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QUASISTATIC AND DYNAMIC EVOLUTION PROBLEMS FOR THIN BODIES IN PERFECT PLASTICITY

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Introduction

This thesis is devoted to the rigorous derivation of lower dimensional models for *thin bodies*, in the framework of linearised elasto-plasticity. A thin structure, such as a plate or a shell, is a three-dimensional body whose thickness is very small with respect to the other dimensions. Understanding the laws governing their motion is very important, since thin structures comprise a growing proportion of engineering constructions, like aircrafts, boats, bridges, and oil rigs. To describe the mechanical behaviour of a thin structure, it is usual to replace three-dimensional theories with lower dimensional theories, since they are simpler to treat analitically and numerically.

A crucial question is how to mathematically justify lower dimensional models, starting from the three-dimensional ones. In the classical approach these models are usually deduced via formal asymptotic expansions, which are based on some kinematical and geometrical restrictions on the class of deformations. For this reason the validity and the generality of these models is not always clear. It is therefore important to tackle this question by a rigorous approach. To this aim, a useful mathematical tool is Γ-convergence, a powerful theory introduced by Ennio De Giorgi in the 70's (see [15]). Roughly speaking, Γ-convergence is a variational convergence which ensures the convergence of minima and minimisers of a sequence of functionals, to the minima and minimisers of the reduced models, respectively. This approach has been successfully applied to the stationary case: for instance, in the framework of nonlinear elasticity to plates [25,26,31], beams [3,43,44,48,49], and shells [24,32,33]. More recently, an increasing interest has been given to evolutionary problems, where the scope is to understand the change in time of the state of the material. The approach based on Γ-convergence has been adapted also to the evolutionary setting: in nonlinear elastodynamics [1,2], crack evolution [6,23], plasticity [13,34,35,45], and delamination problems [42]. For the abstract theory of evolutionary Γ-convergence we refer to [41].

In this thesis we consider thin structures that exhibit an elasto-plastic behaviour. These are bodies, whose response is elastic as long as the applied loads do not exceed the yield stress of the material (that is, the deformation undergone by the body is reversible). When the yield stress is reached, the material undergoes a plastic deformation, that is, a permanent deformation in response to the applied forces. To be more specific, we will focus on the theory of linearised perfect plasticity. Perfect plasticity means that the yield stress remains constant during the evolution, and hardening and softening effects are neglected. We suppose that the plastic response of the body is governed by the associative Prandtl-Reuss flow rule, which is typically used to describe the plastic behaviour of metals. Furthermore, we assume the material to be homogeneous and isotropic.

The dynamic evolution problem in linearised perfect plasticity can be described as follows. Let $U \subset \mathbb{R}^3$ be the reference configuration of a body, let $u(t)$ be the displacement vectorfield at time t, and let $\operatorname{sym} Du(t)$ be the symmetric gradient of $u(t)$. The linearised strain $\operatorname{sym} Du(t)$ is decomposed as the sum of two symmetric matrices: the elastic strain $e(t)$ and the plastic strain $p(t)$. In the modelling of plastic behaviour of metals, plastic deformation is usually assumed to be volume preserving: for this reason, we assume $p(t,x)$ to be a deviatoric matrix for every $x \in U$ and every time t. We further suppose that the evolution is driven by a time-dependent boundary displacement $w(t)$ prescribed on a portion $\partial_d U$ of the boundary of U, by a time-dependent body force $f(t)$, and by a time-dependent surface force $g(t)$ applied on $\partial U \setminus \partial_d U$. The dynamic evolution problem consists in finding a triplet (u, e, p) such that the following conditions hold for every $t \geq 0$:

(c1) *kinematic admissibility:* $\operatorname{sym} Du(t) = e(t) + p(t)$ in U and $u(t) = w(t)$ on $\partial_d U$;

(c2) *constitutive law:* $\sigma(t) := \mathbb{C}e(t)$ in U, where $\sigma(t)$ is the stress field at time t and $\mathbb{C}$ is the elasticity tensor;

(c3) *equation of motion:* $\ddot{u}(t) - \operatorname{div} \sigma(t) = f(t)$ in U and $\sigma(t)\nu_{\partial\Omega} = g(t)$ on $\partial U \setminus \partial_d U$, where $\nu_{\partial U}$ is the outer unit normal to ∂U;

(c4) *stress constraint:* $\sigma_D(t) \in K$ in U, where σ_D is the deviatoric part of σ and K is a given convex and compact set in the space of deviatoric matrices $\mathbb{M}^{3\times 3}_D$;

(c5) *flow rule:* $\dot{p}(t,x)$ belongs to the normal cone to K at $\sigma_D(t,x)$ for every $x \in U$.

We are interested both in *dynamic evolutions*, and in *quasistatic evolutions*, where inertial effects are neglected. More precisely, in quasistatic evolutions the rate of change in time of the applied loads is so slow that one can assume the system to be at equilibrium at each time during the evolution. Thus, in the quasistatic framework, the equation of motion (c3) is replaced with

(c3)′ *equilibrium equation:* $-\operatorname{div} \sigma(t) = f(t)$ in U and $\sigma(t)\nu_{\partial U} = g(t)$ on $\partial U \setminus \partial_d U$,

for every $t \geq 0$. We remark that system (c1), (c2), (c3)′, (c4), and (c5) gives rise to a rate-independent process, while the dynamic evolution model (c1)–(c5) is not rate-independent, because of the inertial term. For the general theory of rate-independent systems we refer to [40]. Under suitable assumptions on the data, existence and uniqueness of solutions to system (c1)–(c5) in U has been proved in [5], while the existence of a solution for (c1), (c2), (c3)′, (c4), and (c5) in U, was originally established in the seminal paper [50], and more recently revisited in [12] by means of a variational approach.

This thesis consists of two parts. In the first one, we rigorously derive a dynamic evolution model for a plastic thin plate. In the second part, we consider a plastic shallow shell in the quasistatic framework. In both cases, we deduce models that belong to the framework of *Kirchhoff-Love theory*. This theory was introduced in 1888 (see [29]), and it is based on the kinematic assumption that straight lines normal to the mid-surface remain straight and normal after the deformation, within the first order. We underline that in our results these kinematic properties are not assumed a priori, but they are obtained at the limit by means of a rigorous convergence argument.

The first result of this thesis, which is discussed in Chapter 2, is the rigorous derivation of a dynamic evolution model for a thin plate in perfect plasticity. The corresponding result in the quasistatic case was established in [13].

Let $\omega \subset \mathbb{R}^2$ be a domain with a C^2 boundary and let $h > 0$. We consider a plate, whose reference configuration is given by the set

$$\Omega_h := \omega \times (-\tfrac{h}{2}, \tfrac{h}{2}).$$

Here ω represents the mid-surface of the plate, while the parameter h denotes its thickness.

We suppose that the body load $f_h(s)$ at every time s is purely vertical and the surface load is zero at every time. We denote by $w_h(s)$ the time-dependent boundary displacement prescribed on a portion $\Gamma_{d,h} := \partial_d\omega \times \left(-\frac{h}{2}, \frac{h}{2}\right)$ of the lateral boundary of the plate.

Let (u_h, e_h, p_h) be a solution of the dynamic evolution problem (c1)–(c5) in Ω_h with this choice of data. The solutions (u_h, e_h, p_h) provided by the existence results belong to the space

$$BD(\Omega_h) \times L^2(\Omega_h; \mathbb{M}^{3\times3}_{sym}) \times M_b(\Omega_h \cup \Gamma_{d,h}; \mathbb{M}^{3\times3}_D),$$

where $BD(\Omega_h)$ is the space of functions with bounded deformation on Ω_h, and $M_b(\Omega_h \cup \Gamma_{d,h}; \mathbb{M}^{3\times3}_D)$ is the space of $\mathbb{M}^{3\times3}_D$-valued bounded Radon measures on $\Omega_h \cup \Gamma_{d,h}$. From a mechanical point of view this formulation is consistent with the well known fact that displacements in perfect plasticity can develop jump discontinuities along so-called slip-surfaces, on which plastic strain concentrates. Furthermore, the Dirichlet boundary condition on $\Gamma_{d,h}$ is relaxed and takes the form

$$p_h(s) = (w_h(s) - u_h(s)) \odot \nu_{\partial\Omega_h}\mathcal{H}^2 \quad \text{ on } \Gamma_{d,h},$$

where $\mathcal{H}^2$ denotes the two-dimensional Hausdorff measure and $\odot$ is the symmetrised tensor product. The mechanical interpretation of this condition is the following: if the prescribed boundary displacement is not attained at time s, a plastic slip develops at the boundary with a strength proportional to $w_h(s) - u_h(s)$.

Because of the weak regularity of p_h (p_h and $\dot{p}_h$ are only measures in the space variable), the meaning of condition (c5) has to be clarified. In [5] this issue is overcome by expressing (c5) as a variational inequality involving only the stress variable σ_h and the velocity $\dot{u}_h$. In [7] the authors replace condition (c5) by its equivalent form

(c5)$'$ *maximum dissipation principle:* $H(\dot{p}_h(s)) = (\sigma_h)_D(s) : \dot{p}_h(s)$ in $\Omega_h \cup \Gamma_{d,h}$,

where $H(\xi) := \sup_{\eta\in K} \xi : \eta$ is the support function of K. The advantage of condition (c5)$'$, compared to (c5), is that the equality in (c5)$'$ has a meaning in a measure sense. This relies on a notion of duality beween stresses and plastic strains that was introduced in [30] and further developed in [12] and [22]. However, the definition of the duality requires some regularity of $\partial\Omega_h$ and of the relative boundary of $\Gamma_{d,h}$ in $\partial\Omega_h$. Since in our framework $\partial\Omega_h$ has only Lipschitz regularity, we prefer not to dwell on duality and we formulate (c5) as an energy inequality:

(c5)$''$ *energy inequality:* for every $0 \le t_1 \le t_2$

$$\begin{aligned}
\mathcal{Q}_h(e_h(t_2)) + \frac{1}{2}\|\dot{u}_h(t_2)\|^2_{L^2} + \int_{t_1}^{t_2} \mathcal{H}_h(\dot{p}_h(s))\, ds \le \mathcal{Q}_h(e_h(t_1)) + \frac{1}{2}\|\dot{u}_h(t_1)\|^2_{L^2} \\
+ \int_{t_1}^{t_2}\int_{\Omega_h} (\sigma_h(s) : \operatorname{sym} D\dot{w}_h(s) + \ddot{u}_h(s)\cdot\dot{w}_h(s))\, dx\, ds \\
+ \int_{t_1}^{t_2}\int_{\Omega_h} f_h(s)e_3 \cdot (\dot{u}_h(s) - \dot{w}_h(s))\, dx\, ds,
\end{aligned}$$

where

$$\mathcal{Q}_h(e_h(s)) := \frac{1}{2}\int_{\Omega_h} \mathbb{C}e_h(s,x) : e_h(s,x)\,dx$$

is the stored elastic energy at time s, while $\mathcal{H}_h(\dot{p}_h(s))$ is the plastic dissipation potential at time s, defined according to the theory of convex functions of measure (see Section 1.2). When the stress-strain duality is defined and (c1)–(c4) are satisfied, one can prove that conditions (c5)$'$ and (c5)$''$ are in fact equivalent. For the reader's convenience the proof of the existence for system (c1)–(c4), and (c5)$''$ is sketched in Section 2.3. In view of the subsequent analysis, a particular attention is paid to the dependence of the involved quantities on the thickness parameter h.

Existence of a dynamic evolution (u_h, e_h, p_h) in Ω_h is therefore established for every $h > 0$. Our main goal is to study the asymptotic behaviour of (u_h, e_h, p_h), as h tends to 0, and characterise its limit as a solution of a suitable limiting problem. This is the subject of Section 2.4.

To discuss the limiting behaviour of (u_h, e_h, p_h) it is convenient to rescale Ω_h to a domain Ω independent of h and to rescale time by setting $t := hs$. According to this change of variables, we define the rescaled displacement u^h on $[0,+\infty) \times \Omega$ as

$$u^h(t,x) := \big(u_h(\tfrac{t}{h},(x',hx_3)) \cdot e_\alpha\,,\; hu_h(\tfrac{t}{h},(x',hx_3)) \cdot e_3\big) \tag{1}$$

for $x = (x', x_3)$, $\alpha = 1, 2$. The spatial scaling of u_h is consistent with that of dimension reduction problems in linearised elasticity. In particular, the ratio of order h between the vertical and the tangential displacements can be rigorously justified starting from nonlinear elasticity, under the small strain assumption (see [26]). Note, however, that in linearised elasticity the problem is invariant under further scalings of u^h, while this is not the case in plasticity, because of the different homogeneity of the elastic energy and the dissipation potential. The scaling (1) is the correct one to see both elastic and plastic contributions in the limit as $h \to 0$ (see also [13]).

The time scaling of u_h is also consistent with the results in the context of elasticity (see, e.g., [1]): oscillations in Ω_h occur at a slow time scale, so that a time scaling is needed to observe oscillations in the limit as $h \to 0$.

The scaling for e_h and p_h is chosen in such a way that the sequence of the rescaled triplets $(u^h(t), e^h(t), p^h(t))$ still satisfies the additive decomposition $\operatorname{sym} Du^h(t) = e^h(t) + p^h(t)$ in Ω for every t. Finally, we perform the same scaling as in (1) on the boundary datum w_h, while for the body load we set

$$f^h(t,x) := \tfrac{1}{h} f_h(\tfrac{t}{h},(x',hx_3)).$$

In Theorem 2.4.1 we prove that, under suitable assumptions on the initial data and on the rescaled boundary condition and body load, the rescaled triplets $(u^h(t), e^h(t), p^h(t))$ converge, up to subsequences, to a limiting triplet $(u(t), e(t), p(t))$ for every time $t \geq 0$.

We now describe the conditions satisfied by the limiting triplet. For every $t \geq 0$ we have

(d1)* *reduced kinematic admissibility:* $u(t)$ is a Kirchhoff-Love displacement, that is,

$$u(t,x) = (\bar{u}_\alpha(t,x') - x_3\partial_\alpha u_3(t,x'), u_3(t,x')) \quad \text{for } x = (x',x_3),\ \alpha = 1,2,$$

where $\bar{u}(t) \in BD(\omega)$ and $u_3(t) \in BH(\omega)$, the space of functions with bounded Hessian. The strains $e(t)$ and $p(t)$ satisfy

$$\operatorname{sym} Du(t) = e(t) + p(t) \quad \text{in } \Omega, \qquad p(t) = (w(t) - u(t)) \odot \nu_{\partial\Omega}\mathcal{H}^2 \quad \text{on } \partial_d\Omega,$$

$$e_{i3}(t) = 0 \quad \text{in } \Omega, \qquad p_{i3}(t) = 0 \quad \text{in } \Omega \cup \partial_d\Omega, \qquad i = 1,2,3.$$

We note that the averaged tangential displacement $\bar{u}(t)$ may have jump discontinuities, while, because of the embedding of $BH(\omega)$ into $C(\overline{\omega})$, the normal displacement $u_3(t)$ is continuous, but its gradient may have jump discontinuities. In particular, the discontinuity sets of $u(t)$, that is, the limiting slip surfaces, are vertical surfaces. Condition (d1)* does not imply, in general, that $e(t)$ and $p(t)$ are affine with respect to x_3. However, they admit the following decomposition:

$$e(t) = \bar{e}(t) + x_3\hat{e}(t) + e_\perp(t), \qquad p(t) = \bar{p}(t) \otimes \mathcal{L}^1 + \hat{p}(t) \otimes x_3\mathcal{L}^1 - e_\perp(t), \tag{2}$$

where the components $\bar{e}(t), \hat{e}(t) \in L^2(\omega; \mathbb{M}^{2\times2}_{sym})$, $e_\perp(t) \in L^2(\Omega; \mathbb{M}^{2\times2}_{sym})$, $\bar{p}(t), \hat{p}(t) \in M_b(\omega \cup \partial_d\omega; \mathbb{M}^{2\times2}_{sym})$ satisfy

$$\operatorname{sym} D\bar{u}(t) = \bar{e}(t) + \bar{p}(t) \quad \text{in } \omega, \qquad \bar{p}(t) = (\bar{w}(t) - \bar{u}(t)) \odot \nu_{\partial\omega}\mathcal{H}^1 \quad \text{on } \partial_d\omega,$$

and

$$-D^2u_3(t) = \hat{e}(t) + \hat{p}(t) \quad \text{in } \omega, \qquad \hat{p}(t) = (\nabla u_3(t) - \nabla w_3(t)) \odot \nu_{\partial\omega}\mathcal{H}^1 \quad \text{on } \partial_d\omega.$$

Moreover, the vertical displacement $u_3(t)$ attains the boundary condition $u_3(t) = w_3(t)$ on $\partial_d\omega$. Here, $\bar{w}(t)$ and $w_3(t)$ are the Kirchhoff-Love components of the limiting displacement $w(t)$.

Since the component $e_\perp(t)$ has a non trivial dependence on the variable x_3, the limiting problem has a genuinely three-dimensional nature and in general cannot be reduced to a purely two-dimensional setting. This feature was already observed in the quasistatic case (see [13]) and is in contrast with the purely elastic case (see [46]).

In addition, the limiting triplet $(u(t), e(t), p(t))$ satisfies the following conditions for every $t \geq 0$:

(d2)* *reduced constitutive law:* $\sigma(t) := \mathbb{C}^*e(t)$ in Ω, where $\mathbb{C}^*$ is the reduced elasticity tensor, which is defined through a suitable minimisation formula (see (2.2.9));

(d3)* *equations of motion:* setting

$$\bar{f}(t, x') = \int_{-1/2}^{1/2} f(t, x)\, dx_3,$$

we have

$$\operatorname{div} \bar{\sigma}(t) = 0 \quad \text{in } \omega, \qquad \ddot{u}_3(t) - \tfrac{1}{12}\operatorname{div}\operatorname{div} \hat{\sigma}(t) = \bar{f}(t) \quad \text{in } \omega,$$

with corresponding Neumann boundary conditions on $\partial\omega \setminus \partial_d\omega$, where $\bar{\sigma}(t) := \mathbb{C}^*\bar{e}(t)$ and $\hat{\sigma}(t) := \mathbb{C}^*\hat{e}(t)$;

(d4)* *reduced stress constraint:* $\sigma(t) \in K^*$ in Ω, where $K^* := \partial H^*(0)$ is the subdifferential of the reduced dissipation potential H^* (whose expression is given in (2.2.11) through a minimisation formula) at 0;

(d5)* *reduced maximum dissipation principle:*

$$\mathcal{H}^*(\dot{p}(t)) = \langle \sigma(t), \dot{p}(t) \rangle_r.$$

In (d3)* we denoted the limiting vertical body load by f. The left-hand side in (d5)* is defined using the theory of convex functions of measures, while the right-hand side involves an ad-hoc notion of "reduced" stress-strain duality, introduced in [13, Section 7] for the study of the quasistatic case. We refer to Section 2.2 for the definition of the duality.

We note that the stretching component $\bar{\sigma}(t)$ and the bending component $\hat{\sigma}(t)$ of the stress decouple in the equations of motion (d3)*, while the whole stress $\sigma(t)$ is involved in the stress constraint (d4)* and in the maximum dissipation principle (d5)*. Thus, the component $\sigma_\perp(t)$ will in general play a role in satisfying these two conditions, leading to a non trivial dependence of the solutions on the thickness variable x_3. As mentioned earlier, this behaviour is not peculiar of the dynamic case, but was already observed in the quasistatic case. Indeed, an explicit example in [14] shows that the yielding threshold may be reached at different times along the vertical fibers of the plate, thus giving rise to a solution with $\sigma_\perp \neq 0$. The emergence of this multiyield behaviour was also observed in [28], where a formal asymptotic expansion of small strain oscillations in an elastoplastic plate with hardening was considered.

The proof of Theorem 2.4.1 is based on two main steps: first we deduce suitable compactness estimates for the three-dimensional evolutions, and then we pass to the limit in the equations via Γ-convergence arguments. Compactness estimates are obtained from the energy inequality (d5)$''$ and from some a posteriori regularity estimates for the three-dimensional problem (see (2.3.8) and (2.3.9)). Clearly the dependence of these inequalities on h is crucial in order to obtain meaningful bounds. While the behaviour of the energy inequality under scaling is relatively straightforward, dealing with the a posteriori estimate is more delicate. At this stage it is essential to have a purely vertical body load. Once these bounds are established, compactness is granted via Ascoli-Arzelà Theorem.

To pass to the limit in the equations, we cannot rely directly on Γ-convergence techniques, because of the inertial term. However, the key ideas of the proof are borrowed from this theory. More precisely, to deduce the limiting equations of motion we construct suitable sequences of test functions for the three-dimensional problems. This is reminiscent of the recovery sequence construction in Γ-convergence. To pass to the limit in (d5)$''$ we apply a Γ-liminf inequality satisfied by $\mathcal{Q}^*$ and $\mathcal{H}^*$. Once we have a limiting energy inequality, condition (d5)* follows by using the reduced stress-strain duality and its properties.

The last section of Chapter 2 is devoted to the study of some properties of solutions to the limiting system (d1)*–(d5)*. In Proposition 2.5.1 we prove uniqueness of the normal displacement and of the elastic strain. This does not ensure uniqueness of the solution to the limiting problem. Indeed, in Proposition 2.5.2 we show that for "tangential" initial and boundary data system (d1)*–(d5)* reduces to a two-dimensional quasistatic evolution, whose solutions are in general not unique (see, e.g., [50]).

The second part of this thesis, which corresponds to Chapter 3, is devoted to the rigorous justification of a quasistatic evolution model for a *shallow shell*, in the framework of linearised perfect plasticity. Roughly speaking, a shallow shell is a shell where the amount of deviation from a plane, measured normally to the plane, is very small. More precisely, we assume the deviation to be of the same order of the thickness of the shell. Hence, our analysis is reminiscent of that developed in [13] for an elasto-plastic thin plate, but the nontrivial geometry of the shell gives rise to a substantial amount of additional difficulties.

We consider a three-dimensional shallow shell occupying the reference configuration $\Sigma_h := \Psi_h(\Omega)$. Here $\Omega := \omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, where $\omega \subset \mathbb{R}^2$ is a C^2 domain, and $0 < h \ll 1$. The map $\Psi_h : \overline{\Omega} \to \overline{\Sigma}_h$ is given by

$$\Psi_h(x) := (x', h\theta(x')) + hx_3\nu_{S_h}(x') \quad \text{for every } x = (x', x_3) \in \overline{\Omega},$$

where ν_{S_h} is the outer unit normal to the two-dimensional surface

$$S_h := \{(x', h\theta(x')) : x' \in \omega\},$$

and $\theta : \overline{\omega} \to \mathbb{R}$ is a scalar function.

Let $T > 0$. Let $w_h(t)$ be a time-dependent displacement prescribed on a subset $\partial_d\Sigma_h := \Psi_h(\partial_d\Omega)$ of the lateral boundary of Σ_h (where $\partial_d\Omega$ is a portion of the lateral boundary of Ω), and assume there are no external loads.

Let (u_h, e_h, p_h) be a solution of (c1), (c2), (c3)$'$, (c4), and (c5) in Σ_h with this choice of data. The scope of Chapter 3 is to characterise the limiting behaviour of $(u_h(t), e_h(t), p_h(t))$, as h tends to 0. As we did in Chapter 2, it is convenient to rescale the triplet $(u_h(t), e_h(t), p_h(t))$, in such a way to have it defined on Ω. In particular, we define the rescaled displacement u^h on $[0, T] \times \Omega$ as

$$u^h(t, x) := (u_h(t, \Psi_h(x)) \cdot e_\alpha, hu_h(t, \Psi_h(x)) \cdot e_3). \tag{3}$$

We note that here no time-scaling is performed. This would be superflous, since the problem is rate-independent. In Theorem 3.5.3 we show the convergence of the rescaled triplets (under suitable assumptions on the initial data and on the rescaled boundary condition) to a limiting triplet $(u(t), e(t), p(t))$ in the space

$$BD(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times3}_{sym}) \times M_b(\Omega \cup \partial_d\Omega, \mathbb{M}^{3\times3}_{sym}),$$

which is a solution of the following limiting problem: for every $t \in [0, T]$ we have

(e1)* *reduced kinematic admissibility:* $u(t)$ is a Kirchhoff-Love displacement, and

$$\operatorname{sym} Du(t) + \nabla\theta \odot \nabla u_3(t) = e(t) + p(t) \quad \text{in } \Omega,$$
$$p(t) = (w(t) - u(t)) \odot \nu_{\partial\Omega} \quad \text{on } \partial_d\Omega,$$
$$e_{i3}(t) = 0 \quad \text{in } \Omega, \qquad p_{i3}(t) = 0 \quad \text{in } \Omega \cup \partial_d\Omega, \qquad i = 1, 2, 3,$$

where $\nu_{\partial\Omega}$ is the outer unit normal to $\partial\Omega$, and $w(t)$ is the limiting boundary diplacement;

(e2)* *reduced constitutive law:* $\sigma(t) := \mathbb{C}^* e(t)$, where $\mathbb{C}^*$ is the reduced elasticity tensor defined in (d2)* ;

(e3)* *equilibrium equations:*

$$\operatorname{div} \bar{\sigma}(t) = 0 \quad \text{in } \omega, \qquad \tfrac{1}{12}\operatorname{div}\operatorname{div} \hat{\sigma}(t) + \bar{\sigma}(t) : D^2\theta = 0 \quad \text{in } \omega,$$

with corresponding Neumann boundary conditions on $\partial\omega \setminus \partial_d\omega$, where $\partial_d\omega$ is the projection of $\partial_d\Omega$ on the plane $\{x_3 = 0\}$;

(e4)* *reduced stress constraint:* $\sigma(t) \in K^*$ in Ω, where K^* is the set introduced in (d4)* ;

(e5)* *reduced maximum dissipation principle:*

$$\mathcal{H}^*(\dot p(t)) = \langle \sigma(t), \dot p(t)\rangle^* .$$

The dissipation potential $\mathcal{H}^*$ in (e5)* coincides with that of (d5)*, while the duality at the right-hand side is slightly different from that used in (d5)*, because of the different kinematic admissibility of the limiting triplet. For $\theta \equiv 0$, the model above coincides with that of [13], that also corresponds to (d1)*–(d5)* when the inertial term is neglected. When θ is different from 0, curvature effects need to be taken into account in the limit. This results in the appearance of the term $\nabla\theta \odot \nabla u_3(t)$ in the kinematic admissibility condition and into a coupling of the stretching component $\bar\sigma(t)$ and the bending component $\hat\sigma(t)$ in the equilibrium equations.

Since $u(t)$ is a Kirchhoff-Love displacement, its symmetrised gradient is affine with respect to x_3, and the strains $e(t)$ and $p(t)$ can be still decomposed as in (2). Because of the new kinematic admissibility condition, the zeroth and first order moments of $e(t)$ and $p(t)$ now satisfy

$$\operatorname{sym} D\bar u(t) + \nabla\theta \odot \nabla u_3(t) = \bar e(t) + \bar p(t) \quad \text{in } \omega, \quad \bar p(t) = (\bar w(t) - \bar u(t)) \odot \nu_{\partial\omega}\mathcal{H}^1 \quad \text{on } \partial_d\omega,$$

and

$$D^2 u_3(t) = -(\hat e(t) + \hat p(t)) \quad \text{in } \omega, \quad \hat p(t) = (\nabla u_3(t) - \nabla w_3(t)) \odot \nu_{\partial\omega}\mathcal{H}^1 \quad \text{on } \partial_d\omega,$$

for every $t \in [0,T]$, where $\nu_{\partial\omega}$ is the outer unit normal to $\partial\omega$. As for the system (d1)*–(d5)*, the limiting problem (e1)*–(e5)* is genuinely three-dimensional, because the component $e_\perp(t)$, that has a non trivial dependence on the variable x_3, may in general play a role in satisfying (e4)* and (e5)*.

Now we describe the strategy of the proof of Theorem 3.5.3. The abstract theory of evolutionary Γ-convergence for rate-independent processes developed in [41] cannot be applied directly here. Indeed, this theory consists in considering separately the Γ-limit of the stored energy functionals and of the dissipation distances, and in coupling them through the construction of a joint recovery sequence. This approach is not applicable to our case, since in perfect plasticity the stored elastic energy and the plastic dissipation must be considered together to get the right compactness properties. As a first step, we focus on the static case. We study the Γ-limit, as h tends to 0, of the total energy functional

$$\mathcal{E}_h(v,\eta,q) := \frac{1}{2}\int_{\Sigma_h} \mathbb{C}\eta(x) : \eta(x)\,dx + \int_{\Sigma_h\cup\partial_d\Sigma_h} H\left(\frac{dq}{d|q|}\right) d|q|$$

defined on all triplets (v,η,q) satisfying

$$\operatorname{sym} Dv = \eta + q \quad \text{in } \Sigma_h, \qquad q = (w_h - v)\odot \nu_{\partial\Sigma_h}\mathcal{H}^2 \text{ on } \partial_d\Sigma_h.$$

In Theorem 3.4.3, we show that the Γ-limit of $\mathcal{E}_h$ (rescaled to the domain Ω) is the functional

$$\mathcal{I}(u,e,p) := \int_\Omega Q^*(e(x))\,dx + \mathcal{H}^*(p)$$

for every $(u,e,p) \in \mathcal{A}_{GKL}(w)$, that is, the class of all (u,e,p) satisfying the kinematic admissibility condition (e1)*. The main difficulty in the proof of this result, compared with that of [13], is the following: the scaled sisplacement u in (3), which can be also written as

$$u = R_h^{-1} v \circ \Psi_h,$$

where

$$R_h := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{h} \end{pmatrix},$$

does not belong to $BD(\Omega)$, since we only know that

$$\operatorname{sym}(R_h Du R_h F_h^{-1}) \in M_b(\Omega; \mathbb{M}^{3\times 3}_{sym}), \tag{4}$$

where $F_h := D\Psi_h R_h$. Furthermore, we cannot rely on the classical Korn-Poincaré inequality for BD functions, as it was done in [13]. Indeed, if we expand F_h^{-1} with respect to h (see Lemma 3.2.1), we obtain

$$\begin{aligned} \operatorname{sym}(R_h Du R_h F_h^{-1})_{\alpha\beta} &= (\operatorname{sym} Du - \partial_3 u \odot \nabla\theta)_{\alpha\beta} + O(h^2)\|u\|_{BV}, \\ \operatorname{sym}(R_h Du R_h F_h^{-1})_{\alpha 3} &= \tfrac{1}{h}\left((\operatorname{sym} Du - \partial_3 u \odot \nabla\theta)_{\alpha 3} + O(h^2)\|u\|_{BV}\right), \\ \operatorname{sym}(R_h Du R_h F_h^{-1})_{33} &= \tfrac{1}{h^2}\left(\partial_3 u_3 (1 + O(h^2)) + h^2 \nabla u_3 \cdot \nabla\theta + O(h^4)\|u\|_{BV}\right), \end{aligned}$$

where $O(h^p)$ is a quantity uniformly bounded by h^p in $\overline{\Omega}$ and $BV(\Omega)$ is the space of functions with bounded variation on Ω. We note that the remainders are controlled by the BV-norm of u, which is not a priori bounded. Therefore, a bound on $\operatorname{sym}(R_h Du R_h F_h^{-1})$ does not provide, in general, any bound on $\operatorname{sym} Du$. To overcome this obstacle, it is convenient to express the scaled displacement in intrinsic curvilinear coordinates, i.e., we define the vectorfield

$$u(h) := (D\Psi_h)^T R_h u.$$

The advantage is that the quantity (4), written in these coordinates (see Proposition 3.3.1), has a simpler form; namely, it is related to

$$(R_h \operatorname{sym} Du(h) R_h)_{ij} - \Gamma^k_{ij}(h) u_k(h),$$

where $\Gamma^k_{ij}(h)$ are the scaled Christoffel symbols of Σ_h. In this expression, the first term is a rescaled symmetrised gradient, while the second term depends only on the displacement, and not on its derivatives. This allows us to show, for the vectorfield of curvilinear coordinates $u(h)$, an ad-hoc Korn-Poincaré inequality on a shallow shell (Theorem 3.3.4). We underline that in this proof, the order of the coefficients $\Gamma^k_{ij}(h)$ with respect to h is crucial, and it is a consequence of the shallowness assumption (that is, of the fact that the amount of the deviation is of order h).

Theorem 3.3.4, together with a compactness result (Lemma 3.4.1), is the key ingredient to deduce compactness for the sequence of scaled triplets. Another delicate point of the proof is to show that the limiting triplet (u, e, p) belongs to the class $\mathcal{A}_{GKL}(w)$: indeed, it is not straightforward to establish the Dirichlet boundary condition

$$p = (w - u) \odot \nu_{\partial\Omega} \mathcal{H}^2 \quad \text{ on } \partial_d\Omega.$$

The idea is to extend the scaled triplets by using the boundary datum w_h, to an open set V such that $V \cap \partial\Omega = \partial_d\Omega$. To ensure the necessary bounds, it is again convenient to express the scaled triplets in their curvilinear coordinates. Finally, the contruction of a recovery sequence is based on Lemma 3.2.7, an approximation result which ensures the density of smooth triplets in $\mathcal{A}_{GKL}(w)$. This is a technical lemma, whose proof is analogous to that of [13, Theorem 4.7].

Once Γ-convergence is established in the static case, the proof of the convergence of the quasistatic evolutions is rather standard. We consider the three-dimensional problem and the limiting problem in terms of their variational formulation, where the equilibrium equations are replaced by a stability condition, and the maximum dissipation principle by an energy balance. To deduce the global stability in the reduced problem, we use as test functions in the three-dimensional problem the recovery sequence provided by Theorem 3.4.3. The energy equality follows from the Γ-liminf inequality provided again by Theorem 3.4.3.

In the last part of Chapter 3 we extend the result about the convergence of quasistatic evolutions to the case of nonzero external loads (Theorem 3.6.7). As usual in perfect plasticity, we require a safe load condition that is uniform with respect to h, to guarantee the coercivity of the total energy functional, and to overcome the lack of continuity of the work of external loads with respect to the convergence of the displacements. Moreover, a key result is a semicontinuity property for the plastic dissipation and the stress-strain duality (Proposition 3.6.8).

The content of Chapter 2 corresponds to the article [37], while the results of Chapter 3 are contained in [38]. These two papers have been both obtained in collaboration with Maria Giovanna Mora.

Chapter 1

Preliminary results

In this Chapter we collect the main notations, definitions and classical results that we will use in the present thesis.

We will assume that *Latin indices* like i, j, k take their values in the set $\{1, 2, 3\}$ and *Greek indices* like α, β, γ in the set $\{1, 2\}$.

Moreover we will adopt the *repeated index summation convention*, for example

$$A_{ij}x_j$$

means

$$\sum_{j=1}^{3} A_{ij}x_j.$$

1.1 Notations

Vectors and matrices

- $u \cdot v := \sum_{i=1}^{n} u_i v_i$: scalar (or inner) product in $\mathbb{R}^n$;
- $|u| := \sqrt{u \cdot u}$: Euclidean norm in $\mathbb{R}^n$;
- $\delta_{ij} := \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$: Kronecker symbol;
- $\{e_1, e_2, \ldots, e_n\}$: canonical basis of $\mathbb{R}^n$;
- $\mathbb{M}^{m\times n}$: set of all real matrices with m rows and n columns;
- $\mathbb{M}^{n\times n}_{\text{sym}}$: set of all symmetric matrices of order n;
- A^T: transpose of a matrix $A \in \mathbb{M}^{m\times n}$;
- $\operatorname{sym} A := \frac{A+A^T}{2}$: symmetric part of a square matrix $A \in \mathbb{M}^{n\times n}$;
- $\text{tr}A := \sum_{i=1}^{n} a_{ii}$: trace of a square matrix $A \in \mathbb{M}^{n\times n}$;
- $\mathbb{M}^{n\times n}_{D}$: set of all deviatoric matrices, i.e., symmetric matrices of order n with zero trace;
- $I_{n\times n}$: identity matrix of order n;

- $A_D := A - \frac{1}{n}\,(\mathrm{tr}A)\, I_{n\times n}$: deviatoric part of a matrix $A \in \mathbb{M}^{n\times n}$;
- $\det A$: determinant of a square matrix $A \in \mathbb{M}^{n\times n}$;
- $\mathrm{cof}A$: cofactor matrix of a square matrix $A \in \mathbb{M}^{n\times n}$;
- $(a\otimes b)_{ij} := a_i b_j$: tensor product of two vectors $a, b \in \mathbb{R}^n$;
- $(a\odot b)_{ij} := \frac{1}{2}\,(a_i b_j + a_j b_i)$: symmetrised tensor product of two vectors $a, b \in \mathbb{R}^n$;

Functional spaces.

- $\mathcal{L}(X;Y)$: space of all linear and continuous functionals between two normed spaces X and Y;
- $X' := \mathcal{L}(X;\mathbb{R})$: dual of X;
- $X([a,b];Y)$: space of all functions from $[a,b]$ into Y which belong to X, where X, Y are two Banach spaces.

Let Ω be a subset of $\mathbb{R}^m$.

