# QUASISTATIC EVOLUTION OF PERFECTLY PLASTIC SHALLOW SHELLS: A RIGOROUS VARIATIONAL DERIVATION

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Abstract. In this paper we rigorously deduce a quasistatic evolution model for shallow shells by means of Γ-convergence. The starting point of the analysis is the three-dimensional model of Prandlt-Reuss elasto-plasticity. We study the asymptotic behaviour of the solutions, as the thickness of the shell tends to zero. As in the case of plates, the limiting model is genuinely three-dimensional, limiting displacements are of Kirchhoff-Love type, and the stretching and bending components of the stress are coupled in the flow rule and in the stress constraint. However, in contrast with the case of plates, the equilibrium equations are not decoupled, because of the presence of curvature terms. An equivalent formulation of the limiting problem in rate form is also discussed.

## 1. Introduction

In this paper we rigorously derive a quasistatic evolution model for perfectly plastic *shallow shells*. Roughly speaking, a shallow shell is a shell in which the amount of deviation from a plane, measured normally to the plane, is very small. More precisely, we will assume the deviation to be of the same order of the thickness of the shell. Our analysis is thus reminiscent of that developed in [11] for elasto-plastic thin plates, but the adaptation to the nontrivial geometry of the shells gives rise to additional difficulties.

Understanding the relation between lower dimensional theories and their three-dimensional counterparts for thin bodies (such as beams, plates, or shells) is a classical question in mechanics. In recent years this problem has been successfully studied by means of a rigorous approach based on Γ-convergence, both in the stationary case (see, e.g., [3, 35, 36, 37, 38] for nonlinearly elastic beams, [20, 21, 24] for nonlinearly elastic plates, [19, 25, 26, 41] for nonlinearly elastic shells) and in the evolutionary setting (see, e.g., [1, 2] for nonlinear elasto-dynamics, [6, 18] for crack evolution, [11, 27, 28, 30] for elasto-plasticity, [33] for delamination problems).

In this paper we focus on the model of small-strain perfect plasticity. We consider a three-dimensional shallow shell made of a homogenous and isotropic material and occupying the reference configuration $\Sigma_h := \Psi_h(\Omega)$. Here $\Omega := \omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, where $\omega$ is a bounded domain in $\mathbb{R}^2$, and $0 < h \ll 1$. The map $\Psi_h : \overline{\Omega} \to \overline{\Sigma}_h$ is given by

$$\Psi_h(x) := (x', h\theta(x')) + hx_3\nu_{S_h}(x') \quad \text{for every } x = (x', x_3) \in \overline{\Omega},$$

where $\nu_{S_h}$ is the unit normal to the two-dimensional surface

$$S_h := \{(x', h\theta(x')) : x' \in \omega\}$$

and $\theta : \overline{\omega} \to \mathbb{R}$ is a scalar function.

The classical formulation of the quasistatic evolution problem of perfect plasticity in $\Sigma_h$ can be described as follows. At a given time $t$ the unknowns of the problem are the *displacement* $u_h(t) : \Sigma_h \to \mathbb{R}^3$, the *elastic strain* $e_h(t) : \Sigma_h \to \mathbb{M}^{3\times3}_{sym}$, and the *plastic strain* $p_h(t) : \Sigma_h \to \mathbb{M}^{3\times3}_D$. Here $\mathbb{M}^{3\times3}_D$ denotes the space of three-dimensional symmetric matrices with zero trace. The assumption $p_h(t) \in \mathbb{M}^{3\times3}_D$ corresponds to the requirement of volume preserving plastic deformations, which is usual in the description of the plastic behaviour in metals. Given a time-dependent displacement $w_h(t)$ prescribed on a subset $\partial_d\Sigma_h := \Psi_h(\partial_d\Omega)$

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of the lateral boundary of $\Sigma_h$ (where $\partial_d\Omega$ is a portion of the lateral boundary of $\Omega$), and assuming there are no external loads, we look for a triplet $(u_h(t), e_h(t), p_h(t))$ satisfying the following conditions for every $t \in [0, T]$:

(d1) *kinematic admissibility:* $\operatorname{sym} Du_h(t) = e_h(t) + p_h(t)$ in $\Sigma_h$ and $u_h(t) = w_h(t)$ on $\partial_d\Sigma_h$, where $\operatorname{sym} Du_h(t) := \frac{1}{2}(Du_h(t) + Du_h(t)^T)$;

(d2) *constitutive law:* $\sigma_h(t) := \mathbb{C}e_h(t)$ in $\Sigma_h$, where $\sigma_h(t)$ is the stress field at time $t$ and $\mathbb{C}$ is the elasticity tensor;

(d3) *equilibrium equation:* $\operatorname{div} \sigma_h(t) = 0$ in $\Sigma_h$ and $\sigma_h(t)\nu_{\partial\Omega_h} = 0$ on $\partial\Sigma_h \setminus \partial_d\Sigma_h$, where $\nu_{\partial\Sigma_h}$ is the outer unit normal to $\partial\Sigma_h$;

(d4) *stress constraint:* $(\sigma_h(t))_D \in K$ in $\Sigma_h$, where $(\sigma_h)_D$ is the deviatoric part of $\sigma_h$ and $K$ is a given convex and compact set in the space of deviatoric matrices $\mathbb{M}^{3\times3}_D$;

(d5) *flow rule:* $\dot p_h(t)$ belongs to the normal cone to $K$ at $(\sigma_h)_D(t)$ in $\Sigma_h$.

The existence of a solution to (d1)–(d5) was originally established in [39] and revisited in [10] within the variational framework for rate-independent processes developed in [31]. In this approach solutions are found in the space

$$BD(\Sigma_h) \times L^2(\Sigma_h; \mathbb{M}^{3\times3}_{sym}) \times M_b(\Sigma_h \cup \partial_d\Sigma_h, \mathbb{M}^{3\times3}_D),$$

where $BD(\Sigma_h)$ denotes the set of functions with bounded deformation on $\Sigma_h$ and $M_b(\Sigma_h \cup \partial_d\Sigma_h; \mathbb{M}^{3\times3}_D)$ is the set of bounded measures on $\Sigma_h \cup \partial_d\Sigma_h$. This functional setting can be also justified in terms of a relaxation process [5, 34]. The variational formulation of (d1)–(d5) is then written in terms of two conditions: a *global stability* condition and an *energy balance* (see Definition 6.1).

The scope of this article is to characterise the limiting behaviour of a sequence of solutions $(u_h(t), e_h(t), p_h(t))$, as $h$ tends to 0. In our main result (Theorem 6.3) we show the convergence, up to scaling, to a limiting triplet $(u(t), e(t), p(t))$, that is characterised as a solution of the following problem. For every $t \in [0, T]$ the displacement $u(t)$ is of *Kirchhoff-Love* type, that is, there exist $\bar u(t) : \omega \to \mathbb{R}^2$ and $u_3(t) : \omega \to \mathbb{R}$ such that

$$u(t, x) = (\bar u_\alpha(t, x') - x_3\partial_\alpha u_3(t, x'), u_3(t, x')) \quad \text{for } x = (x', x_3) \in \Omega,\ \alpha = 1, 2. \tag{1.1}$$

The physical interpretation of this condition is that straight lines normal to the mid-surface, remain straight and normal after the deformation, within the first order. Furthermore, the following equations (in their strong formulation) are satisfied: for every $t \in [0, T]$

(d1)* *reduced kinematic admissibility:* $u(t)$ is a Kirchhoff-Love displacement and

$$\operatorname{sym} Du(t) + \nabla\theta \odot \nabla u_3(t) = e(t) + p(t) \quad \text{in } \Omega, \qquad u(t) = w(t) \quad \text{on } \partial_d\Omega,$$
$$e_{i3}(t) = p_{i3}(t) = 0 \quad \text{in } \Omega, \qquad i = 1, 2, 3,$$

where $w(t)$ is the limit of $w_h(t)$, up to scaling;

(d2)* *reduced constitutive law:* $\sigma(t) := \mathbb{C}^* e(t)$ in $\Omega$, where $\mathbb{C}^*$ is the reduced elasticity tensor, which is defined through a suitable minimisation formula (see (3.19));

(d3)* *equilibrium equations:* denoting by $\bar\sigma(t)$ and $\hat\sigma(t)$ the zero-th and the first order moments of $\sigma(t)$, respectively (see Definition 3.3), we have

$$\operatorname{div} \bar\sigma(t) = 0 \quad \text{in } \omega, \qquad \tfrac{1}{12}\operatorname{div}\operatorname{div} \hat\sigma(t) + \bar\sigma(t) : D^2\theta = 0 \quad \text{in } \omega,$$

with corresponding Neumann boundary conditions on $\partial\omega \setminus \partial_d\omega$, where $\partial_d\omega$ is the projection of $\partial_d\Omega$ on the plane $\{x_3 = 0\}$;

(d4)* *reduced stress constraint:* $\sigma(t) \in K^*$ in $\Omega$, where $K^* := \partial H^*(0)$ is the subdifferential of the reduced dissipation $H^*$ (whose expression is given in (3.21) through a minimisation formula) at 0;

(d5)* *reduced flow rule:* $\dot p(t)$ belongs to the normal cone to $K^*$ at $\sigma(t)$ in $\Omega$.

As in the three-dimensional case, a variational formulation of (d1)*–(d5)* can be given in terms of a *reduced global stability* condition and of a *reduced energy balance* in the space

$$BD(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times3}_{sym}) \times M_b(\Omega \cup \partial_d\Omega, \mathbb{M}^{3\times3}_{sym}),$$

(see Definition 6.2).

If $\theta \equiv 0$, the model above coincides exactly with that derived in [11] for a thin plate. When $\theta$ is different from 0, curvature effects are taken into account in the limit. In the kinematic admissibility condition the linearised strain $\operatorname{sym} Du(t)$ is augmented by the quantity $\nabla\theta \odot \nabla u_3(t)$, which is due to the contribution of the vertical displacement along the tangential directions to the shallow shell. Also, the curvature tensor of the shallow shell, which is approximately given by the Hessian of $\theta$, contributes to the equilibrium equations. In particular, in contrast with the plate model of [11], here the two equilibrium equations do not decouple.

Since $u(t) \in BD(\Omega)$, the Kirchhoff-Love condition (1.1) implies that $\bar{u}(t) \in BD(\omega)$ and $u_3(t) \in BH(\omega)$, where $BH(\omega)$ is the space of functions with bounded Hessian on $\omega$. Moreover, we have that

$$(\operatorname{sym} Du(t))_{\alpha\beta} = (\operatorname{sym} D\bar{u}(t))_{\alpha\beta} - x_3 \partial^2_{\alpha\beta} u_3(t), \quad \alpha, \beta = 1, 2.$$

We note that the horizontal displacement $\bar{u}(t)$ may exhibit jump discontinuities, while, due to the continuous embedding of $BH(\omega)$ into $C(\overline{\omega})$, the vertical displacement $u_3$ is continuous, with a possibly discontinuous gradient. Since the dependence of $u$ on $x_3$ is affine, the discontinuity set of $u$ (that mechanically describes the so-called slip surfaces) is the vertical surface whose projection on $\omega$ is the union of the jump sets of $\bar{u}$ and of $\nabla u_3$.

Condition (d1)* does not imply, in general, that $e(t)$ and $p(t)$ have an affine dependence on $x_3$. However, they admit the following decomposition:

$$e(t) = \bar{e}(t) + x_3\hat{e}(t) + e_\perp(t), \qquad p(t) = \bar{p}(t) + x_3\hat{p}(t) - e_\perp(t),$$

where the zero order moments $\bar{e}(t) \in L^2(\omega; \mathbb{M}^{2\times2}_{sym})$ and $\bar{p}(t) \in M_b(\omega \cup \partial_d\omega; \mathbb{M}^{2\times2}_{sym})$ satisfy

$$\operatorname{sym} D\bar{u}(t) + \nabla\theta \odot \nabla u_3(t) = \bar{e}(t) + \bar{p}(t) \quad \text{in } \omega,$$

the first order moments $\hat{e}(t) \in L^2(\omega; \mathbb{M}^{2\times2}_{sym})$ and $\hat{p}(t) \in M_b(\omega \cup \partial_d\omega; \mathbb{M}^{2\times2}_{sym})$ satisfy

$$D^2 u_3(t) = -(\hat{e}(t) + \hat{p}(t)) \quad \text{in } \omega,$$

and $e_\perp(t) \in L^2(\Omega; \mathbb{M}^{2\times2}_{sym})$.

Explicit examples in the case of plates (see [12, Section 5]) show that in general $e_\perp(t) \not\equiv 0$. Since this component has a nontrivial dependence on $x_3$, the limiting model has a genuinely three-dimensional nature and cannot be fully reduced to a two-dimensional setting.

We now describe our proof strategy and discuss the additional difficulties due to the nontrivial geometry of the shell. The abstract theory of evolutionary $\Gamma$-convergence for rate-independent processes [32] cannot be directly applied here. Indeed, this theory consists in studying separately the $\Gamma$-convergence of the stored energy functionals and of the dissipation potentials, and in coupling the two $\Gamma$-limits by means of a so-called joint recovery sequence. This approach is not applicable to our case, since in perfect plasticity the stored elastic energy and the plastic dissipation must be considered together to get the right compactness properties. For this reason, to identify the correct limiting energy we first study the $\Gamma$-convergence of the total energy functional, given by the sum of the stored energy with the dissipation potential (see also [7, Chapter 3] for a similar setting in the context of a damage problem). More precisely, we focus on the static case, that is, we consider a boundary displacement independent of time and study the $\Gamma$-limit, as $h \to 0$, of the functional

$$\mathcal{E}_h(v, \eta, q) := \int_{\Sigma_h} Q(\eta(x))\, dx + \int_{\Sigma_h \cup \partial_d \Sigma_h} H\Big(\frac{dq}{d|q|}\Big)\, d|q|$$

defined for all triplets $(v, \eta, q)$ such that $\operatorname{sym} Dv = \eta + q$ in $\Sigma_h$ and satisfying the Dirichlet boundary condition on $\partial_d \Sigma_h$. Here $Q(\eta) := \frac{1}{2}\mathbb{C}\eta : \eta$ and $H$ is the support function of the set $K$.

As usual in dimension reduction problems, a scaling of the admissible triplets $(v, \eta, q)$ is introduced. In particular, the scaled displacement is defined in $\Omega$ as

$$u := R_h^{-1} v \circ \Psi_h,$$

where

$$R_h := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{h} \end{pmatrix}.$$

We prove (Theorem 5.2) that the $\Gamma$-limit of $\mathcal{E}_h$ (rescaled to the domain $\Omega$ and in terms of the scaled triplets) is the functional

$$\mathcal{I}(u, e, p) := \int_\Omega Q^*(e(x))\, dx + \int_{\Omega \cup \partial_d \Omega} H^*\Big(\frac{dp}{d|p|}\Big)\, d|p|$$

defined for all triplets $(u, e, p)$ satisfying the reduced kinematic admissibility condition (qs1)$^*$. Here $Q^*(\eta) := \frac{1}{2}\mathbb{C}^*\eta : \eta$ is the reduced elastic energy density and $H^*$ is the reduced dissipation.

The main difficulty in the proof of this result, compared with [11], is that the scaled displacement $u$ does not belong to $BD(\Omega)$, since we only know that

$$\operatorname{sym}(R_h D u R_h F_h^{-1}) \in M_b(\Omega; \mathbb{M}^{3\times 3}_{sym}), \tag{1.2}$$

where $F_h := D\Psi_h R_h$. Furthermore, we cannot rely on the classical Korn-Poincaré inequality for $BD$ functions, as it was done in [11]. Indeed, the expansion of $F_h^{-1}$ for $h$ small (see Lemma 3.1) yields

$$\begin{aligned}
\operatorname{sym}(R_h D u R_h F_h^{-1})_{\alpha\beta} &= (\operatorname{sym} Du - \partial_3 u \odot \nabla\theta)_{\alpha\beta} + O(h^2)\|u\|_{BV}, \\
\operatorname{sym}(R_h D u R_h F_h^{-1})_{\alpha 3} &= \tfrac{1}{h}\left((\operatorname{sym} Du - \partial_3 u \odot \nabla\theta)_{\alpha 3} + O(h^2)\|u\|_{BV}\right), \\
\operatorname{sym}(R_h D u R_h F_h^{-1})_{33} &= \tfrac{1}{h^2}\left(\partial_3 u_3(1 + O(h^2)) + h^2 \nabla u_3 \cdot \nabla\theta + O(h^4)\|u\|_{BV}\right),
\end{aligned}$$

where $O(h^p)$ is a quantity uniformly bounded by $h^p$ in $\overline{\Omega}$ and $\|\cdot\|_{BV}$ denotes the norm in the space $BV(\Omega)$ of functions with bounded variation on $\Omega$. We note that the remainders are controlled by the $BV$-norm, which is not a priori bounded. Therefore, a bound on $\operatorname{sym}(R_h D u R_h F_h^{-1})$ does not provide, in general, any bound on $\operatorname{sym} Du$. To overcome this difficulty it is convenient to express the scaled displacement in intrinsic curvilinear coordinates, that is, we consider the vectorfield

$$u(h) := (D\Psi_h)^T R_h u.$$

The advantage is that the quantity (1.2), written in these coordinates, has a simpler form; namely, it is related to

$$(R_h \operatorname{sym} Du(h) R_h)_{ij} - \Gamma^k_{ij}(h) u_k(h),$$

where $\Gamma^k_{ij}(h)$ are the scaled Christoffel symbols of $\Sigma_h$ (see Proposition 4.1). In this expression the first term is a rescaled symmetrised gradient, while the second term depends only on the displacement $u(h)$, and not on its derivatives. This allows us to prove, for the vectorfield of curvilinear coordinates, an ad-hoc Korn-Poincaré inequality on shallow shells (Theorem 4.4). In this proof the scaling of the coefficients $\Gamma^k_{ij}(h)$ in terms of $h$ is crucial, and it is a consequence of the shallowness assumption (that is, of the fact that the amount of deviation from a plane is of order $h$).

The Korn-Poincaré inequality on shallow shells is the key ingredient to deduce compactness for sequences of scaled triplets with equibounded energy. We then prove their convergence to limiting triplets $(u, e, p)$ satisfying condition (qs1)$^*$. A delicate point here is to show that the limiting triplets $(u, e, p)$ satisfy the Dirichlet boundary condition, that in the $BD$ framework has to be relaxed as

$$p = (w - u) \odot \nu_{\partial\Omega}\mathcal{H}^2 \quad \text{on } \partial_d\Omega,$$

where $\nu_{\partial\Omega}$ is the outer unit normal to $\partial\Omega$. The idea is to extend the scaled triplets by using the boundary datum $w_h$, to an open set $U$ such that $U \cap \partial\Omega = \partial_d\Omega$. To obtain the necessary bounds it is again convenient to express the scaled triplets in their curvilinear coordinates. Finally, the contruction of a recovery sequence is based on an approximation result (Lemma 3.7), which ensures the density of smooth triplets in the class of kinematically admissible triplets for the reduced problem. This is a technical lemma, whose proof is analogous to that of [11, Theorem 4.7].

Once $\Gamma$-convergence is established in the static case, the proof of the convergence of the quasistatic evolutions is rather standard. We consider the three-dimensional problem and the reduced problem in terms of their variational formulations. To deduce the global stability in the reduced problem, we use as test functions in the three-dimensional problem the recovery sequence provided by the $\Gamma$-convergence result. The energy balance follows from the $\Gamma$-liminf inequality and a standard minimality argument.

Finally, we discuss (Section 6.1) how to write a strong fomulation of the reduced quasistatic evolution problem in the $BD$ framework, and in particular how to give a meaning to the flow rule (d5)$^*$ in this context. To this aim we define an ad-hoc notion of stress-strain duality, in the spirit of [23] and [11].

For an extension of these results to the case of nonzero applied loads we refer to [29].

The plan of the paper is as follows. Section 2 contains some preliminary results. In Section 3 we describe the setting of the problem. In Section 4 we prove the Korn-Poincaré inequality on shallow shells. Section 5 is devoted to the $\Gamma$-convergence of the static functionals, while the convergence of the quasistatic evolutions is studied in Section 6.

## 2. Preliminaries

In this section we collect some mathematical preliminaries that will be used throughout the paper.

In this work *Latin indices*, as $i, j, k$, are assumed to take their values in the set $\{1, 2, 3\}$ and *Greek indices*, as $\alpha, \beta, \gamma$, in the set $\{1, 2\}$. We will adopt the *Einstein summation convention*: for instance, the expression $A_{ij}x_j$ stands for

$$\sum_{j=1}^{3} A_{ij}x_j.$$

*Matrices.* The spaces of $n \times n$ matrices and of $n \times n$ symmetric matrices are denoted by $\mathbb{M}^{n\times n}$ and $\mathbb{M}^{n\times n}_{sym}$, respectively. They are endowed with the euclidean scalar product $\xi : \zeta := \sum_{i,j} \xi_{ij}\zeta_{ij}$. The orthogonal complement of the subspace $\mathbb{R}I_{n\times n}$ spanned by the identity matrix $I_{n\times n}$ is the subspace $\mathbb{M}^{n\times n}_D$ of all symmetric matrices with zero trace. For every $\xi \in \mathbb{M}^{n\times n}_{sym}$ we have the orthogonal decomposition

$$\xi = \xi_D + \frac{1}{n}(\operatorname{tr}\xi)I_{n\times n},$$

where $\xi_D \in \mathbb{M}^{n\times n}_D$ is the deviatoric part of $\xi$. The symmetrised tensor product $a \odot b$ of two vectors $a, b \in \mathbb{R}^n$ is the symmetric matrix with entries $(a \otimes b)_{ij} = \frac{1}{2}(a_ib_j + a_jb_i)$. We denote the determinant of a matrix $A$ by $\det A$ and the cofactor of $A$ by $\operatorname{cof} A$.

