vector measure the total variation of which is exactly $\|D^c u\|$.

Lemma 2.7.4 *Let $u \in GBV\left(\Omega;\mathbf{R}^N\right)$ be such that $\|D^c u\|$ is a finite measure, then the sequence $\left(D^c u^i\right)$ pointwise converges to $\lambda \in \mathcal{M}\left(\Omega;\mathbf{R}^{N\times n}\right)$ such that for every $B \in \mathcal{B}(\Omega)$*

$$\|\lambda\|(B) = \|D^c u\|(B).$$

Moreover, λ does not depend on the particular truncations chosen.

Definition 2.7.5 *Let $u \in GBV\left(\Omega;\mathbf{R}^N\right)$ be such that $\|D^c u\|$ is a finite measure, then we define $D^c u := \lambda$.*

Proof. Set

$$\Omega_\infty := \{x \in \Omega \setminus S_u : \tilde{u}(x) = \infty\} \tag{2.7.2}$$

and note that $\|D^c u\|(\Omega_\infty) = 0$. Indeed, for every $i \in \mathbf{N}$, $\Omega_\infty \subseteq \{x \in \Omega \setminus S_u : \tilde{u}^i(x) = 0\}$, thus, by Remark 2.6.9, $\|D^c u^i\|(\Omega_\infty) = 0$. If we set $\Omega_j := \{x \in \Omega \setminus S_u : |\tilde{u}(x)| < a_j\}$, we then have that $\Omega = (\cup_{j\ge 1}\Omega_j) \cup N$, with $\|D^c u\|(N) = 0$. Let $i \ge j$, notice that $\tilde{u}^i \equiv \tilde{u}^j$ on Ω_j, and so, again by Remark 2.6.9,

$$D^c u^i \llcorner \Omega_j = D^c u^j \llcorner \Omega_j. \tag{2.7.3}$$

Let us point out that since $\|D^c u\|$ is a finite measure then $(\|D^c u\|(\Omega \setminus \Omega_j))$ is infinitesimal.

Consider the set function $\lambda : \mathcal{B}(\Omega) \to \mathbf{R}^{N\times n}$ defined as

$$\lambda(B) := \lim_i D^c u^i(B).$$

Let us first notice that the limit above exists since

$$\begin{aligned}&|D^c u^j(B) - D^c u^i(B)|\\ &\quad \leq \|D^c u^j\|\,(B\setminus\Omega_j) + \|D^c u^i\|\,(B\setminus\Omega_j) \leq 2\|D^c u\|\,(B\setminus\Omega_j)\,,\end{aligned}$$

and one can easily check that $\lambda \in \mathcal{M}\left(\Omega; \mathbf{R}^{N\times n}\right)$.

In particular, Proposition 2.2.2 (3) implies that $(D^c u^i)$ is weakly $*$ convergent in the sense of measures to the vector measure λ.

We claim that $\|D^c u\| \equiv \|\lambda\|$. First notice that since $(\|D^c u^i\|)$ converges to $\|D^c u\|$ weakly $*$ in the sense of measures then Proposition 2.2.2 (1) yields $\|D^c u\|(A) \geq \|\lambda\|(A)$ for every $A \in \mathcal{A}(\Omega)$. Moreover, with fixed $j \in \mathbf{N}$ for every $i \geq j$ by (2.7.3)

$$\|\lambda\| \llcorner \Omega_j = \|\lambda \llcorner \Omega_j\| = \|D^c u^i \llcorner \Omega_j\| = \|D^c u^i\| \llcorner \Omega_j = \|D^c u\| \llcorner \Omega_j,$$

from which there follows $\|\lambda\|(\Omega) = \|D^c u\|(\Omega)$ by passing to the limit on $j \to +\infty$. Hence, by Proposition 2.2.2 (3), $(\|D^c u^i\|)$ converges weakly $*$ in the sense of measures to $\|\lambda\|$ and so the conclusion follows.

Eventually, it is easy to check that the argument used does not depend on the particular family of truncating functions chosen. □

Let us now recall an alternative characterization of functions in $(GBV(\Omega))^N$ through their one-dimensional sections, which extends a classical result in BV. The so called 'slicing techniques' have been intensively exploited to prove variational approximations of free-discontinuity problems since one may reduce n-dimensional problems to the one-dimensional case (see [37]).

Let us recall the Slicing Theorem (see [10]), the notation we use here has already been fixed before Lemma 2.4.2.

Theorem 2.7.6 *(a) Let $u \in (GBV(\Omega))^N$, then $u_{\xi,y} \in \left(GBV\left(\Omega_y^\xi\right)\right)^N$ for all $\xi \in \mathbf{S}^{n-1}$ and $\mathcal{H}^{n-1}$ a.e. $y \in \Omega_\xi$. For such y we have*

(i) $\dot{u}_{\xi,y}(t) = \nabla u\,(y + t\xi)\,\xi$ *for $\mathcal{L}^1$ a.e. $t \in \Omega_y^\xi$;*

(ii) $J_{u_{\xi,y}} = (J_u)_y^\xi$*;*

(iii) $u_{\xi,y}^\pm(t) = u^\pm(y+t\xi)$ *or* $u_{\xi,y}^\pm(t) = u^\mp(y+t\xi)$ *according to the cases $\langle \nu_u, \xi\rangle > 0$, $\langle \nu_u, \xi\rangle < 0$ (the case $\langle \nu_u, \xi\rangle = 0$ being negligible).*

(b) Conversely, let $\{\xi_1, \ldots, \xi_n\}$ be a basis of $\mathbf{R}^n$ and $u = (u^\alpha)_{1 \le \alpha \le N} \in \mathcal{B}\left(\Omega; \mathbf{R}^N\right)$. If for every ξ_i, $1 \le i \le n$, $u_{\xi_i, y} \in \left(GBV\left(\Omega_y^{\xi_i}\right)\right)^N$ for $\mathcal{H}^{n-1}$ a.e. $y \in \Omega_{\xi_i}$, and

$$\int_{\Omega_{\xi_i}} \|D((-T) \vee u^\alpha \wedge T)_{\xi_i, y}\| \left(\Omega_y^{\xi_i}\right) d\mathcal{H}^{n-1}(y) < +\infty,$$

for every $T > 0$ and $1 \le \alpha \le N$, then $u \in (GBV(\Omega))^N$.

Let $u \in BV\left(\Omega; \mathbf{R}^N\right)$, then by Remark 2.6.10, the set

$$J_u^\infty := \{x \in J_u : u^+(x) = \infty \text{ or } u^-(x) = \infty\}$$

is $\mathcal{H}^{n-1}$ negligible. On the contrary, for GBV functions it may happen that $\mathcal{H}^{n-1}(J_u^\infty) > 0$. In the following theorem we show that for GBV functions satisfying suitable a priori bounds, J_u^∞ is $\mathcal{H}^{n-1}$ negligible (see Theorem 4.40 [20] for a sharper result under more specific assumptions).

Theorem 2.7.7 *Let $u \in GBV(\Omega)$ be such that*

$$\int_\Omega |\nabla u|\, dx + \int_{J_u} \theta\left(|u^+ - u^-|\right) d\mathcal{H}^{n-1} + \|D^c u\|(\Omega) < +\infty, \tag{2.7.4}$$

where $\theta : [0, +\infty) \to [0, +\infty)$ satisfies

$$\delta > 0 \Rightarrow \inf_{|t| > \delta} \theta > 0; \quad \liminf_{t \to 0^+} \frac{\theta(t)}{t} > 0. \tag{2.7.5}$$

Then

$$\mathcal{H}^{n-1}(J_u^\infty) = 0.$$

Proof. Assume first $n = 1$, in such a case we prove that $u \in BV(\Omega)$, and so the conclusion is a well known property of such functions.

Indeed, let $J_u^\delta = \{t \in J_u : |u^+(t) - u^-(t)| \le \delta\}$, then (2.7.4) and $(2.7.5)_1$ yield

$$\left(\inf_{|t| > \delta} \theta\right) \mathcal{H}^0\left(J_u \setminus J_u^\delta\right) \le \sum_{t \in J_u \setminus J_u^\delta} \theta\left(|u^+ - u^-|\right) < +\infty,$$

hence $M_\delta = \sup_{J_u \setminus \left(J_u^\delta \cup J_u^\infty\right)} |u^+ - u^-|$ is finite and actually it is a maximum. Thus, by $(2.7.5)_2$, we get

$$\sum_{t \in J_u \setminus J_u^\infty} |u^+ - u^-| \le c \sum_{t \in J_u^\delta} \theta\left(|u^+ - u^-|\right) + M_\delta \mathcal{H}^0\left(J_u \setminus J_u^\delta\right) < +\infty.$$

By (2.7.4) $\mathcal{H}^0(J_u^\infty) < +\infty$ and let $J_u^\infty = \{t_i\}_{1 \le i \le r}$ with $t_i < t_{i+1}$. Then, with fixed i, for every $x, y \in (t_i, t_{i+1})$ we get

$$|u_T(x) - u_T(y)| \le \left|\int_x^y |\nabla u|\, dt\right| + \sum_{t \in J_u \setminus J_u^\infty} |u^+ - u^-| + \|D^c u\|((x, y)),$$

where $u_T = ((-T) \vee u \wedge T) \in BV(\Omega)$, $T \in \mathbf{N}$. By choosing $y \in (t_i, t_{i+1}) \setminus \Omega_\infty$, where Ω_∞ is defined in (2.7.2), it follows that there exists a positive constant λ_i such that $|u_T(x)| \le \lambda_i$, and so by passing to the supremum on T we get

$$\sup_{(t_i, t_{i+1})} |u(x)| \le \lambda_i.$$

Hence, $J_u^\infty = \emptyset$ and $u \in L^\infty(\Omega)$, so that $u \in BV(\Omega)$.

In case $n > 1$, notice that by the discussion above for every $\xi \in \mathbf{S}^{n-1}$ and $\mathcal{H}^{n-1}$ a.e. $y \in (J_u^\infty)_\xi$ the set $(J_u^\infty)_y^\xi$ is empty. Hence, the projection of J_u^∞ onto Π^ξ is $\mathcal{L}^{n-1}$ negligible. Eventually, since J_u is countably $\mathcal{H}^{n-1}$ rectifiable, Proposition 2.4.3 implies that J_u^∞ is $\mathcal{H}^{n-1}$ negligible. □

Remark 2.7.8 *If $u \in (GBV(\Omega))^N$, one can show that $J_u^\infty = \cup_{i=1}^N J_{u_i}^\infty \cup N$, with $\mathcal{H}^{n-1}(N) = 0$. Then, from Theorem 2.7.7 we deduce that, if u_i satisfies (2.7.4) for each $1 \le i \le N$, then $\mathcal{H}^{n-1}(J_u^\infty) = 0$.*

2.7.1 Generalized Special functions of Bounded Variation

As in the BV case we may consider the sub-class of GBV functions for which $D^c u = 0$, which now makes sense by Lemma 2.7.4.

Definition 2.7.9 *Given $u \in \mathcal{B}\left(\Omega; \mathbf{R}^N\right)$, we say that u is a* Generalized Special function of Bounded Variation in Ω, *we write $u \in GSBV\left(\Omega; \mathbf{R}^N\right)$, if $g(u) \in SBV(\Omega)$ for every $g \in C^1\left(\mathbf{R}^N\right)$ such that ∇g has compact support.*

Notice that $SBV\left(\Omega; \mathbf{R}^N\right) \subset GSBV\left(\Omega; \mathbf{R}^N\right)$, and $GSBV \cap L^\infty\left(\Omega; \mathbf{R}^N\right) = SBV \cap L^\infty\left(\Omega; \mathbf{R}^N\right)$.

The main features of the space $GSBV\left(\Omega; \mathbf{R}^N\right)$ are the following closure and compactness theorems which turned out to be the essential tools in order to state an existence theory for free-discontinuity problems taking into account only volume and surface terms (see [3],[10],[14]).

Theorem 2.7.10 *Let $\phi : [0, +\infty) \to [0, +\infty)$ be a convex non-decreasing function such that $\frac{\phi(t)}{t} \to +\infty$ as $t \to +\infty$, let $\theta : [0, +\infty) \to [0, +\infty]$ be a concave function such that $\frac{\theta(t)}{t} \to +\infty$ as $t \to 0^+$.*

Let $(u_j) \subset GSBV\left(\Omega; \mathbf{R}^N\right)$ and assume that

$$\sup_j \left\{ \int_\Omega \phi\left(|\nabla u_j|\right) dx + \int_{S_{u_j}} \theta\left(\left|u_j^+ - u_j^-\right|\right) d\mathcal{H}^{n-1} \right\} < +\infty. \tag{2.7.6}$$

If $u_j \to u$ in measure on Ω, then $u \in GSBV\left(\Omega; \mathbf{R}^N\right)$ and *(i)* $\nabla u_j \to \nabla u$ *weakly in* $L^1\left(\Omega; \mathbf{R}^{N\times n}\right)$*;*

(ii) $D^j u_j \to D^j u$ *weakly* $*$ *in the sense of measures;*

(iii)

$$\int_\Omega \phi\left(|\nabla u|\right)\, dx \le \liminf_j \int_\Omega \phi\left(|\nabla u_j|\right)\, dx;$$

(iv)

$$\int_{S_u} \theta\left(|u^+ - u^-|\right)\, d\mathcal{H}^{n-1} \le \liminf_j \int_{S_{u_j}} \theta\left(\left|u_j^+ - u_j^-\right|\right)\, d\mathcal{H}^{n-1}.$$

Theorem 2.7.11 *Consider a sequence* $(u_j) \subset GSBV\left(\Omega; \mathbf{R}^N\right)$ *satisfying (2.7.6), with* ϕ*,* θ *as in Theorem 2.7.10, and assume in addition that* $\|u_j\|_{L^q(\Omega;\mathbf{R}^N)}$ *is uniformly bounded in* j *for some* $q \in (0, +\infty]$*.*

Then there exists a subsequence (u_{j_k}) *and a function* $u \in GSBV\left(\Omega; \mathbf{R}^N\right)$*, such that* $u_{j_k} \to u$ *a.e. in* Ω*.*

Moreover, in case $q = \infty$ $u \in SBV\left(\Omega; \mathbf{R}^N\right)$*.*

Remark 2.7.12 *The original proofs of Theorem 2.7.10 and Theorem 2.7.11 make use of the one-dimensional sections of GBV functions introduced in Theorem 2.8.5.*

Indeed, GSBV functions can be characterized through their one-dimensional sections as follows: $u \in (GSBV(\Omega))^N$ *if and only if* $u \in (GBV(\Omega))^N$ *and* $u_{\xi_i,y} \in SBV\left(\Omega_y^{\xi_i}; \mathbf{R}^N\right)$ *for* $\mathcal{H}^{n-1}$ *a.e.* $y \in \Omega_{\xi_i}$*, where* $\{\xi_1, \ldots, \xi_n\}$ *is a basis of* $\mathbf{R}^n$*.*

Let us now introduce a sub-class of $GSBV$ which can be approximated by piecewise smooth functions. Given $p \in (1, +\infty)$, define the vector spaces

$$\begin{aligned}&(G)SBV^p\left(\Omega; \mathbf{R}^N\right)\\ &\quad := \left\{u \in (G)SBV\left(\Omega; \mathbf{R}^N\right) : \mathcal{H}^{n-1}\left(J_u\right) < +\infty, \nabla u \in L^p\left(\Omega; \mathbf{R}^{N\times n}\right)\right\}.\end{aligned}$$

In analogy with the $W^{1,p}$ case, $GSBV^p$ functions can be approximated by functions which have a regular jump set and which are smooth outside.

Definition 2.7.13 *Let* $\mathcal{W}\left(\Omega; \mathbf{R}^N\right)$ *be the space of all* $u \in SBV\left(\Omega; \mathbf{R}^N\right)$ *such that*

(i) S_u *is* essentially closed*, i.e.,* $\mathcal{H}^{n-1}\left(\overline{S_u} \setminus S_u\right) = 0$*;*

(ii) $\overline{S_u}$ *is a* polyhedral set*, i.e.,* $\overline{S_u}$ *is the intersection of* Ω *with the union of a finite number of* $(n-1)$ *-dimensional simplexes;*

(iii) $u \in W^{k,\infty}\left(\Omega \setminus \overline{S_u}; \mathbf{R}^N\right)$ *for every* $k \in \mathbf{N}$*.*

The following theorem proved by Cortesani and Toader [51] provides a density result of the class $\mathcal{W}\left(\Omega; \mathbf{R}^N\right)$ in $SBV^p \cap L^\infty\left(\Omega; \mathbf{R}^N\right)$ with respect to anisotropic surface energies.

Theorem 2.7.14 *Let $\Omega \subset \mathbf{R}^n$ be an open set with Lipschitz boundary and $u \in SBV^p \cap L^\infty\left(\Omega; \mathbf{R}^N\right)$, for some $p \in (1, +\infty)$. Then there exists a sequence $(u_j) \subset \mathcal{W}\left(\Omega; \mathbf{R}^N\right)$ such that*

(i) $u_j \to u$ *strongly in* $L^1\left(\Omega; \mathbf{R}^N\right)$;

(ii) $\nabla u_j \to \nabla u$ *strongly in* $L^p\left(\Omega; \mathbf{R}^{N\times n}\right)$;

(iii) $\limsup_j \|u_j\|_{L^\infty(\Omega;\mathbf{R}^N)} \leq \|u\|_{L^\infty(\Omega;\mathbf{R}^N)}$;

(iv) for every $A \subset\subset \Omega$ and for every upper semicontinuous function $\varphi : \Omega \times \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{S}^{n-1} \to [0, +\infty)$ such that $\varphi(x, a, b, \nu) = \varphi(x, b, a, -\nu)$ for every $x \in \Omega$, $a, b \in \mathbf{R}^N$ and $\nu \in \mathbf{S}^{n-1}$ there holds

$$\limsup_j \int_{\overline{A}\cap S_{u_j}} \varphi\left(x, u_j^-, u_j^+, \nu_{u_j}\right) d\mathcal{H}^{n-1} \leq \int_{\overline{A}\cap S_u} \varphi\left(x, u^-, u^+, \nu_u\right) d\mathcal{H}^{n-1}. \tag{2.7.7}$$

Remark 2.7.15 *The sequence (u_j) can be chosen such that (2.7.7) holds for every open set $A \subseteq \Omega$ if the following additional condition is satisfied*

$$\limsup_{\substack{(y,a',b',\mu)\to(x,a,b,\nu)\\ y\in\Omega}} \varphi\left(y, a', b', \mu\right) < +\infty$$

for every $x \in \partial\Omega$, $a, b \in \mathbf{R}^N$ and $\nu \in \mathbf{S}^{n-1}$. In this case, $\overline{A}$ must be replaced by the relative closure of A in Ω (see Remark 3.2 [51]).

Remark 2.7.16 *Let $u \in GSBV^p\left(\Omega; \mathbf{R}^N\right)$, then by a truncation argument and a diagonalization procedure, it is easy to infer from Theorem 2.7.14 the existence of a sequence $(u_j) \subset \mathcal{W}\left(\Omega; \mathbf{R}^N\right)$ for which conditions (i) and (ii) of the same result hold true.*

Hence, $\mathcal{W}\left(\Omega; \mathbf{R}^N\right)$ is dense in $GSBV^p\left(\Omega; \mathbf{R}^N\right)$ in the sense of the GSBV Closure Theorem 2.7.10 (see Remark 3.4 [51] for a detailed discussion).

2.7.2 Lower semicontinuity in $GSBV$

In this section we recall some lower semicontinuity results for variational functionals defined on $GSBV$. The first result in this direction is provided by Theorem 2.7.10, where a separate convergence property holds for the volume and surface terms. The same problem can be posed for more general functionals.

The next result, proved by Kristensen [78] (see also [14]), ensures lower semicontinuity for volume integrals exactly in the setting prescribed by the $GSBV$ Closure Theorem 2.7.10.

Theorem 2.7.17 *Let $f : \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times n} \to [0, +\infty)$ be a Carathéodory integrand quasiconvex in z satisfying*

$$c_1 |z|^p + b(u) - a(x) \leq f(x, u, z) \leq c_2(|z|^p + b(u) + a(x))$$

for every $(x,u,z) \in \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times n}$ *with* $p \in (1,+\infty)$, c_1 *and* c_2 *positive constants,* $a \in L^1(\Omega)$, *and* $b \in C^0\left(\mathbf{R}^N\right)$ *a non negative function.*

Let u_j, $u \in GSBV\left(\Omega; \mathbf{R}^N\right)$ *be such that* $u_j \to u$ *in measure on* Ω *and* (u_j) *satisfies (2.7.6) with* $\phi(t) = t^p$.

Then

$$\int_\Omega f(x,u,\nabla u)\, dx \leq \liminf_j \int_\Omega f(x,u_j,\nabla u_j)\, dx. \tag{2.7.8}$$

Eventually, we end this subsection stating some results concerning the lower semicontinuity of surface integrals. The first result follows straightforward from a more general theorem proved by Ambrosio (see Theorem 3.5 [11]).

Theorem 2.7.18 *Let* $\varphi : \mathbf{R}^n \to [0,+\infty)$ *be a norm, let* u_j, $u \in GSBV\left(\Omega; \mathbf{R}^N\right)$ *be such that*

(i) $u_j \to u$ *in measure on* Ω,

(ii) there exists $p \in (1,+\infty)$ *such that*

$$\sup_j \|\nabla u_j\|_{L^p(\Omega;\mathbf{R}^N)} < +\infty.$$

Then

$$\int_{S_u} \varphi(\nu_u)\, dx \leq \liminf_j \int_{S_{u_j}} \varphi\left(\nu_{u_j}\right) dx.$$

In order to state an analogous result for surface energies depending on the one-sided traces $u^\pm$ we need to introduce the notion of subadditivity.

Definition 2.7.19 *Let* $\triangle := \{(z,z) : z \in \mathbf{R}^N\}$. *We say that a function* $\vartheta : \mathbf{R}^N \times \mathbf{R}^N \setminus \triangle \to [0,+\infty]$ *is* subadditive *if for all distinct* $z_i \in \mathbf{R}^N$, $i = 1,2,3$, *we have*

$$\vartheta(z_1,z_2) \leq \vartheta(z_1,z_3) + \vartheta(z_3,z_2).$$

We extend ϑ to the whole $\mathbf{R}^N \times \mathbf{R}^N$ setting $\vartheta \equiv 0$ on $\triangle$.

Notice that, in case we consider traslation invariant functions, that is $\vartheta(z_1,z_2) = \vartheta(z_1 - z_2, 0) =: \theta(z_1 - z_2)$ where $\theta : \mathbf{R}^N \setminus \{0\} \to [0,+\infty]$ and $\theta(0) = 0$, the subadditivity condition on ϑ can be rewritten for θ as

$$\theta(z_1 + z_2) \leq \theta(z_1) + \theta(z_2)$$

for every $z_i \in \mathbf{R}^N$, $i = 1,2$.

The following result is an easy generalization to the vector-valued case of Theorem 4.3 [39].

Theorem 2.7.20 *Let $\Omega \subset \mathbf{R}$ be a bounded open set, let $\vartheta : \mathbf{R}^N \times \mathbf{R}^N \to [0, +\infty]$ be a symmetric, subadditive and lower semicontinuous function.*

Let $u_j \in SBV\left(\Omega; \mathbf{R}^N\right)$ be such that $u_j \to u$ in measure on Ω and (u_j) satisfies (2.7.6), for some ϕ, θ as in Theorem 2.7.10. Then, $u \in SBV\left(\Omega; \mathbf{R}^N\right)$ and

$$\int_{S_u} \vartheta\left(u^+, u^-\right) \, d\mathcal{H}^0 \leq \liminf_j \int_{S_{u_j}} \vartheta\left(u_j^+, u_j^-\right) \, d\mathcal{H}^0.$$

Let us point out that the results stated in Theorem 2.7.20 heavily depend on the one-dimensional setting. Indeed, being $(u_j) \subset SBV\left(\Omega; \mathbf{R}^N\right)$ not equi-bounded in L^∞ a priori, by Theorem 2.7.10 we can only infer that $u \in (GSBV(\Omega))^N$. On the other hand, the super-linearity and the concavity assumptions on the function θ of (2.7.6) and the choice $n = 1$ imply $u \in SBV\left(\Omega; \mathbf{R}^N\right)$ (see the proof of Theorem 2.7.7).

2.8 Functions of Bounded Deformation

We recall some definitions and basic results on functions with bounded deformation. For the general theory of this subject we refer to [17] (see also [32],[60],[91]).

Definition 2.8.1 *Let $u \in L^1(\Omega; \mathbf{R}^n)$; we say that u is a* function of Bounded Deformation *in Ω, and we write $u \in BD(\Omega)$, if the symmetric part of the distributional derivative of u, $Eu := \frac{1}{2}\left(Du + D^t u\right)$, is a $n \times n$ matrix-valued Radon measure on Ω.*

It is easy to check that $BV(\Omega; \mathbf{R}^n) \subset BD(\Omega)$, and the inclusion is strict as shown in [17],[32].

For every $\xi \in \mathbf{R}^n$, let D_ξ be the distributional derivative in the direction ξ defined by $D_\xi v = \langle Dv, \xi \rangle$. For every function $u : \Omega \to \mathbf{R}^n$ let us define the function $u^\xi : \Omega \to \mathbf{R}$ by $u^\xi(x) := \langle u(x), \xi \rangle$.

Theorem 2.8.2 *If $u \in BD(\Omega)$, then $D_\xi u^\xi \in \mathcal{M}(\Omega)$ and*

$$D_\xi u^\xi = \langle Eu\xi, \xi \rangle.$$

Conversely, let $\{\xi_1, \ldots, \xi_n\}$ be a basis of $\mathbf{R}^n$ and let $u \in L^1(\Omega; \mathbf{R}^n)$; then $u \in BD(\Omega)$ if $D_\xi u^\xi \in \mathcal{M}(\Omega)$ for every ξ of the form $\xi_i + \xi_j$, $i, j = 1, \ldots, n$.

As in the BV case, we can decompose Eu as

$$Eu = E^a u + E^j u + E^c u,$$

where $E^a u$ is the absolutely continuous part of Eu with respect to $\mathcal{L}^n$, $E^j u$ is the restriction of Eu to $J_u^\star$, called *jump part* of Eu, and $E^c u$ is the restriction of $E^s u$ to $\Omega \setminus J_u^\star$, called *Cantor part* of Eu.

The following theorem is analogous to Theorem 2.6.8.

Theorem 2.8.3 *Let $u \in BD(\Omega)$, then*

(i) u is approximately differentiable at a.e. x in Ω and for every $B \in \mathcal{B}(\Omega)$

$$E^a u(B) = \int_B \mathcal{E}u(x)\,dx,$$

where

$$\mathcal{E}u := \frac{1}{2}\left(\nabla u + \nabla^t u\right);$$

(ii) $\|E^c u\|$ vanishes on sets $B \in \mathcal{B}(\Omega)$ such that $\mathcal{H}^{n-1}(B) < +\infty$;

(iii) $J_u^\star$ is countably $\mathcal{H}^{n-1}$ rectifiable and for every $B \in \mathcal{B}(\Omega)$

$$E^j u(B) = \int_{B\cap J_u^\star} (u^+ - u^-) \odot \nu_u\, d\mathcal{H}^{n-1},$$

where $a \odot b := \frac{1}{2}(a \otimes b + b \otimes a)$.

One of the main open problem in BD theory is to establish whether $\mathcal{H}^{n-1}(S_u^\star \setminus J_u^\star) = 0$ or not. So far, it has been proved that $\mathcal{H}^{n-1+\varepsilon}(S_u^\star \setminus J_u^\star) = 0$ for every $\varepsilon > 0$, i.e., the Hausdorff dimension of $S_u^\star \setminus J_u^\star$ is at most $n-1$.

Definition 2.8.4 *Let $u \in BD(\Omega)$; we say that u is a* Special function of Bounded Deformation *in Ω, and we write $u \in SBD(\Omega)$, if $E^c u = 0$.*

It is easy to check that $SBV(\Omega;\mathbf{R}^n) \subset SBD(\Omega)$, and the inclusion is strict as shown in [17],[32].

In analogy with the BV functions, we may characterize the spaces $BD(\Omega)$ and $SBD(\Omega)$ by means of suitable one-dimensional sections, for which we introduce an appropriate notation (see [17]). Let $\xi \in \mathbf{R}^n \setminus \{0\}$ and let Π^ξ, E_y^ξ, E_ξ be defined as before Lemma 2.4.2, with $E \subset \mathbf{R}^n$ and $y \in E_\xi$. Moreover, given $u : E \to \mathbf{R}^n$, we define $u^{\xi,y} : E_y^\xi \to \mathbf{R}$ by

$$u^{\xi,y}(t) := \langle u\,(y + t\xi)\,, \xi\rangle.$$

If $u \in BD(\Omega)$ we set

$$J_u^\xi := \left\{x \in J_u^\star : \langle u^+(x) - u^-(x), \xi\rangle \neq 0\right\}.$$

Note that, since $\mathcal{H}^{n-1}\left(\{\xi \in \mathbf{S}^{n-1} : \langle u^+(x) - u^-(x), \xi\rangle = 0\}\right) = 0$ for every $x \in J_u^\star$, by Fubini's Theorem we have for $\mathcal{H}^{n-1}$ a.e. $\xi \in \mathbf{S}^{n-1}$.

$$\mathcal{H}^{n-1}\left(J_u^\star \setminus J_u^\xi\right) = 0.$$

Theorem 2.8.5 (a) *Let $u \in BD(\Omega)$ and let $\xi \in \mathbf{S}^{n-1}$. Then $u^{\xi,y} \in BV(\Omega_y^\xi)$ for $\mathcal{H}^{n-1}$ a.e. $y \in \Omega_\xi$, for such y the following equalities hold*

(i) $\dot{u}^{\xi,y}(t) = \langle \mathcal{E}u\,(y + t\xi)\,\xi, \xi\rangle$ for $\mathcal{L}^1$ a.e. $t \in \Omega_y^\xi$;

(ii) $J_{u^{\xi,y}} = \left(J_u^\xi\right)_y^\xi$.

Moreover, we have

$$\int_{\Omega_\xi} \|Du^{\xi,y}\|(B_y^\xi)\, d\mathcal{H}^{n-1}(y) = \|D_\xi u^\xi\|(B) < +\infty \tag{2.8.1}$$

for every $B \in \mathcal{B}(\Omega)$.

(b) *Conversely, let* $u \in L^1(\Omega; \mathbf{R}^n)$ *and let* $\{\xi_1, \ldots, \xi_n\}$ *be a basis of* $\mathbf{R}^n$. *If for every* ξ *of the form* $\xi_i + \xi_j$, $u^{\xi,y} \in BV(\Omega_y^\xi)$ *for* $\mathcal{H}^{n-1}$ *a.e.* $y \in \Omega_\xi$ *and*

$$\int_{\Omega_\xi} \left\|Du^{\xi,y}\right\| (\Omega_y^\xi)\, d\mathcal{H}^{n-1}(y) < +\infty,$$

then $u \in BD(\Omega)$.

Moreover, if $u \in BD(\Omega)$, *then* $u \in SBD(\Omega)$ *if and only if* $u^{\xi,y} \in SBV(\Omega_y^\xi)$ *for every* ξ *of the form* $\xi_i + \xi_j$ *and for* $\mathcal{H}^{n-1}$ *a.e.* $y \in \Omega_\xi$.

The following compactness result in $SBD(\Omega)$, analogous to the $GSBV$ Compactness Theorem 2.7.11, is due to Bellettini, Coscia and Dal Maso (see [32]) and its proof is based on slicing techniques and on the characterization of $SBD(\Omega)$ provided by Theorem 2.8.5.

Theorem 2.8.6 *Let* $(u_j) \subset SBD(\Omega)$ *be such that*

$$\sup_j \left(\int_\Omega |\mathcal{E}u_j|^2\, dx + \mathcal{H}^{n-1}\left(J_{u_j}^\star\right) + \|u_j\|_{L^\infty(\Omega;\mathbf{R}^n)} \right) < +\infty.$$

Then, there exists a subsequence (u_{j_k}) *converging in* $L^1_{loc}(\Omega; \mathbf{R}^n)$ *to a function* $u \in SBD(\Omega)$. *Moreover,* $\mathcal{E}u_{j_k} \to \mathcal{E}u$ *weakly in* $L^2\left(\Omega; \mathbf{R}^{n^2}\right)$ *and*

$$\liminf_{k\to+\infty} \mathcal{H}^{n-1}\left(J_{u_{j_k}}^\star\right) \geq \mathcal{H}^{n-1}\left(J_u^\star\right).$$

Actually, a sharper lower semicontinuity result in SBD can be proved by following the same ideas and strategy of the proof of Theorem 2.8.6.

Theorem 2.8.7 *Let* $u_j, u \in SBD(\Omega)$ *be such that* $u_j \to u$ *in* $L^1(\Omega; \mathbf{R}^n)$ *and*

$$\sup_j \int_\Omega \left(|\langle \mathcal{E}u_j(x)\xi, \xi\rangle|^2\, dx + \int_{J_{u_j}^\xi} |\langle \nu_{u_j}, \xi\rangle|\, d\mathcal{H}^{n-1} \right) < +\infty \tag{2.8.2}$$

for $\xi \in \mathbf{R}^n \setminus \{0\}$. *Then* $\langle \mathcal{E}u_j(x)\xi, \xi\rangle \to \langle \mathcal{E}u(x)\xi, \xi\rangle$ *weakly in* $L^2(\Omega)$ *and*

$$\int_{J_u^\xi} |\langle \nu_u, \xi\rangle|\, d\mathcal{H}^{n-1} \leq \liminf_j \int_{J_{u_j}^\xi} |\langle \nu_{u_j}, \xi\rangle|\, d\mathcal{H}^{n-1}.$$

In particular, if (2.8.2) *holds for every* $\xi \in \{\xi_1, \ldots, \xi_n\}$ *orthogonal basis in* $\mathbf{R}^n$, *then* $\operatorname{div} u_j \to \operatorname{div} u$ *weakly in* $L^2(\Omega)$.

Eventually, we introduce the following subspace of $SBD(\Omega)$

$$SBD^2(\Omega) := \left\{ u \in SBD(\Omega) : \mathcal{H}^1(J_u^\star) < +\infty;\ \mathcal{E}u \in L^2\left(\Omega; \mathbf{R}^{n^2}\right) \right\}.$$

Chapter 3

Variational Approximation of Energies with Superlinear Growth

3.1 Statement of the Γ-Convergence Result

In this Chapter[1] we prove a variational approximation for integral functionals defined on $(GSBV(\Omega))^N$ having the form

$$\mathcal{F}(u) = \int_{\Omega} f(x, u, \nabla u)\, dx + \int_{J_u} \varphi(\nu_u)\, d\mathcal{H}^{n-1}, \tag{3.1.1}$$

where $\varphi : \mathbf{R}^n \to [0, +\infty)$ is a norm; and $f : \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times n} \to [0, +\infty)$ is a Carathéodory integrand, quasiconvex in z, satisfying

$$c_1 |z|^p + b(u) - a(x) \le f(x, u, z) \le c_2(|z|^p + b(u) + a(x)) \tag{3.1.2}$$

for every $(x, u, z) \in \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times n}$ with $p \in (1, +\infty)$, c_1,$c_2 > 0$, $a \in L^1(\Omega)$, and $b \in C^0\left(\mathbf{R}^N\right)$ a non negative function.

Set M, m for $\max_{\mathbf{S}^{n-1}} \varphi$ and $\min_{\mathbf{S}^{n-1}} \varphi$, respectively. Notice that $m > 0$, thus for every $\nu \in \mathbf{R}^n$ there holds

$$m|\nu| \le \varphi(\nu) \le M|\nu|. \tag{3.1.3}$$

To perform the approximation we add a formal extra variable v to $\mathcal{F}$, defining F : $\mathcal{B}\left(\Omega; \mathbf{R}^{N+1}\right) \to [0, +\infty]$ by

$$F(u, v) = \begin{cases} \mathcal{F}(u) & u \in (GSBV^p(\Omega))^N, \ \ v = 1 \ \text{ a.e. in } \Omega \\ \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1.4}$$

[1] The contents of this Chapter are contained in the paper *On the variational approximation of free-discontinuity problems in the vectorial case*, published on Math. Models Methods Appl. Sci. **11** (2001), 663–684.

The approximating functionals $F_\varepsilon : \mathcal{B}\left(\Omega;\mathbf{R}^{N+1}\right) \to [0,+\infty]$ have the form

$$F_\varepsilon(u,v) = \begin{cases} \displaystyle\int_\Omega \left((\psi(v)+\eta_\varepsilon) f(x,u,\nabla u) + \frac{1}{\varepsilon p'} W(v) + \frac{\varepsilon^{p-1}}{p}\varphi^p(\nabla v) \right) dx \\ \qquad\qquad (u,v) \in W^{1,p}\left(\Omega;\mathbf{R}^{N+1}\right),\ 0 \le v \le 1 \text{ a.e. in } \Omega \\ +\infty \qquad\qquad \text{otherwise,} \end{cases} \tag{3.1.5}$$

where $\psi : [0,1] \to [0,1]$ is any increasing lower semicontinuous function such that $\psi(0) = 0$, $\psi(1) = 1$, and $\psi(t) > 0$ if $t > 0$; $p' = \frac{p}{p-1}$; η_ε is any positive infinitesimal faster than ε^{p-1} for $\varepsilon \to 0^+$; $W(t) = c_W(1-t)^p$; $\Phi : [0,1] \to [0,+\infty)$ is the auxiliary function defined by

$$\Phi(t) := \int_0^t (W(s))^{\frac{1}{p'}}\, ds, \tag{3.1.6}$$

where c_W is chosen such that $\Phi(1) = \frac{1}{2}$, i.e., $c_W = \left(2\int_0^1 (1-s)^{\frac{p}{p'}}\, ds\right)^{-p'}$.

Let us state and prove the main result of the Chapter.

Theorem 3.1.1 *Let $\Omega \subset \mathbf{R}^n$ be an open set with Lipschitz boundary. Let (F_ε) be as above, then (F_ε) Γ-converges with respect to the convergence in measure to the functional F given by (3.1.4).*

Remark 3.1.2 *The apriori condition of quasiconvexity on f is assumed only for simplicity, as in the general case it would suffices to replace in the formula of the effective energy the function f by its quasiconvexification, i.e., the greatest quasiconvex function less or equal to f.*

Remark 3.1.3 *Let us point out that the assumption $v \in [0,1]$ a.e. in (3.1.5) is not restrictive since the functionals F_ε are decreasing by truncations in the v variable.*

Remark 3.1.4 *For the sake of simplicity we have chosen explicitely the potential W. The same result can be proved, with similar techniques, for any continuous function $W : [0,1] \to [0,+\infty)$ such that $W(1) = 0$ and $W(t) > 0$ if $t \neq 1$ (see [37],[39]).*

We divide the proof of Theorem 3.1.1 into two parts, each corresponding to the two inequalities of Definition 2.3.1.

3.2 Lower bound inequality

Lemma 3.2.1 *For any $(u,v) \in \mathcal{B}\left(\Omega;\mathbf{R}^{N+1}\right)$*

$$\Gamma(meas)\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon(u,v) \ge F(u,v). \tag{3.2.1}$$

Proof. Let $\varepsilon_j \to 0^+$, $(u_j, v_j) \in \mathcal{B}\left(\Omega; \mathbf{R}^{N+1}\right)$ be such that $(u_j, v_j) \to (u, v)$ in measure on Ω. Without loss of generality we may suppose

$$\liminf_j F_{\varepsilon_j}(u_j, v_j) < +\infty. \tag{3.2.2}$$

Notice that condition (3.2.2) implies the convergence of (v_j) to 1 in measure on Ω, hence $v = 1$ a.e. in Ω.

We further divide the proof of the lower bound inequality (3.2.1) into two steps corresponding to the estimate on the bulk term and on the surface term, respectively.

Step 1:(Bulk energy inequality) We prove the following inequality

$$\liminf_j \int_\Omega \psi(v_j) f(x, u_j, \nabla u_j)\, dx \geq \int_\Omega f(x, u, \nabla u)\, dx. \tag{3.2.3}$$

First suppose to extract a subsequence, not relabelled for convenience, such that $(u_j, v_j) \to (u, 1)$ a.e. in Ω and

$$\liminf_j \int_\Omega \psi(v_j) f(x, u_j, \nabla u_j)\, dx = \lim_j \int_\Omega \psi(v_j) f(x, u_j, \nabla u_j)\, dx.$$

Consider the auxiliary function Φ introduced in (3.1.6), we claim that $(\Phi(v_j))$ is bounded in $BV(\Omega)$. Indeed, (3.1.3), Young's inequality and (3.2.2) yield

$$\begin{aligned}\sup_j \|D(\Phi(v_j))\|(\Omega) &= \sup_j \int_\Omega |\nabla(\Phi(v_j))|\, dx \\ &\leq \frac{1}{c} \sup_j \int_\Omega \left(\frac{1}{\varepsilon_j p'} W(v_j) + \frac{\varepsilon_j^{p-1}}{p} \varphi^p(\nabla v_j) \right) dx < +\infty,\end{aligned} \tag{3.2.4}$$

for a suitable positive constant c.

To prove $u \in (GSBV^p(\Omega))^N$, let $0 < \gamma < \gamma' < \Phi(1)$ and set

$$U_{j,t} := \{x \in \Omega : \Phi(v_j(x)) > t\}. \tag{3.2.5}$$

By the BV Coarea formula (see Theorem 2.6.6), $U_{j,t}$ has finite perimeter for $\mathcal{L}^1$ a.e. $t \in \mathbf{R}$. Set $p_j(t) = \|D\mathcal{X}_{U_{j,t}}\|(\Omega)$, by the Mean Value Theorem there exists $t_j \in (\gamma, \gamma')$ such that

$$(\gamma' - \gamma)\, p_j(t_j) \leq \int_\gamma^{\gamma'} p_j(t)\, dt \leq \int_0^{\Phi(1)} p_j(t)\, dt = \|D(\Phi(v_j))\|(\Omega). \tag{3.2.6}$$

Let $U_j := U_{j,t_j}$ and define $w_j = u_j \mathcal{X}_{U_j}$, then by a truncation argument and the BV Chain Rule formula (see Theorem 2.6.12) it is easy to check that $w_j \in (GSBV(\Omega))^N$, and also that

$$\begin{aligned}&\liminf_j \int_\Omega \psi(v_j) f(x, u_j, \nabla u_j)\, dx \\ &\quad \geq \liminf_j \psi\left(\Phi^{-1}(\gamma)\right) \int_{U_j} f(x, u_j, \nabla u_j)\, dx \\ &\quad \geq \liminf_j \psi\left(\Phi^{-1}(\gamma)\right) \int_{U_j} f(x, w_j, \nabla w_j)\, dx \\ &\quad = \liminf_j \psi\left(\Phi^{-1}(\gamma)\right) \int_\Omega f(x, w_j, \nabla w_j)\, dx\end{aligned} \tag{3.2.7}$$

where the last equality follows from (3.1.2).

Hence, by (3.1.2), (3.2.2), (3.2.4), (3.2.6) and (3.2.7) the sequence (w_j) satisfies all the assumptions of the $GSBV$ Closure Theorem (see Theorem 2.7.10) with $\phi(t) = t^p$ and $\theta(t) \equiv 1$. Then, since $w_j \to u$ a.e. in Ω, $u \in GSBV^p\left(\Omega; \mathbf{R}^N\right)$. Actually, we can apply the same argument componentwise to deduce $u \in \left(GSBV^p(\Omega)\right)^N$.

Eventually, by using the $GSBV$ lower semicontinuity Theorem 2.7.17 from (3.2.7) we get

$$\liminf_j \int_\Omega \psi(v_j) f(x, u_j, \nabla u_j)\, dx \geq \psi\left(\Phi^{-1}(\gamma)\right) \int_\Omega f(x, u, \nabla u)\, dx.$$

The lower semicontinuity of ψ yields inequality (3.2.3), since, by letting $\gamma \to \Phi(1)$, we have $\Phi^{-1}(\gamma) \to 1$.

Step 2:(Surface energy inequality) We prove the following inequality

$$\liminf_j \int_\Omega \left(\frac{1}{\varepsilon_j p'} W(v_j) + \frac{\varepsilon_j^{p-1}}{p} \varphi^p(\nabla v_j)\right) dx \geq \int_{J_u} \varphi(\nu_u)\, d\mathcal{H}^{n-1}. \tag{3.2.8}$$

It is useful to introduce the dual norm $\varphi_\circ : \mathbf{R}^n \to [0, +\infty)$ of φ defined as

$$\varphi_\circ(\nu) := \sup_{\xi \in \mathbf{S}^{n-1}} \left(\frac{1}{\varphi(\xi)} |\langle \nu, \xi \rangle|\right), \tag{3.2.9}$$

and it is easy to see that $(\varphi_\circ)_\circ \equiv \varphi$.

Notice that, for every $j \in \mathbf{N}$, Young's inequality yields

$$\begin{aligned} &\int_\Omega \left(\frac{1}{\varepsilon_j p'} W(v_j) + \frac{\varepsilon_j^{p-1}}{p} \varphi^p(\nabla v_j)\right) dx \geq \int_\Omega (W(v_j))^{\frac{1}{p'}} \varphi(\nabla v_j)\, dx \\ &= \int_\Omega \varphi(\nabla(\Phi(v_j)))\, dx = \int_0^{\Phi(1)} \left(\int_{J_{\chi_{U_{j,t}}}} \varphi\left(\nu_{\chi_{U_{j,t}}}\right) d\mathcal{H}^{n-1}\right) dt, \end{aligned} \tag{3.2.10}$$

with $U_{j,t}$ defined in (3.2.5) and the last equality following from the BV Coarea formula (see Theorem 2.6.6)

With fixed $A \in \mathcal{A}(\Omega)$ and $\xi \in \mathbf{S}^{n-1}$, we claim that for $\mathcal{H}^{n-1}$ a.e. $y \in A_\xi$

$$\liminf_j \mathcal{H}^0\left(J_{\left(\chi_{U_{j,t}}\right)_{\xi,y}} \cap A\right) \geq 2\mathcal{H}^0\left(J_{u_{\xi,y}} \cap A\right) \tag{3.2.11}$$

for every $t \in (0, \Phi(1))$.