- $C^k(\Omega;\mathbb{R}^n)$: space of all continuously differentiable functions of order k from $\Omega\subseteq\mathbb{R}^m$ into $\mathbb{R}^n$, in particular $C^k(\Omega) := C^k(\Omega;\mathbb{R})$;
- $C^\infty(\Omega;\mathbb{R}^n) := \{u\in C^k(\Omega;\mathbb{R}^n)$ for every $k\in\mathbb{N}\}$: space of smooth functions;
- $C^\infty_c(\Omega;\mathbb{R}^n) := \{u \in C^\infty(\Omega;\mathbb{R}^n) : \mathrm{supp}\, u$ is compact $\}$: space of smooth functions with compact support;
- $L^p(\Omega;\mathbb{R}^n) := \{u : \Omega\to\mathbb{R}^n : u$ Lebesgue measurable , $\|u\|_{L^p(\Omega)} < +\infty\}$: Lebesgue spaces, where

$$\|u\|_{L^p} := \begin{cases} \left\{\int_\Omega \mid u(x)\mid^p \ dx\right\}^{\frac{1}{p}} & \text{if } p\in[1,+\infty),\\ \inf\{M\geq 0 :\mid u(x)\mid\leq M \text{ for a.e. } x\in\Omega\} & \text{if } p = +\infty.\end{cases}$$

  Let $\alpha := (\alpha_1,\dots,\alpha_k)\in\mathbb{N}^k$ be a multi-index and let $\mid\alpha\mid = \alpha_1+\dots+\alpha_m$. Then let us define:

- $W^{k,p}(\Omega;\mathbb{R}^n) := \{u\in L^p(\Omega;\mathbb{R}^n) : \partial_\alpha u\in L^p(\Omega;\mathbb{R}^n),$ for $\mid\alpha\mid\leq k\}$: Sobolev space, provided with norm

$$\|u\|_{W^{k,p}} := \begin{cases} \left\{\int_\Omega \sum_{|\alpha|\leq k} \mid \partial_\alpha u(x)\mid^p \ dx\right\}^{\frac{1}{p}} & \text{if } p\in[1,+\infty),\\ \max\limits_{|\alpha|\leq k} \|\ \partial_\alpha u\ \|_{L^\infty} & \text{if } p = +\infty.\end{cases}$$

- $W^{k,p}_0(\Omega;\mathbb{R}^n) := \overline{C^\infty_c(\Omega;\mathbb{R}^n)}^{\|\cdot\|_{W^{k,p}}}$: closure of $C_c(\Omega;\mathbb{R}^n)$ in $W^{k,p}(\Omega;\mathbb{R}^n)$;
- $H^k(\Omega;\mathbb{R}^n) := W^{k,2}(\Omega;\mathbb{R}^n)$ and $H^k_0(\Omega;\mathbb{R}^n) := W^{k,2}_0(\Omega;\mathbb{R}^n)$;
- $\to$ denotes the strong convergence of a sequence in a functional space;
- $\rightharpoonup$ denotes the weak (or weak*) convergence of a sequence in a functional space.

1.2 Functions of bounded deformation and bounded Hessian

In this section we recall some notions of measure theory, and the most important properties of functions with bounded deformation and bounded Hessian.

Measures

The Lebesgue measure on $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$ and the $(n-1)$-dimensional Hausdorff measure by $\mathcal{H}^{n-1}$. Given a Borel set $B \subset \mathbb{R}^n$ and a finite dimensional Hilbert space X, $M_b(B;X)$ denotes the space of bounded Borel measures on B with values in X, endowed with the norm $\|\mu\|_{M_b} := |\mu|(B)$, where $|\mu| \in M_b(B;\mathbb{R})$ is the variation of the measure μ. For every $\mu \in M_b(B;X)$ we consider the Lebesgue decomposition $\mu = \mu^a + \mu^s$, where μ^a is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^n$ and μ^s is singular with respect to $\mathcal{L}^n$. If $\mu^s = 0$, we always identify μ with its density with respect to $\mathcal{L}^n$. If the relative topology of B is locally compact, by Riesz Representation Theorem $M_b(B;X)$ can be identified with the dual of $C_0(B;X)$, which is the space of continuous functions $\varphi : B \to X$ such that the set $\{\varphi \geq \varepsilon\}$ is compact for every $\varepsilon > 0$. The weak* topology of $M_b(B;X)$ is defined using this duality. The duality between measures and continuous functions, as well as between other pairs of spaces, according to the context, is denoted by $\langle \cdot, \cdot \rangle$.

Convex functions of measures

Let U be an open set of $\mathbb{R}^n$ and let Γ_0 an open subset (in the relative topology) of ∂U. For every $\mu \in M_b(U \cup \Gamma_0; X)$ let $d\mu/d|\mu|$ be the Radon-Nikodým derivative of μ with respect to its variation $|\mu|$. Let $H_0 : X \to [0, +\infty)$ be a convex and positively one-homogeneous function such that

$$r|\xi| \leq H_0(\xi) \leq R|\xi| \quad \text{for every } \xi \in X,$$

where r and R are two constants, with $0 < r \leq R$. According to the theory of convex functions of measures, developed in [27], we introduce the nonnegative Radon measure $H_0(\mu) \in M_b(U \cup \Gamma_0)$ defined by

$$H_0(\mu)(A) := \int_A H_0\Big(\frac{d\mu}{d|\mu|}\Big)\, d|\mu|$$

for every Borel set $A \subset U \cup \Gamma_0$. We also consider the functional $\mathcal{H}_0 : M_b(U \cup \Gamma_0; X) \to [0, +\infty)$ defined by

$$\mathcal{H}_0(\mu) := H_0(\mu)(U \cup \Gamma_0) = \int_{U \cup \Gamma_0} H_0\Big(\frac{d\mu}{d|\mu|}\Big)\, d|\mu|$$

for every $\mu \in M_b(U \cup \Gamma_0; X)$. One can prove that $\mathcal{H}_0$ is lower semicontinuous on $M_b(U \cup \Gamma_0; X)$ with respect to weak* convergence (see, e.g., [4, Theorem 2.38]).

Lipschitz functions with values into a Banach space

Let $T > 0$ and let X be the dual of a separable Banach space. We denote by $\mathrm{Lip}([0,T];X)$ the space of Lipschitz functions on $[0,T]$ with values in X. If

$$f \in \mathrm{Lip}([0,T];X),$$

then the weak* limit

$$\dot{f}(t) := w^*\text{-}\lim_{s\to t} \frac{f(s)-f(t)}{s-t} \tag{1.2.1}$$

exists for a.e. $t \in [0,T]$ (see, e.g., Theorem 7.1 in [12]). If in addition X is separable, then for every $f \in \mathrm{Lip}([0,T];X)$ the limit in (1.2.1) is actually in the strong topology of X, the map $t \mapsto \dot{f}(t)$ is measurable by Pettis Theorem, and

$$\mathrm{Lip}([0,T];X) = W^{1,\infty}([0,T];X).$$

Functions with bounded deformation

Let $U \subset \mathbb{R}^n$ be an open set. The space $BD(U)$ of functions with bounded deformation is the space of all $u \in L^1(U;\mathbb{R}^n)$, whose symmetric gradient (in the sense of distributions) $\mathrm{sym}\, Du$ belongs to the space $M_b(U;\mathbb{M}^{n\times n}_{sym})$. It is easy to see that $BD(U)$ is a Banach space with the norm

$$\|u\|_{BD} := \|u\|_{L^1} + \|\,\mathrm{sym}\, Du\|_{M_b}.$$

We say that a sequence $(u_k)_k$ converges to u weakly* in $BD(U)$ if $u_k \rightharpoonup u$ weakly in $L^1(U;\mathbb{R}^n)$ and $\mathrm{sym}\, Du_k \rightharpoonup \mathrm{sym}\, Du$ weakly* in $M_b(U;\mathbb{M}^{n\times n}_{sym})$. Every bounded sequence in $BD(U)$ has a weakly* converging subsequence. Moreover, if U is bounded and has a Lipschitz boundary, then $BD(U)$ can be continuously embedded in $L^{n/(n-1)}(U;\mathbb{R}^n)$ and compactly embedded in $L^p(U;\mathbb{R}^n)$ for every $p < n/(n-1)$. Moreover, every function $u \in BD(U)$ has a trace, still denoted by u, which belongs to $L^1(\partial U;\mathbb{R}^n)$. Now we recall the classical Korn-Poincaré inequality in BD.

Theorem 1.2.1. *Let Γ be a nonempty open subset of ∂U. Then there exists a constant $C > 0$, depending on U and Γ, such that*

$$\|u\|_{BD} \le C(\|u\|_{L^1(\Gamma)} + \|\,\mathrm{sym}\, Du\|_{M_b}). \tag{1.2.2}$$

for every $u \in BD(U)$.

For the general properties of $BD(U)$ we refer to [51].

Functions with bounded Hessian

The space $BH(U)$ of functions with bounded Hessian is the space of all functions $u \in W^{1,1}(U)$, whose Hessian D^2u (in the sense of distributions) belongs to $M_b(U;\mathbb{M}^{n\times n}_{sym})$. It is easy to see that $BH(U)$ is a Banach space endowed with the norm

$$\|u\|_{BH} := \|u\|_{W^{1,1}} + \|D^2u\|_{M_b}.$$

If U has the cone property, then $BH(U)$ coincides with the space of functions in $L^1(U)$ whose Hessian belongs to $M_b(U;\mathbb{M}^{n\times n}_{sym})$. If U is bounded and has a Lipschitz boundary, $BH(U)$ can be embedded into $W^{1,n/(n-1)}(U)$. If U is bounded and has a C^2 boundary, then for every function $u \in BH(U)$ one can define the traces of u and ∇u, still denoted by u and ∇u: they satisfy $u \in W^{1,1}(\partial U)$, $\nabla u \in L^1(\partial U;\mathbb{R}^n)$, and $\frac{\partial u}{\partial \tau} = \nabla u \cdot \tau \in L^1(\partial U)$ for every τ tangent vector to ∂U. If in addition $n = 2$, then $BH(U)$ embeds into $C(\overline{U})$, which is the space of continuous functions on $\overline{U}$. For the general properties of $BH(U)$ we refer to [17].

Finally, we recall the Poincaré inequality in $BH(\omega)$ (see, e.g., [17, Proposition 1.3]).

Theorem 1.2.2. *Let γ be a nonempty open subset of $\partial\omega$. Then there exists a constant $C > 0$ depending on ω and γ such that*

$$\|u\|_{BH} \leq C(\|u\|_{L^1(\gamma)} + \|\nabla u\|_{L^1(\gamma)} + \|D^2 u\|_{M_b}) \tag{1.2.3}$$

for every $u \in BH(\omega)$.

1.3 Γ-convergence

In this Section we provide the definition and some properties of Γ-convergence.

Definition 1.3.1. Let X be a metric space, let $\varepsilon > 0$ and let $F_\varepsilon, F : X \to \mathbb{R} \cup \{\pm\infty\}$. Then F_ε Γ-converges to F if the following hold:

(i) **(liminf inequality)** for every sequence (x_ε) converging to x

$$F(x) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon); \tag{1.3.1}$$

(ii) **(existence of a recovery sequence)** there exists a sequence (x_ε) converging to x such that

$$F(x) = \lim_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon). \tag{1.3.2}$$

We recall also the definition of equi-coercivity.

Definition 1.3.2. A sequence $F_\varepsilon : X \to \mathbb{R} \cup \{\pm\infty\}$ is equi-coercive if for every $s \in \mathbb{R}$ there exists a compact set K_s such that $\{x \in X : F_\varepsilon(x) \leq s\} \subseteq K_s$.

Now we can state the main convergence result of Γ-convergence.

Theorem 1.3.3. *Let X be a metric space and let (F_ε) be a equi-coercive sequence of functions on X and assume that F_ε Γ-converges to F. Then there exists*

$$\min_X F = \lim_{\varepsilon\to 0} \inf_X F_\varepsilon.$$

Moreover, if (x_ε) is a precompact sequence such that $\lim_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) = \lim_{\varepsilon\to 0} \inf_X F_\varepsilon$, then every limit of a subsequence of (x_ε) is a minimum point for F.

Chapter 2

A dynamic evolution model for perfectly plastic plates

2.1 Overview of the chapter

In this Chapter we consider the dynamic evolution of a linearly elastic-perfectly plastic thin plate subject to a purely vertical body load. As the thickness of the plate goes to zero, we prove that the three-dimensional evolutions converge to a solution of a certain reduced model. In the limiting model admissible displacements are of Kirchhoff-Love type. Moreover, the motion of the body is governed by an equilibrium equation for the stretching stress, a hyperbolic equation involving the vertical displacement and the bending stress, and a rate-independent plastic flow rule. Some further properties of the reduced model are also discussed.

The chapter is organized as follows. In Section 2.2 we describe the formulation of the problem. In Section 2.3, we prove the existence of three-dimensional dynamic evolutions. Section 2.4 concerns the convergence of dynamic evolutions. Finally, in Section 2.5 we discuss some properties of the reduced problem.

2.2 Setting of the problem

2.2.1 The three-dimensional problem

In this section we describe the setting of the three-dimensional problem.

The reference configuration. Let $h > 0$ and let $\omega \subset \mathbb{R}^2$ be a domain (that is, an open, connected, and bounded set) with a C^2 boundary. We consider a thin plate whose reference configuration is given by

$$\Omega_h := \omega \times (-\tfrac{h}{2}, \tfrac{h}{2}).$$

We set $\Omega := \Omega_1$ and for $x \in \Omega$ we write $x = (x', x_3)$, where $x' \in \omega$ and $x_3 \in (-1/2, 1/2)$. We suppose that the boundary of ω is partitioned into two disjoint open sets $\partial_d\omega$, $\partial_n\omega$ (which are the Dirichlet and the Neumann part of $\partial\omega$, respectively) and their common boundary $\partial_{|\partial\omega}\partial_d\omega$, that is,

$$\partial\omega = \partial_d\omega \cup \partial_n\omega \cup \partial_{|\partial\omega}\partial_d\omega.$$

We assume that $\partial_{|\partial\omega}\partial_d\omega = \{P_1, P_2\}$, where P_1 and P_2 are two points of $\partial\omega$. Moreover, we define $\Gamma_{d,h} := \partial_d\omega \times (-\frac{h}{2}, \frac{h}{2})$ and $\Gamma_{n,h} := \partial\Omega_h \setminus \overline{\Gamma}_{d,h}$. We also set $\partial_d\Omega := \Gamma_{d,1}$ and

$\partial_n\Omega := \Gamma_{n,1}$. We will denote the outer unit normal to $\partial\Omega_h$ and to $\partial\omega$ by $\nu_{\partial\Omega_h}$ and by $\nu_{\partial\omega}$, respectively.

The stored elastic energy. Let $\mathbb{C}$ be the three-dimensional elasticity tensor, considered as a symmetric positive definite linear operator $\mathbb{C} : \mathbb{M}^{3\times3}_{sym} \to \mathbb{M}^{3\times3}_{sym}$, and let $Q : \mathbb{M}^{3\times3}_{sym} \to [0,+\infty)$ be the quadratic form associated with $\mathbb{C}$, defined by

$$Q(\xi) := \frac{1}{2}\mathbb{C}\xi : \xi \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{sym}.$$

It turns out that there exists two positive constants $\alpha_\mathbb{C}$ and $\beta_\mathbb{C}$, with $\alpha_\mathbb{C} \le \beta_\mathbb{C}$, such that

$$\alpha_\mathbb{C}|\xi|^2 \le Q(\xi) \le \beta_\mathbb{C}|\xi|^2 \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{sym}. \tag{2.2.1}$$

These inequalities imply that

$$|\mathbb{C}\xi| \le 2\beta_\mathbb{C}|\xi| \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{sym}. \tag{2.2.2}$$

It is convenient to introduce the quadratic form $\mathcal{Q}_h : L^2(\Omega_h;\mathbb{M}^{3\times3}_{sym}) \to [0,+\infty)$ given by

$$\mathcal{Q}_h(e) := \int_{\Omega_h} Q(e(x))\,dx$$

for every $e \in L^2(\Omega_h;\mathbb{M}^{3\times3}_{sym})$. It describes the stored elastic energy of a configuration of Ω_h, whose elastic strain is e. Since $\mathcal{Q}_h$ is a convex functional, it is lower semicontinuous with respect to the weak convergence of $L^2(\Omega_h;\mathbb{M}^{3\times3}_{sym})$. We set $\mathcal{Q} := \mathcal{Q}_1$.

The plastic dissipation. Let K be a convex and compact set in $\mathbb{M}^{3\times3}_D$, whose boundary ∂K is interpreted as the yield surface. We assume that there exist two positive constants r_K and R_K, with $r_K \le R_K$, such that

$$B(0,r_K) \subset K \subset B(0,R_K), \tag{2.2.3}$$

where $B(0,r) := \{\xi \in \mathbb{M}^{3\times3}_D : |\xi| \le r\}$. The support function of K, which represents the three-dimensional plastic dissipation potential, is the function $H : \mathbb{M}^{3\times3}_D \to \mathbb{R}$ given by

$$H(\xi) := \sup_{\tau\in K} \xi : \tau \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_D.$$

It is easy to see that H is convex, positively 1-homogeneous, and satisfies the triangle inequality. Moreover, by (2.2.3) one deduces that

$$r_K|\xi| \le H(\xi) \le R_K|\xi| \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_D. \tag{2.2.4}$$

From standard convex analysis we also have that the set K coincides with the subdifferential $\partial H(0)$ of H at 0.

Let $\mu \in M_b(\Omega_h \cup \Gamma_{d,h};\mathbb{M}^{3\times3}_D)$ and let $d\mu/d|\mu|$ be the Radon-Nikodým derivative of μ with respect to its variation $|\mu|$. According to the theory of convex functions of measures (see [27]), we define the nonnegative Radon measure $H_h(\mu)$ as

$$H_h(\mu)(B) := \int_B H\Big(\frac{d\mu}{d|\mu|}\Big)\,d|\mu|$$

for every Borel set $B \subset \Omega_h \cup \Gamma_{d,h}$. We also consider the functional

$$\mathcal{H}_h : M_b(\Omega_h \cup \Gamma_{d,h};\mathbb{M}^{3\times3}_D) \to [0,+\infty)$$

defined by

$$\mathcal{H}_h(\mu) := H_h(\mu)(\Omega_h \cup \Gamma_{d,h}).$$

One can prove (see, e.g., Chapter II–Section 4 in [51]) that

$$\mathcal{H}_h(\mu) = \sup\Big\{ \int_{\Omega_h\cup\Gamma_{d,h}} \tau : d\mu \ : \\ \tau \in C_0(\Omega_h \cup \Gamma_{d,h}; \mathbb{M}^{3\times3}_D),\ \tau(x) \in K \text{ for a.e. } x \in \Omega_h \cup \Gamma_{d,h} \Big\}.$$

From this characterisation it is clear that $\mathcal{H}_h$ is lower semicontinuous with respect to the weak* convergence of $M_b(\Omega_h \cup \Gamma_{d,h}; \mathbb{M}^{3\times3}_D)$.

We also define the total variation of a function $\mu : [0,T] \to M_b(\Omega_h \cup \Gamma_{d,h}; \mathbb{M}^{3\times3}_D)$ in an interval $[a,b] \subset [0,T]$ as

$$\mathcal{V}_h(\mu; a, b) := \sup\Big\{ \sum_{j=1}^{N} \|\mu(s_j) - \mu(s_{j-1})\|_{M_b} :\ a = s_0 \le s_1 \le \cdots \le s_N = b,\ N \in \mathbb{N} \Big\},$$

and the dissipation of μ in $[a,b]$ as

$$\mathcal{D}_h(\mu; a, b) := \sup\Big\{ \sum_{j=1}^{N} \mathcal{H}_h(\mu(s_j) - \mu(s_{j-1})) :\ a = s_0 \le s_1 \le \cdots \le s_N = b,\ N \in \mathbb{N} \Big\}.$$

It follows from (2.2.4) that

$$r_K \mathcal{V}_h(\mu; a, b) \le \mathcal{D}_h(\mu; a, b) \le R_K \mathcal{V}_h(\mu; a, b).$$

Moreover, if μ is absolutely continuous on $[a,b]$ with values in $M_b(\Omega_h \cup \Gamma_{d,h}; \mathbb{M}^{3\times3}_D)$, then one has

$$\mathcal{D}_h(\mu; a, b) = \int_a^b \mathcal{H}_h(\dot\mu(s))\, ds \tag{2.2.5}$$

(see Theorem 7.1 in [12]). We set $\mathcal{H} := \mathcal{H}_1$, $\mathcal{V} := \mathcal{V}_1$, and $\mathcal{D} := \mathcal{D}_1$.

Kinematic admissibility. Given a boundary datum $w \in H^1(\Omega_h; \mathbb{R}^3)$, we define the class $\mathcal{A}_h(w)$ of admissible displacements and strains, as the set of all triplets $(u, e, p) \in BD(\Omega_h) \times L^2(\Omega_h; \mathbb{M}^{3\times3}_{sym}) \times M_b(\Omega_h \cup \Gamma_{d,h}; \mathbb{M}^{3\times3}_D)$ such that

$$\operatorname{sym} Du = e + p \quad \text{in } \Omega_h, \qquad p = (w - u) \odot \nu_{\partial\Omega_h} \mathcal{H}^2 \quad \text{on } \Gamma_{d,h}.$$

We set $\mathcal{A}(w) := \mathcal{A}_1(w)$ for every $w \in H^1(\Omega; \mathbb{R}^3)$.

The trace of stresses. We recall that, if $\sigma \in L^2(\Omega_h; \mathbb{M}^{3\times3}_{sym})$ with $\operatorname{div}\sigma \in L^2(\Omega_h; \mathbb{R}^3)$, we can define the trace $[\sigma\nu_{\partial\Omega_h}] \in H^{-1/2}(\partial\Omega_h; \mathbb{R}^3)$ of its normal component through the formula

$$\langle [\sigma\nu_{\partial\Omega_h}], \varphi\rangle := \int_{\Omega_h} \sigma : \operatorname{sym} D\varphi\, dx + \int_{\Omega_h} \operatorname{div}\sigma \cdot \varphi\, dx$$

for every $\varphi \in H^1(\Omega_h; \mathbb{R}^3)$. In the following we say that $[\sigma\nu_{\partial\Omega_h}] = 0$ on $\Gamma_{n,h}$ if $\langle [\sigma\nu_{\partial\Omega_h}], \varphi\rangle = 0$ for every $\varphi \in H^1(\Omega_h; \mathbb{R}^3)$ with $\varphi = 0$ on $\Gamma_{d,h}$.

2.2.2 The reduced problem

In this section we introduce the setting of the limiting problem.

The reduced stored elastic energy. Let $\mathbb{M} : \mathbb{M}^{2\times 2}_{sym} \to \mathbb{M}^{3\times 3}_{sym}$ be the operator given by

$$\mathbb{M}\xi := \begin{pmatrix} \xi_{11} & \xi_{12} & \lambda_1(\xi) \\ \xi_{12} & \xi_{22} & \lambda_2(\xi) \\ \lambda_1(\xi) & \lambda_2(\xi) & \lambda_3(\xi) \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym}, \tag{2.2.6}$$

where the triplet $(\lambda_1(\xi), \lambda_2(\xi), \lambda_3(\xi))$ is the unique solution of the minimum problem

$$\min_{\lambda_i \in \mathbb{R}} Q \begin{pmatrix} \xi_{11} & \xi_{12} & \lambda_1 \\ \xi_{12} & \xi_{22} & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}.$$

We observe that $(\lambda_1(\xi), \lambda_2(\xi), \lambda_3(\xi))$ can be characterised as the unique solution of the linear system

$$\mathbb{C}\mathbb{M}\xi : \begin{pmatrix} 0 & 0 & \zeta_1 \\ 0 & 0 & \zeta_2 \\ \zeta_1 & \zeta_2 & \zeta_3 \end{pmatrix} = 0 \tag{2.2.7}$$

for every $\zeta_i \in \mathbb{R}$, $i = 1, 2, 3$. This implies that $\mathbb{M}$ is a linear map and

$$(\mathbb{C}\mathbb{M}\xi)_{i3} = (\mathbb{C}\mathbb{M}\xi)_{3i} = 0 \quad \text{for every } i = 1, 2, 3. \tag{2.2.8}$$

Let $Q^* : \mathbb{M}^{2\times 2}_{sym} \to \mathbb{R}$ be the quadratic form given by

$$Q^*(\xi) := Q(\mathbb{M}\xi) \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym}.$$

It follows from (2.2.1) that

$$\alpha_{\mathbb{C}}|\xi|^2 \le Q^*(\xi) \le \beta_{\mathbb{C}}|\xi|^2 \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym}.$$

We define the reduced elasticity tensor as the linear operator $\mathbb{C}^* : \mathbb{M}^{2\times 2}_{sym} \to \mathbb{M}^{3\times 3}_{sym}$ given by

$$\mathbb{C}^*\xi := \mathbb{C}\mathbb{M}\xi \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym}. \tag{2.2.9}$$

Note that we can always identify $\mathbb{C}^*\xi$ with an element of $\mathbb{M}^{2\times 2}_{sym}$ in view of (2.2.8). Moreover, by (2.2.7) we have

$$\mathbb{C}^*\xi : \zeta = \mathbb{C}^*\xi : \begin{pmatrix} \zeta_{11} & \zeta_{12} & 0 \\ \zeta_{12} & \zeta_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym},\ \zeta \in \mathbb{M}^{3\times 3}_{sym}. \tag{2.2.10}$$

This implies that

$$Q^*(\xi) = \frac{1}{2}\mathbb{C}^*\xi : \begin{pmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{12} & \xi_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym}.$$

Finally, we introduce the functional $\mathcal{Q}^* : L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \to [0, +\infty)$, defined as

$$\mathcal{Q}^*(e) := \int_\Omega Q^*(e(x))\, dx$$

for every $e \in L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$. It describes the reduced elastic energy of a configuration, whose elastic strain is e.

The reduced plastic dissipation. In the reduced problem the plastic dissipation potential is given by the function $H^* : \mathbb{M}^{2\times 2}_{sym} \to [0, +\infty)$, defined as

$$H^*(\xi) := \min_{\lambda_i \in \mathbb{R}} H \begin{pmatrix} \xi_{11} & \xi_{12} & \lambda_1 \\ \xi_{12} & \xi_{22} & \lambda_2 \\ \lambda_1 & \lambda_2 & -(\xi_{11} + \xi_{22}) \end{pmatrix} \tag{2.2.11}$$

for every $\xi \in \mathbb{M}^{2\times 2}_{sym}$. From the properties of H it follows that H^* is convex, positively 1-homogeneous, and satisfies

$$r_K|\xi| \le H^*(\xi) \le \sqrt{3} R_K |\xi| \text{ for every } \xi \in \mathbb{M}^{2\times 2}_{sym}.$$

The set $K^* := \partial H^*(0)$ represents the set of admissible stresses in the reduced problem and can be characterised as follows:

$$\xi \in K^* \quad \Leftrightarrow \quad \begin{pmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{12} & \xi_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{3}(\operatorname{tr}\xi) I_{3\times 3} \in K, \tag{2.2.12}$$

(see Section 3.2 in [13]).

For every $\mu \in M_b(\Omega \cup \partial_d \Omega; \mathbb{M}^{2\times 2}_{sym})$ we define the functional

$$\mathcal{H}^*(\mu) := \int_{\Omega \cup \partial_d \Omega} H^*\Big(\frac{d\mu}{d|\mu|}\Big)\, d|\mu|.$$

We also define the reduced dissipation of a function $\mu : [0, T] \to M_b(\Omega \cup \partial_d \Omega; \mathbb{M}^{2\times 2}_{sym})$ in an interval $[a, b] \subset [0, T]$ as

$$\mathcal{D}^*(\mu; a, b) := \sup\Big\{ \sum_{j=1}^{N} \mathcal{H}^*(\mu(s_j) - \mu(s_{j-1})) : \; a = s_0 \le s_1 \le \cdots \le s_N = b, \; N \in \mathbb{N} \Big\}.$$

If μ is absolutely continuous on $[a, b]$ with values in $M_b(\Omega \cup \partial_d \Omega; \mathbb{M}^{2\times 2}_{sym})$, then

$$\mathcal{D}^*(\mu; a, b) = \int_a^b \mathcal{H}^*(\dot\mu(s))\, ds \tag{2.2.13}$$

(see Theorem 7.1 in [12]).

Reduced kinematic admissibility. We introduce the set $KL(\Omega)$ of Kirchhoff-Love displacements, defined as

$$KL(\Omega) := \{u \in BD(\Omega) : \; (\operatorname{sym} Du)_{i3} = 0, \; i = 1, 2, 3\}.$$

We note that $u \in KL(\Omega)$ if and only if $u_3 \in BH(\omega)$ and there exists $\bar u \in BD(\omega)$ such that

$$u_\alpha(x) = \bar u_\alpha(x') - x_3 \partial_\alpha u_3(x') \quad \text{ for } x = (x', x_3) \in \Omega, \; \alpha = 1, 2.$$

We call $\bar u$, u_3 the *Kirchhoff-Love components* of u.

Given a prescribed displacement $w \in H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega)$, we introduce the set $\mathcal{A}_{KL}(w)$ of *Kirchhoff-Love admissible triplets*, defined as the class of all triplets

$$(u, e, p) \in KL(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times3}_{sym}) \times M_b(\Omega \cup \partial_d\Omega;\mathbb{M}^{3\times3}_{sym})$$

such that

$$\operatorname{sym} Du = e + p \quad \text{in } \Omega, \qquad p = (w-u) \odot \nu_{\partial\Omega}\mathcal{H}^2 \quad \text{on } \partial_d\Omega,$$
$$e_{i3} = 0 \quad \text{in } \Omega, \qquad p_{i3} = 0 \quad \text{in } \Omega \cup \partial_d\Omega, \qquad i = 1,2,3.$$

The linear space $\{\xi \in \mathbb{M}^{3\times3}_{sym} : \ \xi_{i3} = 0, \ i = 1,2,3\}$ is isomorphic to $\mathbb{M}^{2\times2}_{sym}$. Thus, in the following, given $(u,e,p) \in \mathcal{A}_{KL}(w)$, we will always identify e with a function in $L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$ and p with a measure in $M_b(\Omega \cup \partial_d\Omega;\mathbb{M}^{2\times2}_{sym})$.

The following closure property holds.

Lemma 2.2.1. *Let $(w_n)_n$ be a sequence in $H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ and let $(u_n, e_n, p_n) \in \mathcal{A}_{KL}(w_n)$ be a sequence of admissible triplets. Assume that $u_n \rightharpoonup u$ weakly* in $BD(\Omega)$, $e_n \rightharpoonup e$ weakly in $L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$, $p_n \rightharpoonup p$ weakly* in $M_b(\Omega \cup \partial_d\Omega;\mathbb{M}^{2\times2}_{sym})$, and $w_n \rightharpoonup w$ weakly in $H^1(\Omega;\mathbb{R}^3)$. Then $(u,e,p) \in \mathcal{A}_{KL}(w)$.*

Proof. The result easily follows by adapting the proof of Lemma 2.1 in [12]. □

We now give a characterisation of the class of Kirchhoff-Love admissible triplets. To this purpose, we introduce the following definitions.

Definition 2.2.2. Let $f \in L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$. We denote by $\bar{f}$, $\hat{f} \in L^2(\omega;\mathbb{M}^{2\times2}_{sym})$ and by $f_\perp \in L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$ the following orthogonal components (in the sense of $L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$) of f:

$$\bar{f}(x') := \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x',x_3)\,dx_3, \qquad \hat{f}(x') := 12\int_{-\frac{1}{2}}^{\frac{1}{2}} x_3 f(x',x_3)\,dx_3$$

for a.e. $x' \in \omega$, and

$$f_\perp(x) := f(x) - \bar{f}(x') - x_3\hat{f}(x')$$

for a.e. $x \in \Omega$. We call $\bar{f}$ the *zeroth order moment* of f and $\hat{f}$ the *first order moment* of f.

Definition 2.2.3. Let $q \in M_b(\Omega \cup \partial_d\Omega;\mathbb{M}^{2\times2}_{sym})$. We denote by $\bar{q}$, $\hat{q} \in M_b(\omega \cup \partial_d\omega;\mathbb{M}^{2\times2}_{sym})$ and by $q_\perp \in M_b(\Omega \cup \partial_d\Omega;\mathbb{M}^{2\times2}_{sym})$ the following measures:

$$\int_{\omega\cup\partial_d\omega} \varphi : d\bar{q} := \int_{\Omega\cup\partial_d\Omega} \varphi : dq, \qquad \int_{\omega\cup\partial_d\omega} \varphi : d\hat{q} := 12\int_{\Omega\cup\partial_d\Omega} x_3\varphi : dq$$

for every $\varphi \in C_0(\omega \cup \partial_d\omega;\mathbb{M}^{2\times2}_{sym})$, and

$$q_\perp := q - \bar{q} \otimes \mathcal{L}^1 - \hat{q} \otimes x_3\mathcal{L}^1,$$

where $\otimes$ denotes the usual product of measures. We call $\bar{q}$ the *zeroth order moment* of q and $\hat{q}$ the *first order moment* of q.

With these definitions at hand one can prove the following characterisation of the class $\mathcal{A}_{KL}(w)$.

Proposition 2.2.4. *Let $w \in H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ and let $(u,e,p) \in KL(\Omega) \times L^2(\Omega;\mathbb{M}^{2\times2}_{sym}) \times M_b(\Omega \cup \partial_d\Omega;\mathbb{M}^{2\times2}_{sym})$. Then $(u,e,p) \in \mathcal{A}_{KL}(w)$ if and only if the following three conditions are satisfied:*

- $\operatorname{sym} D\bar{u} = \bar{e} + \bar{p}$ *in* ω *and* $\bar{p} = (\bar{w} - \bar{u}) \odot \nu_{\partial\omega}\mathcal{H}^1$ *on* $\partial_d\omega$*;*
- $D^2 u_3 = -(\hat{e} + \hat{p})$ *in* ω*,* $u_3 = w_3$ *on* $\partial_d\omega$*, and* $\hat{p} = (\nabla u_3 - \nabla w_3) \odot \nu_{\partial\omega}\mathcal{H}^1$ *on* $\partial_d\omega$*;*
- $p_\perp = -e_\perp$ *in* Ω *and* $p_\perp = 0$ *on* $\partial_d\Omega$*.*

Proof. The statement easily follows from the definition of moments and from the formula $(\operatorname{sym} Du)_{\alpha\beta} = (\operatorname{sym} D\bar{u})_{\alpha\beta} - x_3\partial^2_{\alpha\beta}u_3$ for $\alpha, \beta = 1, 2$. □

Stress-strain duality. In the reduced model, we shall consider the set $\Sigma(\Omega)$ of *admissible stresses*, defined as

$$\Sigma(\Omega) := \{\sigma \in L^\infty(\Omega; \mathbb{M}^{2\times2}_{sym}) : \ \operatorname{div}\bar{\sigma} \in L^2(\omega; \mathbb{R}^2), \ \operatorname{div}\operatorname{div}\hat{\sigma} \in L^2(\omega)\}.$$

For every $\sigma \in \Sigma(\Omega)$ we can define the trace $[\bar{\sigma}\nu_{\partial\omega}] \in L^\infty(\partial\omega; \mathbb{R}^2)$ of its zeroth order moment normal component as

$$\langle[\bar{\sigma}\nu_{\partial\omega}], \psi\rangle := \int_\omega \bar{\sigma} : \operatorname{sym} D\psi \, dx' + \int_\omega \operatorname{div}\bar{\sigma} \cdot \psi \, dx' \tag{2.2.14}$$

for every $\psi \in W^{1,1}(\omega; \mathbb{R}^2)$. Note that, since $\bar{\sigma} \in L^\infty(\omega; \mathbb{M}^{2\times2}_{sym})$ and $W^{1,1}(\omega; \mathbb{R}^2)$ embeds into $L^2(\omega; \mathbb{R}^2)$, all terms at the right-hand side of (2.2.14) are well defined.

Let $T(W^{2,1}(\omega))$ be the space of all traces of functions in $W^{2,1}(\omega)$ and let $(T(W^{2,1}(\omega)))'$ be its dual space. For every $\sigma \in \Sigma(\Omega)$ we can define the traces $b_0(\hat{\sigma}) \in (T(W^{2,1}(\omega)))'$ and $b_1(\hat{\sigma}) \in L^\infty(\partial\omega)$ of its first order moment as

$$-\langle b_0(\hat{\sigma}), \psi\rangle + \big\langle b_1(\hat{\sigma}), \frac{\partial\psi}{\partial\nu_{\partial\omega}}\big\rangle := \int_\omega \hat{\sigma} : D^2\psi \, dx' - \int_\omega \psi \operatorname{div}\operatorname{div}\hat{\sigma} \, dx' \tag{2.2.15}$$

for every $\psi \in W^{2,1}(\omega)$. Note that the right-hand side of (2.2.15) is well defined since $\hat{\sigma} \in L^\infty(\omega; \mathbb{M}^{2\times2}_{sym})$.

If $\hat{\sigma} \in C^2(\overline{\omega}, \mathbb{M}^{2\times2}_{sym})$, one can prove that

$$\begin{aligned} b_0(\hat{\sigma}) &= \operatorname{div}\hat{\sigma} \cdot \nu_{\partial\omega} + \frac{\partial}{\partial\tau_{\partial\omega}}\left(\hat{\sigma}\tau_{\partial\omega} \cdot \nu_{\partial\omega}\right), \\ b_1(\hat{\sigma}) &= \hat{\sigma}\nu_{\partial\omega} \cdot \nu_{\partial\omega}, \end{aligned}$$

where $\tau_{\partial\omega}$ is the tangent vector to $\partial\omega$.