*Measures.* The Lebesgue measure on $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$ and the $(n-1)$-dimensional Hausdorff measure by $\mathcal{H}^{n-1}$. Given a Borel set $B \subset \mathbb{R}^n$ and a finite dimensional Hilbert space $X$, $M_b(B; X)$ denotes the space of bounded Borel measures on $B$ with values in $X$, endowed with the norm $\|\mu\|_{M_b} := |\mu|(B)$, where $|\mu| \in M_b(B; \mathbb{R})$ is the variation of the measure $\mu$. For every $\mu \in M_b(B; X)$ we consider the Lebesgue decomposition $\mu = \mu^a + \mu^s$, where $\mu^a$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^n$ and $\mu^s$ is singular with respect to $\mathcal{L}^n$. If $\mu^s = 0$, we always identify $\mu$ with its density with respect to $\mathcal{L}^n$. If the relative topology of $B$ is locally compact, by the Riesz Representation Theorem $M_b(B;X)$ can be identified with the dual of $C_0(B;X)$, which is the space of continuous functions $\varphi : B \to X$ such that the set $\{\varphi \geq \varepsilon\}$ is compact for every $\varepsilon > 0$. The weak$^*$ topology of $M_b(B;X)$ is defined using this duality. The duality between measures and continuous functions, as well as between other pairs of spaces, according to the context, is denoted by $\langle \cdot, \cdot \rangle$.

*Convex functions of measures.* Let $U \subset \mathbb{R}^n$ be an open set and let $\Gamma$ an open subset (in the relative topology) of $\partial U$. Let $X$ be a finite dimensional Hilbert space. For every $\mu \in M_b(U \cup \Gamma; X)$ let $d\mu/d|\mu|$ be the Radon-Nikodým derivative of $\mu$ with respect to its variation $|\mu|$. Let $H_0 : X \to [0, +\infty)$ be a convex and positively one-homogeneous function such that

$$r|\xi| \leq H_0(\xi) \leq R|\xi| \quad \text{for every } \xi \in X,$$

where $r$ and $R$ are two constants, with $0 < r \leq R$. According to the theory of convex functions of measures (see [22]), we introduce the nonnegative Radon measure $H_0(\mu) \in M_b(U \cup \Gamma)$ defined by

$$H_0(\mu)(A) := \int_A H_0\Big(\frac{d\mu}{d|\mu|}\Big)\, d|\mu|$$

for every Borel set $A \subset U \cup \Gamma$. We consider the functional $\mathcal{H}_0 : M_b(U \cup \Gamma; X) \to [0, +\infty)$ defined by

$$\mathcal{H}_0(\mu) := H_0(\mu)(U \cup \Gamma) = \int_{U \cup \Gamma} H_0\Big(\frac{d\mu}{d|\mu|}\Big)\, d|\mu|$$

for every $\mu \in M_b(U \cup \Gamma; X)$. One can prove that $\mathcal{H}_0$ is lower semicontinuous on $M_b(U \cup \Gamma; X)$ with respect to the weak$^*$ convergence (see, e.g., [4, Theorem 2.38]).

*Functions with bounded deformation.* Let $U \subset \mathbb{R}^n$ be an open set. The space $BD(U)$ of functions with bounded deformation is the space of all $u \in L^1(U;\mathbb{R}^n)$, whose symmetric gradient (in the sense of distributions) $\operatorname{sym} Du := \frac{1}{2}(Du + Du^T)$ belongs to the space $M_b(U; \mathbb{M}^{n\times n}_{sym})$. It is easy to see that $BD(U)$ is a Banach space with the norm

$$\|u\|_{BD} := \|u\|_{L^1} + \|\operatorname{sym} Du\|_{M_b}.$$

We say that a sequence $(u_k)$ converges to $u$ weakly$^*$ in $BD(U)$ if $u_k \rightharpoonup u$ weakly in $L^1(U;\mathbb{R}^n)$ and $\operatorname{sym} Du_k \rightharpoonup \operatorname{sym} Du$ weakly$^*$ in $M_b(U;\mathbb{M}^{n\times n}_{sym})$. Every bounded sequence in $BD(U)$ has a weakly$^*$ converging subsequence. If $U$ is bounded and has a Lipschitz boundary, then $BD(U)$ can be continuously embedded in $L^{n/(n-1)}(U;\mathbb{R}^n)$ and compactly embedded in $L^p(U;\mathbb{R}^n)$ for every $p < n/(n-1)$. Moreover, every function $u \in BD(U)$ has a trace, still denoted by $u$, which belongs to $L^1(\partial U;\mathbb{R}^n)$. If $\Gamma$ is a nonempty open subset of $\partial U$, there exists a constant $C > 0$, depending on $U$ and $\Gamma$, such that

$$\|u\|_{BD} \leq C(\|u\|_{L^1(\Gamma)} + \|\operatorname{sym} Du\|_{M_b}) \tag{2.1}$$

for every $u \in BD(U)$. For the general properties of $BD(U)$ we refer to [40].

*Functions with bounded Hessian.* Let $U \subset \mathbb{R}^n$ be an open set. The space $BH(U)$ of functions with bounded Hessian is the space of all functions $u \in W^{1,1}(U)$, whose Hessian $D^2u$ (in the sense of distributions) belongs to $M_b(U;\mathbb{M}^{n\times n}_{sym})$. It is easy to see that $BH(U)$ is a Banach space endowed with the norm

$$\|u\|_{BH} := \|u\|_{W^{1,1}} + \|D^2 u\|_{M_b}.$$

If $U$ has the cone property, then $BH(U)$ coincides with the space of functions in $L^1(U)$ whose Hessian belongs to $M_b(U;\mathbb{M}^{n\times n}_{sym})$. If $U$ is bounded and has a Lipschitz boundary, $BH(U)$ can be embedded into $W^{1,n/(n-1)}(U)$. If $U$ is bounded and has a $C^2$ boundary, then for every function $u \in BH(U)$ one can define the traces of $u$ and $\nabla u$, still denoted by $u$ and $\nabla u$: they satisfy $u \in W^{1,1}(\partial U)$, $\nabla u \in L^1(\partial U;\mathbb{R}^n)$, and $\frac{\partial u}{\partial \tau} = \nabla u \cdot \tau \in L^1(\partial U)$ for every $\tau$ tangent vector to $\partial U$. If in addition $n = 2$, then $BH(U)$ embeds into $C(\overline{U})$, which is the space of continuous functions on $\overline{U}$. Finally, if $U$ has a $C^2$ boundary and $\Gamma$ is a nonempty open subset of $\partial U$, then there exists a constant $C > 0$, depending on $U$ and $\Gamma$, such that

$$\|u\|_{BH} \leq C(\|u\|_{L^1(\Gamma)} + \|\nabla u\|_{L^1(\Gamma)} + \|D^2 u\|_{M_b}) \tag{2.2}$$

for every $u \in BH(U)$. For the general properties of $BH(U)$ we refer to [14].

## 3. Setting of the problem

3.1. **The three-dimensional problem.** We start by describing the setting of the three-dimensional problem.

*The reference configuration.* Let $\omega \subset \mathbb{R}^2$ be a bounded domain with a $C^2$ boundary. Let $\partial_d\omega$ and $\partial_n\omega$ be two disjoint open subsets of $\partial\omega$ such that

$$\overline{\partial_d\omega} \cup \overline{\partial_n\omega} = \partial\omega \qquad \text{and} \qquad \overline{\partial_d\omega} \cap \overline{\partial_n\omega} = \{P_1, P_2\},$$

where $P_1$ and $P_2$ are two points of $\partial\omega$ (here topological notions refer to the relative topology of $\partial\omega$). The set $\partial_d\omega$ is the Dirichlet boundary of $\omega$ and $\partial_n\omega$ is the Neumann boundary. We also consider the set

$$\Omega := \omega \times (-\tfrac{1}{2}, \tfrac{1}{2})$$

and its Dirichlet boundary

$$\partial_d\Omega := \partial_d\omega \times (-\tfrac{1}{2}, \tfrac{1}{2}).$$

Let $\theta \in C^3(\overline{\omega})$. For every $0 < h \ll 1$ we consider the two-dimensional surface

$$S_h := \big\{(x', h\theta(x')) : \ x' \in \omega\big\}.$$

A *shallow shell* of thickness $h$ is a three-dimensional body whose reference configuration is given by the set

$$\Sigma_h := \Psi_h(\Omega),$$

where $\Psi_h : \overline{\Omega} \to \mathbb{R}^3$ is the function

$$\Psi_h(x) := (x', h\theta(x')) + hx_3\nu_{S_h}(x') \quad \text{for every } x = (x', x_3) \in \overline{\Omega} \tag{3.1}$$

and $\nu_{S_h}$ is the unit normal to $S_h$ given by

$$\nu_{S_h}(x') = \frac{1}{\sqrt{1 + h^2|\nabla\theta(x')|^2}}(-h\nabla\theta(x'), 1) \quad \text{for every } x' \in \omega.$$

The Dirichlet boundary of $\Sigma_h$ is given by the set

$$\partial_d\Sigma_h := \Psi_h\left(\partial_d\Omega\right).$$

For every $0 < h \ll 1$ we introduce the diagonal matrix

$$R_h := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{h} \end{pmatrix} \tag{3.2}$$

and we define

$$F_h(x) := D\Psi_h(x)R_h \tag{3.3}$$

for every $x \in \overline{\Omega}$. The elementary properties of the determinant give

$$\det D\Psi_h(x) = h \det F_h(x) \tag{3.4}$$

for every $x \in \overline{\Omega}$. The asymptotic behaviour of $F_h$, as $h \to 0$, is made explicit by the following result.

**Lemma 3.1.** *As $h \to 0$, the following expansions hold:*

$$(F_h)_{\alpha\beta} = \delta_{\alpha\beta} - h^2 x_3 \partial^2_{\alpha\beta}\theta + O(h^3), \qquad (F_h)_{\alpha 3} = -h\partial_\alpha \theta + O(h^3),$$
$$(F_h)_{3\beta} = h\partial_\beta\theta + O(h^3), \qquad (F_h)_{33} = 1 - \tfrac{1}{2}h^2|\nabla\theta|^2 + O(h^3),$$

*where $O(h^3)$ denotes a quantity that is uniformly bounded by $h^3$ in $\overline{\Omega}$. Moreover, $F_h$ is invertible for $h$ small enough and the following expansions hold:*

$$(F_h^{-1})_{\alpha\beta} = \delta_{\alpha\beta} + h^2(x_3\partial^2_{\alpha\beta}\theta - \partial_\alpha\theta\partial_\beta\theta) + O(h^3), \qquad (F_h^{-1})_{\alpha 3} = h\partial_\alpha\theta + O(h^3),$$
$$(F_h^{-1})_{3\beta} = -h\partial_\beta\theta + O(h^3), \qquad (F_h^{-1})_{33} = 1 - \tfrac{1}{2}h^2|\nabla\theta|^2 + O(h^3),$$

*and*

$$\det F_h = 1 + O(h^2).$$

*Proof.* See, e.g., [8, Theorem 3.3-1]. □

*The stored elastic energy.* Let $\mathbb{C}$ be the three-dimensional elasticity tensor, considered as a symmetric positive definite linear operator $\mathbb{C} : \mathbb{M}^{3\times 3}_{sym} \to \mathbb{M}^{3\times 3}_{sym}$, and let $Q : \mathbb{M}^{3\times 3}_{sym} \to [0, +\infty)$ be the quadratic form associated with $\mathbb{C}$, defined by

$$Q(\xi) := \tfrac{1}{2}\mathbb{C}\xi : \xi \quad \text{for every } \xi \in \mathbb{M}^{3\times 3}_{sym}.$$

It turns out that there exists two positive constants $\alpha_{\mathbb{C}}$ and $\beta_{\mathbb{C}}$, with $\alpha_{\mathbb{C}} \leq \beta_{\mathbb{C}}$, such that

$$\alpha_{\mathbb{C}}|\xi|^2 \leq Q(\xi) \leq \beta_{\mathbb{C}}|\xi|^2 \quad \text{for every } \xi \in \mathbb{M}^{3\times 3}_{sym}. \tag{3.5}$$

These inequalities imply that

$$|\mathbb{C}\xi| \leq 2\beta_{\mathbb{C}}|\xi| \quad \text{for every } \xi \in \mathbb{M}^{3\times 3}_{sym}. \tag{3.6}$$

The integral

$$\int_{\Sigma_h} Q(\eta(x))\, dx$$

describes the *stored elastic energy* of a configuration of the shallow shell $\Sigma_h$ with elastic strain $\eta \in L^2(\Sigma_h; \mathbb{M}^{3\times 3}_{sym})$.

*The plastic dissipation.* Let $K$ be a convex and compact set in $\mathbb{M}^{3\times 3}_D$, whose boundary $\partial K$ is interpreted as the yield surface. We assume that there exist two positive constants $r_K$ and $R_K$, with $r_K \leq R_K$, such that

$$B(0, r_K) \subset K \subset B(0, R_K), \tag{3.7}$$

where $B(0,r) := \{\xi \in \mathbb{M}^{3\times 3}_D : |\xi| \leq r\}$. Let $H : \mathbb{M}^{3\times 3}_D \to \mathbb{R}$ be the support function of $K$, that is,

$$H(\xi) := \sup_{\tau \in K} \xi : \tau \quad \text{for every } \xi \in \mathbb{M}^{3\times 3}_D.$$

It is easy to see that $H$ is convex, positively 1-homogeneous, and satisfies the triangle inequality. Moreover, by (3.7) one deduces that

$$r_K|\xi| \leq H(\xi) \leq R_K|\xi| \quad \text{for every } \xi \in \mathbb{M}^{3\times 3}_D. \tag{3.8}$$

From standard convex analysis we also have that the set $K$ coincides with the subdifferential $\partial H(0)$ of $H$ at 0.

Let $q \in M_b(\Sigma_h \cup \partial_d\Sigma_h; \mathbb{M}^{3\times 3}_D)$ and let $dq/d|q|$ be the Radon-Nikodým derivative of $q$ with respect to its variation $|q|$. The integral

$$\int_{\Sigma_h \cup \partial_d \Sigma_h} H\Big(\frac{dq}{d|q|}\Big)\, d|q|$$

describes the *plastic dissipation potential* on a configuration of the shallow shell $\Sigma_h$ with plastic strain $q$. The component of $q$ on $\partial_d\Sigma_h$ accounts for plastic slips at the boundary, which may develop when the prescribed boundary condition on $\partial_d\Sigma_h$ is not attained (see condition (3.9) below).

*Kinematic admissibility and energy.* Given a boundary datum $z \in H^1(\Sigma_h;\mathbb{R}^3)$, we define the class $\mathcal{A}(\Sigma_h, z)$ of admissible displacements and strains, as the set of all triplets $(v,\eta,q) \in BD(\Sigma_h) \times L^2(\Sigma_h;\mathbb{M}^{3\times3}_{sym}) \times M_b(\Sigma_h \cup \partial_d\Sigma_h;\mathbb{M}^{3\times3}_D)$ such that

$$\operatorname{sym} Dv = \eta + q \quad \text{in } \Sigma_h, \qquad q = (z-v)\odot \nu_{\partial\Sigma_h}\mathcal{H}^2 \quad \text{on } \partial_d\Sigma_h, \tag{3.9}$$

where $\nu_{\partial\Sigma_h}$ is the outer unit normal to $\partial\Sigma_h$. We define the total energy as

$$\mathcal{E}_h(v,\eta,q) := \int_{\Sigma_h} Q(\eta(x))\,dx + \int_{\Sigma_h\cup\partial_d\Sigma_h} H\Big(\frac{dq}{d|q|}\Big)\,d|q|$$

for every admissible triplet $(v,\eta,q) \in \mathcal{A}(\Sigma_h, z)$.

3.2. **The rescaled problem.** In this section we introduce a suitable scaling of the admissible triplets and of the total energy.

Let $z \in H^1(\Sigma_h;\mathbb{R}^3)$. To any triplet $(v,\eta,q) \in \mathcal{A}(\Sigma_h, z)$ we associate a triplet $(u,e,p)$ defined as follows:

$$u := R_h^{-1} v \circ \Psi_h, \qquad e := \eta \circ \Psi_h, \qquad p := \frac{1}{\det D\Psi_h}\Psi_h^{\#}(q), \tag{3.10}$$

where $\Psi_h$ and $R_h$ are defined in (3.1) and (3.2), and $\Psi_h^{\#}(q)$ is the pull-back measure of $q$, defined as

$$\int_{\Omega\cup\partial_d\Omega} \varphi : d\Psi_h^{\#}(q) = \int_{\Sigma_h\cup\partial_d\Sigma_h} \varphi\circ\Psi_h^{-1} : dq$$

for every $\varphi \in C_0(\Omega\cup\partial_d\Omega;\mathbb{M}^{3\times3}_D)$. It is clear that $u \in L^1(\Omega;\mathbb{R}^3)$, $e \in L^2(\Omega;\mathbb{M}^{3\times3}_{sym})$, and $p \in M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{3\times3}_D)$. Moreover, we have that

$$\operatorname{sym}(R_h D u R_h F_h^{-1}) \in M_b(\Omega;\mathbb{M}^{3\times3}_{sym}) \tag{3.11}$$

and

$$\int_\Omega \varphi : d\operatorname{sym}(R_h D u R_h F_h^{-1}) = \int_{\Sigma_h} (\det D\Psi_h^{-1})\,\varphi\circ\Psi_h^{-1} : d(\operatorname{sym} Dv) \tag{3.12}$$

for every $\varphi \in C_0(\Omega;\mathbb{M}^{3\times3}_{sym})$. Indeed, if $v$ is smooth, then by direct computations and by (3.3) we obtain

$$(\operatorname{sym} Dv)\circ\Psi_h = \operatorname{sym}(R_h D u R_h F_h^{-1}),$$

so that (3.11) and (3.12) follow by an approximation argument.

We also introduce the rescaled boundary datum $w \in H^1(\Omega;\mathbb{R}^3)$, defined as

$$w := R_h^{-1} z\circ\Psi_h$$

and we note that

$$\begin{aligned}\int_{\partial_d\Sigma_h} \varphi\circ\Psi_h^{-1} : dq &= \int_{\partial_d\Sigma_h} \varphi\circ\Psi_h^{-1} : ((z-v)\odot\nu_{\partial\Sigma_h})\,d\mathcal{H}^2 \\ &= h\int_{\partial_d\Omega} \varphi : (R_h(w-u)\odot(\operatorname{cof} F_h)R_h\nu_{\partial\Omega})\,d\mathcal{H}^2 \end{aligned} \tag{3.13}$$

for every $\varphi \in C(\overline{\Omega};\mathbb{M}^{3\times3}_{sym})$, where $\nu_{\partial\Omega}$ is the outer unit normal to $\partial\Omega$.

Since $(v,\eta,q) \in \mathcal{A}(\Sigma_h, z)$, we deduce by (3.9), (3.10), (3.12), and (3.13), that

$$\begin{aligned} \operatorname{sym}(R_h D u R_h F_h^{-1}) = e + p \quad \text{in } \Omega, \\ p = \frac{1}{\det F_h} R_h(w-u)\odot(\operatorname{cof} F_h)R_h\nu_{\partial\Omega}\mathcal{H}^2 \quad \text{on } \partial_d\Omega. \end{aligned} \tag{3.14}$$

Motivated by the results above, we introduce the space

$$V_h(\Omega) := \left\{u \in L^1(\Omega;\mathbb{R}^3) :\ \operatorname{sym}(R_h D u R_h F_h^{-1}) \in M_b(\Omega;\mathbb{M}^{3\times3}_{sym})\right\}.$$

For every $w \in H^1(\Omega;\mathbb{R}^3)$ we denote by $\mathcal{A}_h(\Omega, w)$ the class of all triplets

$$(u,e,p) \in V_h(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times3}_{sym}) \times M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{3\times3}_D)$$

satisfying (3.14). According to the scaling (3.10) and to (3.4), the total energy can be written as

$$\mathcal{E}_h(v,\eta,q) = h\int_\Omega Q(e(x))\det F_h(x)\,dx + h\,\mathcal{H}_h(p),$$

where

$$\mathcal{H}_h(p) := \int_{\Omega\cup\partial_d\Omega} H\left(\frac{dp}{d|p|}\right)\det F_h\,d|p|.$$

We thus define the scaled energy as

$$\mathcal{I}_h(u,e,p) := \int_\Omega Q(e(x))\det F_h(x)\,dx + \mathcal{H}_h(p)$$

for every $(u,e,p)\in\mathcal{A}_h(\Omega,w)$. This will be the starting point of the asymptotic analysis of Sections 5 and 6.