Assume (3.2.11) proven, then, with fixed $t \in (0, \Phi(1))$, (3.2.9), (2.4.1) of Lemma 2.4.2 and Fatou's lemma yield the following lower semicontinuity estimate

$$\begin{aligned} &\liminf_j \int_{J_{\chi_{U_{j,t}}} \cap A} \varphi\left(\nu_{\chi_{U_{j,t}}}\right) d\mathcal{H}^{n-1} \geq \liminf_j \frac{1}{\varphi_\circ(\xi)} \int_{J_{\chi_{U_{j,t}}} \cap A} |\langle \nu_{\chi_{U_{j,t}}}, \xi \rangle|\, d\mathcal{H}^{n-1} \\ &= \liminf_j \frac{1}{\varphi_\circ(\xi)} \int_{A_\xi} \mathcal{H}^0\left(J_{\left(\chi_{U_{j,t}}\right)_{\xi,y}} \cap A\right) d\mathcal{H}^{n-1} \\ &\geq \frac{2}{\varphi_\circ(\xi)} \int_{A_\xi} \mathcal{H}^0\left(J_{u_{\xi,y}} \cap A\right) d\mathcal{H}^{n-1} = \frac{2}{\varphi_\circ(\xi)} \int_{J_u \cap A} |\langle \nu_u, \xi \rangle|\, d\mathcal{H}^{n-1}. \end{aligned} \tag{3.2.12}$$

Consider the superadditive function on disjoint open sets $\lambda_t : \mathcal{A}(\Omega) \to [0,+\infty)$ defined by

$$\lambda_t(A) := \liminf_j \int_{J_{\chi_{U_{j,t}}} \cap A} \varphi\left(\nu_{\chi_{U_{j,t}}}\right) \, d\mathcal{H}^{n-1},$$

then, by Lemma 2.2.4 and the very definition of the dual norm $\varphi_\circ$, we get

$$\lambda_t(\Omega) = \liminf_j \int_{J_{\chi_{U_{j,t}}}} \varphi\left(\nu_{\chi_{U_{j,t}}}\right) \, d\mathcal{H}^{n-1} \geq 2\int_{J_u} \varphi\left(\nu_u\right) \, d\mathcal{H}^{n-1} \tag{3.2.13}$$

by passing to the supremum in (3.2.12) on a sequence (ξ_j) dense in $\mathbf{S}^{n-1}$. Finally, by (3.2.10) and (3.2.13), Fatou's lemma yields (3.2.8).

Thus, to conclude we have only to prove (3.2.11). Notice that, by Theorem 2.7.6 for $\mathcal{H}^{n-1}$ a.e. $y \in \Omega_\xi$ there holds

$$u_{\xi,y} \in \left(GSBV^p\left(\Omega_y^\xi\right)\right)^N, \ \left((u_j)_{\xi,y}, (v_j)_{\xi,y}\right) \to \left(u_{\xi,y}, 1\right) \tag{3.2.14}$$

in measure on Ω_y^ξ. Moreover, by (3.1.2), (3.2.2), (3.2.9) and Fatou's lemma for $\mathcal{H}^{n-1}$ a.e. $y \in \Omega_\xi$ there holds

$$\begin{aligned}\liminf_j \int_{\Omega_y^\xi} \Big(&\psi\left((v_j)_{\xi,y}\right) \left|(\dot{u}_j)_{\xi,y}\right|^p \\ &+\frac{1}{\varepsilon_j p'} W\left((v_j)_{\xi,y}\right) + \frac{\varepsilon_j^{p-1}}{p}\frac{1}{\varphi_\circ^p(\xi)} \left|(\dot{v}_j)_{\xi,y}\right|^p\Big) dt < +\infty. \end{aligned} \tag{3.2.15}$$

Fix $y \in \Omega_\xi$ be satisfying (3.2.14), (3.2.15), and assume also $\mathcal{H}^0\left(J_{u_{\xi,y}} \cap A\right) > 0$ since otherwise (3.2.11) is trivial. Let $\{t_1, ..., t_r\}$ be an arbitrary subset of $J_{u_{\xi,y}} \cap A$, and $(I_i)_{1 \leq i \leq r}$ be a family of pairwise disjoint open intervals such that $t_i \in I_i$, $I_i \subset\subset A_y^\xi$. Then, for every $1 \leq i \leq r$, we claim that

$$s_i = \limsup_j \left(\inf_{I_i} \psi\left((v_j)_{\xi,y}\right)\right) = 0.$$

Indeed, if s_h was strictly positive for some $h \in \{1, ..., r\}$, then

$$\inf_{I_h} \psi\left((v_{j_k})_{\xi,y}\right) \geq \frac{s_h}{2}$$

for a suitable subsequence, and thus (3.2.15) would give

$$\frac{s_h}{2} \liminf_{k \to +\infty} \int_{I_h} \left|(\dot{u}_{j_k})_{\xi,y}\right|^p \leq c.$$

Hence, Rellich-Kondrakov's theorem and (3.2.14) would imply the slice $u_{\xi,y} \in W^{1,1}\left(I_h, \mathbf{R}^N\right)$, which is a contradiction since $\mathcal{H}^0\left(J_{u_{\xi,y}} \cap I_h\right) > 0$.

So let $t_j^i \in I_i$ be such that

$$\lim_j (v_j)_{\xi,y}\left(t_j^i\right) = 0,$$

and α_i, $\beta_i \in I_i$, with $\alpha_i < t_j^i < \beta_i$, be such that

$$\lim_j (v_j)_{\xi,y}(\alpha_i) = \lim_j (v_j)_{\xi,y}(\beta_i) = 1.$$

Then, for all $t \in (0, \Phi(1))$, since $\left(\mathcal{X}_{U_{j,t}}\right)_{\xi,y} = \mathcal{X}_{(U_{j,t})_y^\xi}$, there follows

$$\liminf_j \mathcal{H}^0 \left(J_{\left(\mathcal{X}_{U_{j,t}}\right)_{\xi,y}} \cap I_i \right) \geq 2.$$

Hence, the subadditivity of the inferior limit yields

$$\liminf_j \mathcal{H}^0 \left(J_{\left(\mathcal{X}_{U_{j,t}}\right)_{\xi,y}} \cap A \right) \geq 2r,$$

and by the arbitrariness of r we get (3.2.11). $\square$

3.3 Upper bound inequality

To prove the upper bound inequality, we have to construct a recovery sequence for any function u in $(GSBV^p(\Omega))^N$.

First notice that using an approximation procedure we can reduce ourselves to consider the case in which the limit u belongs to $\mathcal{W}\left(\Omega; \mathbf{R}^N\right)$. Indeed, without loss of generality we may assume $v = 1$ a.e. in Ω and $\mathcal{H}^{n-1}(J_u) < +\infty$, the cases $\mathcal{L}^n(\{x \in \Omega : v(x) < 1\}) > 0$ and $\mathcal{H}^{n-1}(J_u) = +\infty$ being trivial, and suppose the upper bound inequality proven for functions in $\mathcal{W}\left(\Omega; \mathbf{R}^N\right)$.

Let u belong to $SBV^p \cap L^\infty\left(\Omega; \mathbf{R}^N\right)$, take $(u_j) \subset \mathcal{W}\left(\Omega; \mathbf{R}^N\right)$ to be the sequence provided by Theorem 2.7.14, then (2.7.7), Remark 2.7.15 and Theorem 2.7.18 yield

$$\lim_j \int_{J_{u_j}} \varphi\left(\nu_{u_j}\right) d\mathcal{H}^{n-1} = \int_{J_u} \varphi\left(\nu_u\right) d\mathcal{H}^{n-1}.$$

Moreover, Theorem 2.7.17 and Fatou's lemma yield

$$\lim_j \int_\Omega f\left(x, u_j, \nabla u_j\right) dx = \int_\Omega f\left(x, u, \nabla u\right) dx.$$

By a simple diagonal argument and the lower semicontinuity of the upper Γ-limit the upper bound inequality then follows for any u in $SBV^p \cap L^\infty\left(\Omega; \mathbf{R}^N\right)$.

Furthermore, if u belongs to $(GSBV^p(\Omega))^N$, fix $i \in \mathbf{N}$ and consider the auxiliary functions Ψ_i defined by

$$\Psi_i(u) := \begin{cases} u & \text{if } |u| \leq a_i \\ & \\ 0 & \text{if } |u| \geq a_{i+1} \end{cases}, \tag{3.3.1}$$

where $(a_i) \subset (0,+\infty)$ is a strictly increasing and diverging sequence, and for every $i \in \mathbf{N}$ $\Psi_i \in C^1\left(\mathbf{R}^N;\mathbf{R}^N\right)$ and $\|\nabla\Psi_i\|_{L^\infty(\mathbf{R}^N;\mathbf{R}^{N\times N})} \leq 1$. Notice that $u^i = \Psi_i(u)$ belongs to $SBV^p \cap L^\infty\left(\Omega;\mathbf{R}^N\right)$. Lebesgue's Dominated Convergence Theorem yields

$$\lim_i \int_{J_{u^i}} \varphi\left(\nu_{u^i}\right)\, d\mathcal{H}^{n-1} = \int_{J_u} \varphi\left(\nu_u\right)\, d\mathcal{H}^{n-1}.$$

Moreover, Theorem 2.7.17 and Fatou's lemma yield

$$\lim_i \int_\Omega f\left(x,u^i,\nabla u^i\right)\, dx = \int_\Omega f\left(x,u,\nabla u\right)\, dx,$$

then we may use again a standard diagonal argument and the lower semicontinuity of the upper Γ-limit to conclude.

Thus, we have reduced ourselves to prove the following lemma.

Lemma 3.3.1 *Let $u \in \mathcal{W}\left(\Omega;\mathbf{R}^N\right)$, there exists a sequence $(u_j,v_j) \to (u,1)$ in measure on Ω such that*

$$\limsup_j F_{\varepsilon_j}\left(u_j,v_j\right) \leq F\left(u,1\right).$$

Proof. Assumption $u \in \mathcal{W}\left(\Omega;\mathbf{R}^N\right)$ implies that we can find a finite number of polyhedral sets K^i such that

(i) $\overline{S_u} = \Omega \cap \bigcup_{i=1}^r K^i$;

(ii) for every $1 \leq i \leq r$ the set K^i is contained in a $(n-1)$-dimensional plane Π^{ν_i}, with normal ν_i, and $\Pi^{\nu_i} \neq \Pi^{\nu_j}$ for $i \neq j$.

Let a_ε^i, b_ε, d_ε be positive infinitesimals for $\varepsilon \to 0^+$; let γ_ε^i be a minimizer of the one-dimensional problem

$$\int_{b_\varepsilon}^{a_\varepsilon^i+b_\varepsilon} \left(\frac{1}{\varepsilon p'}W\left(v\right) + \frac{\varepsilon^{p-1}}{p}\varphi^p\left(\nu_i\right)|\dot{v}|^p\right)dt, \tag{3.3.2}$$

with the conditions $v\left(b_\varepsilon\right) = 0$, $v\left(a_\varepsilon^i + b_\varepsilon\right) = 1 - d_\varepsilon$, $v \in W^{1,1}\left(b_\varepsilon, a_\varepsilon^i + b_\varepsilon\right)$.

According to [83], the minimum value in (3.3.2) is exactly $\Phi(1-d_\varepsilon)\varphi\left(\nu_i\right)$, and it is achieved by those functions γ_ε^i for which Young's inequality holds with an equality sign, i.e., γ_ε^i is the unique solution of the Cauchy's problem

$$\begin{cases} \dot{\gamma}_\varepsilon^i = \frac{1}{\varphi(\nu_i)\varepsilon}\left(W\left(\gamma_\varepsilon^i\right)\right)^{\frac{1}{p}} \\ \gamma_\varepsilon^i\left(b_\varepsilon\right) = 0. \end{cases}$$

Thus, γ_ε^i is increasing, $0 \leq \gamma_\varepsilon^i \leq 1-d_\varepsilon$, and an explicit computation yields $a_\varepsilon^i = -\varepsilon\varphi\left(\nu_i\right)\ln d_\varepsilon$. Hence, d_ε is chosen such that the a_ε^i is infinitesimal for $\varepsilon \to 0^+$.

Define the functions $\alpha_\varepsilon^i : [0,+\infty) \to [0, 1-d_\varepsilon]$ by

$$\alpha_\varepsilon^i(t) = \begin{cases} 0 & 0 \le t \le b_\varepsilon \\ \\ \gamma_\varepsilon^i(t) & b_\varepsilon \le t \le a_\varepsilon^i + b_\varepsilon \\ \\ 1-d_\varepsilon & t \ge a_\varepsilon^i + b_\varepsilon. \end{cases} \tag{3.3.3}$$

Denote by $\pi_i : \mathbf{R}^n \to \Pi^{\nu_i}$ the orthogonal projection onto Π^{ν_i} and set $d_i(x) := d(x, \Pi^{\nu_i})$. It is well known that if $x \in \mathbf{R}^n \setminus \Pi^{\nu_i}$ there holds

$$\nabla d_i(x) = \frac{x - \pi_i(x)}{|x - \pi_i(x)|} = \pm \nu_i. \tag{3.3.4}$$

For any $\delta > 0$ set $K_\delta^i := \{y \in \Pi^{\nu_i} : d(y, K^i) \le \delta\}$, fix $\varepsilon > 0$, let β_ε^i be a cut-off function between K_ε^i and $K_{2\varepsilon}^i$, i.e., satisfying

$$\beta_\varepsilon^i \in C_c^\infty\left(K_{2\varepsilon}^i\right), \ 0 \le \beta_\varepsilon^i \le 1, \ \beta_\varepsilon^i|_{K_\varepsilon^i} = 1, \ \left\|\nabla \beta_\varepsilon^i\right\|_{L^\infty(\Pi^{\nu_i})} \le \frac{2}{d\left(K_\varepsilon^i, K_{2\varepsilon}^i\right)}.$$

Then define

$$v_\varepsilon^i(x) := \beta_\varepsilon^i(\pi_i(x))\, \alpha_\varepsilon^i(d_i(x)) + \left(1 - \beta_\varepsilon^i(\pi_i(x))\right)(1 - d_\varepsilon), \tag{3.3.5}$$

and

$$B_\varepsilon^i := \left\{x \in \mathbf{R}^n : \pi_i(x) \in K_\varepsilon^i, \ d_i(x) \le b_\varepsilon\right\};$$
$$C_\varepsilon^i := \left\{x \in \mathbf{R}^n : \pi_i(x) \in K_{2\varepsilon}^i, \ d_i(x) \le a_\varepsilon^i + b_\varepsilon\right\}.$$

By the very definition of v_ε^i there holds (see Figure 1 below)

$$v_\varepsilon^i = \begin{cases} 1 - d_\varepsilon & \Omega \setminus C_\varepsilon^i \\ \\ 0 & B_\varepsilon^i, \end{cases} \tag{3.3.6}$$

and since $\pi_i \in W^{1,\infty}(\mathbf{R}^n, \mathbf{R}^n)$ with $\|\nabla \pi_i\|_{L^\infty(\mathbf{R}^n, \mathbf{R}^{n\times n})} \le 1$, $\|\nabla \beta_\varepsilon^i\|_{L^\infty(\Pi^{\nu_i})} \le c\varepsilon^{-1}$ and $|\nabla d_i| = 1$ a.e., there exists a positive constant c such that

$$\left\|\nabla v_\varepsilon^i\right\|_{L^\infty(\mathbf{R}^n, \mathbf{R}^n)} \le \frac{c}{\varepsilon}. \tag{3.3.7}$$

Thus, $0 \le v_\varepsilon^i \le 1$, $v_\varepsilon^i \in W^{1,\infty}(\Omega)$ and $v_\varepsilon^i \to 1$ a.e. in Ω.

Define $H_\varepsilon^i = C_\varepsilon^i \setminus B_\varepsilon^i$, let us estimate the integral

$$I_\varepsilon^i = \int_{H_\varepsilon^i} \left(\frac{1}{\varepsilon p'} W\left(v_\varepsilon^i\right) + \frac{\varepsilon^{p-1}}{p} \varphi^p\left(\nabla v_\varepsilon^i\right)\right) dx. \tag{3.3.8}$$

To do this, consider the sets

$$H_\varepsilon^{i,1} := \left\{x \in \mathbf{R}^n : \pi_i(x) \in K_{2\varepsilon}^i \setminus K_\varepsilon^i, \ d_i(x) \le a_\varepsilon^i + b_\varepsilon\right\},$$
$$H_\varepsilon^{i,2} := \left\{x \in \mathbf{R}^n : \pi_i(x) \in K_\varepsilon^i, \ b_\varepsilon \le d_i(x) \le a_\varepsilon^i + b_\varepsilon\right\},$$

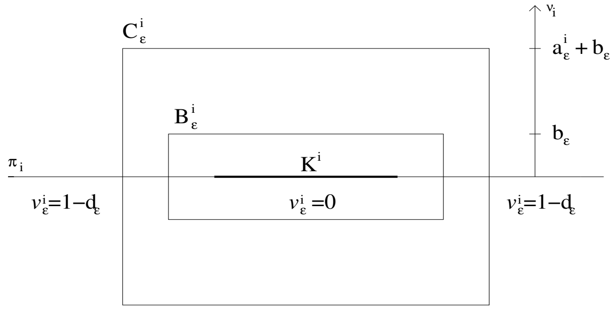

Figure 3.1: construction of v^i_ε

then $H^i_\varepsilon = H^{i,1}_\varepsilon \cup H^{i,2}_\varepsilon$, and setting

$$I^{i,j}_\varepsilon = \int_{H^{i,j}_\varepsilon} \left(\frac{1}{\varepsilon p'} W\left(v^i_\varepsilon\right) + \frac{\varepsilon^{p-1}}{p} \varphi^p \left(\nabla v^i_\varepsilon\right)\right) dx, \tag{3.3.9}$$

there follows $I^i_\varepsilon = I^{i,1}_\varepsilon + I^{i,2}_\varepsilon$. We estimate the $I^{i,j}_\varepsilon$ separately.

By (3.3.7), and since $\mathcal{H}^{n-1}\left(K^i_{2\varepsilon} \setminus K^i_\varepsilon\right) = O\left(\varepsilon\right)$ for $\varepsilon \to 0^+$, we get

$$I^{i,1}_\varepsilon \leq \frac{c}{\varepsilon} \mathcal{L}^n \left(H^{i,1}_\varepsilon\right) = 2c \frac{a^i_\varepsilon + b_\varepsilon}{\varepsilon} \mathcal{H}^{n-1} \left(K^i_{2\varepsilon} \setminus K^i_\varepsilon\right) = o(1). \tag{3.3.10}$$

Moreover, by the definition of v^i_ε on $H^{i,2}_\varepsilon$ there holds

$$\nabla v^i_\varepsilon \left(x\right) = \dot{\gamma}^i_\varepsilon \left(d_i \left(x\right)\right) \nabla d_i \left(x\right),$$

thus, by (3.3.2) and (3.3.4) we get

$$\begin{aligned} I^{i,2}_\varepsilon &= 2 \int_{K^i_\varepsilon} d\mathcal{H}^{n-1} \int_{b_\varepsilon}^{a^i_\varepsilon + b_\varepsilon} \left(\frac{1}{\varepsilon p'} W\left(\gamma^i_\varepsilon\left(t\right)\right) + \frac{\varepsilon^{p-1}}{p} \varphi^p\left(\nu_i\right) \left|\dot{\gamma}^i_\varepsilon(t)\right|^p \right) dt \\ &= 2\Phi(1-d_\varepsilon) \varphi\left(\nu_i\right) \mathcal{H}^{n-1}\left(K^i_\varepsilon\right) \leq \int_{K^i} \varphi\left(\nu_i\right)\, d\mathcal{H}^{n-1} + o\left(1\right). \end{aligned} \tag{3.3.11}$$

Eventually, by adding (3.3.10) and (3.3.11) we get

$$I^i_\varepsilon \leq \int_{K^i} \varphi\left(\nu_i\right)\, d\mathcal{H}^{n-1} + o(1). \tag{3.3.12}$$

Now we define the recovery sequence for the v variable by "gluing up" together the v^i_ε, $1 \leq i \leq r$, as to minimize the surface energy. This will be done defining a function which, on every C^i_ε, coincides with v^i_ε up to a region of very small area. More precisely, let

$$v_\varepsilon := \min_{1 \leq i \leq r} v^i_\varepsilon,$$

then $0 \leq v_\varepsilon \leq 1$, $v_\varepsilon \in W^{1,\infty}(\Omega)$ and $v_\varepsilon \to 1$ a.e. in Ω. Setting $B_\varepsilon = \bigcup_{i=1}^r B_\varepsilon^i$ and $C_\varepsilon = \bigcup_{i=1}^r C_\varepsilon^i$ there holds

$$v_\varepsilon = \begin{cases} 1 - d_\varepsilon & \mathbf{R}^n \setminus C_\varepsilon \\ \\ 0 & B_\varepsilon \end{cases}, \tag{3.3.13}$$

and also

$$\nabla v_\varepsilon = \nabla v_\varepsilon^i \quad \text{a.e. in } \mathcal{V}_{i,\varepsilon} := \bigcap_{j \neq i} \left\{ v_\varepsilon^i \leq v_\varepsilon^j \right\}, \tag{3.3.14}$$

so that by (3.3.7) it follows

$$\|\nabla v_\varepsilon\|_{L^\infty(\mathbf{R}^n, \mathbf{R}^n)} \leq \frac{c}{\varepsilon}. \tag{3.3.15}$$

Since $\Omega = (\Omega \setminus C_\varepsilon) \cup (\Omega \cap C_\varepsilon \setminus B_\varepsilon) \cup (\Omega \cap B_\varepsilon)$, (3.3.13) yields

$$\begin{aligned} \int_\Omega \left(\frac{1}{\varepsilon p'} W(v_\varepsilon) + \frac{\varepsilon^{p-1}}{p} \varphi^p(\nabla v_\varepsilon) \right) dx &\leq c \frac{d_\varepsilon^p}{\varepsilon} \mathcal{L}^n(\Omega \setminus C_\varepsilon) \\ &+ \int_{\Omega \cap (C_\varepsilon \setminus B_\varepsilon)} \left(\frac{1}{\varepsilon p'} W(v_\varepsilon) + \frac{\varepsilon^{p-1}}{p} \varphi^p(\nabla v_\varepsilon) \right) dx + c \frac{b_\varepsilon}{\varepsilon} \\ &= \int_{\Omega \cap (C_\varepsilon \setminus B_\varepsilon)} \left(\frac{1}{\varepsilon p'} W(v_\varepsilon) + \frac{\varepsilon^{p-1}}{p} \varphi^p(\nabla v_\varepsilon) \right) dx + o(1) = R_\varepsilon + o(1), \end{aligned} \tag{3.3.16}$$

choosing d_ε such that $d_\varepsilon^p = o(\varepsilon)$ as well as $\varepsilon \ln d_\varepsilon = o(1)$, and also $b_\varepsilon = o(\varepsilon)$.

To estimate R_ε, notice that $C_\varepsilon \setminus B_\varepsilon = \cup_{i=1}^r (H_\varepsilon^i \setminus B_\varepsilon)$, thus

$$R_\varepsilon \leq \sum_{i=1}^r \int_{\Omega \cap (H_\varepsilon^i \setminus B_\varepsilon)} \left(\frac{1}{\varepsilon p'} W(v_\varepsilon) + \frac{\varepsilon^{p-1}}{p} \varphi^p(\nabla v_\varepsilon) \right) dx,$$

and consider the inclusion $H_\varepsilon^i \setminus B_\varepsilon \subseteq \cup_{j \neq i} (H_\varepsilon^i \cap H_\varepsilon^j) \cup \cap_{j \neq i} (H_\varepsilon^i \setminus C_\varepsilon^j)$. Since $\cap_{j \neq i} (H_\varepsilon^i \setminus C_\varepsilon^j) \subseteq \mathcal{V}_{i,\varepsilon}$, arguing like in (3.3.12), by (3.3.14) we have

$$\begin{aligned} &\int_{\cap_{j \neq i} (H_\varepsilon^i \setminus C_\varepsilon^j) \cap \Omega} \left(\frac{1}{\varepsilon p'} W(v_\varepsilon) + \frac{\varepsilon^{p-1}}{p} \varphi^p(\nabla v_\varepsilon) \right) dx \\ &\quad = \int_{\cap_{j \neq i} (H_\varepsilon^i \setminus C_\varepsilon^j) \cap \Omega} \left(\frac{1}{\varepsilon p'} W\left(v_\varepsilon^i\right) + \frac{\varepsilon^{p-1}}{p} \varphi^p\left(\nabla v_\varepsilon^i\right) \right) dx \\ &\quad \leq \int_{\Omega \cap K^i} \varphi(\nu_i)\, d\mathcal{H}^{n-1} + o(1). \end{aligned} \tag{3.3.17}$$

Moreover, by (3.1.3) and (3.3.15) we have

$$\sum_{j \neq i} \int_{H_\varepsilon^i \cap H_\varepsilon^j} \left(\frac{1}{\varepsilon p'} W(v_\varepsilon) + \frac{\varepsilon^{p-1}}{p} \varphi^p(\nabla v_\varepsilon) \right) dx \leq \frac{c}{\varepsilon} \sum_{j \neq i} \mathcal{L}^n\left(H_\varepsilon^i \cap H_\varepsilon^j\right). \tag{3.3.18}$$

We claim that for every $i, j \in \{1, ..., r\}$, $i \neq j$, there holds

$$\mathcal{L}^n\left(H_\varepsilon^i \cap H_\varepsilon^j\right) = o(\varepsilon). \tag{3.3.19}$$

Indeed, we may assume $K^i \cap K^j \neq \emptyset$, since otherwise for ε sufficiently small it follows $H_\varepsilon^i \cap H_\varepsilon^j = \emptyset$ and then $\mathcal{L}^n \left(H_\varepsilon^i \cap H_\varepsilon^j\right) = 0$. Notice that

$$H_\varepsilon^i \cap H_\varepsilon^j \subseteq \left\{x \in \mathbf{R}^n : \ d_i(x) \leq a_\varepsilon^i + b_\varepsilon\right\} \cap \left\{x \in \mathbf{R}^n : \ d_j(x) \leq a_\varepsilon^j + b_\varepsilon\right\}, \tag{3.3.20}$$

and since condition $\Pi^{\nu_i} \neq \Pi^{\nu_j}$ implies that $K^i \cap K^j$ is contained in an $(n-2)$-dimensional affine subspace of $\mathbf{R}^n$, from (3.3.20) we deduce

$$\mathcal{L}^n \left(H_\varepsilon^i \cap H_\varepsilon^j\right) \leq c(a_\varepsilon^i + b_\varepsilon)(a_\varepsilon^j + b_\varepsilon) = c_1 \varepsilon^2 \ln^2 d_\varepsilon + o(\varepsilon),$$

where c, c_1 depend on $\mathcal{H}^{n-2}\left(K^i \cap K^j\right)$ and on the angle between Π^{ν_i} and Π^{ν_j}. Thus, assertion (3.3.19) is proved if d_ε is such that $\varepsilon \ln^2 d_\varepsilon = o(1)$ as well as $d_\varepsilon^p = o(\varepsilon)$. The choice $d_\varepsilon = \exp\left(-\varepsilon^{-\frac{1}{4}}\right)$ fulfils all the conditions required.

Eventually, (3.3.16)-(3.3.19) yield

$$\begin{aligned} &\int_\Omega \left(\frac{1}{\varepsilon p'} W\left(v_\varepsilon\right) + \frac{\varepsilon^{p-1}}{p} \varphi^p\left(\nabla v_\varepsilon\right)\right) dx \\ &\qquad \leq \sum_{i=1}^r \int_{\Omega \cap K^i} \varphi\left(\nu_i\right)\, d\mathcal{H}^{n-1} + o(1) = \int_{S_u} \varphi\left(\nu_u\right)\, d\mathcal{H}^{n-1} + o(1), \end{aligned} \tag{3.3.21}$$

the last equality holding thanks to the first condition in Definition 2.7.13.

To prove the upper bound inequality set

$$D_\varepsilon = \bigcup_{i=1}^r \left\{x \in \mathbf{R}^n : \pi_i(x) \in K_{\frac{\varepsilon}{2}}^i \text{ and } d_i(x) \leq \frac{b_\varepsilon}{2}\right\},$$

and let φ_ε be a cut-off function between D_ε and B_ε. Define

$$u_\varepsilon := \left(1 - \varphi_\varepsilon\right) u, \tag{3.3.22}$$

$u \in \mathcal{W}\left(\Omega; \mathbf{R}^N\right)$ implies that $u_\varepsilon \in W^{1,\infty}\left(\Omega; \mathbf{R}^N\right)$, moreover $u_\varepsilon \to u$ a.e. in Ω. Finally, (3.3.21) and (3.3.22) yield

$$\begin{aligned} F_\varepsilon\left(u_\varepsilon, v_\varepsilon\right) = &\int_{\Omega \setminus B_\varepsilon} \left(\psi\left(v_\varepsilon\right) + \eta_\varepsilon\right) f\left(x, u, \nabla u\right)\, dx \\ &+ \eta_\varepsilon \int_{B_\varepsilon} f\left(x, u_\varepsilon, \nabla u_\varepsilon\right)\, dx + \int_\Omega \left(\frac{1}{\varepsilon p'} W\left(v_\varepsilon\right) + \frac{\varepsilon^{p-1}}{p} \varphi^p\left(\nabla v_\varepsilon\right)\right) dx \\ &\leq \int_\Omega f\left(x, u, \nabla u\right)\, dx + \int_{J_u} \varphi\left(\nu_u\right)\, d\mathcal{H}^{n-1} + c\eta_\varepsilon b_\varepsilon^{-p+1} + o(1), \end{aligned}$$

the upper bound inequality follows choosing $b_\varepsilon = \left(\eta_\varepsilon \varepsilon\right)^{\frac{1}{p}}$. □

Remark 3.3.2 *The function which associates to u in $\mathcal{B}\left(\Omega; \mathbf{R}^N\right)$ the value $\int_\Omega |u|^q\, dx$, $q \in (0, +\infty)$, is only lower semicontinuous with respect to convergence in measure, thus we cannot* *deduce directly from Theorem 3.1.1 and statement (ii) of Lemma 2.3.2 the* Γ*-convergence of* $(F_\varepsilon(\cdot,\cdot) + \int_\Omega |\cdot|^q\, dx)$ *to* $F(\cdot,\cdot) + \int_\Omega |\cdot|^q\, dx$.

Nevertheless, the result still holds since all the arguments and the constructions we used to prove the lower and upper bound inequalities in Theorem 3.1.1 can be directly applied to such family of approximating functionals.

Remark 3.3.3 *Notice that if we restrict the approximating functionals* F_ε *and the limit one* F *to* $L^1\left(\Omega;\mathbf{R}^{N+1}\right)$*, the* Γ*-convergence result holds also with respect to* $L^1\left(\Omega;\mathbf{R}^{N+1}\right)$ *convergence. Indeed, the lower bound inequality can be deduced straightforward by Lemma 3.2.1 since the* $L^1\left(\Omega;\mathbf{R}^{N+1}\right)$ *convergence is stronger than the one in measure; while the upper bound inequality follows by applying the same arguments and constructions performed in Lemma 3.3.1 above, and noticing that the* $L^1\left(\Omega;\mathbf{R}^{N+1}\right)$ *convergence of the recovery sequence occurs.*

Remark 3.3.4 *We prove that, given* $\varepsilon_j \to 0^+$*, if* (u_j, v_j) *is a recovery sequence for* $(u,1)$*, with* $F(u,1) < +\infty$*, then*

$$\left(\frac{1}{\varepsilon_j p'} W(v_j) + \frac{\varepsilon_j^{p-1}}{p}\varphi^p(\nabla v_j)\right)\mathcal{L}^n \to \varphi(\nu_u)\,\mathcal{H}^{n-1} \llcorner J_u$$

weakly $*$ *in the sense of measures on* Ω.

Indeed, consider the measures with traces on $\mathcal{A}(\Omega)$ *defined by*

$$\begin{aligned}
\lambda_j(A) &:= \int_A \left(\psi(v_j) + \eta_{\varepsilon_j}\right) f(x, u_j, \nabla u_j)\, dx,\\
\mu_j(A) &:= \left(\frac{1}{\varepsilon_j p'} W(v_j) + \frac{\varepsilon_j^{p-1}}{p}\varphi^p(\nabla v_j)\right)\mathcal{L}^n,\\
\lambda(A) &:= \int_A f(x,u,\nabla u)\, dx,\\
\mu(A) &:= \varphi(\nu_u)\,\mathcal{H}^{n-1} \llcorner (J_u \cap A).
\end{aligned}$$

Proposition 3.2.1 yields for every open set $A \in \mathcal{A}(\Omega)$

$$\liminf_j \lambda_j(A) \geq \lambda(A), \quad \liminf_j \mu_j(A) \geq \lambda(A).$$

Moreover, Theorem 3.1.1 yields

$$\lim_j (\lambda_j(\Omega) + \mu_j(\Omega)) = \lambda(\Omega) + \mu(\Omega),$$

from which there follows

$$\lim_j \lambda_j(\Omega) = \lambda(\Omega), \quad \lim_j \mu_j(\Omega) = \mu(\Omega).$$

Hence, Proposition 2.2.2 (3) yields the conclusion.

3.4 Convergence of Minimizers

Let us state an equicoercivity result for the family of functionals defined in Remark 3.3.2.

Lemma 3.4.1 *Let $(u_j, v_j) \in \mathcal{B}\left(\Omega; \mathbf{R}^{N+1}\right)$ be such that*

$$\liminf_j \left(F_{\varepsilon_j}(u_j, v_j) + \int_\Omega |u_j|^q \, dx \right) < +\infty, \tag{3.4.1}$$

with $q \in (0, +\infty)$. Then there exist a subsequence (u_{j_k}, v_{j_k}) and a function $u \in \left(GSBV^p(\Omega)\right)^N$ such that $(u_{j_k}, v_{j_k}) \to (u, 1)$ in measure on Ω.

Proof. We may suppose $\psi \in W^{1,\infty}([0,1])$. This is not restrictive up to an increasing approximation argument using the Yosida's transforms. Indeed, let $\lambda > 0$ and consider the Yosida's transform of ψ, i.e.,

$$\psi_\lambda(t) := \inf_{r \in [0,1]} \left\{ \psi(r) + \lambda |t - r| \right\},$$

then $\psi_\lambda \in W^{1,\infty}([0,1])$, (ψ_λ) is increasing in λ and converges pointwise to ψ. Moreover, since ψ is increasing also ψ_λ satisfies that property. Notice that $\psi_\lambda(1) \leq 1$, with the possibility of strict inequality, but $\psi_\lambda(1) \to 1$ as $\lambda \to +\infty$. Nevertheless, we may substitute ψ with ψ_λ in the definition of F_ε and apply the argument in the sequel to the new sequence of functionals.

Condition (3.4.1) and the bound $\|v_j\|_{L^\infty(\Omega)} \leq 1$ imply that $v_j \to 1$ in $L^1(\Omega)$. Fix $i \in \mathbf{N}$, consider the sequence $\left(\psi\left(\frac{\Phi(v_j)}{\|\Phi'\|_{L^\infty([0,1])}} \right) u_j^i \right) \subset W^{1,1}\left(\Omega; \mathbf{R}^N\right)$, where $u_j^i := \Psi_i(u_j)$ with Ψ_i the auxiliary functions defined in (3.3.1) and Φ is the one defined in (3.1.6). Let us show that $\left(\psi\left(\frac{\Phi(v_j)}{\|\Phi'\|_{L^\infty([0,1])}} \right) u_j^i \right)$ is bounded in $BV\left(\Omega; \mathbf{R}^N\right)$. Indeed, by the Lipschitz continuity and the monotonicity of both Φ and ψ, Young's inequality yields

$$\begin{aligned}
&\int_\Omega \left| \psi\left(\frac{\Phi(v_j)}{\|\Phi'\|_{L^\infty([0,1])}} \right) u_j^i \right| dx + \int_\Omega \left| \nabla \left(\psi\left(\frac{\Phi(v_j)}{\|\Phi'\|_{L^\infty([0,1])}} \right) u_j^i \right) \right| dx \\
&\quad \leq c\, i \mathcal{L}^n(\Omega) + \int_\Omega \psi(v_j) |\nabla u_j| \, dx + \frac{\|\psi'\|_{L^\infty([0,1])}}{\|\Phi'\|_{L^\infty([0,1])}} i \int_\Omega |\nabla(\Phi(v_j))| \, dx \\
&\quad \leq c\, i (1 + F_{\varepsilon_j}(u_j, v_j)),
\end{aligned}$$

denoting by c a positive constant independent of i.

By (3.4.1) and the convergence $v_j \to 1$ in $L^1(\Omega)$, by applying the BV Compactness Theorem 2.6.2 and a diagonal argument we may suppose that, up to a subsequence not relabelled for convenience, for every $i \in \mathbf{N}$ there exists $w^i : \Omega \to \mathbf{R}^N$, with $\|w^i\|_{L^\infty(\Omega)} \leq i$, such that for a.e. in Ω

$$\lim_j u_j^i(x) = w^i(x). \tag{3.4.2}$$

Let us prove that for a.e. x in Ω there exists $u : \Omega \to \mathbf{R}^N$ such that

$$\lim_i w^i(x) = u(x). \tag{3.4.3}$$

Indeed, let $x \in \Omega$ be such that (3.4.2) holds, then either $|u_j(x)| \to +\infty$ or there exist $w \in \mathbf{R}^N$ and $(u_{j_k}) \subseteq (u_j)$ such that $u_{j_k}(x) \to w$. In the first case $w^i(x) = 0$ for every $i \in \mathbf{N}$, and then (3.4.3) holds with $u(x) = 0$; while in the second case $u^i_{j_k}(x) \to w$ as $k \to +\infty$ for every $i > |w|$, thus $u(x) = w$ by (3.4.2).

Let us prove the convergence of (u_j) to u in measure on Ω. Indeed, condition (3.4.1) yields

$$\mathcal{L}^n(\{x \in \Omega : |u_j(x)| > i\}) \leq c\, i^{-q},$$

thus for every $\varepsilon > 0$, since the decomposition

$$\begin{aligned}\{x \in \Omega : |u_j(x) - u(x)| > \varepsilon\} = \left\{x \in \Omega : |u^i_j(x) - u(x)| > \varepsilon\right\} \cup \\ (\{x \in \Omega : |u_j(x) - u(x)| > \varepsilon\} \cap \{x \in \Omega : |u_j(x)| > i\}),\end{aligned}$$

we have

$$\begin{aligned}&\mathcal{L}^n(\{x \in \Omega : |u_j(x) - u(x)| > \varepsilon\}) \\ &\quad \leq \mathcal{L}^n(\left\{x \in \Omega : |u^i_j(x) - u(x)| > \varepsilon\right\}) + c\, i^{-q},\end{aligned}$$

and the claimed convergence follows by (3.4.2) and (3.4.3). By (3.4.1) and the same argument used in *Step 1* of Lemma 3.2.1 we deduce that $u \in (GSBV^p(\Omega))^N$. □

We are now able to state the following result on the convergence of minimum problems.

Theorem 3.4.2 *For every $g \in L^q\left(\Omega; \mathbf{R}^N\right)$, $q \in (0, +\infty)$, and every $\gamma > 0$, there exists a minimizing pair $(u_\varepsilon, v_\varepsilon)$ for the problem*

$$m_\varepsilon = \inf\left\{F_\varepsilon(u, v) + \gamma \int_\Omega |u - g|^q\, dx : (u, v) \in \mathcal{B}\left(\Omega; \mathbf{R}^{N+1}\right)\right\}. \tag{3.4.4}$$

Moreover, every cluster point of (u_ε) is a solution of the minimum problem

$$m = \inf\left\{\mathcal{F}(u) + \gamma \int_\Omega |u - g|^q\, dx : u \in GSBV\left(\Omega; \mathbf{R}^N\right)\right\}, \tag{3.4.5}$$

and $m_\varepsilon \to m$ as $\varepsilon \to 0^+$.

Proof. The existence of $(u_\varepsilon, v_\varepsilon)$ for every $\varepsilon > 0$ follows by (3.1.2) and the very definition of F_ε which ensure its coercivity and lower semicontinuity with respect to convergence in measure (see for instance Theorem 2.7.17).

Assumption $g \in L^q\left(\Omega; \mathbf{R}^N\right)$ yields

$$\sup_\varepsilon \left\{F_\varepsilon(u_\varepsilon, v_\varepsilon) + \gamma \int_\Omega |u_\varepsilon - g|^q\, dx\right\} < +\infty,$$

thus Lemma 3.4.1 ensures the existence of a subsequence $\left(u_{\varepsilon_j}, v_{\varepsilon_j}\right)$ converging in measure on Ω to $(u, 1)$ with $u \in (GSBV^p(\Omega))^N$.

Eventually, statement (iii) of Lemma 2.3.2 and Remark 3.3.2 yield the conclusion. □

Chapter 4

Variational Approximation of Energies with Linear Growth

4.1 Statement of the Γ-Convergence Result

In this Chapter[1] we prove a variational approximation for integral functionals defined on $(GBV(\Omega))^N \cap L^1\left(\Omega;\mathbf{R}^N\right)$ as

$$\begin{aligned}\mathcal{F}(u) = &\int_\Omega f(x,u,\nabla u)\,dx \\ &+\int_\Omega f^\infty(x,\tilde{u},dD^c u) + \int_{J_u} K\left(x,u^+,u^-,\nu_u\right)\,d\mathcal{H}^{n-1},\end{aligned} \tag{4.1.1}$$

where the assumptions on all the quantities appearing above are specified in the sequel.

Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set, and let $f : \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times n} \to [0,+\infty)$ be a Borel integrand satisfying

(f1) there exist three constants $c_0 \geq 0$, c_1 and $c_2 > 0$ such that

$$c_1|z| - c_0 \leq f(x,u,z) \leq c_2(|z|+1) \tag{4.1.2}$$

for every $(x,u,z) \in \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times n}$;

(f2) $f(x,u,\cdot)$ is quasiconvex in z for every $(x,u) \in \Omega \times \mathbf{R}^N$, and either f is Carathéodory or $f(\cdot,\cdot,z)$ is upper semicontinuous for every $z \in \mathbf{R}^{N\times n}$;

(f3) for every $(x_o,u_o) \in \Omega \times \mathbf{R}^N$ and $\eta > 0$ there exists δ, depending on (x_o,u_o) and η, such that

$$f(x_o,u_o,z) - f(x,u,z) \leq \eta\,(1 + f(x,u,z)) \tag{4.1.3}$$

for every $(x,u) \in \Omega \times \mathbf{R}^N$ with $|x-x_o| + |u-u_o| \leq \delta$ and for every $z \in \mathbf{R}^{N\times n}$;

[1]The contents of this Chapter were obtained by the Author in collaboration with R. Alicandro, and are contained in the paper *Variational approximation of free-discontinuity energies with linear growth*, to appear on Comm. Cont. Math.. The paper is also downloadable at `http://cvgmt.sns.it/papers/alifoc01/`.

(f4) for every $x_o \in \Omega$ and $\eta > 0$ there exist $\delta, L > 0$ (depending on x_o and η) such that

$$\left| f^\infty(x,u,z) - \frac{f(x,u,tz)}{t} \right| \le \eta \left(1 + \frac{f(x,u,tz)}{t} \right), \tag{4.1.4}$$

for every $t > L$ and $x \in \Omega$ with $|x - x_o| \le \delta$ and for every $(u,z) \in \mathbf{R}^N \times \mathbf{R}^{N\times n}$;

(f5) for every $x_o \in \Omega$ and $\eta > 0$ there exists δ (depending on x_o and η) such that

$$f^\infty(x_o,u,z) - f^\infty(x,u,z) \le \eta f^\infty(x,u,z), \tag{4.1.5}$$

for every $x \in \Omega$ with $|x - x_o| \le \delta$ and for every $(u,z) \in \mathbf{R}^N \times \mathbf{R}^{N\times n}$;

(f6) $f^\infty(\cdot,\cdot,z)$ is upper semicontinuous for every $(x,u) \in \Omega \times \mathbf{R}^N$.

Remark 4.1.1 *Recall that $f^\infty(x,u,\cdot)$ inherits from $f(x,u,\cdot)$ the quasiconvexity property in z. Moreover, by the growth condition (4.1.2) for every $(x,u,z) \in \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times n}$ there holds*

$$c_1|z| \le f^\infty(x,u,z) \le c_2|z|. \tag{4.1.6}$$

To perform the approximation we introduce an extra variable v and define the functional $F : L^1\left(\Omega; \mathbf{R}^{N+1}\right) \to [0,+\infty]$ by

$$F(u,v) := \begin{cases} \mathcal{F}(u) & \text{if } u \in (GBV(\Omega))^N,\ v = 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise,} \end{cases} \tag{4.1.7}$$

which is equivalent to $\mathcal{F}$ as far as minimum problems are concerned. The approximating functionals $F_\varepsilon : L^1\left(\Omega; \mathbf{R}^{N+1}\right) \to [0,+\infty]$ have the form

$$F_\varepsilon(u,v) := \begin{cases} \displaystyle\int_\Omega f_\varepsilon\left(x,(u,v),\nabla(u,v)\right)\,dx & \text{if } (u,v) \in W^{1,1}\left(\Omega; \mathbf{R}^{N+1}\right), \\ & 0 \le v \le 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise,} \end{cases} \tag{4.1.8}$$

where $f_\varepsilon : \Omega \times \mathbf{R}^{N+1} \times \mathbf{R}^{(N+1)\times n} \to [0,+\infty)$ is defined by

$$f_\varepsilon(x,(u,v),(z,\zeta)) := \psi(v) f(x,u,z) + \frac{1}{\varepsilon} W(v) + \varepsilon|\zeta|^2,$$

with $\psi : [0,1] \to [0,1]$ any lower semicontinuous increasing function such that $\psi(0) = 0$, $\psi(1) = 1$ and $\psi(t) > 0$ if $t > 0$; and $W : [0,1] \to [0,+\infty)$ is any continuous function such that $W(1) = 0$ and $W(t) > 0$ if $t \in [0,1)$.