Since $[\bar{\sigma}\nu_{\partial\omega}] \in L^\infty(\partial\omega; \mathbb{R}^2)$, the expression $[\bar{\sigma}\nu_{\partial\omega}] = 0$ on $\partial_n\omega$ has a clear meaning. The same applies to $b_1(\hat{\sigma})$. As for $b_0(\hat{\sigma})$, in the following we say that $b_0(\hat{\sigma}) = 0$ on $\partial_n\omega$ if $\langle b_0(\hat{\sigma}), \psi\rangle = 0$ for every $\psi \in W^{2,1}(\omega)$ with $\psi = 0$ on $\partial_d\omega$.

We also consider the space of *admissible plastic strains* $\Pi_{\partial_d\Omega}(\Omega)$, which is the set of all measures $p \in M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{2\times2}_{sym})$ for which there exists $(u, e, w) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times2}_{sym}) \times (H^1(\Omega; \mathbb{R}^3) \cap KL(\Omega))$ such that $(u, e, p) \in \mathcal{A}_{KL}(w)$.

For every $\sigma \in \Sigma(\Omega)$ and $\xi \in BD(\omega)$ we define the distribution $[\bar{\sigma} : \operatorname{sym} D\xi]$ on ω as

$$\langle[\bar{\sigma} : \operatorname{sym} D\xi], \varphi\rangle := -\int_\omega \varphi \operatorname{div}\bar{\sigma} \cdot \xi \, dx' - \int_\omega \bar{\sigma} : (\nabla\varphi \odot \xi) \, dx'$$

for every $\varphi \in C^\infty_c(\omega)$. It follows from Theorem 3.2 in [30] that $[\bar{\sigma} : \operatorname{sym} D\xi] \in M_b(\omega)$ and its variation satisfies

$$|[\bar{\sigma} : \operatorname{sym} D\xi]| \leq \|\bar{\sigma}\|_{L^\infty} |\operatorname{sym} D\xi| \quad \text{in } \omega.$$

Given $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\partial_d\Omega}(\Omega)$, we define the measure $[\bar\sigma : \bar p] \in M_b(\omega \cup \partial_d\omega)$ as

$$[\bar\sigma : \bar p] := \begin{cases} [\bar\sigma : \operatorname{sym} D\bar u] - \bar\sigma : \bar e & \text{in } \omega, \\ [\bar\sigma \nu_{\partial\omega}] \cdot (\bar w - \bar u)\, \mathcal{H}^1 & \text{on } \partial_d\omega. \end{cases}$$

For every $\sigma \in \Sigma(\Omega)$ and $v \in BH(\omega)$ we define the distribution $[\hat\sigma : D^2 v]$ on ω as

$$\langle [\hat\sigma : D^2 v], \psi \rangle := \int_\omega \psi v \operatorname{div}\operatorname{div} \hat\sigma \, dx' - 2 \int_\omega \hat\sigma : (\nabla v \odot \nabla \psi)\, dx' - \int_\omega v \hat\sigma : D^2 \psi \, dx'$$

for every $\psi \in C_c^\infty(\omega)$. From Proposition 2.1 in [18] it follows that $[\hat\sigma : D^2 v] \in M_b(\omega)$ and its variation satisfies

$$|[\hat\sigma : D^2 v]| \le \|\hat\sigma\|_{L^\infty} |D^2 v| \quad \text{in } \omega.$$

Given $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\partial_d\Omega}(\Omega)$, we define the measure $[\hat\sigma : \hat p] \in M_b(\omega \cup \partial_d\omega)$ as

$$[\hat\sigma : \hat p] := \begin{cases} -[\hat\sigma : D^2 u_3] - \hat\sigma : \hat e & \text{in } \omega, \\ b_1(\hat\sigma) \dfrac{\partial (u_3 - w_3)}{\partial \nu_{\partial\omega}} \mathcal{H}^1 & \text{on } \partial_d\omega. \end{cases}$$

We are now in a position to introduce a duality between $\Sigma(\Omega)$ and $\Pi_{\partial_d\Omega}(\Omega)$. For every $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\partial_d\Omega}(\Omega)$ we define the measure $[\sigma : p]_r \in M_b(\Omega \cup \partial_d\Omega)$ as

$$[\sigma : p]_r := [\bar\sigma : \bar p] \otimes \mathcal{L}^1 + \frac{1}{12} [\hat\sigma : \hat p] \otimes \mathcal{L}^1 - \sigma_\perp : e_\perp.$$

We also introduce the duality pairings

$$\langle \bar\sigma, \bar p \rangle := [\bar\sigma : \bar p](\omega \cup \partial_d\omega), \qquad \langle \hat\sigma, \hat p \rangle := [\hat\sigma : \hat p](\omega \cup \partial_d\omega)$$

and

$$\langle \sigma, p \rangle_r := [\sigma : p]_r(\Omega \cup \partial_d\Omega) = \langle \bar\sigma, \bar p \rangle + \frac{1}{12} \langle \hat\sigma, \hat p \rangle - \int_\Omega \sigma_\perp : e_\perp \, dx. \tag{2.2.16}$$

One can show (see Proposition 7.8 in [13]) that

$$\mathcal{H}^*(p) = \sup \big\{ \langle \sigma, p \rangle_r : \ \sigma \in \Sigma(\Omega),\ \sigma(x) \in K^* \text{ for a.e. } x \in \Omega \big\}. \tag{2.2.17}$$

Finally, the following integration by parts formula holds (see Proposition 3.5 in [14]).

Proposition 2.2.5. *Let $\sigma \in \Sigma(\Omega)$, $w \in H^1(\Omega; \mathbb{R}^3) \cap KL(\Omega)$, and $(u, e, p) \in \mathcal{A}_{KL}(w)$. Then*

$$\begin{aligned} &\int_{\Omega \cup \partial_d\Omega} \varphi \, d[\sigma : p]_r + \int_\Omega \varphi \sigma : (e - \operatorname{sym} Dw)\, dx \\ &= \ - \int_\omega \bar\sigma : (\nabla\varphi \odot (\bar u - \bar w))\, dx' - \int_\omega \operatorname{div} \bar\sigma \cdot \varphi (\bar u - \bar w)\, dx' \\ &\quad + \int_{\partial_n\omega} [\bar\sigma \nu_{\partial\omega}] \cdot \varphi(\bar u - \bar w)\, d\mathcal{H}^1 + \frac{1}{12} \int_\omega \hat\sigma : (u_3 - w_3) D^2 \varphi \, dx' \\ &\quad + \frac{1}{6} \int_\omega \hat\sigma : (\nabla\varphi \odot (\nabla u_3 - \nabla w_3))\, dx' - \frac{1}{12} \int_\omega \varphi (u_3 - w_3) \operatorname{div}\operatorname{div} \hat\sigma \, dx' \\ &\quad + \frac{1}{12} \langle b_0(\hat\sigma), \varphi(u_3 - w_3) \rangle - \frac{1}{12} \int_{\partial_n\omega} b_1(\hat\sigma) \frac{\partial (\varphi (u_3 - w_3))}{\partial \nu_{\partial\omega}} \, d\mathcal{H}^1 \end{aligned}$$

for every $\varphi \in C^2(\overline{\omega})$.

2.3 Existence of three-dimensional dynamic evolutions

In this section we adapt the existence result of [5] of a dynamic evolution for perfectly plastic bodies to the context of a thin plate. Indeed, in view of the dimension reduction analysis of the next section, it is crucial to understand the dependence of all the involved quantities on the thickness parameter h.

We start by describing the assumptions on the data of the problem.

Forces. We assume the applied body loads to be purely vertical and with the following regularity:

$$f_h \in W^{1,1}_{loc}([0,+\infty); L^2(\Omega_h)). \tag{2.3.1}$$

We assume there are no traction forces on the Neumann part of the boundary $\Gamma_{n,h}$.

Boundary displacement. On $\Gamma_{d,h}$ we prescribe a boundary displacement

$$w_h \in H^2_{loc}([0,+\infty); H^1(\Omega_h;\mathbb{R}^3)) \cap W^{3,1}_{loc}([0,+\infty); L^2(\Omega_h;\mathbb{R}^3)). \tag{2.3.2}$$

Initial data. Let

$$\begin{gathered}(u_{0,h}, e_{0,h}, p_{0,h}) \in \mathcal{A}_h(w_h(0)) \cap (H^1(\Omega_h;\mathbb{R}^3) \times L^2(\Omega_h;\mathbb{M}^{3\times3}_{sym}) \times L^2(\Omega_h;\mathbb{M}^{3\times3}_D)),\\ v_{0,h} \in H^1(\Omega_h;\mathbb{R}^3),\end{gathered} \tag{2.3.3}$$

be the initial data. Setting $\sigma_{0,h} := \mathbb{C}e_{0,h}$, we assume that

$$\begin{gathered}-\operatorname{div}\sigma_{0,h} = f_h(0)e_3 \text{ in } \Omega_h, \quad [\sigma_{0,h}\nu_{\partial\Omega_h}] = 0 \text{ on } \Gamma_{n,h},\\ (\sigma_{0,h})_D \in K \text{ a.e. in } \Omega_h,\end{gathered} \tag{2.3.4}$$

and

$$v_{0,h} = \dot{w}_h(0) \text{ on } \Gamma_{d,h}. \tag{2.3.5}$$

Theorem 2.3.1. *Assume* (2.3.1)–(2.3.5). *Then there exists a triplet* (u_h, e_h, p_h), *with*

$$\begin{gathered}u_h \in W^{2,\infty}_{loc}([0,+\infty); L^2(\Omega_h;\mathbb{R}^3)) \cap \mathrm{Lip}_{loc}([0,+\infty); BD(\Omega_h)),\\ e_h \in W^{1,\infty}_{loc}([0,+\infty); L^2(\Omega_h;\mathbb{M}^{3\times3}_{sym})),\\ p_h \in \mathrm{Lip}_{loc}([0,+\infty); M_b(\Omega_h \cup \Gamma_{d,h};\mathbb{M}^{3\times3}_D)),\end{gathered}$$

satisfying the following system of equations:

(i) kinematic admissibility: $(u_h(t), e_h(t), p_h(t)) \in \mathcal{A}_h(w_h(t))$ *for every* $t \geq 0$;

(ii) initial conditions: $(u_h(0), e_h(0), p_h(0)) = (u_{0,h}, e_{0,h}, p_{0,h})$ *and* $\dot{u}_h(0) = v_{0,h}$;

(iii) stress constraint: $(\sigma_h)_D(t) \in K$ *a.e. in* Ω_h *for every* $t \geq 0$, *where* $\sigma_h(t) := \mathbb{C}e_h(t)$;

(iv) equation of motion: *for a.e.* $t \geq 0$

$$\begin{cases}\ddot{u}_h(t) - \operatorname{div}\sigma_h(t) = f_h(t)e_3 \text{ in } \Omega_h,\\ [\sigma_h(t)\nu_{\partial\Omega_h}] = 0 \text{ on } \Gamma_{n,h};\end{cases} \tag{2.3.6}$$

(v) energy inequality: *for every* $0 \leq t_1 \leq t_2$

$$\begin{aligned}\mathcal{Q}_h(e_h(t_2)) + \frac{1}{2}\|\dot{u}_h(t_2)\|^2_{L^2} + \int_{t_1}^{t_2} \mathcal{H}_h(\dot{p}_h(s))\,ds \leq \mathcal{Q}_h(e_h(t_1)) + \frac{1}{2}\|\dot{u}_h(t_1)\|^2_{L^2} \\ + \int_{t_1}^{t_2}\int_{\Omega_h} (\sigma_h(s) : \operatorname{sym} D\dot{w}_h(s) + \ddot{u}_h(s)\cdot \dot{w}_h(s))\,dx\,ds \\ + \int_{t_1}^{t_2}\int_{\Omega_h} f_h(s)\,((\dot{u}_h)_3(s) - (\dot{w}_h)_3(s))\,dx\,ds. \quad (2.3.7)\end{aligned}$$

Moreover, the following estimates hold:

- *there exists a constant* $C > 0$*, independent of* h*, such that*

$$\begin{aligned}\|\ddot{u}_h\|_{L^\infty([0,t];L^2)} + \|\dot{e}_h\|_{L^\infty([0,t];L^2)} \leq C(\|\operatorname{sym} Dv_{0,h}\|_{L^2} + \|\dot{f}_h\|_{L^1([0,t];L^2)} \\ + \|\dddot{w}_h\|_{L^1([0,t];L^2)} + \|\ddot{w}_h\|_{L^\infty([0,t];L^2)} + \sqrt{t}\,\|\operatorname{sym} D\ddot{w}_h\|_{L^2([0,t];L^2)}) \quad (2.3.8)\end{aligned}$$

  for every $t > 0$*;*

- *there exists a constant* $C' > 0$*, independent of* h*, such that*

$$\begin{aligned}\|p_h(t_2) - p_h(t_1)\|_{M_b} \leq C'\Big(\|e_h\|_{L^\infty([0,T];L^2)}\|e_h(t_2) - e_h(t_1)\|_{L^2} \\ + \|\dot{u}_h\|_{L^\infty([0,T];L^2)}\|\dot{u}_h(t_2) - \dot{u}_h(t_1)\|_{L^2} \\ + \|e_h\|_{L^\infty([0,T];L^2)} \int_{t_1}^{t_2} \|\operatorname{sym} D\dot{w}_h(t)\|_{L^2}\,dt \\ + \|\ddot{u}_h\|_{L^\infty([0,T];L^2)} \int_{t_1}^{t_2} \|\dot{w}_h(t)\|_{L^2}\,dt \\ + \|\dot{u}_h - \dot{w}_h\|_{L^\infty([0,T];L^2)} \int_{t_1}^{t_2} \|f_h(t)\|_{L^2}\,dt\Big) \quad (2.3.9)\end{aligned}$$

  for every $T > 0$ *and every* $t_1, t_2 \in [0, T]$.

Remark 2.3.2. The energy inequality (v) *formally* corresponds to the inequality

$$\int_{\Omega_h \cup \Gamma_{d,h}} (\sigma_h)_D(t) : \dot{p}_h(t)\,dx \geq \mathcal{H}_h(\dot{p}_h(t)) \quad (2.3.10)$$

for a.e. $t \geq 0$. Indeed, it is enough to choose $t_1 = t$ and $t_2 = t + \delta$ in (v), divide the inequality by δ, and pass to the limit as δ tends to zero. Using the kinematic admissibility $\dot{e}_h(t) = \operatorname{sym} D\dot{u}_h(t) - \dot{p}_h(t)$, integration by parts, and (2.3.6), eventually yield (2.3.10). Note, however, that the left-handside of (2.3.10) is in general not well defined, since $(\sigma_h)_D(t) \in L^\infty(\Omega_h; \mathbb{M}^{3\times3}_D)$ and $\dot{p}_h(t) \in M_b(\Omega_h \cup \Gamma_{d,h}; \mathbb{M}^{3\times3}_D)$.

The formal equivalence of (v) and (2.3.10) suggests that (v) contains all the relevant information stored in the Prandtl-Reuss flow rule (see equation (d5)′ in the introduction). Indeed, the converse inequality

$$(\sigma_h)_D(t) : \dot{p}_h(t) \leq H_h(\dot{p}_h(t)) \quad \text{in } \Omega_h \cup \Gamma_{d,h}$$

is an immediate consequence of (iii) and of the definition of H_h (if the left-handside is well defined). Moreover, as we will see in the proof of Theorem 2.4.1, condition (v) is enough to recover the limiting flow rule in the dimension reduction analysis of next section.

If the stress-strain duality in the sense of [12, 22, 30] is defined, the formal arguments above can be rigorously justified; thus, one can show that any solution to (i)–(v) satisfies condition (2.3.7) with an equality and this energy balance is equivalent to the Prandtl-Reuss flow rule (see, e.g., [7]). In this case uniqueness of solutions for the system (i)–(v) can also be proved by standard methods.

In the absence of a notion of stress-strain duality or of additional regularity, the energy balance could in principle fail. This difficulty in establishing the energy equality under general assumptions was already noted in [47] (Remark 6).

Proof. Proof of Theorem 2.3.1 We give here only a sketch of the proof (all the details can be found in Theorem 1.3 of [5] or in Theorem 4.1 of [7] in a slightly different setting). In order to simplify the notation we omit the dependence of the fields on h. Moreover, we denote the space $\{u \in H^1(\Omega;\mathbb{R}^3) : u = 0 \text{ on } \partial_d\Omega\}$ by $H^1_{\partial_d\Omega}(\Omega;\mathbb{R}^3)$ and its dual by $H^{-1}_{\partial_d\Omega}(\Omega;\mathbb{R}^3)$.

We first prove existence for a visco-elastic regularisation of the problem. We start by regularising the initial velocities. More precisely, let $\varepsilon > 0$ and let $v_0^\varepsilon \in H^1(\Omega;\mathbb{R}^3)$ be the solution of the boundary value problem

$$\begin{cases} -\varepsilon \operatorname{div} \operatorname{sym} Dv_0^\varepsilon + v_0^\varepsilon = v_0 \text{ in } \Omega, \\ v_0^\varepsilon = \dot{w}(0) \text{ on } \partial_d\Omega, \\ [\operatorname{sym} Dv_0^\varepsilon \nu_{\partial\Omega}] = 0 \text{ on } \partial_n\Omega. \end{cases} \tag{2.3.11}$$

The standard theory of linear elliptic equations gives

$$v_0^\varepsilon \to v_0 \quad \text{in } H^1(\Omega;\mathbb{R}^3)$$

and

$$\varepsilon \operatorname{div} \operatorname{sym} Dv_0^\varepsilon \to 0 \quad \text{in } L^2(\Omega;\mathbb{R}^3).$$

Using a time-discretisation procedure and arguing exactly as in Theorem 3.1 of [7], one can prove the existence and uniqueness of a triplet $(u_\varepsilon, e_\varepsilon, p_\varepsilon)$, with $u_\varepsilon \in H^1_{loc}([0,+\infty); H^1(\Omega;\mathbb{R}^3)) \cap W^{1,\infty}_{loc}([0,+\infty); L^2(\Omega;\mathbb{R}^3)) \cap H^2_{loc}([0,+\infty); H^{-1}_{\partial_d\Omega}(\Omega;\mathbb{R}^3))$, $e_\varepsilon \in H^1_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{3\times3}_{sym}))$, and $p_\varepsilon \in H^1_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{3\times3}_D))$, satisfying the following conditions:

- *kinematic admissibility:* $(u_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)) \in \mathcal{A}(w(t))$ for every $t \geq 0$;
- *initial conditions:* $(u_\varepsilon(0), e_\varepsilon(0), p_\varepsilon(0)) = (u_0, e_0, p_0)$ and $\dot{u}_\varepsilon(0) = v_0^\varepsilon$;
- *stress constraint:* $(\sigma_\varepsilon)_D(t) \in K$ a.e. in Ω for every $t \geq 0$, where $\sigma_\varepsilon(t) := \mathbb{C}e_\varepsilon(t)$;
- *equation of motion:* for every $t \geq 0$ and every $\varphi \in L^2(0,t; H^1_{\partial_d\Omega}(\Omega;\mathbb{R}^3))$

$$\int_0^t \langle \ddot{u}_\varepsilon(s), \varphi(s)\rangle\, ds + \int_0^t \int_\Omega (\sigma_\varepsilon(s) + \varepsilon \operatorname{sym} D\dot{u}_\varepsilon(s)) : \operatorname{sym} D\varphi(s)\, dx\, ds = \int_0^t \int_\Omega f(s)\varphi_3(s)\, dx\, ds; \tag{2.3.12}$$

- *flow rule:* for a.e. $t \geq 0$

$$H(\dot{p}_\varepsilon(t)) = (\sigma_\varepsilon)_D(t) : \dot{p}_\varepsilon(t) \quad \text{a.e. in } \Omega. \tag{2.3.13}$$

We now prove the following bound: for every $t > 0$

$$\begin{aligned}
&\|\ddot{u}_\varepsilon\|^2_{L^\infty([0,t];L^2)} + \|\dot{e}_\varepsilon\|^2_{L^\infty([0,t];L^2)} + \varepsilon \|\operatorname{sym} D\ddot{u}_\varepsilon\|^2_{L^2([0,t];L^2)} \\
&\leq C(\varepsilon^2 \|\operatorname{div} \operatorname{sym} Dv_0^\varepsilon\|^2_{L^2} + \|\operatorname{sym} Dv_0^\varepsilon\|^2_{L^2} + \|\dot{f}\|^2_{L^1([0,t];L^2)} + \|\dddot{w}\|^2_{L^1([0,t];L^2)} \\
&+ \|\ddot{w}\|^2_{L^\infty([0,t];L^2)} + (\varepsilon + t)\|\operatorname{sym} D\ddot{w}\|^2_{L^2([0,t];L^2)}),
\end{aligned} \tag{2.3.14}$$

where $C > 0$ is a constant independent of t and h.

To prove (2.3.14), we extend continuously the fields involved by setting for $s < 0$

$$u_\varepsilon(s) = u_0 + sv_0^\varepsilon, \quad w(s) = w(0) + s\dot{w}(0), \quad e_\varepsilon(s) = e_0, \quad p_\varepsilon(s) = p_0, \quad f(s) = f(0).$$

We introduce the time incremental quotient

$$D^\delta a(t) := \frac{a(t) - a(t-\delta)}{\delta}.$$

Let $T > 0$, $t \in (0, T]$, and $\delta > 0$. Using the equation of motion, for every test function $\varphi \in L^2([0, t+\delta]; H^1_{\partial_d\Omega}(\Omega; \mathbb{R}^3))$ we have (2.3.12) and

$$\begin{aligned}
\int_\delta^{t+\delta} \langle \ddot{u}_\varepsilon(s-\delta), \varphi(s)\rangle \, ds + \int_\delta^{t+\delta} \int_\Omega (\sigma_\varepsilon(s-\delta) + \varepsilon \operatorname{sym} D\dot{u}_\varepsilon(s-\delta)) : \operatorname{sym} D\varphi(s) \, dx \, ds \\
= \int_\delta^{t+\delta} \int_\Omega f(s-\delta)\varphi_3(s) \, dx \, ds.
\end{aligned}$$

Subtracting (2.3.12) from the previous equation and choosing $\varphi = \frac{1}{\delta}\chi_{(0,t)} D^\delta(\dot{u}_\varepsilon - \dot{w})$ yield

$$\begin{aligned}
&\int_0^t \langle D^\delta(\ddot{u}_\varepsilon - \ddot{w})(s), D^\delta(\dot{u}_\varepsilon - \dot{w})(s)\rangle \, ds \\
&\quad + \int_0^t \int_\Omega D^\delta(\sigma_\varepsilon + \varepsilon \operatorname{sym} D\dot{u}_\varepsilon)(s) : D^\delta \operatorname{sym} D(\dot{u}_\varepsilon - \dot{w})(s) \, dx \, ds \\
&\quad + \int_0^t \int_\Omega D^\delta \ddot{w}(s) \cdot D^\delta(\dot{u}_\varepsilon - \dot{w})(s) \, dx \, ds - \int_0^t \int_\Omega D^\delta f(s) D^\delta((\dot{u}_\varepsilon)_3 - \dot{w}_3)(s) \, dx \, ds \\
&= \frac{1}{\delta} \int_0^\delta \int_\Omega f(0) D^\delta((\dot{u}_\varepsilon)_3 - \dot{w}_3)(s) \, dx \, ds \\
&\quad - \frac{1}{\delta} \int_0^\delta \int_\Omega (\sigma_0(s) + \varepsilon \operatorname{sym} Dv_0^\varepsilon(s)) : D^\delta \operatorname{sym} D(\dot{u}_\varepsilon - \dot{w})(s) \, dx \, ds.
\end{aligned}$$

Integrating by parts, the right-hand side can be rewritten as

$$\begin{aligned}
&\frac{1}{\delta} \int_0^\delta \int_\Omega f(0) D^\delta((\dot{u}_\varepsilon)_3 - \dot{w}_3)(s) \, dx \, ds \\
&\quad + \frac{1}{\delta} \int_0^\delta \int_\Omega \operatorname{div}(\sigma_0(s) + \varepsilon \operatorname{sym} Dv_0^\varepsilon(s)) \cdot D^\delta(\dot{u}_\varepsilon - \dot{w})(s) \, dx \, ds \\
&\quad - \frac{1}{\delta} \int_0^\delta \langle [(\sigma_0(s) + \varepsilon \operatorname{sym} Dv_0^\varepsilon(s))\nu_{\partial\Omega}], D^\delta(\dot{u}_\varepsilon - \dot{w})(s)\rangle \, ds
\end{aligned}$$

$$\begin{aligned}
&= \frac{\varepsilon}{\delta}\int_0^\delta \int_\Omega \operatorname{div}\,\operatorname{sym} Dv_0^\varepsilon(s)\cdot D^\delta(\dot u_\varepsilon - \dot w)(s)\,dx\,ds \\
&\leq \varepsilon\,\|\operatorname{div}\,\operatorname{sym} Dv_0^\varepsilon\|_{L^2}\|D^\delta(\dot u_\varepsilon - \dot w)\|_{L^\infty([0,T];L^2)},
\end{aligned}$$

where the equality follows from (2.3.4) and (2.3.11). We now focus on the term

$$\int_0^t \int_\Omega D^\delta \sigma_\varepsilon(s) : D^\delta \operatorname{sym} D\dot u_\varepsilon(s)\,dx\,ds.$$

Using the kinematic admissibility $\operatorname{sym} D\dot u_\varepsilon = \dot e_\varepsilon + \dot p_\varepsilon$ a.e. in $[0,+\infty)\times\Omega$, we have that for every $\tau \in L^2([0,t+\delta];L^2(\Omega;\mathbb{M}^{3\times 3}_{sym}))$

$$\int_0^t \int_\Omega \operatorname{sym} D\dot u_\varepsilon(s) : \tau(s)\,dx\,ds = \int_0^t \int_\Omega \dot e_\varepsilon(s) : \tau(s)\,dx\,ds + \int_0^t \int_\Omega \dot p_\varepsilon(s) : \tau(s)\,dx\,ds$$

and

$$\begin{aligned}
\int_\delta^{t+\delta} \int_\Omega \operatorname{sym} D\dot u_\varepsilon(s-\delta) : \tau(s)\,dx\,ds = \int_\delta^{t+\delta} \int_\Omega \dot e_\varepsilon(s-\delta) : \tau(s)\,dx\,ds \\
+ \int_\delta^{t+\delta} \int_\Omega \dot p_\varepsilon(s-\delta) : \tau(s)\,dx\,ds.
\end{aligned}$$

Testing the difference of the two previous equations by $\tau = \frac{1}{\delta}\chi_{(0,t)} D^\delta \sigma_\varepsilon$, we obtain

$$\begin{aligned}
&\int_0^t \int_\Omega D^\delta \operatorname{sym} D\dot u_\varepsilon(s) : D^\delta \sigma_\varepsilon(s)\,dx\,ds \\
&\quad = \int_0^t \int_\Omega D^\delta \dot e_\varepsilon(s) : D^\delta \sigma_\varepsilon(s)\,dx\,ds + \int_0^t \int_\Omega D^\delta \dot p_\varepsilon(s) : D^\delta (\sigma_\varepsilon)_D(s)\,dx\,ds \\
&\qquad - \frac{1}{\delta}\int_0^\delta \int_\Omega \operatorname{sym} Dv_0^\varepsilon : D^\delta \sigma_\varepsilon(s)\,dx\,ds \\
&\quad \geq \mathcal{Q}(D^\delta e_\varepsilon(t)) - \frac{1}{\delta}\int_0^\delta \int_\Omega \operatorname{sym} Dv_0^\varepsilon : D^\delta \sigma_\varepsilon(s)\,dx\,ds,
\end{aligned}$$

where we used that $D^\delta e_\varepsilon(0) = 0$ and

$$\int_0^t \int_\Omega D^\delta \dot p_\varepsilon(s) : D^\delta (\sigma_\varepsilon)_D(s)\,dx\,ds \geq 0$$

as a consequence of the stress constraint and of the flow rule (2.3.13). Since $D^\delta(\dot u_\varepsilon - \dot w)(0) = 0$, applying the Holder inequality we deduce

$$\begin{aligned}
&\frac{1}{2}\|D^\delta(\dot u_\varepsilon - \dot w)(t)\|_{L^2}^2 + \mathcal{Q}(D^\delta e_\varepsilon(t)) + \varepsilon\|D^\delta \operatorname{sym} D\dot u_\varepsilon\|^2_{L^2([0,t];L^2)} \\
&\quad \leq \varepsilon\,\|\operatorname{div}\,\operatorname{sym} Dv_0^\varepsilon\|_{L^2}\|D^\delta(\dot u_\varepsilon - \dot w)\|_{L^\infty([0,T];L^2)} + 2\beta_{\mathbb{C}}\|\operatorname{sym} Dv_0^\varepsilon\|_{L^2}\|D^\delta e_\varepsilon\|_{L^\infty([0,T];L^2)} \\
&\qquad + \|D^\delta f\|_{L^1([0,T];L^2)}\|D^\delta((\dot u_\varepsilon)_3 - \dot w_3)\|_{L^\infty([0,T];L^2)} \\
&\qquad + \big(2\beta_{\mathbb{C}}\sqrt{t}\|D^\delta e_\varepsilon\|_{L^\infty([0,T];L^2)} + \varepsilon\|D^\delta \operatorname{sym} D\dot u_\varepsilon\|_{L^2([0,T];L^2)}\big)\|D^\delta \operatorname{sym} D\dot w\|_{L^2([0,T];L^2)} \\
&\qquad + \|D^\delta(\dot u_\varepsilon - \dot w)\|_{L^\infty([0,T];L^2)}\|D^\delta \ddot w\|_{L^1([0,T];L^2)}
\end{aligned}$$

for every $T > 0$ and $t \in [0,T]$. By Young inequality and passing to the limit as δ tends to 0, we obtain (2.3.14).

As a consequence of (2.3.14), we deduce, in particular, that

$$u_\varepsilon \in W^{2,\infty}_{loc}([0,+\infty); L^2(\Omega;\mathbb{R}^3)),$$

so that the equation of motion (2.3.12) can be written in the strong formulation

$$\begin{cases} \ddot{u}_\varepsilon(t) - \operatorname{div}(\sigma_\varepsilon(t) + \varepsilon \operatorname{sym} D\dot{u}_\varepsilon(t)) = f(t)e_3 \text{ in } \Omega, \\ [(\sigma_\varepsilon(t) + \varepsilon \operatorname{sym} D\dot{u}_\varepsilon(t))\nu_{\partial\Omega}] = 0 \text{ on } \partial_n\Omega \end{cases} \tag{2.3.15}$$

for a.e. $t \geq 0$.

We now discuss how to pass to the limit, as $\varepsilon \to 0$. Arguing as in Proposition 4.3 of [7], from the equation of motion and the flow rule we obtain the following energy balance:

$$\begin{aligned} \mathcal{Q}(e_\varepsilon(t)) + \frac{1}{2}\|\dot{u}_\varepsilon(t)\|^2_{L^2} + \int_0^t \mathcal{H}(\dot{p}_\varepsilon(s))\,ds + \varepsilon \int_0^t \int_\Omega |\operatorname{sym} D\dot{u}_\varepsilon(s)|^2\,dx\,ds = \mathcal{Q}(e_0) + \frac{1}{2}\|v_0^\varepsilon\|^2_{L^2} \\ + \int_0^t \int_\Omega ((\sigma_\varepsilon(s) + \varepsilon \operatorname{sym} D\dot{u}_\varepsilon(s)) : \operatorname{sym} D\dot{w}(s) + \ddot{u}_\varepsilon(s)\cdot \dot{w}(s))\,dx\,ds \\ + \int_0^t \int_\Omega f(s)\,((\dot{u}_\varepsilon)_3(s) - \dot{w}_3(s))\,dx\,ds \end{aligned} \tag{2.3.16}$$

for every $\varepsilon > 0$ and every $t > 0$. Combining this inequality with (2.3.14) and using Ascoli-Arzelà and Helly Theorem, we deduce the existence of

$$u \in W^{2,\infty}_{loc}([0,+\infty); L^2(\Omega;\mathbb{R}^3)) \cap BV_{loc}([0,+\infty); BD(\Omega)),$$
$$e \in W^{1,\infty}_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{3\times3}_{sym})), \qquad p \in BV_{loc}([0,+\infty); M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{3\times3}_D))$$

such that, up to subsequences,

$$\begin{aligned} u_\varepsilon(t) &\rightharpoonup u(t) \quad \text{weakly* in } BD(\Omega), \\ \dot{u}_\varepsilon(t) &\rightharpoonup \dot{u}(t) \quad \text{weakly in } L^2(\Omega;\mathbb{R}^3), \\ e_\varepsilon(t) &\rightharpoonup e(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}_{sym}), \\ p_\varepsilon(t) &\rightharpoonup p(t) \quad \text{weakly* in } M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{3\times3}_D) \end{aligned} \tag{2.3.17}$$

for every $t \in [0,T]$. From these convergences we immediately deduce that (u,e,p) satisfies conditions (i)–(iii). By (2.3.16) we have that

$$\varepsilon \operatorname{sym} D\dot{u}_\varepsilon \to 0 \quad \text{strongly in } L^2_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{3\times3}_{sym})).$$

Since $\ddot{u}_\varepsilon \rightharpoonup \ddot{u}$ weakly* in $L^\infty_{loc}([0,+\infty); L^2(\Omega;\mathbb{R}^3))$, we can pass to the limit in the weak formulation of (2.3.15) and, thus, deduce condition (iv).

Taking the difference of the equations of motion (2.3.15) and (2.3.6) and testing by $\dot{u}_\varepsilon - \dot{w}$ on $[0,t]\times\Omega$, one can prove (see Lemma 4.5 in [7]) that

$$\begin{aligned} \dot{u}_\varepsilon &\to \dot{u} \quad \text{strongly in } L^\infty_{loc}([0,+\infty); L^2(\Omega;\mathbb{R}^3)), \\ e_\varepsilon &\to e \quad \text{strongly in } L^\infty_{loc}([0,+\infty); L^2(\Omega;\mathbb{R}^3)). \end{aligned} \tag{2.3.18}$$

We now write the energy balance (2.3.16) between two times $t_1 \leq t_2$ and using the previous convergences and the lower semicontinuity of the elastic energy and of the dissipation, we obtain

$$\mathcal{Q}(e(t_2)) + \frac{1}{2}\|\dot{u}(t_2)\|^2_{L^2} + \mathcal{D}(p;t_1,t_2) \le \mathcal{Q}(e(t_1)) + \frac{1}{2}\|\dot{u}(t_1)\|^2_{L^2} + \int_{t_1}^{t_2}\int_\Omega (\sigma(s) : \operatorname{sym} D\dot{w}(s) + \ddot{u}(s)\cdot\dot{w}(s))\,dx\,ds + \int_{t_1}^{t_2}\int_\Omega f(s)\,(\dot{u}_3(s) - \dot{w}_3(s))\,dx\,ds. \tag{2.3.19}$$

Let $T > 0$. Using the inequality

$$r_K\|p(t_2) - p(t_1)\|_{M_b} \le \mathcal{D}(p;t_1,t_2)$$

in (2.3.19), we deduce that

$$r_K\|p(t_2) - p(t_1)\|_{M_b} \le 2\beta_{\mathbb{C}}\|e\|_{L^\infty([0,T];L^2)}\|e(t_2) - e(t_1)\|_{L^2} + \|\dot{u}\|_{L^\infty([0,T];L^2)}\|\dot{u}(t_2) - \dot{u}(t_1)\|_{L^2} + 2\beta_{\mathbb{C}}\|e\|_{L^\infty([0,T];L^2)}\int_{t_1}^{t_2}\|\operatorname{sym} D\dot{w}(t)\|_{L^2}\,dt + \|\ddot{u}\|_{L^\infty([0,T];L^2)}\int_{t_1}^{t_2}\|\dot{w}(t)\|_{L^2}\,dt + \|\dot{u}_3 - \dot{w}_3\|_{L^\infty([0,T];L^2)}\int_{t_1}^{t_2}\|f(t)\|_{L^2}\,dt \tag{2.3.20}$$

for every $t_1, t_2 \in [0,T]$ with $t_1 \le t_2$. Hence, p is locally Lipschitz continuous on $[0,+\infty)$ with values in $M_b(\Omega\cup\Gamma_d;\mathbb{M}^{3\times3}_D)$, inequality (2.3.9) is satisfied, and (2.3.19) gives condition (v). Using the kinematic admissibility, one can prove that u is locally Lipschitz continuous on $[0,+\infty)$ with values in $BD(\Omega)$. Finally, inequality (2.3.8) easily follows from (2.3.14). □

2.4 Convergence of dynamic evolutions

In this section we discuss the convergence of three-dimensional dynamic evolutions, when the parameter h tends to 0. As it is usual in dimension reduction problems, we perform a change of variable in order to set the problem on a fixed domain Ω. We also perform a rescaling of the time variable (as done, e.g., in [1] in the context of nonlinear elasticity). We thus consider the change of variable $\phi_h : \overline{\Omega} \to \overline{\Omega}_h$ given by

$$\phi_h(x) := (x', hx_3)$$

for every $x = (x', x_3) \in \overline{\Omega}$. We define the linear operator $\Lambda_h : \mathbb{M}^{3\times3}_{sym} \to \mathbb{M}^{3\times3}_{sym}$ as

$$\Lambda_h\xi := \begin{pmatrix} \xi_{11} & \xi_{12} & \frac{1}{h}\xi_{13} \\ \xi_{12} & \xi_{22} & \frac{1}{h}\xi_{23} \\ \frac{1}{h}\xi_{13} & \frac{1}{h}\xi_{23} & \frac{1}{h^2}\xi_{33} \end{pmatrix}$$

for every $\xi \in \mathbb{M}^{3\times3}_{sym}$.