3.3. **The limiting problem.** In this section we introduce the limiting functional, that describes the asymptotic behaviour of the rescaled energy $\mathcal{I}_h$, as $h$ tends to 0.

*The reduced stored elastic energy.* Let $\mathbb{M}:\mathbb{M}^{2\times2}_{sym}\to\mathbb{M}^{3\times3}_{sym}$ be the operator given by

$$\mathbb{M}\xi := \begin{pmatrix}\xi_{11} & \xi_{12} & \lambda_1(\xi)\\ \xi_{12} & \xi_{22} & \lambda_2(\xi)\\ \lambda_1(\xi) & \lambda_2(\xi) & \lambda_3(\xi)\end{pmatrix}\quad\text{for every } \xi\in\mathbb{M}^{2\times2}_{sym}, \tag{3.15}$$

where the triplet $(\lambda_1(\xi),\lambda_2(\xi),\lambda_3(\xi))$ is the unique solution of the minimum problem

$$\min_{\lambda_i\in\mathbb{R}} Q\begin{pmatrix}\xi_{11} & \xi_{12} & \lambda_1\\ \xi_{12} & \xi_{22} & \lambda_2\\ \lambda_1 & \lambda_2 & \lambda_3\end{pmatrix}.$$

We observe that $(\lambda_1(\xi),\lambda_2(\xi),\lambda_3(\xi))$ can be characterised as the unique solution of the linear system

$$\mathbb{C}\mathbb{M}\xi : \begin{pmatrix}0 & 0 & \zeta_1\\ 0 & 0 & \zeta_2\\ \zeta_1 & \zeta_2 & \zeta_3\end{pmatrix} = 0 \tag{3.16}$$

for every $\zeta_i\in\mathbb{R}$. This implies that $\mathbb{M}$ is a linear map and

$$(\mathbb{C}\mathbb{M}\xi)_{i3} = (\mathbb{C}\mathbb{M}\xi)_{3i} = 0. \tag{3.17}$$

Let $Q^*:\mathbb{M}^{2\times2}_{sym}\to\mathbb{R}$ be the quadratic form given by

$$Q^*(\xi) := Q(\mathbb{M}\xi)\quad\text{for every } \xi\in\mathbb{M}^{2\times2}_{sym}. \tag{3.18}$$

It follows from (3.5) that

$$\alpha_{\mathbb{C}}|\xi|^2 \le Q^*(\xi) \le \beta_{\mathbb{C}}|\xi|^2\quad\text{for every } \xi\in\mathbb{M}^{2\times2}_{sym}.$$

We define the reduced elasticity tensor as the linear operator $\mathbb{C}^*:\mathbb{M}^{2\times2}_{sym}\to\mathbb{M}^{3\times3}_{sym}$ given by

$$\mathbb{C}^*\xi := \mathbb{C}\mathbb{M}\xi\quad\text{for every } \xi\in\mathbb{M}^{2\times2}_{sym}. \tag{3.19}$$

Note that we can always identify $\mathbb{C}^*\xi$ with an element of $\mathbb{M}^{2\times2}_{sym}$ in view of (3.17). Moreover, by (3.16) we have

$$\mathbb{C}^*\xi:\zeta = \mathbb{C}^*\xi : \begin{pmatrix}\zeta_{11} & \zeta_{12} & 0\\ \zeta_{12} & \zeta_{22} & 0\\ 0 & 0 & 0\end{pmatrix}\quad\text{for every } \xi\in\mathbb{M}^{2\times2}_{sym},\ \zeta\in\mathbb{M}^{3\times3}_{sym}. \tag{3.20}$$

This implies that

$$Q^*(\xi) = \frac{1}{2}\mathbb{C}^*\xi : \begin{pmatrix}\xi_{11} & \xi_{12} & 0\\ \xi_{12} & \xi_{22} & 0\\ 0 & 0 & 0\end{pmatrix}\quad\text{for every } \xi\in\mathbb{M}^{2\times2}_{sym}.$$

Finally, we introduce the functional $\mathcal{Q}^* : L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}) \to [0,+\infty)$, defined as

$$\mathcal{Q}^*(e) := \int_\Omega Q^*(e(x))\,dx$$

for every $e \in L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$.

*The reduced plastic dissipation.* In the reduced problem the plastic dissipation potential is given by the function $H^* : \mathbb{M}^{2\times 2}_{sym} \to [0,+\infty)$, defined as

$$H^*(\xi) := \min_{\lambda_i\in\mathbb{R}} H\begin{pmatrix} \xi_{11} & \xi_{12} & \lambda_1 \\ \xi_{12} & \xi_{22} & \lambda_2 \\ \lambda_1 & \lambda_2 & -(\xi_{11}+\xi_{22}) \end{pmatrix} \tag{3.21}$$

for every $\xi \in \mathbb{M}^{2\times 2}_{sym}$. From the properties of $H$ it follows that $H^*$ is convex, positively 1-homogeneous, and satisfies

$$r_K|\xi| \le H^*(\xi) \le \sqrt{3}R_K|\xi| \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym}.$$

The set $K^* := \partial H^*(0)$ represents the set of admissible stresses in the reduced problem and can be characterised as follows:

$$\xi \in K^* \quad \Leftrightarrow \quad \begin{pmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{12} & \xi_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{3}(\operatorname{tr}\xi)I_{3\times 3} \in K,$$

(see [11, Section 3.2]). For every $p \in M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{2\times 2}_{sym})$ we define the functional

$$\mathcal{H}^*(p) := \int_{\Omega\cup\partial_d\Omega} H^*\Big(\frac{dp}{d|p|}\Big)\,d|p|.$$

*Generalised Kirchhoff-Love triplets and limiting energy.* We consider the set $KL(\Omega)$ of Kirchhoff-Love displacements, defined as

$$KL(\Omega) := \{u \in BD(\Omega) : \ (\operatorname{sym} Du)_{i3} = 0\}.$$

We note that $u \in KL(\Omega)$ if and only if $u_3 \in BH(\omega)$ and there exists $\bar{u} \in BD(\omega)$ such that

$$u_\alpha(x) = \bar{u}_\alpha(x') - x_3\partial_\alpha u_3(x')$$

for a.e. $x = (x', x_3) \in \Omega$. We call $\bar{u}$, $u_3$ the *Kirchhoff-Love components* of $u$.

For every $u \in KL(\Omega)$ we define the measure

$$E^*u := \operatorname{sym} Du + \nabla\theta \odot \nabla u_3. \tag{3.22}$$

Given a prescribed displacement $w \in H^1(\Omega;\mathbb{R}^3)\cap KL(\Omega)$, the set $\mathcal{A}_{\rm gKL}(w)$ of *generalised Kirchhoff-Love triplets* is defined as the class of all triplets

$$(u,e,p) \in KL(\Omega)\times L^2(\Omega;\mathbb{M}^{3\times 3}_{sym})\times M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{3\times 3}_{sym})$$

such that

$$\begin{aligned} E^*u = e + p \quad \text{in } \Omega, \qquad & p = (w-u)\odot\nu_{\partial\Omega}\mathcal{H}^2 \quad \text{on } \partial_d\Omega, \\ e_{i3} = 0 \quad \text{in } \Omega, \qquad & p_{i3} = 0 \quad \text{in } \Omega\cup\partial_d\Omega. \end{aligned} \tag{3.23}$$

We observe that the class $\mathcal{A}_{\rm gKL}(w)$ is nonempty since it contains $(w, E^*w, 0)$.

Because of the last two conditions in (3.23), if $(u,e,p) \in \mathcal{A}_{\rm gKL}(w)$, $e$ can be always identified with a function in $L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$ and $p$ with a measure in $M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{2\times 2}_{sym})$. In the following we will tacitly make these identifications.

Finally, the limiting energy will be given by the functional $\mathcal{I} : \mathcal{A}_{\rm gKL}(w) \to [0,+\infty)$, defined as

$$\mathcal{I}(u,e,p) := \mathcal{Q}^*(e) + \mathcal{H}^*(p)$$

for every $(u,e,p) \in \mathcal{A}_{\rm gKL}(w)$.

We conclude this section by collecting some properties of the class $\mathcal{A}_{\rm gKL}(w)$. The following closure property holds.

**Lemma 3.2.** *Let $(w^k)$ be a sequence in $H^1(\Omega;\mathbb{R}^3)\cap KL(\Omega)$ and let $(u^k,e^k,p^k)$ be a sequence of triplets such that $(u^k,e^k,p^k)\in\mathcal{A}_{\rm gKL}(w^k)$ for every $k$. Assume that $u^k\rightharpoonup u$ weakly* in $BD(\Omega)$, $e^k\rightharpoonup e$ weakly in $L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$, $p^k\rightharpoonup p$ weakly* in $M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{2\times2}_{sym})$, and $w^k\rightharpoonup w$ weakly in $H^1(\Omega;\mathbb{R}^3)$, as $k\to\infty$. Then $(u,e,p)\in\mathcal{A}_{\rm gKL}(w)$.*

*Proof.* The result easily follows by adapting the proof of [10, Lemma 2.1]. □

A characterisation of $\mathcal{A}_{\rm gKL}(w)$ can be given in terms of moments, whose definition is recalled below.

**Definition 3.3.** Let $f\in L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$. We denote by $\bar f$, $\hat f\in L^2(\omega;\mathbb{M}^{2\times2}_{sym})$ and by $f_\perp\in L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$ the following orthogonal components (in the sense of $L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$) of $f$:

$$\bar f(x'):=\int_{-\frac12}^{\frac12} f(x',x_3)\,dx_3,\qquad \hat f(x'):=12\int_{-\frac12}^{\frac12} x_3 f(x',x_3)\,dx_3$$

for a.e. $x'\in\omega$, and

$$f_\perp(x):=f(x)-\bar f(x')-x_3\hat f(x')$$

for a.e. $x\in\Omega$. We call $\bar f$ the *zeroth order moment* of $f$ and $\hat f$ the *first order moment* of $f$.

**Definition 3.4.** Let $q\in M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{2\times2}_{sym})$. We denote by $\bar q$, $\hat q\in M_b(\omega\cup\partial_d\omega;\mathbb{M}^{2\times2}_{sym})$ and by $q_\perp\in M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{2\times2}_{sym})$ the following measures:

$$\int_{\omega\cup\partial_d\omega}\varphi:d\bar q:=\int_{\Omega\cup\partial_d\Omega}\varphi:dq,\qquad \int_{\omega\cup\partial_d\omega}\varphi:d\hat q:=12\int_{\Omega\cup\partial_d\Omega}x_3\varphi:dq$$

for every $\varphi\in C_0(\omega\cup\partial_d\omega;\mathbb{M}^{2\times2}_{sym})$, and

$$q_\perp:=q-\bar q\otimes\mathcal{L}^1-\hat q\otimes x_3\mathcal{L}^1,$$

where $\otimes$ denotes the usual product of measures. We call $\bar q$ the *zeroth order moment* of $q$ and $\hat q$ the *first order moment* of $q$.

With these definitions at hand one can prove the following result.

**Proposition 3.5.** *Let $w\in H^1(\Omega;\mathbb{R}^3)\cap KL(\Omega)$ and let $(u,e,p)\in KL(\Omega)\times L^2(\Omega;\mathbb{M}^{2\times2}_{sym})\times M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{2\times2}_{sym})$. Then $(u,e,p)\in\mathcal{A}_{\rm gKL}(w)$ if and only if the following three conditions are satisfied:*

(i) $\operatorname{sym}D\bar u+\nabla\theta\odot\nabla u_3=\bar e+\bar p$ *in* $\omega$ *and* $\bar p=(\bar w-\bar u)\odot\nu_{\partial\omega}\mathcal{H}^1$ *on* $\partial_d\omega$;

(ii) $D^2u_3=-(\hat e+\hat p)$ *in* $\omega$, $u_3=w_3$ *on* $\partial_d\omega$, *and* $\hat p=(\nabla u_3-\nabla w_3)\odot\nu_{\partial\omega}\mathcal{H}^1$ *on* $\partial_d\omega$;

(iii) $p_\perp=-e_\perp$ *in* $\Omega$ *and* $p_\perp=0$ *on* $\partial_d\Omega$,

*where $\nu_{\partial\omega}$ is the outer unit normal to $\partial\omega$.*

*Proof.* The proof is analogous to that of [11, Proposition 4.3]. □

Finally, we prove an approximation result in terms of smooth triplets. First of all, we give a definition.

**Definition 3.6.** The space $L^2_{\infty,c}(\Omega;\mathbb{M}^{2\times2}_{sym})$ is the set of all $p\in L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$ satisfying:

(i) $\partial^i_\alpha\partial^j_\beta p\in L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$ for every $i,j\in\mathbb{N}\cup\{0\}$;

(ii) there exists a set $U\subset\subset\omega\cup\partial_n\omega$ such that $p=0$ a.e. on $\omega\setminus\overline{U}\times\left(-\frac12,\frac12\right)$.

We note that functions in $L^2_{\infty,c}(\Omega;\mathbb{M}^{2\times2}_{sym})$ have a smooth dependence on the variable $x'$; namely, if $p\in L^2_{\infty,c}(\Omega;\mathbb{M}^{2\times2}_{sym})$, then $p(\cdot,x_3)\in C^\infty_c(\omega\cup\partial_n\omega;\mathbb{M}^{2\times2}_{sym})$ for a.e. $x_3\in\left(-\frac12,\frac12\right)$.

**Lemma 3.7.** *Let $w\in H^1(\Omega;\mathbb{R}^3)\cap KL(\Omega)$ and let $(u,e,p)\in\mathcal{A}_{\rm gKL}(w)$. Then there exists a sequence of triplets*

$$(u^k,e^k,p^k)\in\left(H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathbb{M}^{2\times2}_{sym})\times L^2_{\infty,c}(\Omega;\mathbb{M}^{2\times2}_{sym})\right)\cap\mathcal{A}_{\rm gKL}(w)$$

*such that $u^k\rightharpoonup u$ weakly* in $BD(\Omega)$, $e^k\to e$ strongly in $L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$, $p^k\rightharpoonup p$ weakly* in $M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{2\times2}_{sym})$, and $\|p^k\|_{M_b}\to\|p\|_{M_b}$, as $k\to\infty$.*

*Proof.* The proof is analogous to [11, Lemma 4.5] and [11, Theorem 4.7]. The only difference is in the definition of the zeroth order moment of $e^k$, that we detail below. Following the same notation as in [11], we replace $\bar{e}^k$ on page 629 with

$$\bar{e}^k := \sum_{j=1}^{\infty} \big((\varphi_j \bar{e}) * \rho_{\delta_j} + (\nabla\varphi_j \odot \bar{u}) * \rho_{\delta_j} - (\varphi_j \nabla\theta \odot \nabla u_3) * \rho_{\delta_j}\big) + \nabla\theta \odot \sum_{j=1}^{\infty} \big((\varphi_j \nabla u_3 + \nabla\varphi_j u_3) * \rho_{\delta_j}\big),$$

and $\bar{e}^{\delta,1}$ on page 632 with

$$\bar{e}^{\delta,1} = (\bar{u} \circ \phi_\delta) \circ \nabla\varphi_1 + \varphi_1 \operatorname{sym}((\bar{e} \circ \phi_\delta) D\phi_\delta) - \varphi_1 \operatorname{sym}\big(((\nabla u_3 \odot \nabla\theta) \circ \phi_\delta) D\phi_\delta\big) + (u_3 \circ \phi_\delta)\nabla\theta \odot \nabla\varphi_1 + \varphi_1 \nabla\theta \odot (D\phi_\delta)^T (\nabla u_3 \circ \phi_\delta).$$

Using this definition, equation (4.38) in [11] is replaced by

$$\bar{e}^{\delta,1} \to \bar{u} \odot \nabla\varphi_1 + \varphi_1 \bar{e} + u_3 \nabla\varphi_1 \odot \nabla\theta \quad \text{strongly in } L^2(\omega; \mathbb{M}^{2\times 2}_{sym}).$$

With respect to the argument on page 633 of [11], we replace $e^\delta$ with

$$e^\delta := e - (\varphi_1 + \varphi_2)(\bar{e} + x_3 \hat{e}) + \bar{e}^{\delta,1} + \bar{e}^{\delta,2} + x_3(\hat{e}^{\delta,1} + \hat{e}^{\delta,2}) + \sum_{\alpha=1}^{2} (-\bar{u} \odot \nabla\varphi_\alpha - u_3 \nabla\theta \odot \nabla\varphi_\alpha + x_3 u_3 D^2\varphi_\alpha + 2x_3 \nabla\varphi_\alpha \odot \nabla u_3)$$

and formula (4.55) on page 634 with

$$\bar{e}^k := \sum_{i=1}^{m} (\varphi_i \bar{e}) \circ \tau_{i,k} + \varphi_0 \bar{e} + \sum_{i=1}^{m} (\nabla\varphi_i \odot \bar{u}) \circ \tau_{i,k} + \nabla\varphi_0 \odot \bar{u} - \sum_{i=1}^{m} (\varphi_i \nabla\theta \odot \nabla u_3) \circ \tau_{i,k} + \nabla\theta \odot \sum_{i=1}^{m} \big((u_3 \nabla\varphi_i) \circ \tau_{i,k} + (\nabla\varphi_i u_3) \circ \tau_{i,k})\big) + u_3 \nabla\theta \odot \nabla\varphi_0.$$

By implementing these changes the same construction as in [11, Lemma 4.5] and [11, Theorem 4.7] provides the desired approximating sequence. ☐

## 4. A Korn-Poincaré inequality on shallow shells

In this section we prove an *ad hoc* version of the Korn-Poincaré inequality for shallow shells. To this purpose it is useful to express displacements in intrinsic curvilinear coordinates. More precisely, to any displacement $u : \Omega \to \mathbb{R}^3$ we associate the vectorfield $u(h) : \Omega \to \mathbb{R}^3$ defined by

$$u(h) := (D\Psi_h)^T R_h u, \tag{4.1}$$

whose components are the scaled curvilinear coordinates of $u$ with respect to the contravariant basis of $\Sigma_h$. In particular, from (3.3) and (4.1) it follows immediately that

$$R_h u(h) = F_h^T R_h u. \tag{4.2}$$

In the following proposition we express the strain in terms of the curvilinear coordinates.

**Proposition 4.1.** *Let $0 < h \ll 1$. Let $u \in V_h(\Omega)$ and let $u(h)$ be defined by* (4.1)*. Then $u(h) \in BD(\Omega)$ and the following equality holds:*

$$F_h^T \operatorname{sym}(R_h Du R_h F_h^{-1}) F_h = E(h, u(h)), \tag{4.3}$$

*where*

$$E(h, u(h))_{ij} := (R_h(\operatorname{sym} Du(h))R_h)_{ij} - \Gamma^k_{ij}(h) u_k(h) \tag{4.4}$$

*and the quantities* $\Gamma^k_{ij}(h)$ *are given by*

$$\begin{aligned}\Gamma^\sigma_{\alpha i}(h)=\Gamma^\sigma_{i\alpha}(h):=(\partial_\alpha(F_h^T)F_h^{-T})_{i\sigma},\qquad \Gamma^3_{\alpha i}(h)=\Gamma^3_{i\alpha}(h):=\frac{1}{h}(\partial_\alpha(F_h^T)F_h^{-T})_{i3},\\ \Gamma^\alpha_{33}(h):=\frac{1}{h}(\partial_3(F_h^T)F_h^{-T})_{3\alpha},\qquad \Gamma^3_{33}(h):=\frac{1}{h^2}(\partial_3(F_h^T)F_h^{-T})_{33}.\end{aligned}\tag{4.5}$$

*Proof.* Assume $u$ smooth. Differentiating (4.2) yields

$$(R_hDu)_{ij}=(F_h^{-T}R_hDu(h))_{ij}+\partial_j(F_h^{-T})_{ik}(R_h)_{kl}u(h)_l.$$

This implies that

$$\begin{aligned}&\operatorname{sym}(R_hDuR_hF_h^{-1})_{ij}=\operatorname{sym}(F_h^{-T}R_hDu(h)R_hF_h^{-1})_{ij}\\ &\quad+\frac{1}{2}\Big(\partial_m(F_h^{-T})_{ik}(R_h)_{kl}u(h)_l(R_h)_{mn}(F_h^{-1})_{nj}+\partial_p(F_h^{-T})_{jk}(R_h)_{kr}u(h)_r(R_h)_{pq}(F_h^{-1})_{qi}\Big).\end{aligned}$$

Using the equality

$$F_h^T\partial_m(F_h^{-T})=-\partial_m(F_h^T)F_h^{-T},$$

direct computations lead to

$$\begin{aligned}&\big(F_h^T\operatorname{sym}(R_hDuR_hF_h^{-1})F_h\big)_{ij}=\operatorname{sym}(R_hDu(h)R_h)_{ij}\\ &\qquad+\frac{1}{2}\Big(\big(\partial_l(F_h^T)F_h^{-T}R_h\big)_{ik}(R_h)_{lj}+\big(\partial_m(F_h^T)F_h^{-T}R_h\big)_{jk}(R_h)_{mi}\Big)u_k(h).\end{aligned}$$

To deduce (4.3) it remains to show that, if we set

$$2\Gamma^k_{ij}(h):=\big(\partial_l(F_h^T)F_h^{-T}R_h\big)_{ik}(R_h)_{lj}+\big(\partial_m(F_h^T)F_h^{-T}R_h\big)_{jk}(R_h)_{mi},$$

then $\Gamma^k_{ij}(h)$ satisfies (4.5). By (3.2) and (3.3) we have that

$$\partial_\alpha(Fe_\beta)=\partial_\beta(Fe_\alpha),\qquad \partial_\alpha(Fe_3)=\frac{1}{h}\partial_3(Fe_\alpha).$$

Using these equalities and again (3.2), we obtain

$$2\Gamma^\sigma_{\alpha\beta}(h)=\big(\partial_\beta(F_h^T)F_h^{-T}\big)_{\alpha\sigma}+\big(\partial_\alpha(F_h^T)F_h^{-T}\big)_{\beta\sigma}=2\big(\partial_\beta(F_h^T)F_h^{-T}\big)_{\alpha\sigma}$$

and

$$2\Gamma^\sigma_{\alpha 3}(h)=\frac{1}{h}\big(\partial_3(F_h^T)F_h^{-T}\big)_{\alpha\sigma}+\big(\partial_\alpha(F_h^T)F_h^{-T}\big)_{3\sigma}=2\big(\partial_\alpha(F_h^T)F_h^{-T}\big)_{3\sigma}.$$

The other equalities in (4.5) can be proved similarly.