Let us fix and recall some notations. If $\nu \in \mathbf{S}^{n-1}$, recall that $\Pi^\nu \subset \mathbf{R}^n$ denotes the orthogonal space to ν, i.e., $\Pi^\nu = \{x \in \mathbf{R}^n : \langle \nu, x\rangle = 0\}$. With fixed $\{\nu_i\}_{1\le i\le n-1}$ an orthogonal base of Π^ν, set $Q'_\nu := \left\{\sum_{1\le i\le n-1} \lambda_i \nu_i : |\lambda_i| < \frac{1}{2}\right\}$, and $Q_\nu := \left\{y + \lambda\nu : y \in Q'_\nu, |\lambda| < \frac{1}{2}\right\}$. In case $\nu = e_n$ we take $\nu_i = e_i$, drop the subscript and use the notation $Q_{e_n} = Q$.

Let us state and prove the main result of the Chapter.

Theorem 4.1.2 *Let (F_ε) be as above, then (F_ε) Γ-converges with respect to the $L^1\left(\Omega;\mathbf{R}^{N+1}\right)$ convergence to the functional F given by (4.1.7), and the function $K:\Omega\times\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{n-1}\to[0,+\infty)$ is defined by*

$$
\begin{aligned}
&K\left(x_o,a,b,\nu\right):=\\
&\quad\inf\left\{\int_{Q_\nu}\left(\psi(v)f^\infty\left(x_o,u,\nabla u\right)+L\,W(v)+\frac{1}{L}\left|\nabla v\right|^2\right)dx:\right.\\
&\quad\left.(u,v)\in\mathcal{A}(a,b,\nu),L>0\right\},
\end{aligned}
\tag{4.1.9}
$$

with

$$
\mathcal{A}(a,b,\nu):=\left\{(u,v)\in W^{1,1}\left(Q_\nu;\mathbf{R}^{N+1}\right):(u,v)=\left(u_{a,b,\nu},1\right)\text{ on }\partial Q_\nu\right\},
\tag{4.1.10}
$$

where for every $(a,b,\nu)\in\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{n-1}$

$$
u_{a,b,\nu}(x):=\begin{cases}b & \text{if }\langle x,\nu\rangle\geq 0\\ a & \text{if }\langle x,\nu\rangle<0\end{cases}.
\tag{4.1.11}
$$

In the rest of the Chapter we will denote $\Gamma\left(L^1\left(\Omega;\mathbf{R}^{N+1}\right)\right)$ by $\Gamma\left(L^1\right)$ for simplicity of notation.

Remark 4.1.3 *The apriori condition of quasiconvexity on f is assumed only for simplicity, as in the general case it would suffices to replace in the formula of the effective energy the function f by its quasiconvexification.*

Remark 4.1.4 *We will prove Theorem 4.1.2 in case (4.1.2) of (f1) is substituted by*

$$
c_1\left|z\right|\leq f(x,u,z)\leq c_2(\left|z\right|+1),
\tag{4.1.12}
$$

for every $(x,u,z)\in\Omega\times\mathbf{R}^N\times\mathbf{R}^{N\times n}$. This is not restrictive, by considering the approximating functionals obtained by substituting f with $f_1=f+c_0$, which now satisfies (4.1.12) above, and by noting that, calling F_1 their Γ-limit, $F_1=F+c_0\mathcal{L}^n(\Omega)$.

Remark 4.1.5 *The result of Theorem 4.1.2 generalizes that of Theorem 5.1 in [6], in which it is considered the particular case $N=1$ and $f(x,u,z)=\tilde{f}(|z|)$, where $\tilde{f}:[0,+\infty)\to[0,+\infty)$ is convex, increasing and $\lim_{t\to+\infty}\frac{\tilde{f}(t)}{t}=1$, that is $f^\infty(x,u,z)=|z|$. In Section 4.1.1, under these assumptions on f and for all $N\geq 1$, we will show that $K(x_o,a,b,\nu)=g(|b-a|)$, where $g:[0,+\infty)\to[0,+\infty)$ is the concave function defined in [6] by*

$$
g(t):=\inf_{r\in[0,1]}\left\{\psi(r)t+4\int_r^1\sqrt{W(s)}\,ds\right\}.
\tag{4.1.13}
$$

So we recover the results of [6] also in the vector-valued setting.

Remark 4.1.6 *Let us notice that by a comparison argument and by the* Γ*-convergence result of [6], we immediately derive a bound for the lower and upper* Γ*-limits of the family* (F_ε)*. Indeed, consider the scalar functional*

$$I(u,v) := \begin{cases} \|Du\|(\Omega \setminus S_u) + \displaystyle\int_{J_u} g\left(|u^+ - u^-|\right)\, d\mathcal{H}^{n-1} & \text{if } u \in GBV(\Omega), \\ & v = 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise,} \end{cases}$$

g being given by (4.1.13). Then, by the growth condition (4.1.2) and by virtue of Theorem 5.1 [6], there exist three positive constants c_0, c_1 and c_2 such that

$$\begin{aligned} c_1 \sum_{i=1}^{N} I(u_i, v) - c_0 &\le \Gamma\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon(u,v) \\ &\le \Gamma\text{-}\limsup_{\varepsilon\to 0^+} F_\varepsilon(u,v) \le c_2 \sum_{i=1}^{N} I(u_i, v) + c_2. \end{aligned}$$

In particular, we deduce that the domains of the lower and upper Γ*-limits of the family* (F_ε) *coincide and are contained in* $(GBV(\Omega))^N$*.*

Remark 4.1.7 *We provide an equivalent characterization of the jump energy density K defined in (4.1.9) which will be useful in the sequel (see Section 4.1.1).*

Let $\tilde{K}$ be the function obtained by substituting in the minimization formula (4.1.9) defining K the class $\mathcal{A}(a,b,\nu)$ with

$$\begin{aligned} \tilde{\mathcal{A}}(a,b,\nu) := \Big\{ &(u,v) \in W^{1,1}_{loc}\left(S_\nu; \mathbf{R}^{N+1}\right) : (u,v) \text{ 1-periodic in } \nu_i, 1 \le i \le n-1, \\ &(u,v) = (a,1) \text{ on } \langle x,\nu\rangle = -\frac{1}{2},\ (u,v) = (b,1) \text{ on } \langle x,\nu\rangle = \frac{1}{2} \Big\} \end{aligned}$$

where $S_\nu := \left\{ x \in \mathbf{R}^N : |\langle x,\nu\rangle| < \frac{1}{2} \right\}$ (see [30]).

Then, since $\mathcal{A}(a,b,\nu) \subseteq \tilde{\mathcal{A}}(a,b,\nu)$ we have $\tilde{K} \le K$. The opposite inequality can be proved by exploiting the same arguments we will use in Lemma 4.2.2. However, we will obtain it as a consequence of Proposition 4.2.1 and inequality (4.3.6) in the proof of Proposition 4.3.3.

First notice that assumption (f5) implies that with fixed $(x_o,a,b,\nu) \in \Omega \times \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{S}^{n-1}$ and $\eta > 0$ there exists $\delta > 0$ such that

$$K(x_o,a,b,\nu) - K(x,a,b,\nu) \le \eta K(x,a,b,\nu) \tag{4.1.14}$$

for every $x \in \Omega$ with $|x - x_o| \le \delta$.

Let $u_{a,b,\nu}$ be the function defined in (4.1.11), then by (4.1.14), Proposition 4.2.1 and inequality (4.3.6), we get

$$\begin{aligned} \frac{1}{1+\eta} K(x_o,a,b,\nu) &\le \liminf_{\delta\to 0^+} \frac{1}{\delta^{n-1}} \int_{(x_o+\Pi_\nu)\cap(x_o+\delta Q_\nu)} K(x,a,b,\nu)\, d\mathcal{H}^{n-1} \\ &\le \limsup_{\delta\to 0^+} \frac{1}{\delta^{n-1}} \left(\Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon\left(u_{a,b,\nu}(\cdot - x_o), 1; x_o + \delta Q_\nu\right) \right) \le \tilde{K}(x_o,a,b,\nu). \end{aligned}$$

The conclusion then follows by letting $\eta \to 0^+$.

Remark 4.1.8 *Assumption (f4) can be dropped in order to state the Γ-convergence of the family (F_ε). Indeed, by exploiting the same arguments of Theorem 4.1.2 below, and with obvious changes in Subsection 4.2.2 and Proposition 4.3.3, one can prove the Γ-convergence of (F_ε) to the functional F defined in (4.1.7), with the formula (4.1.9) defining the surface energy density K substituted by*

$$K\left(x_o, a, b, \nu\right) := \limsup_{t \to 0^+} \left(\inf \left\{ \int_{Q_\nu} \left(\psi(v)\, t\, f\left(x_o, u, \frac{1}{t} \nabla u\right) + L\, W(v) + \frac{1}{L} |\nabla v|^2 \right) dy : (u, v) \in \mathcal{A}(a, b, \nu), L > 0 \right\} \right). \tag{4.1.15}$$

Let us introduce the localized versions of the approximating and limiting functionals. For every $A \in \mathcal{A}(\Omega)$ set

$$F\left(u, v; A\right) := \begin{cases} \mathcal{F}\left(u; A\right) & \text{if } u \in \left(GBV(A)\right)^N,\ v = 1 \text{ a.e. in } A \\ +\infty & \text{otherwise in } L^1\left(A; \mathbf{R}^{N+1}\right), \end{cases}$$

where $\mathcal{F}\left(\cdot; A\right)$ is defined as $\mathcal{F}(\cdot)$ in (4.1.1) by taking A as domain of integration in place of Ω. Moreover, let

$$F_\varepsilon\left(u, v; A\right) := \begin{cases} \displaystyle\int_A f_\varepsilon(x, (u, v), \nabla(u, v))\, dx & \text{if } (u, v) \in W^{1,1}\left(\Omega; \mathbf{R}^{N+1}\right), \\ & 0 \le v \le 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise in } L^1\left(\Omega; \mathbf{R}^{N+1}\right), \end{cases}$$

and

$$G_\varepsilon\left(v; A\right) := \begin{cases} \displaystyle\int_A \left(\frac{1}{\varepsilon} W(v) + \varepsilon \left|\nabla v\right|^2 \right) dx & \text{if } v \in W^{1,1}(\Omega), \\ & 0 \le v \le 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

Eventually, with fixed $x_o \in \Omega$, denote by $F_\varepsilon\left(x_o; \cdot, \cdot; A\right)$, $F_\varepsilon^\infty\left(x_o; \cdot, \cdot; A\right)$ the functionals defined analogously to $F_\varepsilon\left(\cdot, \cdot; A\right)$ and obtained by substituting in the definition of f_ε the function f with $f(x_o, \cdot, \cdot)$, $f^\infty(x_o, \cdot, \cdot)$, respectively. With this notation we get

$$K\left(x_o, a, b, \nu\right) = \inf \left\{ F_{\frac{1}{L}}^\infty\left(x_o; u, v; Q_\nu\right) : (u, v) \in \mathcal{A}(a, b, \nu), L > 0 \right\}. \tag{4.1.16}$$

4.1.1 Properties of the surface density function

Before proving Theorem 4.1.2 we state some properties of the surface energy density K, we will need in the sequel, and we show a more explicit characterization of it in some particular cases. The proofs are in the spirit of the papers [29],[30],[68],[69].

Lemma 4.1.9 *Let $K : \Omega \times \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{S}^{n-1} \to [0, +\infty)$ be defined as in (4.1.9), then*

(a) for every (x_o, a, b, ν), $(x_o, a', b', \nu) \in \Omega \times \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{S}^{n-1}$ there holds

$$|K(x_o, a, b, \nu) - K(x_o, a', b', \nu)| \leq c(|a - a'| + |b - b'|);$$

(b) for every $(x_o, a, b, \nu) \in \Omega \times \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{S}^{n-1}$ there holds

$$c_1 g(|b - a|) \leq K(x_o, a, b, \nu) \leq c_2 g(|b - a|).$$

where c_1, c_2 are positive constants, and g is given by (4.1.13).

Proof. (a) We use the different characterization of K discussed in Remark 4.1.7. Let $(u, v) \in \tilde{\mathcal{A}}(a, b, \nu)$, let $\varphi \in C^\infty(\mathbf{R})$ be a function such that $0 \leq \varphi \leq 1$, $\varphi = 1$ for $t \leq \frac{1}{4}$, $\varphi = 0$ for $t \geq \frac{1}{2}$, then define

$$(\tilde{u}, \tilde{v})(x) := \begin{cases} \varphi(-x \cdot \nu)(a, 1) + (1 - \varphi(-x \cdot \nu))(a', 1) & \text{if } -\frac{1}{2} \leq x \cdot \nu < -\frac{1}{4} \\ (u(2x), v(2x)) & \text{if } |x \cdot \nu| \leq \frac{1}{4} \\ \varphi(x \cdot \nu)(b, 1) + (1 - \varphi(x \cdot \nu))(b', 1) & \text{if } \frac{1}{4} < x \cdot \nu \leq \frac{1}{2} \end{cases}.$$

Then $(\tilde{u}, \tilde{v}) \in \tilde{\mathcal{A}}(a', b', \nu)$ and, for $L > 0$, we get

$$\begin{aligned} K(x_o, a', b', \nu) &\leq F^\infty_{\frac{1}{L}}(x_o; \tilde{u}, \tilde{v}; Q_\nu) \\ &= \int_{Q_\nu \cap \{|x \cdot \nu| < \frac{1}{4}\}} \psi(v(2x)) f^\infty(x_o, u(2x), 2\nabla u(2x))\, dx \\ &+ \int_{Q_\nu \cap \{|x \cdot \nu| < \frac{1}{4}\}} \left(L\, W(v(2x)) + \frac{4}{L} |\nabla v(2x)|^2 \right) dx \\ &+ \int_{Q_\nu \cap \{\frac{1}{4} < x \cdot \nu < \frac{1}{2}\}} f^\infty(x_o, \varphi(x \cdot \nu)\, b + (1 - \varphi(x \cdot \nu))\, b', (b - b') \otimes \varphi'(x \cdot \nu)\, \nu)\, dx \\ &+ \int_{Q_\nu \cap \{\frac{1}{4} < x \cdot \nu < \frac{1}{2}\}} f^\infty(x_o, \varphi(x \cdot \nu)\, a + (1 - \varphi(x \cdot \nu))\, a', (a' - a) \otimes \varphi'(x \cdot \nu)\, \nu)\, dx. \end{aligned}$$

Since the periodicity of (u, v) and by the growth assumption (4.1.6), there follows

$$K(x_o, a', b', \nu) \leq F^\infty_{\frac{2}{L}}(x_o; u, v; Q_\nu) + c(|a - a'| + |b - b'|)$$

and so by taking the infimum on $(u, v) \in \tilde{\mathcal{A}}(a, b, \nu)$ and $L > 0$ we conclude that

$$K(x_o, a', b', \nu) \leq K(x_o, a, b, \nu) + c(|a - a'| + |b - b'|).$$

Analogously, we can prove the opposite inequality.

(b) Use the growth condition (4.1.6) and consider the characterization of K given by Lemma 4.1.10 (b) and Remark 4.1.11 below when $f^\infty(x,u,z) = |z|$. □

In the following we characterize the function K under isotropy assumptions on f^∞. In such a case we show that K can be calculated by restricting the infimum to functions (u,v) with one-dimensional profile. For instance, isotropy always occours in the unidimensional case.

Lemma 4.1.10 *Let $K : \Omega \times \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{S}^{n-1} \to [0,+\infty)$ and $g : [0,+\infty) \to [0,+\infty)$ be defined by (4.1.9) and (4.1.13), respectively. Then*

(a) for every $(x_o, a, b, \nu) \in \Omega \times \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{S}^{n-1}$ there holds

$$K(x_o, a, b, \nu) \leq g(K_f(x_o, a, b, \nu)),$$

where $K_f : \Omega \times \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{S}^{n-1} \to [0,+\infty)$ is defined by

$$K_f(x_o, a, b, \nu) := \inf\left\{ \int_{Q_\nu} f^\infty(x_o, u, \nabla u)\, dy : u \in W^{1,1}\left(Q_\nu; \mathbf{R}^N\right), \right.$$
$$\left. u = u_{a,b,\nu} \text{ on } \partial Q_\nu \right\}; \tag{4.1.17}$$

(b) if f^∞ is isotropic, i.e., for every $(x_o, u, z) \in \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times n}$ and $\nu \in \mathbf{S}^{n-1}$ there holds

$$f^\infty(x_o, u, z\nu \otimes \nu) \leq f^\infty(x_o, u, z),$$

then $K = g(K_f)$.

Proof. (a) With fixed $r \in [0,1)$ and $\eta > 0$, let $T_\eta > 0$, $v_\eta \in W^{1,1}(0, T_\eta)$ be such that $v_\eta(0) = r$, $v_\eta(T_\eta) = 1$ and

$$\int_0^{T_\eta} \left(W(v_\eta) + |v'_\eta|^2 \right) dt \leq 2 \int_r^1 \sqrt{W(s)}\, ds + \eta.$$

Then define

$$v_{\eta,L}(y) := \begin{cases} v_\eta\left(\frac{-y\cdot\nu - \alpha_L}{\beta_L}\right) & -\frac{1}{2} \leq y\cdot\nu < -\alpha_L \\ r & |y\cdot\nu| \leq \alpha_L \\ v_\eta\left(\frac{y\cdot\nu - \alpha_L}{\beta_L}\right) & \alpha_L < y\cdot\nu \leq \frac{1}{2} \end{cases},$$

where α_L is any positive infinitesimal as $L \to +\infty$, and $\beta_L := \frac{\frac{1}{2} - \alpha_L}{T_\eta}$. If $r = 1$ simply take $v_{\eta,L} \equiv 1$.

Let u be admissible for K_f and extend it by periodicity to $\mathbf{R}^n$, then set

$$u_L(y) := \begin{cases} a & -\frac{1}{2} \leq y \cdot \nu < -\alpha_L \\ u\left(\frac{y}{2\alpha_L}\right) & |y \cdot \nu| \leq \alpha_L \\ b & \alpha_L < y \cdot \nu \leq \frac{1}{2} \end{cases} .$$

Notice that $(u_L, v_{\eta,L}) \in \tilde{\mathcal{A}}(a,b,\nu)$. Let us compute $F^{\infty}_{\frac{1}{\beta_L}}(u_L, v_L; Q_\nu)$. Let R^ν be a rotation such that $R^\nu Q = Q_\nu$. Then, since $f^\infty(x,u,\cdot)$ is positively one-homogeneous, we get by simple changes of variables and by Fubini's Theorem

$$\begin{aligned}
&\int_{Q_\nu} \psi(v_L) f^\infty(x_o, u_L, \nabla u_L)\, dy \\
&= \psi(r) \int_{Q_\nu \cap \{|y\cdot\nu| < \alpha_L\}} f^\infty\left(x_o, u\left(\frac{y}{2\alpha_L}\right), \frac{1}{2\alpha_L}\nabla u\left(\frac{y}{2\alpha_L}\right)\right) dy \\
&= \psi(r) \int_{-\frac{1}{2}}^{\frac{1}{2}} dt \int_{Q'} f^\infty\left(x_o, u\left(R^\nu\left(\frac{y'}{2\alpha_L}, t\right)\right), \nabla u\left(R^\nu\left(\frac{y'}{2\alpha_L}, t\right)\right)\right) dy' \\
&= \psi(r) \int_Q f^\infty(x_o, u(R^\nu y), \nabla u(R^\nu y))\, dy + o(1) \\
&= \psi(r) \int_{Q_\nu} f^\infty(x_o, u(y), \nabla u(y))\, dy + o(1),
\end{aligned}$$

where the last equality follows by Riemann-Lebesgue's Lemma. Moreover, there holds

$$\begin{aligned}
G_{\frac{1}{\beta_L}}(v_{\eta,L}; Q_\nu) &= \int_{-\frac{1}{2}}^{-\alpha_L} \frac{1}{\beta_L}\left(W\left(v_\eta\left(\frac{-t-\alpha_L}{\beta_L}\right)\right) + \left|v'_\eta\left(\frac{-t-\alpha_L}{\beta_L}\right)\right|^2\right) dt \\
&+2\frac{\alpha_L}{\beta_L} W(r) + \int_{\alpha_L}^{\frac{1}{2}} \frac{1}{\beta_L}\left(W\left(v_\eta\left(\frac{t-\alpha_L}{\beta_L}\right)\right) + \left|v'_\eta\left(\frac{t-\alpha_L}{\beta_L}\right)\right|^2\right) dt \\
&= 2\int_0^{T_\eta}\left(W(v_\eta) + |v'_\eta|^2\right) dt + o(1) \leq 4\int_r^1 \sqrt{W(s)}\, ds + c\eta + o(1).
\end{aligned}$$

Hence, there follows

$$\begin{aligned}
K(x_o, a, b, \nu) &\leq F^\infty_{\frac{1}{\beta_L}}(u_L, v_{\eta,L}; Q_\nu) \\
&\leq \psi(r) \int_{Q_\nu} f^\infty(x_o, u(y), \nabla u(y))\, dy + 4\int_r^1 \sqrt{W(s)}\, ds + c\eta + o(1),
\end{aligned}$$

and so by letting $L \to +\infty$, $\eta \to 0^+$ and by passing to the infimum on u we get for every $r \in [0,1]$

$$K(x_o, a, b, \nu) \leq \psi(r) K_f(x_o, a, b, \nu) + 4\int_r^1 \sqrt{W(s)}\, ds.$$

Finally, the desired inequality follows by the very definition of g.
(b) Following [21] define

$$\begin{aligned} D_f\left(x_o,a,b,\nu\right) := \inf\Bigg\{ & \int_{Q_\nu} f^\infty\left(x_o,u,\nabla u\right) dy : u \in W^{1,1}\left(Q_\nu;\mathbf{R}^N\right), \\ & u(y)=\xi(y\cdot\nu), \xi\left(-\frac{1}{2}\right)=a, \xi\left(\frac{1}{2}\right)=b\Bigg\} \\ = \inf\Bigg\{ & \int_{-\frac{1}{2}}^{\frac{1}{2}} f^\infty\left(x_o,\xi,\dot{\xi}\otimes\nu\right) dt : \xi \in W^{1,1}\left(\left(-\frac{1}{2},\frac{1}{2}\right);\mathbf{R}^N\right), \\ & \xi\left(-\frac{1}{2}\right)=a,\ \xi\left(\frac{1}{2}\right)=b\Bigg\}; \end{aligned}$$

then it is obvious that $K_f \le D_f$. Moreover, in case f^∞ is isotropic, $D_f = K_f$ (see Proposition 2.6 [69]). Hence, by (a), we have to prove only that $K \ge g\left(D_f\right)$.

The isotropy condition on f^∞ implies that for every $(u,v)\in\mathcal{A}(a,b,\nu)$, $L>0$ there holds

$$\begin{aligned} & F^\infty_{\frac{1}{L}}\left(u,v;Q_\nu\right) \ge I(u,v;Q_\nu) \\ & := \int_{Q_\nu}\left(\psi(v) f^\infty\left(x_o,u,\nabla u\,\nu\otimes\nu\right) + 2\sqrt{W(v)}\,|\nabla v\,\nu\otimes\nu|\right) dy. \end{aligned}$$

For every $y'\in Q'_\nu$ and $t\in[-\frac{1}{2},\frac{1}{2}]$, let as usual, $(u_{\nu,y'}(t),v_{\nu,y'}(t)) = (u(y'+t\nu),v(y'+t\nu))$, then by Fubini's Theorem there holds

$$\begin{aligned} & I(u,v;Q_\nu) \\ & = \int_{Q'_\nu} dy' \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\psi\left(v_{\nu,y'}\right) f^\infty\left(x_o,u_{\nu,y'},\dot{u}_{\nu,y'}\otimes\nu\right) + 2\sqrt{W\left(v_{\nu,y'}\right)}\,\left|\dot{v}_{\nu,y'}\right|\right) dt, \end{aligned}$$

and thus

$$\begin{aligned} K\left(x_o,a,b,\nu\right) \ge \inf\Bigg\{ & \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\psi\left(\xi_2\right) f^\infty\left(x_o,\xi_1,\dot{\xi}_1\otimes\nu\right) + 2\sqrt{W\left(\xi_2\right)}\left|\dot{\xi}_2\right|\right) dt : \\ & (\xi_1,\xi_2)\in W^{1,1}\left(\left(-\frac{1}{2},\frac{1}{2}\right);\mathbf{R}^{N+1}\right), (\xi_1,\xi_2)\left(-\frac{1}{2}\right)=(a,1), (\xi_1,\xi_2)\left(\frac{1}{2}\right)=(b,1)\Bigg\}. \end{aligned}$$

In order to conclude, with fixed (ξ_1,ξ_2) as above, let $m=\inf_{[-\frac{1}{2},\frac{1}{2}]}\xi_2$, then

$$\begin{aligned} & \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\psi\left(\xi_2\right) f^\infty\left(x_o,\xi_1,\dot{\xi}_1\otimes\nu\right) + 2\sqrt{W\left(\xi_2\right)}\left|\dot{\xi}_2\right|\right) dt \\ & \quad \ge \psi(m)\int_{-\frac{1}{2}}^{\frac{1}{2}} f^\infty\left(x_o,\xi_1,\dot{\xi}_1\otimes\nu\right) dt + 4\int_m^1\sqrt{W(s)}\,ds \\ & \quad \ge \psi(m) D_f\left(x_o,a,b,\nu\right) + 4\int_m^1\sqrt{W(s)}\,ds \ge g\left(D_f\left(x_o,a,b,\nu\right)\right). \end{aligned}$$

$\square$

Remark 4.1.11 *The characterization of K in the isotropic case, given in Lemma 4.1.10 (b), is relevant when an explicit expression of K_f is given, for instance in the autonomous and scalar case.*

Indeed, if $f = f(x,z)$ then $K_f(x_o,a,b,\nu) = f^\infty(x_o,(b-a)\otimes\nu)$ (see Remark 2.17 [68]).

In the scalar setting $N=1$, since f satisfies conditions (f1), (f4)-(f6), Corollary 1.4 and Theorem 1.10 [66] (see also [54]) yield the equality

$$K_f(x_o,a,b,\nu) = \begin{cases} \displaystyle\int_b^a f^\infty(x_o,u,\nu)\,du & \text{if } a > b \\ \displaystyle\int_a^b f^\infty(x_o,u,-\nu)\,du & \text{if } a < b \end{cases}.$$

In particular, if $f(z) = |z|$ then $K(x_o,a,b,\nu) = g(|b-a|)$, thus we recover the surface energy density of [6].

4.2 Lower bound inequality

In this section we establish the lower bound inequality when restricting the target functional to $BV\left(\Omega;\mathbf{R}^N\right)\times L^1(\Omega)$. We treat separately the diffuse and jump part. Indeed, we recover straightforward the estimate on the diffuse part by using the semicontinuity result Theorem 2.6.18, while we apply the blow-up argument of Fonseca-Müller to estimate the surface energy density.

Proposition 4.2.1 *For every $(u,v)\in BV\left(\Omega;\mathbf{R}^N\right)\times L^1(\Omega)$ we have*

$$\Gamma\left(L^1\right)\text{-}\liminf_{\varepsilon\to0^+} F_\varepsilon(u,v) \geq F(u,v).$$

Proof. Let $\varepsilon_j\to0^+$ and $(u_j,v_j)\to(u,v)$ in $L^1\left(\Omega;\mathbf{R}^{N+1}\right)$. Without loss of generality we may assume the inferior limit $\liminf_j F_{\varepsilon_j}(u_j,v_j)$ to be finite and to be a limit. Then, we get

$$\liminf_j \int_\Omega W(v_j)\,dx \leq \liminf_j\left(\varepsilon_j G_{\varepsilon_j}(v_j;\Omega)\right) = 0,$$

so that by Fatou's lemma there follows

$$W(v) \leq \liminf_j W(v_j) = 0$$

for a.e. $x\in\Omega$, and then $v=1$ for a.e. $x\in\Omega$.

Since $f_{\varepsilon_j}\geq0$, up to passing to a subsequence, we may assume that there exists a non-negative finite Radon measure μ on Ω such that

$$f_{\varepsilon_j}(\cdot,(u_j(\cdot),v_j(\cdot)),\nabla(u_j(\cdot),v_j(\cdot))\mathcal{L}^n\llcorner\Omega\to\mu$$

weakly $*$ in the sense of measures. Using the Radon-Nikodým's Theorem we decompose μ in the sum of four mutually orthogonal measures

$$\mu = \mu_a\mathcal{L}^n + \mu_c\|D^cu\| + \mu_J|u^+-u^-|\mathcal{H}^{n-1}\llcorner J_u + \mu_s,$$

we claim that

$$\mu_a(x_o) \geq f\left(x_o, u(x_o), \nabla u(x_o)\right) \tag{4.2.1}$$

for a.e. $x_o \in \Omega$;

$$\mu_c(x_o) \geq f^\infty\left(x_o, \tilde{u}(x_o), \frac{dD^c u}{d\|D^c u\|}(x_o)\right) \tag{4.2.2}$$

for $\|D^c u\|$ a.e. $x_o \in \Omega$;

$$\mu_J(x_o) \geq \frac{1}{|u^+(x_o) - u^-(x_o)|} K\left(x_o, u^+(x_o), u^-(x_o), \nu_u(x_o)\right) \tag{4.2.3}$$

for $|u^+ - u^-|\mathcal{H}^{n-1} \llcorner J_u$ a.e. $x_o \in \Omega$.

Assuming the previous inequalities shown, to conclude consider an increasing sequence of smooth cut-off functions $(\varphi_i) \subset C_0^\infty(\Omega)$ such that $0 \leq \varphi_i \leq 1$ and $\sup_i \varphi_i(x) = 1$ on Ω, then for every $i \in \mathbf{N}$ we have

$$\begin{aligned}
\lim_j F_{\varepsilon_j}(u_j, v_j; \Omega) &\geq \liminf_j \int_\Omega f_{\varepsilon_j}\left(x, (u_j, v_j), \nabla(u_j, v_j)\right) \varphi_i \, dx \\
&= \int_\Omega \varphi_i d\mu \geq \int_\Omega f(x, u, \nabla u) \varphi_i \, dx + \int_\Omega f^\infty\left(x, \tilde{u}, \frac{dD^c u}{d\|D^c u\|}\right) \varphi_i d\|D^c u\| \\
&+ \int_{J_u} K\left(x, u^+, u^-, \nu_u\right) \varphi_i \, d\mathcal{H}^{n-1}.
\end{aligned}$$

Eventually, let $i \to +\infty$ and apply the Monotone Convergence Theorem. □

In the following subsections we prove (4.2.1), (4.2.2), (4.2.3).

4.2.1 The density of the diffuse part

Consider the auxiliary function $\Phi : [0,1] \to [0, +\infty)$ defined by

$$\Phi(t) = 2\int_0^t \sqrt{W(s)}\, ds, \tag{4.2.4}$$

then notice that Φ is increasing, $\Phi(t) = 0$ if and only if $t = 0$ and $\Phi \in W^{1,\infty}([0,1])$.

Define the function $\tilde{f} : \Omega \times \mathbf{R}^{N+1} \times \mathbf{R}^{(N+1)\times n} \to [0, +\infty)$ by

$$\tilde{f}(x, (u, v), (z, \zeta)) := \psi\left(\Phi^{-1}\left(v \vee 0 \wedge \Phi(1)\right)\right)\left(f(x, u, z) + |\zeta|\right),$$

then notice that for every $(u, v) \in W^{1,1}\left(\Omega; \mathbf{R}^{N+1}\right)$, $\varepsilon > 0$ and $A \in \mathcal{A}(\Omega)$ Young's inequality yields

$$G_\varepsilon(v; A) \geq 2\int_A \sqrt{W(v)}\, |\nabla v|\, dx = \int_A |\nabla(\Phi(v))|\, dx,$$

from which we infer that

$$F_\varepsilon(u, v; A) \geq \int_A \tilde{f}\left(x, (u, \Phi(v)), \nabla(u, \Phi(v))\right)\, dx.$$

It can be easily seen, by the hypotheses on f and ψ, that $\tilde{f}$ satisfies all the assumptions of Theorem 2.6.18.

Moreover, if $v_j \to 1$ in $L^1(\Omega;[0,1])$, then $\Phi(v_j) \to \Phi(1)$ in $L^1(\Omega;[0,\Phi(1)])$. Hence, given (u_j, v_j) as in the proof of Proposition 4.2.1, for every $A \in \mathcal{A}(\Omega)$ there holds

$$\begin{aligned}
\liminf_j F_{\varepsilon_j}(u_j, v_j; A) &\geq \liminf_j \int_A \tilde{f}(x,(u_j,\Phi(v_j)),\nabla(u_j,\Phi(v_j)))\, dx \\
&\geq \int_A \tilde{f}(x,(u,\Phi(1)),\nabla(u,\Phi(1)))\, dx + \int_A \tilde{f}^\infty(x,(\tilde{u},\Phi(1)),dD^c(u,\Phi(1))) \\
&= \int_A f(x,u,\nabla u)\, dx + \int_A f^\infty(x,\tilde{u},dD^c u).
\end{aligned}$$

From this, it is easy to infer (4.2.1) and (4.2.2).

4.2.2 The density of the jump part

To prove (4.2.3) recall that (2.5.3), Lemma 2.6.11, Remark 2.6.10 and Radon-Nikodým's Theorem yield for $\mathcal{H}^{n-1}$ a.e. $x_o \in J_u$

$$\lim_{t\to 0^+} \frac{1}{t^{n-1}} \int_{J_u \cap \left(x_o + tQ_{\nu_u(x_o)}\right)} \left|u^+(x) - u^-(x)\right| d\mathcal{H}^{n-1} = \left|u^+(x_o) - u^-(x_o)\right|, \tag{4.2.5}$$

$$\lim_{t\to 0^+} \frac{1}{t^n} \int_{x_o + tQ^{\pm}_{\nu_u(x_o)}} \left|u(x) - u^{\pm}(x_o)\right| dx = 0, \tag{4.2.6}$$

$$\mu_J(x_o) = \lim_{t\to 0^+} \frac{\mu\left(x_o + tQ_{\nu_u(x_o)}\right)}{|u^+ - u^-|\,\mathcal{H}^{n-1}\left(J_u \cap \left(x_o + tQ_{\nu_u(x_o)}\right)\right)} \tag{4.2.7}$$

exists and is finite.

By (4.2.5) and (4.2.7), and since the function $\mathcal{X}_{x_o + t\overline{Q}_{\nu_u(x_o)}}$ is upper semicontinuous and with compact support in Ω if t is sufficiently small, Proposition 2.2.2 (2) yields

$$\begin{aligned}
&\left|u^+(x_o) - u^-(x_o)\right| \mu_J(x_o) = \lim_{t\to 0^+} \frac{1}{t^{n-1}} \int_{x_o + t\overline{Q}_{\nu_u(x_o)}} d\mu(x) \\
&\geq \limsup_{t\to 0^+} \limsup_j \frac{1}{t^{n-1}} \int_{x_o + tQ_{\nu_u(x_o)}} f_{\varepsilon_j}(x,(u_j,v_j),\nabla(u_j,v_j))\, dx \\
&= \limsup_{t\to 0^+} \limsup_j \int_{Q_{\nu_u(x_o)}} t f_{\varepsilon_j}(x_o + ty,(u_j,v_j)(x_o+ty),\nabla(u_j,v_j)(x_o+ty))\, dy \\
&= \limsup_{t\to 0^+} \limsup_j \int_{Q_{\nu_u(x_o)}} \left(t\psi\left(v_j^t(y)\right) f\left(x_o + ty, u_j^t(y), \frac{1}{t}\nabla u_j^t(y)\right)\right. \\
&\qquad\qquad \left. + \frac{t}{\varepsilon_j} W\left(v_j^t(y)\right) + \frac{\varepsilon_j}{t}\left|\nabla v_j^t(y)\right|^2\right) dy,
\end{aligned} \tag{4.2.8}$$

where $(u_j^t(y), v_j^t(y)) := (u_j(x_o+ty), v_j(x_o+ty))$. Notice that $(u_j^t(y), v_j^t(y)) \to (u(x_o+ty),1)$ in $L^1\left(Q_{\nu_u(x_o)};\mathbf{R}^{N+1}\right)$ as $j\to+\infty$, and by (4.2.6) there follows $(u(x_o+ty),1) \to (u_o(x),1)$

in $L^1\left(Q_{\nu_u(x_o)};\mathbf{R}^{N+1}\right)$ as $t\to 0^+$, where

$$u_o(x) := \begin{cases} u^+(x_o) & \langle x-x_o,\nu_u(x_o)\rangle \geq 0 \\ \\ u^-(x_o) & \langle x-x_o,\nu_u(x_o)\rangle < 0 \end{cases}.$$

With fixed $\eta>0$, let δ, $L>0$ be given by (f4) and (f5). Then, by (4.1.4) of (f4), if $t<\frac{1}{L}\wedge\frac{2}{\sqrt{n}}\delta$ we get

$$\begin{aligned}
&\int_{Q_{\nu_u(x_o)}} \psi\left(v_j^t(y)\right) tf\left(x_o+ty,u_j^t(y),\frac{1}{t}\nabla u_j^t(y)\right)\,dx \\
&\quad\geq \frac{1}{1+\eta}\int_{Q_{\nu_u(x_o)}} \psi\left(v_j^t(y)\right) f^\infty\left(x_o+ty,u_j^t(y),\nabla u_j^t(y)\right)\,dx - \frac{c\eta}{1+\eta}.
\end{aligned}$$

On the other hand, by (4.1.5) of (f5) there follows

$$\begin{aligned}
&\int_{Q_{\nu_u(x_o)}} \psi\left(v_j^t(y)\right) f^\infty\left(x_o+ty,u_j^t(y),\nabla u_j^t(y)\right)\,dy \\
&\quad\geq \frac{1}{1+\eta}\int_{Q_{\nu_u(x_o)}} \psi\left(v_j^t(y)\right) f^\infty\left(x_o,u_j^t(y),\nabla u_j^t(y)\right)\,dy.
\end{aligned}$$

Therefore, letting $\eta\to 0$, from (4.2.8) we obtain

$$\begin{aligned}
&\left|u^+(x_o)-u^-(x_o)\right|\mu_J(x_o) \\
&\quad\geq \limsup_{t\to 0^+}\limsup_{j}\int_{Q_{\nu_u(x_o)}}\Bigg(\psi\left(v_j^t(y)\right) f^\infty\left(x_o,u_j^t(y),\nabla u_j^t(y)\right) \\
&\qquad\qquad +\frac{t}{\varepsilon_j}W\left(v_j^t(y)\right)+\frac{\varepsilon_j}{t}\left|\nabla v_j^t(y)\right|^2\Bigg)\,dy \\
&\quad= \limsup_{t\to 0^+}\limsup_{j} F^\infty_{\frac{\varepsilon_j}{t}}\left(x_o;u_j^t,v_j^t;Q_{\nu_u(x_o)}\right).
\end{aligned} \tag{4.2.9}$$

By using a diagonal argument for every $h\in\mathbf{N}$ there exist indexes $j_h\in\mathbf{N}$ and $t_h\in(0,+\infty)$ such that $\gamma_h:=\frac{\varepsilon_{j_h}}{t_h}\leq\frac{1}{h}$, the sequence $\left(u_{j_h}^{t_h},v_{j_h}^{t_h}\right)\to(u_o,1)$ in $L^1\left(Q_{\nu_u(x_o)};\mathbf{R}^{N+1}\right)$, and

$$\left|u^+(x_o)-u^-(x_o)\right|\mu_J(x_o)\geq \lim_h F^\infty_{\gamma_h}\left(x_o;u_{j_h}^{t_h},v_{j_h}^{t_h};Q_{\nu_u(x_o)}\right). \tag{4.2.10}$$

In order to establish (4.2.3) and to take into account the definition of K, we need to modify $\left(u_{j_h}^{t_h},v_{j_h}^{t_h}\right)$ near $\partial Q_{\nu_u(x_o)}$ without increasing the energy in the limit and in such a way that the new sequence belongs to $\mathcal{A}\left(u^+(x_o),u^-(x_o),\nu_u(x_o)\right)$. Assuming Lemma 4.2.2 below proved, we are done. $\square$

Let us prove the following De Giorgi's type averaging-slicing lemma.

Lemma 4.2.2 *For every $x_o \in \Omega$, $(a,b,\nu) \in \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{S}^{n-1}$, $\gamma_j \to 0^+$, $(u_j,v_j) \to (u_{a,b,\nu},1)$ in $L^1\left(Q_\nu;\mathbf{R}^{N+1}\right)$ there exists $(\hat{u}_j,\hat{v}_j) \in \mathcal{A}(a,b,\nu)$ such that $(\hat{u}_j,\hat{v}_j) \to (u_{a,b,\nu},1)$ in $L^1\left(Q_\nu;\mathbf{R}^{N+1}\right)$ and*

$$\limsup_j F^\infty_{\gamma_j}(x_o;\hat{u}_j,\hat{v}_j;Q_\nu) \le \liminf_j F^\infty_{\gamma_j}(x_o;u_j,v_j;Q_\nu) \tag{4.2.11}$$

Proof. Without loss of generality we may assume the inferior limit in (4.2.11) to be finite and to be a limit. Moreover, we denote by c a generic positive constant which may vary from line to line.