Let $t \mapsto (u_h(t), e_h(t), p_h(t))$ be a dynamic evolution in Ω_h with boundary datum w_h, force term f_h, and initial conditions $(u_{0,h}, e_{0,h}, p_{0,h})$ and $v_{0,h}$, as in Theorem 2.3.1. We associate with it an h-rescaled dynamic evolution in Ω, defined as follows:

$$t \mapsto (u^h(t), e^h(t), p^h(t)) \in BD(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times3}_{sym}) \times M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{3\times3}_{sym}),$$

where for every $t \ge 0$ and a.e. $x \in \Omega$

$$u^h_\alpha(t,x) := (u_h)_\alpha\big(\tfrac{t}{h}, \phi_h(x)\big) \quad \alpha = 1,2, \qquad u^h_3(t,x) := h(u_h)_3\big(\tfrac{t}{h}, \phi_h(x)\big), \qquad e^h(t,x) := \Lambda_h^{-1}e_h\big(\tfrac{t}{h}, \phi_h(x)\big), \tag{2.4.1}$$

and for every $t \geq 0$

$$p^h(t) := \tfrac{1}{h}\Lambda_h^{-1}\phi_h^{\#}p_h\big(\tfrac{t}{h}\big). \tag{2.4.2}$$

Here $\phi_h^{\#}q \in M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times 3}_D)$ denotes the pull-back measure of q, defined as

$$\int_{\Omega\cup\partial_d\Omega} \psi : d\phi_h^{\#}q := \int_{\Omega_h\cup\Gamma_{d,h}} \psi\circ\phi_h^{-1} : dq$$

for every $\psi \in C_0(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times 3}_D)$. Finally, we rescale the boundary datum w_h as

$$w^h_\alpha(t,x) := (w_h)_\alpha\big(\tfrac{t}{h}, \phi_h(x)\big) \quad \alpha = 1,2, \qquad w^h_3(t,x) := h(w_h)_3\big(\tfrac{t}{h}, \phi_h(x)\big) \tag{2.4.3}$$

for every $t \geq 0$ and a.e. $x \in \Omega$, and the vertical force f_h as

$$f^h(t,x) := \tfrac{1}{h}f_h\big(\tfrac{t}{h}, \phi_h(x)\big) \tag{2.4.4}$$

for every $t \geq 0$ and a.e. $x \in \Omega$.

The rescaled triplet satisfies the following conditions:

- *kinematic admissibility:* for every $t \geq 0$ we have

$$\begin{gathered} \operatorname{sym} Du^h(t) = e^h(t) + p^h(t) \text{ in } \Omega, \\ p^h(t) = (w^h(t) - u^h(t)) \odot \nu_{\partial\Omega}\mathcal{H}^2 \text{ on } \partial_d\Omega, \\ p^h_{11}(t) + p^h_{22}(t) + \tfrac{1}{h^2}p^h_{33}(t) = 0 \text{ in } \Omega \cup \partial_d\Omega; \end{gathered} \tag{2.4.5}$$

- *stress constraint:* $\sigma^h_D(t) \in K$ a.e. in Ω for every $t \geq 0$, where $\sigma^h(t) := \mathbb{C}\Lambda_h e^h(t)$;

- *equation of motion:* for a.e. $t \geq 0$

$$\begin{cases} \begin{pmatrix} h^2\ddot{u}^h_\alpha(t) \\ \ddot{u}^h_3(t) \end{pmatrix} - \operatorname{div}\Lambda_h\sigma^h(t) = f^h(t)e_3 \text{ in } \Omega, \\ [\Lambda_h\sigma^h(t)\nu_{\partial\Omega}] = 0 \text{ on } \partial_n\Omega; \end{cases} \tag{2.4.6}$$

- *energy inequality:* for every $0 \leq t_1 \leq t_2$

$$\begin{aligned} \mathcal{Q}(\Lambda_h e^h(t_2)) + \frac{1}{2}\Big\| \begin{pmatrix} h\dot{u}^h_\alpha(t_2) \\ \dot{u}^h_3(t_2) \end{pmatrix} \Big\|^2_{L^2} + \int_{t_1}^{t_2} \mathcal{H}(\Lambda_h \dot{p}^h(s))\, ds \\ \leq \mathcal{Q}(\Lambda_h e^h(t_1)) + \frac{1}{2}\Big\| \begin{pmatrix} h\dot{u}^h_\alpha(t_1) \\ \dot{u}^h_3(t_1) \end{pmatrix} \Big\|^2_{L^2} \\ + \int_{t_1}^{t_2}\int_\Omega \Big(\sigma^h(s) : \Lambda_h \operatorname{sym} D\dot{w}^h(s) + \begin{pmatrix} h\ddot{u}^h_\alpha(s) \\ \ddot{u}^h_3(s) \end{pmatrix} \cdot \begin{pmatrix} h\dot{w}^h_\alpha(s) \\ \dot{w}^h_3(s) \end{pmatrix} \Big)\, dx\, ds \\ + \int_{t_1}^{t_2}\int_\Omega f^h(s)\,(\dot{u}^h_3(s) - \dot{w}^h_3(s))\, dx\, ds. \end{aligned} \tag{2.4.7}$$

We now state the assumptions on the rescaled data of the problem.

Forces. We consider a sequence of vertical loads $(f^h) \subset W^{1,1}_{loc}([0,+\infty); L^2(\Omega))$ such that for every $T > 0$ there exists a constant $C(T) > 0$ for which

$$\|f^h\|_{W^{1,1}([0,T];L^2)} \le C(T) \tag{2.4.8}$$

for every $h > 0$. We also assume that there exists $f \in L^\infty_{loc}([0,+\infty); L^2(\Omega))$ such that

$$f^h(t) \to f(t) \quad \text{strongly in } L^2(\Omega) \tag{2.4.9}$$

for every $t \ge 0$.

Boundary displacements. We consider a sequence of boundary displacements

$$(w^h) \subset H^2_{loc}([0,+\infty); H^1(\Omega;\mathbb{R}^3)) \cap W^{3,1}_{loc}([0,+\infty); L^2(\Omega;\mathbb{R}^3)) \tag{2.4.10}$$

such that for every $T > 0$ there exists a constant $C(T) > 0$ for which

$$\Big\| \begin{pmatrix} h\dot{w}^h_\alpha \\ \dot{w}^h_3 \end{pmatrix} \Big\|_{W^{2,1}([0,T];L^2)} + \|\Lambda_h \operatorname{sym} Dw^h\|_{H^2([0,T];L^2)} \le C(T) \tag{2.4.11}$$

for every $h > 0$. We assume that for every $t \ge 0$

$$w^h(t) \rightharpoonup w(t) \quad \text{weakly in } L^2(\Omega;\mathbb{R}^3), \tag{2.4.12}$$

$$\Lambda_h \operatorname{sym} D\dot{w}^h(t) \to \eta(t) \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}_{sym}), \tag{2.4.13}$$

and

$$\begin{pmatrix} h\dot{w}^h_\alpha \\ \dot{w}^h_3 \end{pmatrix} \to \dot{w}_3 e_3 \quad \text{strongly in } L^1_{loc}([0,+\infty); L^2(\Omega;\mathbb{R}^3)) \tag{2.4.14}$$

for some $w \in H^2_{loc}([0,+\infty); H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega))$ and some $\eta \in H^1_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{3\times3}_{sym}))$.

Initial data. We fix a triplet $(u^h_0, e^h_0, p^h_0) \in H^1(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{M}^{3\times3}_{sym}) \times L^2(\Omega;\mathbb{M}^{3\times3}_{sym})$ satisfying the kinematic admissibility conditions (2.4.5) and an initial velocity $v^h_0 \in H^1(\Omega;\mathbb{R}^3)$ such that, setting $\sigma^h_0 := \mathbb{C}\Lambda_h e^h_0$, we have

$$-\operatorname{div} \Lambda_h \sigma^h_0 = f^h(0)e_3 \text{ in } \Omega, \quad [\Lambda_h \sigma^h_0 \nu_{\partial\Omega}] = 0 \text{ on } \partial_n\Omega, \quad (\sigma^h_0)_D \in K \text{ a.e. in } \Omega, \tag{2.4.15}$$

and

$$v^h_0 = \dot{w}^h(0) \text{ on } \partial_d\Omega \tag{2.4.16}$$

for every $h > 0$. Moreover, we suppose that

$$\begin{pmatrix} h(v^h_0)_\alpha \\ (v^h_0)_3 \end{pmatrix} \to v_0 e_3 \text{ strongly in } L^2(\Omega;\mathbb{R}^3), \tag{2.4.17}$$

$$\Lambda_h e^h_0 \to \tilde{e}_0 \text{ strongly in } L^2(\Omega;\mathbb{M}^{3\times3}_{sym}), \tag{2.4.18}$$

$$\|\Lambda_h \operatorname{sym} Dv^h_0\|_{L^2} + \|\Lambda_h p^h_0\|_{M_b} \le C \tag{2.4.19}$$

for some $v_0 \in H^1(\Omega;\mathbb{R}^3)$, $\tilde{e}_0 \in L^2(\Omega;\mathbb{M}^{3\times3}_{sym})$, and some constant C independent of h.

We are now in a position to state the main result of this paper.

Theorem 2.4.1. *Assume* (2.4.8)–(2.4.19) *and let* (u^h, e^h, p^h) *be an* h*-rescaled dynamic evolution for the boundary datum* w^h*, the force term* f^h*, and the initial data* (u_0^h, e_0^h, p_0^h) *and* v_0^h*. Then there exists a map* $t \mapsto (u(t), e(t), p(t))$ *of class*

$$\mathrm{Lip}_{loc}([0, +\infty); KL(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}) \times M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times 3}_{sym}))$$

with $u_3 \in W^{2,\infty}_{loc}([0, +\infty); L^2(\omega))$*, such that, up to subsequences,*

$$u^h(t) \rightharpoonup u(t) \quad \textit{weakly* in } BD(\Omega), \tag{2.4.20}$$

$$\dot{u}_3^h(t) \to \dot{u}_3(t) \quad \textit{strongly in } L^2(\Omega), \tag{2.4.21}$$

$$e^h(t) \to e(t) \quad \textit{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}), \tag{2.4.22}$$

$$\Lambda_h e^h(t) \to \mathbb{M}e(t) \quad \textit{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}), \tag{2.4.23}$$

$$p^h(t) \rightharpoonup p(t) \quad \textit{weakly* in } M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times 3}_{sym}) \tag{2.4.24}$$

for every $t \geq 0$*. The map* $t \mapsto (u(t), e(t), p(t))$ *satisfies the following system of equations:*

(i) kinematic admissibility: $(u(t), e(t), p(t)) \in \mathcal{A}_{KL}(w(t))$ *for every* $t \geq 0$*;*

(ii) initial conditions: $(u(0), e(0), p(0)) = (u_0, e_0, p_0)$ *and* $\dot{u}_3(0) = (v_0)_3$*, where* $u_0^h \rightharpoonup u_0$ *weakly* in* $BD(\Omega)$*,* $e_0^h \to e_0$ *strongly in* $L^2(\Omega; \mathbb{M}^{3\times 3}_{sym})$*, and* $p_0^h \rightharpoonup p_0$ *weakly* in* $M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times 3}_{sym})$ *(these limits exist, up to subsequences);*

(iii) stress constraint: $\sigma(t) \in K^*$ *a.e. in* Ω *for every* $t \geq 0$*, where* $\sigma(t) := \mathbb{C}_r e(t)$*;*

(iv) equations of motion: *for every* $t \geq 0$

$$\begin{cases} \mathrm{div}\, \bar{\sigma}(t) = 0 & \textit{in } \omega, \\ [\bar{\sigma}(t)\nu_{\partial\omega}] = 0 & \textit{on } \partial_n\omega, \end{cases} \tag{2.4.25}$$

and for a.e. $t \geq 0$

$$\begin{cases} \ddot{u}_3(t) - \frac{1}{12}\mathrm{div}\,\mathrm{div}\, \hat{\sigma}(t) = \bar{f}(t) \ \textit{in } \omega, \\ b_0(\hat{\sigma}(t)) = b_1(\hat{\sigma}(t)) = 0 \quad \textit{on } \partial_n\omega, \end{cases} \tag{2.4.26}$$

where

$$\bar{f}(x') := \int_{-1/2}^{1/2} f(x', x_3)\, dx_3 \quad \textit{for a.e. } x' \in \omega;$$

(v) flow rule: *for a.e.* $t \geq 0$

$$\mathcal{H}^*(\dot{p}(t)) = \langle \sigma(t), \dot{p}(t) \rangle_r \,. \tag{2.4.27}$$

Proof. The proof of Theorem 2.4.1 is subdivided into six steps.

Step 1: Compactness estimates. We first deduce some a priori estimates. Writing the estimate (2.3.8) on $[0, t/h]$ and performing the scaling, we obtain

$$\begin{aligned} \Big\| \begin{pmatrix} h\ddot{u}_\alpha^h \\ \ddot{u}_3^h \end{pmatrix} \Big\|_{L^\infty([0,t];L^2)} + \|\Lambda_h \dot{e}^h\|_{L^\infty([0,t];L^2)} &\leq C\Big(\|\Lambda_h \,\mathrm{sym}\, Dv_0^h\|_{L^2} + \|\dot{f}^h\|_{L^1([0,t];L^2)} \\ + \Big\| \begin{pmatrix} h\dddot{w}_\alpha^h \\ \dddot{w}_3^h \end{pmatrix} \Big\|_{L^1([0,t];L^2)} &+ \Big\| \begin{pmatrix} h\ddot{w}_\alpha^h \\ \ddot{w}_3^h \end{pmatrix} \Big\|_{L^\infty([0,t];L^2)} + \sqrt{t}\, \|\Lambda_h \,\mathrm{sym}\, D\ddot{w}^h\|_{L^2([0,t];L^2)} \Big) \end{aligned} \tag{2.4.28}$$

for every $t > 0$. By the assumptions on the data we deduce that for every $T > 0$ there exists a constant $C(T) > 0$, depending on T but independent of h, such that

$$\Big\| \begin{pmatrix} h\ddot{u}^h_\alpha \\ \ddot{u}^h_3 \end{pmatrix} \Big\|_{L^\infty([0,T];L^2)} + \|\Lambda_h \dot{e}^h\|_{L^\infty([0,T];L^2)} \le C(T). \tag{2.4.29}$$

We now write the rescaled energy inequality (2.4.7) with $t_1 = 0$ and $t_2 \in [0,t]$. By (2.2.1) and (2.2.2) we have

$$\begin{aligned} &\alpha_{\mathbb{C}} \|\Lambda_h e^h\|^2_{L^\infty([0,t];L^2)} + \frac{1}{2}\Big\| \begin{pmatrix} h\dot{u}^h_\alpha \\ \dot{u}^h_3 \end{pmatrix} \Big\|^2_{L^\infty([0,t];L^2)} \\ &\le \beta_{\mathbb{C}} \|\Lambda_h e^h_0\|^2_{L^2} + \frac{1}{2}\Big\| \begin{pmatrix} h(v^h_0)_\alpha \\ (v^h_0)_3 \end{pmatrix} \Big\|^2_{L^2} + 2\beta_{\mathbb{C}} \|\Lambda_h e^h\|_{L^\infty([0,t];L^2)} \int_0^t \|\Lambda_h \operatorname{sym} D\dot{w}^h(s)\|_{L^2}\, ds \\ &\qquad + \Big\| \begin{pmatrix} h\ddot{u}^h_\alpha \\ \ddot{u}^h_3 \end{pmatrix} \Big\|_{L^\infty([0,t];L^2)} \int_0^t \Big\| \begin{pmatrix} h\dot{w}^h_\alpha \\ \dot{w}^h_3 \end{pmatrix} \Big\|_{L^2} ds \\ &\qquad\qquad + \left(\|\dot{u}^h_3\|_{L^\infty([0,t];L^2)} + \|\dot{w}^h_3\|_{L^\infty([0,t];L^2)}\right) \int_0^t \|f^h(s)\|_{L^2}\, ds. \end{aligned}$$

By the Cauchy inequality, the assumptions on the data and (2.4.29), we deduce that for every $T > 0$ there exists a constant $C(T) > 0$, depending on T but independent of h, such that

$$\|\Lambda_h e^h\|_{L^\infty([0,T];L^2)} + \Big\| \begin{pmatrix} h\dot{u}^h_\alpha \\ \dot{u}^h_3 \end{pmatrix} \Big\|_{L^\infty([0,T];L^2)} \le C(T). \tag{2.4.30}$$

Finally, we perform the scaling in (2.3.9) and by (2.4.29) and (2.4.30) we get

$$\begin{aligned} \|\Lambda_h p^h(t_2) - \Lambda_h p^h(t_1)\|_{M_b} &\le C(T)\Big(\|\Lambda_h e^h(t_2) - \Lambda_h e^h(t_1)\|_{L^2} \\ &+ \Big\| \begin{pmatrix} h\dot{u}^h_\alpha(t_2) - h\dot{u}^h_\alpha(t_1) \\ \dot{u}^h_3(t_2) - \dot{u}^h_3(t_1) \end{pmatrix} \Big\|_{L^2} + \int_{t_1}^{t_2} \|\Lambda_h \operatorname{sym} D\dot{w}^h(t)\|_{L^2}\, dt \\ &+ \int_{t_1}^{t_2} \Big\| \begin{pmatrix} h\dot{w}^h_\alpha(t) \\ \dot{w}^h_3(t) \end{pmatrix} \Big\|_{L^2} dt + \int_{t_1}^{t_2} \|f^h(t)\|_{L^2}\, dt \Big) \end{aligned} \tag{2.4.31}$$

for every $T > 0$ and every $t_1, t_2 \in [0,T]$.

We now deduce some compactness properties for the triplets (u^h, e^h, p^h), as $h \to 0$. By (2.4.29), (2.4.30), and the Ascoli-Arzelà Theorem we infer the existence of

$$e, \tilde{e} \in W^{1,\infty}_{loc}([0,+\infty); L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}))$$

with $e_{\alpha\beta} = \tilde{e}_{\alpha\beta}$ for $\alpha, \beta = 1,2$ and $e_{i3} = 0$ for $i = 1,2,3$, such that, up to subsequences,

$$e^h(t) \rightharpoonup e(t) \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}), \tag{2.4.32}$$

$$\Lambda_h e^h(t) \rightharpoonup \tilde{e}(t) \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}) \tag{2.4.33}$$

for every $t \ge 0$. Moreover, by (2.4.29) and (2.4.31) the functions $\Lambda_h p^h$ are equi-Lipschitz continuous in time with values in $M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times 3}_D)$. Therefore, again by the Ascoli-Arzelà Theorem and by (2.4.19) there exist

$$p \in \operatorname{Lip}_{loc}([0,+\infty); M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times 3}_{sym})), \quad \tilde{p} \in \operatorname{Lip}_{loc}([0,+\infty); M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times 3}_D)),$$

with $p_{\alpha\beta} = \tilde{p}_{\alpha\beta}$ for $\alpha, \beta = 1, 2$ and $p_{i3} = 0$ for $i = 1, 2, 3$, such that, up to subsequences,

$$p^h(t) \rightharpoonup p(t) \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times3}_{sym}), \tag{2.4.34}$$

$$\Lambda_h p^h(t) \rightharpoonup \tilde{p}(t) \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times3}_{D}) \tag{2.4.35}$$

for every $t \geq 0$.

We now prove the weak* compactness in $BD(\Omega)$ of the sequence of displacements (u^h). Since $\partial_d\omega$ is open in $\partial\omega$, there exists an open set $A \subset \mathbb{R}^2$ such that $\partial_d\omega = A \cap \partial\omega$. Let $\Omega' := (\omega \cup A) \times (-\frac{1}{2}, \frac{1}{2})$. By (2.4.11) and (2.4.12) we have that

$$\operatorname{sym} Dw^h(t) \rightharpoonup \operatorname{sym} Dw(t) \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{3\times3}_{sym}) \tag{2.4.36}$$

for every $t \geq 0$. Thus, for every $t \geq 0$ we can extend $w^h(t)$ and $w(t)$ to Ω' in such a way that $w^h(t) \rightharpoonup w(t)$ weakly in $L^2(\Omega'; \mathbb{R}^3)$ and $\operatorname{sym} Dw^h(t) \rightharpoonup \operatorname{sym} Dw(t)$ weakly in $L^2(\Omega'; \mathbb{M}^{3\times3}_{sym})$ for every $t \geq 0$.

We now extend the triplets (u^h, e^h, p^h) to Ω' by setting

$$u^h(t) := w^h(t) \text{ in } \Omega' \backslash \Omega, \qquad e^h(t) := \operatorname{sym} Dw^h(t) \text{ in } \Omega' \backslash \Omega, \qquad p^h(t) := 0 \text{ in } \Omega' \backslash (\Omega \cup \partial_d\Omega)$$

and we note that $\operatorname{sym} Du^h(t) = e^h(t) + p^h(t)$ in Ω'. Similarly, we set

$$e(t) := \operatorname{sym} Dw(t) \text{ in } \Omega' \setminus \Omega, \qquad p(t) := 0 \text{ in } \Omega' \setminus (\Omega \cup \partial_d\Omega).$$

By (2.4.32) and (2.4.34) we deduce that $e^h(t) \rightharpoonup e(t)$ weakly in $L^2(\Omega'; \mathbb{M}^{3\times3}_{sym})$ and $p^h(t) \rightharpoonup p(t)$ weakly* in $M_b(\Omega'; \mathbb{M}^{3\times3}_{sym})$, for every $t \geq 0$. Thus,

$$\operatorname{sym} Du^h(t) = e^h(t) + p^h(t) \rightharpoonup e(t) + p(t) \quad \text{weakly}^* \text{ in } M_b(\Omega'; \mathbb{M}^{3\times3}_{sym}).$$

Since $u^h(t) = w^h(t)$ in $\Omega' \setminus \Omega$ and $w^h(t)$ is bounded in $L^2(\Omega'; \mathbb{R}^3)$, the Korn-Poincaré inequality implies that the sequence $(u^h(t))$ is uniformly bounded in $BD(\Omega')$. Consequently, there exist $u(t) \in BD(\Omega')$ and a subsequence $u^{h_j}(t)$ such that $u^{h_j}(t) \rightharpoonup u(t)$ weakly* in $BD(\Omega')$. Since

$$u(t) = w(t) \text{ in } \Omega' \setminus \Omega \quad \text{ and } \quad \operatorname{sym} Du(t) = e(t) + p(t) \text{ in } \Omega',$$

the Korn-Poincaré inequality ensures that the limit $u(t)$ is uniquely determined. Therefore, the whole sequence converges in Ω' and in particular,

$$u^h(t) \rightharpoonup u(t) \quad \text{weakly}^* \text{ in } BD(\Omega) \tag{2.4.37}$$

for every $t \geq 0$.

Since $e_{i3}(t) = p_{i3}(t) = 0$, it is easy to see that

$$(u(t), e(t), p(t)) \in \mathcal{A}_{KL}(w(t))$$

for every $t \geq 0$. Moreover, $u \in Lip_{loc}([0, +\infty); KL(\Omega))$, owing to the time regularity of e, p, and w, and as a consequence of Lemma 2.2.1,

$$(\dot{u}(t), \dot{e}(t), \dot{p}(t)) \in \mathcal{A}_{KL}(\dot{w}(t)) \tag{2.4.38}$$

for a.e. $t \geq 0$.

Finally, combining (2.4.29), (2.4.30), (2.4.37), together with the Ascoli-Arzelà Theorem, we conclude that

$$u_3 \in W^{2,\infty}_{loc}([0,+\infty); L^2(\Omega))$$

and

$$h\dot{u}^h_\alpha(t) \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega) \quad \text{for } \alpha = 1,2, \tag{2.4.39}$$

$$\dot{u}^h_3(t) \rightharpoonup \dot{u}_3(t) \quad \text{weakly in } L^2(\Omega) \tag{2.4.40}$$

for every $t \geq 0$. Moreover, we also have that

$$h\dot{u}^h_\alpha \rightharpoonup 0 \quad \text{weakly* in } W^{1,\infty}([0,T]; L^2(\Omega)) \quad \text{for } \alpha = 1,2, \tag{2.4.41}$$

$$\dot{u}^h_3 \rightharpoonup \dot{u}_3 \quad \text{weakly* in } W^{1,\infty}([0,T]; L^2(\Omega)) \tag{2.4.42}$$

for every $T > 0$.

The previous arguments also prove that, up to subsequences, $u^h_0 \rightharpoonup u_0$ weakly* in $BD(\Omega)$, $e^h_0 \to e_0$ strongly in $L^2(\Omega; \mathbb{M}^{3\times3}_{sym})$, and $p^h_0 \rightharpoonup p_0$ weakly* in $M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times3}_{sym})$, for some $(u_0, e_0, p_0) \in \mathcal{A}_{KL}(w(0))$, and the initial conditions are satisfied.

Step 2: Identification of the limiting elastic strain. We claim that

$$\tilde{e}(t) = \mathbb{M}e(t) \tag{2.4.43}$$

for every $t \geq 0$, where $\tilde{e}$ satisfies (2.4.33) and $\mathbb{M}$ is the operator defined in (2.2.6).

We first show that (2.4.43) holds for a.e. $t \geq 0$. Owing to (2.2.7), this is equivalent to prove that for a.e. $t \geq 0$

$$\mathbb{C}\tilde{e}(t,x) : \begin{pmatrix} 0 & 0 & \lambda_1 \\ 0 & 0 & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} = 0$$

for every $\lambda_i \in \mathbb{R}$ and a.e. $x \in \Omega$. Let $(a,b) \subset (-\frac{1}{2}, \frac{1}{2})$ and let $U \subset \omega$ be an open set. Let $(\ell_n) \subset C^1([-\frac{1}{2}, \frac{1}{2}])$ and $(\lambda^i_n) \subset C^1_c(\omega)$ be two sequences such that $\ell'_n \to \chi_{(a,b)}$ strongly in $L^4(-\frac{1}{2}, \frac{1}{2})$ and $\lambda^i_n \to \lambda_i\chi_U$ strongly in $L^4(\omega)$ for every $i = 1,2,3$, as $n \to \infty$.

We define

$$\phi^h_n(t,x) := \psi(t) \begin{pmatrix} 2h\lambda^1_n(x')\ell_n(x_3) \\ 2h\lambda^2_n(x')\ell_n(x_3) \\ h^2\lambda^3_n(x')\ell_n(x_3) \end{pmatrix},$$

where $\psi \in L^2(0,+\infty)$. Testing (2.4.6) by ϕ^h_n yields

$$\int_0^{+\infty}\int_\Omega \begin{pmatrix} h^2\ddot{u}^h_\alpha(t) \\ \ddot{u}^h_3(t) \end{pmatrix} \cdot \phi^h_n(t)\,dx\,dt + \int_0^{+\infty}\int_\Omega \mathbb{C}\Lambda_h e^h(t) : \Lambda_h \operatorname{sym} D\phi^h_n(t)\,dx\,dt$$

$$= \int_0^{+\infty}\int_\Omega f^h(t)(\phi^h_n)_3(t)\,dx\,dt.$$

Owing to (2.4.30), (2.4.33), (2.4.41), and (2.4.42), we can pass to the limit as $h \to 0$ and then, as $n \to +\infty$. This yields

$$\int_0^{+\infty}\int_{U\times(a,b)} \psi(t)\,\mathbb{C}\tilde{e}(t,x) : \begin{pmatrix} 0 & 0 & \lambda_1 \\ 0 & 0 & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} dx\,dt = 0.$$

Since the sets (a,b), U and the function ψ are arbitrary, we deduce that for a.e. $t \geq 0$ $\tilde{e}(t) = \mathbb{M}e(t)$ a.e. in Ω. Since $\tilde{e}$ and $\mathbb{M}e$ are continuous functions of time, this implies (2.4.43).

This argument also proves that $\tilde{e}_0 = \mathbb{M}e_0$, where $\tilde{e}_0$ is the limit in (2.4.18).

Step 3: Equations of motions. Let $T > 0$. Let $\varphi \in L^2([0,T]; KL(\Omega) \cap H^1(\Omega;\mathbb{R}^3))$ with $\varphi = 0$ on $\partial_d\Omega$. We test the rescaled equation of motion (2.4.6) by φ on $[0,T] \times \Omega$. This yields

$$\int_0^T \int_\Omega \begin{pmatrix} h^2 \ddot{u}^h_\alpha(t) \\ \ddot{u}^h_3(t) \end{pmatrix} \cdot \varphi(t)\, dx\, dt + \int_0^T \int_\Omega \mathbb{C}\Lambda_h e^h(t) : \operatorname{sym} D\varphi(t)\, dx\, dt = \int_0^T \int_\Omega f^h(t)\varphi_3(t)\, dx\, dt,$$

where we used that $\Lambda_h \operatorname{sym} D\varphi(t) = \operatorname{sym} D\varphi(t)$ since $\varphi(t) \in KL(\Omega)$. As a consequence of (2.4.30), (2.4.33), (2.4.41), and (2.4.42), we can pass to the limit in the previous equation and obtain

$$\int_0^T \int_\omega \ddot{u}_3(t) \cdot \varphi_3(t)\, dx'\, dt + \int_0^T \int_\Omega \sigma(t) : \begin{pmatrix} \operatorname{sym} D\bar{\varphi}(t) - x_3 D^2\varphi_3(t) & 0 \\ 0 & 0 \end{pmatrix} dx\, dt = \int_0^T \int_\omega \bar{f}(t)\varphi_3(t)\, dx'\, dt, \quad (2.4.44)$$

where $\sigma(t) := \mathbb{C}^* e(t) = \mathbb{C}\mathbb{M}e(t)$.

By choosing $\varphi = (\bar{\varphi}, 0)$ with $\bar{\varphi} \in L^2([0,T]; H^1(\omega;\mathbb{R}^2))$, $\bar{\varphi}(t) = 0$ on $\partial_d\omega$, in (2.4.44) we deduce that

$$\int_0^T \int_\omega \bar{\sigma}(t) : \operatorname{sym} D\bar{\varphi}(t)\, dx'\, dt = 0.$$

This implies that for a.e. $t \geq 0$

$$\int_\omega \bar{\sigma}(t) : \operatorname{sym} D\bar{\varphi}\, dx' = 0$$

for every $\bar{\varphi} \in H^1(\omega;\mathbb{R}^2)$, $\bar{\varphi} = 0$ on $\partial_d\omega$. The continuity of $\bar{\sigma}$ with respect to time implies that the above equation is actually satisfied for every $t \geq 0$. By Lemma 7.10–(i) in [13] we conclude that

$$\operatorname{div} \bar{\sigma}(t) = 0 \text{ in } \omega, \quad [\bar{\sigma}(t)\nu_{\partial\omega}] = 0 \text{ on } \partial_n\omega$$

for every $t \geq 0$.

We now choose φ in (2.4.44) of the form $\varphi = \varphi_3 e_3$, with $\varphi_3 \in L^2([0,T]; H^2(\omega))$, $\varphi_3(t) = 0$ and $\nabla\varphi_3(t) = 0$ on $\partial_d\omega$ and obtain

$$\int_0^T \int_\omega \ddot{u}_3(t)\varphi_3(t)\, dx'\, dt - \frac{1}{12}\int_0^T \int_\omega \hat{\sigma}(t) : D^2\varphi_3(t)\, dx'\, dt = \int_0^T \int_\omega \bar{f}(t)\varphi_3(t)\, dx'\, dt.$$

By Lemma 7.10–(ii) in [13] this implies that

$$\ddot{u}_3(t) - \frac{1}{12}\operatorname{div}\operatorname{div} \hat{\sigma}(t) = \bar{f}(t) \quad \text{in } [0,+\infty) \times \omega,$$

together with the corresponding Neumann boundary conditions.

Step 4: Stress constraint. We recall that $(\mathbb{C}\Lambda_h e^h)_D(t) \in K$ a.e. in Ω for every $t \geq 0$ and every h. Since $\mathbb{C}\Lambda_h e^h(t) \rightharpoonup \sigma(t)$ weakly in $L^2(\Omega; \mathbb{M}^{3\times 3}_{sym})$ for every $t \geq 0$ and K is a closed and convex set, we have that $\sigma_D(t) \in K$ a.e. in Ω. By (2.2.12) this is equivalent to saying that $\sigma(t) \in K^*$ a.e. in Ω for every $t \geq 0$.

Step 5: Flow rule. We first observe that

$$\mathcal{H}^*(\dot{p}(t)) \geq \langle \sigma(t), \dot{p}(t) \rangle_r \tag{2.4.45}$$

for a.e. $t \geq 0$. This follows from (2.2.17) combined with the fact that $\sigma(t) \in K^*$ a.e. in Ω for every $t \geq 0$. Moreover, as a consequence of Proposition 2.2.5, (2.4.25), (2.4.26), and (2.4.38), we have that

$$\begin{aligned}\langle \sigma(t), \dot{p}(t) \rangle_r &= \int_\Omega \sigma(t) : (\operatorname{sym} D\dot{w}(t) - \dot{e}(t))\, dx - \frac{1}{12}\int_\omega \operatorname{div}\operatorname{div}\hat{\sigma}(t)(\dot{u}_3(t) - \dot{w}_3(t))\, dx' \\ &= \int_\Omega \sigma(t) : (\operatorname{sym} D\dot{w}(t) - \dot{e}(t))\, dx + \int_\omega (\bar{f}(t) - \ddot{u}_3(t))(\dot{u}_3(t) - \dot{w}_3(t))\, dx'\end{aligned} \tag{2.4.46}$$

for a.e. $t \geq 0$.

On the other hand, we can pass to the limit in the rescaled energy inequality arguing as follows. By (2.4.35), the lower semicontinuity of the dissipation and the definition of $\mathcal{D}^*$, it turns out that

$$\mathcal{D}^*(p; 0, T) \leq \liminf_{h\to 0} \mathcal{D}(\Lambda_h p^h; 0, T)$$

for every $T > 0$. Combining this inequality with the regularity of p, (2.2.5), and (2.2.13), we have that

$$\int_0^T \mathcal{H}^*(\dot{p}(t))\, dt \leq \liminf_{h\to 0} \int_0^T \mathcal{H}(\Lambda_h \dot{p}^h(t))\, dt \tag{2.4.47}$$

for every $T > 0$. We now write the rescaled energy inequality (2.4.7) with $t_1 = 0$ and $t_2 = T$. Using the lower semicontinuity of $\mathcal{Q}$, the definition of $\mathcal{Q}^*$, and the assumptions on the data (3.5.6) and (2.4.13)–(2.4.18), we deduce that

$$\begin{aligned}&\mathcal{Q}^*(e(T)) + \frac{1}{2}\|\dot{u}_3(T)\|^2_{L^2} + \int_0^T \mathcal{H}^*(\dot{p}(t))\, dt \leq \mathcal{Q}^*(e(0)) + \frac{1}{2}\|\dot{u}_3(0)\|^2_{L^2} \\ &\quad + \int_0^T \int_\Omega \big(\mathbb{C}^* e(t) : \operatorname{sym} D\dot{w}(t) + \ddot{u}_3(t)\dot{w}_3(t)\big)\, dx\, dt + \int_0^T \int_\omega \bar{f}(t)\,(\dot{u}_3(t) - \dot{w}_3(t))\, dx'\, dt\end{aligned} \tag{2.4.48}$$

for every $T > 0$. Here we used that $\eta_{\alpha\beta}(t) = \operatorname{sym} Dw_{\alpha\beta}(t)$ by (2.4.13) and (2.4.36). By the time regularity of e and u, inequality (2.4.48) can be rewritten as

$$\begin{aligned}\int_0^T \mathcal{H}^*(\dot{p}(t))\, dt \quad &\leq \quad \int_0^T \int_\Omega \sigma(t) : (\operatorname{sym} D\dot{w}(t) - \dot{e}(t))\, dx\, dt \\ &\quad + \int_0^T \int_\omega (\bar{f}(t) - \ddot{u}_3(t))(\dot{u}_3(t) - \dot{w}_3(t))\, dx'\, dt.\end{aligned}$$

Hence, by (2.4.46)

$$\int_0^T \mathcal{H}^*(\dot{p}(t))\, dt \leq \int_0^T \langle \sigma(t), \dot{p}(t) \rangle_r\, dt. \tag{2.4.49}$$

Combining the above inequality with (2.4.45), we deduce the flow rule (2.4.27).

We also note, for future references, that the flow rule implies that the inequality in (2.4.49) is actually an equality. Therefore, by (2.4.46) inequality (2.4.48) is an equality, as well. In other words, the following energy balance holds:

$$\mathcal{Q}^*(e(T)) + \frac{1}{2}\|\dot{u}_3(T)\|_{L^2}^2 + \int_0^T \mathcal{H}^*(\dot{p}(t))\,dt = \mathcal{Q}^*(e(0)) + \frac{1}{2}\|\dot{u}_3(0)\|_{L^2}^2$$
$$+ \int_0^T \int_\Omega \mathbb{C}^* e(t) : \mathrm{sym}\, D\dot{w}(t)\,dx\,dt + \int_0^T \int_\omega \big(\ddot{u}_3(t)\dot{w}_3(t) + \bar{f}(t)\,(\dot{u}_3(t) - \dot{w}_3(t))\big)\,dx'\,dt \tag{2.4.50}$$

for every $T > 0$.