The general case follows by an approximation argument. □

**Remark 4.2.** Note that (4.4) coincides, up to a scaling, with the quantity considered in [9, Theorem 1.3.1]. Moreover, the coefficients $\Gamma^k_{ij}(h)$ are the suitably scaled Christoffel symbols of $\Sigma_h$. In particular, for $h=1$ (that is, when $R_h$ is replaced by the identity matrix and thus, $F_h$ is equal to $D\Psi_h$) they exactly coincide with the Christoffel symbols of $\Sigma_h$. Indeed, following the notation of [9, Section 1.2], let $g_i:=F_he_i=\partial_i\Psi_h$ (where $e_i$ is the canonical basis of $\mathbb{R}^3$), and let $g^j:=F_h^{-T}e_j$, so that $g_i\cdot g^j=\delta_{ij}$. Then

$$\Gamma^k_{ij}(h)=g^k\cdot\partial_jg_i,$$

which is the usual definition of the Christoffel symbols in differential geometry.

In the following lemma we study the dependence of $\Gamma^k_{ij}(h)$ on the thickness parameter $h$.

**Lemma 4.3.** *The following expansions hold:*

$$\Gamma^\sigma_{\alpha\beta}(h)=h^2\partial^2_{\alpha\beta}\theta\,\partial_\sigma\theta-h^2x_3\partial^3_{\alpha\beta\sigma}\theta+O(h^3),\tag{4.6}$$

$$\Gamma^3_{\alpha\beta}(h)=\partial^2_{\alpha\beta}\theta+O(h^2),\tag{4.7}$$

$$\Gamma^\sigma_{\alpha 3}(h)=-h\partial^2_{\alpha\sigma}\theta+O(h^2),\tag{4.8}$$

$$\Gamma^i_{33}(h)=\Gamma^3_{\alpha 3}(h)=0,\tag{4.9}$$

*where* $O(h^p)$ *denotes a quantity uniformly bounded by* $h^p$*, as* $h\to 0$*.*

*Proof.* Let $g_i^h := F_h e_i$ and $g^{h,i} := F_h^{-T} e_i$. These definitions, (3.2), and (4.5) lead to

$$\begin{aligned} \Gamma^\sigma_{\alpha i}(h) &= g^{h,\sigma} \cdot \partial_\alpha g_i^h, & \Gamma^3_{\alpha i}(h) &= \frac{1}{h} g^{h,3} \cdot \partial_\alpha g_i^h, \\ \Gamma^\alpha_{33}(h) &= \frac{1}{h} g^{h,\alpha} \cdot \partial_3 g_3^h, & \Gamma^3_{33}(h) &= \frac{1}{h^2} g^{h,3} \cdot \partial_3 g_3^h. \end{aligned} \tag{4.10}$$

By direct computations we have that

$$g_\alpha^h = e_\alpha + h\partial_\alpha \theta\, e_3 + h x_3 \partial_\alpha \nu_{S_h}, \qquad g_3^h = \nu_{S_h}.$$

Since $g_i^h \cdot g^{h,j} = \delta_{ij}$, we immediately deduce that

$$g^{h,3} = \nu_{S_h},$$

while by applying Lemma 3.1 we obtain

$$g^{h,\alpha} = e_\alpha + h\partial_\alpha \theta\, e_3 + O(h^2).$$

Since

$$\begin{aligned} \nu_{S_h} &= e_3 - h\partial_1\theta\, e_1 - h\partial_2\theta\, e_2 + O(h^2), \\ \partial_\alpha \nu_{S_h} &= -h\partial^2_{1\alpha}\theta\, e_1 - h\partial^2_{2\alpha}\theta\, e_2 + O(h^2), \\ \partial^2_{\alpha\beta} \nu_{S_h} &= -h\partial^3_{1\alpha\beta}\theta\, e_1 - h\partial^3_{2\alpha\beta}\theta\, e_2 + O(h^2), \end{aligned}$$

we deduce (4.6)–(4.8) from (4.10). Equalities (4.9) follow again from (4.10) by observing that $\partial_3 g_3^h = 0$ and

$$g^{h,3} \cdot \partial_\alpha g_3^h = \frac{1}{2} \partial_\alpha (\nu_{S_h} \cdot \nu_{S_h}) = 0.$$

This concludes the proof of the lemma. □

We are ready to prove the *Korn-Poincaré inequality on shallow shells.*

**Theorem 4.4.** *There exist $h_0 > 0$ and $C > 0$, depending on $\Omega$ and $\partial_d\Omega$, such that*

$$\|u\|_{L^1} + \|R_h(\operatorname{sym} Du)R_h\|_{M_b} \le C\left(\|E(h,u)\|_{M_b} + \|u\|_{L^1(\partial_d\Omega)}\right)$$

*for every $0 < h \le h_0$ and every $u \in BD(\Omega)$, where $E(h,u)$ is defined in* (4.4).

*Proof.* Assume for contradiction that for every $n \in \mathbb{N}$ there exist $h_n \to 0^+$ and $(u^n) \subset BD(\Omega)$ such that

$$\|u^n\|_{L^1} + \|R_{h_n}(\operatorname{sym} Du^n)R_{h_n}\|_{M_b} = 1 \tag{4.11}$$

and

$$\|E(h_n, u^n)\|_{M_b} + \|u^n\|_{L^1(\partial_d\Omega)} \to 0. \tag{4.12}$$

By (4.11) the sequence $(u^n)$ is uniformly bounded in $BD(\Omega)$; therefore, there exists $u \in BD(\Omega)$ such that $u^n \rightharpoonup u$ weakly* in $BD(\Omega)$ and strongly in $L^1(\Omega;\mathbb{R}^3)$, up to subsequences. On the other hand, it follows from (4.4) and (4.9) that

$$\begin{aligned} (R_{h_n}(\operatorname{sym} Du^n)R_{h_n})_{\alpha\beta} &= (\operatorname{sym} Du^n)_{\alpha\beta} = E(h_n,u_n)_{\alpha\beta} + \Gamma^i_{\alpha\beta}(h_n)u^n_i, \\ (R_{h_n}(\operatorname{sym} Du^n)R_{h_n})_{\alpha 3} &= \frac{1}{h_n}(\operatorname{sym} Du^n)_{\alpha 3} = E(h_n,u_n)_{\alpha 3} + \Gamma^\sigma_{\alpha 3}(h_n)u^n_\sigma, \\ (R_{h_n}(\operatorname{sym} Du^n)R_{h_n})_{33} &= \frac{1}{h_n^2}(\operatorname{sym} Du^n)_{33} = E(h_n,u_n)_{33}. \end{aligned}$$

Using (4.12), Lemma 4.3, and the strong convergence of $(u^n)$ in $L^1(\Omega;\mathbb{R}^3)$, we deduce that

$$\begin{aligned} (\operatorname{sym} Du^n)_{\alpha\beta} &\to u_3 \partial^2_{\alpha\beta}\theta = (\operatorname{sym} Du)_{\alpha\beta} \quad \text{strongly in } M_b(\Omega), \\ (\operatorname{sym} Du^n)_{i3} &\to 0 = (\operatorname{sym} Du)_{i3} \quad \text{strongly in } M_b(\Omega), \end{aligned}$$

and

$$R_{h_n}(\operatorname{sym} Du^n)R_{h_n} \to \operatorname{sym} Du \quad \text{strongly in } M_b(\Omega;\mathbb{M}^{3\times 3}_{sym}). \tag{4.13}$$

Thus, $u \in KL(\Omega)$ and

$$u^n \to u \quad \text{strongly in } BD(\Omega). \tag{4.14}$$

Together with (4.12), this implies that $u = 0$ on $\partial_d\Omega$.

Let now $\bar{u} \in BD(\omega)$ and $u_3 \in BH(\omega)$ be the Kirchhoff-Love components of $u$. Since

$$(\operatorname{sym} D\bar{u})_{\alpha\beta} - x_3 \partial^2_{\alpha\beta} u_3 = u_3 \partial^2_{\alpha\beta}\theta, \tag{4.15}$$

we obtain that $\partial^2_{\alpha\beta} u_3 = 0$. Moreover, the boundary condition $u = 0$ on $\partial_d\Omega$ implies that $\bar{u} - x_3\nabla u_3 = 0$ on $\partial_d\Omega$, hence $\nabla u_3 = 0$ on $\partial_d\omega$, and $u_3 = 0$ on $\partial_d\omega$. By (2.2) we deduce that $u_3 = 0$ in $\omega$. Thus, $\operatorname{sym} D\bar{u} = 0$ in $\omega$ by (4.15) and, in turn, $\operatorname{sym} Du = 0$ in $\Omega$. Since $u = 0$ on $\partial_d\Omega$, it follows from (2.1) that $u = 0$ in $\Omega$. Since $\|u\|_{BD} = 1$ by (4.11), (4.13), and (4.14), we obtain a contradiction. $\square$

## 5. Γ-convergence of the static functionals

In this section we study the asymptotic behaviour of minimisers of the rescaled energies $\mathcal{I}_h$, as $h$ tends to 0. We begin with a compactness result for scaled displacements.

**Lemma 5.1.** *Let $(w^h) \subset H^1(\Omega;\mathbb{R}^3)$ be such that $\|w^h\|_{L^2(\partial_d\Omega)} \le C$ for every $0 < h \ll 1$. Let $(u^h)$ be a sequence in $V_h(\Omega)$ such that*

$$\|\operatorname{sym}(R_h Du^h R_h F_h^{-1})\|_{M_b} + \|R_h(w^h - u^h) \odot (\operatorname{cof} F_h) R_h \nu_{\partial\Omega}\|_{L^1(\partial_d\Omega)} \le C \tag{5.1}$$

*for every $0 < h \ll 1$. Then there exists $u \in KL(\Omega)$ such that, up to subsequences,*

$$u^h \to u \quad \text{strongly in } L^1(\Omega;\mathbb{R}^3) \tag{5.2}$$

*and*

$$\operatorname{sym}(R_h Du^h R_h F_h^{-1})_{\alpha\beta} \rightharpoonup (E^*u)_{\alpha\beta} \quad \text{weakly}^* \text{ in } M_b(\Omega), \tag{5.3}$$

*as $h \to 0$, where $E^*u$ is defined in* (3.22).

*Proof.* For every $h$ we consider the vectorfield $u^h(h)$ given by the curvilinear coordinates of $u^h$, defined according to (4.1). For simplicity of notation we write $u(h)$ instead of $u^h(h)$.

By Lemma 3.1 the sequence $(F_h)$ is uniformly bounded with respect to $h$. Thus, by (4.3) and (5.1) we deduce that

$$\|E(h, u(h))\|_{M_b} \le C$$

for every $0 < h \ll 1$. Since $|a \odot b| \ge \frac{1}{\sqrt{2}}|a||b|$ for every $a, b \in \mathbb{R}^n$, it follows from (5.1) that

$$\int_{\partial_d\Omega} |R_h(w^h - u^h)||(\operatorname{cof} F_h) R_h \nu_{\partial\Omega}|\, d\mathcal{H}^2 \le C$$

for every $0 < h \ll 1$. Moreover,

$$|(\operatorname{cof} F_h) R_h \nu_{\partial\Omega}| \ge \frac{|R_h \nu_{\partial\Omega}|}{|\operatorname{cof} F_h^{-1}|} \ge C|R_h\nu_{\partial\Omega}| \ge C,$$

where we used that $\operatorname{cof} F_h^{-1} \to I_{3\times 3}$ uniformly by Lemma 3.1. Therefore, we conclude that

$$\|R_h(w^h - u^h)\|_{L^1(\partial_d\Omega)} \le C.$$

In particular, we have that $\|w^h - u^h\|_{L^1(\partial_d\Omega)} \le C$, hence $\|u^h\|_{L^1(\partial_d\Omega)} \le C$ for every $h$ small enough. By Lemma 3.1 we can write

$$(D\Psi_h)^T R_h = I_{3\times 3} + \begin{pmatrix} 0 & 0 & \partial_1\theta \\ 0 & 0 & \partial_2\theta \\ 0 & 0 & 0 \end{pmatrix} + O(h), \tag{5.4}$$

hence by (4.1) we obtain that $\|u(h)\|_{1,\partial_d\Omega} \le C$ for every $h$.

By applying Theorem 4.4 to the sequence $(u(h))$, we deduce that

$$\|u(h)\|_{L^1} + \|R_h(\operatorname{sym} Du(h))R_h\|_{M_b} \le C.$$

Thus, there exists $\tilde{u} \in KL(\Omega)$ such that $u(h) \rightharpoonup \tilde{u}$ weakly* in $BD(\Omega)$ and strongly in $L^1(\Omega;\mathbb{R}^3)$, up to subsequences. We deduce that (5.2) holds with $u \in KL(\Omega)$ defined by

$$u_\alpha := \tilde{u}_\alpha - \partial_\alpha\theta\, \tilde{u}_3, \qquad u_3 := \tilde{u}_3. \tag{5.5}$$

Indeed, by (4.1) and (5.4) we have that

$$u^h = ((D\Psi_h)^T R^h)^{-1} u(h) = u(h) + \begin{pmatrix} 0 & 0 & -\partial_1\theta \\ 0 & 0 & -\partial_2\theta \\ 0 & 0 & 0 \end{pmatrix} u(h) + u^h_*, \tag{5.6}$$

where

$$\|u^h_*\|_{L^1} \le Ch\|u(h)\|_{L^1} \le Ch,$$

with $C$ independent of $h$. Passing to the limit in (5.6), we obtain (5.2).

Since $F_h \to I_{3\times 3}$ uniformly, as $h$ tends to 0, equality (4.3) implies that $E(h, u(h))$ and $\operatorname{sym}(R_h Du^h R_h F_h^{-1})$ have the same weak* limit in $M_b(\Omega; \mathbb{M}^{3\times 3}_{sym})$. In particular, by (4.6) and (4.7) we obtain

$$E(h, u(h))_{\alpha\beta} \rightharpoonup (\operatorname{sym} D\tilde{u})_{\alpha\beta} - \tilde{u}_3 \partial^2_{\alpha\beta}\theta \quad \text{weakly* in } M_b(\Omega),$$

and by (5.5) we have

$$\begin{aligned}(\operatorname{sym} D\tilde{u})_{\alpha\beta} - \tilde{u}_3\partial^2_{\alpha\beta}\theta &= (\operatorname{sym} Du)_{\alpha\beta} + \operatorname{sym}(D(u_3\nabla\theta))_{\alpha\beta} - u_3\partial^2_{\alpha\beta}\theta \\ &= (\operatorname{sym} Du)_{\alpha\beta} + (\nabla\theta \odot \nabla u_3)_{\alpha\beta} \;=\; (E^* u)_{\alpha\beta}.\end{aligned}$$

This proves (5.3) and concludes the proof. □

The following theorem is the main result of this section. The proof is in the spirit of Γ-convergence.

**Theorem 5.2.** *Let $(w^h) \subset H^1(\Omega; \mathbb{R}^3)$ be such that*

$$\|w^h\|_{L^2(\partial_d\Omega)} \le C \quad \text{for every } 0 < h \ll 1, \tag{5.7}$$

$$\operatorname{sym}(R_h Dw^h R_h F_h^{-1}) \to \zeta \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}), \tag{5.8}$$

*where $C > 0$ is independent of $h$ and $\zeta \in L^2(\Omega; \mathbb{M}^{3\times 3}_{sym})$. For every $0 < h \ll 1$ let $(u^h, e^h, p^h) \in \mathcal{A}_h(\Omega, w^h)$ be a minimiser of $\mathcal{I}_h$. Then there exist $w \in KL(\Omega) \cap H^1(\Omega; \mathbb{R}^3)$ and a triplet $(u, e, p) \in \mathcal{A}_{\mathrm{gKL}}(w)$ such that $(E^* w)_{\alpha\beta} = \zeta_{\alpha\beta}$ and, up to subsequences,*

$$w^h \to w \quad \text{strongly in } H^1(\Omega; \mathbb{R}^3), \tag{5.9}$$

$$u^h \to u \quad \text{strongly in } L^1(\Omega; \mathbb{R}^3), \tag{5.10}$$

$$\operatorname{sym}(R_h Du^h R_h F_h^{-1})_{\alpha\beta} \rightharpoonup (E^* u)_{\alpha\beta} \quad \text{weakly* in } M_b(\Omega), \tag{5.11}$$

$$e^h \to \mathbb{M}e \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}), \tag{5.12}$$

$$p^h_{\alpha\beta} \rightharpoonup p_{\alpha\beta} \quad \text{weakly* in } M_b(\Omega \cup \partial_d\Omega). \tag{5.13}$$

*Moreover, $(u, e, p)$ is a minimiser of $\mathcal{I}$ and*

$$\lim_{h\to 0} \mathcal{I}_h(u^h, e^h, p^h) = \mathcal{I}(u, e, p). \tag{5.14}$$

**Remark 5.3.** By the definition (3.15) of the operator $\mathbb{M}$ convergence (5.12) implies that $e^h_{\alpha\beta} \to e_{\alpha\beta}$ strongly in $L^2(\Omega)$.

*Proof of Theorem 5.2.* The proof is subdivided into four steps. First of all, as a consequence of Lemma 3.1, we note that the following expansions hold:

$$\begin{aligned}\operatorname{sym}(R_h Dv R_h F_h^{-1})_{\alpha\beta} &= (\operatorname{sym} Dv - \partial_3 v \odot \nabla\theta)_{\alpha\beta} + O(h^2)\|v\|_{H^1}, \\ \operatorname{sym}(R_h Dv R_h F_h^{-1})_{\alpha 3} &= \tfrac{1}{h}\left((\operatorname{sym} Dv - \partial_3 v \odot \nabla\theta)_{\alpha 3} + O(h^2)\|v\|_{H^1}\right), \\ \operatorname{sym}(R_h Dv R_h F_h^{-1})_{33} &= \tfrac{1}{h^2}\left(\partial_3 v_3(1 + O(h^2)) + h^2 \nabla v_3 \cdot \nabla\theta + O(h^4)\|v\|_{H^1}\right)\end{aligned} \tag{5.15}$$

for every $v \in H^1(\Omega; \mathbb{R}^3)$.