Let $(w_j) \subset W^{1,1}\left(Q_\nu;\mathbf{R}^N\right)$ be such that $w_j \to u_{a,b,\nu}$ in $L^1\left(Q_\nu;\mathbf{R}^N\right)$, $w_j = u_{a,b,\nu}$ on ∂Q_ν and $\|Dw_j\|(Q_\nu) \to \|Du_{a,b,\nu}\|(Q_\nu)$, which is provided by Lemma 2.6.4.
Let $a_j \to 0^+$, $b_j \in \mathbf{N}$ to be chosen suitably and such that $s_j := \frac{a_j}{b_j} \to 0$, then set $Q_\nu^{j,i} := (1-a_j+is_j)Q_\nu$, $0 \le i \le b_j$. Let $(\varphi_{j,i}) \subset C_0^\infty\left(Q_\nu^{j,i}\right)$, $1 \le i \le b_j$, be a family of cut-off functions such that $0 \le \varphi_{j,i} \le 1$, $\varphi_{j,i} = 1$ on $Q_\nu^{j,i-1}$, $\|\nabla\varphi_{j,i}\|_{L^\infty(Q_\nu)} = O(s_j^{-1})$. Define

$$u_j^i := \varphi_{j,i-1}u_j + (1-\varphi_{j,i-1})w_j; \quad v_j^i := \varphi_{j,i}v_j + (1-\varphi_{j,i}),$$

then $\left(u_j^i,v_j^i\right) \in \mathcal{A}(a,b,\nu)$ and converges to $(u_{a,b,\nu},1)$ in $L^1\left(\Omega;\mathbf{R}^{N+1}\right)$ as $j \to +\infty$ for every $i \in \mathbf{N}$. Moreover

$$\begin{aligned}
F^\infty_{\gamma_j}\left(x_o;u_j^i,v_j^i;Q_\nu\right) &\le F^\infty_{\gamma_j}\left(x_o;u_j,v_j;Q_\nu^{j,i-2}\right)\\
&+ \int_{Q_\nu^{j,i-1}\setminus Q_\nu^{j,i-2}} \psi(v_j) f^\infty(x_o,u_j^i,\nabla u_j^i)\,dx + G_{\gamma_j}(v_j;Q_\nu^{j,i-1}\setminus Q_\nu^{j,i-2})\\
&+ \int_{Q_\nu\setminus Q_\nu^{j,i-1}} \psi\left(v_j^i\right) f^\infty(x_o;w_j,\nabla w_j)\,dx + G_{\gamma_j}\left(v_j^i;Q_\nu^{j,i}\setminus Q_\nu^{j,i-1}\right).
\end{aligned} \tag{4.2.12}$$

We estimate separately the terms appearing above. To begin with, we have that

$$F^\infty_{\gamma_j}\left(x_o;u_j,v_j;Q_\nu^{j,i-2}\right) + G_{\gamma_j}(v_j;Q_\nu^{j,i-1}\setminus Q_\nu^{j,i-2}) \le F^\infty_{\gamma_j}(x_o,u_j,v_j;Q_\nu). \tag{4.2.13}$$

Moreover, since $\nabla u_j^i = \varphi_{j,i-1}\nabla u_j + (1-\varphi_{j,i-1})\nabla w_j + \nabla\varphi_{j,i-1}\otimes(u_j-w_j)$, by the growth condition (4.1.2) we have

$$\begin{aligned}
&\int_{Q_\nu^{j,i-1}\setminus Q_\nu^{j,i-2}} \psi(v_j) f^\infty(x_o,u_j^i,\nabla u_j^i)\,dx\\
&\quad\le c\int_{Q_\nu^{j,i-1}\setminus Q_\nu^{j,i-2}} \psi(v_j)\left(|\nabla u_j| + |\nabla w_j| + |\nabla\varphi_{j,i-1}|\,|u_j-w_j|\right)dx\\
&\quad\le c\int_{Q_\nu^{j,i-1}\setminus Q_\nu^{j,i-2}} \psi(v_j) f^\infty(x_o,u_j,\nabla u_j)\,dx + c\int_{Q_\nu^{j,i-1}\setminus Q_\nu^{j,i-2}} |\nabla w_j|\,dx\\
&\quad+\frac{c}{s_j}\int_{Q_\nu^{j,i-1}\setminus Q_\nu^{j,i-2}} |u_j-w_j|.
\end{aligned} \tag{4.2.14}$$

Analogously, there follows

$$\int_{Q_\nu\setminus Q_\nu^{j,i-1}} \psi\left(v_j^i\right) f^\infty(x_o,w_j,\nabla w_j)\,dx \le c\int_{Q_\nu\setminus Q_\nu^{j,i-1}} |\nabla w_j|\,dx. \tag{4.2.15}$$

In addition, since $\nabla v_j^i = \varphi_{j,i}\nabla v_j + (v_j - 1)\nabla\varphi_{j,i}$, we get

$$\begin{aligned} G_{\gamma_j}\left(v_j^i; Q_\nu^{j,i} \setminus Q_\nu^{j,i-1}\right) &\leq c\int_{Q_\nu^{j,i}\setminus Q_\nu^{j,i-1}} \left(\frac{1}{\gamma_j} + \gamma_j\left|\nabla v_j\right|^2 + \frac{\gamma_j}{s_j^2}|v_j - 1|^2\right) dx \\ &\leq \frac{c}{\gamma_j}\mathcal{L}^n\left(Q_\nu^{j,i} \setminus Q_\nu^{j,i-1}\right) + c\, G_{\gamma_j}\left(v_j; Q_\nu^{j,i} \setminus Q_\nu^{j,i-1}\right) + c\frac{\gamma_j}{s_j^2}\int_{Q_\nu^{j,i}\setminus Q_\nu^{j,i-1}} |v_j - 1|^2\, dx. \end{aligned} \tag{4.2.16}$$

By collecting (4.2.13), (4.2.14), (4.2.15) and (4.2.16) in (4.2.12) above, by adding up on i and averaging, we have that there exists an index $i_j \in \mathbf{N}$, $2 \leq i_j \leq b_j$, such that

$$\begin{aligned} F_{\gamma_j}^\infty\left(x_o; u_j^{i_j}, v_j^{i_j}; Q_\nu\right) &\leq \frac{1}{b_j - 1}\sum_{i=2}^{b_j} F_{\gamma_j}^\infty\left(x_o; u_j^i, v_j^i; Q_\nu\right) \\ &\leq \left(1 + \frac{c}{b_j}\right) F_{\gamma_j}^\infty\left(x_o; u_j, v_j; Q_\nu\right) \\ &+ c\int_{Q_\nu\setminus Q_\nu^{j,0}} |\nabla w_j|\, dx + \frac{c}{a_j}\int_{Q_\nu\setminus Q_\nu^{j,0}} |u_j - w_j|\, dx \\ &+ \frac{c}{\gamma_j b_j}\mathcal{L}^n\left(Q_\nu \setminus Q_\nu^{j,0}\right) + c\frac{\gamma_j b_j}{a_j^2}\int_{Q_\nu\setminus Q_\nu^{j,0}} |v_j - 1|^2\, dx. \end{aligned} \tag{4.2.17}$$

Eventually, choose $a_j = \|u_j - w_j\|_{L^1(Q_\nu;\mathbf{R}^N)}^{\frac{1}{2}} + \|v_j - 1\|_{L^2(Q_\nu)}^{\frac{1}{2}}$, $b_j = \left[\gamma_j^{-1}\right]$ and set $(\hat{u}_j, \hat{v}_j) = \left(u_j^{i_j}, v_j^{i_j}\right)$. The conclusion then follows by passing to the limit on $j \to +\infty$ in (4.2.17), and noticing that $\mathcal{L}^n\left(Q_\nu \setminus Q_\nu^{j,0}\right) = O(a_j)$ and $\|Dw_j\|\left(Q_\nu \setminus Q_\nu^{j,0}\right) \to 0$. The last assertion follows since $(\|Dw_j\|)$ weakly $*$ converges to $\|Du_{a,b,\nu}\|$ in the sense of measures, $\|Dw_j\|(Q_\nu) \to \|Du_{a,b,\nu}\|(Q_\nu)$ and $\|Du_{a,b,\nu}\|\left(\partial Q_\nu^{j,0}\right) = 0$ for every $j \in \mathbf{N}$. □

4.3 Upper bound inequality

In order to prove Theorem 4.1.2 for functions u in $BV\left(\Omega; \mathbf{R}^N\right)$, we follow an abstract approach (see [16],[53]). Indeed, first we prove that for any $\overline{\Gamma}\left(L^1\right)$-converging subsequence of (F_ε), the limit, as a set function, is a Borel measure and, by Proposition 2.3.9, coincides with its Γ-limit. Then, by using Theorems 2.6.20 and 2.6.15, we provide, in Proposition 4.3.3 in the sequel, an upper estimate of the limiting functional, which, combined with the lower estimate of Proposition 4.2.1, allows us to conclude that the $\Gamma\left(L^1\right)$-limit does not depend on the chosen subsequence and it is equal to F. Hence, by Urysohn's property (see Proposition 2.3.4), the whole family (F_ε) $\Gamma\left(L^1\right)$-converges to F.

As a first step we prove the following crucial lemma, in which we establish the so called weak subadditivity for $F''(u, 1, \cdot)$ (see [53],[55]).

The argument used is a careful modification of well known techniques in this kind of problems, and it is strictly related to the ones exploited in Lemma 4.2.2.

Lemma 4.3.1 *Let $u \in BV\left(\Omega;\mathbf{R}^N\right)$, let $A', A, B \in \mathcal{A}(\Omega)$ with $A' \subset\subset A$, then*

$$F''(u,1;A'\cup B) \le F''(u,1;A) + F''(u,1;B).$$

Proof. Let $(w_j) \subset C^\infty\left(\Omega;\mathbf{R}^N\right)$ be strictly converging to u, i.e., such that $w_j \to u$ in $L^1\left(\Omega;\mathbf{R}^N\right)$ and $\|Dw_j\|(\Omega) \to \|Du\|(\Omega)$, and let $\left(u_j^A, v_j^A\right)$, $\left(u_j^B, v_j^B\right)$ be converging to $(u,1)$ in $L^1\left(\Omega;\mathbf{R}^{N+1}\right)$ and such that

$$\begin{aligned}
\limsup_j F_{\varepsilon_j}\left(u_j^A, v_j^A; A\right) &= F''(u,1;A),\\
\limsup_j F_{\varepsilon_j}\left(u_j^B, v_j^B; B\right) &= F''(u,1;B),
\end{aligned}$$

respectively. Set $\delta := d(A',\partial A)$, let $M \in \mathbf{N}$ and define

$$\begin{cases}
A_i^M := \left\{x \in A : d(x,A') \le \frac{\delta}{M} i\right\} & 1 \le i \le M\\
A_0^M := A'
\end{cases}.$$

Let $(\varphi_i) \subset C_0^\infty(A_i^M)$, $1 \le i \le M$, be a family of cut-off functions such that $0 \le \varphi_i \le 1$, $\varphi_i = 1$ on A_{i-1}^M, $\|\nabla\varphi_i\|_{L^\infty(A)} \le \frac{2M}{\delta}$. Define

$$u_j^i := \begin{cases}
\varphi_{i-1} u_j^A + (1-\varphi_{i-1}) w_j & A_{i-1}^M\\
w_j & A_i^M \setminus A_{i-1}^M\\
(1-\varphi_{i+1}) u_j^B + \varphi_{i+1} w_j & \Omega \setminus A_i^M
\end{cases},$$

and

$$v_j^i := \varphi_i v_j^A + (1-\varphi_i) v_j^B,$$

then $\left(u_j^i, v_j^i\right) \to (u,1)$ in $L^1\left(\Omega;\mathbf{R}^{N+1}\right)$ for every $1 \le i \le M$. Moreover, there follows

$$\begin{aligned}
F_{\varepsilon_j}\left(u_j^i, v_j^i; A'\cup B\right) &\le F_{\varepsilon_j}\left(u_j^A, v_j^A; A_{i-2}^M\right)\\
&+ \int_{\left(A_{i-1}^M \setminus \overline{A_{i-2}^M}\right)\cap B} \psi\left(v_j^A\right) f\left(x, u_j^i, \nabla u_j^i\right) dx + G_{\varepsilon_j}\left(v_j^A; \left(A_{i-1}^M \setminus \overline{A_{i-2}^M}\right)\cap B\right)\\
&+ c\int_{\left(A_i^M \setminus \overline{A_{i-1}^M}\right)\cap B} (1+|\nabla w_j|)\, dx + G_{\varepsilon_j}\left(v_j^i; \left(A_i^M \setminus \overline{A_{i-1}^M}\right)\cap B\right)\\
&+ \int_{\left(A_{i+1}^M \setminus \overline{A_i^M}\right)\cap B} \psi\left(v_j^B\right) f\left(x, u_j^i, \nabla u_j^i\right) dx + G_{\varepsilon_j}\left(v_j^B; \left(A_{i+1}^M \setminus \overline{A_i^M}\right)\cap B\right)\\
&+ F_{\varepsilon_j}\left(u_j^B, v_j^B; B \setminus \overline{A_{i+1}^M}\right).
\end{aligned}$$

Let us estimate only the terms above depending on the superscript A, analogous computations holds for the one with B. First, it is easy to check that

$$F_{\varepsilon_j}\left(u_j^A, v_j^A; A_{i-2}^M\right) + G_{\varepsilon_j}\left(v_j^A; \left(A_{i-1}^M \setminus \overline{A_{i-2}^M}\right) \cap B\right) \leq F_{\varepsilon_j}\left(u_j^A, v_j^A; A\right), \tag{4.3.1}$$

and

$$\begin{aligned}\int_{\left(A_{i-1}^M \setminus \overline{A_{i-2}^M}\right)\cap B} \psi\left(v_j^A\right) f\left(x, u_j^i, \nabla u_j^i\right) dx &\leq c\, F_{\varepsilon_j}\left(u_j^A, v_j^A; \left(A_{i-1}^M \setminus \overline{A_{i-2}^M}\right) \cap B\right) \\ +c \int_{\left(A_{i-1}^M \setminus \overline{A_{i-2}^M}\right)\cap B} (1 + |\nabla w_j|)\, dx &+ \int_{\left(A_{i-1}^M \setminus \overline{A_{i-2}^M}\right)\cap B} |\nabla \varphi_{i-1}| \left|u_j^A - w_j\right| dx.\end{aligned} \tag{4.3.2}$$

Moreover, there holds

$$\begin{aligned}&G_{\varepsilon_j}\left(v_j^i; \left(A_i^M \setminus \overline{A_{i-1}^M}\right) \cap B\right) \\ &\quad\leq c\, G_{\varepsilon_j}\left(v_j^A; \left(A_i^M \setminus \overline{A_{i-1}^M}\right) \cap B\right) + c\, G_{\varepsilon_j}\left(v_j^B; \left(A_i^M \setminus \overline{A_{i-1}^M}\right) \cap B\right) \\ &\quad+c\, \varepsilon_j \int_{\left(A_i^M \setminus \overline{A_{i-1}^M}\right)\cap B} |\nabla \varphi_i|^2 \left|v_j^A - v_j^B\right|^2 dx + \frac{c}{\varepsilon_j} \mathcal{L}^n\left(\left(A_i^M \setminus \overline{A_{i-1}^M}\right) \cap B\right).\end{aligned} \tag{4.3.3}$$

Then, from (4.3.1), (4.3.2), (4.3.3), by adding up on i and averaging, there exists an index $2 \leq i_j \leq M-1$ such that

$$\begin{aligned}F_{\varepsilon_j}\left(u_j^{i_j}, v_j^{i_j}; A' \cup B\right) &\leq \frac{1}{M-2} \sum_{i=2}^{M-1} F_{\varepsilon_j}\left(u_j^i, v_j^i; A' \cup B\right) \\ &\leq \left(1 + \frac{c}{M-2}\right)\left(F_{\varepsilon_j}\left(u_j^A, v_j^A; A\right) + F_{\varepsilon_j}\left(u_j^B, v_j^B; B\right)\right) \\ &+\frac{cM}{(M-2)\delta}\left(\int_{(A \setminus A')\cap B} \left|u_j^A - w_j\right| dx + \int_{(A \setminus A')\cap B} \left|u_j^B - w_j\right| dx\right) \\ &+\frac{c\, \varepsilon_j M^2}{(M-2)\delta^2} \int_{(A \setminus A')\cap B} \left|v_j^A - v_j^B\right|^2 dx + \frac{c}{M-2} \int_{(A \setminus A')\cap B} |\nabla w_j|\, dx \\ &+\frac{c}{\varepsilon_j (M-2)} \mathcal{L}^n\left((A \setminus A') \cap B\right).\end{aligned}$$

Now choose $M_j = \left[\varepsilon_j^{-1} \left\|v_j^A - v_j^B\right\|_{L^2(\Omega)}^{-1}\right]$, then by passing to the superior limit on $j \to +\infty$ and by the definition of F'' we get the conclusion. □

By virtue of Lemma 4.3.1 we get the following.

Corollary 4.3.2 *Assume that* $\left(F_{\varepsilon_j}\right)$ $\overline{\Gamma}\left(L^1\right)$ *-converges to* $\hat{F}$*, then for every* $u \in BV\left(\Omega; \mathbf{R}^N\right)$ *the set function* $\hat{F}(u, 1; \cdot)$ *is a Borel measure.*

Moreover, for every $A \in \mathcal{A}(\Omega)$

$$\hat{F}(u,1;A) \leq c\left(\mathcal{L}^n(A) + \|Du\|(A)\right),$$

and

$$\hat{F}(u,1;A) = \Gamma\left(L^1\right)\text{-}\lim_j F_{\varepsilon_j}(u,1;A).$$

Proof. It suffices to take into account that the growth assumptions (4.1.2) on f and to apply Propositions 2.3.8 and 2.3.9. □

We now are able to prove Theorem 4.1.2 in the BV case.

Proposition 4.3.3 *For every $u \in BV\left(\Omega;\mathbf{R}^N\right)$ we have*

$$\Gamma\left(L^1\right)\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon(u,1) = F(u,1).$$

Proof. Let $\varepsilon_j \to 0^+$ be such that for every $u \in BV\left(\Omega;\mathbf{R}^N\right)$ and $A \in \mathcal{A}(\Omega)$ there exists $\hat{F}(u,1;A) := \Gamma\left(L^1\right)\text{-}\lim_j F_{\varepsilon_j}(u,v;A)$.

Then, by Proposition 4.2.1, we are done if we show that

$$\hat{F}(u,1;\Omega) \leq F(u,1).$$

Since by Corollary 4.3.2 $\hat{F}(u,1;\cdot)$ is a Borel measure, it suffices to prove that

$$\hat{F}(u,1;\Omega\setminus J_u) \leq \int_\Omega f(x,u,\nabla u)\,dx + \int_\Omega f^\infty(x,\tilde{u},dD^c u), \tag{4.3.4}$$

and

$$\hat{F}(u,1;J_u) \leq \int_{J_u} K\left(x,u^+,u^-,\nu_u\right)\,d\mathcal{H}^{n-1}. \tag{4.3.5}$$

To prove (4.3.4), note that for every $j \in \mathbf{N}$

$$F_{\varepsilon_j}(u,1;A) \equiv F_0(u;A),$$

with

$$F_0(u;A) = \begin{cases} \displaystyle\int_A f(x,u,\nabla u)\,dx & \text{if } u \in W^{1,1}\left(\Omega;\mathbf{R}^N\right) \\ +\infty & \text{if } u \in L^1(\Omega;\mathbf{R}^N)\setminus W^{1,1}\left(\Omega;\mathbf{R}^N\right). \end{cases}$$

Hence, for every $B \in \mathcal{B}(\Omega)$

$$\hat{F}(u,1;B) \leq \overline{F}_0(u;B).$$

By Theorems 2.6.18 and 2.6.20, we get that for every $u \in BV\left(\Omega;\mathbf{R}^N\right)$

$$\overline{F}_0(u;\Omega\setminus J_u) = \int_\Omega f(x,u,\nabla u)\,dx + \int_\Omega f^\infty(x,\tilde{u},dD^c u),$$

from which (4.3.4) is easily deduced.

By virtue of (2.6.2) of Theorem 2.6.15, to prove (4.3.5), it suffices to show that for every $(x_o, a, b, \nu) \in \Omega \times \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{S}^{n-1}$

$$\limsup_{\delta \to 0^+} \frac{\hat{F}\left(u_{a,b,\nu}(\cdot - x_o), 1; x_o + \delta Q_\nu\right)}{\delta^{n-1}} \le K(x_o, a, b, \nu). \tag{4.3.6}$$

Without loss of generality we prove (4.3.6) assuming $x_o = 0$ and $\nu = e_n$ (recall that Q_{e_n} is denoted by Q).

With given $\gamma > 0$, let $(u, v) \in \mathcal{A}(a, b, e_n)$ and $L > 0$ be such that

$$F^{\infty}_{\frac{1}{L}}(0, u, v; Q) \le K(0, a, b, e_n) + \gamma,$$

and define $(u_j, v_j) \in W^{1,1}\left(Q; \mathbf{R}^{N+1}\right)$ by

$$(u_j, v_j)(x) = \begin{cases} (b, 1) & \text{if } x_n > \frac{\varepsilon_j L}{2} \\ (u, v)\left(\frac{x}{\varepsilon_j L}\right) & \text{if } |x_n| \le \frac{\varepsilon_j L}{2} \\ (a, 1) & \text{if } x_n < -\frac{\varepsilon_j L}{2} \end{cases},$$

where (u, v) is extended to $\mathbf{R}^n$ by periodicity. Hence, $(u_j, v_j) \to (u_{a,b,e_n}, 1)$ in $L^1\left(Q; \mathbf{R}^{N+1}\right)$, and thus

$$\hat{F}(u_{a,b,e_n}, 1; \delta Q) \le \limsup_j F_{\varepsilon_j}\left(u_j, v_j; \delta Q\right). \tag{4.3.7}$$

Set $Q^j_\delta := \delta Q \cap \left\{|x_n| \le \frac{\varepsilon_j L}{2}\right\}$ and $Q'_\delta = \delta Q \cap \{x_n = 0\}$, then we have

$$\begin{aligned} F_{\varepsilon_j}\left(u_j, v_j; \delta Q\right) = &\int_{\delta Q \cap \left\{x_n < -\frac{\varepsilon_j L}{2}\right\}} f(x, a, 0)\, dx \\ &+ \int_{\delta Q \cap \left\{x_n > \frac{\varepsilon_j L}{2}\right\}} f(x, b, 0)\, dx + F_{\varepsilon_j}\left(u_j, v_j; Q^j_\delta\right). \end{aligned} \tag{4.3.8}$$

The change of variables $t = \frac{x_n}{\varepsilon_j L}$ yields for j large

$$\begin{aligned} &F_{\varepsilon_j}\left(u_j, v_j; Q^j_\delta\right) = \\ &\int_{-1/2}^{1/2} \varepsilon_j L\, dt \int_{Q'_\delta} \psi\left(v\left(\frac{x'}{\varepsilon_j L}, t\right)\right) f\left(\left(x', \varepsilon_j L t\right), u\left(\frac{x'}{\varepsilon_j L}, t\right), \frac{1}{\varepsilon_j L} \nabla u\left(\frac{x'}{\varepsilon_j L}, t\right)\right) dx' \\ &+ \int_{-1/2}^{1/2} dt \int_{Q'_\delta} \left(L\, W\left(v\left(\frac{x'}{\varepsilon_j L}, t\right)\right) + \frac{1}{L}\left|\nabla\left(v\left(\frac{x'}{\varepsilon_j L}, t\right)\right)\right|^2\right) dx' \\ &=: I^1_{j,\delta} + I^2_{j,\delta}. \end{aligned} \tag{4.3.9}$$

With fixed $\eta > 0$, by (4.1.4), we can choose δ small enough such that for j large we have

$$I^1_{j,\delta} \leq \frac{1}{1-\eta}\Bigg(\eta\delta^{n-1} \tag{4.3.10}$$
$$+\int_{-1/2}^{1/2} dt \int_{Q'_\delta} \psi\left(v\left(\frac{x'}{\varepsilon_j L}, t\right)\right) f^\infty\left((x', \varepsilon_j L t), u\left(\frac{x'}{\varepsilon_j L}, t\right), \nabla u\left(\frac{x'}{\varepsilon_j L}, t\right)\right) dx'\Bigg).$$

Now consider the Yosida's Transform of f^∞ defined, for $\lambda > 0$, as

$$f^\infty_\lambda(x, u, z) := \sup_{y \in \mathbf{R}^n} \{f^\infty(y, u, z) - \lambda|y - x|\}.$$

Recall that

$$f^\infty(x, u, z) \leq f^\infty_{\lambda_1}(x, u, z) \leq f^\infty_{\lambda_2}(x, u, z) \tag{4.3.11}$$

if $0 < \lambda_2 \leq \lambda_1$ and, since $f^\infty(\cdot, u, z)$ is upper semicontinuous, $f^\infty_\lambda(\cdot, u, z) \to f^\infty(\cdot, u, z)$ pointwise as $\lambda \to +\infty$. Moreover, f^∞_λ is λ-Lipschitz, i.e.,

$$|f^\infty_\lambda(x_1, u, z) - f^\infty_\lambda(x_2, u, z)| \leq \lambda|x_1 - x_2| \tag{4.3.12}$$

and, by (4.1.2), for every $z \in \mathbf{R}^N$ there holds

$$0 < f^\infty_\lambda(x, u, z) \leq c(1 + |z|).$$

Thus, given $\lambda > 0$, by (4.3.10), (4.3.11) and (4.3.12), we get

$$I^1_{j,\delta} \leq \frac{1}{1-\eta}\Bigg(\eta\delta^{n-1} + 2\lambda\delta^n$$
$$+\int_{-1/2}^{1/2} dt \int_{Q'_\delta} \psi\left(v\left(\frac{x'}{\varepsilon_j L}, t\right)\right) f^\infty_\lambda\left(0, u\left(\frac{x'}{\varepsilon_j L}, t\right), \nabla u\left(\frac{x'}{\varepsilon_j L}, t\right)\right) dx'\Bigg). \tag{4.3.13}$$

Let now $j \to +\infty$ in (4.3.8) and take into account the inequalities (4.3.9) and (4.3.13); then by virtue of the Riemann-Lebesgue's Lemma we have

$$\limsup_j F_{\varepsilon_j}(u_j, v_j; \delta Q) \leq \frac{1}{1-\eta}\delta^{n-1}\int_Q \psi(v) f^\infty_\lambda(0, u, \nabla u)\, dx$$
$$+\delta^{n-1}\int_Q \left(LW(v) + \frac{1}{L}|\nabla v|^2\right) dx + \frac{\eta}{1-\eta}\delta^{n-1} + \left(\frac{2\lambda}{1-\eta} + c\right)\delta^n.$$

Thus, by (4.3.7), we get

$$\limsup_{\delta \to 0^+} \frac{\hat{F}(u_{a,b,e_n}, 1; \delta Q)}{\delta^{n-1}} \leq \frac{1}{1-\eta}\int_Q \psi(v) f^\infty_\lambda(0, u, \nabla u)\, dx$$
$$+\int_Q \left(LW(v) + \frac{1}{L}|\nabla v|^2\right) dx + \frac{\eta}{1-\eta}.$$

Eventually, by letting $\eta \to 0^+$ and $\lambda \to +\infty$, by Lebesgue's Theorem we get

$$\begin{aligned}
&\limsup_{\delta \to 0^+} \frac{\hat{F}\left(u_{a,b,e_n}, 1; \delta Q\right)}{\delta^{n-1}} \\
&\quad \le \int_Q \left(\psi(v) f^\infty\left(0, u, \nabla u\right) + LW(v) + \frac{1}{L}|\nabla v|^2\right) dx \\
&\quad = F^\infty_{\frac{1}{L}}(0; u, v; Q) \le K(0, a, b, e_n) + \gamma,
\end{aligned}$$

and by the arbitrariness of $\gamma > 0$ we obtain (4.3.6). □

4.4 The GBV case

In this section we prove the full result stated in Theorem 4.1.2. We recall that we have already shown the Γ-convergence result if the target function $u \in BV\left(\Omega; \mathbf{R}^N\right)$, here we extend the proof to all functions $u \in L^1\left(\Omega; \mathbf{R}^N\right)$, and we identify the domain of the limit functional in a subset of $(GBV(\Omega))^N \times \{1\}$.

We first state and prove a preliminary lemma on the continuity of $F(\cdot, 1)$ with respect to truncations. To do that consider the auxiliary functions Ψ_i defined by

$$\Psi_i(u) := \begin{cases} u & \text{if } |u| \le a_i \\ \\ 0 & \text{if } |u| \ge a_{i+1} \end{cases}, \tag{4.4.1}$$

where $(a_i) \subset (0, +\infty)$ is a strictly increasing and diverging sequence, and for every $i \in \mathbf{N}$ $\Psi_i \in C^1\left(\mathbf{R}^N; \mathbf{R}^N\right)$ and $\|\nabla \Psi_i\|_{L^\infty(\mathbf{R}^N; \mathbf{R}^{N \times N})} \le 1$.

Lemma 4.4.1 *Let $u \in (GBV(\Omega))^N$ with $F(u, 1; \Omega) < +\infty$ and let $u^i := \Psi_i(u)$, $i \in \mathbf{N}$. Then*

$$\lim_i F\left(u^i, 1\right) = F\left(u, 1\right).$$

Proof. We prove separately the convergence of the different terms of F.

Since $\nabla u(x) = \nabla u^i(x)$ for a.e. $x \in \Omega_i := \{x \in \Omega : |\tilde{u}(x)| < a_i\}$, we have

$$\int_\Omega f(x, u^i, \nabla u^i)\, dx = \int_{\Omega_i} f(x, u, \nabla u)\, dx + \int_{\Omega \setminus \Omega_i} f(x, u^i, \nabla u^i)\, dx.$$

By the growth assumption (4.1.2), we get

$$\left| \int_{\Omega \setminus \Omega_i} f\left(x, u^i, \nabla u^i\right)\, dx \right| \le c \int_{\Omega \setminus \Omega_i} \left(1 + |\nabla u|\right)\, dx,$$

and so, being the term on the right hand side above infinitesimal, we deduce that

$$\lim_i \int_\Omega f(x, u^i, \nabla u^i)\, dx = \int_\Omega f\left(x, u, \nabla u\right)\, dx.$$

Let us prove the convergence of the Cantor part of the energy. Since the measures $D^c u^i$ are absolutely continuous with respect to $\|D^c u\|$ and $D^c u^i \llcorner \Omega_i \equiv D^c u \llcorner \Omega_i$, we have

$$\begin{aligned}
\int_\Omega f^\infty\left(x, \tilde{u}^i, dD^c u^i\right) &= \int_\Omega f^\infty\left(x, \tilde{u}^i, \frac{dD^c u^i}{d\|D^c u\|}\right) d\|D^c u\| \\
&= \int_{\Omega_i} f^\infty\left(x, \tilde{u}, dD^c u\right) + \int_{\Omega \setminus \Omega_i} f^\infty\left(x, \tilde{u}^i, \frac{dD^c u^i}{d\|D^c u\|}\right) d\|D^c u\|. \qquad (4.4.2)
\end{aligned}$$

Moreover, by (4.1.6), we have

$$\left|\int_{\Omega \setminus \Omega_i} f^\infty\left(x, \tilde{u}^i, \frac{dD^c u^i}{d\|D^c u\|}\right) d\|D^c u\|\right| \leq c\|D^c u\|\left(\Omega \setminus \Omega_i\right),$$

and thus, since $\|D^c u\|\left(\Omega \setminus \Omega_i\right) \to 0$ as $i \to +\infty$, from (4.4.2) we conclude that

$$\lim_i \int_\Omega f^\infty\left(x, \tilde{u}^i, dD^c u^i\right) = \int_\Omega f^\infty\left(x, \tilde{u}, dD^c u\right).$$

Furthermore, for what the surface energy is concerned, note that $\mathcal{H}^{n-1}\left(J_u^\infty\right) = 0$ (see Theorem 2.7.7, Remark 2.7.8 and Remark 4.1.6) and $J_{u^i} \subseteq J_u$ for every $i \in \mathbf{N}$ with $\nu_{u^i} = \nu_u$ for $\mathcal{H}^{n-1}$ a.e. $x \in J_{u^i}$. Then, $\left(u^i\right)^\pm \to u^\pm$, $\mathcal{X}_{J_{u^i}} \to \mathcal{X}_{J_u}$ for $\mathcal{H}^{n-1}$ a.e. $x \in J_u$ as $i \to +\infty$. Hence, there follows

$$\begin{aligned}
&\lim_i \int_{J_{u^i}} K\left(x, \left(u^i\right)^+, \left(u^i\right)^-, \nu_{u^i}\right) d\mathcal{H}^{n-1} \\
&= \lim_i \int_{J_u} K\left(x, \left(u^i\right)^+, \left(u^i\right)^-, \nu_u\right) \mathcal{X}_{J_{u^i}}\, d\mathcal{H}^{n-1} \\
&= \int_{J_u} K\left(x, u^+, u^-, \nu_u\right) d\mathcal{H}^{n-1},
\end{aligned}$$

by Lebesgue's Theorem and taking into account properties (a) and (b) of Lemma 4.1.9. □

The idea of the proof of the Γ-liminf inequality in the next proposition is based again on De Giorgi's averaging-slicing method but now the truncations are performed on the range rather than on the domain (see Lemma 3.7 [29], Lemma 3.5 [42]).

Proposition 4.4.2 *For every $(u, v) \in L^1\left(\Omega; \mathbf{R}^{N+1}\right)$ we have*

$$\Gamma\left(L^1\right)\text{-}\lim_{\varepsilon \to 0^+} F_\varepsilon(u, v) = F(u, v).$$

Proof. We divide the proof in two steps, dealing with the Γ-liminf and the Γ-limsup inequality separately.

Step 1:(Lower Bound inequality): for every $(u, v) \in L^1\left(\Omega; \mathbf{R}^{N+1}\right)$ there holds

$$\Gamma\left(L^1\right)\text{-}\liminf_{\varepsilon \to 0^+} F_\varepsilon(u, v) \geq F(u, v). \qquad (4.4.3)$$

Let $(u_j, v_j) \to (u, v)$ in $L^1\left(\Omega; \mathbf{R}^{N+1}\right)$ be such that

$$\lim_j F_{\varepsilon_j}(u_j, v_j) = \Gamma\left(L^1\right)\text{-}\liminf_j F_{\varepsilon_j}(u, v). \tag{4.4.4}$$

We may also assume such a limit to be finite; hence, as already shown in Proposition 4.2.1, we have that $v_j \to 1$ in $L^1(\Omega)$, and, as observed in Remark 4.1.6, $u \in (GBV(\Omega))^N$.

Define $u_j^i := \Psi_i(u_j)$, $u^i := \Psi_i(u)$, where Ψ_i are the auxiliary functions in (4.4.1), then $u_j^i \in W^{1,1}\left(\Omega; \mathbf{R}^N\right)$, $u^i \in BV\left(\Omega; \mathbf{R}^N\right)$ and $u_j^i \to u^i$ in $L^1\left(\Omega; \mathbf{R}^N\right)$ for every $i \in \mathbf{N}$. Moreover, notice that

$$F_{\varepsilon_j}\left(u_j^i, v_j\right) = \int_\Omega \psi(v_j) f\left(x, u_j^i, \nabla u_j^i\right) dx + G_{\varepsilon_j}(v_j; \Omega). \tag{4.4.5}$$

Fix $j \in \mathbf{N}$, then we have

$$\begin{aligned}
\int_\Omega \psi(v_j) f\left(x, u_j^i, \nabla u_j^i\right) dx &= \int_{\{|u_j| < a_i\}} \psi(v_j) f(x, u_j, \nabla u_j)\, dx \\
&+ \int_{\{a_i \le |u_j| \le a_{i+1}\}} \psi(v_j) f\left(x, u_j^i, \nabla u_j^i\right) dx + \int_{\{|u_j| > a_{i+1}\}} \psi(v_j) f(x, 0, 0)\, dx \\
&\le \int_\Omega \psi(v_j) f(x, u_j, \nabla u_j)\, dx + c \int_{\{a_i \le |u_j| \le a_{i+1}\}} \psi(v_j)(1 + |\nabla u_j|)\, dx \\
&+ c\mathcal{L}^n(\{|u_j| > a_{i+1}\}).
\end{aligned}$$

With fixed $\eta > 0$ there exists $i_o \in \mathbf{N}$, $i_o \ge \frac{1}{\eta}$, such that $c\mathcal{L}^n(\{|u_j| \ge a_{i_o}\}) \le \eta$. Let $M \in \mathbf{N}$, then for every $j \in \mathbf{N}$ there exists $i_j \in \{i_o, i_o + 1, \ldots, i_o + M - 1\}$ such that

$$\begin{aligned}
\int_\Omega \psi(v_j) f\left(x, u_j^{i_j}, \nabla u_j^{i_j}\right) dx &\le \frac{1}{M} \sum_{i=i_o}^{i_o+M-1} \int_\Omega \psi(v_j) f\left(x, u_j^i, \nabla u_j^i\right) dx \\
&\le \int_\Omega \psi(v_j) f(x, u_j, \nabla u_j)\, dx + \frac{c}{M} \int_{\{|u_j| \ge a_{i_o}\}} \psi(v_j)(1 + |\nabla u_j|)\, dx + \eta \\
&\le \int_\Omega \psi(v_j) f(x, u_j, \nabla u_j)\, dx + 2\eta,
\end{aligned} \tag{4.4.6}$$

by (4.1.2), (4.4.4) and by choosing $M \in \mathbf{N}$ suitably. Note that M is independent of j and depends only on η. Moreover, (4.4.5) and (4.4.6) yield

$$F_{\varepsilon_j}\left(u_j^{i_j}, v_j\right) \le F_{\varepsilon_j}(u_j, v_j) + 2\eta. \tag{4.4.7}$$

Since $i_j \in \{i_o, i_o + 1, \ldots, i_o + M - 1\}$ for every $j \in \mathbf{N}$, up to extracting a subsequence not relabelled for convenience, we may assume $i_j \equiv i_\eta$ to be constant. Hence, $u_j^{i_\eta} \to u^{i_\eta}$ in $L^1\left(\Omega; \mathbf{R}^N\right)$ and so by (4.4.4), (4.4.7) and Subsection 4.2.1 there follows

$$\begin{aligned}
F\left(u^{i_\eta}, v\right) &\le \lim_j F_{\varepsilon_j}\left(u_j^{i_\eta}, v_j\right) \\
&\le \Gamma\left(L^1\right)\text{-}\liminf_j F_{\varepsilon_j}(u, v) + 2\eta.
\end{aligned} \tag{4.4.8}$$

Eventually, letting $\eta \to 0^+$ in (4.4.8), by Lemma 4.4.1 we obtain (4.4.3).

Step 2: (Upper Bound inequality): for every $(u,v) \in L^1\left(\Omega;\mathbf{R}^{N+1}\right)$ we have

$$\Gamma\left(L^1\right)\text{-}\limsup_{\varepsilon\to 0^+} F_\varepsilon(u,v) \le F(u,v). \tag{4.4.9}$$

It suffices to prove (4.4.9) for $u \in (GBV(\Omega))^N$ with $F(u,1) < +\infty$ and $v \equiv 1$. Let u^i be the truncation of u defined before, then, since $u^i \in BV\left(\Omega;\mathbf{R}^N\right)$, Proposition 4.3.3 yields

$$\Gamma\left(L^1\right)\text{-}\limsup_{\varepsilon\to 0^+} F_\varepsilon\left(u^i,1\right) = F\left(u^i,1\right). \tag{4.4.10}$$

Letting $i \to +\infty$ in (4.4.10), the conclusion follows by Lemma 4.4.1 and the lower semicontinuity of $\Gamma\left(L^1\right)\text{-}\limsup_{\varepsilon\to 0^+} F_\varepsilon$. □

4.5 Compactness and Convergence of Minimizers

Let us state an equicoercivity result for the approximating functionals defined in (4.1.8).

Lemma 4.5.1 *Let $(u_j,v_j) \in L^1\left(\Omega;\mathbf{R}^{N+1}\right)$ be such that*

$$\liminf_j\left(F_{\varepsilon_j}(u_j,v_j) + \int_\Omega |u_j|^q\,dx\right) < +\infty, \tag{4.5.1}$$

with $q \in (1,+\infty)$. Then there exists a subsequence (u_{j_k},v_{j_k}) and $u \in (GBV(\Omega))^N$ such that $(u_{j_k},v_{j_k}) \to (u,1)$ in $L^1\left(\Omega;\mathbf{R}^{N+1}\right)$.

Proof. By arguing exactly as in the proof of Lemma 3.4.1 we deduce that there exist a subsequence (u_{j_k},v_{j_k}) and a function $u \in \mathcal{B}\left(\Omega;\mathbf{R}^N\right)$ such that $(u_{j_k},v_{j_k}) \to (u,1)$ in measure on Ω.

Moreover, since $q \in (1,+\infty)$, by (4.5.1) we have that the sequence (u_j) is equi-integrable and so the strong $L^1\left(\Omega;\mathbf{R}^{N+1}\right)$ convergence follows by Vitali's Theorem.

By (4.5.1) and by Remark 4.1.6 we deduce that $u \in (GBV(\Omega))^N$. □

We are now able to state the following result on the convergence of minimum problems.

Theorem 4.5.2 *For every $g \in L^q\left(\Omega;\mathbf{R}^N\right)$, $q \in (1,+\infty)$, and every $\gamma > 0$, define*

$$m_\varepsilon := \inf\left\{F_\varepsilon(u,v) + \gamma\int_\Omega |u-g|^q\,dx : (u,v) \in L^1\left(\Omega;\mathbf{R}^{N+1}\right)\right\},$$

and let $(u_\varepsilon,v_\varepsilon)$ be asymptotically minimizing, i.e.,

$$F_\varepsilon(u_\varepsilon,v_\varepsilon) - m_\varepsilon \to 0.$$

Then, every cluster point of (u_ε) is a solution of the minimum problem

$$m := \inf\left\{\mathcal{F}(u) + \gamma\int_\Omega |u-g|^q\,dx : u \in (GBV(\Omega))^N\right\},$$

and $m_\varepsilon \to m$ as $\varepsilon \to 0^+$.

4.6 Generalizations

In this section we discuss further extensions of the model obtained by slightly varying the approximating functionals (4.1.8).

4.6.1 Anisotropic singular perturbations

In this subsection we consider spatially and directionally anisotropic singular perturbation terms in the definition of the approximating functionals (4.1.8), and we obtain a generalization of Theorem 4.1.2.

With fixed $p \in (1, +\infty)$, let $h : \Omega \times \mathbf{R}^n \to [0, +\infty)$ be a Borel integrand satisfying the following set of assumptions:

(h1) there exist three constants $c_3 \geq 0$ and c_4, $c_5 > 0$ such that

$$c_4 |\zeta| - c_3 \leq h(x, \zeta) \leq c_5 (|\zeta| + 1)$$

for every $(x, \zeta) \in \Omega \times \mathbf{R}^n$;

(h2) $h(x, \cdot)$ is locally Lipschitz for every $x \in \Omega$;

(h3) for every $x_o \in \Omega$ and for every $\eta > 0$ there exists $\delta > 0$, depending on x_o and η, such that

$$|(h^\infty)^p (x_o, \zeta) - (h^\infty)^p (x, \zeta)| \leq \eta (h^\infty)^p (x, \zeta)$$

for every $x \in \Omega$ with $|x - x_o| \leq \delta$ and for every $\zeta \in \mathbf{R}^n$;

(h4) for every $x_o \in \Omega$ and for every $\eta > 0$ there exist $\delta, L > 0$, depending on x_o and η, such that

$$\left| (h^\infty)^p (x, \zeta) - \frac{h^p (x, t\zeta)}{t^p} \right| \leq \eta \left(1 + \frac{h^p (x, t\zeta)}{t^p} \right)$$

for every $t > L$ and $x \in \Omega$ with $|x - x_o| \leq \delta$ and for every $\zeta \in \mathbf{R}^n$.

Let

$$h_\varepsilon(x, (u, v), (z, \zeta)) := \psi(v) f (x, u, z) + \frac{W(v)}{p'\varepsilon} + \frac{\varepsilon^{p-1}}{p} h^p (x, \zeta) ,$$

with f, ψ and W as in Section 4.1, $p' = \frac{p}{p-1}$. Then, consider the family of functionals $H_\varepsilon : L^1 \left(\Omega; \mathbf{R}^{N+1}\right) \to [0, +\infty]$ defined by

$$H_\varepsilon (u, v) := \begin{cases} \displaystyle\int_\Omega h_\varepsilon (x, (u, v), \nabla(u, v)) \, dx & \text{if } (u, v) \in W^{1,1} \left(\Omega; \mathbf{R}^{N+1}\right) \\ & 0 \leq v \leq 1 \text{ a.e.,} \\ +\infty & \text{otherwise.} \end{cases}$$

The proof of the Γ-convergence for the family (H_ε) follows by exploiting the same arguments used to prove Theorem 4.1.2 with some minor changes.

Theorem 4.6.1 *Let (H_ε) be as above, then (H_ε) Γ-converges with respect to the strong $L^1\left(\Omega;\mathbf{R}^{N+1}\right)$ convergence to the functional F defined in (4.1.7) with surface energy density $K:\Omega\times\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{n-1}\to[0,+\infty)$ given by*

$$\begin{aligned}&K\left(x_o,a,b,\nu\right):=\\&\quad\inf\left\{\int_{Q_\nu}\left(\psi(v)f^\infty\left(x_o,u,\nabla u\right)+\frac{L}{p'}W(v)+\frac{1}{pL^{p-1}}\left(h^\infty\right)^p\left(x_o,\nabla v\right)\right)dy:\right.\\&\quad\left.(u,v)\in\mathcal{A}(a,b,\nu),L>0\right\}.\end{aligned}\tag{4.6.1}$$

Let us remark that Lemma 4.1.9 still holds true. Moreover, (a) of Lemma 4.1.10 is valid provided the function g appearing in the statement is substituted by

$$g_h\left(x_o,\nu,t\right):=\inf_{r\in[0,1]}\left\{\psi(r)t+\left(h^\infty(x_o,\nu)+h^\infty(x_o,-\nu)\right)\int_r^1\left(W(s)\right)^{\frac{1}{p'}}ds\right\}.$$

Eventually, in case f^∞ is isotropic assume, in addition, either h^∞ to be isotropic or $h^\infty(x,\cdot)$ to be a norm for every $x\in\Omega$. Then K can be characterized as in Lemma 4.1.10 (b) with the function g substituted by g_h defined above. Indeed, the first claim follows by using the isotropy assumption on f^∞ and h^∞ to reduce the minimization formula (4.6.1) to one-dimensional profile functions; while the second by using the characterization of a norm via its dual norm (see formula (3.2.9)).

4.6.2 Approximation of superlinear energies via linear ones

In this subsection we use the results of Theorem 4.6.1 to approximate energies with superlinear bulk term, as considered in (3.1.1) of Chapter 3, by linear functionals as in (4.1.8).

For the sake of simplicity we deal only with the autonomous case, from which one can recover the general one of an integrand $f=f(x,u,z)$ satisfying (f3), by applying the blow-up method.

Thus, we consider $f:\mathbf{R}^{N\times n}\to\mathbf{R}$ such that for some $c_0\geq 0$, c_1, $c_2>0$ and $p\in(1,+\infty)$

$$c_1|z|^p-c_0\leq f(z)\leq c_2\left(|z|^p-1\right)\tag{4.6.2}$$

holds for all z.

We recall the following approximation result proved by Kristensen [78], which shows that quasiconvex functions with linear growth play in the quasiconvex setting the same role of affine functions in the convex case.

Lemma 4.6.2 *Let $f:\mathbf{R}^{N\times n}\to\mathbf{R}$ be a quasiconvex function, satisfying (4.6.2).*

Then there exist $f_j:\mathbf{R}^{N\times n}\to\mathbf{R}$ that are quasiconvex and satisfy

(1) $f_j(z)\leq f_{j+1}(z)$;

(2) $f_j(z)\to f(z)$ *as* $j\to+\infty$;

(3) there exists a_j,$r_j > 0$, $b_j \in \mathbf{R}$ such that if $|z| \geq r_j$

$$f_j(z) = f_j^{**}(z) = a_j|z| + b_j,$$

where f_j^{**} denotes the convex envelope of f_j.