Step 6: Strong convergence of the stress and the velocity. We conclude the proof by showing the strong convergence of the sequences $(\dot{u}_3^h(t))$, $(e^h(t))$, and $(\Lambda_h e^h(t))$.

By (2.4.7), (2.4.50), and the assumptions on the data we have

$$\begin{aligned}
&\limsup_{h\to 0} \Big\{ \mathcal{Q}(\Lambda_h e^h(T)) + \frac{1}{2}\Big\| \begin{pmatrix} h\dot{u}_\alpha^h(T) \\ \dot{u}_3^h(T) \end{pmatrix} \Big\|_{L^2}^2 + \int_0^T \mathcal{H}(\Lambda_h \dot{p}^h(t))\,dt \Big\} \\
\le\;& \mathcal{Q}_r(e(0)) + \frac{1}{2}\|\dot{u}_3(0)\|_{L^2}^2 + \int_0^T \int_\Omega \sigma(t) : \mathrm{sym}\, D\dot{w}(t)\,dx\,dt \\
&+ \int_0^t \int_\omega (\ddot{u}_3(t)\dot{w}_3(t) + \bar{f}(t)(\dot{u}_3(t) - \dot{w}_3(t))\,dx'\,dt \\
=\;& \mathcal{Q}^*(e(T)) + \frac{1}{2}\|\dot{u}_3(T)\|_{L^2}^2 + \int_0^T \mathcal{H}^*(\dot{p}(t))\,dx\,dt.
\end{aligned}$$

Recalling (2.4.47) and

$$\mathcal{Q}^*(e(T)) \le \liminf_{h\to 0} \mathcal{Q}(\Lambda_h e^h(T)), \qquad \|\dot{u}_3(T)\|_{L^2}^2 \le \liminf_{h\to 0} \|\dot{u}_3^h(T)\|_{L^2}^2,$$

the inequality above implies that $\dot{u}_3^h(t) \to \dot{u}_3(t)$ strongly in $L^2(\Omega)$ and

$$\mathcal{Q}(\Lambda_h e^h(t)) \to \mathcal{Q}^*(e(t)) = \mathcal{Q}(\mathbb{M}e(t))$$

for every $t \ge 0$. Since

$$\mathcal{Q}(\Lambda_h e^h(t) - \mathbb{M}e(t)) = \mathcal{Q}(\Lambda_h e^h(t)) + \mathcal{Q}(\mathbb{M}e(t)) - \int_\Omega \mathbb{C}\Lambda_h e^h(t) : \mathbb{M}e(t)\,dx,$$

equations (2.4.33) and (2.4.43) imply that

$$\lim_{h\to 0} \mathcal{Q}(\Lambda_h e^h(t) - \mathbb{M}e(t)) = 0$$

for every $t \ge 0$. Hence, by (2.2.1) we conclude that $\Lambda_h e^h(t) \to \mathbb{M}e(t)$ strongly in $L^2(\Omega; \mathbb{M}^{3\times 3}_{sym})$ for every $t \ge 0$. As an immediate consequence, we deduce that $e^h(t) \to e(t)$ strongly in $L^2(\Omega; \mathbb{M}^{3\times 3}_{sym})$ for every $t \ge 0$.

This concludes the proof of Theorem 2.4.1. □

Remark 2.4.2. A key ingredient in the proof of Theorem 2.4.1 is given by the higher regularity estimates (2.3.8) and (2.3.9). Using (2.4.1)–(2.4.4) these estimates can be written in the scaled variables. This leads to inequalities (2.4.28) and (2.4.31), which are instrumental to deduce compactness of the three-dimensional evolutions in the energy space. At this point, is it crucial to have a purely vertical body load. Indeed, in the presence of a nontrivial tangential force, the regularity estimates (2.3.8) and (2.3.9) do not have the right invariance property with respect to scaling in h, because of the different order of magnitude of the horizontal and vertical loads in terms of h (which is, in turn, due to the different order of magnitude of the horizontal and vertical displacements), so that the simple scaling argument described above does not allow to make the dependence on h fully explicit in estimates (2.4.28) and (2.4.31).

2.5 Some properties of the reduced model

In this section we collect some results about uniqueness for the reduced dynamic model, that has been derived in the previous section. We first prove uniqueness of the vertical displacement, of the elastic strain, and of some components of the plastic strain.

Proposition 2.5.1. *Let $t \mapsto (u(t), e(t), p(t))$ be a reduced dynamic evolution, that is, a solution to system* (i)–(v) *in Theorem 2.4.1. Then the vertical displacement u_3, the elastic strain e, and the plastic strain components $\hat{p}$ and $p_\perp$ are unique.*

Proof. Let (u, e, p) and (v, η, q) be two solutions. Let $\sigma(t) := \mathbb{C}^* e(t)$ and $\tau(t) := \mathbb{C}^* \eta(t)$. Subtracting the two equations of motion for u_3 and v_3 leads to

$$\ddot{u}_3(t) - \ddot{v}_3(t) - \frac{1}{12}\operatorname{div}\operatorname{div}(\hat{\sigma}(t) - \hat{\tau}(t)) = 0 \quad \text{in } \omega$$

for a.e. $t \geq 0$. Multiplying this equation by $\dot{u}_3(t) - \dot{v}_3(t)$ and integrating on $[0, T] \times \omega$ yields

$$\int_0^T \int_\omega (\ddot{u}_3(t) - \ddot{v}_3(t))(\dot{u}_3(t) - \dot{v}_3(t))\, dx'\, dt \\ - \frac{1}{12}\int_0^T \int_\omega \operatorname{div}\operatorname{div}(\hat{\sigma}(t) - \hat{\tau}(t))(\dot{u}_3(t) - \dot{v}_3(t))\, dx'\, dt = 0. \quad (2.5.1)$$

Since $\dot{u}_3(0) = \dot{v}_3(0)$, we have

$$\int_0^T \int_\omega (\ddot{u}_3(t) - \ddot{v}_3(t))(\dot{u}_3(t) - \dot{v}_3(t))\, dx'\, dt = \frac{1}{2}\|\dot{u}_3(T) - \dot{v}_3(T)\|_{L^2}^2. \quad (2.5.2)$$

On the other hand, by Proposition 2.2.5, (2.4.25), and (2.4.26), we obtain

$$- \frac{1}{12}\int_0^T \int_\omega \operatorname{div}\operatorname{div}(\hat{\sigma}(t) - \hat{\tau}(t))(\dot{u}_3(t) - \dot{v}_3(t))\, dx'\, dt \\ = \int_0^T \int_\Omega (\sigma(t) - \tau(t)) : (\dot{e}(t) - \dot{\eta}(t))\, dx\, dt + \int_0^T \langle \sigma(t) - \tau(t), \dot{p}(t) - \dot{q}(t)\rangle_r\, dt, \quad (2.5.3)$$

where we have also used that $(\dot{u}(t) - \dot{v}(t), \dot{e}(t) - \dot{\eta}(t), \dot{p}(t) - \dot{q}(t)) \in \mathcal{A}_{KL}(0)$ for a.e. $t \geq 0$. Since $e(0) = \eta(0)$, we have

$$\int_0^T \int_\Omega (\sigma(t) - \tau(t)) : (\dot{e}(t) - \dot{\eta}(t))\, dx\, dt = \mathcal{Q}^*(e(T) - \eta(T)). \quad (2.5.4)$$

Moreover, using the flow rule, (2.2.17), and the fact that $\tau(t) \in K^*$ a.e. in Ω, we infer that

$$\langle \sigma(t) - \tau(t), \dot{p}(t) \rangle_r \geq 0$$

for a.e. $t \geq 0$. Similarly,

$$\langle \tau(t) - \sigma(t), \dot{q}(t) \rangle_r \geq 0$$

for a.e. $t \geq 0$. Summing up the previous inequalities and integrating in time yields

$$\int_0^T \langle \sigma(t) - \tau(t), \dot{p}(t) - \dot{q}(t) \rangle_r \, dt \geq 0. \tag{2.5.5}$$

Gathering (2.5.1)–(2.5.5) we deduce that

$$\frac{1}{2} \| \dot{u}_3(T) - \dot{v}_3(T) \|_{L^2}^2 + \mathcal{Q}^*(e(T) - \eta(T)) \leq 0.$$

By (2.2.1) we conclude that $\dot{u}_3 = \dot{v}_3$, hence $u_3 = v_3$, and that $e = \eta$. Finally, by Proposition 2.2.4 we deduce that $\hat{p} = \hat{q}$ and $p_\perp = q_\perp$. □

The following proposition gives a two-dimensional characterisation of the reduced dynamic evolution model for a specific choice of the data.

Proposition 2.5.2. *For every $t \geq 0$ let*

$$w(t, x) = \begin{pmatrix} \bar{w}(t, x') \\ 0 \end{pmatrix} \qquad \textit{for a.e. } x \in \Omega,$$

where $\bar{w} \in H^2_{loc}([0, +\infty); H^1(\omega; \mathbb{R}^2))$. Let $(u_0, e_0, p_0) \in \mathcal{A}_{KL}(w(0))$ be of the form

$$u_0(x) = \begin{pmatrix} \bar{u}_0(x') \\ 0 \end{pmatrix}, \qquad e_0(x) = \bar{e}_0(x') \quad \textit{for a.e. } x \in \Omega, \qquad p_0 = \bar{p}_0 \otimes \mathcal{L}^1.$$

Then a map $t \mapsto (u(t), e(t), p(t))$ is a reduced dynamic evolution, that is, a solution to (i)–(v) *in Theorem 2.4.1, with boundary datum w, force term $\bar{f} = 0$, and initial conditions $(u(0), e(0), p(0)) = (u_0, e_0, p_0)$ and $\dot{u}_3(0) = 0$, if and only if*

$$u(t, x) = \begin{pmatrix} \bar{u}(t, x') \\ 0 \end{pmatrix}, \qquad e(t, x) = \bar{e}(t, x') \quad \textit{for a.e. } x \in \Omega, \qquad p(t) = \bar{p}(t) \otimes \mathcal{L}^1 \tag{2.5.6}$$

for every $t \geq 0$, where

$$t \mapsto (\bar{u}(t), \bar{e}(t), \bar{p}(t)) \in BD(\omega) \times L^2(\omega; \mathbb{M}^{2\times2}_{sym}) \times M_b(\omega \cup \partial_d \omega; \mathbb{M}^{2\times2}_{sym})$$

satisfies the following conditions:

(a) $\operatorname{sym} D\bar{u}(t) = \bar{e}(t) + \bar{p}(t)$ *in ω, $\bar{p}(t) = (\bar{w}(t) - \bar{u}(t)) \odot \nu_{\partial\omega} \mathcal{H}^1$ on $\partial_d \omega$ for every $t \geq 0$;*

(b) $(\overline{u}(0), \overline{e}(0), \overline{p}(0)) = (\overline{u}_0, \overline{e}_0, \overline{p}_0)$*;*

(c) $\bar{\sigma}(t) \in K^*$ *a.e. in ω for every $t \geq 0$;*

(d) *for every $t \geq 0$*

$$\begin{cases} \operatorname{div} \bar{\sigma}(t) = 0 & \textit{in } \omega, \\ [\bar{\sigma}(t) \nu_{\partial\omega}] = 0 & \textit{on } \partial_n \omega; \end{cases}$$

(e) $\mathcal{H}^*(\dot{\bar{p}}(t)) = \langle \bar{\sigma}(t), \dot{\bar{p}}(t)\rangle_r$ *for a.e.* $t \geq 0$.

Proof. Assume that $t \mapsto (u(t), e(t), p(t))$ is a reduced dynamic evolution with the given data. We have to prove that (2.5.6) and (a)–(e) are satisfied. To do this we argue as in Proposition 7.16 in [13]. The theory of convex functions of measure ensures that

$$\mathcal{H}^*(\dot{p}(t)) = \mathcal{H}^*(\dot{p}^a(t)) + \mathcal{H}^*(\dot{p}^s(t)). \tag{2.5.7}$$

By the Fubini-Tonelli Theorem and the Jensen inequality we have

$$\begin{aligned}\mathcal{H}^*(\dot{p}^a(t)) &= \int_{\omega\cup\partial_d\omega}\int_{-\frac{1}{2}}^{\frac{1}{2}} H^*(\dot{\bar{p}}^a(t) + x_3\dot{\hat{p}}^a(t) - \dot{e}_\perp(t))\,dx_3\,dx' \\ &\geq \int_{\omega\cup\partial_d\omega} H^*\Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} (\dot{\bar{p}}^a(t) + x_3\dot{\hat{p}}^a(t) - \dot{e}_\perp(t))\,dx_3\Big)\,dx' \\ &= \mathcal{H}^*(\dot{\bar{p}}^a(t))\end{aligned} \tag{2.5.8}$$

for a.e. $t \geq 0$. Let $\lambda(t) := |\dot{\bar{p}}^s(t)| + |\dot{\hat{p}}^s(t)|$ for a.e. $t \geq 0$. Then the measure $\dot{\bar{p}}^s(t) + x_3\dot{\hat{p}}^s(t)$ is absolutely continuous with respect to $\lambda(t)$ for every $x_3 \in (-\frac{1}{2}, \frac{1}{2})$. Thus, by the Radon-Nikodým Theorem we can write

$$\dot{p}^s(t) = \Big(\frac{d\dot{\bar{p}}^s(t)}{d\lambda(t)} + x_3\frac{d\dot{\hat{p}}^s(t)}{d\lambda(t)}\Big)\lambda(t) \overset{gen.}{\otimes} \mathcal{L}^1,$$

where $\overset{gen.}{\otimes}$ denotes the generalised product of measures (see, e.g., Definition 2.27 in [4]). By the Fubini-Tonelli Theorem and the Jensen inequality, we obtain

$$\begin{aligned}\mathcal{H}^*(\dot{p}^s(t)) &= \int_{\omega\cup\partial_d\omega}\int_{-\frac{1}{2}}^{\frac{1}{2}} H^*\Big(\frac{d\dot{\bar{p}}^s(t)}{d\lambda(t)} + x_3\frac{d\dot{\hat{p}}^s(t)}{d\lambda(t)}\Big)\,dx_3\,d\lambda(t) \\ &\geq \int_{\omega\cup\partial_d\omega} H^*\Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} \Big(\frac{d\dot{\bar{p}}^s(t)}{d\lambda(t)} + x_3\frac{d\dot{\hat{p}}^s(t)}{d\lambda(t)}\Big)\,dx_3\Big)\,d\lambda(t) \\ &= \mathcal{H}^*(\dot{\bar{p}}^s(t))\end{aligned} \tag{2.5.9}$$

for a.e. $t \geq 0$. Combining (2.5.7)–(2.5.9), we conclude that

$$\mathcal{H}^*(\dot{p}(t)) \geq \mathcal{H}^*(\dot{\bar{p}}^a(t)) + \mathcal{H}^*(\dot{\bar{p}}^s(t)) = \mathcal{H}^*(\dot{\bar{p}}(t)) \tag{2.5.10}$$

for a.e. $t \geq 0$.

On the other hand, by (2.2.16), (2.2.17), (2.4.26), (2.4.27), and Proposition 2.2.5, we deduce that

$$\begin{aligned}\mathcal{H}^*(\dot{p}(t)) &= \langle \sigma(t), \dot{p}(t)\rangle_r = \langle \bar{\sigma}(t), \dot{\bar{p}}(t)\rangle + \frac{1}{12}\langle \hat{\sigma}(t), \dot{\hat{p}}(t)\rangle - \int_\Omega \sigma_\perp(t) : \dot{e}_\perp(t)\,dx \\ &\leq \mathcal{H}^*(\dot{\bar{p}}(t)) - \int_\Omega \sigma_\perp(t) : \dot{e}_\perp(t)\,dx - \frac{1}{12}\int_\omega \hat{\sigma}(t) : \dot{\hat{e}}(t)\,dx' - \int_\omega \dot{u}_3(t)\ddot{u}_3(t)\,dx'.\end{aligned} \tag{2.5.11}$$

Here we used that $w_3(t) = 0$ and $\bar{f}(t) = 0$ for every $t \geq 0$.

Therefore, by (2.5.10) we have

$$\int_\Omega \sigma_\perp(t) : \dot{e}_\perp(t)\,dx + \frac{1}{12}\int_\omega \hat{\sigma}(t) : \dot{\hat{e}}(t)\,dx' + \int_\omega \dot{u}_3(t)\ddot{u}_3(t)\,dx' \\ = \frac{d}{dt}\big(\mathcal{Q}^*(e_\perp(t)) + \frac{1}{12}\mathcal{Q}^*(\hat{e}(t)) + \frac{1}{2}\|\dot{u}_3(t)\|^2_{L^2}\big) \leq 0$$

for a.e. $t \geq 0$. Integrating with respect to time, this inequality yields

$$\mathcal{Q}^*(e_\perp(t)) + \frac{1}{12}\mathcal{Q}^*(\hat{e}(t)) + \frac{1}{2}\|\dot{u}_3(t)\|^2_{L^2} \\ \leq \mathcal{Q}^*(e_\perp(0)) + \frac{1}{12}\mathcal{Q}^*(\hat{e}(0)) + \frac{1}{2}\|\dot{u}_3(0)\|^2_{L^2} = 0.$$

Since $u_3(0) = 0$, this implies that $u_3 = 0$ and $\hat{e} = e_\perp = 0$. By Proposition 2.2.4 we deduce that $\hat{p} = p_\perp = 0$. In other words, (2.5.6) is satisfied.

Condition (a) follows immediately from Proposition 2.2.4, (b) is straightforward, and (d) follows from (2.4.25). Since $\sigma(t) \in K^*$ a.e. in Ω, it is easy to check that $\bar{\sigma}(t) \in K^*$ a.e. in ω, that is, (c) holds. Finally, (2.5.11) and (2.5.10) yield (e).

Conversely, if $t \mapsto (u(t), e(t), p(t))$ is of the form (2.5.6) and conditions (a)–(e) are satisfied, it is trivial to check that $t \mapsto (u(t), e(t), p(t))$ is a reduced dynamic evolution. □

Remark 2.5.3. The previous proposition suggests that, in general, one cannot expect uniqueness for the components $\bar{u}$ and $\bar{p}$ of a reduced dynamic evolution. Indeed, Proposition 2.5.2 shows that for some specific choice of the data the reduced dynamic evolution coincides with a two-dimensional quasistatic model, for which uniqueness of displacement and plastic strain in general fails (see, e.g., [50]).

Chapter 3

A quasistatic evolution model for perfectly plastic shallow shells

3.1 Overview of the chapter

In this Chapter we rigorously deduce a quasistatic evolution model for shallow shells by means of Γ-convergence. The starting point of the analysis is the three-dimensional model of Prandlt-Reuss elastoplasticity. We study the asymptotic behaviour of the solutions, as the thickness of the shallow shell tends to 0. As in the case of plates, the limiting model is genuinely three-dimensional. Limiting displacements are of Kirchhoff-Love type, and the stretching and bending components of the stress are coupled in the flow rule and in the stress constraint. Moreover, in contrast with the case of plates, the equilibrium equations are not decoupled, because of the presence of curvature terms. An equivalent formulation of the limiting problem in rate form is also deduced. We discuss the case of external loads.

Let us now briefly outline the content of this Chapter. In Section 3.2 we describe the setting of the problem. In Section 3.3 we prove a Korn Poincaré inequality on a shallow shell. Section 3.4 is devoted to the Γ-convergence of the static functionals, while in Section 3.5 we study the convergence of the quasistatic evolutions. Finally, Section 3.6 we consider the general case where external loads are applied to the shallow shell.

3.2 Setting of the problem

3.2.1 The three-dimensional problem

We start by describing the setting of the three-dimensional problem.

The reference configuration

Let $\omega \subset \mathbb{R}^2$ be a domain (that is, an open, connected, and bounded set) with a C^2 boundary. Let $\partial_d\omega$ and $\partial_n\omega$ be two disjoint open subsets of $\partial\omega$ such that

$$\overline{\partial_d\omega} \cup \overline{\partial_n\omega} = \partial\omega \qquad \text{and} \qquad \overline{\partial_d\omega} \cap \overline{\partial_n\omega} = \{P_1, P_2\},$$

where P_1 and P_2 are two points of $\partial\omega$ (here topological notions refer to the relative topology of $\partial\omega$). The set $\partial_d\omega$ is the Dirichlet boundary of ω and $\partial_n\omega$ is the Neumann boundary. We also consider the set

$$\Omega := \omega \times (-\tfrac{1}{2}, \tfrac{1}{2})$$

and its Dirichlet boundary

$$\partial_d\Omega := \partial_d\omega \times (-\tfrac{1}{2}, \tfrac{1}{2}).$$

Let $\theta \in C^3(\overline{\omega})$. For every $0 < h \ll 1$ we consider the two-dimensional surface

$$S_h := \big\{(x', h\theta(x')) : \ x' \in \omega\big\}.$$

A *shallow shell* of thickness h is a three-dimensional body whose reference configuration is given by the set

$$\Sigma_h := \Psi_h(\Omega),$$

where $\Psi_h : \overline{\Omega} \to \mathbb{R}^3$ is the function

$$\Psi_h(x) := (x', h\theta(x')) + hx_3\nu_{S_h}(x') \quad \text{for every } x = (x', x_3) \in \overline{\Omega} \tag{3.2.1}$$

and ν_{S_h} is the outer unit normal to S_h given by

$$\nu_{S_h}(x') = \frac{1}{\sqrt{1 + h^2|\nabla\theta(x')|^2}}(-h\partial_\alpha\theta(x')e_\alpha + e_3) \quad \text{for every } x' \in \omega.$$

Here $\{e_i\}$ denotes the canonical basis of $\mathbb{R}^3$. The Dirichlet boundary of Σ_h is given by the set

$$\partial_d\Sigma_h := \Psi_h\left(\partial_d\Omega\right).$$

For every $0 < h \ll 1$ we introduce the diagonal matrix

$$R_h := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{h} \end{pmatrix} \tag{3.2.2}$$

and we define

$$F_h(x) := D\Psi_h(x)R_h \tag{3.2.3}$$

for every $x \in \overline{\Omega}$. The elementary properties of the determinant give

$$\det D\Psi_h(x) = h \det F_h(x) \tag{3.2.4}$$

for every $x \in \overline{\Omega}$. The asymptotic behaviour of F_h, as $h \to 0$, is made explicit by the following result.

Lemma 3.2.1. *As $h \to 0$, the following expansions hold:*

$$(F_h)_{\alpha\beta} = \delta_{\alpha\beta} - h^2 x_3 \partial^2_{\alpha\beta}\theta + O(h^3), \qquad (F_h)_{\alpha 3} = -h\partial_\alpha\theta + O(h^3),$$
$$(F_h)_{3\beta} = h\partial_\beta\theta + O(h^3), \qquad (F_h)_{33} = 1 - \tfrac{1}{2}h^2|\nabla\theta|^2 + O(h^3),$$

where $O(h^3)$ denotes a quantity that is uniformly bounded by h^3 in $\overline{\Omega}$. Moreover, F_h is invertible for h small enough and the following expansions hold:

$$(F_h^{-1})_{\alpha\beta} = \delta_{\alpha\beta} + h^2(x_3\partial^2_{\alpha\beta}\theta - \partial_\alpha\theta\partial_\beta\theta) + O(h^3), \qquad (F_h^{-1})_{\alpha 3} = h\partial_\alpha\theta + O(h^3),$$
$$(F_h^{-1})_{3\beta} = -h\partial_\beta\theta + O(h^3), \qquad (F_h^{-1})_{33} = 1 - \tfrac{1}{2}h^2|\nabla\theta|^2 + O(h^3),$$

and

$$\det F_h = 1 + O(h^2).$$

Proof. See, e.g., [10, Theorem 3.3-1]. □

The stored elastic energy

Let $\mathbb{C}$ be the three-dimensional elasticity tensor, considered as a symmetric positive definite linear operator $\mathbb{C} : \mathbb{M}^{3\times3}_{sym} \to \mathbb{M}^{3\times3}_{sym}$, and let $Q : \mathbb{M}^{3\times3}_{sym} \to [0, +\infty)$ be the quadratic form associated with $\mathbb{C}$, defined by

$$Q(\xi) := \tfrac{1}{2}\mathbb{C}\xi : \xi \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{sym}.$$

It turns out that there exists two positive constants $\alpha_{\mathbb{C}}$ and $\beta_{\mathbb{C}}$, with $\alpha_{\mathbb{C}} \leq \beta_{\mathbb{C}}$, such that

$$\alpha_{\mathbb{C}}|\xi|^2 \leq Q(\xi) \leq \beta_{\mathbb{C}}|\xi|^2 \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{sym}. \tag{3.2.5}$$

These inequalities imply that

$$|\mathbb{C}\xi| \leq 2\beta_{\mathbb{C}}|\xi| \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{sym}. \tag{3.2.6}$$

The integral

$$\int_{\Sigma_h} Q(\eta(x))\, dx$$

describes the *stored elastic energy* of a configuration of the shallow shell Σ_h with elastic strain $\eta \in L^2(\Sigma_h; \mathbb{M}^{3\times3}_{sym})$.

The plastic dissipation

Let K be a convex and compact set in $\mathbb{M}^{3\times3}_D$, whose boundary ∂K is interpreted as the yield surface. We assume that there exist two positive constants r_K and R_K, with $r_K \leq R_K$, such that

$$B(0, r_K) \subset K \subset B(0, R_K), \tag{3.2.7}$$

where $B(0, r) := \{\xi \in \mathbb{M}^{3\times3}_D : |\xi| \leq r\}$. Let $H : \mathbb{M}^{3\times3}_D \to \mathbb{R}$ be the support function of K, that is,

$$H(\xi) := \sup_{\tau\in K} \xi : \tau \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_D.$$

It is easy to see that H is convex, positively 1-homogeneous, and satisfies the triangle inequality. Moreover, by (3.2.7) one deduces that

$$r_K|\xi| \leq H(\xi) \leq R_K|\xi| \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_D. \tag{3.2.8}$$

From standard convex analysis we also have that the set K coincides with the subdifferential $\partial H(0)$ of H at 0.

Let $q \in M_b(\Sigma_h \cup \partial_d\Sigma_h; \mathbb{M}^{3\times3}_D)$ and let $dq/d|q|$ be the Radon-Nikodým derivative of q with respect to its variation $|q|$. The integral

$$\int_{\Sigma_h\cup\partial_d\Sigma_h} H\Big(\frac{dq}{d|q|}\Big)\, d|q|$$

describes the *plastic dissipation potential* on a configuration of the shallow shell Σ_h with plastic strain q.

Kinematic admissibility and energy

Given a boundary datum $z \in H^1(\Sigma_h;\mathbb{R}^3)$, we define the class $\mathcal{A}(\Sigma_h, z)$ of admissible displacements and strains, as the set of all triplets $(v,\eta,q) \in BD(\Sigma_h) \times L^2(\Sigma_h;\mathbb{M}^{3\times3}_{sym}) \times M_b(\Sigma_h \cup \partial_d\Sigma_h;\mathbb{M}^{3\times3}_D)$ such that

$$\operatorname{sym} Dv = \eta + q \quad \text{in } \Sigma_h, \qquad q = (z-v)\odot \nu_{\partial\Sigma_h}\mathcal{H}^2 \quad \text{on } \partial_d\Sigma_h, \tag{3.2.9}$$

where $\nu_{\partial\Sigma_h}$ is the outer unit normal to $\partial\Sigma_h$. We define the total energy as

$$\mathcal{E}_h(v,\eta,q) := \int_{\Sigma_h} Q(\eta(x))\,dx + \int_{\Sigma_h\cup\partial_d\Sigma_h} H\Big(\frac{dq}{d|q|}\Big)\,d|q| \tag{3.2.10}$$

for every admissible triplet $(v,\eta,q)\in\mathcal{A}(\Sigma_h,w)$.

3.2.2 The rescaled problem

In this section we introduce a suitable scaling of the admissible triplets and of the total energy.

Let $z \in H^1(\Sigma_h;\mathbb{R}^3)$. To any triplet $(v,\eta,q)\in\mathcal{A}(\Sigma_h,z)$ we associate a triplet (u,e,p) defined as follows:

$$u := R_h^{-1} v\circ\Psi_h, \qquad e := \eta\circ\Psi_h, \qquad p := \frac{1}{\det D\Psi_h}\Psi_h^{\#}(q), \tag{3.2.11}$$

where Ψ_h and R_h are defined in (3.2.1) and (3.2.2), and $\Psi_h^{\#}(q)$ is the pull-back measure of q, that is,

$$\int_{\Omega\cup\partial_d\Omega} \varphi : d\Psi_h^{\#}(q) = \int_{\Sigma_h\cup\partial_d\Sigma_h} \varphi\circ\Psi_h^{-1} : dq$$

for every $\varphi \in C_0(\Omega\cup\partial_d\Omega;\mathbb{M}^{3\times3}_D)$. It is clear that $u\in L^1(\Omega;\mathbb{R}^3)$, $e\in L^2(\Omega;\mathbb{M}^{3\times3}_{sym})$, and $p\in M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{3\times3}_D)$. Moreover, we have that

$$\operatorname{sym}(R_h D u R_h F_h^{-1}) \in M_b(\Omega;\mathbb{M}^{3\times3}_{sym}) \tag{3.2.12}$$

and

$$\int_\Omega \varphi : d\operatorname{sym}(R_h D u R_h F_h^{-1}) = \int_{\Sigma_h} (\det D\Psi_h^{-1})\,\varphi\circ\Psi_h^{-1} : d(\operatorname{sym} Dv) \tag{3.2.13}$$

for every $\varphi\in C_0(\Omega;\mathbb{M}^{3\times3}_{sym})$. Indeed, if v is smooth, then by direct computations and by (3.2.3) we obtain

$$(\operatorname{sym} Dv)\circ\Psi_h = \operatorname{sym}(R_h D u R_h F_h^{-1}), \tag{3.2.14}$$

so that (3.2.12) and (3.2.13) follow by an approximation argument.

We also introduce the rescaled boundary datum $w\in H^1(\Omega;\mathbb{R}^3)$, defined as

$$w := R_h^{-1} z\circ\Psi_h \tag{3.2.15}$$

and we note that

$$\begin{aligned}\int_{\partial_d\Sigma_h} \varphi\circ\Psi_h^{-1} : dq &= \int_{\partial_d\Sigma_h} \varphi\circ\Psi_h^{-1} : ((z-v)\odot\nu_{\partial\Sigma_h})\,d\mathcal{H}^2 \\ &= h\int_{\partial_d\Omega} \varphi : (R_h(w-u)\odot(\operatorname{cof} F_h)R_h\nu_{\partial\Omega})\,d\mathcal{H}^2 \end{aligned} \tag{3.2.16}$$

for every $\varphi \in C(\overline{\Omega}; \mathbb{M}^{3\times 3}_{sym})$, where $\nu_{\partial\Omega}$ is the outer unit normal to $\partial\Omega$.

Since $(v, \eta, q) \in \mathcal{A}(\Sigma_h, w)$, we deduce by (3.2.9), (3.2.11), (3.2.13), and (3.2.16), that

$$\begin{aligned} \operatorname{sym}(R_h D u R_h F_h^{-1}) &= e + p \quad \text{ in } \Omega, \\ p &= \frac{1}{\det F_h} R_h (w - u) \odot (\operatorname{cof} F_h) R_h \nu_{\partial\Omega} \mathcal{H}^2 \quad \text{ on } \partial_d\Omega. \end{aligned} \tag{3.2.17}$$

Motivated by the results above, we introduce the space

$$V_h(\Omega) := \left\{ u \in L^1(\Omega; \mathbb{R}^3) : \ \operatorname{sym}(R_h D u R_h F_h^{-1}) \in M_b(\Omega; \mathbb{M}^{3\times 3}_{sym}) \right\}.$$

For every $w \in H^1(\Omega; \mathbb{R}^3)$ we denote by $\mathcal{A}_h(\Omega, w)$ the class of all triplets

$$(u, e, p) \in V_h(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}) \times M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times 3}_D)$$

satisfying (3.2.17). According to the scaling (3.2.11) and to (3.2.4), the total energy can be written as

$$\mathcal{E}_h(v, \eta, q) = h \int_\Omega Q(e(x)) \det F_h(x)\, dx + h\, \mathcal{H}_h(p),$$

where

$$\mathcal{H}_h(p) := \int_{\Omega \cup \partial_d\Omega} H\left(\frac{dp}{d|p|}\right) \det F_h \, d|p|.$$

We thus define the scaled energy as

$$\mathcal{I}_h(u, e, p) := \int_\Omega Q(e(x)) \det F_h(x)\, dx + \mathcal{H}_h(p) \tag{3.2.18}$$

for every $(u, e, p) \in \mathcal{A}_h(\Omega, w)$. This will be the starting point of the asymptotic analysis of Sections 3.4 and 3.5.

3.2.3 The limiting problem

In this section we introduce the limiting functional, that describes the asymptotic behaviour of the rescaled energy $\mathcal{I}_h$, as h tends to 0.

The reduced stored elastic energy

Let $\mathbb{M} : \mathbb{M}^{2\times 2}_{sym} \to \mathbb{M}^{3\times 3}_{sym}$ be the operator given by

$$\mathbb{M}\xi := \begin{pmatrix} \xi_{11} & \xi_{12} & \lambda_1(\xi) \\ \xi_{12} & \xi_{22} & \lambda_2(\xi) \\ \lambda_1(\xi) & \lambda_2(\xi) & \lambda_3(\xi) \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym}, \tag{3.2.19}$$

where the triplet $(\lambda_1(\xi), \lambda_2(\xi), \lambda_3(\xi))$ is the unique solution of the minimum problem

$$\min_{\lambda_i \in \mathbb{R}} Q \begin{pmatrix} \xi_{11} & \xi_{12} & \lambda_1 \\ \xi_{12} & \xi_{22} & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}.$$

We observe that $(\lambda_1(\xi), \lambda_2(\xi), \lambda_3(\xi))$ can be characterised as the unique solution of the linear system

$$\mathbb{C}\mathbb{M}\xi : \begin{pmatrix} 0 & 0 & \zeta_1 \\ 0 & 0 & \zeta_2 \\ \zeta_1 & \zeta_2 & \zeta_3 \end{pmatrix} = 0 \tag{3.2.20}$$

for every $\zeta_i \in \mathbb{R}$. This implies that $\mathbb{M}$ is a linear map and

$$(\mathbb{C}\mathbb{M}\xi)_{i3} = (\mathbb{C}\mathbb{M}\xi)_{3i} = 0. \tag{3.2.21}$$

Let $Q^* : \mathbb{M}^{2\times 2}_{sym} \to \mathbb{R}$ be the quadratic form given by

$$Q^*(\xi) := Q(\mathbb{M}\xi) \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym}. \tag{3.2.22}$$

It follows from (3.2.5) that

$$\alpha_{\mathbb{C}}|\xi|^2 \leq Q^*(\xi) \leq \beta_{\mathbb{C}}|\xi|^2 \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym}.$$

We define the reduced elasticity tensor as the linear operator $\mathbb{C}^* : \mathbb{M}^{2\times 2}_{sym} \to \mathbb{M}^{3\times 3}_{sym}$ given by

$$\mathbb{C}^*\xi := \mathbb{C}\mathbb{M}\xi \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym}. \tag{3.2.23}$$

Note that we can always identify $\mathbb{C}^*\xi$ with an element of $\mathbb{M}^{2\times 2}_{sym}$ in view of (3.2.21). Moreover, by (3.2.20) we have

$$\mathbb{C}^*\xi : \zeta = \mathbb{C}^*\xi : \begin{pmatrix} \zeta_{11} & \zeta_{12} & 0 \\ \zeta_{12} & \zeta_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym},\ \zeta \in \mathbb{M}^{3\times 3}_{sym}. \tag{3.2.24}$$

This implies that

$$Q^*(\xi) = \frac{1}{2}\mathbb{C}^*\xi : \begin{pmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{12} & \xi_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym}.$$

Finally, we introduce the functional $\mathcal{Q}^* : L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \to [0, +\infty)$, defined as

$$\mathcal{Q}^*(e) := \int_\Omega Q^*(e(x))\, dx$$

for every $e \in L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$. It describes the reduced elastic energy of a configuration, whose elastic strain is e.