*Step 1: Convergence of $(w^h)$.* By (5.15) and the fact that $\partial_3\theta = 0$ we deduce that

$$\|\operatorname{sym}(R_h Dw^h R_h F_h^{-1})\|_{L^2} \ge \|\operatorname{sym} Dw^h - \partial_3 w^h \odot \nabla\theta\|_{L^2} - O(h^2)\|w^h\|_{H^1}.$$

This implies that for $h$ small enough

$$\begin{aligned}
&\|w^h\|_{L^2(\partial_d\Omega)} + \|\operatorname{sym}(R_h Dw^h R_h F_h^{-1})\|_{L^2} \\
&\quad\geq \|w^h\|_{L^2(\partial_d\Omega)} + \|\operatorname{sym} Dw^h - \partial_3 w^h \odot \nabla\theta\|_{L^2} - O(h^2)\|w^h\|_{H^1} \\
&\quad\geq C\|w^h\|_{H^1},
\end{aligned} \tag{5.16}$$

where the last estimate follows from the generalised Korn inequality in $H^1$ for shallow shells (see, e.g., [8, Theorem 3.4-1]). By (5.7) and (5.8) we conclude that the sequence $(w^h)$ is uniformly bounded in $H^1(\Omega;\mathbb{R}^3)$ for $h$ small enough. Thus, there exists $w \in H^1(\Omega;\mathbb{R}^3)$ such that

$$w^h \rightharpoonup w \quad \text{weakly in } H^1(\Omega;\mathbb{R}^3), \tag{5.17}$$

up to subsequences. Convergence (5.17) yields

$$\operatorname{sym} Dw^h - \partial_3 w^h \odot \nabla\theta \rightharpoonup \operatorname{sym} Dw - \partial_3 w \odot \nabla\theta \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{sym}).$$

On the other hand, owing to (5.8) and (5.15), we also have that $(\operatorname{sym} Dw^h - \partial_3 w^h \odot \nabla\theta)_{\alpha\beta} \to \zeta_{\alpha\beta}$ and $(\operatorname{sym} Dw^h - \partial_3 w^h \odot \nabla\theta)_{i3} \to 0$ strongly in $L^2(\Omega)$. Therefore, we deduce that

$$\operatorname{sym} Dw^h - \partial_3 w^h \odot \nabla\theta \to \operatorname{sym} Dw - \partial_3 w \odot \nabla\theta \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{sym}), \tag{5.18}$$

with

$$(\operatorname{sym} Dw - \partial_3 w \odot \nabla\theta)_{\alpha\beta} = \zeta_{\alpha\beta} \tag{5.19}$$

and $(\operatorname{sym} Dw - \partial_3 w \odot \nabla\theta)_{i3} = 0$. Since $\partial_3\theta = 0$, this last equality implies that

$$(\operatorname{sym} Dw - \partial_3 w \odot \nabla\theta)_{33} = \partial_3 w_3 = 0,$$

and consequently

$$(\operatorname{sym} Dw - \partial_3 w \odot \nabla\theta)_{\alpha 3} = (\operatorname{sym} Dw)_{\alpha 3} = 0.$$

In other words, $(\operatorname{sym} Dw)_{i3} = 0$, that is, $w \in KL(\Omega)$. In particular, we have that $\partial_3 w_\alpha = -\partial_\alpha w_3$, hence $\partial_3 w \odot \nabla\theta = -\nabla w_3 \odot \nabla\theta$, so that (5.19) gives the equality $(E^* w)_{\alpha\beta} = \zeta_{\alpha\beta}$.

To conclude it remains to show that the convergence in (5.17) is strong. By applying again [8, Theorem 3.4-1] we obtain

$$\begin{aligned}
&\|w^h - w^{h'}\|_{H^1} \\
&\leq C(\|w^h - w^{h'}\|_{L^2(\partial_d\Omega)} + \|\operatorname{sym} Dw^h - \partial_3 w^h \odot \nabla\theta - \operatorname{sym} Dw^{h'} + \partial_3 w^{h'} \odot \nabla\theta\|_{L^2})
\end{aligned} \tag{5.20}$$

for every $0 < h, h' \ll 1$. By (5.17) and the compactness of the trace operator we have that $w^h \to w$ strongly in $L^2(\partial_d\Omega;\mathbb{R}^3)$. Thus, by (5.18) and (5.20) we conclude that $(w^h)$ is a Cauchy sequence in $H^1(\Omega;\mathbb{R}^3)$, hence (5.9) holds.

*Step 2: Compactness.* Since

$$(w^h, \operatorname{sym}(R_h Dw^h R_h F_h^{-1}), 0) \in \mathcal{A}_h(\Omega, w^h),$$

the minimality of $(u^h, e^h, p^h)$ implies that

$$\mathcal{I}_h(u^h, e^h, p^h) \leq \mathcal{I}_h(w^h, \operatorname{sym}(R_h Dw^h R_h F_h^{-1}), 0) \leq C \tag{5.21}$$

for every $0 < h \ll 1$, where the last inequality is a consequence of (3.5), (5.8), and Lemma 3.1. Using again Lemma 3.1, (3.5), and (3.8), the bound (5.21) yields

$$\|e^h\|_{L^2} + \|p^h\|_{M_b} \leq C \tag{5.22}$$

for every $0 < h \ll 1$. Thus, there exist $\tilde{e} \in L^2(\Omega;\mathbb{M}^{3\times 3}_{sym})$ and $\tilde{p} \in M_b(\Omega \cup \partial_d\Omega;\mathbb{M}^{3\times 3}_{sym})$ such that, up to subsequences,

$$e^h \rightharpoonup \tilde{e} \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{sym}), \tag{5.23}$$

$$p^h \rightharpoonup \tilde{p} \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \partial_d\Omega;\mathbb{M}^{3\times 3}_D). \tag{5.24}$$

We introduce $e \in L^2(\Omega;\mathbb{M}^{3\times 3}_{sym})$ and $p \in M_b(\Omega \cup \partial_d\Omega;\mathbb{M}^{3\times 3}_{sym})$ defined by $e_{\alpha\beta} := \tilde{e}_{\alpha\beta}$, $e_{i3} := 0$, and $p_{\alpha\beta} := \tilde{p}_{\alpha\beta}$, $p_{i3} := 0$, respectively.

Since $Q$ is convex and $\det F_h \to 1$ uniformly, as $h \to 0$, by Lemma 3.1, we have

$$\liminf_{h\to 0} \int_\Omega Q(e^h) \det F_h \, dx \geq \int_\Omega Q(\tilde{e}) \, dx \geq \mathcal{Q}^*(e), \tag{5.25}$$

where the last inequality follows from the definition of $Q^*$. Analogously, by the Reshetnyak Theorem and the definition of $H^*$ we deduce

$$\liminf_{h\to 0} \mathcal{H}_h(p^h) \geq \int_{\Omega\cup\partial_d\Omega} H\left(\frac{d\tilde{p}}{d|\tilde{p}|}\right) d|\tilde{p}| \geq \mathcal{H}^*(p). \tag{5.26}$$

By (3.14) and (5.22) we can apply Lemma 5.1. Thus, there exists $u \in KL(\Omega)$ such that, up to subsequences,

$$u^h \to u \quad \text{strongly in } L^1(\Omega;\mathbb{R}^3), \tag{5.27}$$

$$\operatorname{sym}(R_h Du^h R_h F_h^{-1})_{\alpha\beta} \rightharpoonup (E^*u)_{\alpha\beta} \quad \text{weakly}^* \text{ in } M_b(\Omega). \tag{5.28}$$

We claim that $(u, e, p) \in \mathcal{A}_{\mathrm{gKL}}(w)$. Combining (5.23), (5.24), and (5.28), we deduce that $E^*u = e + p$ in $\Omega$.

To conclude it remains to show that $p = (w - u) \odot \nu_{\partial\Omega}\mathcal{H}^2$ on $\partial_d\Omega$. We argue as in [10, Lemma 2.1]. Since $\gamma_d$ is open in $\partial\omega$, there exists an open set $A \subseteq \mathbb{R}^2$ such that $\gamma_d = A \cap \partial\omega$. We set $U := (\omega \cup A) \times \left(-\frac{1}{2}, \frac{1}{2}\right)$. We extend $\theta$ to $\omega \cup A$ in such a way that $\theta \in C^3(\overline{\omega \cup A})$. Consequently, $\Psi_h \in C^2(\overline{U};\mathbb{R}^3)$ and $F_h \in C^1(\overline{U};\mathbb{M}^{3\times 3})$ for every $0 < h \ll 1$. Let $u^h(h)$ and $w^h(h)$ be the vectorfields given by the curvilinear coordinates of $u^h$ and $w^h$, defined according to (4.1). For simplicity we write $u(h)$ and $w(h)$ instead of $u^h(h)$ and $w^h(h)$. By (4.1), (5.4), and (5.9) we have that

$$w(h) \to \tilde{w} := w + w_3\nabla\theta \quad \text{strongly in } L^2(\Omega;\mathbb{R}^3). \tag{5.29}$$

By Proposition 4.1, Lemma 4.3, and (5.8), the sequence $(\operatorname{sym} Dw(h))$ is also strongly converging in $L^2(\Omega;\mathbb{M}^{3\times 3}_{sym})$. Thus, by the Korn inequality the convergence in (5.29) is strong in $H^1(\Omega;\mathbb{R}^3)$. Moreover, we can extend $w(h)$ and $\tilde{w}$ to $U$ in such a way that

$$w(h) \rightharpoonup \tilde{w} \quad \text{weakly in } H^1(U;\mathbb{R}^3). \tag{5.30}$$

We now define the triplet $(v(h), \eta(h), q(h)) \in BD(U) \times L^2(U;\mathbb{M}^{3\times 3}_{sym}) \times M_b(U;\mathbb{M}^{3\times 3}_{sym})$ as

$$v(h) := \begin{cases} u(h) & \text{in } \Omega, \\ w(h) & \text{in } U \setminus \Omega, \end{cases} \qquad \eta(h) := \begin{cases} R_h^{-1} F_h^T e^h F_h R_h^{-1} & \text{in } \Omega, \\ R_h^{-1} E(h, w(h)) R_h^{-1} & \text{in } U \setminus \Omega, \end{cases}$$

and

$$q(h) := \begin{cases} R_h^{-1} F_h^T p^h F_h R_h^{-1} & \text{in } \Omega \cup \partial_d\Omega, \\ 0 & \text{in } U \setminus (\Omega \cup \partial_d\Omega), \end{cases}$$

where $E(h, w(h))$ is defined as in (4.4). We have that

$$(\operatorname{sym} Dv(h))_{ij} = \eta(h)_{ij} + q(h)_{ij} + (R_h^{-1})_{ik}\Gamma^m_{kl}(h) v_m(h) (R_h^{-1})_{lj} \quad \text{in } U. \tag{5.31}$$

Indeed, this equality holds in $\Omega$ and in $U \setminus \overline{\Omega}$ as a consequence of (3.14), (4.3), and (4.4), while on $\partial_d\Omega$ it follows from (3.14), (4.2), and the definition of the cofactor.

By (4.1), (5.4), (5.27), and (5.30) we deduce that

$$v(h) \to v \quad \text{strongly in } L^1(U;\mathbb{R}^3), \tag{5.32}$$

where

$$v := \begin{cases} u + u_3\nabla\theta & \text{in } \Omega, \\ \tilde{w} & \text{in } U \setminus \Omega. \end{cases}$$

Since $(\eta(h))$ is uniformly bounded in $L^2(U;\mathbb{M}^{3\times 3}_{sym})$ by (5.23), Lemma 3.1, (4.4), and (5.30), there exists $\eta \in L^2(U;\mathbb{M}^{3\times 3}_{sym})$ such that

$$\eta(h) \rightharpoonup \eta \quad \text{weakly in } L^2(U;\mathbb{M}^{3\times 3}_{sym}), \tag{5.33}$$

up to subsequences. Finally, it follows from Lemma 3.1 and (5.24) that

$$q(h) \rightharpoonup q \quad \text{weakly}^* \text{ in } M_b(U;\mathbb{M}^{3\times 3}_{sym}), \tag{5.34}$$

where

$$q := \begin{cases} p & \text{in } \Omega \cup \partial_d\Omega, \\ 0 & \text{in } U \setminus (\Omega \cup \partial_d\Omega). \end{cases}$$

Passing to the limit in (5.31) by (5.32)–(5.34) and Lemma 4.3, we obtain

$$\operatorname{sym} Dv = \eta + q + v_3 D^2\theta \quad \text{in } U.$$

In particular, since $\tilde{w} = w + w_3\nabla\theta$, the previous equality on $\partial_d\Omega$ reads as

$$p = (w - u + (w_3 - u_3)\nabla\theta) \odot \nu_{\partial\Omega}\mathcal{H}^2 \quad \text{on } \partial_d\Omega.$$

Since $p_{\alpha 3} = 0$, $\nu_{\partial\Omega} \cdot e_3 = 0$ on $\partial_d\Omega$, and $\partial_3\theta = 0$, this implies that $u_3 = w_3$ on $\partial_d\Omega$ and, in turn, the desired equality.

*Step 3: Existence of a recovery sequence.* We show that for every $(v, \eta, q) \in \mathcal{A}_{\mathrm{gKL}}(w)$ there exists a sequence of triplets $(v^h, \eta^h, q^h) \in \mathcal{A}_h(\Omega, w^h)$ such that

$$v^h \to v \quad \text{strongly in } L^1(\Omega;\mathbb{R}^3), \tag{5.35}$$

$$\operatorname{sym}(R_h Dv^h R_h F_h^{-1})_{\alpha\beta} \rightharpoonup (E^*v)_{\alpha\beta} \quad \text{weakly* in } M_b(\Omega), \tag{5.36}$$

$$\eta^h \to \mathbb{M}\eta \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{sym}), \tag{5.37}$$

$$q^h_{\alpha\beta} \rightharpoonup q_{\alpha\beta} \quad \text{weakly* in } M_b(\Omega \cup \partial_d\Omega), \tag{5.38}$$

$$\mathcal{H}_h(q^h) \to \mathcal{H}^*(q), \tag{5.39}$$

and

$$\lim_{h\to 0} \mathcal{I}_h(v^h, \eta^h, q^h) = \mathcal{I}(v, \eta, q). \tag{5.40}$$

Owing to Lemma 3.7, it is enough to construct an approximating sequence for a triplet

$$(v, \eta, q) \in \left(H^1(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{M}^{3\times 3}_{sym}) \times L^2_{\infty,c}(\Omega;\mathbb{M}^{3\times 3}_{sym})\right) \cap \mathcal{A}_{\mathrm{gKL}}(w). \tag{5.41}$$

In the general case one can argue by density as in [11, Theorem 5.4].

Let $(v, \eta, q)$ be as in (5.41). Since $q \in L^2(\Omega;\mathbb{M}^{3\times 3}_{sym})$, we have that $q = 0$ on $\partial_d\Omega$ and $v = w$ on $\partial_d\Omega$. Let $\phi_1, \phi_2, \phi_3 \in L^2(\Omega)$ be such that

$$\mathbb{M}\eta = \begin{pmatrix} \eta_{11} & \eta_{12} & \phi_1 \\ \eta_{21} & \eta_{22} & \phi_2 \\ \phi_1 & \phi_2 & \phi_3 \end{pmatrix}. \tag{5.42}$$

As $q \in L^2(\Omega;\mathbb{M}^{3\times 3}_{sym})$, by the measurable selection Lemma (see, e.g., [16]) and by the definition of $H^*$ there exist $\psi_1, \psi_2 \in L^2(\Omega)$ such that

$$H^*(q) = H\begin{pmatrix} q_{11} & q_{12} & \psi_1 \\ q_{21} & q_{22} & \psi_2 \\ \psi_1 & \psi_2 & -(q_{11} + q_{22}) \end{pmatrix}. \tag{5.43}$$

We approximate the functions $\phi_i$ and $\psi_\alpha$ by means of elliptic regularisations; namely, for every $0 < h \ll 1$ we consider the solutions $\phi_i^h \in H^1_0(\Omega)$ and $\psi_\alpha^h \in H^1_0(\Omega)$ of the problems

$$\begin{cases} -h\Delta\phi_i^h + \phi_i^h = \phi_i & \text{in } \Omega, \\ \phi_i^h = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -h\Delta\psi_\alpha^h + \psi_\alpha^h = \psi_\alpha & \text{in } \Omega, \\ \psi_\alpha^h = 0 & \text{on } \partial\Omega. \end{cases}$$

Similarly, for every $0 < h \ll 1$ we define $\xi_i^h \in H^1_0(\Omega)$ as the solutions of the problems

$$\begin{cases} -h\Delta\xi_\alpha^h + \xi_\alpha^h = -\zeta_{3\alpha} & \text{in } \Omega, \\ \xi_\alpha^h = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -h\Delta\xi_3^h + \xi_3^h = \nabla(w_3 - v_3)\cdot\nabla\theta - \zeta_{33} & \text{in } \Omega, \\ \xi_3^h = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\zeta_{3i}$ are the components of the function $\zeta$ in (5.8). The standard theory of elliptic equations implies that

$$\begin{gathered} \phi_i^h \to \phi_i \quad \text{strongly in } L^2(\Omega), \qquad \psi_\alpha^h \to \psi_\alpha \quad \text{strongly in } L^2(\Omega), \\ \xi_\alpha^h \to -\zeta_{3\alpha} \quad \text{strongly in } L^2(\Omega), \\ \xi_3^h \to \nabla(w_3 - v_3)\cdot\nabla\theta - \zeta_{33} \quad \text{strongly in } L^2(\Omega), \end{gathered} \tag{5.44}$$

as $h \to 0$, and

$$\|\nabla \phi_i^h\|_{L^2} + \|\nabla \psi_\alpha^h\|_{L^2} + \|\nabla \xi_i^h\|_{L^2} \leq C h^{-\frac{1}{2}} \tag{5.45}$$

for every $0 < h \ll 1$. We also introduce the function $k^h \in L^2(\Omega; \mathbb{M}^{3\times 3})$, defined componentwise as

$$\begin{aligned} k_{\alpha\beta}^h(x', x_3) &:= 2h \int_0^{x_3} \big(\partial_\beta \phi_\alpha^h(x', s) + \partial_\beta \psi_\alpha^h(x', s) + \partial_\beta \xi_\alpha^h(x', s)\big)\, ds, \\ k_{3\beta}(x', x_3) &:= h^2 \int_0^{x_3} \big(\partial_\beta \phi_3^h(x', s) + \partial_\beta \xi_3^h(x', s) - \partial_\beta q_{11}(x', s) - \partial_\beta q_{22}(x', s)\big)\, ds, \\ k_{\alpha 3}^h &:= 2h(\phi_\alpha^h + \psi_\alpha^h + \xi_\alpha^h), \qquad k_{33}^h := h^2(\phi_3^h + \xi_3^h - q_{11} - q_{22}). \end{aligned}$$

We are now in a position to define the recovery sequence. We set

$$\begin{aligned} v_\alpha^h &:= v_\alpha + w_\alpha^h - w_\alpha + 2h \int_0^{x_3} \big(\phi_\alpha^h(x', s) + \psi_\alpha^h(x', s) + \xi_\alpha^h(x', s)\big)\, ds, \\ v_3^h &:= v_3 + w_3^h - w_3 + h^2 \int_0^{x_3} \big(\phi_3^h(x', s) + \xi_3^h(x', s) - q_{11}(x', s) - q_{22}(x', s)\big)\, ds. \end{aligned}$$

It is straightforward to check that

$$Dv^h = Dv + Dw^h - Dw + k^h.$$

This leads us to define

$$\begin{aligned} q^h &:= q + \begin{pmatrix} 0 & 0 & \psi_1^h \\ 0 & 0 & \psi_2^h \\ \psi_1^h & \psi_2^h & -(q_{11} + q_{22}) \end{pmatrix}, \\ \eta^h &:= \operatorname{sym}(R_h(Dv + Dw^h - Dw)R_h F_h^{-1}) + \operatorname{sym}(R_h k^h R_h F_h^{-1}) - q^h. \end{aligned}$$

Since $\phi_i^h, \psi_\alpha^h, \xi_i^h \in H_0^1(\Omega)$, $q \in L^2_{\infty,c}(\Omega; \mathbb{M}^{2\times 2}_{sym})$, and $v = w$ on $\partial_d \Omega$, we have that $v^h = w^h$ on $\partial_d \Omega$. Hence, $(v^h, \eta^h, q^h) \in \mathcal{A}_h(\Omega, w^h)$.

It follows from (5.9) and (5.44) that $v^h \to v$ strongly in $L^2(\Omega; \mathbb{R}^3)$. In particular, (5.35) holds. By the definition of $q^h$ we immediately deduce (5.38). Owing to (5.44), we obtain that

$$q^h \to q + \begin{pmatrix} 0 & 0 & \psi_1 \\ 0 & 0 & \psi_2 \\ \psi_1 & \psi_2 & -(q_{11} + q_{22}) \end{pmatrix} \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}). \tag{5.46}$$

Convergence (5.46), together with (5.43) and Lemma 3.1, implies (5.39).