Let (ε_j) be a positive infinitesimal sequence, then set

$$h_j(x,(u,v),(z,\zeta)) := \psi(v) f_j(z) + \frac{W(v)}{p'\varepsilon_j} + \frac{\varepsilon_j^{p-1}}{p}\varphi^p(\zeta),$$

with f a quasiconvex function satisfying (4.6.2), ψ and W as in Section 4.1, $p \in (1,+\infty)$ and $p' = \frac{p}{p-1}$, and $\varphi : \mathbf{R}^n \to [0,+\infty)$ a norm. Consider the family of functionals $H_j : L^1\left(\Omega;\mathbf{R}^{N+1}\right) \to [0,+\infty]$ defined by

$$H_j(u,v) := \begin{cases} \displaystyle\int_\Omega h_j(x,(u,v),\nabla(u,v))\, dx & \text{if } (u,v) \in W^{1,1}\left(\Omega;\mathbf{R}^{N+1}\right) \\ & 0 \leq v \leq 1 \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.6.3}$$

Theorem 4.6.3 *(H_j) Γ-converges with respect to the $L^1\left(\Omega;\mathbf{R}^{N+1}\right)$ convergence to the functional $H : L^1\left(\Omega;\mathbf{R}^{N+1}\right) \to [0,+\infty]$ defined by*

$$H(u,v) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\, dx + c_W \int_{J_u} \varphi(\nu_u)\, d\mathcal{H}^{n-1} & \text{if } u \in \left(GSBV^p(\Omega)\right)^N \\ +\infty & \text{otherwise} \end{cases},$$

where $c_W = 2\int_0^1 (W(s))^{\frac{1}{p'}}\, ds$.

Proof. Without loss of generality assume $(u,v) \in L^1\left(\Omega;\mathbf{R}^{N+1}\right)$ to be such that

$$\Gamma\text{-}\liminf_j H_j(u,v) < +\infty.$$

To prove the lower bound inequality, with fixed $k \in \mathbf{N}$, consider the functions

$$h_j^k(x,(u,v),(z,\zeta)) := \psi(v) f_k(z) + \frac{W(v)}{p'\varepsilon_j} + \frac{{\varepsilon_j}^{p-1}}{p}\varphi^p(\zeta),$$

and the corresponding functionals H_j^k obtained by substituting h_j with h_j^k in (4.6.3). Then, since $H_j \geq H_j^k$ for $j \geq k$, Theorem 4.6.1 yields for any $k \in \mathbf{N}$

$$\Gamma\text{-}\liminf_j H_j(u,v) \geq F_k(u,v),$$

F_k being the Γ-limit of (H_j^k). Hence, $u \in (GBV(\Omega))^N$ and $v = 1$ a.e. in Ω.

Moreover, Lemma 4.6.2 (2) implies that (F_k) is increasing, so that

$$\Gamma\text{-}\liminf_j H_j(u,v) \geq \sup_k F_k(u,v). \tag{4.6.4}$$

By Lemma 4.6.2 (3) $f_k^\infty(z) = a_k|z|$, and thus there follows

$$F_k(u,v) = \int_\Omega f_k(\nabla u)\, dx + a_k \|D^c u\|(\Omega) + \int_{J_u} K_k\left(x, u^+, u^-, \nu_u\right)\, d\mathcal{H}^{n-1}, \tag{4.6.5}$$

where

$$K_k(x_o, a, b, \nu) := \inf\left\{\int_{Q_\nu}\left(a_k\psi(v)|\nabla u| + \frac{L}{p'}W(v) + \frac{1}{pL^{p-1}}\varphi^p(\nabla v)\right) dy :\right.$$
$$\left.(u,v) \in \mathcal{A}(a,b,\nu), L > 0\right\}.$$

It is easy to check that $a_k \to +\infty$ as $k \to +\infty$, so that by passing to the supremum on k, by (4.6.4) and (4.6.5), we infer that $\|D^c u\|(\Omega) = 0$ and then $u \in (GSBV(\Omega))^N$. Moreover, the final remarks in Subsection 4.6.1 yield

$$K_k(x_o, a, b, \nu) = a_k \min_{r\in[0,1]}\left\{\psi(r)|z| + 2\varphi\left(\frac{\nu}{a_k}\right)\int_r^1 (W(s))^{\frac{1}{p'}}\, ds\right\}.$$

Hence, there follows

$$\lim_k K_k(x_o, a, b, \nu) = 2\varphi(\nu)\int_0^1 (W(s))^{\frac{1}{p'}}\, ds = c_W\varphi(\nu). \tag{4.6.6}$$

By Lemma 4.6.2 (1), we can apply the Monotone Convergence Theorem and deduce from (4.6.4), (4.6.5) and (4.6.6) the lower bound inequality and $u \in (GSBV^p(\Omega))^N$.

To prove the upper bound inequality it suffices to notice that

$$H_j(u,v) \leq F_j(u,v),$$

where F_j is obtained by substituting in definition (4.6.3) the function h_j with

$$f_j(x,(u,v),(z,\zeta)) := \psi(v)f(z) + \frac{W(v)}{p'\varepsilon_j} + \frac{\varepsilon_j^{p-1}}{p}\varphi^p(\zeta),$$

and then to apply arguments analogous to those used in Section 3.3 to conclude. $\square$

Chapter 5

Finite Differences Approximation of Energies in Fracture Mechanics

5.1 Statement of the Γ-convergence result

In this Chapter[1] we provide a discrete and continuous approximation of linearized-elasticity energies for brittle materials, i.e., defined on $SBD(\Omega)$ by

$$\mu \int_{\Omega} |\mathcal{E}u(x)|^2 \, dx + \frac{\lambda}{2} \int_{\Omega} |\operatorname{div} u(x)|^2 \, dx + \gamma \mathcal{H}^{n-1} \left(J_u^{\star} \right), \tag{5.1.1}$$

in dimension $n = 2, 3$. Let us notice that since our analysis will be restricted to bounded deformations, i.e., $u \in L^{\infty}(\Omega; \mathbf{R}^n)$, $J_u \equiv J_u^{\star}$ (see the discussion after Definition 2.5.2). Hence, in the sequel we will use the symbol J_u for the sake of simplicity.

We introduce first a discretization of the divergence. If $\xi = (\xi^1, \xi^2) \in \mathbf{R}^2$, we denote by $\xi^{\perp}$ the vector in $\mathbf{R}^2$ orthogonal to ξ defined by $\xi^{\perp} := (-\xi^2, \xi^1)$. Fix $\xi, \zeta \in \mathbf{R}^2 \setminus \{0\}$; for $\varepsilon > 0$ and for any $u : \mathbf{R}^2 \to \mathbf{R}^2$ define

$$\begin{aligned}
&D_{\varepsilon}^{\xi} u(x) := \langle u(x + \varepsilon\xi) - u(x), \xi \rangle, \\
&\operatorname{div}_{\varepsilon}^{\xi,\zeta} u(x) := D_{\varepsilon}^{\xi} u(x) + D_{\varepsilon}^{\zeta} u(x), \\
&|D_{\varepsilon,\xi} u(x)|^2 := |D_{\varepsilon}^{\xi} u(x)|^2 + |D_{\varepsilon}^{-\xi} u(x)|^2, \\
&|\operatorname{Div}_{\varepsilon,\xi} u(x)|^2 := |\operatorname{div}_{\varepsilon}^{\xi,\xi^{\perp}} u(x)|^2 + |\operatorname{div}_{\varepsilon}^{\xi,-\xi^{\perp}} u(x)|^2 \\
&\qquad\qquad + |\operatorname{div}_{\varepsilon}^{-\xi,\xi^{\perp}} u(x)|^2 + |\operatorname{div}_{\varepsilon}^{-\xi,-\xi^{\perp}} u(x)|^2.
\end{aligned} \tag{5.1.2}$$

Starting from this definition we will provide discrete and continuous approximation results for functionals of type (5.1.1). We underline that this is only one possible definition of discretized divergence that seems to agree with mechanical models of neighbouring atomic interactions.

[1]The contents of this Chapter were obtained by the Author in collaboration with R. Alicandro and M.S. Gelli, and are contained in the paper *Finite-difference approximation of energies in fracture mechanics*, published on Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **29** (2000), 671–709.

We can give also the following alternative definition

$$
\begin{aligned}
&D_\varepsilon^\xi u(x) := \langle u(x+\varepsilon\xi) - u(x-\varepsilon\xi), \xi\rangle,\\
&|D_{\varepsilon,\xi} u(x)|^2 := \frac{1}{2}|D_\varepsilon^\xi u(x)|^2\\
&|\mathrm{Div}_{\varepsilon,\xi} u(x)|^2 := |D_\varepsilon^\xi u(x) + D_\varepsilon^{\xi^\perp} u(x)|^2.
\end{aligned}
\tag{5.1.3}
$$

This second definition can be motivated by the fact that from a numerical point of view it gives a more accurate approximation of the divergence as $\varepsilon \to 0^+$, although the centered differences usually have other drawbacks.

For the sake of simplicity, in the proofs of Sections 5.3 and 5.4, we will assume that the "finite-difference terms" involved in the approximating functionals are defined by (5.1.2). The arguments we will use can be easily adapted when considering definition (5.1.3).

5.1.1 Discrete approximation result

Let Ω be a bounded open set of $\mathbf{R}^2$ and, for $\varepsilon > 0$, define

$$
\mathcal{A}_\varepsilon(\Omega) := \left\{ u : \Omega \to \mathbf{R}^2 \ : \ u \equiv const \text{ on } (\alpha + [0,\varepsilon)^2) \cap \Omega \text{ for any } \alpha \in \varepsilon\mathbf{Z}^2 \right\}.
$$

Let $f : [0,+\infty) \to [0,+\infty)$ be an increasing function, such that $a, b > 0$ exist with

$$
a := \lim_{t\to 0+} \frac{f(t)}{t}, \quad b := \lim_{t\to+\infty} f(t) \tag{5.1.4}
$$

and $f(t) \le (at) \wedge b$ for any $t \ge 0$. For $u \in \mathcal{A}_\varepsilon(\Omega)$ and $\xi \in \mathbf{Z}^2$, set

$$
\mathcal{F}_\varepsilon^{d,\xi}(u) := \sum_{\alpha\in R_\varepsilon^\xi} \varepsilon f\left(\frac{1}{\varepsilon}\left(|D_{\varepsilon,\xi} u(\alpha)|^2 + \theta|\mathrm{Div}_{\varepsilon,\xi} u(\alpha)|^2\right)\right), \tag{5.1.5}
$$

where θ is a strictly positive parameter and

$$
R_\varepsilon^\xi := \left\{ \alpha \in \varepsilon\mathbf{Z}^2 : \ [\alpha - \varepsilon\xi, \alpha + \varepsilon\xi] \cup [\alpha - \varepsilon\xi^\perp, \alpha + \varepsilon\xi^\perp] \subset \Omega \right\}.
$$

Then consider the functional $F_\varepsilon^d : L^1(\Omega; \mathbf{R}^2) \to [0,+\infty]$ defined as

$$
F_\varepsilon^d(u) = \begin{cases} \displaystyle\sum_{\xi\in\mathbf{Z}^2} \rho(\xi)\mathcal{F}_\varepsilon^{d,\xi}(u) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega) \\ +\infty & \text{otherwise} \end{cases} \tag{5.1.6}
$$

where $\rho : \mathbf{Z}^2 \to [0,+\infty)$ is such that $\sum_{\xi\in\mathbf{Z}^2} |\xi|^4 \rho(\xi) < +\infty$ and $\rho(\xi) > 0$ for $\xi \in \{\mathrm{e}_1, \mathrm{e}_1 + \mathrm{e}_2\}$.

Theorem 5.1.1 *Let Ω be a convex bounded open set of $\mathbf{R}^2$. Then, (F_ε^d) Γ-converges on $L^\infty(\Omega; \mathbf{R}^2)$ to the functional $F^d : L^\infty(\Omega; \mathbf{R}^2) \to [0,+\infty]$ given by*

$$
F^d(u) = \begin{cases} \displaystyle\sum_{\xi\in\mathbf{Z}^2} \rho(\xi)\mathcal{F}^\xi(u) & \textit{if } u \in SBD(\Omega) \\ +\infty & \textit{otherwise} \end{cases} \tag{5.1.7}
$$

with respect to both the $L^1(\Omega;\mathbf{R}^2)$-convergence and the convergence in measure, where

$$\begin{aligned}\mathcal{F}^\xi(u) := {} & 2a\int_\Omega |\langle \mathcal{E}u(x)\xi,\xi\rangle|^2\,dx + 4a\theta|\xi|^4\int_\Omega |\mathrm{div}\,u(x)|^2\,dx \\ & +2b\int_{J_u}\Phi^\xi(u^+-u^-,\nu_u)\,d\mathcal{H}^1,\end{aligned}$$

with the function $\Phi^\xi:\mathbf{R}^2\to[0,+\infty)$ defined by

$$\Phi^\xi(z,\nu) := \psi^\xi(z,\nu)\vee\psi^{\xi^\perp}(z,\nu),$$

where for $\eta\in\mathbf{R}^2$

$$\psi^\eta(z,\nu) := \begin{cases} |\langle \nu,\eta\rangle| & \text{if } \langle z,\eta\rangle\neq 0 \\ 0 & \text{otherwise.}\end{cases}$$

The proof of the theorem above will be given in Section 5.3, as a consequence of some propositions, which deal with lower and upper Γ-limits separately.

Remark 5.1.2 *Notice that the surface term can be written explicitly as*

$$\begin{aligned}\int_{J_u}\Phi^\xi(u^+-u^-,\nu_u)\,d\mathcal{H}^1 = {} & \int_{J_u^\xi\setminus J_u^{\xi^\perp}}|\langle\nu_u,\xi\rangle|\,d\mathcal{H}^1 \\ & +\int_{J_u^{\xi^\perp}\setminus J_u^\xi}|\langle\nu_u,\xi^\perp\rangle|\,d\mathcal{H}^1+\int_{J_u^\xi\cap J_u^{\xi^\perp}}|\langle\nu_u,\xi\rangle|\vee|\langle\nu_u,\xi^\perp\rangle|\,d\mathcal{H}^1.\end{aligned} \tag{5.1.8}$$

Remark 5.1.3 *We point out that the assumption Ω convex will be used only in the proof of the Γ-$\limsup$ inequality. This assumption can be weakened (see Remark 5.3.5).*

Remark 5.1.4 *Notice that the domain of F^d is $L^\infty(\Omega;\mathbf{R}^2)\cap SBD^2(\Omega)$. Indeed, taking into account the assumption on ρ, an easy computation shows that*

$$F^d(u)\geq\sum_{\xi=\mathrm{e}_1,\mathrm{e}_1+\mathrm{e}_2}\rho(\xi)\mathcal{F}^\xi(u)\geq c\left(\int_\Omega|\mathcal{E}u(x)|^2\,dx+\mathcal{H}^1(J_u)\right).$$

Remark 5.1.5 *Note that, by a suitable choice of the discrete function ρ, the limit functional is isotropic in the volume term, i.e.,*

$$F^d(u)=\mu_1\int_\Omega|\mathcal{E}u(x)|^2\,dx+\lambda_1\int_\Omega|\mathrm{div}\,u(x)|^2\,dx+\int_{J_u}\Phi(u^+-u^-,\nu_u)\,d\mathcal{H}^1. \tag{5.1.9}$$

Choose, for example, $\rho(\mathrm{e}_1)=\rho(\mathrm{e}_2)=2\rho(\mathrm{e}_2\pm\mathrm{e}_1)\neq 0$ and $\rho(\xi)=0$ elsewhere. Moreover, for suitable choices of f and θ, it is possible to approximate functionals of type (5.1.9) for any strictly positive μ_1,λ_1.

By dropping the divergence term in (5.1.5) (i.e. $\theta = 0$), one can consider the functional $G^d_\varepsilon : L^1(\Omega;\mathbf{R}^2) \to [0,+\infty]$ defined as

$$G^d_\varepsilon(u) = \begin{cases} \displaystyle\sum_{\xi\in\mathbf{Z}^2} \rho(\xi) \sum_{\alpha\in\widetilde{R}^\xi_\varepsilon} \varepsilon f\left(\frac{1}{\varepsilon}|D_{\varepsilon,\xi}u(\alpha)|^2\right) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{5.1.10}$$

where $\widetilde{R}^\xi_\varepsilon := \{\alpha \in \varepsilon\mathbf{Z}^2 : [\alpha - \varepsilon\xi, \alpha + \varepsilon\xi] \subset \Omega\}$ and ρ is as above and satisfies also the condition $\rho(\mathrm{e}_2) \neq 0$.

Theorem 5.1.6 *Let Ω be a convex bounded open set of $\mathbf{R}^2$. Then $\left(G^d_\varepsilon\right)$ Γ-converges on $L^\infty(\Omega;\mathbf{R}^2)$ to the functional $G^d : L^\infty(\Omega;\mathbf{R}^2) \to [0,+\infty]$ given by*

$$G^d(u) = \begin{cases} \displaystyle\sum_{\xi\in\mathbf{Z}^2} \rho(\xi) G^\xi(u) & \text{if } u \in SBD(\Omega) \\ +\infty & \text{otherwise} \end{cases} \tag{5.1.11}$$

with respect to both the $L^1(\Omega;\mathbf{R}^2)$-convergence and the convergence in measure, where

$$G^\xi(u) := 2a \int_\Omega |\langle \mathcal{E}u(x)\xi, \xi\rangle|^2 \, dx + 2b \int_{J^\xi_u} |\langle \nu_u, \xi\rangle| \, d\mathcal{H}^1.$$

The proof of this theorem can be recovered from the proof of Theorem 5.1.1, up to slight modifications. We only remark that the further hypothesis $\rho(\mathrm{e}_2) \neq 0$ is needed in order to have good coercivity properties of the family G^d_ε.

Remark 5.1.7 *Notice that, although the definition of G^d_ε corresponds in some sense to taking $\theta = 0$ in (5.1.6), its Γ-limit G^d differs from F^d for $\theta = 0$ in the surface term.*

5.1.2 Continuous approximation result

Let Ω be a bounded open set of $\mathbf{R}^2$ and let $f : [0,+\infty) \to [0,+\infty)$ be as in the previous section.

For $\varepsilon > 0$, define $F^c_\varepsilon : L^1(\Omega;\mathbf{R}^2) \to [0,+\infty]$ as

$$F^c_\varepsilon(u) := \int_{\mathbf{R}^2} \rho(\xi) \mathcal{F}^{c,\xi}_\varepsilon(u) \, d\xi,$$

where

$$\mathcal{F}^{c,\xi}_\varepsilon(u) := \frac{1}{\varepsilon} \int_{\Omega^\xi_\varepsilon} f\left(\frac{1}{\varepsilon}\left(|D_{\varepsilon,\xi}u(x)|^2 + \theta|\mathrm{Div}_{\varepsilon,\xi}u(x)|^2\right)\right) dx \tag{5.1.12}$$

with

$$\Omega^\xi_\varepsilon := \left\{x \in \mathbf{R}^2 : [x - \varepsilon\xi, x + \varepsilon\xi] \cup [x - \varepsilon\xi^\perp, x + \varepsilon\xi^\perp] \subset \Omega\right\},$$

$\theta > 0$ and $\rho(\xi) = \psi(|\xi|)$ where $\psi : [0,+\infty) \to [0,+\infty)$ is such that for some $M > 0$ $\operatorname{ess\,inf}_{|t|\le M} \psi(t) > 0$ and $\int_0^{+\infty} t^5 \psi(t) \, dt < +\infty$.

Theorem 5.1.8 *(F^c_ε) Γ-converges on $L^\infty(\Omega;\mathbf{R}^2)$ with respect to the $L^1(\Omega;\mathbf{R}^2)$-convergence to the functional $F^c : L^\infty(\Omega;\mathbf{R}^2) \to [0,+\infty]$ given by*

$$F^c(u) := \begin{cases} \mu \displaystyle\int_\Omega |\mathcal{E}u(x)|^2\,dx + \lambda \int_\Omega |\mathrm{div}\,u(x)|^2\,dx + \gamma\mathcal{H}^1(J_u) & \textit{if } u \in SBD(\Omega) \\ +\infty & \textit{otherwise,} \end{cases}$$

where

$$\begin{aligned} \mu &:= a\int_{\mathbf{R}^2} \rho(y)\left(|y|^4 - 4y_1^2y_2^2\right)\,dy, \\ \lambda &:= a\int_{\mathbf{R}^2} \rho(y)\left(4\theta|y|^4 + 2y_1^2y_2^2\right)\,dy, \\ \gamma &:= 2b\int_{\mathbf{R}^2} \rho(y)\,(|y_1| \vee |y_2|)\,dy. \end{aligned}$$

Moreover, F^c_ε converges to F^c pointwise on $L^\infty(\Omega;\mathbf{R}^2)$.

The proof of the theorem above will be consequence of propositions in Sections 5.3 and 5.4.

Remark 5.1.9 *Notice that $\mu = a\int_{\mathbf{R}^2}\rho(y)\left(y_1^2 - y_2^2\right)^2 dy$, so that μ, λ and γ are all positive. Moreover, the summability assumption on ψ easily yields the finiteness of such constants.*

Remark 5.1.10 *We underline that for any positive coefficients μ, λ and γ, we can choose f, ρ and θ such that the limit functional has the form*

$$\mu\int_\Omega |\mathcal{E}u(x)|^2\,dx + \lambda\int_\Omega |\mathrm{div}\,u(x)|^2\,dx + \gamma\mathcal{H}^1(J_u).$$

Analogously to the discrete case, we may drop in (5.1.12) the divergence term and consider the sequence of functionals $G^c_\varepsilon : L^1(\Omega;\mathbf{R}^2) \to [0,+\infty]$ defined by

$$G^c_\varepsilon(u) := \int_{\mathbf{R}^2} \rho(\xi)\frac{1}{\varepsilon}\int_{\widetilde{\Omega}^\xi_\varepsilon} f\left(\frac{1}{\varepsilon}|D_{\varepsilon,\xi}u(x)|^2\right)\,dx\,d\xi$$

with $\widetilde{\Omega}^\xi_\varepsilon := \{x \in \mathbf{R}^2 : [x - \varepsilon\xi, x + \varepsilon\xi] \subset \Omega\}$ and ρ as above. By applying the same slicing techniques of [75] the following result can be proved.

Theorem 5.1.11 *(G^c_ε) Γ-converges on $L^\infty(\Omega;\mathbf{R}^2)$ with respect to the $L^1(\Omega;\mathbf{R}^2)$-convergence to the functional $G^c : L^\infty(\Omega;\mathbf{R}^2) \to [0,+\infty]$ given by*

$$G^c(u) := \begin{cases} \mu'\displaystyle\int_\Omega |\mathcal{E}u(x)|^2\,dx + \lambda'\int_\Omega |\mathrm{div}\,u(x)|^2\,dx + \gamma'\mathcal{H}^1(J_u) & \\ & \textit{if } u \in SBD(\Omega) \\ +\infty & \textit{otherwise,} \end{cases}$$

where

$$\mu' := a \int_{\mathbf{R}^2} \rho(y) \left(|y|^4 - 4y_1^2 y_2^2\right) dy,$$
$$\lambda' := 2a \int_{\mathbf{R}^2} \rho(y) y_1^2 y_2^2 \, dy,$$
$$\gamma' := 2b \int_{\mathbf{R}^2} \rho(y) |y_1| \, dy.$$

Remark 5.1.12 *As in the discrete case, the* Γ*-limit* G^c *does not correspond to* F^c *for* $\theta = 0$.

Remark 5.1.13 *The restriction to* $L^\infty(\Omega; \mathbf{R}^2)$ *in Theorems 5.1.1 and 5.1.8 is technical in order to characterize the* Γ*-limit. For a function* u *in* $L^1(\Omega; \mathbf{R}^2) \setminus L^\infty(\Omega; \mathbf{R}^2)$*, by following the procedure of the proof of Proposition 5.3.1 below, one can deduce from the finiteness of the* Γ*-limits that the one-dimensional sections of* u *belong to* $SBV(\Omega_y^\xi)$*. Anyway, since condition (2.8.1) of Theorem 2.8.5 is not satisfied in general, one cannot conclude that* $u \in SBD(\Omega)$*. On the other hand this condition is satisfied if* $u \in BD(\Omega)$*, so that Theorems 5.1.1, 5.1.6, 5.1.8, 5.1.11 still hold if we replace* $L^\infty(\Omega; \mathbf{R}^2)$ *by* $BD(\Omega)$ *and* J_u *by* $J_u^\star$.

5.1.3 Discrete functions and their continuous counterparts

In the previous sections, in order to study the Γ-convergence of discrete energies, we have identified a function u defined on a lattice with a suitable "piecewise-constant" interpolation, i.e., a function which takes on each cell of the lattice the value of u in one node of the cell itself. Then, fixed a discretization step length, we treated the convergence (in measure or L^1 strong) of discrete functions through this association.

This choice is not arbitrarily done. Indeed, the convergence of piecewise-constant interpolations ensures the convergence of any other "piecewise-affine" ones, the values of which on each cell are obtained as a convex combination of the values of the discrete function in the nodes of the cell itself. Actually, the converse result also holds true, as the following proposition shows.

Proposition 5.1.14 *Let* ε *be a positive parameter tending to* 0 *and let* $\mathcal{T}_\varepsilon = (T_\varepsilon^i)_{i\in\mathbf{N}}$ *be a family of* n*-simplices in* $\mathbf{R}^n$ *such that* $\operatorname{int}(T_\varepsilon^i) \cap \operatorname{int}(T_\varepsilon^j) = \emptyset$ *if* $i \neq j$, $\bigcup_i T_\varepsilon^i = \mathbf{R}^n$ *and assume also that* $\sup_i \operatorname{diam} T_\varepsilon^i \to 0$ *as* $\varepsilon \to 0$*. Let* $u_\varepsilon \in L^1(\mathbf{R}^n)$ *be a family of functions which are affine on the interior of each simplex* T_ε^i*. Consider the two piecewise constant functions* $\underline{u}_\varepsilon, \overline{u}_\varepsilon \in L^1(\mathbf{R}^n)$*, defined on every simplex* T_ε^i *by*

$$\underline{u}_\varepsilon := \text{ess-inf}_{\,T_\varepsilon^i} u_\varepsilon, \qquad \overline{u}_\varepsilon := \text{ess-sup}_{\,T_\varepsilon^i} u_\varepsilon.$$

Then, $u_\varepsilon \to u$ *in* $L^1(\mathbf{R}^n)$ *implies* $\underline{u}_\varepsilon, \overline{u}_\varepsilon \to u$ *in* $L^1(\mathbf{R}^n)$*. The same holds if* $L^1(\mathbf{R}^n)$ *convergence is replaced by* $L^1_{\text{loc}}(\mathbf{R}^n)$ *convergence or local convergence in measure.*

Proof. We prove the result for the $L^1(\mathbf{R}^n)$ convergence. With fixed ε and $i \in \mathbf{N}$ let $u_{\varepsilon,i} : T_\varepsilon^i \to \mathbf{R}$ be the unique continuous extension of $u_{\varepsilon|\operatorname{int}(T_\varepsilon^i)}$ to the closed simplex T_ε^i and let $y_{\varepsilon,i}^-, y_{\varepsilon,i}^+$ be two vertices of T_ε^i such that

$$u_{\varepsilon,i}(y_{\varepsilon,i}^-) = \min_{T_\varepsilon^i} u_{\varepsilon,i} \qquad u_{\varepsilon,i}(y_{\varepsilon,i}^+) = \max_{T_\varepsilon^i} u_{\varepsilon,i}.$$

If $u_{\varepsilon,i}$ is constant on T^i_ε, we suppose in addition that $y^-_{\varepsilon,i} \neq y^+_{\varepsilon,i}$ and define $\tau^i_\varepsilon := y^+_{\varepsilon,i} - y^-_{\varepsilon,i}$. Let A^i_ε the n-simplex homothetic to T^i_ε of ratio $\frac{1}{3}$ and with homothety center in $y^-_{\varepsilon,i}$ and let $B^i_\varepsilon := A^i_\varepsilon + \frac{1}{3}\tau^i_\varepsilon$. It is easy to see that $B^i_\varepsilon \subset T^i_\varepsilon$.

We will proceed as follows: first we will construct a function v_ε on $B_\varepsilon := \cup_{i\in\mathbf{N}} B^i_\varepsilon$ which is affine on the interior of each set B^i_ε and whose distance from u in the $L^1(B_\varepsilon)$-norm tends to 0. Afterwards we will estimate two particular convex combinations of $\underline{u}_\varepsilon$ and $\overline{u}_\varepsilon$ that will allow us to estimate the oscillation $\overline{u}_\varepsilon - \underline{u}_\varepsilon$. Let v_ε be defined in $x \in \mathrm{int}\,(B^i_\varepsilon)$ as $u_\varepsilon\left(x - \frac{1}{3}\tau^i_\varepsilon\right)$. In order to prove that

$$\lim_{\varepsilon\to 0} \int_{B_\varepsilon} |v_\varepsilon - u|\, dx = 0, \tag{5.1.13}$$

we observe that

$$\lim_{\varepsilon\to 0} \sum_i \int_{A^i_\varepsilon} \left| u(x) - u\left(x + \frac{1}{3}\tau^i_\varepsilon\right)\right| \, dx = 0.$$

This would be trivial if u were continuous with compact support, and can be proved by a standard approximation argument for a general $u \in L^1(\mathbf{R}^n)$. Then we have

$$\begin{aligned}
&\sum_i \int_{B^i_\varepsilon} |v_\varepsilon(x) - u(x)|\, dx = \sum_i \int_{A^i_\varepsilon} \left| u_\varepsilon(x) - u\left(x + \frac{1}{3}\tau^i_\varepsilon\right)\right| \, dx \\
&\leq \sum_i \int_{A^i_\varepsilon} |u_\varepsilon(x) - u(x)|\, dx + \sum_i \int_{A^i_\varepsilon} \left| u(x) - u\left(x + \frac{1}{3}\tau^i_\varepsilon\right)\right| \, dx \to 0,
\end{aligned}$$

which proves (5.1.13).

We note that $\frac{2}{3}y^-_{\varepsilon,i} + \frac{1}{3}y^+_{\varepsilon,i}$ is a maximum point of $u_{\varepsilon,i}$ on A^i_ε and a minimum point of $u_{\varepsilon,i}$ on B^i_ε. This gives

$$v_\varepsilon \leq \frac{2}{3}u_\varepsilon(y^-_{\varepsilon,i}) + \frac{1}{3}u_\varepsilon(y^+_{\varepsilon,i}) \leq u_\varepsilon \qquad \text{on } B^i_\varepsilon.$$

Hence

$$\left|\frac{2}{3}u_\varepsilon(y^-_{\varepsilon,i}) + \frac{1}{3}u_\varepsilon(y^+_{\varepsilon,i}) - u\right| \leq \max\{|u_\varepsilon - u|, |v_\varepsilon - u|\} \qquad \text{on } B^i_\varepsilon,$$

and

$$\lim_{\varepsilon\to 0} \sum_i \int_{B^i_\varepsilon} \left|\frac{2}{3}\underline{u}_\varepsilon + \frac{1}{3}\overline{u}_\varepsilon - u\right| \, dx = 0.$$

By a similar argument, we may prove also that

$$\lim_{\varepsilon\to 0} \sum_i \int_{B^i_\varepsilon} \left|\frac{1}{3}\underline{u}_\varepsilon + \frac{2}{3}\overline{u}_\varepsilon - u\right| \, dx = 0.$$

Since $\overline{u}_\varepsilon$ and $\underline{u}_\varepsilon$ are constant on each simplex T^i_ε, and $|B^i_\varepsilon| = 3^{-n}|T^i_\varepsilon|$, we conclude that

$$\lim_{\varepsilon\to 0} \int_{\mathbf{R}^n} |\overline{u}_\varepsilon - \underline{u}_\varepsilon|\, dx = 3^n \sum_i \lim_\varepsilon \int_{B^i_\varepsilon} |\overline{u}_\varepsilon - \underline{u}_\varepsilon|\, dx = 0$$

and, finally, by using the following inequalities,

$$|\underline{u}_\varepsilon - u| \leq |\underline{u}_\varepsilon - u_\varepsilon| + |u_\varepsilon - u| \leq |\underline{u}_\varepsilon - \overline{u}_\varepsilon| + |u_\varepsilon - u|,$$

we get that $\underline{u}_\varepsilon \to u$ in $L^1(\mathbf{R}^n)$ and analogously for $\overline{u}_\varepsilon$.

If $u_\varepsilon \to u$ in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ or locally in measure, it suffices to repeat the constructions and arguments above, localizing each integral. For the local convergence in measure, one has also to replace the L^1 norm with the distance in (2.1.1) inducing the convergence in measure on a bounded set. □

Remark 5.1.15 *The functions $\overline{u}_\varepsilon$, $\underline{u}_\varepsilon$ of Proposition 5.1.14 don't coincide with the piecewise-constant ones we considered in Subsections 5.1.1,5.1.2. Nevertheless, from Proposition 5.1.14, one can easily deduce that the convergence ($L^1(\mathbf{R}^n)$, $L^1_{loc}(\mathbf{R}^n)$ or locally in measure) of piecewise-affine functions implies the convergence of the piecewise-constant ones in $\mathcal{A}_\varepsilon(\Omega)$.*

5.2 Preliminary lemmata

In this subsection we state and prove some preliminary results, that will be used in the sequel.

Let $\mathcal{B} := \{\xi_1, \ldots, \xi_n\}$ an orthogonal basis of $\mathbf{R}^n$. Then for any measurable function $u : \mathbf{R}^n \to \mathbf{R}^n$ and $y \in \mathbf{R}^n \setminus \{0\}$ define

$$T_y^{\varepsilon,\mathcal{B}}u(x) := u\left(\varepsilon y + \varepsilon\left[\frac{x}{\varepsilon}\right]_{\mathcal{B}}\right) \tag{5.2.1}$$

where $[z]_{\mathcal{B}} := \sum\limits_{i=1}^{n}\left[\frac{\langle z,\xi_i\rangle}{|\xi_i|^2}\right]\xi_i$.

Notice that $T_y^{\varepsilon,\mathcal{B}}u$ is constant on each cell $\alpha + \varepsilon Q_{\mathcal{B}}$, $\alpha \in \varepsilon \bigoplus\limits_{i=1}^{n} \xi_i\mathbf{Z}$, where $Q_{\mathcal{B}} := \{x \in \mathbf{R}^n : 0 < \langle x, \xi_i\rangle \leq |\xi_i|^2\}$. The following result generalizes Lemma 3.36 in [37].

Lemma 5.2.1 *Let $u_\varepsilon \to u$ in $L^1_{loc}(\mathbf{R}^n;\mathbf{R}^n)$, then for any compact set K of $\mathbf{R}^n$ it holds*

$$\lim_{\varepsilon\to 0}\int_{Q_{\mathcal{B}}} \|T_y^{\varepsilon,\mathcal{B}}u_\varepsilon - u\|_{L^1(K,\mathbf{R}^n)}\,dy = 0. \tag{5.2.2}$$

Proof. For the sake of simplicity we assume $\mathcal{B} = \{\mathrm{e}_1, \ldots, \mathrm{e}_n\}$. With fixed a compact set K, call I_ε the double integral in (5.2.2). By Fubini's Theorem and the change of variable $\varepsilon y + \varepsilon\left[\frac{x}{\varepsilon}\right]_{\mathcal{B}} \to y$ we get

$$\begin{aligned}
I_\varepsilon &= \int_K \int_{(0,1)^n} \left|u_\varepsilon\left(\varepsilon y + \varepsilon\left[\frac{x}{\varepsilon}\right]_{\mathcal{B}}\right) - u(x)\right|\,dy\,dx \\
&\leq \int_K \frac{1}{\varepsilon^n}\int_{x+\varepsilon(-1,1)^n} |u_\varepsilon(y) - u(x)|\,dy\,dx \\
&\leq \int_K \frac{1}{\varepsilon^n}\int_{x+\varepsilon(-1,1)^n} \left(|u_\varepsilon(y) - u_\varepsilon(x)| + |u_\varepsilon(x) - u(x)|\right)\,dy\,dx.
\end{aligned}$$

The further change of variable $y \to x + \varepsilon y$ and Fubini's Theorem yield

$$I_\varepsilon \leq \int_{(-1,1)^n}\int_K |u_\varepsilon(x+\varepsilon y) - u_\varepsilon(x)|\,dx\,dy + 2^n\int_K |u_\varepsilon(x) - u(x)|\,dx,$$

thus the conclusion follows by the uniform continuity of the translation operator for strongly converging families in $L^1_{loc}(\mathbf{R}^n;\mathbf{R}^n)$. $\square$

Remark 5.2.2 *Let $C_\varepsilon \subset Q_{\mathcal{B}}$ a family of sets such that*

$$\liminf_{\varepsilon\to 0^+} \mathcal{L}^n(C_\varepsilon) \geq c > 0. \tag{5.2.3}$$

Then under the hypothesis of the previous lemma, for any compact set K of $\mathbf{R}^n$ we can choose $y_\varepsilon \in C_\varepsilon$ such that $T^{\varepsilon,\mathcal{B}}_{y_\varepsilon} u_\varepsilon \to u$ in $L^1(K;\mathbf{R}^n)$. Indeed, by the Mean Value Theorem, there exist $y_\varepsilon \in C_\varepsilon$ such that

$$\begin{aligned}\|T^{\varepsilon,\mathcal{B}}_{y_\varepsilon} u_\varepsilon - u\|_{L^1(K,\mathbf{R}^n)}\mathcal{L}^n(C_\varepsilon) \leq\\ \int_{C_\varepsilon} \|T^{\varepsilon,\mathcal{B}}_{y} u_\varepsilon - u\|_{L^1(K,\mathbf{R}^n)}\, dy \leq \int_{Q_{\mathcal{B}}} \|T^{\varepsilon,\mathcal{B}}_{y} u_\varepsilon - u\|_{L^1(K,\mathbf{R}^n)}\, dy.\end{aligned}$$

Then the conclusion easily follows from (5.2.2) and (5.2.3). This property will be used in the proof of Propositions 5.3.4 and 5.4.1.

In the sequel for $n = 2$, $\xi \in \mathbf{R}^2 \setminus \{0\}$ and $\mathcal{B} = \{\xi, \xi^\perp\}$, we will denote the operators $T^{\varepsilon,\mathcal{B}}_y$ and $[\,\cdot\,]_{\mathcal{B}}$ by $T^{\varepsilon,\xi}_y$ and $[\,\cdot\,]_\xi$, respectively.

Lemma 5.2.3 *Let J be a countably $\mathcal{H}^{n-1}$ rectifiable set and define*

$$J^\xi_\varepsilon := \{x \in \mathbf{R}^n : x = y + t\xi \text{ with } t \in (-\varepsilon,\varepsilon) \text{ and } y \in J\} \tag{5.2.4}$$

for $\xi \in \mathbf{R}^n$ and

$$J^{\xi_1,..,\xi_r}_\varepsilon := \bigcup_{i=1}^{r} J^{\xi_i}_\varepsilon \tag{5.2.5}$$

for $\xi_1,\dots,\xi_r \in \mathbf{R}^n$, $r \in \mathbf{N}$. Then, if $\mathcal{H}^{n-1}(J) < +\infty$

$$\limsup_{\varepsilon\to 0} \frac{\mathcal{L}^n\left(J^{\xi_1,..,\xi_r}_\varepsilon\right)}{\varepsilon} \leq 2\int_J \sup_i |\langle \nu_J, \xi_i\rangle|\, d\mathcal{H}^{n-1}, \tag{5.2.6}$$

where $\nu_J(x)$ is the unitary normal vector to J at x, according to Lemma 2.4.2.