The reduced plastic dissipation

In the reduced problem the plastic dissipation potential is given by the function $H^* : \mathbb{M}^{2\times 2}_{sym} \to [0, +\infty)$, defined as

$$H^*(\xi) := \min_{\lambda_i \in \mathbb{R}} H \begin{pmatrix} \xi_{11} & \xi_{12} & \lambda_1 \\ \xi_{12} & \xi_{22} & \lambda_2 \\ \lambda_1 & \lambda_2 & -(\xi_{11} + \xi_{22}) \end{pmatrix} \tag{3.2.25}$$

for every $\xi \in \mathbb{M}^{2\times 2}_{sym}$. From the properties of H it follows that H^* is convex, positively 1-homogeneous, and satisfies

$$r_K|\xi| \leq H^*(\xi) \leq \sqrt{3}R_K|\xi| \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym}.$$

The set $K^* := \partial H^*(0)$ represents the set of admissible stresses in the reduced problem and can be characterised as follows:

$$\xi \in K^* \quad \Leftrightarrow \quad \begin{pmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{12} & \xi_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{3}(\operatorname{tr}\xi)I_{3\times 3} \in K, \tag{3.2.26}$$

(see [13, Section 3.2]). For every $p \in M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{2\times 2}_{sym})$ we define the functional

$$\mathcal{H}^*(p) := \int_{\Omega \cup \partial_d\Omega} H^*\Big(\frac{dp}{d|p|}\Big)\, d|p|.$$

Generalised Kirchhoff-Love triplets and limiting energy

We consider the set $KL(\Omega)$ of Kirchhoff-Love displacements, defined as

$$KL(\Omega) := \{u \in BD(\Omega) : \ (\operatorname{sym} Du)_{i3} = 0\}.$$

We note that $u \in KL(\Omega)$ if and only if $u_3 \in BH(\omega)$ and there exists $\bar{u} \in BD(\omega)$ such that

$$u_\alpha(x) = \bar{u}_\alpha(x') - x_3 \partial_\alpha u_3(x')$$

for every $x = (x', x_3) \in \Omega$. We call $\bar{u}$, u_3 the *Kirchhoff-Love components* of u.

For every $u \in KL(\Omega)$ we define the measure

$$\bar{E}u := \operatorname{sym} Du + \nabla\theta \odot \nabla u_3.$$

Given a prescribed displacement $w \in H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega)$, the set $\mathcal{A}_{GKL}(w)$ of *generalised Kirchhoff-Love triplets* is defined as the class of all triplets

$$(u, e, p) \in KL(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times3}_{sym}) \times M_b(\Omega \cup \partial_d\Omega;\mathbb{M}^{3\times3}_{sym})$$

such that

$$\bar{E}u = e + p \quad \text{in } \Omega, \qquad p = (w-u) \odot \nu_{\partial\Omega}\mathcal{H}^2 \quad \text{on } \partial_d\Omega,$$
$$e_{i3} = 0 \quad \text{in } \Omega, \qquad p_{i3} = 0 \quad \text{in } \Omega \cup \partial_d\Omega.$$

The linear space $\{\xi \in \mathbb{M}^{3\times3}_{sym} : \ \xi_{i3} = 0, \ i = 1,2,3\}$ is isomorphic to $\mathbb{M}^{2\times2}_{sym}$. Thus, in the following, given $(u, e, p) \in \mathcal{A}_{GKL}(w)$, we will always identify e with a function in $L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$, $\bar{E}u$ with a measure in $M_b(\Omega;\mathbb{M}^{2\times2}_{sym})$, and p with a measure in $M_b(\Omega \cup \partial_d\Omega;\mathbb{M}^{2\times2}_{sym})$. We observe that the class $\mathcal{A}_{GKL}(w)$ is nonempty as it contains $(w, E^*w, 0)$.

Finally, the limiting energy will be given by the functional $\mathcal{I} : \mathcal{A}_{GKL}(w) \to [0, +\infty)$, defined as

$$\mathcal{I}(u, e, p) := \mathcal{Q}^*(e) + \mathcal{H}^*(p) \tag{3.2.27}$$

for every $(u, e, p) \in \mathcal{A}_{GKL}(w)$.

We conclude this section by collecting some properties of the class $\mathcal{A}_{GKL}(w)$. The following closure property holds.

Lemma 3.2.2. *Let (w_n) be a sequence in $H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ and let (u_n, e_n, p_n) be a sequence of triplets such that $(u_n, e_n, p_n) \in \mathcal{A}_{GKL}(w_n)$ for every n. Assume that $u_n \rightharpoonup u$ weakly* in $BD(\Omega)$, $e_n \rightharpoonup e$ weakly in $L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$, $p_n \rightharpoonup p$ weakly* in $M_b(\Omega \cup \partial_d\Omega;\mathbb{M}^{2\times2}_{sym})$, and $w_n \rightharpoonup w$ weakly in $H^1(\Omega;\mathbb{R}^3)$. Then $(u, e, p) \in \mathcal{A}_{GKL}(w)$.*

Proof. The result easily follows by adapting the proof of [12, Lemma 2.1]. □

A characterisation of triplets in $\mathcal{A}_{GKL}(w)$ can be given in terms of moments, whose definition is recalled below.

Definition 3.2.3. Let $f \in L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$. We denote by $\bar{f}$, $\hat{f} \in L^2(\omega;\mathbb{M}^{2\times2}_{sym})$ and by $f_\perp \in L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$ the following orthogonal components (in the sense of $L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$) of f:

$$\bar{f}(x') := \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x', x_3)\, dx_3, \qquad \hat{f}(x') := 12 \int_{-\frac{1}{2}}^{\frac{1}{2}} x_3 f(x', x_3)\, dx_3$$

for a.e. $x' \in \omega$, and

$$f_\perp(x) := f(x) - \bar{f}(x') - x_3 \hat{f}(x')$$

for a.e. $x \in \Omega$. We call $\bar{f}$ the *zeroth order moment* of f and $\hat{f}$ the *first order moment* of f.

Definition 3.2.4. Let $q \in M_b(\Omega \cup \partial_d \Omega; \mathbb{M}^{2\times 2}_{sym})$. We denote by $\bar{q}$, $\hat{q} \in M_b(\omega \cup \partial_d \omega; \mathbb{M}^{2\times 2}_{sym})$ and by $q_\perp \in M_b(\Omega \cup \partial_d \Omega; \mathbb{M}^{2\times 2}_{sym})$ the following measures:

$$\int_{\omega \cup \partial_d \omega} \varphi : d\bar{q} := \int_{\Omega \cup \partial_d \Omega} \varphi : dq, \qquad \int_{\omega \cup \partial_d \omega} \varphi : d\hat{q} := 12 \int_{\Omega \cup \partial_d \Omega} x_3 \varphi : dq$$

for every $\varphi \in C_0(\omega \cup \partial_d \omega; \mathbb{M}^{2\times 2}_{sym})$, and

$$q_\perp := q - \bar{q} \otimes \mathcal{L}^1 - \hat{q} \otimes x_3 \mathcal{L}^1,$$

where $\otimes$ denotes the usual product of measures. We call $\bar{q}$ the *zeroth order moment* of q and $\hat{q}$ the *first order moment* of q.

With these definitions at hand one can prove the following result.

Proposition 3.2.5. *Let $w \in H^1(\Omega; \mathbb{R}^3) \cap KL(\Omega)$ and let $(u, e, p) \in KL(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times M_b(\Omega \cup \partial_d \Omega; \mathbb{M}^{2\times 2}_{sym})$. Then $(u, e, p) \in \mathcal{A}_{GKL}(w)$ if and only if the following three conditions are satisfied:*

(i) $\operatorname{sym} D\bar{u} + \nabla\theta \odot \nabla u_3 = \bar{e} + \bar{p}$ *in* ω *and* $\bar{p} = (\bar{w} - \bar{u}) \odot \nu_{\partial\omega} \mathcal{H}^1$ *on* $\partial_d \omega$*;*

(ii) $D^2 u_3 = -(\hat{e} + \hat{p})$ *in* ω*,* $u_3 = w_3$ *on* $\partial_d \omega$*, and* $\hat{p} = (\nabla u_3 - \nabla w_3) \odot \nu_{\partial\omega} \mathcal{H}^1$ *on* $\partial_d \omega$*;*

(iii) $p_\perp = -e_\perp$ *in* Ω *and* $p_\perp = 0$ *on* $\partial_d \Omega$*,*

where $\nu_{\partial\omega}$ is the outer unit normal to $\partial\omega$.

Proof. The proof is analogous to that of [13, Proposition 3.4]. □

Finally, we prove an approximation result in terms of smooth triplets. First of all, we give a definition.

Definition 3.2.6. The space $L^2_{\infty,c}(\Omega; \mathbb{M}^{2\times 2}_{sym})$ is the set of all $p \in L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$ satisfying:

(i) $\partial^i_\alpha \partial^j_\beta p \in L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$ for every $i, j \in \mathbb{N} \cup \{0\}$,

(ii) there exists a set $U \subset\subset \omega \cup \partial_n \omega$ such that $p = 0$ a.e. on $\omega \setminus \overline{U} \times \left(-\frac{1}{2}, \frac{1}{2}\right)$.

We note that functions in $L^2_{\infty,c}(\Omega; \mathbb{M}^{2\times 2}_{sym})$ have a smooth dependence on the variable x'; namely, if $p \in L^2_{\infty,c}(\Omega; \mathbb{M}^{2\times 2}_{sym})$, then $p(\cdot, x_3) \in C^\infty_c(\omega \cup \partial_n \omega; \mathbb{M}^{2\times 2}_{sym})$ for a.e. $x_3 \in \left(-\frac{1}{2}, \frac{1}{2}\right)$.

Lemma 3.2.7. *Let $w \in H^1(\Omega; \mathbb{R}^3) \cap KL(\Omega)$ and let $(u, e, p) \in \mathcal{A}_{GKL}(w)$. Then there exists a sequence of triplets*

$$(u^k, e^k, p^k) \in \left(H^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times L^2_{\infty,c}(\Omega; \mathbb{M}^{2\times 2}_{sym})\right) \cap \mathcal{A}_{GKL}(w)$$

such that

$$u^k \rightharpoonup u \quad \text{weakly* in } BD(\Omega), \tag{3.2.28}$$

$$e^k \to e \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}), \tag{3.2.29}$$

$$p^k \rightharpoonup p \quad \text{weakly* in } M_b(\Omega \cup \partial_d \Omega; \mathbb{M}^{2\times 2}_{sym}), \tag{3.2.30}$$

$$\|p^k\|_{M_b} \to \|p\|_{M_b}, \tag{3.2.31}$$

as $k \to +\infty$.

Proof. The proof is analogous to [13, Lemma 4.5] and [13, Theorem 4.7]. The only difference is in the definition of the zeroth order moment of e^k, that we detail below. Following the same notation of [13], we replace $\bar{e}^k$ on page 629 with

$$\bar{e}^k := \sum_{j=1}^{\infty} \big((\varphi_j \bar{e}) * \rho_{\delta_j} + (\nabla\varphi_j \odot \bar{u}) * \rho_{\delta_j} - (\varphi_j \nabla\theta \odot \nabla u_3) * \rho_{\delta_j}\big) \\ + \nabla\theta \odot \sum_{j=1}^{\infty} \big((\varphi_j \nabla u_3 + \nabla\varphi_j u_3) * \rho_{\delta_j}\big),$$

and $\bar{e}^{\delta,1}$ on page 632 with

$$\bar{e}^{\delta,1} = (\bar{u} \circ \phi_\delta) \circ \nabla\varphi_1 + \varphi_1 \operatorname{sym}((\bar{e} \circ \phi_\delta) D\phi_\delta) - \varphi_1 \operatorname{sym}\big(((\nabla u_3 \odot \nabla\theta) \circ \phi_\delta) D\phi_\delta\big) \\ + (u_3 \circ \phi_\delta) \nabla\theta \odot \nabla\varphi_1 + \varphi_1 \nabla\theta \odot (D\phi_\delta)^T (\nabla u_3 \circ \phi_\delta).$$

Using this definition, equation (4.38) in [13] is replaced by

$$\bar{e}^{\delta,1} \to \bar{u} \odot \nabla\varphi_1 + \varphi_1 \bar{e} + u_3 \nabla\varphi_1 \odot \nabla\theta \quad \text{strongly in } L^2(\omega; \mathbb{M}^{2\times2}_{sym}).$$

On page 633 of [13] we replace e^δ with

$$e^\delta := e - (\varphi_1 + \varphi_2)(\bar{e} + x_3 \hat{e}) + \bar{e}^{\delta,1} + \bar{e}^{\delta,2} + x_3(\hat{e}^{\delta,1} + \hat{e}^{\delta,2}) \\ + \sum_{\alpha=1}^{2} (-\bar{u} \odot \nabla\varphi_\alpha - u_3 \nabla\theta \odot \nabla\varphi_\alpha + x_3 u_3 D^2\varphi_\alpha + 2x_3 \nabla\varphi_\alpha \odot \nabla u_3).$$

and formula (4.55) on page 634 with

$$\bar{e}^k := \sum_{i=1}^{m} (\varphi_i \bar{e}) \circ \tau_{i,k} + \varphi_0 \bar{e} + \sum_{i=1}^{m} (\nabla\varphi_i \odot \bar{u}) \circ \tau_{i,k} + \nabla\varphi_0 \odot \bar{u} - \sum_{i=1}^{m} (\varphi_i \nabla\theta \odot \nabla u_3) \circ \tau_{i,k} \\ + \nabla\theta \odot \sum_{i=1}^{m} \big((u_3 \nabla\varphi_i) \circ \tau_{i,k} + (\nabla\varphi_i u_3) \circ \tau_{i,k}\big) + u_3 \nabla\theta \odot \nabla\varphi_0$$

By implementing these changes the construction of [13, Lemma 4.5] and [13, Theorem 4.7] provides the desired approximating sequence. □

3.3 A Korn-Poincaré inequality on a shallow shell

In this section we prove an *ad hoc* version of the Korn-Poincaré inequality for shallow shells. To this purpose it is useful to express displacements in intrinsic curvilinear coordinates. More precisely, to any displacement $u : \Omega \to \mathbb{R}^3$ we associate the vectorfield $u(h) : \Omega \to \mathbb{R}^3$ defined by

$$u(h) := (D\Psi_h)^T R_h u, \tag{3.3.1}$$

whose components are the scaled curvilinear coordinates of u with respect to the contravariant basis of Σ_h. In particular, from (3.2.3) and (3.3.1) it follows immediately that

$$R_h u(h) = F_h^T R_h u. \tag{3.3.2}$$

In the following proposition we express the strain in terms of the curvilinear coordinates.

Proposition 3.3.1. *Let $0 < h \ll 1$. Let $u \in V_h(\Omega)$ and let $u(h)$ be defined by (3.3.1). Then $u(h) \in BD(\Omega)$ and the following equality holds:*

$$F_h^T \operatorname{sym}(R_h D u R_h F_h^{-1}) F_h = E(h, u(h)), \tag{3.3.3}$$

where

$$E(h, u(h))_{ij} := (R_h(\operatorname{sym} Du(h))R_h)_{ij} - \Gamma_{ij}^k(h) u_k(h) \tag{3.3.4}$$

and the quantities $\Gamma_{ij}^k(h)$ are given by

$$\begin{aligned} \Gamma_{\alpha i}^\sigma(h) = \Gamma_{i\alpha}^\sigma(h) &:= (\partial_\alpha(F_h^T)F_h^{-T})_{i\sigma}, \qquad & \Gamma_{\alpha i}^3(h) = \Gamma_{i\alpha}^3(h) &:= \frac{1}{h}(\partial_\alpha(F_h^T)F_h^{-T})_{i3}, \\ \Gamma_{33}^\alpha(h) &:= \frac{1}{h}(\partial_3(F_h^T)F_h^{-T})_{3\alpha}, \qquad & \Gamma_{33}^3(h) &:= \frac{1}{h^2}(\partial_3(F_h^T)F_h^{-T})_{33}. \end{aligned} \tag{3.3.5}$$

Proof. Assume u smooth. Differentiating (3.3.2) yields

$$(R_h D u)_{ij} = (F_h^{-T} R_h D u(h))_{ij} + \partial_j (F_h^{-T})_{ik} (R_h)_{kl} u(h)_l.$$

This implies that

$$\begin{aligned} &\operatorname{sym}(R_h D u R_h F_h^{-1})_{ij} = \operatorname{sym}(F_h^{-T} R_h D u(h) R_h F_h^{-1})_{ij} \\ &+ \frac{1}{2}\Big(\partial_m (F_h^{-T})_{ik} (R_h)_{kl} u(h)_l (R_h)_{mn} (F_h^{-1})_{nj} + \partial_p (F_h^{-T})_{jk} (R_h)_{kr} u(h)_r (R_h)_{pq} (F_h^{-1})_{qi}\Big). \end{aligned}$$

Using the equality

$$F_h^T \partial_m (F_h^{-T}) = -\partial_m (F_h^T) F_h^{-T},$$

direct computations lead to

$$\begin{aligned} \big(F_h^T \operatorname{sym}(R_h D u R_h F_h^{-1}) F_h\big)_{ij} &= \operatorname{sym}(R_h D u(h) R_h)_{ij} \\ &+ \frac{1}{2}\Big(\big(\partial_l(F_h^T)F_h^{-T} R_h\big)_{ik} (R_h)_{lj} + \big(\partial_m(F_h^T)F_h^{-T} R_h\big)_{jk} (R_h)_{mi}\Big) u_k(h). \end{aligned}$$

To deduce (3.3.3) it remains to show that, if we set

$$2\Gamma_{ij}^k(h) := \big(\partial_l(F_h^T)F_h^{-T} R_h\big)_{ik} (R_h)_{lj} + \big(\partial_m(F_h^T)F_h^{-T} R_h\big)_{jk} (R_h)_{mi},$$

then $\Gamma_{ij}^k(h)$ satisfies (3.3.5). By (3.2.2) and (3.2.3) we have that

$$\partial_\alpha(F e_\beta) = \partial_\beta(F e_\alpha), \qquad \partial_\alpha(F e_3) = \frac{1}{h}\partial_3(F e_\alpha).$$

Using these equalities and again (3.2.2), we obtain

$$2\Gamma_{\alpha\beta}^\sigma(h) = \big(\partial_\beta(F_h^T)F_h^{-T}\big)_{\alpha\sigma} + \big(\partial_\alpha(F_h^T)F_h^{-T}\big)_{\beta\sigma} = 2\big(\partial_\beta(F_h^T)F_h^{-T}\big)_{\alpha\sigma}$$

and

$$2\Gamma_{\alpha 3}^\sigma(h) = \frac{1}{h}\big(\partial_3(F_h^T)F_h^{-T}\big)_{\alpha\sigma} + \big(\partial_\alpha(F_h^T)F_h^{-T}\big)_{3\sigma} = 2\big(\partial_\alpha(F_h^T)F_h^{-T}\big)_{3\sigma}.$$

The other equalities in (3.3.5) can be proved similarly.

The general case follows by an approximation argument. □

Remark 3.3.2. Note that (3.3.4) coincides, up to a scaling, with the quantity considered in [11, Theorem 1.3.1]. Moreover, the coefficients $\Gamma^k_{ij}(h)$ are the suitably scaled Christoffel symbols of Σ_h. In particular, for $h = 1$ (that is, when R_h is replaced by the identity matrix and thus, F_h is equal to $D\Psi_h$) they exactly coincide with the Christoffel symbols of Σ_h. Indeed, following the notation of [11, Section 1.2], let $g_i := F_h e_i = \partial_i \Psi_h$ (where e_i is the canonical basis of $\mathbb{R}^3$), and let $g^j := F_h^{-T} e_j$, so that $g_i \cdot g^j = \delta_{ij}$. Then

$$\Gamma^k_{ij}(h) = g^k \cdot \partial_j g_i,$$

which is the usual definition of the Christoffel symbols in differential geometry.

In the following lemma we study the dependence of $\Gamma^k_{ij}(h)$ on the thickness parameter h.

Lemma 3.3.3. *The following expansions hold:*

$$\Gamma^\sigma_{\alpha\beta}(h) = h^2 \partial^2_{\alpha\beta}\theta \, \partial_\sigma \theta - h^2 x_3 \partial^3_{\alpha\beta\sigma}\theta + O(h^3), \tag{3.3.6}$$

$$\Gamma^3_{\alpha\beta}(h) = \partial^2_{\alpha\beta}\theta + O(h^2), \tag{3.3.7}$$

$$\Gamma^\sigma_{\alpha 3}(h) = -h\partial^2_{\alpha\sigma}\theta + O(h^2), \tag{3.3.8}$$

$$\Gamma^i_{33}(h) = \Gamma^3_{\alpha 3}(h) = 0, \tag{3.3.9}$$

where $O(h^p)$ denotes a quantity uniformly bounded by h^p, as $h \to 0$.

Proof. Let $g^h_i := F_h e_i$ and $g^{h,i} := F_h^{-T} e_i$. These definitions, (3.2.2), and (3.3.5) lead to

$$\begin{aligned}
\Gamma^\sigma_{\alpha i}(h) &= g^{h,\sigma} \cdot \partial_\alpha g^h_i, \\
\Gamma^3_{\alpha i}(h) &= \frac{1}{h} g^{h,3} \cdot \partial_\alpha g^h_i, \\
\Gamma^\alpha_{33}(h) &= \frac{1}{h} g^{h,\alpha} \cdot \partial_3 g^h_3, \\
\Gamma^3_{33}(h) &= \frac{1}{h^2} g^{h,3} \cdot \partial_3 g^h_3.
\end{aligned} \tag{3.3.10}$$

By direct computations we have that

$$g^h_\alpha = e_\alpha + h\partial_\alpha \theta \, e_3 + h x_3 \partial_\alpha \nu_{S_h}, \qquad g^h_3 = \nu_{S_h}.$$

Since $g^h_i \cdot g^{h,j} = \delta_{ij}$, we immediately deduce that

$$g^{h,3} = \nu_{S_h},$$

while by applying Lemma 3.2.1 we obtain

$$g^{h,\alpha} = e_\alpha + h\partial_\alpha \theta \, e_3 + O(h^2).$$

Since

$$\begin{aligned}
\nu_{S_h} &= e_3 - h\partial_1\theta \, e_1 - h\partial_2 \theta \, e_2 + O(h^2), \\
\partial_\alpha \nu_{S_h} &= -h\partial^2_{1\alpha}\theta \, e_1 - h\partial^2_{2\alpha}\theta \, e_2 + O(h^2), \\
\partial^2_{\alpha\beta} \nu_{S_h} &= -h\partial^3_{1\alpha\beta}\theta \, e_1 - h\partial^3_{2\alpha\beta}\theta \, e_2 + O(h^2),
\end{aligned}$$

we deduce (3.3.6)–(3.3.8) from (3.3.10). Equalities (3.3.9) follow again from (3.3.10) by observing that $\partial_3 g^h_3 = 0$ and

$$g^{h,3} \cdot \partial_\alpha g^h_3 = \frac{1}{2} \partial_\alpha (\nu_{S_h} \cdot \nu_{S_h}) = 0.$$

This concludes the proof of the lemma. □

We are ready to prove the announced *Korn-Poincaré inequality on a shallow shell.*

Theorem 3.3.4. *There exist $h_0 > 0$ and $C > 0$, depending on Ω and $\partial_d\Omega$, such that*

$$\|u\|_{L^1} + \|R_h(\operatorname{sym} Du)R_h\|_{M_b} \leq C\left(\|E(h,u)\|_{M_b} + \|u\|_{L^1(\partial_d\Omega)}\right) \tag{3.3.11}$$

for every $0 < h \leq h_0$ and every $u \in BD(\Omega)$.

Proof. Assume, by contradiction, that for every $n \in \mathbb{N}$ there exist $h_n \to 0^+$ and $(u^n) \subset BD(\Omega)$ such that

$$\|u^n\|_{L^1} + \|R_{h_n}(\operatorname{sym} Du^n)R_{h_n}\|_{M_b} = 1 \tag{3.3.12}$$

and

$$\|E(h_n, u^n)\|_{M_b} + \|u^n\|_{L^1(\partial_d\Omega)} \to 0. \tag{3.3.13}$$

By (3.3.12) the sequence (u^n) is uniformly bounded in $BD(\Omega)$; therefore, there exists $u \in BD(\Omega)$ such that $u^n \rightharpoonup u$ weakly* in $BD(\Omega)$ and strongly in $L^1(\Omega;\mathbb{R}^3)$, up to subsequences. On the other hand, it follows from (3.3.4) and (3.3.9) that

$$\begin{aligned}
(R_{h_n}(\operatorname{sym} Du^n)R_{h_n})_{\alpha\beta} &= (\operatorname{sym} Du^n)_{\alpha\beta} = E(h_n,u_n)_{\alpha\beta} + \Gamma^i_{\alpha\beta}(h_n)u^n_i,\\
(R_{h_n}(\operatorname{sym} Du^n)R_{h_n})_{\alpha 3} &= \frac{1}{h_n}(\operatorname{sym} Du^n)_{\alpha 3} = E(h_n,u_n)_{\alpha 3} + \Gamma^\sigma_{\alpha 3}(h_n)u^n_\sigma,\\
(R_{h_n}(\operatorname{sym} Du^n)R_{h_n})_{33} &= \frac{1}{h_n^2}(\operatorname{sym} Du^n)_{33} = E(h_n,u_n)_{33}.
\end{aligned}$$

Using (3.3.13), Lemma 3.3.3, and the strong convergence of (u^n) in $L^1(\Omega;\mathbb{R}^3)$, we deduce that

$$R_{h_n}(\operatorname{sym} Du^n)R_{h_n} \to \operatorname{sym} Du \quad \text{strongly in } M_b(\Omega;\mathbb{M}^{3\times 3}_{sym}) \tag{3.3.14}$$

with $(\operatorname{sym} Du)_{i3} = 0$ and $(\operatorname{sym} Du)_{\alpha\beta} = u_3\partial^2_{\alpha\beta}\theta$. Thus, $u \in KL(\Omega)$ and

$$u^n \to u \quad \text{strongly in } BD(\Omega). \tag{3.3.15}$$

Together with (3.3.13), this implies that $u = 0$ on $\partial_d\Omega$.

Let now $\bar{u} \in BD(\omega)$ and $u_3 \in BH(\omega)$ be the Kirchhoff-Love components of u. Since

$$(\operatorname{sym} D\bar{u})_{\alpha\beta} - x_3\partial^2_{\alpha\beta}u_3 = u_3\partial^2_{\alpha\beta}\theta, \tag{3.3.16}$$

we obtain that $\partial^2_{\alpha\beta}u_3 = 0$. Moreover, the boundary condition $u = 0$ on $\partial_d\Omega$ implies that $\bar{u} - x_3\nabla u_3 = 0$ on $\partial_d\Omega$, hence $\nabla u_3 = 0$ on $\partial_d\omega$, and $u_3 = 0$ on $\partial_d\omega$. By (1.2.3) we deduce that $u_3 = 0$ in ω. Thus, $\operatorname{sym} D\bar{u} = 0$ in ω by (3.3.16) and, in turn, $\operatorname{sym} Du = 0$ in Ω. Since $u = 0$ on $\partial_d\Omega$, it follows from (1.2.2) that $u = 0$ in Ω. Since $\|u\|_{BD} = 1$ by (3.3.12), (3.3.14), and (3.3.15), we arrive at a contradiction. □

3.4 Γ-convergence of the static functionals

In this section we study the asymptotic behaviour of minimisers of the rescaled energy $\mathcal{I}_h$, as h tends to 0. We begin with a compactness result for scaled displacements.

Lemma 3.4.1. *Let $(w^h) \subset H^1(\Omega;\mathbb{R}^3)$ be such that $\|w^h\|_{L^2(\partial_d\Omega)} \leq C$ for every $0 < h \ll 1$. Let (u^h) be a sequence in $V_h(\Omega)$ such that*

$$\|\operatorname{sym}(R_h Du^h R_h F_h^{-1})\|_{M_b} + \|R_h(w^h - u^h) \odot (\operatorname{cof} F_h)R_h\nu_{\partial\Omega}\|_{L^1(\partial_d\Omega)} \leq C \tag{3.4.1}$$

for every $0 < h \ll 1$*. Then there exists* $u \in KL(\Omega)$ *such that, up to subsequences,*

$$u^h \to u \quad \text{strongly in } L^1(\Omega;\mathbb{R}^3) \tag{3.4.2}$$

and

$$\mathrm{sym}(R_h Du^h R_h F_h^{-1})_{\alpha\beta} \rightharpoonup (\bar{E}u)_{\alpha\beta} \quad \text{weakly* in } M_b(\Omega), \tag{3.4.3}$$

as $h \to 0$.

Proof. For every h we consider the vectorfield $u^h(h)$ given by the curvilinear coordinates of u^h, defined according to (3.3.1). For simplicity of notation, we write $u(h)$ instead of $u^h(h)$.

By Lemma 3.2.1 the sequence (F_h) is uniformly bounded with respect to h. Thus, by (3.3.3) and (3.4.1) we deduce that

$$\|E(h, u(h))\|_{M_b} \leq C$$

for every $0 < h \ll 1$. Since $|a \odot b| \geq \frac{1}{\sqrt{2}}|a||b|$ for every $a, b \in \mathbb{R}^n$, it follows from (3.4.1) that

$$\int_{\partial_d\Omega} |R_h(w^h - u^h)||(\mathrm{cof}\, F_h)R_h\nu_{\partial\Omega}|\, d\mathcal{H}^2 \leq C$$

for every $0 < h \ll 1$. Moreover,

$$|(\mathrm{cof}\, F_h)R_h\nu_{\partial\Omega}| \geq \frac{|R_h\nu_{\partial\Omega}|}{|\,\mathrm{cof}\, F_h^{-1}|} \geq C|R_h\nu_{\partial\Omega}| \geq C,$$

where we used that $\mathrm{cof}\, F_h^{-1} \to I_{3\times 3}$ uniformly by Lemma 3.2.1. Therefore, we conclude that

$$\|R_h(w^h - u^h)\|_{L^1(\partial_d\Omega)} \leq C.$$

In particular, we have that $\|w^h - u^h\|_{L^1(\partial_d\Omega)} \leq C$, hence $\|u^h\|_{L^1(\partial_d\Omega)} \leq C$ for every h small enough. By Lemma 3.2.1 we can write

$$(D\Psi_h)^T R_h = I_{3\times 3} + \begin{pmatrix} 0 & 0 & \partial_1\theta \\ 0 & 0 & \partial_2\theta \\ 0 & 0 & 0 \end{pmatrix} + O(h), \tag{3.4.4}$$

hence by (3.3.1) we have that $\|u(h)\|_{1,\partial_d\Omega} \leq C$ for every h.

By applying Theorem 3.3.4 to the sequence $(u(h))$, we deduce that

$$\|u(h)\|_{L^1} + \|R_h(\mathrm{sym}\, Du(h))R_h\|_{M_b} \leq C.$$

Thus, there exists $\tilde{u} \in KL(\Omega)$ such that $u(h) \rightharpoonup \tilde{u}$ weakly* in $BD(\Omega)$ and strongly in $L^1(\Omega;\mathbb{R}^3)$, up to subsequences. In particular, we deduce that (3.4.2) holds with $u \in KL(\Omega)$ defined by

$$u_\alpha := \tilde{u}_\alpha - \partial_\alpha\theta\, \tilde{u}_3, \qquad u_3 := \tilde{u}_3. \tag{3.4.5}$$

Indeed, by (3.3.1) and (3.4.4) we have that

$$u^h = ((D\Psi_h)^T R^h)^{-1} u(h) = u(h) + \begin{pmatrix} 0 & 0 & -\partial_1\theta \\ 0 & 0 & -\partial_2\theta \\ 0 & 0 & 0 \end{pmatrix} u(h) + u_*^h, \tag{3.4.6}$$

where

$$\|u_*^h\|_{L^1} \leq Ch\|u(h)\|_{L^1} \leq Ch,$$

with C independent of h. Passing to the limit in (3.4.6), we obtain (3.4.2) and (3.4.5).

Since $F_h \to I_{3\times 3}$ uniformly as h tends to 0, equality (3.3.3) implies that $E(h, u(h))$ and $\mathrm{sym}(R_h Du^h R_h F_h^{-1})$ have the same weak* limit in $M_b(\Omega; \mathbb{M}^{3\times 3}_{sym})$. In particular, by (3.3.6) and (3.3.7) we obtain

$$E(h, u(h))_{\alpha\beta} \rightharpoonup (\mathrm{sym}\, D\tilde{u})_{\alpha\beta} - \tilde{u}_3 \partial^2_{\alpha\beta}\theta \quad \text{weakly* in } M_b(\Omega),$$

and by (3.4.5) we have

$$\begin{aligned}(\mathrm{sym}\, D\tilde{u})_{\alpha\beta} - \tilde{u}_3 \partial^2_{\alpha\beta}\theta &= (\mathrm{sym}\, Du)_{\alpha\beta} + \mathrm{sym}(D(u_3\nabla\theta))_{\alpha\beta} - u_3\partial^2_{\alpha\beta}\theta \\ &= (\mathrm{sym}\, Du)_{\alpha\beta} + (\nabla\theta \odot \nabla u_3)_{\alpha\beta} \\ &= (\bar{E}u)_{\alpha\beta}.\end{aligned}$$

This proves (3.4.3) and concludes the proof. □

Remark 3.4.2. As a consequence of the continuous embedding of $BD(\Omega)$ in $L^{3/2}(\Omega; \mathbb{R}^3)$ and of the compact embedding of $BD(\Omega)$ in $L^p(\Omega; \mathbb{R}^3)$ for every $p < \frac{3}{2}$, in Lemma 3.4.1 we also have that $u^h \rightharpoonup u$ weakly in $L^{3/2}(\Omega; \mathbb{R}^3)$ and $u^h \to u$ strongly in $L^p(\Omega; \mathbb{R}^3)$ for every $p < \frac{3}{2}$.

The following theorem is the main result of this section. The proof is in the spirit of Γ-convergence.

Theorem 3.4.3. *Let $(w^h) \subset H^1(\Omega; \mathbb{R}^3)$ be such that*

$$\|w^h\|_{L^2(\partial_d\Omega)} \leq C \quad \text{for every } 0 < h \ll 1, \tag{3.4.7}$$
$$\mathrm{sym}(R_h Dw^h R_h F_h^{-1}) \to z \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}), \tag{3.4.8}$$

where $C > 0$ is independent of h and $z \in L^2(\Omega; \mathbb{M}^{3\times 3}_{sym})$. Let $(u^h, e^h, p^h) \in \mathcal{A}_h(\Omega, w^h)$ be a minimiser of $\mathcal{I}_h$. Then there exist $w \in KL(\Omega) \cap H^1(\Omega; \mathbb{R}^3)$ and a triplet $(u, e, p) \in \mathcal{A}_{GKL}(w)$ such that, up to subsequences,

$$w^h \to w \quad \text{strongly in } H^1(\Omega; \mathbb{R}^3), \tag{3.4.9}$$
$$\mathrm{sym}(R_h Dw^h R_h F_h^{-1})_{\alpha\beta} \to (E^* w)_{\alpha\beta} \quad \text{strongly in } L^2(\Omega), \tag{3.4.10}$$
$$u^h \to u \quad \text{strongly in } L^1(\Omega; \mathbb{R}^3), \tag{3.4.11}$$
$$\mathrm{sym}(R_h Du^h R_h F_h^{-1})_{\alpha\beta} \rightharpoonup (\bar{E}u)_{\alpha\beta} \quad \text{weakly* in } M_b(\Omega), \tag{3.4.12}$$
$$e^h \to \mathbb{M}e \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}), \tag{3.4.13}$$
$$p^h_{\alpha\beta} \rightharpoonup p_{\alpha\beta} \quad \text{weakly* in } M_b(\Omega \cup \partial_d\Omega). \tag{3.4.14}$$

Moreover, (u, e, p) is a minimiser of $\mathcal{I}$ and

$$\lim_{h\to 0} \mathcal{I}_h(u^h, e^h, p^h) = \mathcal{I}(u, e, p). \tag{3.4.15}$$

Remark 3.4.4. By the definition (3.2.19) of the operator $\mathbb{M}$ convergence (3.4.13) implies, in particular, that $e^h_{\alpha\beta} \to e_{\alpha\beta}$ strongly in $L^2(\Omega)$.

Proof of Theorem 3.4.3. The proof is subdivided into four steps. First of all, as a consequence of Lemma 3.2.1, we note that the following expansions hold:

$$\begin{aligned}\operatorname{sym}(R_h Dv R_h F_h^{-1})_{\alpha\beta} &= (\operatorname{sym} Dv - \partial_3 v \odot \nabla\theta)_{\alpha\beta} + O(h^2)\|v\|_{H^1},\\ \operatorname{sym}(R_h Dv R_h F_h^{-1})_{\alpha 3} &= \tfrac{1}{h}\left((\operatorname{sym} Dv - \partial_3 v \odot \nabla\theta)_{\alpha 3} + O(h^2)\|v\|_{H^1}\right),\\ \operatorname{sym}(R_h Dv R_h F_h^{-1})_{33} &= \tfrac{1}{h^2}\left(\partial_3 v_3(1+O(h^2)) + h^2\nabla v_3\cdot\nabla\theta + O(h^4)\|v\|_{H^1}\right)\end{aligned} \tag{3.4.16}$$

for every $v \in H^1(\Omega;\mathbb{R}^3)$.

Step 0: Convergence of (w^h). In this step we prove (3.4.9) and (3.4.10).