We now prove (5.37). Since $v, w \in KL(\Omega)$, expansions (5.15) imply that

$$\begin{aligned} \operatorname{sym}(R_h(Dv - Dw)R_h F_h^{-1})_{\alpha\beta} &= (E^*v - E^*w)_{\alpha\beta} + O(h^2), \\ \operatorname{sym}(R_h(Dv - Dw)R_h F_h^{-1})_{\alpha 3} &= O(h), \\ \operatorname{sym}(R_h(Dv - Dw)R_h F_h^{-1})_{33} &= \nabla\theta \cdot \nabla(v_3 - w_3) + O(h^2). \end{aligned}$$

Thus, by (5.8) and the equality $(E^*w)_{\alpha\beta} = \zeta_{\alpha\beta}$ we deduce that

$$\begin{aligned} \operatorname{sym}(R_h(Dv + Dw^h - Dw)R_h F_h^{-1})_{\alpha\beta} &\to (E^*v)_{\alpha\beta} \quad \text{strongly in } L^2(\Omega), \\ \operatorname{sym}(R_h(Dv + Dw^h - Dw)R_h F_h^{-1})_{\alpha 3} &\to \zeta_{\alpha 3} \quad \text{strongly in } L^2(\Omega), \end{aligned} \tag{5.47}$$

and

$$\operatorname{sym}(R_h(Dv + Dw^h - Dw)R_h F_h^{-1})_{33} \to \zeta_{33} + \nabla\theta \cdot \nabla(v_3 - w_3) \quad \text{strongly in } L^2(\Omega). \tag{5.48}$$

From (5.44) and (5.45) it follows that

$$\begin{aligned} (R_h k^h R_h)_{i\beta} &\to 0 \quad \text{strongly in } L^2(\Omega), \\ (R_h k^h R_h)_{\alpha 3} &\to 2\,(\phi_\alpha + \psi_\alpha - \zeta_{3\alpha}) \quad \text{strongly in } L^2(\Omega), \\ (R_h k^h R_h)_{33} &\to \phi_3 + \nabla(w_3 - v_3) \cdot \nabla\theta - \zeta_{33} - q_{11} - q_{22} \quad \text{strongly in } L^2(\Omega). \end{aligned}$$

This, together with the uniform convergence of $F_h^{-1}$ to $I_{3\times 3}$, implies that

$$\begin{gathered}\mathrm{sym}(R_h k^h R_h F_h^{-1})_{\alpha\beta} \to 0 \quad \text{strongly in } L^2(\Omega),\\ \mathrm{sym}(R_h k^h R_h F_h^{-1})_{\alpha 3} \to \phi_\alpha + \psi_\alpha - \zeta_{3\alpha} \quad \text{strongly in } L^2(\Omega),\\ \mathrm{sym}(R_h k^h R_h F_h^{-1})_{33} \to \phi_3 + \nabla(w_3 - v_3)\cdot\nabla\theta - \zeta_{33} - q_{11} - q_{22} \quad \text{strongly in } L^2(\Omega).\end{gathered}$$

Combining the convergences above with (5.42), (5.46), (5.47), and (5.48), yields (5.37).

Finally, (5.36) follows from (5.37) and (5.38), while (5.40) is a consequence of (3.18), (5.37), and (5.39).

*Step 4: Minimality of* $(u,e,p)$ *and strong convergence of the elastic strains.* Let $(v,\eta,q)\in\mathcal{A}_{\mathrm{gKL}}(w)$. By Step 3 there exists a sequence $(v^h,\eta^h,q^h)$ in $\mathcal{A}_h(\Omega,w^h)$ such that (5.35)–(5.40) hold. Therefore,

$$\mathcal{I}(v,\eta,q) = \lim_{h\to 0}\mathcal{I}_h(v^h,\eta^h,q^h) \geq \limsup_{h\to 0}\mathcal{I}_h(u^h,e^h,p^h), \tag{5.49}$$

where the last inequality follows from the minimality of $(u^h,e^h,p^h)$. On the other hand, by (5.25) and (5.26)

$$\liminf_{h\to 0}\mathcal{I}_h(u^h,e^h,p^h) \geq \mathcal{I}(u,e,p). \tag{5.50}$$

Combining (5.49) and (5.50), we conclude that $(u,e,p)$ is a minimiser of $\mathcal{I}$. Moreover, by choosing $(v,\eta,q)=(u,e,p)$ in (5.49) we deduce (5.14).

It remains to prove (5.12). From (5.25), (5.26), and (5.14) it follows that

$$\lim_{h\to 0}\int_\Omega Q(e^h)\det F_h\,dx = \mathcal{Q}^*(e).$$

Since $\det F_h \to 1$ uniformly, as $h\to 0$, this implies that

$$\lim_{h\to 0}\int_\Omega Q(e^h)\,dx = \mathcal{Q}^*(e). \tag{5.51}$$

On the other hand, by (3.18) we have

$$Q(e^h - \mathbb{M}e) = Q(e^h) + Q^*(e) - \mathbb{C}\mathbb{M}e : e^h.$$

Therefore, owing to (5.23), (5.51), and (3.17), we get

$$\lim_{h\to 0}\int_\Omega Q(e^h - \mathbb{M}e)\,dx = 0.$$

By the coercivity (3.5) of $Q$ this implies (5.12). □

## 6. Convergence of quasistatic evolutions

In this section we discuss the convergence of the quasistatic evolution problems associated with the functionals $\mathcal{I}_h$.

We fix a time interval $[0,T]$ with $T>0$ and we give the following definitions.

**Definition 6.1.** Let $0<h\ll 1$ and let $w^h\in \mathrm{Lip}([0,T];H^1(\Omega;\mathbb{R}^3))$. An *h-quasistatic evolution* for the boundary datum $w^h$ is a function $t\mapsto (u^h(t),e^h(t),p^h(t))$ from $[0,T]$ into $V_h(\Omega)\times L^2(\Omega;\mathbb{M}^{3\times 3}_{sym})\times M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{3\times 3}_D)$ that satisfies the following conditions:

(qs1) *global stability:* for every $t\in[0,T]$ we have that $(u^h(t),e^h(t),p^h(t))\in\mathcal{A}_h(\Omega,w^h(t))$ and

$$\int_\Omega Q(e^h(t))\det F_h\,dx \leq \int_\Omega Q(\eta)\det F_h\,dx + \mathcal{H}_h(q-p^h(t)) \tag{6.1}$$

for every $(v,\eta,q)\in\mathcal{A}_h(\Omega,w^h(t))$;

(qs2) *energy balance:* $p^h \in BV([0,T]; M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times3}_{sym}))$ and for every $t \in [0,T]$

$$\int_\Omega Q(e^h(t)) \det F_h \, dx + \mathcal{D}_h(p^h; 0, t)$$
$$= \int_\Omega Q(e^h(0)) \det F_h \, dx + \int_0^t \int_\Omega \mathbb{C}e^h(s) : \mathrm{sym}(R_h D\dot{w}^h(s) R_h F_h^{-1}) \det F_h \, dx \, ds. \quad (6.2)$$

In (6.2) the notation $\mathcal{D}_h(p^h; 0, t)$ stands for the *dissipation* of $p^h$ in the interval $[0, t]$, defined as

$$\mathcal{D}_h(p; a, b) := \sup\Big\{ \sum_{j=1}^N \mathcal{H}_h(p(s_j) - p(s_{j-1})) : \ a = s_0 \le s_1 \le \cdots \le s_N = b, \ N \in \mathbb{N} \Big\}$$

for every $p \in BV([0,T]; M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times3}_{sym}))$ and every $0 \le a \le b \le T$.

**Definition 6.2.** Let $w \in \mathrm{Lip}([0,T]; H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega))$. A *reduced quasistatic evolution* for the boundary datum $w$ is a function $t \mapsto (u(t), e(t), p(t))$ from $[0,T]$ into $BD(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times2}_{sym}) \times M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{2\times2}_{sym})$ that satisfies the following conditions:

(qs1)$^*$ *reduced global stability:* for every $t \in [0,T]$ we have that $(u(t), e(t), p(t)) \in \mathcal{A}_{\mathrm{gKL}}(w(t))$ and

$$\mathcal{Q}^*(e(t)) \le \mathcal{Q}^*(\eta) + \mathcal{H}^*(q - p(t)) \quad (6.3)$$

for every $(v, \eta, q) \in \mathcal{A}_{\mathrm{gKL}}(w(t))$;

(qs2)$^*$ *reduced energy balance:* $p \in BV([0,T]; M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{2\times2}_{sym}))$ and for every $t \in [0,T]$

$$\mathcal{Q}^*(e(t)) + \mathcal{D}^*(p; 0, t) = \mathcal{Q}^*(e(0)) + \int_0^t \int_\Omega \mathbb{C}^* e(s) : E^*\dot{w}(s) \, dx \, ds. \quad (6.4)$$

In (6.4) the notation $\mathcal{D}^*(p; 0, t)$ stands for the *reduced dissipation* of $p$ in the interval $[0, t]$, defined as

$$\mathcal{D}^*(p; a, b) := \sup\Big\{ \sum_{j=1}^N \mathcal{H}^*(p(s_j) - p(s_{j-1})) : \ a = s_0 \le s_1 \le \cdots \le s_N = b, \ N \in \mathbb{N} \Big\}$$

for every $p \in BV([0,T]; M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{2\times2}_{sym}))$ and every $0 \le a \le b \le T$.

We now prove the convergence of a sequence of $h$-quasistatic evolutions to a reduced quasistatic evolution, as $h \to 0$. This will be proved under the following assumptions on the boundary and initial data.

*Boundary displacements.* We consider a sequence of boundary displacements

$$(w^h) \subset \mathrm{Lip}([0,T]; H^1(\Omega;\mathbb{R}^3)) \quad (6.5)$$

such that for every $0 < h \ll 1$

$$\|w^h\|_{\mathrm{Lip}([0,T]; L^2(\partial_d\Omega;\mathbb{R}^3))} + \|\mathrm{sym}(R_h D w^h R_h F_h^{-1})\|_{\mathrm{Lip}([0,T];L^2)} \le C \quad (6.6)$$

with a constant $C > 0$, independent of $h$. Furthermore, we assume that there exists $\zeta \in \mathrm{Lip}([0,T]; L^2(\Omega; \mathbb{M}^{3\times3}_{sym}))$ such that

$$\mathrm{sym}(R_h D w^h(t) R_h F_h^{-1}) \to \zeta(t) \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times3}_{sym}) \quad (6.7)$$

for every $t \in [0,T]$ and

$$\mathrm{sym}(R_h D \dot{w}^h(t) R_h F_h^{-1}) \to \dot{\zeta}(t) \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times3}_{sym}) \quad (6.8)$$

for a.e. $t \in [0,T]$.

*Initial data.* Let $(u_0^h, e_0^h, p_0^h) \in \mathcal{A}_h(\Omega, w^h(0))$ be such that

$$\int_\Omega Q(e_0^h) \det F_h \, dx \leq \int_\Omega Q(\eta) \det F_h \, dx + \mathcal{H}_h(q - p_0^h) \tag{6.9}$$

for every $(v, \eta, q) \in \mathcal{A}_h(\Omega, w^h(0))$. Moreover, we assume that

$$e_0^h \to \tilde{e}_0 \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}) \tag{6.10}$$

for some $\tilde{e}_0 \in L^2(\Omega; \mathbb{M}^{3\times 3}_{sym})$ and that for every $0 < h \ll 1$

$$\|p_0^h\|_{M_b} \leq C \tag{6.11}$$

for some constant $C > 0$, independent of $h$.

We are now in a position to state the main result of this paper.

**Theorem 6.3.** *Assume* (6.5)–(6.11). *For every* $0 < h \ll 1$ *let* $t \mapsto (u^h(t), e^h(t), p^h(t))$ *be an $h$-quasistatic evolution for the boundary datum* $w^h$ *such that* $(u^h(0), e^h(0), p^h(0)) = (u_0^h, e_0^h, p_0^h)$. *Then there exist* $w \in \mathrm{Lip}([0,T]; H^1(\Omega; \mathbb{R}^3) \cap KL(\Omega))$ *and a reduced quasistatic evolution*

$$(u, e, p) \in \mathrm{Lip}\left([0,T]\,; BD(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{2\times 2}_{sym})\right)$$

*for the boundary datum* $w$ *such that, up to subsequences, for every* $t \in [0,T]$

$$w^h(t) \to w(t) \quad \textit{strongly in } H^1(\Omega; \mathbb{R}^3), \tag{6.12}$$

$$u^h(t) \to u(t) \quad \textit{strongly in } L^1(\Omega; \mathbb{R}^3), \tag{6.13}$$

$$\mathrm{sym}(R_h D u^h(t) R_h F_h^{-1})_{\alpha\beta} \rightharpoonup (E^* u(t))_{\alpha\beta} \quad \textit{weakly}^* \textit{ in } M_b(\Omega), \tag{6.14}$$

$$e^h(t) \to \mathbb{M} e(t) \quad \textit{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}), \tag{6.15}$$

$$p^h_{\alpha\beta}(t) \rightharpoonup p_{\alpha\beta}(t) \quad \textit{weakly}^* \textit{ in } M_b(\Omega \cup \partial_d\Omega), \tag{6.16}$$

*as* $h \to 0$.

**Remark 6.4.** Given a boundary datum $w^h$ and a triplet $(u_0^h, e_0^h, p_0^h) \in \mathcal{A}_h(\Omega, w^h(0))$ satisfying (6.9), the existence of an $h$-quasistatic evolution $t \to (u^h(t), e^h(t), p^h(t))$ with boundary datum $w^h$ and initial condition $(u^h(0), e^h(0), p^h(0)) = (u_0^h, e_0^h, p_0^h)$ follows from [10, Theorem 4.5]. In [10] this result is proven for $\partial\Omega$ of class $C^2$, but, as observed in [17], Lipschitz regularity of the boundary is enough in the absence of external forces. Furthermore, since the problem is rate-independent, one can always assume the data to be Lipschitz continuous in time (and not only absolutely continuous), up to a time scaling, so that solutions are Lipschitz continuous in time (see [10, Theorem 5.2]).

**Remark 6.5.** The assumptions (6.10) and (6.11) on the initial data are crucial to deduce the right compactness estimates for the sequence of $h$-quasistatic evolutions (see Step 2 in the proof of Theorem 6.3). Moreover, the strong convergence in (6.10) is needed to pass to the limit in the energy balance and deduce an energy inequality for the reduced problem (see Step 6 in the proof of Theorem 6.3).

For the proof of Theorem 6.3 we will need some preliminary results. The first one is a characterisation of the global stability condition (qs1)$^*$ of the reduced problem.

**Lemma 6.6.** *Let* $w \in H^1(\Omega; \mathbb{R}^3) \cap KL(\Omega)$ *and let* $(u, e, p) \in \mathcal{A}_{\mathrm{gKL}}(w)$. *The following conditions are equivalent:*

(a) $\mathcal{Q}^*(e) \leq \mathcal{Q}^*(\eta) + \mathcal{H}^*(q - p)$ *for every* $(v, \eta, q) \in \mathcal{A}_{\mathrm{gKL}}(w)$;

(b) $-\mathcal{H}^*(q) \leq \int_\Omega \mathbb{C}^* e : \eta \, dx$ *for every* $(v, \eta, q) \in \mathcal{A}_{\mathrm{gKL}}(0)$.

*Proof.* Assume (a) and let $(v, \eta, q) \in \mathcal{A}_{\mathrm{gKL}}(0)$. For every $\varepsilon > 0$ we have that $(u + \varepsilon v, e + \varepsilon\eta, p + \varepsilon q) \in \mathcal{A}_{\mathrm{gKL}}(w)$. Therefore,

$$\mathcal{Q}^*(e) \leq \mathcal{Q}^*(e + \varepsilon\eta) + \mathcal{H}^*(\varepsilon q).$$

Using the positive homogeneity of $H^*$, dividing by $\varepsilon$ and sending $\varepsilon$ to 0, we deduce (b). Conversely, (b) implies (a) by convexity of $\mathcal{Q}^*$ and $\mathcal{H}^*$. □

Arguing in the same way as in the previous lemma, one can prove the following characterisation of the global stability condition (qs1) of the $h$-quasistatic evolution problem.

**Lemma 6.7.** *Let $0 < h \ll 1$, let $w \in H^1(\Omega;\mathbb{R}^3)$, and let $(u,e,p) \in \mathcal{A}_h(\Omega,w)$. The following conditions are equivalent:*

(a) $\displaystyle\int_\Omega Q(e)\det F_h\,dx \le \int_\Omega Q(\eta)\det F_h\,dx + \mathcal{H}_h(q-p)$ *for every* $(v,\eta,q) \in \mathcal{A}_h(\Omega,w)$*;*

(b) $\displaystyle -\mathcal{H}_h(q) \le \int_\Omega \mathbb{C}e : \eta \det F_h\,dx$ *for every* $(v,\eta,q) \in \mathcal{A}_h(\Omega,0)$*.*

The next lemma concerns a variant of the Gronwall inequality.

**Lemma 6.8.** *Let $\phi,\psi : [0,T] \to [0,+\infty)$ be such that $\phi \in L^\infty(0,T)$ and $\psi \in L^1(0,T)$. Assume that*

$$\phi(t)^2 \le \int_0^t \phi(s)\psi(s)\,ds$$

*for every $t \in [0,T]$. Then*

$$\phi(t) \le \frac{1}{2}\int_0^t \psi(s)\,ds$$

*for every $t \in [0,T]$.*

*Proof.* We define

$$F(t) := \int_0^t \phi(s)\psi(s)\,ds$$

for every $t \in [0,T]$. Thus, $F \in AC([0,T])$ and by assumption $\phi(t)^2 \le F(t)$ for every $t \in [0,T]$. Therefore,

$$F'(t) = \phi(t)\psi(t) \le F(t)^{1/2}\psi(t)$$

for a.e. $t \in [0,T]$. This leads to

$$F(t)^{1/2} \le \frac{1}{2}\int_0^t \psi(s)\,ds$$

for every $t \in [0,T]$, which implies the thesis by using the assumption again. □

We have now all the ingredients to prove Theorem 6.3.

*Proof of Theorem 6.3.* The proof is split into six steps.

*Step 1: Convergence of $w^h$.* Hypothesis (6.6) and estimate (5.16) ensure that

$$\|w^h\|_{\mathrm{Lip}([0,T];H^1)} \le C$$

for every $0 < h \ll 1$. By the Ascoli-Arzelà Theorem there exist $w \in \mathrm{Lip}([0,T];H^1(\Omega;\mathbb{R}^3))$ and a subsequence $(w^h)$, not relabeled, such that

$$w^h(t) \rightharpoonup w(t) \quad \text{weakly in } H^1(\Omega;\mathbb{R}^3)$$

for every $t \in [0,T]$. Arguing as in Step 1 of the proof of Theorem 5.2, we infer that $w(t) \in KL(\Omega)$ and the above convergence is strong, namely (6.12) holds. Moreover,

$$\mathrm{sym}(R_h Dw^h(t) R_h F_h^{-1})_{\alpha\beta} \to (E^* w(t))_{\alpha\beta} \quad \text{strongly in } L^2(\Omega)$$

for every $t \in [0,T]$. In particular, by (6.7) we have that $\zeta_{\alpha\beta}(t) = (E^* w(t))_{\alpha\beta}$.

*Step 2: Compactness estimates.* We claim that there exists $C > 0$, independent of $h$, such that

$$\|e^h(t_2) - e^h(t_1)\|_{L^2} \le C|t_2 - t_1|\,\|\mathrm{sym}(R_h D\dot{w}^h R_h F_h^{-1})\|_{L^\infty([0,T];L^2)} \tag{6.17}$$

$$\|p^h(t_2) - p^h(t_1)\|_{M_b} \le C|t_2 - t_1|\,\|\mathrm{sym}(R_h D\dot{w}^h R_h F_h^{-1})\|_{L^\infty([0,T];L^2)} \tag{6.18}$$

for every $t_1, t_2 \in [0,T]$ and every $0 < h \ll 1$.

From (6.2), (3.5), (3.8), Lemma 3.1, and the Hölder inequality it follows that

$$(\alpha_{\mathbb{C}} + O(h^2))\|e^h(t)\|_{L^2}^2 + (r_K + O(h^2))\|p^h(t) - p_0^h\|_{M_b}$$
$$\leq (\beta_{\mathbb{C}} + O(h^2)) \int_0^t \|e^h(s)\|_{L^2} \|\operatorname{sym}(R_h D\dot{w}^h(s) R_h F_h^{-1})\|_{L^2}\, ds + (\beta_{\mathbb{C}} + O(h^2))\|e_0^h\|_{L^2}^2.$$

Owing to (6.6), (6.10), (6.11), and the Cauchy inequality, we deduce that

$$\sup_{t\in[0,T]} \|e^h(t)\|_{L^2} + \sup_{t\in[0,T]} \|p^h(t)\|_{M_b} \leq C \tag{6.19}$$

for every $h$ sufficiently small.