Proof. First note that by Fubini's Theorem and Lemma 2.4.2

$$\mathcal{L}^n\left(J^\xi_\varepsilon\right) \leq 2\varepsilon \int_{J_\xi} \mathcal{H}^0\left(J^\xi_y\right)\, d\mathcal{H}^{n-1}(y) = 2\varepsilon\int_J |\langle \nu_J, \xi\rangle|\, d\mathcal{H}^{n-1}, \tag{5.2.7}$$

hence

$$\mathcal{L}^n\left(J^{\xi_1,..,\xi_r}_\varepsilon\right) \leq 2\varepsilon \int_J \sum_{i=1}^{r} |\langle \nu_J, \xi_i\rangle|\, d\mathcal{H}^{n-1} \leq 2r\varepsilon \sup_i |\xi_i| \mathcal{H}^{n-1}(J). \tag{5.2.8}$$

By the very definition of rectifiability there exist countably many compact subsets K_i of C^1 graphs such that

$$\mathcal{H}^{n-1}\left(J \setminus \bigcup_{i\geq 1} K_i\right) = 0,$$

and $\mathcal{H}^{n-1}(K_i \cap K_j) = 0$ for $i \neq j$. Thus, by (5.2.8) for any $M \in \mathbf{N}$ we have

$$\frac{\mathcal{L}^n\left(J_\varepsilon^{\xi_1,..,\xi_r}\right)}{\varepsilon} \leq \sum_{1\leq i\leq M} \frac{\mathcal{L}^n\left((K_i)_\varepsilon^{\xi_1,..,\xi_r}\right)}{\varepsilon} + 2r \sup_i |\xi_i| \mathcal{H}^{n-1}\left(J \setminus \bigcup_{1\leq i\leq M} K_i\right),$$

hence, first letting $\varepsilon \to 0$ and then $M \to +\infty$ it follows

$$\limsup_{\varepsilon\to 0} \frac{\mathcal{L}^n\left(J_\varepsilon^{\xi_1,..,\xi_r}\right)}{\varepsilon} \leq \sum_{i\geq 1} \limsup_{\varepsilon\to 0} \frac{\mathcal{L}^n\left((K_i)_\varepsilon^{\xi_1,..,\xi_r}\right)}{\varepsilon}.$$

Thus, it suffices to prove (5.2.6) for J compact subset of a C^1 graph. Up to an outer approximation with open sets we may assume J open. Furthermore, splitting J into its connected components, we can reduce ourselves to prove the inequality for J connected. For such a J (5.2.6) follows by an easy computation. $\square$

5.3 The discrete case

In this section we will prove Theorem 5.1.1. In the sequel we need to "localize" the functionals $\mathcal{F}_\varepsilon^{d,\xi}$ as

$$\mathcal{F}_\varepsilon^{d,\xi}(u, A) := \sum_{\alpha\in R_\varepsilon^\xi(A)} \varepsilon f\left(\frac{1}{\varepsilon}\left(|D_{\varepsilon,\xi}u(\alpha)|^2 + \theta|\mathrm{Div}_{\varepsilon,\xi}u(\alpha)|^2\right)\right),$$

for any $A \in \mathcal{A}(\Omega)$ and $u \in \mathcal{A}_\varepsilon(\Omega)$, where

$$R_\varepsilon^\xi(A) := \left\{\alpha \in \varepsilon\mathbf{Z}^2 : \ [\alpha - \varepsilon\xi, \alpha + \varepsilon\xi] \cup [\alpha - \varepsilon\xi^\perp, \alpha + \varepsilon\xi^\perp] \subset A\right\}.$$

Proposition 5.3.1 *For any $u \in L^\infty(\Omega; \mathbf{R}^2)$,*

$$\Gamma(meas)\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon^d(u) \geq F^d(u).$$

Proof. Step 1: Let us first prove the inequality in the case $f(t) = (at) \wedge b$. Let $\varepsilon_j \to 0$, $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega)$ and $u \in L^\infty(\Omega; \mathbf{R}^2)$ be such that $u_j \to u$ in measure. We can suppose that $\liminf_j F_{\varepsilon_j}^d(u_j) = \lim_j F_{\varepsilon_j}^d(u_j) < +\infty$. In particular, for any $\xi \in \mathbf{Z}^2$ such that $\rho(\xi) \neq 0$, $\liminf_j \mathcal{F}_{\varepsilon_j}^{d,\xi}(u_j) < +\infty$. Using this estimate for $\xi \in \{\mathrm{e}_1, \mathrm{e}_1 + \mathrm{e}_2\}$, we will deduce that $u \in SBD(\Omega)$ and we will obtain the required inequality by proving that, for any $\xi \in \mathbf{Z}^2$ such that $\rho(\xi) \neq 0$,

$$\liminf_j \mathcal{F}^{d,\xi}_{\varepsilon_j}(u_j) \geq \mathcal{F}^{\xi}(u). \tag{5.3.1}$$

To this aim, as in Theorem 4.1 of [44], we will proceed by splitting the lattice $\mathbf{Z}^2$ into similar sub-lattices and reducing ourselves to study the limit of functionals defined on one of these sub-lattices. Indeed, fixed $\xi \in \mathbf{Z}^2$ such that $\rho(\xi) \neq 0$, we split $\mathbf{Z}^2$ into an union of disjoint copies of $|\xi|\mathbf{Z}^2$ as

$$\mathbf{Z}^2 = \bigcup_{i=1}^{|\xi|^2} (z_i + \mathbf{Z}\xi \oplus \mathbf{Z}\xi^{\perp}),$$

where

$$\left\{z_i : i = 1, \dots, |\xi|^2\right\} := \left\{\alpha \in \mathbf{Z}^2 \, : \, 0 \leq \langle \alpha, \xi \rangle < |\xi|, \, 0 \leq \langle \alpha, \xi^{\perp} \rangle < |\xi|\right\}.$$

Then, for any $A \in \mathcal{A}(\Omega)$, we write

$$\mathcal{F}^{d,\xi}_{\varepsilon_j}(u_j, A) = \sum_{i=1}^{|\xi|^2} \mathcal{F}^{d,\xi,i}_{\varepsilon_j}(u_j, A)$$

where

$$\mathcal{F}^{d,\xi,i}_{\varepsilon_j}(u_j, A) := \sum_{R^{\xi,i}_{\varepsilon_j}(A)} \varepsilon_j f\left(\frac{1}{\varepsilon_j}\left(|D_{\varepsilon_j,\xi} u_j(\alpha)|^2 + \theta |\mathrm{Div}_{\varepsilon_j,\xi} u_j(\alpha)|^2\right)\right)$$

with $R^{\xi,i}_{\varepsilon_j}(A) := R^{\xi}_{\varepsilon_j}(A) \cap \varepsilon_j(z_i + \mathbf{Z}\xi \oplus \mathbf{Z}\xi^{\perp})$. We split as well the lattice $\mathbf{Z}\xi \oplus \mathbf{Z}\xi^{\perp}$ into an union of disjoint sub-lattices as

$$\mathbf{Z}\xi \oplus \mathbf{Z}\xi^{\perp} = Z^{\xi} \cup (Z^{\xi} + \xi) \cup (Z^{\xi} + \xi^{\perp}) \cup (Z^{\xi} + (\xi + \xi^{\perp}))$$

where $Z^{\xi} := 2\mathbf{Z}\xi \oplus 2\mathbf{Z}\xi^{\perp}$. We confine now our attention to the sequence

$$\mathcal{F}_j(A) := \sum_{\alpha \in Z_j(A)} \varepsilon_j f\left(\frac{1}{\varepsilon_j}\left(|D_{\varepsilon_j,\xi} u_j(\alpha)|^2 + \theta |\mathrm{Div}_{\varepsilon_j,\xi} u_j(\alpha)|^2\right)\right)$$

where $Z_j(A) := R^{\xi}_{\varepsilon_j}(A) \cap \varepsilon_j Z^{\xi}$. Set

$$I_j := \left\{\alpha \in R^{\xi}_{\varepsilon_j} \, : |D_{\varepsilon_j,\xi} u_j(\alpha)|^2 + \theta |\mathrm{Div}_{\varepsilon_j,\xi} u_j(\alpha)|^2 > \frac{b}{a}\varepsilon_j\right\}$$

and let (v_j) be the sequence in $SBV(\Omega; \mathbf{R}^2)$, whose components are piecewise affine, uniquely determined by

$$\langle v_j(x), \xi \rangle := \begin{cases} \langle u_j(\alpha - \varepsilon_j \xi), \xi \rangle & \begin{array}{l} x \in (\alpha + \varepsilon_j Q_{\xi}) \cap \Omega \\ \alpha \in \varepsilon_j Z^{\xi} \cap I_j \end{array} \\ \\ \langle u_j(\alpha), \xi \rangle + \frac{1}{\varepsilon_j |\xi|^2} D^{\xi}_{\varepsilon_j} u_j(\alpha) \langle x - \alpha, \xi \rangle & \begin{array}{l} x \in (\alpha + \varepsilon_j Q_{\xi,+}) \cap \Omega \\ \alpha \in \varepsilon_j Z^{\xi} \setminus I_j \end{array} \\ \\ \langle u_j(\alpha), \xi \rangle + \frac{1}{\varepsilon_j |\xi|^2} D^{-\xi}_{\varepsilon_j} u_j(\alpha) \langle x - \alpha, \xi \rangle & \begin{array}{l} x \in (\alpha + \varepsilon_j Q_{\xi,-}) \cap \Omega \\ \alpha \in \varepsilon_j Z^{\xi} \setminus I_j \end{array} \end{cases}$$

$$\langle v_j(x), \xi^\perp \rangle := \begin{cases} \langle u_j(\alpha - \varepsilon_j \xi^\perp), \xi^\perp \rangle & x \in (\alpha + \varepsilon_j Q_{\xi^\perp}) \cap \Omega \\ & \alpha \in \varepsilon_j Z^\xi \cap I_j \\ \langle u_j(\alpha), \xi^\perp \rangle + \frac{1}{\varepsilon_j |\xi|^2} D^{\xi^\perp}_{\varepsilon_j} u_j(\alpha) \langle x - \alpha, \xi^\perp \rangle & \\ & x \in (\alpha + \varepsilon_j Q_{\xi^\perp,+}) \cap \Omega \\ & \alpha \in \varepsilon_j Z^\xi \setminus I_j \\ \langle u_j(\alpha), \xi^\perp \rangle + \frac{1}{\varepsilon_j |\xi|^2} D^{-\xi^\perp}_{\varepsilon_j} u_j(\alpha) \langle x - \alpha, \xi^\perp \rangle & \\ & x \in (\alpha + \varepsilon_j Q_{\xi^\perp,-}) \cap \Omega \\ & \alpha \in \varepsilon_j Z^\xi \setminus I_j, \end{cases}$$

where

$$\begin{aligned} &Q_\xi := \left\{ x \in \mathbf{R}^2 \,:\, |\langle x, \xi \rangle| \le |\xi|^2, \, |\langle x, \xi^\perp \rangle| \le |\xi|^2 \right\} \\ &Q_{\xi,\pm} := \{ x \in Q_\xi \,:\, \pm \langle x, \xi \rangle \ge 0 \}. \end{aligned}$$

In order to clarify this construction, we note that, in the case $\xi = \mathrm{e}_1$, $v_j = (v_j^1, v_j^2)$ is the sequence whose component v_j^i is piecewise affine along the direction e_i and piecewise constant along the orthogonal direction, for $i = 1, 2$.
It is easy to check that v_j still converges to u in measure. Let us fix $\eta > 0$ and consider $A_\eta := \{ x \in A \,:\, \mathrm{dist}\,(x, \mathbf{R}^2 \setminus A) > \eta \}$. Note that, by construction, for j large we have

$$\begin{aligned} &\sum_{\alpha \in Z_j(A) \setminus I_j} a \left(|D_{\varepsilon_j, \xi} u_j(\alpha)|^2 + \theta |\mathrm{Div}_{\varepsilon_j, \xi} u_j(\alpha)|^2 \right) \\ &\qquad \ge \frac{a}{2|\xi|^2} \int_{A_\eta} |\langle \mathcal{E} v_j(x) \xi, \xi \rangle|^2 \, dx + a\theta |\xi|^2 \int_{A_\eta} |\mathrm{div}\, v_j(x)|^2 \, dx \end{aligned}$$

and

$$\begin{aligned} &b\varepsilon_j \# \{ Z_j(A) \cap I_j \} \\ &\qquad \ge \frac{b}{2|\xi|^2} \max \left\{ \int_{J^\xi_{v_j} \cap A_\eta} |\langle \nu_{v_j}(y), \xi \rangle| \, d\mathcal{H}^1(y), \int_{J^{\xi^\perp}_{v_j} \cap A_\eta} |\langle \nu_{v_j}(y), \xi^\perp \rangle| \, d\mathcal{H}^1(y) \right\} \end{aligned}$$

Then, for j large and for any fixed $\delta \in [0, 1]$,

$$\begin{aligned} \mathcal{F}_j(A) \ge &\sum_{\alpha \in Z_j(A) \setminus I_j} a \left(|D_{\varepsilon_j, \xi} u_j(\alpha)|^2 + \theta |\mathrm{Div}_{\varepsilon_j, \xi} u_j(\alpha)|^2 \right) \\ &+ b\varepsilon_j \# \{ Z_j(A) \cap I_j \} \\ &\ge \frac{a}{2|\xi|^2} \int_{A_\eta} |\langle \mathcal{E} v_j(x) \xi, \xi \rangle|^2 \, dx + a\theta |\xi|^2 \int_{A_\eta} |\mathrm{div}\, v_j(x)|^2 \, dx \\ &+ \frac{b}{2|\xi|^2} \delta \int_{J^\xi_{v_j} \cap A_\eta} |\langle \nu_{v_j}(y), \xi \rangle| \, d\mathcal{H}^1(y) \\ &+ \frac{b}{2|\xi|^2} (1 - \delta) \int_{J^{\xi^\perp}_{v_j} \cap A_\eta} |\langle \nu_{v_j}(y), \xi^\perp \rangle| \, d\mathcal{H}^1(y). \end{aligned} \tag{5.3.2}$$

In particular by applying a slicing argument and taking into account the notation used in Theorem 2.8.5, by Fatou's lemma, we get

$$\begin{aligned} +\infty &> \liminf_j \mathcal{F}_j(A) \\ &\geq \frac{1}{2|\xi|^2} \int_{\Pi^\xi} \liminf_j \left(a \int_{(A_\eta)_y^\xi} |\dot{v}_j^{\xi,y}|^2 \, dt + b\mathcal{H}^0 \left(J_{v_j^{\xi,y}} \cap A_\eta \right) \right) d\mathcal{H}^1(y). \end{aligned} \tag{5.3.3}$$

Note that, even if $\rho(\xi^\perp) = 0$, taking into account also the divergence term and the second surface term in (5.3.2), we can obtain an analogue of the inequality (5.3.3) for $\xi^\perp$. By the closure and lower semicontinuity Theorem 2.7.10 we deduce that $u^{\zeta,y} \in SBV((A_\eta)_y^\zeta)$, and since $u \in L^\infty(\Omega; \mathbf{R}^2)$ we get

$$c \geq \int_{\Pi^\xi} |Du^{\zeta,y}|((A_\eta)_y^\zeta) \, d\mathcal{H}^1(y) \tag{5.3.4}$$

for $\zeta = \xi, \xi^\perp$. Recall that by assumption $\rho(\mathrm{e}_1), \rho(\mathrm{e}_1 + \mathrm{e}_2) \neq 0$, thus (5.3.4) holds in particular for $\zeta = \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_1 + \mathrm{e}_2$. Then by Theorem 2.8.5, we get that $u \in SBD(A_\eta)$ for any $\eta > 0$. Moreover, since the estimate in (5.3.4) is uniform with respect to η, we conclude that $u \in SBD(A)$.

Going back to (5.3.2), by applying Theorem 2.8.7 and then letting $\eta \to 0$, we get

$$\begin{aligned} \liminf_j \mathcal{F}_j(A) \geq \frac{a}{2|\xi|^2} \int_A |\langle \mathcal{E}u(x)\xi, \xi \rangle|^2 \, dx + a\theta|\xi|^2 \int_A |\operatorname{div} u(x)|^2 \, dx \\ + \frac{b}{2|\xi|^2} \left(\delta \int_{J_u^\xi \cap A} |\langle \nu_u, \xi \rangle| \, d\mathcal{H}^1 + (1-\delta) \int_{J_u^{\xi^\perp} \cap A} |\langle \nu_u, \xi^\perp \rangle| \, d\mathcal{H}^1 \right), \end{aligned}$$

for any $\delta \in [0,1]$.

Note that, using the inequality above with $A = \Omega, \xi = \mathrm{e}_1, \mathrm{e}_1 + \mathrm{e}_2$, it can be easily checked that $\mathcal{E}u \in L^2(\Omega; \mathbf{R}^{2\times 2})$ and $\mathcal{H}^1(J_u) < +\infty$. Then, by Lemma 2.2.4 applied with

$$\lambda(A) = \liminf_j \mathcal{F}_j(A),$$

$$\mu = \frac{a}{2|\xi|^2} \mathcal{L}^2 \llcorner \Omega + \frac{b}{2|\xi|^2} \mathcal{H}^1 \llcorner J_u,$$

$$\psi_h(x) = \begin{cases} (|\langle \mathcal{E}u(x)\xi, \xi \rangle|^2 + \theta|\xi|^2 |\operatorname{div} u(x)|^2) & \text{on } \Omega \setminus J_u \\ \delta_h |\langle \nu_u, \xi \rangle| & \text{on } J_u^\xi \setminus J_u^{\xi^\perp} \\ (1-\delta_h)|\langle \nu_u, \xi^\perp \rangle| & \text{on } J_u^{\xi^\perp} \setminus J_u^\xi \\ \delta_h |\langle \nu_u, \xi \rangle| + (1-\delta_h)|\langle \nu_u, \xi^\perp \rangle| & \text{on } J_u^\xi \cap J_u^{\xi^\perp}, \end{cases}$$

with $\delta_h \in \mathbf{Q} \cap [0,1]$, we get

$$\begin{aligned}\liminf_j \mathcal{F}_j(\Omega) \geq & \frac{a}{2|\xi|^2} \int_\Omega |\langle \mathcal{E}u(x)\xi, \xi\rangle|^2 \, dx + a\theta|\xi|^2 \int_\Omega |\operatorname{div} u(x)|^2 \, dx \\ & + \frac{b}{2|\xi|^2} \Big(\int_{J_u^\xi \setminus J_u^{\xi^\perp}} |\langle \nu_u, \xi\rangle| \, d\mathcal{H}^1 + \int_{J_u^{\xi^\perp} \setminus J_u^\xi} |\langle \nu_u, \xi^\perp\rangle| \, d\mathcal{H}^1 \\ & + \int_{J_u^\xi \cap J_u^{\xi^\perp}} |\langle \nu_u, \xi\rangle| \vee |\langle \nu_u, \xi^\perp\rangle| \, d\mathcal{H}^1 \Big).\end{aligned}$$

Finally, since the argument above is not affected by the choice of the sub-lattices in which $\mathbf{Z}^2$ has been split with respect to ξ, we obtain (5.3.1). The thesis follows by summing over $\xi \in \mathbf{Z}^2$.

Step 2: If f is any increasing positive function satisfying (5.1.4), we can find two sequences of positive numbers (a_i) and (b_i) such that $\sup_i a_i = a$, $\sup_i b_i = b$ and $f(t) \geq (a_i t) \wedge b_i$ for any $t \geq 0$. By *Step 1* we have that $\Gamma(meas)\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon^d(u)$ is finite only if $F^d(u)$ is finite and

$$\begin{aligned}\Gamma(meas)\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon^d(u) \geq & \sum_{\xi\in\mathbf{Z}^2} \rho(\xi) \left(2a_i \int_\Omega |\langle \mathcal{E}u(x)\xi, \xi\rangle|^2 \, dx \right. \\ & \left. + 4a_i\theta|\xi|^4 \int_\Omega |\operatorname{div} u(x)|^2 \, dx + 2b_i \int_{J_u} \Phi^\xi(u^+ - u^-, \nu_u) \, d\mathcal{H}^1 \right).\end{aligned}$$

Then the thesis follows as above from Lemma 2.2.4. □

The following proposition will be crucial for the proof of the Γ-limsup inequality in both the discrete and the continuous case.

Proposition 5.3.2 *Let $u \in SBD^2(\Omega) \cap L^\infty(\Omega, \mathbf{R}^2)$, then*

$$\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{c,\xi}(u) \leq \mathcal{F}^\xi(u).$$

Proof. Using the notation of Lemma 5.2.3, set

$$J_u^\varepsilon := \left(J_u^\xi \setminus J_u^{\xi^\perp}\right)_\varepsilon^\xi \cup \left(J_u^{\xi^\perp} \setminus J_u^\xi\right)_\varepsilon^{\xi^\perp} \cup \left(J_u^\xi \cap J_u^{\xi^\perp}\right)_\varepsilon^{\xi,\xi^\perp}.$$

Since $f(t) \leq b$, by Lemma 5.2.3 there follows

$$\begin{aligned}\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{c,\xi}(u) \leq & \limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{c,\xi}\left(u, \Omega_\varepsilon^\xi \setminus J_u^\varepsilon\right) + b \limsup_{\varepsilon\to 0} \frac{\mathcal{L}^2\left(J_u^\varepsilon\right)}{\varepsilon} \\ \leq & \limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{c,\xi}\left(u, \Omega_\varepsilon^\xi \setminus J_u^\varepsilon\right) \\ & + 2b \left(\int_{J_u^\xi \setminus J_u^{\xi^\perp}} |\langle \nu_u, \xi\rangle| \, d\mathcal{H}^1 + \int_{J_u^{\xi^\perp} \setminus J_u^\xi} |\langle \nu_u, \xi^\perp\rangle| \, d\mathcal{H}^1 \right. \\ & \left. + \int_{J_u^\xi \cap J_u^{\xi^\perp}} |\langle \nu_u, \xi\rangle| \vee |\langle \nu_u, \xi^\perp\rangle| \, d\mathcal{H}^1 \right).\end{aligned}$$

Let us prove that for a.e. $x \in \Omega^\xi_\varepsilon \setminus J^\varepsilon_u$ and for $\zeta \in \{\pm\xi, \pm\xi^\perp\}$

$$D^\zeta_\varepsilon u(x) = \langle u(x+\varepsilon\zeta) - u(x), \zeta\rangle = \int_0^\varepsilon \langle \mathcal{E}u(x+s\zeta)\zeta, \zeta\rangle\, ds. \tag{5.3.5}$$

Let, for instance, $\zeta = \xi$, then using the notation of Theorem 2.8.5 if $x \in \Omega^\xi_\varepsilon \setminus J^\varepsilon_u$ and $x = y + t\xi$, with $y \in \Pi^\xi$, we get

$$\langle u(x+\varepsilon\xi) - u(x), \xi\rangle = u^{\xi,y}(t+\varepsilon) - u^{\xi,y}(t).$$

Since $u \in SBD\,(\Omega)$, for $\mathcal{H}^1$ a.e. $y \in \Pi^\xi$ we have that $u^{\xi,y} \in SBV\left(\left(\Omega^\xi_\varepsilon\right)^\xi_y\right)$, $\dot{u}^{\xi,y}(t) = \langle \mathcal{E}u(y+t\xi)\xi, \xi\rangle$ for $\mathcal{L}^1$ a.e. $t \in \left(\Omega^\xi_\varepsilon\right)^\xi_y$ and $J_{u^{\xi,y}} = \left(J^\xi_u\right)^\xi_y$. Thus

$$\begin{aligned} u^{\xi,y}(t+\varepsilon) - u^{\xi,y}(t) &= \int_t^{t+\varepsilon} \langle \mathcal{E}u(y+s\xi)\xi, \xi\rangle\, ds \\ &+ \sum_{s \in \left(J^\xi_u\right)^\xi_y} \left(\left(u^{\xi,y}\right)^+(s) - \left(u^{\xi,y}\right)^-(s)\right) \operatorname{sgn}\langle \xi, \nu_u\rangle \end{aligned} \tag{5.3.6}$$

and, since $\left(J^\xi_u\right)^\xi_y \cap [t, t+\varepsilon] = \emptyset$, (5.3.5) follows.

Moreover, Jensen's inequality, Fubini's Theorem and (5.3.5) yield

$$\begin{aligned} &\frac{1}{\varepsilon^2}\int_{\Omega^\xi_\varepsilon \setminus J^\varepsilon_u} \left|D^\zeta_\varepsilon u(x)\right|^2 dx = \\ &\int_{\Omega^\xi_\varepsilon \setminus J^\varepsilon_u} \frac{1}{\varepsilon^2}\left|\int_0^\varepsilon \langle \mathcal{E}u(x+s\zeta)\zeta, \zeta\rangle\, ds\right|^2 dx \le \int_\Omega |\langle \mathcal{E}u(x)\zeta, \zeta\rangle|^2\, dx, \end{aligned} \tag{5.3.7}$$

for $\zeta = \pm\xi$.

Let us also prove that

$$\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^2}\int_{\Omega^\xi_\varepsilon \setminus J^\varepsilon_u} |\operatorname{div}^{\xi,\xi^\perp}_\varepsilon u|^2\, dx \le |\xi|^4 \int_\Omega |\operatorname{div} u(x)|^2\, dx. \tag{5.3.8}$$

Setting

$$g(x) := |\xi|^2 \operatorname{div} u(x)$$

and

$$g_\varepsilon(x) := \frac{1}{\varepsilon}\operatorname{div}^{\xi,\xi^\perp}_\varepsilon u(x)\mathcal{X}_{\Omega^\xi_\varepsilon \setminus J^\varepsilon_u}(x),$$

(5.3.8) follows if we prove prove that

$$\|g - g_\varepsilon\|_{L^2(\Omega)} \to 0. \tag{5.3.9}$$

Note that

$$g(x) = \langle \mathcal{E}u(x)\xi, \xi\rangle + \langle \mathcal{E}u(x)\xi^\perp, \xi^\perp\rangle, \tag{5.3.10}$$

and that by (5.3.5) on $\Omega^\xi_\varepsilon \setminus J^\varepsilon_u$ we have

$$\operatorname{div}^{\xi,\xi^\perp}_\varepsilon u(x) = \int_0^\varepsilon \langle \mathcal{E}u(x+s\xi)\xi, \xi\rangle + \langle \mathcal{E}u(x+s\xi^\perp)\xi^\perp, \xi^\perp\rangle\, ds. \tag{5.3.11}$$

Thus, by absolute continuity and Jensen's inequality we get

$$\|g - g_\varepsilon\|^2_{L^2(\Omega)} \le o(1) + 2|\xi|^4 \int_{\Omega^\xi_\varepsilon} \frac{1}{\varepsilon} \int_0^\varepsilon |\mathcal{E}u(x+s\xi) - \mathcal{E}u(x)|^2 \, ds \, dx$$
$$+2|\xi|^4 \int_{\Omega^\xi_\varepsilon} \frac{1}{\varepsilon} \int_0^\varepsilon |\mathcal{E}u(x+s\xi^\perp) - \mathcal{E}u(x)|^2 \, ds \, dx.$$

Applying Fubini's Theorem and then extending $\mathcal{E}u$ to 0 outside Ω yield

$$\|g - g_\varepsilon\|^2_{L^2(\Omega)} \le o(1) + 2|\xi|^4 \frac{1}{\varepsilon} \int_0^\varepsilon \int_\Omega |\mathcal{E}u(x+s\xi) - \mathcal{E}u(x)|^2 \, dx \, ds$$
$$+2|\xi|^4 \frac{1}{\varepsilon} \int_0^\varepsilon \int_\Omega |\mathcal{E}u(x+s\xi^\perp) - \mathcal{E}u(x)|^2 \, dx \, ds,$$

and so (5.3.9) follows by the continuity of the traslation operator in $L^2(\Omega; \mathbf{R}^{2\times 2})$.

Of course, by using the same argument, we can claim that the analogous inequalities of (5.3.8), obtained by replacing $\left(\xi, \xi^\perp\right)$ by one among the pairs $\left(\xi, -\xi^\perp\right)$, $\left(-\xi, \xi^\perp\right)$, $\left(-\xi, -\xi^\perp\right)$, hold true.

Eventually, since $f(t) \le at$, by (5.3.7) and (5.3.8) we get

$$\limsup_{\varepsilon\to 0} \mathcal{F}^{c,\xi}_\varepsilon \left(u, \Omega^\xi_\varepsilon \setminus J^\varepsilon_u\right) \le 2a \int_\Omega |\langle \mathcal{E}u(x)\xi, \xi\rangle|^2 \, dx + 4a\theta|\xi|^4 \int_\Omega |\operatorname{div} u(x)|^2 \, dx$$

and the conclusion follows. □

Remark 5.3.3 *Arguing as in the proof of Proposition* 5.3.2 *we infer that the functionals defined by*

$$\mathcal{G}^\xi_\varepsilon(u) := \frac{1}{\varepsilon} \int_{\Omega^\xi_\varepsilon} g\left(\frac{1}{\varepsilon}|D_{\varepsilon,\xi}u(x)|^2\right) \, dx,$$

where $g(t) := (at) \wedge b$, *satisfy the estimate*

$$\mathcal{G}^\xi_\varepsilon(u) \le 2a \int_\Omega |\langle \mathcal{E}u(x)\xi, \xi\rangle|^2 \, dx + 2b \int_{J^\xi_u} |\langle \nu_u, \xi\rangle| \, d\mathcal{H}^1,$$

for any $u \in SBD(\Omega)$.

Moreover, by the subadditivity of g *and since* $f(t) \le g(t)$ *by hypothesis, there holds*

$$\mathcal{F}^{c,\xi}_\varepsilon(u) \le c\left(\mathcal{G}^\xi_\varepsilon(u) + \mathcal{G}^{\xi^\perp}_\varepsilon(u)\right) \le c\mathcal{F}^\xi(u). \tag{5.3.12}$$

Now we are going to prove the Γ-limsup inequality that concludes the proof of Theorem 5.1.1. We will obtain the recovery sequence for $u \in L^\infty(\Omega; \mathbf{R}^2)$ as suitable interpolations of the function u itself.

Proposition 5.3.4 *For any* $u \in L^\infty(\Omega; \mathbf{R}^2)$,

$$\Gamma(L^1)\text{-}\limsup_{\varepsilon\to 0} F^d_\varepsilon(u) \le F^d(u).$$

Proof. It suffices to prove the inequality above for $u \in SBD^2(\Omega)$. Up to a translation argument we may assume that $0 \in \Omega$. Let $\lambda \in (0,1)$ and define $u_\lambda(x) := u(\lambda x)$ for $x \in \Omega_\lambda := \lambda^{-1}\Omega$. Notice that $\Omega \subset\subset \Omega_\lambda$ and $u_\lambda \in SBD^2(\Omega_\lambda)$. It is easy to check that $u_\lambda \to u$ in $L^1(\Omega; \mathbf{R}^2)$ as $\lambda \to 1$ and

$$\lim_{\lambda \to 1} F^d(u_\lambda, \Omega_\lambda) = F^d(u).$$

Then, by the lower semicontinuity of $\Gamma\text{-}\limsup_{\varepsilon\to 0} F^d_\varepsilon$, it suffices to prove that

$$\Gamma(L^1)\text{-}\limsup_{\varepsilon\to 0} F^d_\varepsilon(u_\lambda) \le F^d(u_\lambda, \Omega_\lambda),$$

for any $\lambda \in (0,1)$.

We now generalize an argument used in [49],[37]. Let $\varepsilon_j \to 0$ and consider u_λ extended to 0 outside Ω_λ. Notice that for $\alpha \in \varepsilon\mathbf{Z}^2$ and $\xi \in \mathbf{Z}^2$ we have $\varepsilon\left[\frac{\alpha}{\varepsilon}\right]_{\mathrm{e}_1} = \alpha$ and $\varepsilon\left[\frac{\alpha+\varepsilon\xi}{\varepsilon}\right]_{\mathrm{e}_1} = \alpha+\varepsilon\xi$, thus we get

$$\begin{aligned}&\int_{(0,1)^2} F^d_{\varepsilon_j}\left(T^{\varepsilon_j}_y u_\lambda\right)\,dy\\ &\quad= \sum_{\xi\in\mathbf{Z}^2}\rho(\xi)\int_{(0,1)^2}\sum_{\alpha\in R^\xi_{\varepsilon_j}} \varepsilon_j f\left(\frac{1}{\varepsilon_j}\left(|D_{\varepsilon_j,\xi}u_\lambda(\varepsilon_j y+\alpha)|^2 + \theta|\mathrm{Div}_{\varepsilon_j,\xi}u_\lambda(\varepsilon_j y+\alpha)|^2\right)\right)dy,\end{aligned}$$

where T^ε_y is given by (5.2.1) for $\mathcal{B} = \{\mathrm{e}_1, \mathrm{e}_2\}$.

Then, using the change of variable $\varepsilon_j y + \alpha \to y$, we obtain

$$\begin{aligned}&\int_{(0,1)^2} F^d_{\varepsilon_j}\left(T^{\varepsilon_j}_y u_\lambda\right)\,dy\\ &= \sum_{\xi\in\mathbf{Z}^2}\rho(\xi)\sum_{\alpha\in R^\xi_{\varepsilon_j}}\int_{\alpha+(0,\varepsilon_j)^2}\frac{1}{\varepsilon_j} f\left(\frac{1}{\varepsilon_j}\left(|D_{\varepsilon_j,\xi}u_\lambda(y)|^2 + \theta|\mathrm{Div}_{\varepsilon_j,\xi}u_\lambda(y)|^2\right)\right)dy\\ &\le \sum_{\xi\in\mathbf{Z}^2}\rho(\xi)\mathcal{F}^{c,\xi}_{\varepsilon_j}(u_\lambda, \Omega_\lambda).\end{aligned}$$

In particular, by Proposition 5.3.2 and Remark 5.3.3, there holds

$$\begin{aligned}&\limsup_j \int_{(0,1)^2} F^d_{\varepsilon_j}(T^{\varepsilon_j}_y u_\lambda)\,dy\\ &\quad\le \sum_{\xi\in\mathbf{Z}^2}\rho(\xi)\limsup_j \mathcal{F}^{c,\xi}_{\varepsilon_j}(u_\lambda, \Omega_\lambda) \le F^d(u_\lambda, \Omega_\lambda) < +\infty.\end{aligned}\tag{5.3.13}$$

Fix $\delta > 0$ and set

$$C^j_\delta := \left\{z \in (0,1)^2 : F^d_{\varepsilon_j}\left(T^{\varepsilon_j}_z u_\lambda\right) \le \int_{(0,1)^2} F^d_{\varepsilon_j}\left(T^{\varepsilon_j}_y u_\lambda\right)\,dy + \delta\right\}.$$

By (5.3.13), we have for j large

$$\mathcal{L}^n\left((0,1)^2\setminus C^j_\delta\right) \le \frac{\displaystyle\int_{(0,1)^2} F^d_{\varepsilon_j}\left(T^{\varepsilon_j}_y u_\lambda\right)\,dy}{\displaystyle\int_{(0,1)^2} F^d_{\varepsilon_j}\left(T^{\varepsilon_j}_y u_\lambda\right)\,dy + \delta} \le c < 1,$$

which implies

$$\mathcal{L}^n\left(C_\delta^j\right) > 1 - c > 0.$$

Then, by Remark 5.2.2, for any $j \in \mathbf{N}$ we can choose $z_j \in C_\delta^j$ such that $T_{z_j}^{\varepsilon_j} u_\lambda \to u_\lambda$ in $L^1(\Omega;\mathbf{R}^2)$ and

$$F_{\varepsilon_j}^d(T_{z_j}^{\varepsilon_j} u_\lambda) \leq \int_{(0,1)^2} F_{\varepsilon_j}^d\left(T_y^{\varepsilon_j} u_\lambda\right) dy + \delta. \tag{5.3.14}$$

Hence, by (5.3.13) and (5.3.14), there holds

$$\limsup_j F_{\varepsilon_j}^d\left(T_{z_j}^{\varepsilon_j} u_\lambda\right) \leq F^d(u_\lambda, \Omega_\lambda) + \delta,$$

from which we infer

$$\Gamma(L^1)\text{-}\limsup_j F_{\varepsilon_j}^d(u_\lambda) \leq F^d(u_\lambda, \Omega_\lambda) + \delta$$

and letting $\delta \to 0$ we get the conclusion. □

Remark 5.3.5 *In the proof of the previous proposition the convexity assumption on Ω is used only to ensure that for any $\lambda > 1$ $\Omega \subset\subset \Omega_\lambda$. This condition is needed in order to justify the existence of some kind of extension of a SBD function outside of Ω with controlled energy. An alternative approach would involve the use of an extension theorem in SBD analogous to the one holding true in SBV (see Theorem 2.6.14). So far, such a result has not been proved.*

Thus, Proposition 5.3.4 and Theorem 5.1.1 can be stated for open sets sharing one of the previous properties.

5.4 The continuous case

In this section we will prove Theorem 5.1.8. We "localize" the functionals $\mathcal{F}_\varepsilon^{c,\xi}$ as

$$\mathcal{F}_\varepsilon^{c,\xi}(u, A) := \frac{1}{\varepsilon}\int_{A_\varepsilon^\xi} f\left(\frac{1}{\varepsilon}\left(|D_{\varepsilon,\xi}u(x)|^2 + \theta|\mathrm{Div}_{\varepsilon,\xi}u(x)|^2\right)\right) dx,$$

for any $u \in L^1(\Omega;\mathbf{R}^2)$, $A \in \mathcal{A}(\Omega)$, with

$$A_\varepsilon^\xi := \{x \in \mathbf{R}^2 : [x - \varepsilon\xi, x + \varepsilon\xi] \cup [x - \varepsilon\xi^\perp, x + \varepsilon\xi^\perp] \subset A\}.$$

Proposition 5.4.1 *For any $u \in L^\infty(\Omega;\mathbf{R}^2)$,*

$$\Gamma\left(L^1\right)\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon^c(u) \geq F^c(u).$$

Proof. Step 1: Let us first assume $f(t) = (at) \wedge b$. Let $\varepsilon_j \to 0$, $u_j \in L^1(\Omega;\mathbf{R}^2)$, $u \in L^\infty(\Omega;\mathbf{R}^2)$ be such that $u_j \to u$ in $L^1(\Omega;\mathbf{R}^2)$ and $\liminf_j F_{\varepsilon_j}^c(u_j) = \lim_j F_{\varepsilon_j}^c(u_j) < +\infty$. In particular for a.e. $\xi \in \mathbf{R}^2$ such that $\rho(\xi) \neq 0$ $\liminf_j \mathcal{F}_{\varepsilon_j}^{c,\xi}(u_j) < +\infty$. Fix such a $\xi \in \mathbf{R}^2$ and $A \in \mathcal{A}(\Omega)$. Up to passing to a subsequence we may assume that $\liminf_j \mathcal{F}^{c,\xi}_{\varepsilon_j}(u_j,A) = \lim_j \mathcal{F}^{c,\xi}_{\varepsilon_j}(u_j,A) < +\infty$. We now adapt to our case a "discretization" argument used in the proof of Proposition 3.38 of [37]. In what follows, when needed, we will consider u_j and u extended to 0 outside Ω. If we define

$$g_j(x) := \begin{cases} f\left(\frac{1}{\varepsilon_j}\left(|D_{\varepsilon_j,\xi}u_j(x)|^2 + \theta|\mathrm{Div}_{\varepsilon_j,\xi}u_j(x)|^2\right)\right) & \text{if } x \in A^{\xi}_{\varepsilon_j} \\ 0 & \text{otherwise in } \mathbf{R}^2, \end{cases}$$

we can write

$$\begin{aligned} \mathcal{F}^{c,\xi}_{\varepsilon_j}(u_j,A) &= \frac{1}{\varepsilon_j}\int_{\mathbf{R}^2} g_j(x)\,dx = \frac{1}{\varepsilon_j}\sum_{\alpha\in\varepsilon_j(\mathbf{Z}\xi\oplus\mathbf{Z}\xi^{\perp})}\int_{\alpha+\varepsilon_j\widetilde{Q}_\xi} g_j(x)\,dx \\ &= \frac{1}{\varepsilon_j}\sum_{\alpha\in\varepsilon_j(\mathbf{Z}\xi\oplus\mathbf{Z}\xi^{\perp})}\int_{\varepsilon_j\widetilde{Q}_\xi} g_j(x+\alpha)\,dx = \int_{\widetilde{Q}_\xi}\phi_j(x)\,dx, \end{aligned}$$

where

$$\begin{aligned} \phi_j(x) &:= \sum_{\alpha\in\varepsilon_j(\mathbf{Z}\xi\oplus\mathbf{Z}\xi^{\perp})} \varepsilon_j g_j(\varepsilon_j x+\alpha), \\ \widetilde{Q}_\xi &:= \{x\in\mathbf{R}^2 : 0\le\langle x,\xi\rangle < |\xi|^2, 0\le\langle x,\xi^{\perp}\rangle < |\xi^{\perp}|^2\}. \end{aligned}$$

Fix $\delta > 0$, then, arguing as in the proof of Proposition 5.3.4, by using Remark 5.2.2, for any $j\in\mathbf{N}$ we can find $x_j\in\widetilde{Q}_\xi$ such that $T^{\varepsilon_j,\xi}_{x_j}u_j \to u$ in $L^1(\Omega;\mathbf{R}^2)$ and

$$\begin{aligned} &\mathcal{F}^{c,\xi}_{\varepsilon_j}(u_j,A) + \delta \ge |\xi|^2\phi_j(x_j) \\ &\ge |\xi|^2\varepsilon_j \sum_{\substack{\alpha\in\varepsilon_j(\mathbf{Z}\xi\oplus\mathbf{Z}\xi^{\perp}) \\ \alpha\in A^{\xi}_{\varepsilon_j}-\varepsilon_j x_j}} f\left(\frac{1}{\varepsilon_j}\left(|D_{\varepsilon_j,\xi}u_j(\alpha+\varepsilon_j x_j)|^2 + \theta|\mathrm{Div}_{\varepsilon_j,\xi}u_j(\alpha+\varepsilon_j x_j)|^2\right)\right) \end{aligned}$$

Now we point out that the functionals on the right hand side is of the same type of those defined in (5.1.5). Hence, up to slight modifications, we can proceed as in the proof of Proposition 5.3.1 to obtain that $u\in SBD(\Omega)$ and

$$\begin{aligned} \liminf_j \mathcal{F}^{c,\xi}_{\varepsilon_j}(u_j) &\ge 2a\int_\Omega |\langle\mathcal{E}u(x)\xi,\xi\rangle|^2\,dx + 4a\theta|\xi|^4\int_\Omega |\mathrm{div}\,u(x)|^2\,dx \\ &+2b\Big(\int_{J^{\xi}_u\setminus J^{\xi^{\perp}}_u} |\langle\nu_u,\xi\rangle|\,d\mathcal{H}^1 + \int_{J^{\xi^{\perp}}_u\setminus J^{\xi}_u} |\langle\nu_u,\xi^{\perp}\rangle|\,d\mathcal{H}^1 \\ &+\int_{J^{\xi}_u\cap J^{\xi^{\perp}}_u} |\langle\nu_u,\xi\rangle|\vee|\langle\nu_u,\xi^{\perp}\rangle|\,d\mathcal{H}^1\Big). \end{aligned}$$

Finally, recalling that $\mathcal{H}^1(J_u\setminus J^{\xi}_u) = 0$ for a.e. $\xi\in\mathbf{R}^2$, by integrating with respect to ξ and by Fatou's lemma, we get

$$\liminf_j F^c_{\varepsilon_j}(u_j) \ge \int_{\mathbf{R}^2}\rho(\xi)\liminf_j \mathcal{F}^{c,\xi}_{\varepsilon_j}(u_j)\,d\xi$$

$$\begin{aligned}
&\geq \int_{\mathbf{R}^2} \rho(\xi) \left(\int_\Omega 2a|\langle \mathcal{E}u(x)\xi, \xi\rangle|^2 + 4a\theta|\xi|^4 |\operatorname{div} u(x)|^2 \, dx \right) d\xi \\
&+ \int_{\mathbf{R}^2} 2b\rho(\xi) \left(\int_{J_u} |\langle \nu_u, \xi\rangle| \vee |\langle \nu_u, \xi^\perp\rangle| \, d\mathcal{H}^1 \right) d\xi \\
&= \int_\Omega \left(\int_{\mathbf{R}^2} \rho(\xi) \left(2a|\langle \mathcal{E}u(x)\xi, \xi\rangle|^2 + 4a\theta|\xi|^4 |\operatorname{div} u(x)|^2 \right) d\xi \right) dx \\
&+ \int_{J_u} \left(\int_{\mathbf{R}^2} 2b\rho(\xi) |\langle \nu_u, \xi\rangle| \vee |\langle \nu_u, \xi^\perp\rangle| \, d\xi \right) d\mathcal{H}^1.
\end{aligned}$$

The expressions for μ, λ, γ follow after a simple computation.

Step 2: If f is any, arguing as in *Step 2* of the proof of Proposition 5.3.1, we have that $\Gamma\left(L^1\right)\text{-}\liminf_{\varepsilon\to 0} F^c_\varepsilon(u)$ is finite only if $F^c(u)$ is finite and

$$\Gamma\left(L^1\right)\text{-}\liminf_{\varepsilon\to 0} F^c_\varepsilon(u) \geq \mu_i \int_\Omega |\mathcal{E}u(x)|^2 \, dx + \lambda_i \int_\Omega |\operatorname{div} u(x)|^2 \, dx + \gamma_i \mathcal{H}^1(J_u)$$

with $\sup_i \mu_i = \mu$, $\sup_i \lambda_i = \lambda$ and $\sup_i \gamma_i = \gamma$. The thesis follows using once more Lemma 2.2.4. □

Let us now prove the Γ-limsup inequality which easily follows by Proposition 5.3.2 and Remark 5.3.3.

Proposition 5.4.2 *For any $u \in L^\infty\left(\Omega; \mathbf{R}^2\right)$,*

$$\Gamma(L^1)\text{-}\limsup_{\varepsilon\to 0} F^c_\varepsilon(u) \leq F^c(u).$$

Proof. As usual we can reduce ourselves to prove the inequality for $u \in SBD^2(\Omega)$. For such a u the recovery sequence is provided by the function itself. Indeed, by Proposition 5.3.2, estimate (5.3.12) and Fatou's lemma, we get

$$\limsup_{\varepsilon\to 0} F^c_\varepsilon(u) \leq \int_{\mathbf{R}^2} \rho(\xi) \limsup_{\varepsilon\to 0} \mathcal{F}^{c,\xi}_\varepsilon(u) \, d\xi \leq \int_{\mathbf{R}^2} \rho(\xi) \mathcal{F}^\xi(u) \, d\xi = F^c(u).$$

□

5.5 Convergence of minimum problems in the discrete case

5.5.1 A compactness lemma

The following lemma will be crucial to derive the convergence of the minimum problems treated in the next section.

Lemma 5.5.1 *Let f, ρ, F^d_ε be as in Theorem 5.1.1; assume in addition that Ω is a bounded Lipschitz open set. Let (u_j) be a sequence in $\mathcal{A}_{\varepsilon_j}(\Omega)$ such that*

$$\sup_j \left(F^d_{\varepsilon_j}(u_j) + \|u_j\|_{L^\infty(\Omega;\mathbf{R}^2)}\right) < +\infty. \tag{5.5.1}$$

Then there exists a subsequence (u_{j_k}) converging in $L^1(\Omega; \mathbf{R}^2)$ to a function $u \in SBD^2(\Omega)$.

Proof. Without loss of generality we may assume $f(t) = (at) \wedge b$. Set

$$C_j := \sum_{\alpha \in R^{e_1}_{\varepsilon_j}} \varepsilon_j f\left(\frac{1}{\varepsilon_j}|D^{e_1}_{\varepsilon_j} u_j(\alpha)|^2\right) + \sum_{\alpha \in R^{e_2}_{\varepsilon_j}} \varepsilon_j f\left(\frac{1}{\varepsilon_j}|D^{e_2}_{\varepsilon_j} u_j(\alpha)|^2\right)$$
$$+ \sum_{\alpha \in R^{e_1+e_2}_{\varepsilon_j}} \varepsilon_j f\left(\frac{1}{\varepsilon_j}|D^{e_1+e_2}_{\varepsilon_j} u_j(\alpha)|^2\right) + \sum_{\alpha \in R^{e_1-e_2}_{\varepsilon_j}} \varepsilon_j f\left(\frac{1}{\varepsilon_j}|D^{e_1-e_2}_{\varepsilon_j} u_j(\alpha)|^2\right),$$

then the monotonicity and subadditivity of f yield

$$\sup_j C_j \le c \ \sup_j F^d_{\varepsilon_j}(u_j) < +\infty. \tag{5.5.2}$$

Let

$$M_j(\alpha) := \max\{\, |D^{e_1}_{\varepsilon_j} u_j(\alpha)|^2;\ |D^{e_2}_{\varepsilon_j} u_j(\alpha)|^2;\ |D^{e_1}_{\varepsilon_j} u_j(\alpha + \varepsilon_j e_2)|^2;$$
$$|D^{e_2}_{\varepsilon_j} u_j(\alpha + \varepsilon_j e_1)|^2;\ |D^{e_1+e_2}_{\varepsilon_j} u_j(\alpha)|^2;\ |D^{e_2-e_1}_{\varepsilon_j} u_j(\alpha + \varepsilon_j e_1)|^2\}$$

and

$$R_j := R^{e_1}_{\varepsilon_j} \cap R^{e_1+e_2}_{\varepsilon_j} \cap \left(R^{e_1}_{\varepsilon_j} - \varepsilon_j e_2\right) \cap \left(R^{e_1}_{\varepsilon_j} - \varepsilon_j e_1\right) \cap \left(R^{e_1+e_2}_{\varepsilon_j} - \varepsilon_j e_1\right),$$

then set

$$I_j := \left\{\alpha \in R_j :\ M_j(\alpha) \le \frac{b}{a}\varepsilon_j\right\}.$$

Consider the (piecewise affine) functions $v_j = (v^1_j, v^2_j)$ defined on $\alpha + \varepsilon_j[0,1)^2,\ \alpha \in I_j$, as

$$v^1_j(x) := \begin{cases} u^1_j(\alpha) + \frac{1}{\varepsilon_j} D^{e_1}_{\varepsilon_j} u_j(\alpha)(x_1 - \alpha_1) \\ +\frac{1}{\varepsilon_j}\left(u^1_j(\alpha + \varepsilon_j e_2) - u^1_j(\alpha)\right)(x_2 - \alpha_2) & x \in (\alpha + \varepsilon_j T^-) \cap \Omega \\ u^1_j(\alpha + \varepsilon_j(e_1 + e_2)) + \frac{1}{\varepsilon_j} D^{e_1}_{\varepsilon_j} u_j(\alpha + \varepsilon_j e_2)(x_1 - \alpha_1 - \varepsilon_j) \\ +\frac{1}{\varepsilon_j}\left((u^1_j(\alpha + \varepsilon_j(e_1 + e_2)) - u^1_j(\alpha + \varepsilon_j e_1)\right)(x_2 - \alpha_2 - \varepsilon_j) \\ & x \in (\alpha + \varepsilon_j T^+) \cap \Omega \end{cases}$$

$$v^2_j(x) := \begin{cases} u^2_j(\alpha) + \frac{1}{\varepsilon_j} D^{e_2}_{\varepsilon_j} u_j(\alpha)(x_2 - \alpha_2) \\ +\frac{1}{\varepsilon_j}\left(u^2_j(\alpha + \varepsilon_j e_1) - u^2_j(\alpha)\right)(x_1 - \alpha_1) & x \in (\alpha + \varepsilon_j T^-) \cap \Omega \\ u^2_j(\alpha + \varepsilon_j(e_1 + e_2)) + \frac{1}{\varepsilon_j} D^{e_2}_{\varepsilon_j} u_j(\alpha + \varepsilon_j e_1)(x_2 - \alpha_2 - \varepsilon_j) \\ +\frac{1}{\varepsilon_j}\left((u^2_j(\alpha + \varepsilon_j(e_1 + e_2)) - u^2_j(\alpha + \varepsilon_j e_2)\right)(x_1 - \alpha_1 - \varepsilon_j) \\ & x \in (\alpha + \varepsilon_j T^+) \cap \Omega \end{cases}$$

where $T^\pm = \{x \in (0,1)^2 :\ \pm(x_1 + x_2 - 1) \ge 0\}$ and $x_i = \langle x, e_i\rangle$, $i = 1, 2$, and $v_j = u_j$ elsewhere in Ω. Notice that on each triangle $\alpha + \varepsilon_j T^\pm$ v_j is an affine interpolation of the values of u_j on the vertices of the triangle.