By (3.4.16) and the fact that $\partial_3\theta = 0$ we deduce that

$$\|\operatorname{sym}(R_h Dw^h R_h F_h^{-1})\|_{L^2} \geq \|\operatorname{sym} Dw^h - \partial_3 w^h \odot \nabla\theta\|_{L^2} - O(h^2)\|w^h\|_{H^1}.$$

This implies that for h small enough

$$\begin{aligned}&\|w^h\|_{L^2(\partial_d\Omega)} + \|\operatorname{sym}(R_h Dw^h R_h F_h^{-1})\|_{L^2}\\ &\geq \|w^h\|_{L^2(\partial_d\Omega)} + \|\operatorname{sym} Dw^h - \partial_3 w^h \odot \nabla\theta\|_{L^2} - O(h^2)\|w^h\|_{H^1}\\ &\geq C\|w^h\|_{H^1},\end{aligned} \tag{3.4.17}$$

where the last estimate follows from a generalised Korn inequality in H^1 for shallow shells (see, e.g, [10, Theorem 3.4-1]). By (3.4.7) and (3.4.8) we conclude that the sequence (w^h) is uniformly bounded in $H^1(\Omega;\mathbb{R}^3)$ for h small enough. Thus, there exists $w \in H^1(\Omega;\mathbb{R}^3)$ such that

$$w^h \rightharpoonup w \quad \text{weakly in } H^1(\Omega;\mathbb{R}^3), \tag{3.4.18}$$

up to subsequences. Convergence (3.4.18) yields

$$\operatorname{sym} Dw^h - \partial_3 w^h \odot \nabla\theta \rightharpoonup \operatorname{sym} Dw - \partial_3 w \odot \nabla\theta \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{sym}).$$

Owing to (3.4.8) and (3.4.16), we also have that $(\operatorname{sym} Dw^h - \partial_3 w^h \odot \nabla\theta)_{\alpha\beta} \to z_{\alpha\beta}$ and $(\operatorname{sym} Dw^h - \partial_3 w^h \odot \nabla\theta)_{i3} \to 0$ strongly in $L^2(\Omega)$. Therefore, we deduce that

$$\operatorname{sym} Dw^h - \partial_3 w^h \odot \nabla\theta \to \operatorname{sym} Dw - \partial_3 w \odot \nabla\theta \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{sym}) \tag{3.4.19}$$

and $(\operatorname{sym} Dw - \partial_3 w \odot \nabla\theta)_{i3} = 0$. Since $\partial_3\theta = 0$, this last equality implies that

$$(\operatorname{sym} Dw - \partial_3 w \odot \nabla\theta)_{33} = \partial_3 w_3 = 0,$$

and consequently

$$(\operatorname{sym} Dw - \partial_3 w \odot \nabla\theta)_{\alpha 3} = (\operatorname{sym} Dw)_{\alpha 3} = 0.$$

In other words, $(\operatorname{sym} Dw)_{i3} = 0$, that is, $w \in KL(\Omega)$. In particular, we have that $\partial_3 w_\alpha = -\partial_\alpha w_3$, hence $\partial_3 w \odot \nabla\theta = -\nabla w_3 \odot \nabla\theta$, so that (3.4.16) and (3.4.19) give (3.4.10).

To conclude it remains to show that convergence (3.4.18) is strong. By applying again [10, Theorem 3.4-1] we obtain

$$\begin{aligned}&\|w^h - w^{h'}\|_{H^1}\\ &\quad\leq C(\|w^h - w^{h'}\|_{L^2(\partial_d\Omega)} + \|\operatorname{sym} Dw^h - \partial_3 w^h \odot \nabla\theta - \operatorname{sym} Dw^{h'} + \partial_3 w^{h'} \odot \nabla\theta\|_{L^2})\end{aligned} \tag{3.4.20}$$

for every $0 < h, h' \ll 1$. By (3.4.18) and the compactness of the trace operator we have that $w^h \to w$ strongly in $L^2(\partial_d\Omega;\mathbb{R}^3)$. Thus, by (3.4.19) and (3.4.20) we conclude that (w^h) is a Cauchy sequence in $H^1(\Omega;\mathbb{R}^3)$, hence (3.4.9) holds.

Step 1: Compactness. Since

$$(w^h, \mathrm{sym}(R_h Dw^h R_h F_h^{-1}), 0) \in \mathcal{A}_h(\Omega, w^h), \tag{3.4.21}$$

the minimality of (u^h, e^h, p^h) implies

$$\mathcal{I}_h(u^h, e^h, p^h) \leq \mathcal{I}_h(w^h, \mathrm{sym}(R_h Dw^h R_h F_h^{-1}), 0) \leq C \tag{3.4.22}$$

for every $0 < h \ll 1$, where the last inequality is a consequence of (3.2.5), (3.4.8), and Lemma 3.2.1. Using again Lemma 3.2.1, (3.2.5), and (3.2.8), the bound (3.4.22) yields

$$\|e^h\|_{L^2} + \|p^h\|_{M_b} \leq C \tag{3.4.23}$$

for every $0 < h \ll 1$. Thus, there exist $\tilde{e} \in L^2(\Omega;\mathbb{M}^{3\times3}_{sym})$ and $\tilde{p} \in M_b(\Omega \cup \partial_d\Omega;\mathbb{M}^{3\times3}_{sym})$ such that, up to subsequences,

$$e^h \rightharpoonup \tilde{e} \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}_{sym}), \tag{3.4.24}$$

$$p^h \rightharpoonup \tilde{p} \quad \text{weakly* in } M_b(\Omega \cup \partial_d\Omega;\mathbb{M}^{3\times3}_D). \tag{3.4.25}$$

We introduce $e \in L^2(\Omega;\mathbb{M}^{3\times3}_{sym})$ and $p \in M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{3\times3}_{sym})$ defined by $e_{\alpha\beta} := \tilde{e}_{\alpha\beta}$, $e_{i3} := 0$, and $p_{\alpha\beta} := \tilde{p}_{\alpha\beta}$, $p_{i3} := 0$, respectively.

Since Q is convex and $\det F_h \to 1$ uniformly, as $h \to 0$, by Lemma 3.2.1, we have

$$\liminf_{h\to0} \int_\Omega Q(e^h) \det F_h \, dx \geq \int_\Omega Q(\tilde{e})\, dx \geq \mathcal{Q}^*(e), \tag{3.4.26}$$

where the last inequality follows from the definition of Q^*. Analogously, by the Reshetnyak Theorem and the definition of H^* we deduce

$$\liminf_{h\to0} \mathcal{H}_h(p^h) \geq \int_{\Omega\cup\partial_d\Omega} H\left(\frac{d\tilde{p}}{d|\tilde{p}|}\right) d|\tilde{p}| \geq \mathcal{H}^*(p). \tag{3.4.27}$$

By (3.2.17) and (3.4.23) we can apply Lemma 3.4.1. Thus, there exists $u \in KL(\Omega)$ such that, up to subsequences,

$$u^h \to u \quad \text{strongly in } L^1(\Omega;\mathbb{R}^3), \tag{3.4.28}$$

$$\mathrm{sym}(R_h Du^h R_h F_h^{-1})_{\alpha\beta} \rightharpoonup (\bar{E}u)_{\alpha\beta} \quad \text{weakly* in } M_b(\Omega). \tag{3.4.29}$$

We claim that $(u, e, p) \in \mathcal{A}_{GKL}(w)$. Combining (3.4.24), (3.4.25), and (3.4.29), we deduce that $\bar{E}u = e + p$ in Ω.

To conclude it remains to show that $p = (w - u) \odot \nu_{\partial\Omega}\mathcal{H}^2$ on $\partial_d\Omega$. We argue as in [12, Lemma 2.1]. Since γ_d is open in $\partial\omega$, there exists an open set $A \subseteq \mathbb{R}^2$ such that $\gamma_d = A \cap \partial\omega$. We set $U := (\omega \cup A) \times \left(-\frac{1}{2}, \frac{1}{2}\right)$. We extend θ to U in such a way that $\theta \in C^3(\overline{U})$. Consequently, $\Psi_h \in C^2(\overline{U};\mathbb{R}^3)$ and $F_h \in C^1(\overline{U};\mathbb{M}^{3\times3})$ for every $0 < h \ll 1$. Let $u^h(h)$ and $w^h(h)$ be the vectorfields given by the curvilinear coordinates of u^h and w^h, defined according to (3.3.1). For simplicity we write $u(h)$ and $w(h)$ instead of $u^h(h)$ and $w^h(h)$. By (3.3.1), (3.4.4), and (3.4.9) we have that

$$w(h) \to \tilde{w} := w + w_3\nabla\theta \quad \text{strongly in } L^2(\Omega;\mathbb{R}^3). \tag{3.4.30}$$

By Proposition 3.3.1, Lemma 3.3.3, and (3.4.8), the sequence ($\operatorname{sym} Dw(h)$) is also strongly convergent in $L^2(\Omega; \mathbb{M}^{3\times3}_{sym})$ for h small enough. Thus, by the Korn inequality the convergence in (3.4.30) is strong in $H^1(\Omega; \mathbb{R}^3)$. Moreover, we can extend $w(h)$ and $\tilde{w}$ to U in such a way that

$$w(h) \rightharpoonup \tilde{w} \quad \text{weakly in } H^1(U; \mathbb{R}^3). \tag{3.4.31}$$

We now define the triplet $(v(h), \eta(h), q(h)) \in BD(U) \times L^2(U; \mathbb{M}^{3\times3}_{sym}) \times M_b(U; \mathbb{M}^{3\times3}_{sym})$ as

$$v(h) := \begin{cases} u(h) & \text{in } \Omega, \\ w(h) & \text{in } U \setminus \Omega, \end{cases} \qquad \eta(h) := \begin{cases} R_h^{-1} F_h^T e^h F_h R_h^{-1} & \text{in } \Omega, \\ R_h^{-1} E(h, w(h)) R_h^{-1} & \text{in } U \setminus \Omega, \end{cases}$$

and

$$q(h) := \begin{cases} R_h^{-1} F_h^T p^h F_h R_h^{-1} & \text{in } \Omega \cup \partial_d \Omega, \\ 0 & \text{in } U \setminus (\Omega \cup \partial_d \Omega)\,, \end{cases}$$

where $E(h, w(h))$ is defined as in (3.3.4). We have that

$$(\operatorname{sym} Dv(h))_{ij} = \eta(h)_{ij} + q(h)_{ij} + (R_h^{-1})_{ik} \Gamma^m_{kl}(h) v_m(h) (R_h^{-1})_{lj} \quad \text{in } U. \tag{3.4.32}$$

Indeed, this equality holds in Ω and in $U \setminus \overline{\Omega}$ as a consequence of (3.2.17), (3.3.3), and (3.3.4), while on $\partial_d \Omega$ it follows from (3.2.17), (3.3.2), and the definition of the cofactor.

By (3.3.1), (3.4.4), (3.4.28), and (3.4.31) we deduce that

$$v(h) \to v \quad \text{strongly in } L^1(U; \mathbb{R}^3), \tag{3.4.33}$$

where

$$v := \begin{cases} u + u_3 \nabla\theta & \text{in } \Omega, \\ \tilde{w} & \text{in } U \setminus \Omega. \end{cases}$$

Since ($\eta(h)$) is uniformly bounded in $L^2(\Omega; \mathbb{M}^{3\times3}_{sym})$ by (3.4.24), Lemma 3.2.1, (3.3.4), and (3.4.31), we have that there exists $\eta \in L^2(U; \mathbb{M}^{3\times3}_{sym})$ such that

$$\eta(h) \rightharpoonup \eta \quad \text{weakly in } L^2(U; \mathbb{M}^{3\times3}_{sym}), \tag{3.4.34}$$

up to subsequences. Finally, it follows from Lemma 3.2.1 and (3.4.25) that

$$q(h) \rightharpoonup q \quad \text{weakly}^* \text{ in } M_b(U; \mathbb{M}^{3\times3}_{sym}), \tag{3.4.35}$$

where

$$q := \begin{cases} p & \text{in } \Omega \cup \partial_d \Omega, \\ 0 & \text{in } U \setminus (\Omega \cup \partial_d \Omega)\,. \end{cases}$$

Passing to the limit in (3.4.32) by (3.4.33)–(3.4.35) and Lemma 3.3.3, we obtain

$$\operatorname{sym} Dv = \eta + q + v_3 D^2\theta \quad \text{in } U.$$

In particular, since $\tilde{w} = w + w_3 \nabla\theta$, the previous equality on $\partial_d \Omega$ reads as

$$p = (w - u + (w_3 - u_3)\nabla\theta) \odot \nu_{\partial\Omega} \mathcal{H}^2 \quad \text{on } \partial_d \Omega.$$

Since $p_{\alpha 3} = 0$, $\nu_{\partial\Omega} \cdot e_3 = 0$ on $\partial_d \Omega$, and $\partial_3 \theta = 0$, this implies that $u_3 = w_3$ on $\partial_d \Omega$ and, in turn, the desired equality.

Step 2: Existence of a recovery sequence. We show that for every $(v, \eta, q) \in \mathcal{A}_{GKL}(w)$ there exists a sequence of triplets $(v^h, \eta^h, q^h) \in \mathcal{A}_h(\Omega, w^h)$ such that

$$v^h \to v \quad \text{strongly in } L^1(\Omega; \mathbb{R}^3), \tag{3.4.36}$$

$$\mathrm{sym}(R_h D v^h R_h F_h^{-1})_{\alpha\beta} \rightharpoonup (\bar{E}v)_{\alpha\beta} \quad \text{weakly* in } M_b(\Omega), \tag{3.4.37}$$

$$\eta^h \to \mathbb{M}\eta \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}), \tag{3.4.38}$$

$$q^h_{\alpha\beta} \rightharpoonup q_{\alpha\beta} \quad \text{weakly* in } M_b(\Omega \cup \partial_d \Omega), \tag{3.4.39}$$

$$\mathcal{H}_h(q^h) \to \mathcal{H}^*(q), \tag{3.4.40}$$

and

$$\lim_{h\to 0} \mathcal{I}_h(v^h, \eta^h, q^h) = \mathcal{I}(v, \eta, q). \tag{3.4.41}$$

Owing to Lemma 3.2.7, it is enough to construct an approximating sequence for a triplet

$$(v, \eta, q) \in \big(H^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}) \times L^2_{\infty,c}(\Omega; \mathbb{M}^{3\times 3}_{sym})\big) \cap \mathcal{A}_{GKL}(w). \tag{3.4.42}$$

Indeed, in the general case one can argue by density as in [13, Theorem 5.4].

Let (v, η, q) be as in (3.4.42). Since $q \in L^2(\Omega; \mathbb{M}^{3\times 3}_{sym})$, we have that $q = 0$ on $\partial_d\Omega$ and $v = w$ on $\partial_d\Omega$. Let $\phi_1, \phi_2, \phi_3 \in L^2(\Omega)$ be such that

$$\mathbb{M}\eta = \begin{pmatrix} \eta_{11} & \eta_{12} & \phi_1 \\ \eta_{21} & \eta_{22} & \phi_2 \\ \phi_1 & \phi_2 & \phi_3 \end{pmatrix}. \tag{3.4.43}$$

As $q \in L^2(\Omega; \mathbb{M}^{3\times 3}_{sym})$, by the measurable selection lemma (see, e.g., [21]) and by the definition of H^* there exist $\xi_1, \xi_2 \in L^2(\Omega)$ such that

$$H^*(q) = H \begin{pmatrix} q_{11} & q_{12} & \xi_1 \\ q_{21} & q_{22} & \xi_2 \\ \xi_1 & \xi_2 & -(q_{11} + q_{22}) \end{pmatrix}. \tag{3.4.44}$$

We approximate the functions ϕ_i and ξ_α by means of elliptic regularisations, namely for every h we consider the solutions $\phi_i^h \in H^1_0(\Omega)$ and $\xi^h_\alpha \in H^1_0(\Omega)$ of the problems

$$\begin{cases} -h\Delta\phi_i^h + \phi_i^h = \phi_i & \text{in } \Omega, \\ \phi_i^h = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -h\Delta\xi_\alpha^h + \xi_\alpha^h = \xi_\alpha & \text{in } \Omega, \\ \xi_\alpha^h = 0 & \text{on } \partial\Omega. \end{cases}$$

Similarly, for every h we define $\mu_i^h \in H^1_0(\Omega)$ as the solutions of the problems

$$\begin{cases} -h\Delta\mu_\alpha^h + \mu_\alpha^h = -z_{3\alpha} & \text{in } \Omega, \\ \mu_\alpha^h = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -h\Delta\mu_3^h + \mu_3^h = \nabla(w_3 - v_3)\cdot\nabla\theta - z_{33} & \text{in } \Omega, \\ \mu_3^h = 0 & \text{on } \partial\Omega, \end{cases}$$

where z_{3i} are the components of the function z in (3.4.8). The standard theory of elliptic equations implies that

$$\phi_i^h \to \phi_i \quad \text{strongly in } L^2(\Omega), \tag{3.4.45}$$

$$\xi_\alpha^h \to \xi_\alpha \quad \text{strongly in } L^2(\Omega), \tag{3.4.46}$$

$$\mu_\alpha^h \to -z_{3\alpha} \quad \text{strongly in } L^2(\Omega), \tag{3.4.47}$$

$$\mu_3^h \to \nabla(w_3 - v_3)\cdot\nabla\theta - z_{33} \quad \text{strongly in } L^2(\Omega), \tag{3.4.48}$$

as $h \to 0$, and

$$\|\nabla \phi_i^h\|_{L^2} + \|\nabla \xi_\alpha^h\|_{L^2} + \|\nabla \mu_i^h\|_{L^2} \leq C h^{-\frac{1}{2}} \tag{3.4.49}$$

for every h. We also introduce the function $k^h \in L^2(\Omega; \mathbb{M}^{3\times 3})$, defined componentwise as

$$\begin{aligned}
k^h_{\alpha\beta}(x', x_3) &:= 2h \int_0^{x_3} \big(\partial_\beta \phi_\alpha^h(x', s) + \partial_\beta \xi_\alpha^h(x', s) + \partial_\beta \mu_\alpha^h(x', s)\big)\, ds,\\
k_{3\beta}(x', x_3) &:= h^2 \int_0^{x_3} \big(\partial_\beta \phi_3^h(x', s) + \partial_\beta \mu_3^h(x', s) - \partial_\beta q_{11}(x', s) - \partial_\beta q_{22}(x', s)\big)\, ds,\\
k^h_{\alpha 3} &:= 2h(\phi^h_\alpha + \xi^h_\alpha + \mu^h_\alpha), \qquad k^h_{33} := h^2(\phi^h_3 + \mu^h_3 - q_{11} - q_{22}).
\end{aligned}$$

We are now in a position to define the recovery sequence. We set

$$\begin{aligned}
v^h_\alpha &:= v_\alpha + w^h_\alpha - w_\alpha + 2h \int_0^{x_3} \big(\phi^h_\alpha(x', s) + \xi^h_\alpha(x', s) + \mu^h_\alpha(x', s)\big)\, ds,\\
v^h_3 &:= v_3 + w^h_3 - w_3 + h^2 \int_0^{x_3} \big(\phi^h_3(x', s) + \mu^h_3(x', s) - q_{11}(x', s) - q_{22}(x', s)\big)\, ds.
\end{aligned}$$

It is straightforward to check that

$$Dv^h = Dv + Dw^h - Dw + k^h.$$

This leads us to define

$$\begin{aligned}
q^h &:= q + \begin{pmatrix} 0 & 0 & \xi^h_1 \\ 0 & 0 & \xi^h_2 \\ \xi^h_1 & \xi^h_2 & -(q_{11} + q_{22}) \end{pmatrix},\\
\eta^h &:= \mathrm{sym}(R_h(Dv + Dw^h - Dw)R_h F_h^{-1}) + \mathrm{sym}(R_h k^h R_h F_h^{-1}) - q^h.
\end{aligned}$$

Since $\phi^h_i, \xi^h_\alpha, \mu^h_i \in H^1_0(\Omega)$, $q \in L^2_{\infty,c}(\Omega; \mathbb{M}^{2\times 2}_{sym})$, and $v = w$ on $\partial_d\Omega$, we have that $v^h = w^h$ on $\partial_d\Omega$. Hence, it is clear that $(v^h, \eta^h, q^h) \in \mathcal{A}_h(\Omega, w^h)$.

It follows from (3.4.9) and (3.4.45)–(3.4.48) that $v^h \to v$ strongly in $L^2(\Omega; \mathbb{R}^3)$. In particular, (3.4.36) holds. By definition of q^h we immediately deduce (3.4.39). Owing to (3.4.46), we obtain that

$$q^h \to q + \begin{pmatrix} 0 & 0 & \xi_1 \\ 0 & 0 & \xi_2 \\ \xi_1 & \xi_2 & -(q_{11} + q_{22}) \end{pmatrix} \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}). \tag{3.4.50}$$

Convergence (3.4.50), together with (3.4.44) and Lemma 3.2.1, implies (3.4.40).

We now prove (3.4.38). Since $v, w \in KL(\Omega)$, expansions (3.4.16) imply that

$$\begin{aligned}
\mathrm{sym}(R_h(Dv - Dw)R_h F_h^{-1})_{\alpha\beta} &= (\bar{E}v - E^* w)_{\alpha\beta} + O(h^2),\\
\mathrm{sym}(R_h(Dv - Dw)R_h F_h^{-1})_{\alpha 3} &= O(h),\\
\mathrm{sym}(R_h(Dv - Dw)R_h F_h^{-1})_{33} &= \nabla\theta \cdot \nabla(v_3 - w_3) + O(h^2).
\end{aligned} \tag{3.4.51}$$

Thus, by (3.4.8) and (3.4.10) we deduce that

$$\begin{aligned}
\mathrm{sym}(R_h(Dv + Dw^h - Dw)R_h F_h^{-1})_{\alpha\beta} &\to (\bar{E}v)_{\alpha\beta} \quad \text{strongly in } L^2(\Omega),\\
\mathrm{sym}(R_h(Dv + Dw^h - Dw)R_h F_h^{-1})_{\alpha 3} &\to z_{\alpha 3} \quad \text{strongly in } L^2(\Omega),
\end{aligned} \tag{3.4.52}$$

and

$$\mathrm{sym}(R_h(Dv+Dw^h-Dw)R_hF_h^{-1})_{33} \to z_{33}+\nabla\theta\cdot\nabla(v_3-w_3) \quad \text{strongly in } L^2(\Omega). \tag{3.4.53}$$

From (3.4.45)–(3.4.49) it follows that

$$(R_hk^hR_h)_{i\beta} \to 0 \quad \text{strongly in } L^2(\Omega),$$
$$(R_hk^hR_h)_{\alpha 3} \to 2\,(\phi_\alpha + \xi_\alpha - z_{3\alpha}) \quad \text{strongly in } L^2(\Omega),$$
$$(R_hk^hR_h)_{33} \to \phi_3 + \nabla(w_3 - v_3)\cdot\nabla\theta - z_{33} - q_{11} - q_{22} \quad \text{strongly in } L^2(\Omega).$$

This, together with the uniform convergence F_h^{-1} to $I_{3\times3}$, implies that

$$\mathrm{sym}(R_hk^hR_hF_h^{-1})_{\alpha\beta} \to 0 \quad \text{strongly in } L^2(\Omega),$$
$$\mathrm{sym}(R_hk^hR_hF_h^{-1})_{\alpha 3} \to \phi_\alpha + \xi_\alpha - z_{3\alpha} \quad \text{strongly in } L^2(\Omega),$$
$$\mathrm{sym}(R_hk^hR_hF_h^{-1})_{33} \to \phi_3 + \nabla(w_3 - v_3)\cdot\nabla\theta - z_{33} - q_{11} - q_{22} \quad \text{strongly in } L^2(\Omega).$$

Combining the convergences above with (3.4.43), (3.4.50), (3.4.52), and (3.4.53), yields (3.4.38).

Finally, (3.4.37) follows from (3.4.38) and (3.4.39), while (3.4.41) is a consequence of (3.2.22), (3.4.38), and (3.4.40).

Step 3: Conclusion. Let $(v,\eta,q) \in \mathcal{A}_{GKL}(w)$. By Step 2 there exists a sequence (v^h,η^h,q^h) in $\mathcal{A}_h(\Omega,w^h)$ such that (3.4.36)–(3.4.41) hold. Therefore,

$$\mathcal{I}(v,\eta,q) \geq \lim_{h\to0}\mathcal{I}_h(v^h,\eta^h,q^h) \geq \limsup_{h\to0}\mathcal{I}_h(u^h,e^h,p^h), \tag{3.4.54}$$

where the last inequality follows from the minimality of (u^h,e^h,p^h). On the other hand, by (3.4.26) and (3.4.27)

$$\liminf_{h\to0}\mathcal{I}_h(u^h,e^h,p^h) \geq \mathcal{I}(u,e,p). \tag{3.4.55}$$

Combining (3.4.54) and (3.4.55), we conclude that (u,e,p) is a minimiser of $\mathcal{I}$. Moreover, by choosing $(v,\eta,q) = (u,e,p)$ in (3.4.54) we deduce (3.4.15).

It remains to prove (3.4.13). From (3.4.26), (3.4.27), and (3.4.15) it follows that

$$\lim_{h\to0}\int_\Omega Q(e^h)\det F_h\,dx = \mathcal{Q}^*(e).$$

Since $\det F_h \to 1$ uniformly, as $h\to0$, this implies that

$$\lim_{h\to0}\int_\Omega Q(e^h)\,dx = \mathcal{Q}^*(e). \tag{3.4.56}$$

On the other hand, by (3.2.22) we have

$$Q(e^h - \mathbb{M}e) = Q(e^h) + Q^*(e) - \mathbb{C}\mathbb{M}e : e^h.$$

Therefore, owing to (3.4.24), (3.4.56), and (3.2.21), we get

$$\lim_{h\to0}\int_\Omega Q(e^h - \mathbb{M}e)\,dx = 0.$$

By the coercivity (3.2.5) of Q this implies (3.4.13). □

Remark 3.4.5. In our framework we cannot rely on the abstract theory of evolutionary Γ-convergence for rate-independent systems, developed in [41]. Indeed, this theory consists in studying separately the Γ-limit of the stored-energy functionals and that of the dissipation potentials, and in coupling them through the construction of a joint recovery sequence. This technique has been successfully applied, for example, in [34] and in [35], where the presence of hardening gives rise to an energy functional that is coercive in the L^2 norm both with respect to e and p. This approach is not suited to our case, since the elastic energy is coercive only with respect to the elastic strain e, while the plastic strain p can be controlled only through the dissipation. For this reason, to identify the correct limiting energy we studied the Γ-convergence of the total energy functional, given by the sum of the stored energy and of the dissipation distance.

3.5 Convergence of quasistatic evolutions

In this section we focus on the convergence of the quasistatic evolution problems associated with the functionals $\mathcal{I}_h$ and $\mathcal{I}$.

We fix a time interval $[0,T]$ with $T>0$ and we give the following definitions.

Definition 3.5.1. Let $0<h\ll 1$ and let $w^h\in \mathrm{Lip}([0,T];H^1(\Omega;\mathbb{R}^3))$. An *h-quasistatic evolution* for the boundary datum w^h is a function $t\mapsto (u^h(t),e^h(t),p^h(t))$ from $[0,T]$ into $V_h(\Omega)\times L^2(\Omega;\mathbb{M}^{3\times3}_{sym})\times M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{3\times3}_D)$ that satisfies the following conditions:

(qs1) *global stability:* for every $t\in[0,T]$ we have that $(u^h(t),e^h(t),p^h(t))\in\mathcal{A}_h(\Omega,w^h(t))$ and

$$\int_\Omega Q(e^h(t))\det F_h\,dx \le \int_\Omega Q(\eta)\det F_h\,dx + \mathcal{H}_h(q-p^h(t)) \tag{3.5.1}$$

for every $(v,\eta,q)\in\mathcal{A}_h(\Omega,w^h(t))$;

(qs2) *energy balance:* $p^h\in BV([0,T];M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{3\times3}_{sym}))$ and for every $t\in[0,T]$

$$\begin{aligned}&\int_\Omega Q(e^h(t))\det F_h\,dx + \mathcal{D}_h(p^h;0,t)\\ &\quad= \int_\Omega Q(e^h(0))\det F_h\,dx + \int_0^t\int_\Omega \mathbb{C}e^h(s):\mathrm{sym}(R_hD\dot{w}^h(s)R_hF_h^{-1})\det F_h\,dx\,ds.\end{aligned} \tag{3.5.2}$$

In the formula (3.5.2) the notation $\mathcal{D}_h(p^h;0,t)$ stands for the *dissipation* of p^h in the interval $[0,t]$, defined as

$$\mathcal{D}_h(\mu;a,b) := \sup\Big\{\sum_{j=1}^N \mathcal{H}_h(\mu(s_j)-\mu(s_{j-1})):\ a=s_0\le s_1\le\cdots\le s_N=b,\ N\in\mathbb{N}\Big\}$$

for every $\mu\in BV([0,T];M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{3\times3}_{sym}))$ and every $0\le a\le b\le T$.

Definition 3.5.2. Let $w\in\mathrm{Lip}([0,T];H^1(\Omega;\mathbb{R}^3)\cap KL(\Omega))$. A *reduced quasistatic evolution* for the boundary datum w is a function $t\mapsto(u(t),e(t),p(t))$ from $[0,T]$ into $BD(\Omega)\times L^2(\Omega;\mathbb{M}^{2\times2}_{sym})\times M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{2\times2}_{sym})$ that satisfies the following conditions:

(qs1)* *reduced global stability:* for every $t \in [0,T]$ we have that

$$(u(t), e(t), p(t)) \in \mathcal{A}_{GKL}(w(t))$$

and

$$\mathcal{Q}^*(e(t)) \leq \mathcal{Q}^*(\eta) + \mathcal{H}^*(q - p(t)) \tag{3.5.3}$$

for every $(v, \eta, q) \in \mathcal{A}_{GKL}(w(t))$;

(qs2)* *reduced energy balance:* $p \in BV([0,T]; M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{2\times 2}_{sym}))$ and for every $t \in [0,T]$

$$\mathcal{Q}^*(e(t)) + \mathcal{D}^*(p; 0, t) = \mathcal{Q}^*(e(0)) + \int_0^t \int_\Omega \mathbb{C}^* e(s) : E^* \dot{w}(s)\, dx\, ds. \tag{3.5.4}$$

In the formula (3.5.4) the notation $\mathcal{D}^*(p; 0, t)$ stands for the *reduced dissipation* of p in the interval $[0,t]$, defined as

$$\mathcal{D}^*(\mu; a, b) := \sup\Big\{ \sum_{j=1}^N \mathcal{H}^*(\mu(s_j) - \mu(s_{j-1})) : \ a = s_0 \leq s_1 \leq \cdots \leq s_N = b, \ N \in \mathbb{N} \Big\}$$

for every $\mu \in BV([0,T]; M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{2\times 2}_{sym}))$ and every $0 \leq a \leq b \leq T$.

In the following we will show the convergence of a sequence of h-quasistatic evolutions to a reduced quasistatic evolution, as $h \to 0$. This will be proved under the following assumptions on the boundary and initial data.

Boundary displacements.

We consider a sequence of boundary displacements

$$(w^h) \subset \mathrm{Lip}([0,T]; H^1(\Omega; \mathbb{R}^3)) \tag{3.5.5}$$

such that for every $0 < h \ll 1$

$$\|w^h\|_{W^{1,\infty}([0,T]; L^2(\partial_d\Omega; \mathbb{R}^3))} + \|\operatorname{sym}(R_h D w^h R_h F_h^{-1})\|_{W^{1,\infty}([0,T]; L^2)} \leq C \tag{3.5.6}$$

with a constant $C > 0$, independent of h. Furthermore, we assume that for every $t \in [0,T]$

$$\operatorname{sym}(R_h D w^h(t) R_h F_h^{-1}) \to z(t) \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}), \tag{3.5.7}$$

$$\operatorname{sym}(R_h D \dot{w}^h(t) R_h F_h^{-1}) \to \dot{z}(t) \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}) \tag{3.5.8}$$

for some $z \in \mathrm{Lip}([0,T]; L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}))$.

Initial data.

Let $(u_0^h, e_0^h, p_0^h) \in \mathcal{A}_h(\Omega, w^h(0))$ such that

$$\int_\Omega Q(e_0^h) \det F_h\, dx \leq \int_\Omega Q(\eta) \det F_h\, dx + \mathcal{H}_h(q - p_0^h) \tag{3.5.9}$$

for every $(v, \eta, q) \in \mathcal{A}_h(\Omega, w^h(0))$. Moreover, we assume that

$$e_0^h \to \tilde{e}_0 \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}) \tag{3.5.10}$$

for some $\tilde{e}_0 \in L^2(\Omega;\mathbb{M}^{3\times3}_{sym})$ and for every $0 < h \ll 1$

$$\|p_0^h\|_{M_b} \leq C \tag{3.5.11}$$

for some constant $C > 0$, independent of h.

We are now in a position to state the main result of this paper.

Theorem 3.5.3. *Assume* (3.5.5)–(3.5.11). *For every* $0 < h \ll 1$ *let*

$$t \mapsto (u^h(t), e^h(t), p^h(t))$$

be an h*-quasistatic evolution for the boundary datum* w^h *such that* $(u^h(0), e^h(0), p^h(0)) = (u_0^h, e_0^h, p_0^h)$. *Then there exist* $w \in \mathrm{Lip}([0,T]; H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega))$ *and a reduced quasistatic evolution*

$$(u, e, p) \in \mathrm{Lip}\left([0,T]\,; BD(\Omega) \times L^2(\Omega;\mathbb{M}^{2\times2}_{sym}) \times M_b(\Omega \cup \partial_d\Omega;\mathbb{M}^{2\times2}_{sym})\right)$$

for the boundary datum w *such that, up to subsequences,*

$$w^h(t) \to w(t) \quad \text{strongly in } H^1(\Omega;\mathbb{R}^3), \tag{3.5.12}$$

$$u^h(t) \to u(t) \quad \text{strongly in } L^1(\Omega;\mathbb{R}^3), \tag{3.5.13}$$

$$\mathrm{sym}(R_h D u^h(t) R_h F_h^{-1})_{\alpha\beta} \rightharpoonup (\bar{E}u(t))_{\alpha\beta} \quad \text{weakly* in } M_b(\Omega), \tag{3.5.14}$$

$$e^h(t) \to \mathbb{M}e(t) \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}_{sym}), \tag{3.5.15}$$

$$p^h_{\alpha\beta}(t) \rightharpoonup p_{\alpha\beta}(t) \quad \text{weakly* in } M_b(\Omega \cup \partial_d\Omega), \tag{3.5.16}$$

as $h \to 0$, *for every* $t \in [0,T]$.

Remark 3.5.4. Given a boundary datum w^h and a triplet $(u_0^h, e_0^h, p_0^h) \in \mathcal{A}_h(\Omega, w^h(0))$ satisfying (3.5.9), the existence of an h-quasistatic evolution $t \to (u^h(t), e^h(t), p^h(t))$ with boundary datum w^h and initial condition $(u^h(0), e^h(0), p^h(0)) = (u_0^h, e_0^h, p_0^h)$ follows from [12, Theorem 4.5]. In [12] this result is proven for $\partial\Omega$ of class C^2, but, as observed in [22], Lipschitz regularity of the boundary is sufficient in absence of external forces. Furthermore, since the problem is rate-independent, one can always assume the data to be Lipschitz continuous in time (and not only absolutely continuous), up to a time scaling.

For the proof of Theorem 3.5.3 we will need some preliminary results. The first one is a characterisation of the global stability condition (qs1)* of the reduced problem.

Lemma 3.5.5. *Let* $w \in H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ *and let* $(u, e, p) \in \mathcal{A}_{GKL}(w)$. *The following conditions are equivalent:*

(a) $\mathcal{Q}^*(e) \leq \mathcal{Q}^*(\eta) + \mathcal{H}^*(q - p)$ *for every* $(v, \eta, q) \in \mathcal{A}_{GKL}(w)$;

(b) $-\mathcal{H}^*(q) \leq \int_\Omega \mathbb{C}^* e : \eta \, dx$ *for every* $(v, \eta, q) \in \mathcal{A}_{GKL}(0)$.

Proof. Assume (a) and let $(v, \eta, q) \in \mathcal{A}_{GKL}(0)$. For every $\varepsilon > 0$ we have that $(u + \varepsilon v, e + \varepsilon\eta, p + \varepsilon q) \in \mathcal{A}_{GKL}(w)$, therefore

$$\mathcal{Q}^*(e) \leq \mathcal{Q}^*(e + \varepsilon\eta) + \mathcal{H}^*(\varepsilon q).$$

Using the positive homogeneity of H^*, dividing by ε and sending ε to 0, we get (b). Conversely, (b) implies (a) by convexity of $\mathcal{Q}^*$ and $\mathcal{H}^*$. □

Arguing in the same way as in the previous lemma, one can prove the following characterisation of the global stability condition (qs1) of the h-quasistatic evolution problem.

Lemma 3.5.6. *Let $0 < h \ll 1$, let $w \in H^1(\Omega;\mathbb{R}^3)$, and let $(u,e,p) \in \mathcal{A}_h(\Omega,w)$. The following conditions are equivalent:*

(a) $\displaystyle\int_\Omega Q(e)\det F_h\,dx \le \int_\Omega Q(\eta)\det F_h\,dx + \mathcal{H}_h(q-p)$ *for every* $(v,\eta,q) \in \mathcal{A}_h(\Omega,w)$*;*

(b) $\displaystyle -\mathcal{H}_h(q) \le \int_\Omega \mathbb{C}e : \eta \det F_h\,dx$ *for every* $(v,\eta,q) \in \mathcal{A}_h(\Omega,0)$*.*

The next lemma concerns a variant of the Gronwall inequality.