We now use condition (qs1) at time $t_1$. Let

$$v = u^h(t_2) - u^h(t_1) - w^h(t_2) + w^h(t_1),$$
$$\eta = e^h(t_2) - e^h(t_1) - \operatorname{sym}(R_h D w^h(t_2) R_h F_h^{-1}) + \operatorname{sym}(R_h D w^h(t_1) R_h F_h^{-1}),$$
$$q = p^h(t_2) - p^h(t_1).$$

Since $(v, \eta, q) \in \mathcal{A}_h(\Omega, 0)$, by Lemma 6.7 we have that

$$-\int_\Omega \mathbb{C}e^h(t_1) : \big(e^h(t_2) - e^h(t_1)\big) \det F_h\, dx$$
$$+ \int_\Omega \mathbb{C}e^h(t_1) : \big(\operatorname{sym}(R_h D w^h(t_2) R_h F_h^{-1}) - \operatorname{sym}(R_h D w^h(t_1) R_h F_h^{-1})\big) \det F_h\, dx$$
$$\leq \ \mathcal{H}_h(p^h(t_2) - p^h(t_1)) \ \leq \ \mathcal{D}_h(p^h; t_1, t_2),$$

where the last inequality is an immediate consequence of the definition of $\mathcal{D}_h$. Using the previous inequality in the energy balance (6.2) written at times $t_1$ and $t_2$, we get

$$\int_\Omega Q(e^h(t_2)) \det F_h\, dx - \int_\Omega Q(e^h(t_1)) \det F_h\, dx - \int_\Omega \mathbb{C}e^h(t_1) : \big(e^h(t_2) - e^h(t_1)\big) \det F_h\, dx$$
$$\leq \int_{t_1}^{t_2} \int_\Omega \mathbb{C}\left(e^h(s) - e^h(t_1)\right) : \operatorname{sym}(R_h D\dot{w}^h(s) R_h F_h^{-1}) \det F_h\, dx\, ds.$$

We observe that the left-hand side of the previous inequality is exactly

$$\int_\Omega Q(e^h(t_2) - e^h(t_1)) \det F_h\, dx.$$

Thus, from (3.5), (3.6), Lemma 3.1, and the Hölder inequality it follows that

$$(\alpha_{\mathbb{C}} + O(h^2))\|e^h(t_2) - e^h(t_1)\|_{L^2}^2$$
$$\leq (2\beta_{\mathbb{C}} + O(h^2)) \int_{t_1}^{t_2} \|e^h(s) - e^h(t_1)\|_{L^2} \|\operatorname{sym}(R_h D\dot{w}^h(s) R_h F_h^{-1})\|_{L^2}\, ds.$$

By Lemma 6.8 we deduce that

$$\|e^h(t_2) - e^h(t_1)\|_{L^2} \leq C \int_{t_1}^{t_2} \|\operatorname{sym}(R_h D\dot{w}^h(s) R_h F_h^{-1})\|_{L^2}\, ds,$$

hence (6.17).

Using again the energy balance (6.2) at times $t_1$ and $t_2$, together with (3.8) and Lemma 3.1, we obtain

$$
\begin{aligned}
&(r_K + O(h^2))\|p^h(t_2) - p^h(t_1)\|_{M_b} \\
&\le \int_\Omega Q(e^h(t_1)) \det F_h \,dx - \int_\Omega Q(e^h(t_2)) \det F_h \,dx \\
&\quad + \int_{t_1}^{t_2} \int_\Omega \mathbb{C} e^h(s) : \operatorname{sym}(R_h D\dot w^h(s) R_h F_h^{-1}) \det F_h \,dx\,ds \\
&\le C \sup_{t\in[0,T]} \|e^h(t)\|_{L^2} \Big( \int_{t_1}^{t_2} \|\operatorname{sym}(R_h D\dot w^h(s) R_h F_h^{-1})\|_{L^2}\,ds + \|e^h(t_2) - e^h(t_1)\|_{L^2} \Big) \\
&\le C|t_2 - t_1|\, \|\operatorname{sym}(R_h D\dot w^h R_h F_h^{-1})\|_{L^\infty([0,T];L^2)},
\end{aligned}
$$

where the last inequality follows from (6.19) and (6.17), and $C > 0$ is a constant independent of $h$. This proves (6.18) and concludes Step 2.

*Step 3: Reduced kinematic admissibility.* By (6.10), (6.11), (6.17), and (6.18) we can apply the Ascoli-Arzelà Theorem to the sequences $(e^h)$ and $(p^h)$ and deduce the existence of $\tilde e \in \mathrm{Lip}([0,T]; L^2(\Omega; \mathbb{M}^{3\times3}_{sym}))$ and $\tilde p \in \mathrm{Lip}([0,T]; M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times3}_D))$ such that, up to subsequences,

$$e^h(t) \rightharpoonup \tilde e(t) \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{3\times3}_{sym}), \tag{6.20}$$

$$p^h(t) \rightharpoonup \tilde p(t) \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times3}_D) \tag{6.21}$$

for every $t \in [0,T]$. We introduce $e \in \mathrm{Lip}([0,T]; L^2(\Omega; \mathbb{M}^{3\times3}_{sym}))$ and $p \in \mathrm{Lip}([0,T]; M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times3}_{sym}))$ defined by $e_{\alpha\beta}(t) := \tilde e_{\alpha\beta}(t)$, $e_{i3}(t) := 0$ for every $t \in [0,T]$, and $p_{\alpha\beta}(t) := \tilde p_{\alpha\beta}(t)$, $p_{i3}(t) := 0$ for every $t \in [0,T]$, respectively.

Since $(u^h(t), e^h(t), p^h(t)) \in \mathcal{A}_h(\Omega; w^h(t))$, and owing to (6.6) and (6.19), we can apply Lemma 5.1 and infer that for every $t \in [0,T]$ there exists $u(t) \in KL(\Omega)$ and a subsequence $u^{h_j}(t)$, possibly depending on $t$, such that

$$u^{h_j}(t) \to u(t) \quad \text{strongly in } L^1(\Omega; \mathbb{R}^3), \tag{6.22}$$

$$\operatorname{sym}(R_h D u^{h_j}(t) R_h F_h^{-1}))_{\alpha\beta} \rightharpoonup (E^* u(t))_{\alpha\beta} \quad \text{weakly}^* \text{ in } M_b(\Omega). \tag{6.23}$$

Furthermore, arguing as in Step 2 of the proof of Theorem 5.2, and using (6.20) and (6.21), we infer that $(u(t), e(t), p(t)) \in \mathcal{A}_{\mathrm{gKL}}(w(t))$. We now prove that $u(t)$ is uniquely determined. Assume that there exist $t \in [0,T]$ and two subsequences $(u^{h_j}(t))$ and $(u^{h'_j}(t))$ with limits $u_1(t)$ and $u_2(t)$, respectively. Set $z(t) := u_1(t) - u_2(t)$. Since both $(u_1(t), e(t), p(t))$ and $(u_2(t), e(t), p(t))$ belong to $\mathcal{A}_{\mathrm{gKL}}(w(t))$, we have that $z(t) \in KL(\Omega)$ and

$$E^* z(t) = 0 \quad \text{in } \Omega, \qquad z(t) = 0 \quad \text{on } \partial_d\Omega.$$

We deduce that

$$\operatorname{sym} D\bar z(t) + \nabla z_3(t) \odot \nabla\theta = x_3 D^2 z_3(t) \quad \text{in } \Omega. \tag{6.24}$$

Thus, $D^2 z_3(t) = 0$ in $\Omega$ and the boundary condition $\bar z(t) - x_3 \nabla z_3(t) = 0$ on $\partial_d\Omega$ gives $\nabla z_3(t) = 0$ on $\partial_d\omega$ and $z_3(t) = 0$ on $\partial_d\omega$. By (2.2) we deduce that $z_3(t) = 0$ in $\omega$. Hence, $\operatorname{sym} D\bar z(t) = 0$ in $\omega$ by (6.24) and, in turn, $\operatorname{sym} Dz(t) = 0$ in $\Omega$. Since $z(t) = 0$ on $\partial_d\Omega$, it follows from (2.1) that $z(t) = 0$ in $\Omega$. This proves that $u(t)$ is uniquely determined, hence convergences (6.22) and (6.23) hold for the whole sequence. Thus, (6.13) and (6.14) are proved.

It remains to check that $u \in \mathrm{Lip}([0,T]; BD(\Omega))$. Since $e, p$, and $w$ are Lipschitz continuous, by kinematic admissibility we infer that

$$(u, E^* u) \in \mathrm{Lip}([0,T]; L^1(\partial_d\Omega; \mathbb{R}^3) \times M_b(\Omega; \mathbb{M}^{2\times2}_{sym})). \tag{6.25}$$

Now let us consider the first order moments of $u$ and $E^* u$. One can prove that

$$\|\hat E^* u(t)\|_{M_b(\omega)} \le C \|E^* u(t)\|_{M_b(\Omega)}, \qquad \|\hat u(t)\|_{L^1(\omega)} \le C \|u(t)\|_{L^1(\Omega)},$$

with $C > 0$. These estimates, together with the relations $\hat{u}_\alpha(t) = -\partial_\alpha u_3(t)$ and $\hat{E}^*u(t) = -D^2u_3(t)$, imply that

$$(u_3, \nabla u_3, D^2u_3) \in \mathrm{Lip}([0,T]; L^1(\partial_d\omega) \times L^1(\partial_d\omega;\mathbb{R}^2) \times M_b(\omega;\mathbb{M}^{2\times 2}_{sym}))$$

and, in turn, owing to (2.2), that $u_3 \in \mathrm{Lip}([0,T]; BH(\omega))$. It follows now from (6.25) and the definition (3.22) of $E^*u$ that $\mathrm{sym}\, Du \in \mathrm{Lip}([0,T]; M_b(\Omega;\mathbb{M}^{3\times 3}_{sym}))$. Therefore it is a consequence of (2.1) that

$$u \in \mathrm{Lip}([0,T]; BD(\Omega)).$$

The previous arguments, together with (6.10) and (6.11), also prove that, up to subsequences, $u_0^h \to u_0$ strongly in $L^1(\Omega;\mathbb{R}^3)$, $(\mathrm{sym}(R_hDu_0^hR_hF_h^{-1})_{\alpha\beta} \rightharpoonup (E^*u_0)_{\alpha\beta}$ weakly* in $M_b(\Omega)$, $(e_0^h)_{\alpha\beta} \to (e_0)_{\alpha\beta}$ strongly in $L^2(\Omega)$, $(p_0^h)_{\alpha\beta} \rightharpoonup (p_0)_{\alpha\beta}$ weakly* in $M_b(\Omega)$, for some $(u_0, e_0, p_0) \in \mathcal{A}_{KL}(w(0))$. Since $(u^h(0), e^h(0), p^h(0)) = (u_0^h, e_0^h, p_0^h)$, we have that $(u(0), e(0), p(0)) = (u_0, e_0, p_0)$.

*Step 4: Reduced global stability.* We prove (6.3). Let $t \in [0,T]$. By Lemma 6.6 condition (6.3) at time $t$ is equivalent to

$$-\mathcal{H}^*(q) \le \int_\Omega \mathbb{C}^*e(t) : \eta \, dx \quad \text{for every } (v,\eta,q) \in \mathcal{A}_{\mathrm{gKL}}(0). \tag{6.26}$$

Let $(v,\eta,q) \in \mathcal{A}_{\mathrm{gKL}}(0)$. By Step 3 in the proof of Theorem 5.2 there exists a sequence $(v^h,\eta^h,q^h) \in \mathcal{A}_h(\Omega, 0)$ such that

$$\eta^h \to \mathbb{M}\eta \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{sym}), \tag{6.27}$$

$$\mathcal{H}_h(q^h) \to \mathcal{H}^*(q). \tag{6.28}$$

By Lemma 6.7 and (6.1) at time $t$ we have that

$$-\mathcal{H}_h(q^h) \le \int_\Omega \mathbb{C}e^h(t) : \eta^h \det F_h \, dx$$

for every $0 < h \ll 1$. By (6.20), (6.27), and (6.28) we can pass to the limit in the previous estimate, as $h$ tends to 0, and deduce that

$$-\mathcal{H}^*(q) \le \int_\Omega \mathbb{C}\tilde{e}(t) : \mathbb{M}\eta \, dx \quad \text{for every } (v,\eta,q) \in \mathcal{A}_{\mathrm{gKL}}(0).$$

Since $\mathbb{C}\tilde{e}(t) : \mathbb{M}\eta = \mathbb{C}\mathbb{M}e(t) : \mathbb{M}\eta = \mathbb{C}^*e(t) : \eta$ by (3.20), this inequality reduces to (6.26).

*Step 5: Identification of the limiting elastic strain.* We now prove that $\tilde{e}(t) = \mathbb{M}e(t)$ for every $t \in [0,T]$.

Let $t \in [0,T]$. For every $\psi \in H^1(\Omega;\mathbb{R}^3)$ with $\psi = 0$ on $\partial_d\Omega$ we consider the triplets $(\pm\psi, \pm\,\mathrm{sym}(R_hD\psi R_hF_h^{-1}), 0)$ as test functions in condition (b) of Lemma 6.6 at time $t$. This leads to

$$\int_\Omega \mathbb{C}e^h(t) : \mathrm{sym}(R_hD\psi R_hF_h^{-1}) \det F_h \, dx = 0$$

for every $0 < h \ll 1$.

Let now $(a,b) \subset (-\frac{1}{2}, \frac{1}{2})$, let $U \subset \omega$ be an open set, and let $\lambda_i \in \mathbb{R}$. Let $(\varphi^n) \subset C^1([-\frac{1}{2}, \frac{1}{2}])$ and $(\lambda_i^n) \subset C^1_c(\omega)$ be sequences such that $(\varphi^n)' \to \chi_{(a,b)}$ strongly in $L^4(-\frac{1}{2}, \frac{1}{2})$ and $\lambda_i^n \to \lambda_i\chi_U$ strongly in $L^4(\omega)$, as $n \to \infty$. For $0 < h \ll 1$ and $n \in \mathbb{N}$ we define

$$\psi^{h,n}(x) := \big(2h\,\varphi^n(x_3)\lambda_\alpha^n(x'), h^2\varphi^n(x_3)\lambda_3^n(x')\big).$$

Since $\psi^{h,n} \in H^1(\Omega;\mathbb{R}^3)$ and $\psi^{h,n} = 0$ on $\partial_d\Omega$, we have

$$\int_\Omega \mathbb{C}e^h(t) : \mathrm{sym}(R_hD\psi^{h,n}R_hF_h^{-1}) \det F_h \, dx = 0. \tag{6.29}$$

Using that $F_h^{-1} = I_{3\times 3} + O(h)$ by Lemma 3.1, we obtain that

$$\mathrm{sym}(R_hD\psi^{h,n}R_hF_h^{-1})_{\alpha\beta} = O(h), \qquad \mathrm{sym}(R_hD\psi^{h,n}R_hF_h^{-1})_{i3} = (\varphi^n)'\lambda_i^n + O(h).$$

These expansions, together with (6.20) and the uniform convergence of $\det F_h$ to 1, allow us to pass to the limit in (6.29), first as $h \to 0$, and then, as $n \to \infty$. This yields

$$\int_{U\times(a,b)} \mathbb{C}\tilde{e}(t) : \begin{pmatrix} 0 & 0 & \lambda_1 \\ 0 & 0 & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} dx = 0.$$

Since the sets $(a,b)$ and $U$ are arbitrary, we conclude from (3.16) that $\tilde{e}(t) = \mathbb{M}e(t)$ a.e. in $\Omega$. In particular, we have that $\tilde{e}_0 = \mathbb{M}e_0$, where $\tilde{e}_0$ is the limit in (6.10).

*Step 6: Reduced energy balance.* The lower semicontinuity of $\mathcal{Q}^*$ and $\mathcal{D}^*$, together with (6.20) and (6.21), imply that

$$\begin{aligned} \mathcal{Q}^*(e(t)) &\le \liminf_{h\to 0} \int_\Omega Q(e^h(t)) \det F_h \, dx, \\ \mathcal{D}^*(p;0,t) &\le \liminf_{h\to 0} \mathcal{D}_h(p^h;0,t) \end{aligned} \tag{6.30}$$

for every $t \in [0,T]$. Passing to the limit in the energy balance (6.2) yields

$$\begin{aligned} &\mathcal{Q}^*(e(t)) + \mathcal{D}^*(p;0,t) \\ &\le \limsup_{h\to 0} \Big\{ \int_\Omega Q(e^h(0)) \det F_h \, dx + \int_0^t \int_\Omega \mathbb{C}e^h(s) : \mathrm{sym}(R_h D\dot{w}^h(s) R_h F_h^{-1}) \det F_h \, dx \, ds \Big\} \\ &= \int_\Omega Q(\tilde{e}_0)\, dx + \int_0^t \int_\Omega \mathbb{C}\tilde{e}(s) : \dot{\zeta}(s) \, dx \, ds, \end{aligned}$$

where the second equality is a consequence of (6.8), (6.6), (6.10), (6.19), (6.20), and the Dominated Convergence Theorem. By Step 5 and the equality $\zeta_{\alpha\beta}(t) = (E^*w(t))_{\alpha\beta}$, we conclude that

$$\mathcal{Q}^*(e(t)) + \mathcal{D}^*(p;0,t) \le \mathcal{Q}^*(e_0) + \int_0^t \int_\Omega \mathbb{C}^* e(s) : E^*\dot{w}(s) \, dx \, ds.$$

As it is standard in the variational theory for rate-independent processes, the converse energy inequality follows from the minimality condition (qs1)$^*$ (see, e.g., [31, Theorem 4.4] or [10, Theorem 4.7]). We have thus proved that $t \mapsto (u(t), e(t), p(t))$ is a reduced quasistatic evolution.

To conclude the proof it remains to show the strong convergence of $e^h(t)$ to $\mathbb{M}e(t)$ for every $t \in [0,T]$. Since we have showed that the right-hand side of (6.2) converges to the right-hand side of (6.4), we have that

$$\lim_{h\to 0} \Big\{ \int_\Omega Q(e^h(t)) \det F_h \, dx + \mathcal{D}_h(p^h;0,t) \Big\} = \mathcal{Q}^*(e(t)) + \mathcal{D}^*(p;0,t)$$

for every $t \in [0,T]$. Thus, by (6.30) and Lemma 3.1 we deduce that

$$\mathcal{Q}^*(e(t)) = \lim_{h\to 0} \int_\Omega Q(e^h(t)) \det F_h \, dx = \lim_{h\to 0} \int_\Omega Q(e^h(t)) \, dx$$

Since

$$\mathcal{Q}^*(e(t)) = \int_\Omega Q(\mathbb{M}e(t)) \, dx,$$

convergence (6.15) follows from (6.20), Step 5, and the coercivity (3.5) of $Q$. The proof of Theorem 6.3 is concluded. □

6.1. **Characterisation of reduced quasistatic evolutions in rate form.** We conclude this section with a characterisation of reduced quasistatic evolutions.

*Stress-strain duality.* In the framework of the reduced problem we introduce a notion of duality between stresses and plastic strains. Here we follow [11, Section 7].

We define the set $\Sigma(\Omega)$ of *admissible stresses* as

$$\Sigma(\Omega) := \{\sigma \in L^\infty(\Omega; \mathbb{M}^{2\times 2}_{sym}) : \ \operatorname{div} \bar\sigma \in L^2(\omega; \mathbb{R}^2), \ \operatorname{div}\operatorname{div} \hat\sigma \in L^2(\omega)\}.$$

For every $\sigma \in \Sigma(\Omega)$ we can define the trace $[\bar\sigma \nu_{\partial\omega}] \in L^\infty(\partial\omega; \mathbb{R}^2)$ of its zeroth order moment normal component as

$$\langle [\bar\sigma\nu_{\partial\omega}], \psi\rangle := \int_\omega \bar\sigma : \operatorname{sym} D\psi \, dx' + \int_\omega \operatorname{div} \bar\sigma \cdot \psi \, dx' \tag{6.31}$$

for every $\psi \in W^{1,1}(\omega; \mathbb{R}^2)$. Note that, since $\bar\sigma \in L^\infty(\omega; \mathbb{M}^{2\times 2}_{sym})$ and $W^{1,1}(\omega; \mathbb{R}^2)$ embeds into $L^2(\omega; \mathbb{R}^2)$, all terms on the right-hand side of (6.31) are well defined.

Let $T(W^{2,1}(\omega))$ be the space of traces of functions in $W^{2,1}(\omega)$ and let $(T(W^{2,1}(\omega)))'$ be its dual space. For every $\sigma \in \Sigma(\Omega)$ we can define the traces $b_0(\hat\sigma) \in (T(W^{2,1}(\omega)))'$ and $b_1(\hat\sigma) \in L^\infty(\partial\omega)$ of its first order moment as

$$-\langle b_0(\hat\sigma), \psi\rangle + \Big\langle b_1(\hat\sigma), \frac{\partial\psi}{\partial\nu_{\partial\omega}}\Big\rangle := \int_\omega \hat\sigma : D^2\psi \, dx' - \int_\omega \psi \operatorname{div}\operatorname{div} \hat\sigma \, dx' \tag{6.32}$$

for every $\psi \in W^{2,1}(\omega)$. Note that the right-hand side of (6.32) is well defined since $\hat\sigma \in L^\infty(\omega; \mathbb{M}^{2\times 2}_{sym})$. If $\hat\sigma \in C^2(\overline\omega, \mathbb{M}^{2\times 2}_{sym})$, one can prove that

$$b_0(\hat\sigma) = \operatorname{div}\hat\sigma \cdot \nu_{\partial\omega} + \frac{\partial}{\partial\tau_{\partial\omega}} (\hat\sigma\tau_{\partial\omega} \cdot \nu_{\partial\omega}),$$
$$b_1(\hat\sigma) = \hat\sigma\nu_{\partial\omega} \cdot \nu_{\partial\omega},$$

where $\tau_{\partial\omega}$ is a unit tangent vector to $\partial\omega$ (see [13, Théorème 2.3]).