By direct computation it is easily seen that for any $\alpha \in I_j$ and $x \in \alpha + \varepsilon_j(0,1)^2$ there holds

$$|\mathcal{E} v_j(x)|^2 \leq c\left(\frac{1}{\varepsilon_j}\right)^2 M_j(\alpha),$$

hence, by taking into account (5.5.2) and the subadditivity of f, we get

$$\sup_j \int_\Omega |\mathcal{E} v_j(x)|^2\,dx \leq c\ \sup_j C_j < +\infty. \tag{5.5.3}$$

Now, we provide an estimate for $\mathcal{H}^1\left(J_{v_j}\right)$. Let

$$\begin{aligned} A_j &:= \{\alpha \in \varepsilon_j \mathbf{Z}^2 :\ \alpha + \varepsilon_j(0,1)^2 \cap \Omega \neq \emptyset\}, \\ D_j &:= \{x \in \mathbf{R}^2 :\ d\,(x, \partial\Omega) < 2\varepsilon_j\}, \end{aligned}$$

and note that

$$\bigcup_{\alpha \in A_j \setminus R_j} \left(\alpha + \varepsilon_j(0,1)^2\right) \subseteq D_j.$$

By the Lipschitz regularity assumption on Ω, it follows

$$\#\,(A_j \setminus R_j) \leq \frac{\mathcal{L}^n\,(D_j)}{{\varepsilon_j}^2} \leq c\ \frac{\mathcal{H}^1\,(\partial\Omega)}{\varepsilon_j},$$

and thus we get

$$\begin{aligned} \mathcal{H}^1\left(J_{v_j}\right) &\leq 4\varepsilon_j \#\,(A_j \setminus I_j) \\ &= 4\varepsilon_j \#\,(A_j \setminus R_j) + 4\varepsilon_j \#\,(R_j \setminus I_j) \leq c\ \{\mathcal{H}^1\,(\partial\Omega) + C_j\} \leq c. \end{aligned} \tag{5.5.4}$$

Since $(v_j) \subset SBD(\Omega)$ is bounded in $L^\infty\,(\Omega; \mathbf{R}^2)$ and by taking into account (5.5.3) and (5.5.4), Theorem 2.8.6 yields the existence of a subsequence (v_{j_k}) converging in $L^1\,(\Omega; \mathbf{R}^2)$ to a function $u \in SBD^2(\Omega)$. The thesis follows noticing that by Proposition 5.1.14 (u_{j_k}) is also converging to u in $L^1\,(\Omega; \mathbf{R}^2)$. □

Thanks to Lemma 5.5.1 and by taking into account Theorems 5.1.1 and 2.3.2, we have the following convergence result for obstacle problems with Neumann boundary conditions.

Theorem 5.5.2 *Let K be a compact subset of $\mathbf{R}^2$ and let $h \in L^1(\Omega; \mathbf{R}^2)$. Then the minimum values*

$$\min\left\{F^d_\varepsilon(u) - \int_\Omega \langle h, u\rangle\,dx :\ u \in L^1(\Omega; \mathbf{R}^2),\ u \in K \ a.e.\right\}, \tag{5.5.5}$$

converge to the minimum value

$$\min\left\{F^d(u) - \int_\Omega \langle h, u\rangle\,dx :\ u \in L^1(\Omega; \mathbf{R}^2),\ u \in K \ a.e.\right\}, \tag{5.5.6}$$

Moreover, for any family of minimizers (u_ε) for (5.5.5) and for any sequence (ε_j) of positive numbers converging to 0, there exists a subsequence (not relabeled) u_{ε_j} converging to a minimizer of (5.5.6).

5.5.2 Boundary value problems

In this section we deal with boundary value problems for discrete energies. Following [44], we separate 'interior interactions' from those 'crossing the boundary'. Let $\Omega \subset \mathbf{R}^2$ be a convex set such that $0 \in \Omega$, let $\eta > 0$ and denote by Ω_η the open set $\{x \in \mathbf{R}^2 : \; dist\,(x, \partial\Omega) < \eta\}$. Let $\varphi : \partial\Omega \to \mathbf{R}^2$ and $p_1, \ldots, p_N \in \partial\Omega$ such that φ is Lipschitz on each connected component of $\partial\Omega \setminus \{p_1, ..., p_N\}$. Then define for $\eta < dist\,(0, \partial\Omega)$ the function $\tilde{\varphi} : \Omega_\eta \to \mathbf{R}^2$ by

$$\tilde{\varphi}(x) := \varphi\,(\tau x)\,,$$

where $\tau \geq 0$ is such that $\{\tau x\} = \{tx\}_{t \geq 0} \cap \partial\Omega$.

We remark that $\tilde{\varphi} \in W^{1,\infty}\left(\Omega_\eta \setminus \cup_{i=1}^N \{tp_i\}_{t\geq 0}\,; \mathbf{R}^2\right)$ and $J_{\tilde{\varphi}} = \cup_{i=1}^N \{tp_i\}_{t \geq 0} \cap \Omega_\eta$. The function $\tilde{\varphi}$ is a possible extension of φ to Ω_η. Other extensions are possible which, under regularity assumptions, yield the same result. Here we examine this 'radial' extension only, for the sake of simplicity.

With given $u \in SBD(\Omega)$, let

$$u^\varphi(x) := \begin{cases} u(x) & \text{if } x \in \Omega \\ \\ \tilde{\varphi}(x) & \text{if } x \in \Omega_\eta \setminus \Omega, \end{cases}$$

then $u^\varphi \in SBD(\Omega_\eta)$ and $J_{u^\varphi} = J_u \cup J_{\tilde{\varphi}} \cup \{x \in \partial\Omega : \; \gamma(u)(x) \neq \varphi(x)\}$, where $\gamma(u)$ denotes the inner trace of u with respect to $\partial\Omega$. Finally, we define a suitable discretization of $\tilde{\varphi}$ by

$$\varphi_\varepsilon(\alpha) := \begin{cases} \tilde{\varphi}(\alpha) & \text{if } \alpha \in \varepsilon\mathbf{Z}^2 \cap \Omega_\eta \\ \\ 0 & \text{if } \alpha \notin \varepsilon\mathbf{Z}^2 \cap \Omega_\eta. \end{cases}$$

Let f, ρ, F_ε^d be as in Theorem 5.1.1 and assume in addition that $\rho(\xi) = 0$ for $|\xi| > M$, with $M \geq 2$. Let $\mathcal{B}_\varepsilon(u) := F_\varepsilon^d(u) + F_\varepsilon^{d,\varphi}(u)$, where

$$F_\varepsilon^{d,\varphi}(u) := \begin{cases} \displaystyle\sum_{|\xi| \leq M} \rho(\xi) \sum_{\alpha \in R_\varepsilon^\xi(\partial\Omega)} \varepsilon f\left(\frac{1}{\varepsilon}\left(|D_{\varepsilon,\xi} u^{\varphi_\varepsilon}(\alpha)|^2 + \theta |\mathrm{Div}_{\varepsilon,\xi} u^{\varphi_\varepsilon}(\alpha)|^2\right)\right) \\ & \text{if } u \in \mathcal{A}_\varepsilon(\Omega) \\ \\ +\infty & \text{otherwise} \end{cases}$$

with

$$R_\varepsilon^\xi(\partial\Omega) := \{\alpha \in \varepsilon\mathbf{Z}^2 \setminus R_\varepsilon^\xi(\Omega) : \; [\alpha - \varepsilon\xi, \alpha + \varepsilon\xi] \cup [\alpha - \varepsilon\xi^\perp, \alpha + \varepsilon\xi^\perp] \cap \Omega \neq \emptyset\}$$

and

$$u^{\varphi_\varepsilon}(\alpha) := \begin{cases} u(\alpha) & \text{if } \alpha \in \varepsilon\mathbf{Z}^2 \cap \Omega \\ \\ \varphi_\varepsilon(\alpha) & \text{if } \alpha \notin \varepsilon\mathbf{Z}^2 \cap \Omega. \end{cases}$$

$R_\varepsilon^\xi(\partial\Omega)$ represents that part of the lattice R_ε^ξ underlying interactions among pairs of points of Ω_η, one inside and the other outside Ω (interactions through the boundary).

Proposition 5.5.3 *$(\mathcal{B}_\varepsilon)$ Γ-converges on $L^\infty(\Omega;\mathbf{R}^2)$ to the functional $\mathcal{B} : L^\infty(\Omega;\mathbf{R}^2) \to [0,+\infty]$ given by*

$$\mathcal{B}(u) := \begin{cases} F^d(u) + 2b \displaystyle\sum_{|\xi|\le M} \rho(\xi) \int_{J_{u^\varphi}\cap\partial\Omega} \Phi^\xi\left(\gamma(u) - \varphi, \nu_{\partial\Omega}\right) \, d\mathcal{H}^1 & \text{if } u \in SBD(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

with respect to both the $L^1(\Omega;\mathbf{R}^2)$-convergence and the convergence in measure, where $\nu_{\partial\Omega}$ is the inner unit normal to $\partial\Omega$ and the function $\Phi^\xi : \mathbf{R}^2 \to [0,+\infty)$ is defined by

$$\Phi^\xi(z,\nu) := \psi^\xi(z,\nu) \vee \psi^{\xi^\perp}(z,\nu),$$

with for $\eta \in \mathbf{R}^2$

$$\psi^\eta(z,\nu) := \begin{cases} |\langle \nu, \eta\rangle| & \text{if } \langle z,\eta\rangle \ne 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Note that if ε is sufficiently small, for every $v \in \mathcal{A}_\varepsilon(\Omega)$ we have

$$\mathcal{B}_\varepsilon(v) \ge F^d_\varepsilon\left(v^{\varphi_\varepsilon}, \Omega\cup\Omega_\eta\right) - F^d_\varepsilon\left(\varphi_\varepsilon, \Omega_\eta \setminus \overline{\Omega}\right). \tag{5.5.7}$$

Moreover, the regularity of φ and the assumptions on f yield

$$\limsup_{\varepsilon\to 0} F^d_\varepsilon\left(\varphi_\varepsilon, \Omega_\eta \setminus \overline{\Omega}\right) \le c\,|\Omega_\eta\setminus\Omega| + c\,\mathcal{H}^1(J_{\tilde\varphi} \cap (\Omega_\eta\setminus\Omega)). \tag{5.5.8}$$

Let $u_\varepsilon \to u$ in measure on Ω, then $u_\varepsilon^{\varphi_\varepsilon} \to u^\varphi$ in measure on $\Omega\cup\Omega_\eta$. Thus, by Theorem 5.1.1 and inequalities (5.5.7) and (5.5.8), we get

$$\liminf_{\varepsilon\to 0} \mathcal{B}_\varepsilon(u_\varepsilon) \ge F^d(u^\varphi, \Omega\cup\Omega_\eta) - \omega(\eta) \ge \mathcal{B}(u) - \omega(\eta),$$

with $\lim_{\eta\to 0}\omega(\eta) = 0$. Then the Γ-$\liminf$ inequality follows by letting $\eta \to 0$.

Let $u \in L^\infty(\Omega;\mathbf{R}^2) \cap SBD^2(\Omega)$, fix $\lambda \in (0,1)$ and define $u^\varphi_\lambda \in SBD^2(\Omega_\eta)$ as

$$u^\varphi_\lambda(x) := \begin{cases} u\left(\lambda^{-1}x\right) & x \in \lambda\Omega \\ \tilde\varphi(x) & x \in \Omega_\eta \setminus \lambda\Omega, \end{cases}$$

then $u^\varphi_\lambda \to u$ in $L^1(\Omega;\mathbf{R}^2)$ and $F^d\left(u^\varphi_\lambda, \Omega\right) \to \mathcal{B}(u)$ for $\lambda \to 1$.
Hence, to prove the Γ-$\limsup$ inequality, it suffices to show that

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} \mathcal{B}_\varepsilon\left(u^\varphi_\lambda\right) \le F^d\left(u^\varphi_\lambda, \Omega\right). \tag{5.5.9}$$

Fix $\delta > 0$, arguing as in the proof of Proposition 5.3.4, we can find v_ε of the form $u^\varphi_\lambda(\cdot + \tau_\varepsilon)$, with $\tau_\varepsilon \le c\varepsilon$ and $v_\varepsilon \to u^\varphi_\lambda$ in $L^1(\Omega;\mathbf{R}^2)$ as $\varepsilon \to 0$, such that

$$\limsup_{\varepsilon\to 0} F^d_\varepsilon\left(v_\varepsilon, \Omega\right) \le F^d\left(u^\varphi_\lambda, \Omega\right) + \delta. \tag{5.5.10}$$

Note that if $\beta = \alpha, \alpha \pm \varepsilon\xi, \alpha \pm \varepsilon\xi^{\perp}$ with $\alpha \in R_{\varepsilon}^{\xi}(\partial\Omega)$, for ε small we have

$$v_{\varepsilon}^{\varphi_{\varepsilon}}(\beta) = \begin{cases} \tilde{\varphi}(\beta + \tau_{\varepsilon}) & \text{if } \beta \in \varepsilon\mathbf{Z}^2 \cap \Omega \\ \\ \tilde{\varphi}(\beta) & \text{if } \beta \notin \varepsilon\mathbf{Z}^2 \cap \Omega. \end{cases}$$

Then, by the regularity of $\tilde{\varphi}$, it can be proved that

$$F_{\varepsilon}^{d,\varphi}(v_{\varepsilon}) = O(\varepsilon),$$

hence, by (5.5.10),

$$\limsup_{\varepsilon \to 0} \mathcal{B}_{\varepsilon}(v_{\varepsilon}) \leq F^{d}(u_{\lambda}^{\varphi}, \Omega) + \delta.$$

Then, inequality (5.5.9) follows letting $\delta \to 0$. $\square$

As a consequence of Lemma 5.5.1 and Proposition 5.5.3, we get the following convergence result for boundary value problems.

Theorem 5.5.4 *Let K be a compact set of $\mathbf{R}^2$ and let $\mathcal{B}_{\varepsilon}$ be as in Proposition 5.5.3. Then the minimum values*

$$\min\{\mathcal{B}_{\varepsilon}(u) : u \in K \text{ a.e.}\} \tag{5.5.11}$$

converge to the minimum value

$$\min\{\mathcal{B}(u) : u \in K \text{ a.e.}\}. \tag{5.5.12}$$

Moreover, for any family of minimizers (u_{ε}) for (5.5.11) and for any sequence (ε_j) of positive numbers converging to 0, there exists a subsequence (not relabeled) u_{ε_j} converging to a minimizer of (5.5.12).

Proof. It easily follows from Lemma 5.5.1, Proposition 5.5.3 and Theorem 2.3.2. $\square$

5.6 Generalizations

By following the approach of Section 5.1, different generalizations to higher dimension can be proposed. We present here one possible extension of the discrete model in $\mathbf{R}^3$ which provides as well an approximation of energies of type (5.1.1).

For any orthogonal pair $(\xi, \zeta) \in \mathbf{R}^3 \setminus \{0\}$ and for any $u : \mathbf{R}^3 \to \mathbf{R}^3$ define

$$\begin{aligned}
&D_{\varepsilon}^{\xi}u(x) := \langle u(x+\varepsilon\xi) - u(x), \xi\rangle, \\
&|D_{\varepsilon,\xi}u(x)|^2 := |D_{\varepsilon}^{\xi}u(x)|^2 + |D_{\varepsilon}^{-\xi}u(x)|^2, \\
&|D_{\varepsilon,\xi,\zeta}u(x)|^2 := |D_{\varepsilon,\xi}u(x)|^2 + |D_{\varepsilon,\zeta}u(x)|^2, \\
&|\mathrm{Div}_{\varepsilon,\xi,\zeta}u(x)|^2 := \\
&\sum_{(\sigma_1,\sigma_2,\sigma_3)\in\{1,-1\}^3} \left(\frac{1}{|\xi|^2}D_{\varepsilon}^{\sigma_1\xi}u(x) + \frac{1}{|\zeta|^2}D_{\varepsilon}^{\sigma_2\zeta}u(x) + \frac{1}{|\xi\times\zeta|^2}D_{\varepsilon}^{\sigma_3\xi\times\zeta}u(x)\right)^2,
\end{aligned}$$

where $\xi \times \zeta$ denotes the external product of ξ and ζ.

Let Ω be a bounded open set of $\mathbf{R}^3$ and $\mathcal{A}^3_\varepsilon(\Omega) := \{u : \Omega \to \mathbf{R}^3 \ : \ u \equiv const$ on $(\alpha + [0,\varepsilon)^3) \cap \Omega$ for any $\alpha \in \varepsilon\mathbf{Z}^3\}$. Then set

$$S := \{(\mathrm{e}_1, \mathrm{e}_2), (\mathrm{e}_1, \mathrm{e}_3), (\mathrm{e}_2, \mathrm{e}_3), (\mathrm{e}_1 + \mathrm{e}_2, \mathrm{e}_1 - \mathrm{e}_2), (\mathrm{e}_1 + \mathrm{e}_3, \mathrm{e}_1 - \mathrm{e}_3), (\mathrm{e}_2 + \mathrm{e}_3, \mathrm{e}_2 - \mathrm{e}_3)\}$$

and consider the sequence of functionals $F^{d,3}_\varepsilon : L^1(\Omega; \mathbf{R}^3) \to [0, +\infty]$ defined by

$$F^{d,3}_\varepsilon u := \begin{cases} \displaystyle\sum_{(\xi,\zeta)\in S} \sum_{\alpha \in R^{\xi,\zeta}_\varepsilon} \varepsilon^2 f\left(\frac{1}{\varepsilon}\left(|D_{\varepsilon,\xi,\zeta} u(\alpha)|^2 + \theta |\mathrm{Div}_{\varepsilon,\xi,\zeta} u(\alpha)|^2\right)\right) & \text{if } u \in \mathcal{A}^3_\varepsilon(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

with

$$R^{\xi,\zeta}_\varepsilon := \{\alpha \in \varepsilon\mathbf{Z}^3 : [\alpha - \varepsilon\xi, \alpha + \varepsilon\xi] \cup [\alpha - \varepsilon\zeta, \alpha + \varepsilon\zeta] \cup [\alpha - \varepsilon\xi \times \zeta, \alpha + \varepsilon\xi \times \zeta] \subset \Omega\}$$

and f, θ as in Section 5.1.

Theorem 5.6.1 *Let Ω be convex. Then $\left(F^{d,3}_\varepsilon\right)$ Γ-converges on $L^\infty(\Omega; \mathbf{R}^3)$ to the functional $F^{d,3} : L^\infty(\Omega; \mathbf{R}^3) \to [0, +\infty]$ given by*

$$F^d(u) = \begin{cases} \displaystyle 8a \int_\Omega |\mathcal{E}u(x)|^2\, dx + 4(1+2\theta)a \int_\Omega |\mathrm{div}\, u(x)|^2\, dx & \\ \displaystyle \qquad\qquad + 2b \sum_{(\xi,\zeta)\in S} \int_{J_u} \Phi^{\xi,\zeta}(u^+ - u^-, \nu_u)\, d\mathcal{H}^2 & \textit{if } u \in SBD(\Omega) \\ +\infty & \textit{otherwise} \end{cases}$$

with respect to both the $L^1(\Omega; \mathbf{R}^3)$-convergence and the convergence in measure, where $\Phi^{\xi,\zeta} : \mathbf{R}^3 \to [0, +\infty)$ is defined by

$$\Phi^{\xi,\zeta}(z,\nu) := \psi^\xi(z,\nu) \vee \psi^\zeta(z,\nu) \vee \psi^{\xi\times\zeta}(z,\nu),$$

with for $\eta \in \mathbf{R}^3$

$$\psi^\eta(z,\nu) := \begin{cases} |\langle \nu, \eta\rangle| & \textit{if } \langle z, \eta\rangle \neq 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. It suffices to proceed as in the proof of Theorem 5.1.1, by extending all the arguments to dimension 3 and taking into account Lemmas 5.2.1 and 5.2.3, that are stated in any dimension. $\square$

Chapter 6

Approximation Results by Difference Schemes of Fracture Energies in the Vectorial Case

6.1 Statement of the Γ-convergence result

In this Chapter[1] we provide a variational approximation, in the three-dimensional case, of energies which takes the form on $(GSBV^p(\Omega))^N$

$$\int_{\Omega} \psi(\nabla u)\, dx + \int_{S_u} g\left(u^+ - u^-, \nu_u\right)\, d\mathcal{H}^2,$$

with

(h1) $\psi : \mathbf{R}^{N\times 3} \to [0, +\infty)$ a quasiconvex function with superlinear growth, i.e., there exist $p \in (1, +\infty)$ and $C_1, C_2 > 0$ such that for every $X \in \mathbf{R}^{N\times 3}$

$$C_1 \left|X\right|^p \leq \psi\left(X\right) \leq C_2 \left(1 + \left|X\right|^p\right); \tag{6.1.1}$$

$g : \mathbf{R}^N \times \mathbf{S}^2 \to [0, +\infty)$ defined by

$$g(z, \nu) := \sum_{\ell=1}^{3} g_\ell\left(z\right) \left|\langle \nu, \mathrm{e}_\ell, \rangle\right| \tag{6.1.2}$$

where

(h2) g_ℓ is a symmetric, subadditive, lower semicontinuous function such that $\inf_{\mathbf{R}^N\setminus\{0\}} g_\ell > 0$;

(h3) g_ℓ is an upper semicontinuous function bounded in a neighbourhood of $z = 0$.

[1]The contents of this Chapter were obtained by the Author in collaboration with M.S. Gelli, and are contained in the paper *Approximation Results by Difference Schemes of Fracture Energies: the Vectorial Case*, to appear on NoDEA. The paper is also downloadable at `http://cvgmt.sns.it/papers/focgel00`.

Notice that the subadditivity and the local boundedness assumptions on g_ℓ imply the existence of a positive constant c_2 such that for every $z \in \mathbf{R}^N \setminus \{0\}$ and for $\ell = 1, 2, 3$ there holds

$$c_1 \leq g_\ell(z) \leq c_2(1 + |z|), \tag{6.1.3}$$

where $c_1 = \min_\ell\{\inf_{\mathbf{R}^N\setminus\{0\}} g_\ell\}$.

Let $u_T := ((-T) \wedge \langle u, e_k\rangle \vee T)_{k=1,\dots,N}$, and assume that

(h4) for every $u \in (GSBV^p(\Omega))^N$ there exists a sequence $(T_j) \subseteq [0, +\infty)$ with $T_j \to +\infty$ such that

$$\limsup_j \int_{J_u} g_\ell\left(\left[u_{T_j}\right]\right) |\langle \nu_u, \mathrm{e}_\ell\rangle|\, d\mathcal{H}^2 = \int_{J_u} g_\ell\left([u]\right) |\langle \nu_u, \mathrm{e}_\ell\rangle|\, d\mathcal{H}^2.$$

Notice that this technical condition is fulfilled in case all the g_ℓ are bounded on $\mathbf{R}^N \setminus \{0\}$. We will make further comments on this assumption in Remark 6.1.2 and Remark 6.1.6.

Let $Q = [0,1]^3$ and consider the triangulation given by the six congruent simplices T_r, $r = 1, \dots, 6$, defined by

$$\begin{aligned}
&T_1 = co\{0, \mathrm{e}_1, \mathrm{e}_3, \mathrm{e}_2 + \mathrm{e}_3\}, \quad T_4 = co\{\mathrm{e}_1, \mathrm{e}_1 + \mathrm{e}_2, \mathrm{e}_2 + \mathrm{e}_3, \mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_3\},\\
&T_2 = co\{\mathrm{e}_1, \mathrm{e}_3, \mathrm{e}_1 + \mathrm{e}_3, \mathrm{e}_2 + \mathrm{e}_3\}, \quad T_5 = co\{\mathrm{e}_1, \mathrm{e}_1 + \mathrm{e}_3, \mathrm{e}_2 + \mathrm{e}_3, \mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_3\},\\
&T_3 = co\{0, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_2 + \mathrm{e}_3\}, \quad T_6 = co\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_1 + \mathrm{e}_2, \mathrm{e}_2 + \mathrm{e}_3\},
\end{aligned}$$

(see Figure 6.1 below).

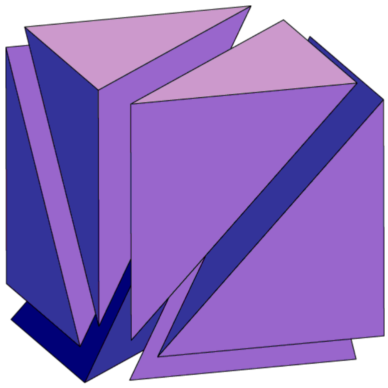

Figure 6.1: the partition $(T_i)_{r=1,\dots,6}$ of the unitary cube

Let $\Omega \subseteq \mathbf{R}^3$ be a bounded open set, for every subset $A \subseteq \Omega$ and for $r = 1, \dots, 6$ define the sets of tetrahedra

$$\begin{aligned}
&\mathcal{T}_\varepsilon^r(A) := \{\alpha + \varepsilon T_r : \alpha + \varepsilon T_r \subseteq A,\ \alpha \in \varepsilon\mathbf{Z}^3\},\\
&T_\varepsilon^r(A) := \{\alpha + \varepsilon T_r : (\alpha + \varepsilon T_r) \cap A \neq \emptyset,\ \alpha \in \varepsilon\mathbf{Z}^3\},
\end{aligned}$$

which identify the simplices properly contained in A and those intersecting A, respectively. Notice that $\cup_{r=1}^{6} T_{\varepsilon}^{r}(A) \supseteq A \supseteq \cup_{r=1}^{6} \mathcal{T}_{\varepsilon}^{r}(A)$. In case $A=\Omega$ we will drop the dependence on Ω in the definitions above.

Moreover, denote by $\mathcal{A}_{\varepsilon}\left(\Omega;\mathbf{R}^N\right)$ the set of functions $u:\Omega\to\mathbf{R}^N$ such that u is continuous on Ω and affine on each simplex belonging to $\cup_{r=1}^{6} T_{\varepsilon}^{r}$.

Let us introduce the approximating functionals. First, extend g_ℓ to $\mathbf{R}^N$ setting $g_\ell(0)=0$, thus preserving its lower semicontinuity property, then define $\psi_{\varepsilon}^{g}:\mathbf{R}^{N\times 3}\to[0,+\infty)$ as

$$\psi_{\varepsilon}^{g}(X) := \begin{cases} \psi(X) & \text{if } |X|\le\lambda_{\varepsilon} \\ \dfrac{1}{\varepsilon}\displaystyle\sum_{\ell=1}^{3} g_\ell\left(\varepsilon X \mathrm{e}_\ell\right) & \text{otherwise,} \end{cases} \tag{6.1.4}$$

where $(\lambda_{\varepsilon})\subset[0,+\infty)$ is such that $\lambda_{\varepsilon}\to+\infty$ as $\varepsilon\to0^+$ and $\sup_{\varepsilon>0}\left(\varepsilon\lambda_{\varepsilon}^{p}\right)<+\infty$. Consider the family of functionals $\mathcal{F}_{\varepsilon}^{g}:L^1\left(\Omega;\mathbf{R}^N\right)\to[0,+\infty]$ given by

$$\mathcal{F}_{\varepsilon}^{g}(u) := \begin{cases} \displaystyle\int_{\cup_{r=1}^{6}\mathcal{T}_{\varepsilon}^{r}} \psi_{\varepsilon}^{g}\left(\nabla u(x)\right)dx & \text{if } u\in\mathcal{A}_{\varepsilon}\left(\Omega;\mathbf{R}^N\right) \\ +\infty & \text{otherwise.} \end{cases}$$

Then the following result holds.

Theorem 6.1.1 *Let $\Omega\subset\mathbf{R}^3$ be a bounded open set with Lipschitz boundary and assume* (h1)-(h4). *Then, $(\mathcal{F}_{\varepsilon}^{g})$ Γ-converges with respect to both the convergence in measure and strong $L^1\left(\Omega;\mathbf{R}^N\right)$ to the functional $\mathcal{F}^{g}:L^1\left(\Omega;\mathbf{R}^N\right)\to[0,+\infty]$ defined by*

$$\mathcal{F}^{g}(u) := \begin{cases} \displaystyle\int_{\Omega}\psi(\nabla u)dx + \int_{J_u} g\left(u^+-u^-,\nu_u\right)\,d\mathcal{H}^2 & \textit{if } u\in\left(GSBV^p(\Omega)\right)^N \\ +\infty & \textit{otherwise,} \end{cases}$$

where $g:\mathbf{R}^N\times\mathbf{S}^2\to[0,+\infty)$ is defined in (6.1.2).

Remark 6.1.2 *For instance, if we assume that*

$$h(|z|)\vee c_1 \le g_\ell(z) \le c_2\left(1+h(|z|)\right) \tag{6.1.5}$$

with $h:[0,+\infty)\to[0,+\infty)$ an increasing function and c_1, c_2 positive constants, then (h4) *is satisfied.*

In case $h(t)=t$ the control from above in (6.1.5) *is automatically satisfied as noticed in (6.1.3). Moreover, the additional control from below implies that the domain of the limit functional $\mathcal{F}^{g}$ is contained in $SBV^p\left(\Omega;\mathbf{R}^N\right)$.*

6.1.1 Lower bound inequality

In this subsection we prove the lower bound inequality for Theorem 6.1.1. It will be deduced by a more general result, proved in Proposition 6.1.3 below, holding true in case the functions g_ℓ, used in the definition of the functionals $\mathcal{F}^g_\varepsilon$, all satisfy the following milder growth condition:

(h5) g_ℓ, $\ell = 1,2,3$, is a lower semicontinuous, symmetric and subadditive function such that for every $z \in \mathbf{R}^N \setminus \{0\}$

$$g_\ell(z) \geq \gamma(|z|), \tag{6.1.6}$$

where $\gamma : (0,+\infty) \to (0,+\infty]$ is a lower semicontinuous, increasing and subadditive function satisfying

$$\lim_{t\to 0^+} \frac{\gamma(t)}{t} = +\infty. \tag{6.1.7}$$

Notice that we recover (h2) choosing γ in (h5) to be constant.

Let us then prove a lower bound inequality in case the functions g_ℓ satisfy (h5) instead of (h2).

Proposition 6.1.3 *Let $\Omega \subset \mathbf{R}^3$ be a bounded open set, assume* (h1) *and* (h5)*. Then, for any $u \in L^1\left(\Omega;\mathbf{R}^N\right)$*

$$\Gamma(meas)\text{-}\liminf_{\varepsilon\to 0} \mathcal{F}^g_\varepsilon(u) \geq \mathcal{F}^g(u).$$

Proof. Let $(u_j) \subset \mathcal{A}_{\varepsilon_j}\left(\Omega;\mathbf{R}^N\right)$ and $u \in L^1\left(\Omega;\mathbf{R}^N\right)$ be such that $u_j \to u$ in measure. Moreover, assume that $\liminf_j \mathcal{F}^g_{\varepsilon_j}(u_j) = \lim_j \mathcal{F}^g_{\varepsilon_j}(u_j) < +\infty$. Consider the sets $\mathcal{N}^r_{\varepsilon_j} \subseteq \mathcal{T}^r_{\varepsilon_j}$ defined by

$$\mathcal{N}^r_{\varepsilon_j} := \{(\alpha + \varepsilon_j T_r) \in \mathcal{T}^r_{\varepsilon_j} :\ |\nabla u_j||_{(\alpha+\varepsilon_j T_r)} > \lambda_{\varepsilon_j}\}; \tag{6.1.8}$$

then, by taking into account the growth condition (6.1.5), the subadditivity and the monotonicity of γ we get

$$\sum_{r=1}^{6} \sup_j \left(\varepsilon_j^2 \gamma\left(\varepsilon_j \lambda_{\varepsilon_j}\right) \#\mathcal{N}^r_{\varepsilon_j}\right) < +\infty. \tag{6.1.9}$$

In order to prove the Γ-liminf inequality we will show that the sets $\mathcal{N}^r_{\varepsilon_j}$ in (6.1.8) detect the jump set of u. Thus, we will divide $\mathcal{F}^g_{\varepsilon_j}$ into two terms contributing separately to the bulk and surface energies of the limit functional.

Step 1:(Bulk energy inequality) According to the scheme stated above we construct a sequence $(v_j) \subset SBV\left(\Omega;\mathbf{R}^N\right)$ such that $v_j \to u$ in measure, (v_j) satisfies locally all the assumptions of Theorem 2.7.10 and, with fixed $\eta > 0$, we have

$$\int_{\cup_{r=1}^6 (\mathcal{T}^r_{\varepsilon_j} \setminus \mathcal{N}^r_{\varepsilon_j})} \psi^g_{\varepsilon_j}\left(\nabla u_j(x)\right)\, dx \geq \int_{\Omega_\eta} \psi(\nabla v_j)\, dx + o(1) \tag{6.1.10}$$

for j sufficiently large. Indeed, let $v_j : \Omega \to \mathbf{R}^N$ be the function whose components are piecewise affine, uniquely determined by

$$v_j(x) := \begin{cases} u_j(x) & x \notin \cup_{r=1}^6 \mathcal{N}_{\varepsilon_j}^r \\ \\ u_j(\alpha) & x \in \alpha + \varepsilon_j T_r, \\ & (\alpha + \varepsilon_j T_r) \in \cup_{r=1}^6 \mathcal{N}_{\varepsilon_j}^r. \end{cases} \tag{6.1.11}$$

It is easy to check that $v_j \to u$ in measure and that there holds

$$\int_{\cup_{r=1}^6 (\mathcal{T}_{\varepsilon_j}^r \setminus \mathcal{N}_{\varepsilon_j}^r)} \psi_{\varepsilon_j}^g (\nabla u_j(x))\, dx = \int_{\cup_{r=1}^6 \mathcal{T}_{\varepsilon_j}^r} \psi(\nabla v_j(x))\, dx - \frac{\varepsilon_j^3}{6} \psi(0) \# (\cup_{r=1}^6 \mathcal{N}_{\varepsilon_j}^r).$$

Since, for j sufficiently large, $\Omega_\eta \subseteq \cup_{r=1}^6 \mathcal{T}_{\varepsilon_j}^r$, by taking into account (6.1.9), the superlinearity of γ in 0 and the choice of λ_{ε_j}, we get (6.1.10).

Conditions (6.1.1) and (6.1.10) yield

$$\int_{\Omega_\eta} |\nabla v_j|^p\, dx \le c \mathcal{F}_{\varepsilon_j}^g(u_j), \tag{6.1.12}$$

for some positive constant c. Moreover, notice that

$$J_{v_j} \subseteq \cup_{r=1}^6 \cup_{\mathcal{N}_{\varepsilon_j}^r} \partial(\alpha + \varepsilon_j T_r)$$

and that $\left(v_j^+ - v_j^-\right)|_{(\alpha+\varepsilon_j T_r)}$ is a convex combination of the finite-differences computed in the nodes of the tetrahedron $(\alpha + \varepsilon_j T_r)$ belonging to $\mathcal{N}_{\varepsilon_j}^r$.

Therefore, by using the subadditivity and the monotonicity of γ, it is easy to check that

$$\int_{\Omega_\eta \cap J_{v_j}} \gamma(|[v_j]|)\, d\mathcal{H}^2 \le c \mathcal{F}_{\varepsilon_j}^g(u_j), \tag{6.1.13}$$

for some positive constant c. Hence, the sequences $(\langle v_j, e_k \rangle)$, $k = 1, \dots, N$, satisfy all the assumptions of Theorem 2.7.10 on Ω_η, so that $u \in (GSBV(\Omega_\eta))^N$ for every $\eta > 0$ and there holds

$$\int_{\Omega_\eta} |\nabla u|^p\, dx \le \liminf_j \int_{\Omega_\eta} |\nabla v_j|^p\, dx,$$
$$\int_{\Omega_\eta \cap J_u} \gamma(|[u]|)\, d\mathcal{H}^2 \le \liminf_j \int_{\Omega_\eta \cap J_{v_j}} \gamma(|[v_j]|)\, d\mathcal{H}^2.$$

The last two inequalities and conditions (6.1.12), (6.1.13) yield $u \in (GSBV^p(\Omega))^N$.

Eventually, by applying the lower semicontinuity result of Theorem 2.7.17 in (6.1.10), and then by passing to the limit on $\eta \to 0^+$, we get

$$\liminf_j \int_{\cup_{r=1}^6 (\mathcal{T}_{\varepsilon_j}^r \setminus \mathcal{N}_{\varepsilon_j}^r)} \psi_{\varepsilon_j}^g (\nabla u(x))\, dx \ge \int_\Omega \psi(\nabla u)\, dx. \tag{6.1.14}$$

Step 2:(Surface energy inequality) With fixed $\ell = 1, 2, 3$ we will prove the following inequality

$$\liminf_j \frac{1}{\varepsilon_j} \int_{\cup_{r=1}^6 \mathcal{N}_{\varepsilon_j}^r} g_\ell\left(\varepsilon_j \nabla u(x) \mathrm{e}_\ell\right) \, dx \geq \int_{J_u} g_\ell\left([u]\right) |\langle \nu_u, \mathrm{e}_\ell \rangle| \, d\mathcal{H}^2. \tag{6.1.15}$$

To this aim, for any $r = 1, \dots, 6$, we construct a sequence $(v_j^{\ell,r}) \subset SBV\left(\Omega; \mathbf{R}^N\right)$ with one-dimensional profile along e_ℓ, which is locally pre-compact in SBV in this given direction, but in general not globally in $GSBV$. More precisely, let $p_{\ell,r}$ be the unique vertex in T_r such that $p_{\ell,r} + \mathrm{e}_\ell \in T_r$ and define

$$v_j^{\ell,r}(x) := \begin{cases} u_j(\alpha + \varepsilon_j p_{\ell,r}) & \begin{array}{l} x \in (\alpha + \varepsilon_j [0,1)^3) \cap \Omega \\ (\alpha + \varepsilon_j T_r) \in \mathcal{N}_{\varepsilon_j}^r \end{array} \\ \\ u_j(\alpha + \varepsilon_j p_{\ell,r}) + \nabla u_j(x) \mathrm{e}_\ell \langle x - \alpha - \varepsilon_j p_{\ell,r}, \mathrm{e}_\ell \rangle & \\ & \begin{array}{l} x \in (\alpha + \varepsilon_j [0,1)^3) \cap \Omega \\ (\alpha + \varepsilon_j T_r) \notin \mathcal{N}_{\varepsilon_j}^r, \end{array} \end{cases}$$

then $(v_j^{\ell,r}) \subset SBV\left(\Omega; \mathbf{R}^N\right)$ and $v_j^{\ell,r} \to u$ in measure. With fixed $\eta > 0$, notice that by (6.1.1) we get

$$\int_{\Omega_\eta} \left| \frac{\partial v_j^{\ell,r}}{\partial \mathrm{e}_\ell} \right|^p \, dx \leq c \mathcal{F}_{\varepsilon_j}^g(u_j), \tag{6.1.16}$$

for some positive constant c. Moreover, since $\nu_{v_j^{\ell,r}} \in \{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\}$ $\mathcal{H}^2$ a.e. in $J_{v_j^{\ell,r}}$ there holds

$$\begin{aligned} \frac{1}{\varepsilon_j} \int_{\mathcal{N}_{\varepsilon_j}^r} g_\ell\left(\varepsilon_j \nabla u_j(x) \mathrm{e}_\ell\right) \, dx &\geq \frac{1}{6} \int_{\Omega_\eta \cap J_{v_j^{\ell,r}}} g_\ell\left(\left[v_j^{\ell,r}\right]\right) |\langle \nu_{v_j^{\ell,r}}, \mathrm{e}_\ell \rangle| \, d\mathcal{H}^2 \\ &= \frac{1}{6} \int_{\Pi^{\mathrm{e}_\ell}} d\mathcal{H}^2 \int_{\Omega_\eta \cap (J_{v_j^{\ell,r}})_y^{\mathrm{e}_\ell}} g_\ell\left(\left[\left(v_j^{\ell,r}\right)^{\mathrm{e}_\ell, y}\right]\right) d\mathcal{H}^0, \end{aligned} \tag{6.1.17}$$

where the last equality follows by using the characterization of BV functions through their one-dimensional sections (see Theorem 2.7.6) and the generalized coarea formula for rectifiable sets (see (2.4.1) of Lemma 2.4.2). By passing to the limit on $j \to +\infty$ in (6.1.17) and by applying Fatou's lemma we have

$$\begin{aligned} &\liminf_j \frac{1}{\varepsilon_j} \int_{\mathcal{N}_{\varepsilon_j}^r} g_\ell\left(\varepsilon_j \nabla u_j(x) \mathrm{e}_\ell\right) \, dx \\ &\geq \frac{1}{6} \int_{\Pi^{\mathrm{e}_\ell}} d\mathcal{H}^2 \liminf_j \int_{\Omega_\eta \cap (J_{v_j^{\ell,r}})_y^{\mathrm{e}_\ell}} g_\ell\left(\left[\left(v_j^{\ell,r}\right)^{\mathrm{e}_\ell, y}\right]\right) d\mathcal{H}^0, \end{aligned}$$

from which we infer that for $\mathcal{H}^2$ a.e. $y \in \Pi^{\mathrm{e}_\ell}$ there holds

$$\liminf_j \int_{\Omega_\eta \cap (J_{v_j^{\ell,r}})_y^{\mathrm{e}_\ell}} g_\ell\left(\left[\left(v_j^{\ell,r}\right)^{\mathrm{e}_\ell, y}\right]\right) d\mathcal{H}^0 < +\infty. \tag{6.1.18}$$

Thus, for $\mathcal{H}^2$ a.e. $y \in \Pi^{\mathrm{e}_\ell}$, up to extracting subsequences depending on such a fixed y, we may assume $\left(v_j^{\ell,r}\right)^{\mathrm{e}_\ell,y} \to u^{\mathrm{e}_\ell,y}$ in measure on $(\Omega_\eta)_y^{\mathrm{e}_\ell}$, the inferior limit in (6.1.18) to be a limit and, by taking into account (6.1.16), also that

$$\sup_j \int_{(\Omega_\eta)_y^{\mathrm{e}_\ell}} \left|\left(\dot{v}_j^{\ell}\right)^{\mathrm{e}_\ell,y}\right|^p \, dt < +\infty.$$

Hence, the slices $\left(\left(v_j^{\ell,r}\right)^{\mathrm{e}_\ell,y}\right)$ satisfy on $(\Omega_\eta)_y^{\mathrm{e}_\ell}$ all the assumptions of Theorem 2.7.10, so that, by Theorem 2.7.20, we have

$$\begin{aligned}
&\liminf_j \frac{1}{\varepsilon_j} \int_{\cup_{r=1}^6 \mathcal{N}_{\varepsilon_j}^r} g_\ell\left(\varepsilon_j \nabla u(x) \mathrm{e}_\ell\right) dx \\
&\quad \geq \frac{1}{6} \int_{\Pi^{\mathrm{e}_\ell}} d\mathcal{H}^2 \liminf_j \int_{\Omega_\eta \cap (J_{v_j^{\ell,r}})_y^{\mathrm{e}_\ell}} g_\ell\left(\left[\left(v_j^{\ell,r}\right)^{\mathrm{e}_\ell,y}\right]\right) d\mathcal{H}^0 \\
&\quad \geq \frac{1}{6} \int_{\Pi^{\mathrm{e}_\ell}} d\mathcal{H}^2 \int_{\Omega_\eta \cap (J_u)_y^{\mathrm{e}_\ell}} g_\ell\left([u^{\mathrm{e}_\ell,y}]\right) d\mathcal{H}^0 = \frac{1}{6} \int_{\Omega_\eta \cap J_u} g_\ell\left([u]\right) \left|\langle \nu_u, \mathrm{e}_\ell \rangle\right| d\mathcal{H}^2.
\end{aligned}$$

We deduce (6.1.15) passing to the limit on $\eta \to 0^+$ and using the subadditivity of the inferior limit.

To conclude it suffices to collect (6.1.14) and (6.1.15). □

Remark 6.1.4 *We claim that, by proceeding as in Step 1 of the proof of Proposition 6.1.3, one can prove that the families of functions* $(u_\varepsilon) \subseteq \mathcal{A}_\varepsilon\left(\Omega; \mathbf{R}^N\right)$ *such that*

$$\sup_{\varepsilon > 0} \left(\mathcal{F}_\varepsilon^g(u_\varepsilon) + \|u_\varepsilon\|_{L^1(\Omega;\mathbf{R}^N)}\right) < +\infty$$

are pre-compact in $L^1\left(\Omega; \mathbf{R}^N\right)$. *Indeed, to get the result it suffices to apply the GSBV Compactness theorem (see Theorem 2.7.11) to the family* (v_ε) *constructed in (6.1.11).*

6.1.2 Upper bound inequality

In this subsection we prove the upper bound inequality for Theorem 6.1.1.

Proposition 6.1.5 *Let* $\Omega \subset \mathbf{R}^3$ *be a bounded open set with Lipschitz boundary, assume* (h1)-(h4). *Then, for any* $u \in L^1\left(\Omega; \mathbf{R}^N\right)$,

$$\Gamma\left(L^1\right)\text{-}\limsup_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon^g(u) \leq \mathcal{F}^g(u).$$

Proof. It suffices to prove the inequality above for $u \in (GSBV^p(\Omega))^N$. We will first prove the inequality for a class of more regular functions.

Step 1: Let Ω' be an open set such that $\Omega' \supset\supset \Omega$ and suppose $u \in \mathcal{W}\left(\Omega'; \mathbf{R}^N\right)$.

Let us first fix some notations. With fixed $m \in \mathbf{N} \setminus \{0\}$, let

$$J_u^m := \left\{ x \in J_u : \left| u^+(x) - u^-(x) \right| \geq \frac{1}{m} \right\},$$

then (J_u^m) is an increasing family of sets such that $J_u = \cup_{m \in \mathbf{N} \setminus \{0\}} J_u^m$ and so

$$\lim_{m \to +\infty} \mathcal{H}^2(J_u^m) = \mathcal{H}^2(J_u).$$

Moreover, let $J := \overline{J_u}$ and define the sets

$$\mathcal{J}_\varepsilon^r := \cup_{\ell=1,2,3} \left\{ \alpha + \varepsilon T_r : \alpha \in \varepsilon \mathbf{Z}^3,\ \alpha + \varepsilon [p_{\ell,r}, p_{\ell,r} + \mathrm{e}_\ell] \cap J \neq \emptyset \right\},$$

and

$$\mathcal{J}_{m,\varepsilon}^r := \cup_{\ell=1,2,3} \left\{ \alpha + \varepsilon T_r : \alpha \in \varepsilon \mathbf{Z}^3,\ \alpha + \varepsilon [p_{\ell,r}, p_{\ell,r} + \mathrm{e}_\ell] \cap J_u^m \neq \emptyset \right\},$$

for $m \in \mathbf{N} \setminus \{0\}$, where the points $p_{\ell,r}$ have been defined in the proof of Proposition 6.1.3. Up to infinitesimal traslations we may assume that $J \cap \varepsilon \mathbf{Z}^3 = \emptyset$ for every $\varepsilon > 0$, then let u_ε be the continuous piecewise affine interpolation of the values $u(\alpha)$ with $\alpha \in \varepsilon \mathbf{Z}^3 \cap \Omega'$. Notice that $u_\varepsilon \in \mathcal{A}_\varepsilon \left(\Omega; \mathbf{R}^N \right)$ and $u_\varepsilon \to u$ strongly in $L^1 \left(\Omega; \mathbf{R}^N \right)$. Denote as usual

$$\mathcal{N}_\varepsilon^r := \left\{ (\alpha + \varepsilon T_r) \in \mathcal{T}_\varepsilon^r : |\nabla u_\varepsilon| |_{(\alpha + \varepsilon T_r)} > \lambda_\varepsilon \right\}.$$

By taking into account Theorem 2.7.6 we have for $\ell = 1, 2, 3$ and for $x \in (\alpha + \varepsilon T_r)$

$$\begin{aligned} &\varepsilon \nabla u_\varepsilon(x) \mathrm{e}_\ell = u(\alpha + \varepsilon p_{\ell,r} + \varepsilon \mathrm{e}_\ell) - u(\alpha + \varepsilon p_{\ell,r}) \\ &= \int_0^\varepsilon \nabla u \left(\alpha + \varepsilon p_{\ell,r} + t \mathrm{e}_\ell \right) \mathrm{e}_\ell \, dt + \sum_{y \in J_u \cap (\alpha + \varepsilon [p_{\ell,r}, p_{\ell,r} + \mathrm{e}_\ell))} [u](y) sgn \left(\langle \nu_u(y), \mathrm{e}_\ell \rangle \right). \end{aligned} \tag{6.1.19}$$

Thus, if $(\alpha + \varepsilon T_r) \in \mathcal{T}_\varepsilon^r \setminus \mathcal{J}_\varepsilon^r$, for any $x \in (\alpha + \varepsilon T_r)$, we have

$$\varepsilon \nabla u_\varepsilon(x) \mathrm{e}_\ell = \int_0^\varepsilon \nabla u \left(\alpha + \varepsilon p_{\ell,r} + t \mathrm{e}_\ell \right) \mathrm{e}_\ell \, dt, \tag{6.1.20}$$

for $\ell = 1, 2, 3$, from which it follows $|\nabla u_\varepsilon| |_{(\alpha + \varepsilon T_r)} \leq \|\nabla u\|_{L^\infty(\Omega'; \mathbf{R}^{N \times 3})}$. Define the vector fields $W_\varepsilon : \Omega \to \mathbf{R}^{N \times 3}$ by

$$W_\varepsilon(x) := \left(\frac{1}{\varepsilon} \int_0^\varepsilon \nabla u \left(\alpha + \varepsilon p_{\ell,r} + t \mathrm{e}_\ell \right) \mathrm{e}_\ell \, dt \right)_{\ell = 1,2,3},$$

if $x \in \cup_{r=1}^6 T_\varepsilon^r$. Then, $W_\varepsilon \to \nabla u$ strongly in $L^p \left(\Omega; \mathbf{R}^{N \times 3} \right)$ and by (6.1.20) there follows

$$\begin{aligned} \limsup_{\varepsilon \to 0^+} \int_{\cup_{r=1}^6 (\mathcal{T}_\varepsilon^r \setminus \mathcal{J}_\varepsilon^r)} \psi_\varepsilon^g \left(\nabla u_\varepsilon(x) \right) \, dx &= \limsup_{\varepsilon \to 0^+} \int_{\cup_{r=1}^6 (\mathcal{T}_\varepsilon^r \setminus \mathcal{J}_\varepsilon^r)} \psi \left(W_\varepsilon(x) \right) \, dx \\ &\leq \lim_{\varepsilon \to 0^+} \int_\Omega \psi \left(W_\varepsilon(x) \right) \, dx = \int_\Omega \psi \left(\nabla u(x) \right) \, dx, \end{aligned} \tag{6.1.21}$$

the last equality holding by the growth condition (6.1.1).

Consider the decomposition $\mathcal{J}^r_\varepsilon = (\mathcal{J}^r_\varepsilon \cap \mathcal{N}^r_\varepsilon) \cup (\mathcal{J}^r_\varepsilon \setminus \mathcal{N}^r_\varepsilon)$, then it follows

$$\begin{aligned}\int_{\cup_{r=1}^6 \mathcal{J}^r_\varepsilon} \psi^g_\varepsilon \left(\nabla u_\varepsilon(x)\right) \, dx &= \sum_{\ell=1}^3 \frac{1}{\varepsilon} \int_{\cup_{r=1}^6 (\mathcal{J}^r_\varepsilon \cap \mathcal{N}^r_\varepsilon)} g_\ell \left(\varepsilon \nabla u_\varepsilon(x) \mathrm{e}_\ell\right) \, dx \\ &+ \int_{\cup_{r=1}^6 (\mathcal{J}^r_\varepsilon \setminus \mathcal{N}^r_\varepsilon)} \psi \left(\nabla u_\varepsilon(x)\right) \, dx. \end{aligned} \tag{6.1.22}$$

Let us estimate separately the two terms in (6.1.22) above.

Let $\overline{B} := \overline{B}_{(0, \|\nabla u\|_{L^\infty(\Omega;\mathbf{R}^{N\times 3})} + 2M\|u\|_{L^\infty(\Omega;\mathbf{R}^N)})}$, then, since $\sup_{\overline{B}} g_\ell < +\infty$, for every $m \in \mathbf{N} \setminus \{0\}$ it follows

$$\begin{aligned}&\sum_{\ell=1}^3 \frac{1}{\varepsilon} \int_{\cup_{r=1}^6 ((\mathcal{J}^r_\varepsilon \setminus \mathcal{J}^r_{m,\varepsilon}) \cap \mathcal{N}^r_\varepsilon)} g_\ell \left(\varepsilon \nabla u_\varepsilon(x) \mathrm{e}_\ell\right) \, dx \\ &\quad \le c \frac{1}{\varepsilon} \mathcal{L}^3 \left(\{ x : \operatorname{dist}(x, J \setminus J^m_u) \le \sqrt{3}\varepsilon \} \right) \le c \mathcal{H}^2 \left(\overline{J_u \setminus J^m_u} \right) + o(1), \end{aligned} \tag{6.1.23}$$

the last term being infinitesimal as $m \to +\infty$.

Moreover, let $\omega_m : [0, +\infty) \to [0, +\infty)$ be the maximum of the moduli of continuity of g_ℓ, $\ell = 1, 2, 3$, on the compact set $\overline{B} \setminus B_{(0, \frac{1}{m})}$, then for ε small enough we get by (6.1.19)

$$\begin{aligned}&\frac{1}{\varepsilon} \int_{\cup_{r=1}^6 (\mathcal{J}^r_{m,\varepsilon} \cap \mathcal{N}^r_\varepsilon)} g_\ell \left(\varepsilon \nabla u_\varepsilon(x) \mathrm{e}_\ell\right) \, dx \\ &\quad \le \frac{1}{\varepsilon} \sum_{r=1}^6 \sum_{\mathcal{J}^r_{m,\varepsilon}} \int_{\alpha + \varepsilon T_r} \left(\sum_{y \in J_u \cap (\alpha + \varepsilon [p_{\ell,r}, p_{\ell,r} + \mathrm{e}_\ell))} g_\ell \left([u](y) \right) \right) dx \\ &\quad + \omega_m \left(\varepsilon \|\nabla u\|_{\infty, \Omega'} \right) \frac{\varepsilon^2}{6} \# \left(\cup_{r=1}^6 \mathcal{J}^r_{m,\varepsilon} \right), \end{aligned} \tag{6.1.24}$$

the last inequality holding by the subadditivity and the symmetry of g_ℓ, $\ell = 1, 2, 3$. It can be proved that, by the regularity assumptions (i)-(iii) on u and the continuity of g_ℓ on $\mathbf{R}^N \setminus \{0\}$, we have

$$\begin{aligned}&\limsup_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{\mathcal{J}^r_{m,\varepsilon}} \left(\sum_{y \in J_u \cap (\alpha + \varepsilon [p_{\ell,r}, p_{\ell,r} + \mathrm{e}_\ell))} g_\ell \left([u](y) \right) \right) dx \\ &\quad \le \frac{1}{6} \int_J g_\ell \left([u] \right) |\langle \nu_u, \mathrm{e}_\ell \rangle| \, d\mathcal{H}^2. \end{aligned} \tag{6.1.25}$$

Hence, by (6.1.24) and (6.1.25) we infer

$$\limsup_{\varepsilon \to 0^+} \sum_{\ell=1}^3 \frac{1}{\varepsilon} \int_{\cup_{r=1}^6 (\mathcal{J}^r_{m,\varepsilon} \cap \mathcal{N}^r_\varepsilon)} g_\ell \left(\varepsilon \nabla u_\varepsilon(x) \mathrm{e}_\ell\right) \, dx \le \int_{J_u} g \left([u], \nu_u \right) \, d\mathcal{H}^2. \tag{6.1.26}$$

By collecting (6.1.23), (6.1.26) and since $\mathcal{H}^2\left(\overline{J_u} \setminus J_u\right) = 0$ by passing to the limit on $m \to +\infty$ we get

$$\limsup_{\varepsilon\to 0^+} \sum_{\ell=1}^{3} \frac{1}{\varepsilon} \int_{\cup_{r=1}^{6}(\mathcal{J}_\varepsilon^r \cap \mathcal{N}_\varepsilon^r)} g_\ell\left(\varepsilon \nabla u_\varepsilon(x) \mathrm{e}_\ell\right) dx \le \int_{J_u} g\left([u], \nu_u\right) d\mathcal{H}^2. \tag{6.1.27}$$

In order to estimate the second term in (6.1.22), notice that by (6.1.1) there holds

$$\int_{\cup_{r=1}^{6}(\mathcal{J}_\varepsilon^r \setminus \mathcal{N}_\varepsilon^r)} \psi\left(\nabla u_\varepsilon(x)\right) dx \le C_2 \frac{\varepsilon^3}{6}\left(1 + \lambda_\varepsilon^p\right) \#\left(\mathcal{J}_\varepsilon^r \setminus \mathcal{N}_\varepsilon^r\right),$$

and the term on the right hand side above is infinitesimal as $\varepsilon \to 0^+$. Indeed, with fixed $m \in \mathbf{N} \setminus \{0\}$, arguing as in (6.1.23) we deduce

$$\limsup_{\varepsilon\to 0^+} \varepsilon^2 \#\left(\mathcal{J}_\varepsilon^r \setminus \mathcal{N}_\varepsilon^r\right) \le \limsup_{\varepsilon\to 0^+} \varepsilon^2 \#\left(\mathcal{J}_\varepsilon^r \setminus \mathcal{J}_{m,\varepsilon}^r\right) \le c\mathcal{H}^2\left(\overline{J_u \setminus J_u^m}\right). \tag{6.1.28}$$

Hence, by assumption $\sup_{\varepsilon>0}\left(\varepsilon \lambda_\varepsilon^p\right) < +\infty$, (6.1.28) and by letting $m \to +\infty$, we have that

$$\limsup_{\varepsilon\to 0^+} \int_{\cup_{r=1}^{6}(\mathcal{J}_\varepsilon^r \setminus \mathcal{N}_\varepsilon^r)} \psi\left(\nabla u_\varepsilon(x)\right) dx = 0 \tag{6.1.29}$$

Finally, by collecting (6.1.21), (6.1.27) and (6.1.29) we get the conclusion, i.e.,

$$\limsup_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon^g(u_\varepsilon) \le \int_\Omega \psi\left(\nabla u(x)\right) dx + \int_{J_u} g\left([u], \nu_u\right) d\mathcal{H}^2.$$

Step 2: Assume $u \in SBV^p \cap L^\infty\left(\Omega; \mathbf{R}^N\right)$. By taking into account the SBV Extension Theorem (see Theorem 2.6.14), with fixed an open and bounded set Ω' with lipschitz boundary and such that $\Omega' \supset\supset \Omega$, there exists a function $\hat{u} \in SBV \cap L^\infty\left(\Omega'; \mathbf{R}^N\right)$ such that $\hat{u}|_\Omega \equiv u$, $\nabla \hat{u} \in L^p\left(\Omega'; \mathbf{R}^{N\times 3}\right)$, $\mathcal{H}^2\left(J_{\hat{u}}\right) < +\infty$ and $\mathcal{H}^2\left(\partial\Omega \cap J_{\hat{u}}\right) = 0$.

By Theorem 2.7.14 there exists a sequence $(u_j) \subset \mathcal{W}\left(\Omega'; \mathbf{R}^N\right)$ such that $u_j \to \hat{u}$ in $L^1\left(\Omega'; \mathbf{R}^N\right)$ and, since the continuity and local boundedness of g, there holds

$$\lim_{j\to+\infty} \int_{\Omega\cap J_{u_j}} g\left([u_j], \nu_{u_j}\right) d\mathcal{H}^2 = \int_{J_u} g\left([u], \nu_u\right) d\mathcal{H}^2.$$

Hence, by *Step 1* and the lower semicontinuity of the upper Γ-limit we conclude.

Step 3: Let $u \in \left(GSBV^p(\Omega)\right)^N$, then for every $T > 0$ the truncated functions $u_T = \left((-T) \wedge \langle u, e_k\rangle \vee T\right)_{k=1,\dots,N}$ are in $SBV^p \cap L^\infty\left(\Omega; \mathbf{R}^N\right)$ and $J_{u_T} \subseteq J_u$.

Moreover, by Theorem 2.7.7 there holds

$$\mathcal{H}^2\left(\{x \in J_u : |u^\pm(x)| = +\infty\}\right) = 0.$$

Hence, $\lim_{T\to+\infty} g_\ell\left([u_T](x)\right) = g_\ell\left([u](x)\right)$ for $\mathcal{H}^2$ a.e. $x \in J_u$, $\ell = 1, 2, 3$. Then by assumption (h4) we may apply the Dominated Convergence Theorem, Step 2 and the lower semicontinuity of the upper Γ-limit to conclude. □

Remark 6.1.6 *Let us point out that the assumption* (h4) *is technical and needed only to recover the limsup estimate on* $(GSBV^p(\Omega))^N \setminus SBV\left(\Omega;\mathbf{R}^N\right)$.

Indeed, assume $u \in SBV\left(\Omega;\mathbf{R}^N\right)$ *to be such that* $\mathcal{F}^g(u) < +\infty$*, by following the notations of Step 3 in Proposition 6.1.5 above and by taking into account (6.1.3) we get*

$$c_1 \leq g\left([u_T], \nu_{u_T}\right) \leq 2c_2\left(1 + |[u]|\right).$$

Moreover, since $u \in SBV\left(\Omega;\mathbf{R}^N\right)$ *implies* $|[u]| \in L^1\left(J_u;\mathcal{H}^2\right)$ *we have*

$$\lim_{T\to+\infty}\int_{J_u} g\left([u_T], \nu_{u_T}\right)\, d\mathcal{H}^2 = \int_{J_u} g\left([u], \nu_u\right)\, d\mathcal{H}^2.$$

Hence, in this case (h4) *is automatically satisfied.*

6.2 Discrete approximations in dimension 2

In this section we treat the two dimensional case. We provide two different approximation results. The first one is the transposition of Theorem 6.1.1 in dimension $n = 2$ for a fixed regular partition of the square $[0,1]^2$. The proof works using the same techniques of Theorem 6.1.1. Actually, the result is independent on the choice of the regular triangulation, indeed one may assign on each square $\alpha + \varepsilon[0,1]^2$, $\alpha \in \varepsilon\mathbf{Z}^2$, one among the two possible partitions (see Figure 6.2 below).

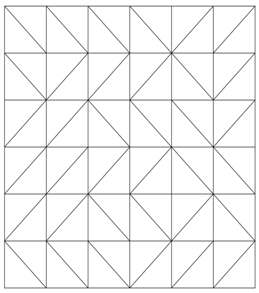

Figure 6.2: a random triangulation of $\mathbf{R}^2$

The second result is a slight variant of Theorem 6.1.1, but the surface term depends heavily on the assigned triangulation (for simplicity we choose the one in Figure 6.3 below).

The anisotropy induced by the model can be computed by means of the function φ of Lemma 6.2.4. To deal with this model more sophisticated tools need to be used.

Let us fix some definitions. Let $S_1 := co\{0, \mathrm{e}_1, \mathrm{e}_2\}$, $S_2 := co\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_1 + \mathrm{e}_2\}$ and define for $r = 1, 2$

$$\mathcal{T}_\varepsilon^r(A) := \{\alpha + \varepsilon S_r : \alpha + \varepsilon S_r \subseteq A,\ \alpha \in \varepsilon\mathbf{Z}^2\},$$

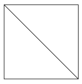

Figure 6.3: regular partition of the square

for $A \in \mathcal{A}(\Omega)$ and Ω a bounded open subset of $\mathbf{R}^2$. In case $A = \Omega$ we will drop the dependence on Ω in the definition above.

In the following we will use the same notations and assumptions (h1)-(h4) of Section 6.1 suitably changed according to the two dimensional setting.

Consider the family of functionals $\mathcal{F}^g_\varepsilon : L^1\left(\Omega;\mathbf{R}^N\right) \to [0,+\infty]$ given by

$$\mathcal{F}^g_\varepsilon(u) := \begin{cases} \displaystyle\int_{\mathcal{T}^1_\varepsilon \cup \mathcal{T}^2_\varepsilon} \psi^g_\varepsilon\left(\nabla u(x)\right) dx & \text{if } u \in \mathcal{A}_\varepsilon\left(\Omega;\mathbf{R}^N\right) \\ +\infty & \text{otherwise.} \end{cases}$$

Then the following result holds.

Theorem 6.2.1 *Let $\Omega \subset \mathbf{R}^2$ be a bounded open set with Lipschitz boundary and assume* (h1)-(h4). *Then $(\mathcal{F}^g_\varepsilon)$ Γ-converges with respect to both the convergence in measure and strong $L^1\left(\Omega;\mathbf{R}^N\right)$ to the functional $\mathcal{F}^g : L^1\left(\Omega;\mathbf{R}^N\right) \to [0,+\infty]$ defined by*

$$\mathcal{F}^g(u) := \begin{cases} \displaystyle\int_\Omega \psi(\nabla u)dx + \int_{J_u} g\left(u^+ - u^-, \nu_u\right) d\mathcal{H}^1 & \textit{if } u \in \left(GSBV^p(\Omega)\right)^N \\ +\infty & \textit{otherwise,} \end{cases}$$

where $g : \mathbf{R}^N \times \mathbf{S}^1 \to [0,+\infty)$ is defined by

$$g(z,\nu) := g_1(z)\left|\langle \nu, \mathrm{e}_1\rangle\right| + g_2(z)\left|\langle \nu, \mathrm{e}_2\rangle\right|.$$

Let us consider now the function $\psi^\beta_\varepsilon : \mathbf{R}^{N\times 2} \to [0,+\infty)$ defined by

$$\psi^\beta_\varepsilon(X) := \begin{cases} \psi(X) & \text{if } |X| \le \lambda_\varepsilon \\ \frac{1}{\varepsilon}\beta & \text{otherwise,} \end{cases}$$

where β is a positive constant. Note that even in case $g_1 = g_2 = \frac{\beta}{2}$ the functions ψ^g_ε, ψ^β_ε are different, since ψ^g_ε takes into account the values $X\mathrm{e}_1$, $X\mathrm{e}_2$ separately. If in the definition of the family $\mathcal{F}^g_\varepsilon$, ψ^g_ε is substituted by ψ^β_ε, we prove the following result for the corresponding family of functionals $(\mathcal{F}^\beta_\varepsilon)$.

Theorem 6.2.2 *Let $\Omega \subset \mathbf{R}^2$ be a bounded open set with Lipschitz boundary and assume* (h1). *Then $(\mathcal{F}^\beta_\varepsilon)$ Γ-converges with respect to both the convergence in measure and strong $L^1\left(\Omega;\mathbf{R}^N\right)$ to the functional $\mathcal{F}^\beta : L^1\left(\Omega;\mathbf{R}^N\right) \to [0,+\infty]$ defined by*

$$\mathcal{F}^\beta(u) := \begin{cases} \displaystyle\int_\Omega \psi(\nabla u)dx + \beta\int_{J_u} \varphi(\nu_u)\, d\mathcal{H}^1 & \textit{if } u \in \left(GSBV^p(\Omega)\right)^N \\ +\infty & \textit{otherwise,} \end{cases} \tag{6.2.1}$$

where $\varphi : \mathbf{S}^1 \to [0,+\infty)$ *is given by*

$$\varphi(\nu) := \begin{cases} |\langle \nu, \mathrm{e}_1\rangle| \vee |\langle \nu, \mathrm{e}_2\rangle| & \textit{if } \langle \nu, \mathrm{e}_1\rangle\langle \nu, \mathrm{e}_2\rangle \geq 0 \\ \\ |\langle \nu, \mathrm{e}_1\rangle| + |\langle \nu, \mathrm{e}_2\rangle| & \textit{if } \langle \nu, \mathrm{e}_1\rangle\langle \nu, \mathrm{e}_2\rangle < 0. \end{cases} \tag{6.2.2}$$

We now prove the lower semicontinuity inequality for the family of functionals $\left(\mathcal{F}_\varepsilon^\beta\right)$. To this aim we need to 'localize' the functionals $\mathcal{F}_\varepsilon^\beta$. For every $A \in \mathcal{A}(\Omega)$ and $u \in \mathcal{A}_\varepsilon\left(\Omega; \mathbf{R}^N\right)$ let

$$\mathcal{F}_\varepsilon^\beta(u, A) := \begin{cases} \displaystyle\int_{\mathcal{T}_\varepsilon^1(A)\cup\mathcal{T}_\varepsilon^2(A)} \psi_\varepsilon^\beta\left(\nabla u(x)\right) dx & \text{if } u \in \mathcal{A}_\varepsilon\left(\Omega; \mathbf{R}^N\right) \\ +\infty & \text{otherwise.} \end{cases}$$

We obtain separate estimates on the bulk and on the surface terms which we 'glue' together by means of Lemma 2.2.4. Besides using the same techniques applied in the proof of Proposition 6.1.3 in the two dimensional case, we will perform an additional construction with profile along the diagonal direction $\mathrm{e}_2 - \mathrm{e}_1$.

Proposition 6.2.3 *Let* $\Omega \subset \mathbf{R}^2$ *be a bounded open set, assume* (h1). *Then, for any* $u \in L^1\left(\Omega; \mathbf{R}^N\right)$,

$$\Gamma(meas)\text{-}\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon^\beta(u) \geq \mathcal{F}^\beta(u).$$

Proof. Let $(u_j) \subset \mathcal{A}_{\varepsilon_j}\left(\Omega; \mathbf{R}^N\right)$ and $u \in L^1\left(\Omega; \mathbf{R}^N\right)$ be such that $u_j \to u$ in measure. By using analogous arguments of those of Proposition 6.1.3 it is possible to show that for every $A \in \mathcal{A}(\Omega)$ the following inequalities hold true:

$$\liminf_j \int_{\cup_{r=1}^2 (\mathcal{T}_\varepsilon^r(A)\setminus\mathcal{N}_{\varepsilon_j}^r)} \psi_\varepsilon^\beta\left(\nabla u(x)\right)\, dx \geq \int_A \psi(\nabla u(x))\, dx \tag{6.2.3}$$

and

$$\liminf_j \beta\varepsilon_j \#\left(\mathcal{T}_\varepsilon^r(A) \cap \mathcal{N}_{\varepsilon_j}^r\right) \geq \beta \int_{A\cap J_u} |\langle \nu_u, \mathrm{e}_\ell\rangle|\, d\mathcal{H}^1, \tag{6.2.4}$$

for $r, \ell = 1, 2$. Thus to conclude it suffices to show that there holds for $r = 1, 2$

$$\liminf_j \beta\varepsilon_j \#\left(\mathcal{T}_\varepsilon^r(A) \cap \mathcal{N}_{\varepsilon_j}^r\right) \geq \beta \int_{A\cap J_u} |\langle \nu_u, \mathrm{e}_2 - \mathrm{e}_1\rangle|\, d\mathcal{H}^1. \tag{6.2.5}$$

Indeed, for $\ell = 1, 2$, let $\left(\delta_h^\ell\right)_h = \mathbf{Q} \cap [0,1]$, $\delta_h^1 + \delta_h^2 \leq 1$, then by using Lemma 2.2.4 with

$$\mu(A) := \liminf_j \mathcal{F}_{\varepsilon_j}^\beta(u_j, A),$$

$$\lambda := \mathcal{L}^2 \llcorner (\Omega \setminus J_u) + \mathcal{H}^1 \llcorner J_u,$$

$$\phi_h(x) := \begin{cases} \psi(\nabla u) & \text{on } \Omega \setminus J_u \\ \\ \beta(\,\delta_h^1|\langle \nu_u, \mathrm{e}_1\rangle| + \delta_h^2|\langle \nu_u, \mathrm{e}_2\rangle| \\ + \left(1 - \delta_h^1 - \delta_h^2\right)|\langle \nu_u, \mathrm{e}_2 - \mathrm{e}_1\rangle|) & \text{on } J_u, \end{cases}$$

the statement follows by noticing that if $x \in J_u$ is such that $\langle \nu_u(x), \mathrm{e}_1\rangle\langle \nu_u(x), \mathrm{e}_2\rangle \geq 0$, then

$$|\langle \nu_u(x), \mathrm{e}_2 - \mathrm{e}_1\rangle| \leq |\langle \nu_u, \mathrm{e}_1\rangle| \vee |\langle \nu_u, \mathrm{e}_2\rangle|,$$

and if $x \in J_u$ is such that $\langle \nu_u(x), \mathrm{e}_1\rangle\langle \nu_u(x), \mathrm{e}_2\rangle < 0$, then

$$|\langle \nu_u(x), \mathrm{e}_2 - \mathrm{e}_1\rangle| = |\langle \nu_u(x), \mathrm{e}_1\rangle| + |\langle \nu_u(x), \mathrm{e}_2\rangle| \geq |\langle \nu_u, \mathrm{e}_1\rangle| \vee |\langle \nu_u, \mathrm{e}_2\rangle|.$$

To prove (6.2.5) we will construct, for $r = 1, 2$, a sequence $(w_j^r) \subset SBV\left(\Omega; \mathbf{R}^N\right)$ with one-dimensional profile in $\mathrm{e}_2 - \mathrm{e}_1$ which is locally pre-compact in SBV in this given direction, but not in general globally in $GSBV$. Suppose $\liminf_j \mathcal{F}^\beta_{\varepsilon_j}(u_j) = \lim_j \mathcal{F}^\beta_{\varepsilon_j}(u_j) < +\infty$, consider the sets of triangles $\mathcal{N}^r_{\varepsilon_j} := \{(\alpha + \varepsilon_j S_r) \in \mathcal{T}^r_{\varepsilon_j} : \ |\nabla u_j||_{(\alpha+\varepsilon_j S_r)} > \lambda_{\varepsilon_j}\}$, for $r = 1, 2$, then we get

$$\sup_j \varepsilon_j \# \left(\mathcal{N}^1_{\varepsilon_j} \cup \mathcal{N}^2_{\varepsilon_j}\right) < +\infty. \tag{6.2.6}$$

Let

$$P_{\varepsilon_j} := \varepsilon_j \left\{x \in \mathbf{R}^2 : x = \lambda(-\mathrm{e}_1) + \mu(\mathrm{e}_2 - \mathrm{e}_1),\ \lambda, \mu \in [0, 1)\right\},$$

and define for $r = 1, 2$ the sequence

$$w_j^r(x) := \begin{cases} u_j(\alpha) & x \in \left(\alpha + P_{\varepsilon_j}\right) \cap \Omega \\ & (\alpha + \varepsilon_j S_r) \in \mathcal{N}^r_{\varepsilon_j} \\ & \\ u_j(\alpha) + \frac{1}{\sqrt{2}} \nabla u_j(x)(\mathrm{e}_2 - \mathrm{e}_1)\langle x - \alpha, \mathrm{e}_2 - \mathrm{e}_1\rangle & \\ & x \in \left(\alpha + P_{\varepsilon_j}\right) \cap \Omega \\ & (\alpha + \varepsilon_j S_r) \notin \mathcal{N}^r_{\varepsilon_j}. \end{cases}$$

Notice the analogy with the construction of $v_j^{\ell,r}$ in Proposition 6.1.3: in this case the cubic cell $\varepsilon_j[0, 1)^2$ is replaced by the slanted one P_{ε_j}.

We have that $(w_j^r) \subset SBV\left(\Omega; \mathbf{R}^N\right)$, $w_j^r \to u$ in measure and, for every $\eta > 0$ and $A \in \mathcal{A}(\Omega)$, by the growth condition of ψ there holds

$$\int_{A_\eta} \left|\frac{\partial w_j^r}{\partial(\mathrm{e}_2 - \mathrm{e}_1)}\right|^p dx \leq c\mathcal{F}^\beta_{\varepsilon_j}(u_j, A), \tag{6.2.7}$$

for some positive constant c. Moreover, since $\nu_{w_j^r} \in \{\mathrm{e}_2, \mathrm{e}_1 + \mathrm{e}_2\}$ $\mathcal{H}^1$ a.e. in $J_{w_j^r}$ we have

$$\int_{A_\eta \cap J_{w_j^r}} \left|\langle \nu_{w_j^r}, \mathrm{e}_2 - \mathrm{e}_1\rangle\right| d\mathcal{H}^1 \leq \varepsilon_j \# \left(\mathcal{T}^r_{\varepsilon_j}(A) \cap \mathcal{N}^r_{\varepsilon_j}\right). \tag{6.2.8}$$

Notice that (6.2.6) together with (6.2.7), (6.2.8) for $A = \Omega$ assure that for $\mathcal{H}^1$ a.e. $y \in \Pi^{\mathrm{e}_2 - \mathrm{e}_1}$, up to subsequences depending on such a fixed y, the slices $\left(\left(w_j^r\right)^{\mathrm{e}_2 - \mathrm{e}_1, y}\right)$ satisfy on $(\Omega_\eta)^{\mathrm{e}_2 - \mathrm{e}_1}_y$

assumption (2.7.6) of Theorem 2.7.10. Thus, by taking into account Fatou's lemma and Theorem 2.7.20, by passing to the inferior limit on $j \to +\infty$ in (6.2.8), we get

$$\frac{\beta}{2} \liminf_j \varepsilon_j \# \left(\mathcal{T}^r_{\varepsilon_j}(A) \cap \mathcal{N}^r_{\varepsilon_j}\right) \geq \frac{\beta}{2} \int_{A_\eta \cap J_u} |\langle \nu_u, \mathrm{e}_2 - \mathrm{e}_1 \rangle| \; d\mathcal{H}^1. \tag{6.2.9}$$

□

The following result will be used in the proof of the limsup inequality. Notice that the ideas and strategy used in the proof are strongly related to the regularity assumptions on the set J.

Lemma 6.2.4 *Let $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2 \setminus \{0\}$ and denote by $\xi^\perp := (-\xi_2, \xi_1)$. Let $\nu \in \mathbf{S}^1$ and let $J \subseteq \Pi^\nu$ be a closed set with $\mathcal{H}^1(J) < +\infty$. Define*

$$J^{\xi,r}_\varepsilon := \left\{\alpha + \varepsilon S^\xi_r : \alpha \in \varepsilon \mathbf{Z}^2, \; \left(\alpha + \varepsilon S^\xi_r\right) \cap J \neq \emptyset\right\},$$

where $S^\xi_1 := co\{0, \xi, \xi^\perp\}$ and $S^\xi_2 := co\{\xi, \xi^\perp, \xi + \xi^\perp\}$. Then, for $r = 1, 2$, we get

$$\limsup_{\varepsilon \to 0^+} \frac{\mathcal{L}^2\left(J^{\xi,r}_\varepsilon\right)}{\varepsilon} \leq \frac{1}{2} \int_J \varphi(\nu) \; d\mathcal{H}^1,$$

where $\varphi : \mathbf{S}^1 \to [0, +\infty)$ is defined as in (6.2.2).

Proof. Let $J^\eta := \{x \in \Pi^\nu : d(x, J) < \eta\}$, then there exists a sequence $(\eta_j) \subseteq (0,1)$ such that $\eta_j \to 0^+$ and $\mathcal{H}^1(J^{\eta_j}) = \mathcal{H}^1\left(\overline{J^{\eta_j}}\right) \to \mathcal{H}^1(J)$. It suffices then to prove the assertion for an open set $A \subseteq \Pi^\nu$ essentially closed, i.e., $\mathcal{H}^1(A) = \mathcal{H}^1\left(\overline{A}\right) < +\infty$.

Let $A = \cup_{s \geq 1} A_s$, where A_s are the connected components of A in Π^ν; since for every $M \in \mathbf{N}$

$$\mathcal{L}^2\left(A^{\xi,r}_\varepsilon\right) \leq \sum_{s=1}^M \mathcal{L}^2\left((A_s)^{\xi,r}_\varepsilon\right) + \mathcal{L}^2\left((\cup_{s \geq M} A_s)^{\xi,r}_\varepsilon\right), \tag{6.2.10}$$

we have that

$$\limsup_{\varepsilon \to 0^+} \frac{\mathcal{L}^2\left(A^{\xi,r}_\varepsilon\right)}{\varepsilon} \leq \sum_{s=1}^M \limsup_{\varepsilon \to 0^+} \frac{\mathcal{L}^2\left((A_s)^{\xi,r}_\varepsilon\right)}{\varepsilon} + 2\sqrt{2}|\xi|\mathcal{H}^1\left(\overline{\cup_{s \geq M} A_s}\right), \tag{6.2.11}$$

being the estimate on the second term in (6.2.10) due to a Minkowski's content argument (see [20]). Since A is supposed to be essentially closed there follows

$$\mathcal{H}^1\left(\cup_{s \geq M} A_s\right) = \mathcal{H}^1\left(\overline{\cup_{s \geq M} A_s}\right).$$

Hence,

$$\sup_{s \geq M} \mathcal{H}^1\left(\overline{\cup_{s \geq M} A_s}\right) = 0,$$

and, by passing to the supremum on M in (6.2.11), we get

$$\limsup_{\varepsilon\to 0+} \frac{\mathcal{L}^2\left(A_\varepsilon^{\xi,r}\right)}{\varepsilon} \leq \sum_{s\geq 1} \limsup_{\varepsilon\to 0+} \frac{\mathcal{L}^2\left((A_s)_\varepsilon^{\xi,r}\right)}{\varepsilon}.$$

Thus, we may assume A to be an open interval in Π^ν and, without loss of generality, we may also assume $\xi = \mathrm{e}_1$. For $\ell = 1, 2$ let us define

$$\mathcal{J}_\varepsilon^{\ell,r}(A) := \left\{\alpha + \varepsilon S_r^\xi : \alpha \in \varepsilon\mathbf{Z}^2,\ \alpha + \varepsilon[p_{\ell,r}, p_{\ell,r} + \mathrm{e}_\ell] \cap A \neq \emptyset\right\},$$

where the points $p_{\ell,r}$ are defined as in the proof of Proposition 6.1.3. Notice that in case $n = 2$ the points $p_{\ell,r}$ satisfy $p_{\ell,1} = 0$ for $\ell = 1, 2$, $p_{1,2} = \mathrm{e}_2$ and $p_{2,2} = \mathrm{e}_1$.

Then it can be easily proved that (see Figure 6.4 (i),(ii))

$$\varepsilon\#\mathcal{J}_\varepsilon^{\ell,r}(A) \leq \mathcal{H}^1(A)|\langle\nu, \mathrm{e}_\ell^\perp\rangle| + 2\varepsilon. \tag{6.2.12}$$

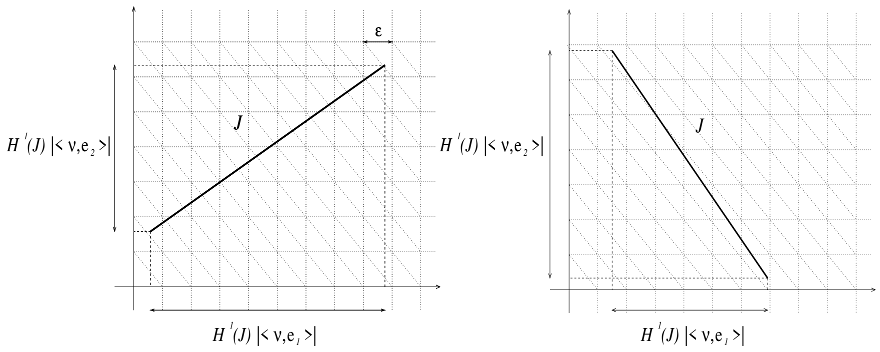

Figure 6.4: (i) case $\langle\nu, \mathrm{e}_1\rangle\langle\nu, \mathrm{e}_2\rangle \leq 0$ (ii) case $\langle\nu, \mathrm{e}_1\rangle\langle\nu, \mathrm{e}_2\rangle > 0$

Note that if $\langle\nu, \mathrm{e}_1\rangle\langle\nu, \mathrm{e}_2\rangle \leq 0$, then $\mathcal{J}_\varepsilon^{1,r} \cap \mathcal{J}_\varepsilon^{2,r} = \emptyset$, while if $\langle\nu, \mathrm{e}_1\rangle\langle\nu, \mathrm{e}_2\rangle > 0$, then either $\mathcal{J}_\varepsilon^{1,r} \subseteq \mathcal{J}_\varepsilon^{2,r}$ or $\mathcal{J}_\varepsilon^{2,r} \subseteq \mathcal{J}_\varepsilon^{1,r}$, according to the cases $|\langle\nu, \mathrm{e}_2\rangle| \geq |\langle\nu, \mathrm{e}_1\rangle|$, $|\langle\nu, \mathrm{e}_1\rangle| \geq |\langle\nu, \mathrm{e}_2\rangle|$.

Hence, we will treat separately the two cases. Assume first that $\langle\nu, \mathrm{e}_1\rangle\langle\nu, \mathrm{e}_2\rangle \leq 0$, then by (6.2.12)

$$\frac{\mathcal{L}^2\left(A_\varepsilon^{\mathrm{e}_1,r}\right)}{\varepsilon} = \frac{\varepsilon}{2}\left(\#\mathcal{J}^{1,r}(A) + \#\mathcal{J}_\varepsilon^{2,r}(A)\right) \leq \frac{1}{2}\mathcal{H}^1(A)\varphi(\nu) + o(1)$$

and the thesis follows.

If, instead, $\langle\nu, \mathrm{e}_1\rangle\langle\nu, \mathrm{e}_2\rangle > 0$ and $|\langle\nu, \mathrm{e}_2\rangle| \geq |\langle\nu, \mathrm{e}_1\rangle|$, then $\mathcal{J}_\varepsilon^{1,r} \subseteq \mathcal{J}_\varepsilon^{2,r}$ and by (6.2.12)

$$\frac{\mathcal{L}^2\left(A_\varepsilon^{\mathrm{e}_1,r}\right)}{\varepsilon} = \frac{\varepsilon}{2}\#\mathcal{J}^{2,r}(A) \leq \frac{1}{2}\mathcal{H}^1(A)\varphi(\nu) + o(1)$$

and the thesis follows. Analogously, we infer the thesis in case $\langle\nu, \mathrm{e}_1\rangle\langle\nu, \mathrm{e}_2\rangle > 0$ and $|\langle\nu, \mathrm{e}_1\rangle| \geq |\langle\nu, \mathrm{e}_2\rangle|$. $\square$

Proposition 6.2.5 *Let $\Omega \subset \mathbf{R}^2$ be a bounded open set with Lipschitz boundary, assume* (h1). *Then, for any $u \in L^1\left(\Omega; \mathbf{R}^N\right)$,*

$$\Gamma\left(L^1\right)\text{-}\limsup_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon^\beta(u) \leq \mathcal{F}^\beta(u).$$

Proof. Let $u \in (GSBV^p(\Omega))^N$ be such that $\mathcal{F}^\beta(u) < +\infty$. Let us first prove the inequality for regular functions. Let Ω' be an open set such that $\Omega' \supset\supset \Omega$ and suppose u regular as in *Step 1* of Proposition 6.1.5. By using analogous notation, the set $\mathcal{J}_\varepsilon^r$ now equals to

$$\cup_{\ell=1,2}\left\{\alpha + \varepsilon S_r : \alpha \in \varepsilon\mathbf{Z}^2,\ \alpha + \varepsilon[p_{\ell,r}, p_{\ell,r} + \mathrm{e}_\ell] \cap J \neq \emptyset\right\},$$

and the points $p_{\ell,r}$, in this case satisfy $p_{\ell,1} = 0$ for $\ell = 1, 2$, $p_{1,2} = \mathrm{e}_2$ and $p_{2,2} = \mathrm{e}_1$.

Hence, we get

$$\begin{aligned}\limsup_{\varepsilon\to 0^+} \int_{\cup_{r=1}^2(\mathcal{T}_\varepsilon^r\setminus\mathcal{J}_\varepsilon^r)} \psi_\varepsilon^\beta\left(\nabla u_\varepsilon(x)\right)\, dx &= \limsup_{\varepsilon\to 0^+} \int_{\cup_{r=1}^2(\mathcal{T}_\varepsilon^r\setminus\mathcal{J}_\varepsilon^r)} \psi_\varepsilon^\beta\left(W_\varepsilon(x)\right)\, dx \\ &\leq \lim_{\varepsilon\to 0^+} \int_\Omega \psi\left(W_\varepsilon(x)\right)\, dx = \int_\Omega \psi\left(\nabla u(x)\right)\, dx. \qquad (6.2.13)\end{aligned}$$

By the very definition of ψ_ε^β and $\mathcal{N}_\varepsilon^r$ we have

$$\begin{aligned}\int_{\cup_{r=1}^2(\mathcal{J}_\varepsilon^r\cap\mathcal{N}_\varepsilon^r)} \psi_\varepsilon^\beta\left(\nabla u_\varepsilon(x)\right)\, dx &= \beta\frac{\varepsilon}{2}\left(\#\left(\mathcal{J}_\varepsilon^1\cap\mathcal{N}_\varepsilon^1\right) + \#\left(\mathcal{J}_\varepsilon^2\cap\mathcal{N}_\varepsilon^2\right)\right) \\ &\leq \frac{\beta}{\varepsilon}\left(\mathcal{L}^2\left(\left(\overline{\Omega\cap J_u}\right)_\varepsilon^{\mathrm{e}_1,1}\right) + \mathcal{L}^2\left(\left(\overline{\Omega\cap J_u}\right)_\varepsilon^{\mathrm{e}_1,2}\right)\right),\end{aligned}$$

and then by Lemma 6.2.4

$$\limsup_{\varepsilon\to 0^+} \int_{\cup_{r=1}^2(\mathcal{J}_\varepsilon^r\cap\mathcal{N}_\varepsilon^r)} \psi_\varepsilon^\beta\left(\nabla u_\varepsilon(x)\right)\, dx \leq \beta \int_{\overline{\Omega\cap J_u}} \varphi(\nu_u)\, d\mathcal{H}^1. \qquad (6.2.14)$$

Moreover, by taking into account (6.1.1), we get

$$\int_{\cup_{r=1}^2(\mathcal{J}_\varepsilon^r\setminus\mathcal{N}_\varepsilon^r)} \psi_\varepsilon^\beta\left(\nabla u_\varepsilon(x)\right)\, dx \leq C_2\frac{\varepsilon^2}{2}\left(1+\lambda_\varepsilon^p\right)\#\left(\cup_{r=1}^2(\mathcal{J}_\varepsilon^r\setminus\mathcal{N}_\varepsilon^r)\right), \qquad (6.2.15)$$

the term on the right hand side above being infinitesimal as proved in (6.1.28) of Proposition 6.1.5. Furthermore, by collecting (6.2.13), (6.2.14) and (6.2.15) we get

$$\limsup_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon^\beta(u_\varepsilon) \leq \int_\Omega \psi\left(\nabla u\right)\, dx + \beta\int_{J_u} \varphi(\nu_u)\, d\mathcal{H}^1.$$

To infer the result for every $u \in L^1\left(\Omega; \mathbf{R}^N\right)$ it suffices to argue like in *Step 2* and *Step 3* of the proof of Proposition 6.1.5. $\square$

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