Let $(h, m_0, m_1) \in L^\infty(\partial\omega; \mathbb{R}^2) \times T(W^{2,1}(\omega)))' \times L^\infty(\partial\omega)$. Since $[\bar\sigma\nu_{\partial\omega}] \in L^\infty(\partial\omega; \mathbb{R}^2)$ and $b_1(\hat\sigma) \in L^\infty(\partial\omega)$, the expressions $[\bar\sigma\nu_{\partial\omega}] = h$ on $\partial_n\omega$ and $b_1(\hat\sigma) = m_1$ on $\partial_n\omega$ have a clear meaning. As for $b_0(\hat\sigma)$, we say that $b_0(\hat\sigma) = m_0$ on $\partial_n\omega$ if $\langle b_0(\hat\sigma) - m_0, \psi\rangle = 0$ for every $\psi \in W^{2,1}(\omega)$ with $\psi = 0$ on $\partial_d\omega$.

We define the space of *admissible plastic strains* $\Pi_{\partial_d\Omega}(\Omega)$ as the set of all measures $p \in M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{2\times 2}_{sym})$ for which there exists $(u, e, w) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times (H^1(\Omega; \mathbb{R}^3) \cap KL(\Omega))$ such that $(u, e, p) \in \mathcal{A}_{\mathrm{gKL}}(w)$.

For every $\sigma \in \Sigma(\Omega)$ and $\xi \in BD(\omega)$ we define the distribution $[\bar\sigma : \operatorname{sym} D\xi]$ on $\omega$ as

$$\langle [\bar\sigma : \operatorname{sym} D\xi], \varphi\rangle := -\int_\omega \varphi \operatorname{div}\bar\sigma \cdot \xi \, dx' - \int_\omega \bar\sigma : (\nabla\varphi \odot \xi) \, dx'$$

for every $\varphi \in C^\infty_c(\omega)$. It follows from [23, Theorem 3.2] that $[\bar\sigma : \operatorname{sym} D\xi] \in M_b(\omega)$ and its variation satisfies

$$|[\bar\sigma : \operatorname{sym} D\xi]| \le \|\bar\sigma\|_{L^\infty} |\operatorname{sym} D\xi| \quad \text{in } \omega.$$

Given $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\partial_d\Omega}(\Omega)$, we define the measure $[\bar\sigma : \bar p] \in M_b(\omega \cup \partial_d\omega)$ as

$$[\bar\sigma : \bar p] := \begin{cases} [\bar\sigma : \operatorname{sym} D\bar u] + \bar\sigma : (\nabla\theta \odot \nabla u_3) - \bar\sigma : \bar e & \text{in } \omega, \\ [\bar\sigma\nu_{\partial\omega}] \cdot (\bar w - \bar u)\, \mathcal{H}^1 & \text{on } \partial_d\omega, \end{cases}$$

where $(u, e, w) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times (H^1(\Omega; \mathbb{R}^3) \cap KL(\Omega))$ are such that $(u, e, p) \in \mathcal{A}_{\mathrm{gKL}}(w)$. Note that since $\nabla u_3 \in BV(\omega; \mathbb{R}^2)$ and $BV(\omega; \mathbb{R}^2)$ embeds into $L^2(\omega; \mathbb{R}^2)$, the term $\bar\sigma : (\nabla\theta \odot \nabla u_3)$ is in $L^1(\Omega)$. Moreover, one can easily check that the definition of $[\bar\sigma : \bar p]$ is independent of the choice of $(u, e, w)$.

For every $\sigma \in \Sigma(\Omega)$ and $v \in BH(\omega)$ we define the distribution $[\hat\sigma : D^2 v]$ on $\omega$ as

$$\langle [\hat\sigma : D^2 v], \psi\rangle := \int_\omega \psi v \operatorname{div}\operatorname{div}\hat\sigma \, dx' - 2\int_\omega \hat\sigma : (\nabla v \odot \nabla\psi) \, dx' - \int_\omega v\hat\sigma : D^2\psi \, dx'$$

for every $\psi \in C^\infty_c(\omega)$. From [15, Proposition 2.1] it follows that $[\hat\sigma : D^2 v] \in M_b(\omega)$ and its variation satisfies

$$|[\hat\sigma : D^2 v]| \le \|\hat\sigma\|_{L^\infty} |D^2 v| \quad \text{in } \omega.$$

Given $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\partial_d\Omega}(\Omega)$, we define the measure $[\hat{\sigma} : \hat{p}] \in M_b(\omega \cup \partial_d\omega)$ as

$$[\hat{\sigma} : \hat{p}] := \begin{cases} -[\hat{\sigma} : D^2 u_3] - \hat{\sigma} : \hat{e} & \text{in } \omega, \\ b_1(\hat{\sigma}) \dfrac{\partial(u_3 - w_3)}{\partial \nu_{\partial\omega}} \, \mathcal{H}^1 & \text{on } \partial_d\omega, \end{cases}$$

where $(u, e, w) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times (H^1(\Omega; \mathbb{R}^3) \cap KL(\Omega))$ are such that $(u, e, p) \in \mathcal{A}_{\rm gKL}(w)$. This definition is independent of the choice of $(u, e, w)$.

We are now in a position to define the duality between $\Sigma(\Omega)$ and $\Pi_{\partial_d\Omega}(\Omega)$. For every $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\partial_d\Omega}(\Omega)$ we define the measure $[\sigma : p]^* \in M_b(\Omega \cup \partial_d\Omega)$ as

$$[\sigma : p]^* := [\bar{\sigma} : \bar{p}] \otimes \mathcal{L}^1 + \frac{1}{12}[\hat{\sigma} : \hat{p}] \otimes \mathcal{L}^1 - \sigma_\perp : e_\perp.$$

We also introduce the duality pairings

$$\langle \bar{\sigma}, \bar{p} \rangle := [\bar{\sigma} : \bar{p}](\omega \cup \partial_d\omega), \qquad \langle \hat{\sigma}, \hat{p} \rangle := [\hat{\sigma} : \hat{p}](\omega \cup \partial_d\omega)$$

and

$$\langle \sigma, p \rangle^* := [\sigma : p]^*(\Omega \cup \partial_d\Omega) = \langle \bar{\sigma}, \bar{p} \rangle + \frac{1}{12}\langle \hat{\sigma}, \hat{p} \rangle - \int_\Omega \sigma_\perp : e_\perp \, dx.$$

The next two results concern some useful properties of the stress-strain duality. We first show that the duality satisfies an integration by parts formula.

**Proposition 6.9.** *Let $\sigma \in \Sigma(\Omega)$, $w \in H^1(\Omega; \mathbb{R}^3) \cap KL(\Omega)$, and $(u, e, p) \in \mathcal{A}_{\rm gKL}(w)$. Then*

$$\begin{aligned} &\int_{\Omega \cup \partial_d\Omega} \varphi \, d[\sigma : p]^* + \int_\Omega \varphi\sigma : (e - E^* w) \, dx \\ &= \; - \int_\omega \bar{\sigma} : (\nabla\varphi \odot (\bar{u} - \bar{w})) \, dx' - \int_\omega \operatorname{div} \bar{\sigma} \cdot \varphi(\bar{u} - \bar{w}) \, dx' + \int_{\partial_n\omega} [\bar{\sigma}\nu_{\partial\omega}] \cdot \varphi(\bar{u} - \bar{w}) \, d\mathcal{H}^1 \\ &\quad + \frac{1}{12} \int_\omega \hat{\sigma} : (u_3 - w_3) D^2\varphi \, dx' + \frac{1}{6} \int_\omega \hat{\sigma} : (\nabla\varphi \odot (\nabla u_3 - \nabla w_3)) \, dx' \\ &\quad - \int_\omega \varphi(u_3 - w_3) \left( \frac{1}{12} \operatorname{div}\operatorname{div} \hat{\sigma} + \bar{\sigma} : D^2\theta + \operatorname{div} \bar{\sigma} \cdot \nabla\theta \right) dx' \\ &\quad - \int_\omega (u_3 - w_3) \bar{\sigma} : (\nabla\varphi \odot \nabla\theta) \, dx' + \int_{\partial_n\omega} \varphi(u_3 - w_3) [\bar{\sigma}\nu_{\partial\omega}] \cdot \nabla\theta \, d\mathcal{H}^1 \\ &\quad + \frac{1}{12} \langle b_0(\hat{\sigma}), \varphi(u_3 - w_3) \rangle - \frac{1}{12} \int_{\partial_n\omega} b_1(\hat{\sigma}) \frac{\partial(\varphi(u_3 - w_3))}{\partial\nu_{\partial\omega}} \, d\mathcal{H}^1 \end{aligned}$$

*for every $\varphi \in C^2(\overline{\omega})$.*

*Proof.* The proof follows from [12, Proposition 4] by observing that

$$\begin{aligned} \int_{\Omega \cup \partial_d\Omega} \varphi \, d[\sigma : p]^* = \int_{\Omega \cup \partial_d\Omega} \varphi \, d[\sigma : (p - \nabla\theta \odot \nabla u_3)]_r &+ \int_\omega \varphi\bar{\sigma} : (\nabla\theta \odot \nabla w_3) \, dx' \\ &+ \int_\omega \varphi\bar{\sigma} : (\nabla\theta \odot \nabla(u_3 - w_3)) \, dx', \end{aligned}$$

where $[\sigma : p]_r$ is the notion of duality introduced in [11, 12]. Moreover, by (6.31) we have

$$\begin{aligned}
&\int_\omega \varphi\bar\sigma : (\nabla\theta \odot \nabla(u_3 - w_3))\,dx' \\
&\quad= \int_\omega \bar\sigma : \operatorname{sym} D(\varphi(u_3 - w_3)\nabla\theta)\,dx' - \int_\omega (u_3 - w_3)\bar\sigma : (\nabla\varphi \odot \nabla\theta)\,dx' \\
&\qquad - \int_\omega \varphi(u_3 - w_3)\bar\sigma : D^2\theta\,dx' \\
&\quad= -\int_\omega \varphi(u_3 - w_3)\operatorname{div}\bar\sigma \cdot \nabla\theta\,dx' + \int_{\partial_n\omega} \varphi(u_3 - w_3)[\bar\sigma\nu_{\partial\omega}] \cdot \nabla\theta\,d\mathcal{H}^1 \\
&\qquad - \int_\omega (u_3 - w_3)\bar\sigma : (\nabla\varphi \odot \nabla\theta)\,dx' - \int_\omega \varphi(u_3 - w_3)\bar\sigma : D^2\theta\,dx',
\end{aligned}$$

where we used that $\varphi(u_3 - w_3)\nabla\theta \in BH(\omega;\mathbb{R}^2)$, hence $\varphi(u_3 - w_3)\nabla\theta \in W^{1,1}(\omega;\mathbb{R}^2)$ and $u_3 = w_3$ on $\partial_d\omega$ by Proposition 3.5. ☐

The next lemma is a characterisation of the dissipation potential $\mathcal{H}^*$ in terms of the duality.

**Lemma 6.10.** *Let $p \in \Pi_{\partial_d\Omega}(\Omega)$. Then the following equalities hold:*

$$\mathcal{H}^*(p) = \sup\{\langle\sigma, p\rangle^* :\ \sigma \in \Sigma(\Omega) \cap \mathcal{K}^*(\Omega)\} = \sup\{\langle\sigma, p\rangle^* :\ \sigma \in \Theta(\Omega)\},$$

*where*

$$\mathcal{K}^*(\Omega) := \big\{\sigma \in L^2(\Omega;\mathbb{M}^{2\times2}_{sym}) :\ \sigma(x) \in K^* \text{ for a.e. } x \in \Omega\big\}$$

*and $\Theta(\Omega)$ is the set of all $\sigma \in \Sigma(\Omega) \cap \mathcal{K}^*(\Omega)$ such that $[\bar\sigma\nu_{\partial\omega}] = 0$ on $\partial_n\omega$ and $b_0(\hat\sigma) = b_1(\hat\sigma) = 0$ on $\partial_n\omega$.*

*Proof.* Let $\Gamma := \big(\partial_n\omega \times \big(-\frac{1}{2}, \frac{1}{2}\big)\big) \cup \big(\omega \times \big(\pm\frac{1}{2}\big)\big)$. From [40, Chapter II, Section 4] it follows that

$$\begin{aligned}
\mathcal{H}^*(p) &= \sup\left\{\int_{\Omega\cup\partial_d\Omega} \sigma : dp :\ \sigma \in C^\infty(\mathbb{R}^3;\mathbb{M}^{2\times2}_{sym}) \cap \mathcal{K}^*(\Omega),\ \operatorname{supp}\sigma \cap \Gamma = \emptyset\right\} \\
&\le \sup\{\langle\sigma, p\rangle^* :\ \sigma \in \Theta(\Omega)\} \ \le\ \sup\{\langle\sigma, p\rangle^* :\ \sigma \in \Sigma(\Omega) \cap \mathcal{K}^*(\Omega)\}.
\end{aligned}$$

The converse inequality can be proved as in [11, Proposition 7.8] by an approximation argument, where the density result is provided in our framework by Lemma 3.7. ☐

Now we are ready to state and prove the main result of this section.

**Theorem 6.11.** *Let $w \in \mathrm{Lip}([0,T]; H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega))$. Let $t \mapsto (u(t), e(t), p(t))$ be a map from $[0,T]$ into $KL(\Omega) \times L^2(\Omega;\mathbb{M}^{2\times2}_{sym}) \times M_b(\Omega \cup \partial_d\Omega;\mathbb{M}^{2\times2}_{sym})$. Let $\sigma(t) := \mathbb{C}^* e(t)$. Then the following conditions are equivalent:*

(a) *$t \mapsto (u(t), e(t), p(t))$ is a reduced quasistatic evolution for the boundary datum $w$;*

(b) *$t \mapsto (u(t), e(t), p(t))$ is Lipschitz continuous and*
  - (b1) *for every $t \in [0,T]$ we have $(u(t), e(t), p(t)) \in \mathcal{A}_{\mathrm{gKL}}(w(t))$, $\sigma(t) \in \Theta(\Omega)$, $\operatorname{div}\bar\sigma(t) = 0$ in $\omega$ and $\frac{1}{12}\operatorname{div}\operatorname{div}\hat\sigma(t) + \bar\sigma(t) : D^2\theta = 0$ in $\omega$;*
  - (b2) *for a.e. $t \in [0,T]$ there holds*
$$\mathcal{H}^*(\dot p(t)) = \langle\sigma(t), \dot p(t)\rangle^*.$$

**Remark 6.12.** In the strong formulation given by condition (b) in the above theorem, the stability condition (qs1)$^*$ is replaced by the equilibrium equations $\operatorname{div}\bar\sigma(t) = 0$ and $\frac{1}{12}\operatorname{div}\operatorname{div}\hat\sigma(t) + \bar\sigma(t) : D^2\theta = 0$, supplemented by Neumann boundary conditions on the complement of $\partial_d\omega$, while the energy balance is replaced by the equality $\mathcal{H}^*(\dot p(t)) = \langle\sigma(t), \dot p(t)\rangle^*$. By Lemma 6.10 this last condition is, in turn, equivalent to the maximum dissipation principle $\langle\tau - \sigma(t), \dot p(t)\rangle^* \le 0$ for every $\tau \in \Theta(\Omega)$. This can be interpreted as an integral version of the pointwise flow rule (d5)$^*$ in the introduction.

**Remark 6.13.** In contrast with the plate model deduced in [11], the two equilibrium equations in (b) are coupled. This implies, in particular, that for a shallow shell subject to "horizontal" initial and boundary data it is in general not possible to write the reduced quasistatic evolution problem purely in terms of the "horizontal" components $\bar{u}$, $\bar{e}$, and $\bar{p}$, as it was instead proven for plates in [11, Proposition 7.6].

*Proof of Theorem 6.11.* Arguing as in [10, Theorem 5.2] one can prove that every reduced quasistatic evolution is Lipschitz continuous.

We first prove the equivalence between (qs1)* and (b1). Let $t \in [0,T]$. By Lemma 6.6 it is enough to show that (b1) is equivalent to the following condition:

$$-\mathcal{H}^*(q) \leq \int_\Omega \sigma(t) : \eta \, dx \quad \text{for every } (v,\eta,q) \in \mathcal{A}_{\mathrm{gKL}}(0). \tag{6.33}$$

Assume (6.33). Let $B \subset \Omega$ be a Borel set and let $\chi_B$ be its characteristic function. Let $\xi \in \mathbb{M}^{2\times 2}_{sym}$ and let $\eta := \chi_B \xi$. By choosing $(0,-\eta,\eta) \in \mathcal{A}_{\mathrm{gKL}}(0)$ as test function in (6.33), we have that

$$\int_B \sigma(t) : \xi \, dx \leq \mathcal{L}^3(B) H^*(\xi).$$

Since $B$ is arbitrary, we conclude that $\sigma(t,x) : \xi \leq H^*(\xi)$ a.e. in $\Omega$, hence $\sigma(t) \in \partial H^*(0) = K^*$ a.e. in $\Omega$.

Let now $v \in H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ be such that $v = 0$ on $\partial_d\Omega$. Since $(\pm v, \pm E^* v, 0) \in \mathcal{A}_{\mathrm{gKL}}(0)$, equation (6.33) implies

$$\int_\Omega \sigma(t) : E^* v \, dx = 0 \tag{6.34}$$

for every $v \in H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ with $v = 0$ on $\partial_d\Omega$. By choosing $v = \psi_\alpha e_\alpha$ with $\psi \in H^1(\omega;\mathbb{R}^2)$ and $\psi = 0$ on $\partial_d\omega$ in (6.34), we deduce that

$$\int_\omega \bar{\sigma}(t) : \mathrm{sym}\, D\psi \, dx' = 0$$

for every $\psi \in H^1(\omega;\mathbb{R}^2)$, $\psi = 0$ on $\partial_d\omega$. Since this holds, in particular, for every $\psi \in C^\infty_c(\omega;\mathbb{R}^2)$, we have

$$\mathrm{div}\, \bar{\sigma}(t) = 0 \quad \text{in } \omega. \tag{6.35}$$

Moreover, by [11, Lemma 7.10-(i)] we obtain

$$[\bar{\sigma}(t)\nu_{\partial\omega}] = 0 \qquad \text{on } \partial_n\omega. \tag{6.36}$$

We now choose $v$ in (6.34) of the form $v = \varphi e_3$, with $\varphi \in H^2(\omega)$, $\varphi = 0$ and $\nabla\varphi = 0$ on $\partial_d\omega$. This leads to

$$\int_\omega \bar{\sigma}(t) : (\nabla\theta \odot \nabla\varphi) \, dx' - \frac{1}{12}\int_\omega \hat{\sigma}(t) : D^2\varphi \, dx' = 0.$$

By (6.35), (6.36), and (6.31) we obtain

$$\int_\omega \bar{\sigma}(t) : (\nabla\theta \odot \nabla\varphi) \, dx' = \int_\omega \bar{\sigma}(t) : \mathrm{sym}\, D(\varphi\nabla\theta) \, dx' - \int_\omega \varphi\bar{\sigma}(t) : D^2\theta \, dx' = -\int_\omega \varphi\bar{\sigma}(t) : D^2\theta \, dx'.$$

Thus, we deduce that

$$\int_\omega \varphi\bar{\sigma}(t) : D^2\theta \, dx' + \frac{1}{12}\int_\omega \hat{\sigma}(t) : D^2\varphi \, dx' = 0$$

for every $\varphi \in H^2(\omega)$, $\varphi = 0$ and $\nabla\varphi = 0$ on $\partial_d\omega$. Since this holds, in particular, for every $\varphi \in C^\infty_c(\omega)$, we have

$$\bar{\sigma}(t) : D^2\theta + \frac{1}{12}\mathrm{div}\,\mathrm{div}\, \hat{\sigma}(t) = 0 \qquad \text{in } \omega.$$

Moreover, by [11, Lemma 7.10-(ii)] we obtain that $b_0(\hat{\sigma}) = b_1(\hat{\sigma}) = 0$ on $\partial_n\omega$. In particular, $\sigma(t) \in \Theta(\Omega)$ and (b1) holds.

Assume now (b1) and let $(v,\eta,q)\in\mathcal{A}_{\mathrm{gKL}}(0)$. Applying Proposition 6.9 to $(v,\eta,q)$ with $\varphi=1$ yields

$$\langle\sigma(t),q\rangle^*=-\int_\Omega\sigma(t):\eta\,dx.$$

Since $\sigma\in\Theta(\Omega)$, we deduce (6.33) by Lemma 6.10.

We now show, that if (b1) holds, then (qs2)* and (b2) are equivalent. Assume (b1). Since $p$ is Lipschitz continuous, [10, Theorem 7.1] guarantees that

$$\mathcal{D}^*(p;0,t)=\int_0^t\mathcal{H}^*(\dot p(s))\,ds \tag{6.37}$$

for every $t\in[0,T]$. Moreover, using Lemma 3.2 one can prove that $(\dot u(t),\dot e(t),\dot p(t))\in\mathcal{A}_{\mathrm{gKL}}(\dot w(t))$ for a.e. $t\in[0,T]$. Applying Proposition 6.9 to $(\dot u(t),\dot e(t),\dot p(t))$ with $\varphi=1$ yields

$$\langle\sigma(t),\dot p(t)\rangle^*=\int_\Omega\sigma(t):(E^*\dot w(t)-\dot e(t))\,dx. \tag{6.38}$$

Differentiation of (qs2)* with respect to time, together with (6.37) and (6.38), yields (b2), and conversely, integration of (b2) with respect to time yields (qs2)*. $\square$

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