Lemma 3.5.7. *Let $\phi,\psi : [0,T] \to [0,+\infty)$ be such that $\phi \in L^\infty(0,T)$ and $\psi \in L^1(0,T)$. Assume that*

$$\phi(t)^2 \le \int_0^t \phi(s)\psi(s)\,ds$$

for every $t \in [0,T]$. Then

$$\phi(t) \le \frac{1}{2}\int_0^t \psi(s)\,ds$$

for every $t \in [0,T]$.

Proof. We define

$$F(t) := \int_0^t \phi(s)\psi(s)\,ds$$

for every $t \in [0,T]$. Thus, $F \in AC([0,T])$ and by assumption $\phi(t)^2 \le F(t)$ for every $t \in [0,T]$. Therefore,

$$F'(t) = \phi(t)\psi(t) \le F(t)^{1/2}\psi(t)$$

for a.e. $t \in [0,T]$. This leads to

$$F(t)^{1/2} \le \frac{1}{2}\int_0^t \psi(s)\,ds$$

for every $t \in [0,T]$, which implies the thesis by using the assumption again. □

We have now all the ingredients to prove Theorem 3.5.3.

Proof of Theorem 3.5.3. The proof is split into five steps.

Step 0: Convergence of w^h. Hypothesis (3.5.6) and estimate (3.4.17) ensure that the sequences $(w^h(t))$ and $(\dot w^h(t))$ are uniformly bounded with respect to h in $H^1(\Omega;\mathbb{R}^3)$, with a constant independent of $t \in [0,T]$, that is,

$$\|w^h\|_{W^{1,\infty}([0,T];H^1)} \le C$$

for every $0 < h \ll 1$. By the Ascoli-Arzelà Theorem there exist $w \in \mathrm{Lip}([0,T];H^1(\Omega;\mathbb{R}^3))$ and a subsequence (w^h), not relabeled, such that

$$w^h(t) \rightharpoonup w(t) \quad \text{weakly in } H^1(\Omega;\mathbb{R}^3),$$

for every $t \in [0,T]$. Arguing as in Step 0 of the proof of Theorem 3.4.3, we infer that $w(t) \in KL(\Omega)$ and the above convergence is strong, namely (3.5.12) holds. Moreover,

$$\operatorname{sym}(R_h Dw^h(t) R_h F_h^{-1})_{\alpha\beta} \to (E^* w(t))_{\alpha\beta} \quad \text{strongly in } L^2(\Omega) \tag{3.5.17}$$

for every $t \in [0,T]$. In particular, by (3.5.7) we have that

$$z_{\alpha\beta}(t) = (E^* w(t))_{\alpha\beta}. \tag{3.5.18}$$

Step 1: Compactness estimates. We claim that there exists $C > 0$, independent of h, such that

$$\|e^h(t_2) - e^h(t_1)\|_{L^2} \leq C|t_2 - t_1| \, \| \operatorname{sym}(R_h D\dot{w}^h R_h F_h^{-1})\|_{L^\infty([0,T];L^2)} \tag{3.5.19}$$

$$\|p^h(t_2) - p^h(t_1)\|_{M_b} \leq C|t_2 - t_1| \, \| \operatorname{sym}(R_h D\dot{w}^h R_h F_h^{-1})\|_{L^\infty([0,T];L^2)} \tag{3.5.20}$$

for every $t_1, t_2 \in [0,T]$ and every $0 < h \ll 1$.

From (3.5.2), (3.2.5), (3.2.8), Lemma 3.2.1, and the Hölder inequality it follows that

$$\begin{aligned}(\alpha_{\mathbb{C}} + O(h^2))\|e^h(t)\|_{L^2}^2 + (r_K + O(h^2))\|p^h(t) - p_0^h\|_{M_b} \\ \leq (\beta_{\mathbb{C}} + O(h^2)) \int_0^t \|e^h(s)\|_{L^2} \| \operatorname{sym}(R_h D\dot{w}^h(s) R_h F_h^{-1})\|_{L^2} \, ds + (\beta_{\mathbb{C}} + O(h^2))\|e_0^h\|_{L^2}^2.\end{aligned}$$

Owing to (3.5.6), (3.5.10), (3.5.11), and the Cauchy inequality, we deduce that

$$\sup_{t\in[0,T]} \|e^h(t)\|_{L^2} + \sup_{t\in[0,T]} \|p^h(t)\|_{M_b} \leq C \tag{3.5.21}$$

for every h sufficiently small.

We now use condition (qs1) at time t_1. Let

$$\begin{aligned}&v = u^h(t_2) - u^h(t_1) - w^h(t_2) + w^h(t_1),\\ &\eta = e^h(t_2) - e^h(t_1) - \operatorname{sym}(R_h Dw^h(t_2) R_h F_h^{-1}) + \operatorname{sym}(R_h Dw^h(t_1) R_h F_h^{-1}),\\ &q = p^h(t_2) - p^h(t_1).\end{aligned}$$

Since $(v, \eta, q) \in \mathcal{A}_h(\Omega, 0)$, by Lemma 3.5.6 we have that

$$\begin{aligned}&-\int_\Omega \mathbb{C}e^h(t_1) : \Big(e^h(t_2) - e^h(t_1)\Big) \det F_h \, dx \\ &\qquad + \int_\Omega \mathbb{C}e^h(t_1) : \Big(\operatorname{sym}(R_h Dw^h(t_2) R_h F_h^{-1}) - \operatorname{sym}(R_h Dw^h(t_1) R_h F_h^{-1})\Big) \det F_h \, dx \\ &\leq \mathcal{H}_h(p^h(t_2) - p^h(t_1)) \leq \mathcal{D}_h(p^h; t_1, t_2),\end{aligned}$$

where the last inequality is an immediate consequence of the definition of $\mathcal{D}_h$. Using the previous inequality in the energy balance (3.5.2) written at times t_1 and t_2, we get

$$\begin{aligned}\int_\Omega Q(e^h(t_2)) \det F_h \, dx - \int_\Omega Q(e^h(t_1)) \det F_h \, dx - \int_\Omega \mathbb{C}e^h(t_1) : \Big(e^h(t_2) - e^h(t_1)\Big) \det F_h \, dx \\ \leq \int_{t_1}^{t_2} \int_\Omega \mathbb{C}\left(e^h(s) - e^h(t_1)\right) : \operatorname{sym}(R_h D\dot{w}^h(s) R_h F_h^{-1}) \det F_h \, dx \, ds.\end{aligned}$$

We observe that the left-hand side of the previous inequality is exactly

$$\int_\Omega Q(e^h(t_2) - e^h(t_1)) \det F_h \, dx.$$

Thus, from (3.2.5), (3.2.6), Lemma 3.2.1, and the Hölder inequality it follows that

$$\begin{aligned}(\alpha_{\mathbb{C}} + O(h^2))&\|e^h(t_2) - e^h(t_1)\|^2_{L^2} \\ &\leq (2\beta_{\mathbb{C}} + O(h^2)) \int_{t_1}^{t_2} \|e^h(s) - e^h(t_1)\|_{L^2} \| \operatorname{sym}(R_h D\dot{w}^h(s) R_h F_h^{-1})\|_{L^2} \, ds.\end{aligned}$$

By Lemma 3.5.7 we deduce that

$$\|e^h(t_2) - e^h(t_1)\|_{L^2} \leq C \int_{t_1}^{t_2} \| \operatorname{sym}(R_h D\dot{w}^h(s) R_h F_h^{-1})\|_{L^2} \, ds,$$

hence (3.5.19).

Using again the energy balance (3.5.2) at times t_1 and t_2, together with (3.2.8) and Lemma 3.2.1, we obtain

$$\begin{aligned}(r_K &+ O(h^2))\|p^h(t_2) - p^h(t_1)\|_{M_b} \\ \leq\ & \int_\Omega Q(e^h(t_1)) \det F_h \, dx - \int_\Omega Q(e^h(t_2)) \det F_h \, dx \\ &+ \int_{t_1}^{t_2} \int_\Omega \mathbb{C} e^h(s) : \operatorname{sym}(R_h D\dot{w}^h(s) R_h F_h^{-1}) \det F_h \, dx \, ds \\ \leq\ & C \sup_{t\in[0,T]} \|e^h(t)\|_{L^2} \Big(\int_{t_1}^{t_2} \| \operatorname{sym}(R_h D\dot{w}^h(s) R_h F_h^{-1})\|_{L^2} \, ds + \|e^h(t_2) - e^h(t_1)\|_{L^2} \Big) \\ \leq\ & C|t_2 - t_1| \, \| \operatorname{sym}(R_h D\dot{w}^h R_h F_h^{-1})\|_{L^\infty([0,T];L^2)},\end{aligned}$$

where the last inequality follows from (3.5.21) and (3.5.19), and $C > 0$ is a constant independent of h. This proves (3.5.20) and concludes Step 1.

Step 2: Reduced kinematic admissibility. By (3.5.6), (3.5.19), and (3.5.20) we can apply the Ascoli-Arzelà Theorem to the sequences (e^h) and (p^h) and deduce the existence of $\tilde{e} \in \mathrm{Lip}([0,T]; L^2(\Omega; \mathbb{M}^{3\times3}_{sym}))$ and $\tilde{p} \in \mathrm{Lip}([0,T]; M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times3}_D))$ such that, up to subsequences,

$$e^h(t) \rightharpoonup \tilde{e}(t) \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{3\times3}_{sym}), \tag{3.5.22}$$

$$p^h(t) \rightharpoonup \tilde{p}(t) \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times3}_D) \tag{3.5.23}$$

for every $t \in [0,T]$. We introduce $e \in \mathrm{Lip}([0,T]; L^2(\Omega; \mathbb{M}^{3\times3}_{sym}))$ and $p \in \mathrm{Lip}([0,T]; M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times3}_{sym}))$ defined by $e_{\alpha\beta}(t) := \tilde{e}_{\alpha\beta}(t)$, $e_{i3}(t) := 0$ for every $t \in [0,T]$, and $p_{\alpha\beta}(t) := \tilde{p}_{\alpha\beta}(t)$, $p_{i3}(t) := 0$ for every $t \in [0,T]$, respectively.

Since $(u^h(t), e^h(t), p^h(t)) \in \mathcal{A}_h(\Omega; w^h(t))$, and owing to (3.5.6) and (3.5.21), we can apply Lemma 3.4.1 and infer that for every $t \in [0,T]$ there exists $u(t) \in KL(\Omega)$ and a subsequence $u^{h_j}(t)$, possibly depending on t, such that

$$u^{h_j}(t) \to u(t) \quad \text{strongly in } L^1(\Omega; \mathbb{R}^3), \tag{3.5.24}$$

$$\operatorname{sym}(R_h D u^{h_j}(t) R_h F_h^{-1}))_{\alpha\beta} \rightharpoonup (\bar{E}u(t))_{\alpha\beta} \quad \text{weakly}^* \text{ in } M_b(\Omega). \tag{3.5.25}$$

Furthermore, arguing as in Step 1 of the proof of Theorem 3.4.3, and using (3.5.22) and (3.5.23), we infer that $(u(t), e(t), p(t)) \in \mathcal{A}_{GKL}(w(t))$. Now we want to prove that $u(t)$ is uniquely determined. Assume that there exist $t \in [0, T]$ and two subsequences $(u^{h_j}(t))$ and $(u^{h'_j}(t))$ with two limits $u_1(t)$ and $u_2(t)$. We set $z(t) := u_1(t) - u_2(t)$. Since

$$(u_1(t), e(t), p(t)), (u_2(t), e(t), p(t)) \in \mathcal{A}_{GKL}(w(t)),$$

we have that $z(t) \in KL(\Omega)$ and

$$E^* z(t) = 0 \text{ in } \Omega, \quad z(t) = 0 \text{ on } \partial_d \Omega.$$

Hence, we have

$$\operatorname{sym} D\bar{z}(t) + \nabla z_3(t) \odot \nabla\theta = x_3 D^2 z_3(t) \quad \text{ in } \Omega. \tag{3.5.26}$$

Thus, $D^2 z_3(t) = 0$ in Ω and the boundary condition $\bar{z}(t) - x_3 \nabla z_3(t) = 0$ on $\partial_d\Omega$ gives $\nabla z_3(t) = 0$ on $\partial_d\omega$ and $z_3(t) = 0$ on $\partial_d\omega$. By (1.2.3) we deduce that $z_3(t) = 0$ in ω. Hence, $\operatorname{sym} D\bar{z}(t) = 0$ in ω by (3.5.26) and, in turn, $\operatorname{sym} Dz(t) = 0$ in Ω. Since $z(t) = 0$ on $\partial_d\Omega$, it follows from (1.2.2) that $z(t) = 0$ in Ω. This proves that $u(t)$ is uniquely determined, hence convergences (3.5.24) and (3.5.25) hold for the whole sequence. Thus, (3.5.13) and (3.5.14) are proved.

It remains to check that $u \in \operatorname{Lip}([0, T]; BD(\Omega))$. Since e, p, and w are Lipschitz continuous, by kinematic admissibility we infer that

$$(u, \bar{E}u) \in \operatorname{Lip}([0, T]; L^1(\partial_d\Omega; \mathbb{R}^3) \times M_b(\Omega; \mathbb{M}^{2\times 2}_{sym})).$$

Now let us consider the first order moments of u and $\bar{E}u$. One can prove that

$$\|\hat{E}^* u(t)\|_{M_b} \leq C\|\bar{E}u(t)\|_{M_b}, \quad \|\hat{u}(t)\|_{L^1} \leq C\|u(t)\|_{L^1},$$

with $C > 0$. These estimates, together with the relations $\hat{u}_\alpha(t) = -\partial_\alpha u_3(t)$ and $\hat{E}^* u(t) = -D^2 u_3(t)$, imply that

$$(u_3, \nabla u_3, D^2 u_3) \in \operatorname{Lip}([0, T]; L^1(\partial_d\Omega) \times L^1(\partial_d\Omega; \mathbb{R}^2) \times M_b(\Omega; \mathbb{M}^{2\times 2}_{sym}))$$

and, in turn, owing to (1.2.3), that $u_3 \in \operatorname{Lip}([0, T]; BH(\omega))$. It follows now from (3.5.26) that $\operatorname{sym} Du \in \operatorname{Lip}([0, T]; BD(\omega))$. Therefore it is a consequence of (1.2.2) that

$$u \in \operatorname{Lip}([0, T]; BD(\Omega)).$$

The previous arguments, together with (3.5.10) and (3.5.11), also prove that, up to subsequences,

$$\begin{aligned} u_0^h &\to u_0 \quad \text{strongly in } L^1(\Omega; \mathbb{R}^3), \\ (\operatorname{sym}(R_h D u_0^h R_h F_h^{-1})_{\alpha\beta} &\rightharpoonup (\bar{E}u_0)_{\alpha\beta} \quad \text{weakly* in } M_b(\Omega), \\ (e_0^h)_{\alpha\beta} &\to (e_0)_{\alpha\beta} \quad \text{strongly in } L^2(\Omega), \\ (p_0^h)_{\alpha\beta} &\rightharpoonup (p_0)_{\alpha\beta} \quad \text{weakly* in } M_b(\Omega), \end{aligned}$$

for some $(u_0, e_0, p_0) \in \mathcal{A}_{KL}(w(0))$. Since $(u^h(0), e^h(0), p^h(0)) = (u_0^h, e_0^h, p_0^h)$, we have that $(u(0), e(0), p(0)) = (u_0, e_0, p_0)$.

Step 3: Reduced global stability. We prove (3.5.3). Let $t \in [0, T]$. By Lemma 3.5.5 condition (3.5.3) at time t is equivalent to

$$-\mathcal{H}^*(q) \leq \int_\Omega \mathbb{C}^* e(t) : \eta \, dx \quad \text{ for every } (v, \eta, q) \in \mathcal{A}_{GKL}(0). \tag{3.5.27}$$

Let $(v,\eta,q)\in\mathcal{A}_{GKL}(0)$. By Step 2 in the proof of Theorem 3.4.3 there exists a sequence $(v^h,\eta^h,q^h)\in\mathcal{A}_h(\Omega,0)$ such that

$$\eta^h\to\mathbb{M}\eta\quad\text{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{sym}),\tag{3.5.28}$$

$$\mathcal{H}_h(q^h)\to\mathcal{H}^*(q).\tag{3.5.29}$$

By Lemma 3.5.6 and (3.5.1) at time t we have that

$$-\mathcal{H}_h(q^h)\le\int_\Omega\mathbb{C}e^h(t):\eta^h\det F_h\,dx$$

for every $0<h\ll 1$. By (3.5.22), (3.5.28), and (3.5.29) we can pass to the limit in the previous estimate, as h tends to 0, and deduce that

$$-\mathcal{H}^*(q)\le\int_\Omega\mathbb{C}\tilde{e}(t):\mathbb{M}\eta\,dx\qquad\text{for every } (v,\eta,q)\in\mathcal{A}_{GKL}(0).$$

Since $\mathbb{C}\tilde{e}(t):\mathbb{M}\eta=\mathbb{C}\mathbb{M}e(t):\mathbb{M}\eta=\mathbb{C}^*e(t):\eta$ by (3.2.24), this inequality reduces to (3.5.27).

Step 4: Identification of the limiting elastic strain. We now prove that $\tilde{e}(t)=\mathbb{M}e(t)$ for every $t\in[0,T]$.

Let $t\in[0,T]$. For every $\psi\in H^1(\Omega;\mathbb{R}^3)$ with $\psi=0$ on $\partial_d\Omega$ we consider the triplets $(\pm\psi,\pm\,\mathrm{sym}(R_hD\psi R_hF_h^{-1}),0)$ as test functions in condition (b) of Lemma 3.5.5 at time t. This leads to

$$\int_\Omega\mathbb{C}e^h(t):\mathrm{sym}(R_hD\psi R_hF_h^{-1})\det F_h\,dx=0$$

for every $0<h\ll 1$. Let $(a,b)\subset(-\frac{1}{2},\frac{1}{2})$ and let $U\subset\omega$ be an open set. Let $(\ell_n)\subset C^1([-\frac{1}{2},\frac{1}{2}])$ and $(\lambda_n^i)\subset C^1_c(\omega)$ be sequences such that $\ell_n'\to\chi_{(a,b)}$ strongly in $L^4(-\frac{1}{2},\frac{1}{2})$ and $\lambda_n^i\to\lambda_i\chi_U$ strongly in $L^4(\omega)$ for every $i=1,2,3$, as $n\to\infty$.

We define

$$\phi_n^h(x):=\begin{pmatrix}2h\lambda_n^1(x')\ell_n(x_3)\\2h\lambda_n^2(x')\ell_n(x_3)\\h^2\lambda_n^3(x')\ell_n(x_3)\end{pmatrix}.\tag{3.5.30}$$

Since $\phi_n^h\in H^1(\Omega;\mathbb{R}^3)$ and $\phi_n^h=0$ on $\partial_d\Omega$, we have

$$\int_\Omega\mathbb{C}e^h(t):\mathrm{sym}(R_hD\phi_n^hR_hF_h^{-1})\det F_h\,dx=0.\tag{3.5.31}$$

Using that $F_h^{-1}=I_{3\times 3}+O(h)$ by Lemma 3.2.1, we obtain that

$$\mathrm{sym}(R_hD\phi_n^hR_hF_h^{-1})_{\alpha\beta}=O(h),\qquad\mathrm{sym}(R_hD\phi_n^hR_hF_h^{-1})_{i3}=\lambda_n^i\ell_n'+O(h).$$

These expansions, together with (3.5.22) and the uniform convergence of $\det F_h$ to 1, allow us to pass to the limit in (3.5.31), first as $h\to 0$, and then, as $n\to\infty$. This yields

$$\int_{U\times(a,b)}\mathbb{C}\tilde{e}(t):\begin{pmatrix}0&0&\lambda_1\\0&0&\lambda_2\\\lambda_1&\lambda_2&\lambda_3\end{pmatrix}dx=0.$$

Since the sets (a,b) and U are arbitrary, we conclude from (3.2.20) that $\tilde{e}(t)=\mathbb{M}e(t)$ a.e. in Ω. In particular, we have that $\tilde{e}_0=\mathbb{M}e_0$, where $\tilde{e}_0$ is the limit in (3.5.10).

Step 5: Reduced energy balance. The lower semicontinuity of $\mathcal{Q}^*$ and $\mathcal{D}^*$, together with (3.5.22) and (3.5.23), imply that

$$\begin{gathered}\mathcal{Q}^*(e(t)) \leq \liminf_{h\to 0} \int_\Omega Q(e^h(t)) \det F_h \, dx, \\ \mathcal{D}^*(p;0,t) \leq \liminf_{h\to 0} \mathcal{D}_h(p^h;0,t)\end{gathered} \tag{3.5.32}$$

for every $t \in [0,T]$. Passing to the limit in the energy balance (3.5.2) yields

$$\begin{aligned}&\mathcal{Q}^*(e(t)) + \mathcal{D}^*(p;0,t) \\ &\quad \leq \limsup_{h\to 0} \Big\{ \int_\Omega Q(e^h(0)) \det F_h \, dx + \int_0^t \int_\Omega \mathbb{C}e^h(s) : \mathrm{sym}(R_h D\dot{w}^h(s) R_h F_h^{-1}) \det F_h \, dx \, ds \Big\} \\ &\quad = \int_\Omega Q(\tilde{e}_0)\, dx + \int_0^t \int_\Omega \mathbb{C}\tilde{e}(s) : \dot{z}(s) \, dx \, ds,\end{aligned}$$

where the second equality is a consequence of (3.5.8), (3.5.6), (3.5.10), (3.5.21), and the Dominated Convergence Theorem. By Step 4 and (3.5.18) we conclude that

$$\mathcal{Q}^*(e(t)) + \mathcal{D}^*(p;0,t) \leq \mathcal{Q}^*(e_0) + \int_0^t \int_\Omega \mathbb{C}^* e(s) : E^* \dot{w}(s) \, dx \, ds.$$

As it is standard in the variational theory for rate-independent processes, the converse energy inequality follows from the minimality condition (qs1*) (see, e.g., [39, Theorem 4.4] or [12, Theorem 4.7]). We have thus proved that $t \mapsto (u(t), e(t), p(t))$ is a reduced quasistatic evolution.

To conclude the proof it remains to show the strong convergence of $e^h(t)$ to $\mathbb{M}e(t)$ for every $t \in [0,T]$. Since we have showed that the right-hand side of (3.5.2) converges to the right-hand side of (3.5.4), we have that

$$\lim_{h\to 0} \Big\{ \int_\Omega Q(e^h(t)) \det F_h \, dx + \mathcal{D}_h(p^h;0,t) \Big\} = \mathcal{Q}^*(e(t)) + \mathcal{D}^*(p;0,t)$$

for every $t \in [0,T]$. Thus, by (3.5.32) and Lemma 3.2.1 we deduce that

$$\mathcal{Q}^*(e(t)) = \lim_{h\to 0} \int_\Omega Q(e^h(t)) \det F_h \, dx = \lim_{h\to 0} \int_\Omega Q(e^h(t)) \, dx$$

Since

$$\mathcal{Q}^*(e(t)) = \int_\Omega Q(\mathbb{M}e(t)) \, dx,$$

convergence (3.5.15) follows from (3.5.22), Step 4, and the coercivity (3.2.5) of Q. The proof of Theorem 3.5.3 is concluded. □

3.5.1 Characterisation of reduced quasistatic evolutions in rate form

We conclude this section with a characterisation of reduced quasistatic evolutions.

Stress-strain duality

In the framework of the reduced problem we introduce a notion of duality between elastic stresses and plastic strains. Here we follow [13, Section 7].

We define the set $\Sigma(\Omega)$ of *admissible stresses* as

$$\Sigma(\Omega) := \{\sigma \in L^\infty(\Omega;\mathbb{M}^{2\times 2}_{sym}) : \ \operatorname{div}\bar\sigma \in L^2(\omega;\mathbb{R}^2),\ \operatorname{div}\operatorname{div}\hat\sigma \in L^2(\omega)\}.$$

For every $\sigma \in \Sigma(\Omega)$ we can define the trace $[\bar\sigma\nu_{\partial\omega}] \in L^\infty(\partial\omega;\mathbb{R}^2)$ of its zeroth order moment normal component as

$$\langle[\bar\sigma\nu_{\partial\omega}],\psi\rangle := \int_\omega \bar\sigma : \operatorname{sym} D\psi\, dx' + \int_\omega \operatorname{div}\bar\sigma\cdot\psi\, dx' \tag{3.5.33}$$

for every $\psi \in W^{1,1}(\omega;\mathbb{R}^2)$. Note that, since $\bar\sigma \in L^\infty(\omega;\mathbb{M}^{2\times 2}_{sym})$ and $W^{1,1}(\omega;\mathbb{R}^2)$ embeds into $L^2(\omega;\mathbb{R}^2)$, all terms on the right-hand side of (3.5.33) are well defined.

Let $T(W^{2,1}(\omega))$ be the space of traces of functions in $W^{2,1}(\omega)$ and let $(T(W^{2,1}(\omega)))'$ be its dual space. For every $\sigma \in \Sigma(\Omega)$ we can define the traces $b_0(\hat\sigma) \in (T(W^{2,1}(\omega)))'$ and $b_1(\hat\sigma) \in L^\infty(\partial\omega)$ of its first order moment as

$$-\langle b_0(\hat\sigma),\psi\rangle + \left\langle b_1(\hat\sigma),\frac{\partial\psi}{\partial\nu_{\partial\omega}}\right\rangle := \int_\omega \hat\sigma : D^2\psi\, dx' - \int_\omega \psi\operatorname{div}\operatorname{div}\hat\sigma\, dx' \tag{3.5.34}$$

for every $\psi \in W^{2,1}(\omega)$. Note that the right-hand side of (3.5.34) is well defined since $\hat\sigma \in L^\infty(\omega;\mathbb{M}^{2\times 2}_{sym})$. If $\hat\sigma \in C^2(\overline{\omega},\mathbb{M}^{2\times 2}_{sym})$, one can prove that

$$\begin{aligned} b_0(\hat\sigma) &= \operatorname{div}\hat\sigma\cdot\nu_{\partial\omega} + \frac{\partial}{\partial\tau_{\partial\omega}}\left(\hat\sigma\tau_{\partial\omega}\cdot\nu_{\partial\omega}\right),\\ b_1(\hat\sigma) &= \hat\sigma\nu_{\partial\omega}\cdot\nu_{\partial\omega}, \end{aligned}$$

where $\tau_{\partial\omega}$ is a unit tangent vector to $\partial\omega$ (see [16, Théorème 2.3]).

Let $(h,m_0,m_1) \in L^\infty(\partial\omega;\mathbb{R}^2)\times T(W^{2,1}(\omega)))'\times L^\infty(\partial\omega)$. Since $[\bar\sigma\nu_{\partial\omega}] \in L^\infty(\partial\omega;\mathbb{R}^2)$, the expressions $[\bar\sigma\nu_{\partial\omega}] = h$ on $\partial_n\omega$ and $b_1(\hat\sigma) = m_1$ on $\partial_n\omega$ have a clear meaning. As for $b_0(\hat\sigma)$, we say that $b_0(\hat\sigma) = m_0$ on $\partial_n\omega$ if $\langle b_0(\hat\sigma) - m_0,\psi\rangle = 0$ for every $\psi \in W^{2,1}(\omega)$ with $\psi = 0$ on $\partial_d\omega$.

We also define the space of *admissible plastic strains* $\Pi_{\partial_d\Omega}(\Omega)$ as the set of all measures $p \in M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{2\times 2}_{sym})$ for which there exists $(u,e,w) \in BD(\Omega)\times L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})\times(H^1(\Omega;\mathbb{R}^3)\cap KL(\Omega))$ such that $(u,e,p) \in \mathcal{A}_{GKL}(w)$.

For every $\sigma \in \Sigma(\Omega)$ and $\xi \in BD(\omega)$ we define the distribution $[\bar\sigma : \operatorname{sym} D\xi]$ on ω as

$$\langle[\bar\sigma : \operatorname{sym} D\xi],\varphi\rangle := -\int_\omega \varphi\operatorname{div}\bar\sigma\cdot\xi\, dx' - \int_\omega \bar\sigma : (\nabla\varphi\odot\xi)\, dx'$$

for every $\varphi \in C^\infty_c(\omega)$. It follows from [30, Theorem 3.2] that $[\bar\sigma : \operatorname{sym} D\xi] \in M_b(\omega)$ and its variation satisfies

$$|[\bar\sigma : \operatorname{sym} D\xi]| \le \|\bar\sigma\|_{L^\infty}|\operatorname{sym} D\xi| \quad \text{in } \omega.$$

Given $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\partial_d\Omega}(\Omega)$, we define the measure $[\bar\sigma : \bar p] \in M_b(\omega\cup\partial_d\omega)$ as

$$[\bar\sigma : \bar p] := \begin{cases} [\bar\sigma : \operatorname{sym} D\bar u] + \bar\sigma : (\nabla\theta\odot\nabla u_3) - \bar\sigma : \bar e & \text{in } \omega,\\ [\bar\sigma\nu_{\partial\omega}]\cdot(\bar w - \bar u)\,\mathcal{H}^1 & \text{on } \partial_d\omega, \end{cases} \tag{3.5.35}$$

where $(u,e,w) \in BD(\Omega) \times L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}) \times (H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega))$ are such that $(u,e,p) \in \mathcal{A}_{GKL}(w)$. Note that since $\nabla u_3 \in BV(\omega;\mathbb{R}^2)$ and $BV(\omega;\mathbb{R}^2)$ embeds into $L^2(\omega;\mathbb{R}^2)$, the term $\bar\sigma : (\nabla\theta \odot \nabla u_3)$ is in $L^1(\Omega)$. Moreover, definition (3.5.35) is independent of the choice of (u,e,w).

For every $\sigma \in \Sigma(\Omega)$ and $v \in BH(\omega)$ we define the distribution $[\hat\sigma : D^2 v]$ on ω as

$$\langle [\hat\sigma : D^2 v], \psi\rangle := \int_\omega \psi v \operatorname{div}\operatorname{div}\hat\sigma \, dx' - 2\int_\omega \hat\sigma : (\nabla v \odot \nabla\psi)\, dx' - \int_\omega v\hat\sigma : D^2\psi\, dx'$$

for every $\psi \in C_c^\infty(\omega)$. From [19, Proposition 2.1] it follows that $[\hat\sigma : D^2 v] \in M_b(\omega)$ and its variation satisfies

$$|[\hat\sigma : D^2 v]| \le \|\hat\sigma\|_{L^\infty} |D^2 v| \quad \text{in } \omega.$$

Given $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\partial_d\Omega}(\Omega)$, we define the measure $[\hat\sigma : \hat p] \in M_b(\omega \cup \partial_d\omega)$ as

$$[\hat\sigma : \hat p] := \begin{cases} -[\hat\sigma : D^2 u_3] - \hat\sigma : \hat e & \text{in } \omega, \\ b_1(\hat\sigma)\dfrac{\partial(u_3 - w_3)}{\partial\nu_{\partial\omega}}\mathcal{H}^1 & \text{on } \partial_d\omega, \end{cases} \tag{3.5.36}$$

where $(u,e,w) \in BD(\Omega) \times L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}) \times (H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega))$ are such that $(u,e,p) \in \mathcal{A}_{GKL}(w)$. Note that definition (3.5.36) is independent of the choice of (u,e,w).

We are now in a position to define the duality between $\Sigma(\Omega)$ and $\Pi_{\partial_d\Omega}(\Omega)$. For every $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\partial_d\Omega}(\Omega)$ we define the measure $[\sigma : p]^* \in M_b(\Omega \cup \partial_d\Omega)$ as

$$[\sigma : p]^* := [\bar\sigma : \bar p] \otimes \mathcal{L}^1 + \frac{1}{12}[\hat\sigma : \hat p] \otimes \mathcal{L}^1 - \sigma_\perp : e_\perp.$$

We also introduce the duality pairings

$$\langle \bar\sigma, \bar p\rangle := [\bar\sigma : \bar p](\omega \cup \partial_d\omega), \qquad \langle \hat\sigma, \hat p\rangle := [\hat\sigma : \hat p](\omega \cup \partial_d\omega)$$

and

$$\langle \sigma, p\rangle^* := [\sigma : p]^*(\Omega \cup \partial_d\Omega) = \langle \bar\sigma, \bar p\rangle + \frac{1}{12}\langle \hat\sigma, \hat p\rangle - \int_\Omega \sigma_\perp : e_\perp \, dx. \tag{3.5.37}$$

The next two results concern some useful properties of the stress-strain duality. We first show that the duality satisfies an integration by parts formula.

Proposition 3.5.8. *Let $\sigma \in \Sigma(\Omega)$, $w \in H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega)$, and $(u,e,p) \in \mathcal{A}_{GKL}(w)$. Then*

$$\begin{aligned}
&\int_{\Omega\cup\partial_d\Omega} \varphi \, d[\sigma : p]^* + \int_\Omega \varphi\sigma : (e - E^*w)\, dx \\
= \; & -\int_\omega \bar\sigma : (\nabla\varphi \odot (\bar u - \bar w))\, dx' - \int_\omega \operatorname{div}\bar\sigma \cdot \varphi(\bar u - \bar w)\, dx' + \int_{\partial_n\omega} [\bar\sigma\nu_{\partial\omega}] \cdot \varphi(\bar u - \bar w)\, d\mathcal{H}^1 \\
& + \frac{1}{12}\int_\omega \hat\sigma : (u_3 - w_3) D^2\varphi \, dx' + \frac{1}{6}\int_\omega \hat\sigma : (\nabla\varphi \odot (\nabla u_3 - \nabla w_3))\, dx' \\
& - \int_\omega \varphi(u_3 - w_3)\left(\frac{1}{12}\operatorname{div}\operatorname{div}\hat\sigma + \bar\sigma : D^2\theta + \operatorname{div}\bar\sigma \cdot \nabla\theta\right) dx' \\
& - \int_\omega (u_3 - w_3)\bar\sigma : (\nabla\varphi \odot \nabla\theta)\, dx' + \int_{\partial_n\omega} \varphi(u_3 - w_3)[\bar\sigma\nu_{\partial\omega}] \cdot \nabla\theta \, d\mathcal{H}^1 \\
& + \frac{1}{12}\langle b_0(\hat\sigma), \varphi(u_3 - w_3)\rangle - \frac{1}{12}\int_{\partial_n\omega} b_1(\hat\sigma)\frac{\partial(\varphi(u_3 - w_3))}{\partial\nu_{\partial\omega}}\, d\mathcal{H}^1
\end{aligned}$$

for every $\varphi \in C^2(\overline{\omega})$.

Proof. The proof follows from [14, Proposition 4] by observing that

$$\int_{\Omega\cup\partial_d\Omega} \varphi\, d[\sigma : p]^* = \int_{\Omega\cup\partial_d\Omega} \varphi\, d[\sigma : (p - \nabla\theta \odot \nabla u_3)]_r + \int_\omega \varphi\bar{\sigma} : (\nabla\theta \odot \nabla w_3)\, dx' + \int_\omega \varphi\bar{\sigma} : (\nabla\theta \odot \nabla(u_3 - w_3))\, dx',$$

where $[\sigma : p]_r$ is the notion of duality introduced in [13,14]. Moreover, by (3.5.33) we have

$$\begin{aligned} &\int_\omega \varphi\bar{\sigma} : (\nabla\theta \odot \nabla(u_3 - w_3))\, dx' \\ &= \int_\omega \bar{\sigma} : \operatorname{sym} D(\varphi(u_3 - w_3)\nabla\theta)\, dx' - \int_\omega (u_3 - w_3)\bar{\sigma} : (\nabla\varphi \odot \nabla\theta)\, dx' \\ &\quad - \int_\omega \varphi(u_3 - w_3)\bar{\sigma} : D^2\theta\, dx' \\ &= -\int_\omega \varphi(u_3 - w_3)\operatorname{div}\bar{\sigma} \cdot \nabla\theta\, dx' + \int_{\partial_n\omega} \varphi(u_3 - w_3)[\bar{\sigma}\nu_{\partial\omega}] \cdot \nabla\theta\, d\mathcal{H}^1 \\ &\quad - \int_\omega (u_3 - w_3)\bar{\sigma} : (\nabla\varphi \odot \nabla\theta)\, dx' - \int_\omega \varphi(u_3 - w_3)\bar{\sigma} : D^2\theta\, dx', \end{aligned}$$

where we used that $\varphi(u_3 - w_3)\nabla\theta \in BH(\omega;\mathbb{R}^2)$, hence $\varphi(u_3 - w_3)\nabla\theta \in W^{1,1}(\omega;\mathbb{R}^2)$ and $u_3 = w_3$ on $\partial_d\omega$ by Proposition 3.2.5. □

The next lemma is a characterisation of the dissipation potential $\mathcal{H}^*$ in terms of the duality.

Lemma 3.5.9. *Let $p \in \Pi_{\partial_d\Omega}(\Omega)$. Then the following equalities hold:*

$$\mathcal{H}^*(p) = \sup\{\langle\sigma, p\rangle^* : \sigma \in \Sigma(\Omega) \cap \mathcal{K}^*(\Omega)\} = \sup\{\langle\sigma, p\rangle^* : \sigma \in \Theta(\Omega)\}, \tag{3.5.38}$$

where

$$\mathcal{K}^*(\Omega) := \left\{\sigma \in L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}) :\ \sigma(x) \in K^* \text{ for a.e. } x \in \Omega\right\}$$

and $\Theta(\Omega)$ is the set of all $\sigma \in \Sigma(\Omega) \cap \mathcal{K}^*(\Omega)$ such that $[\bar{\sigma}\nu_{\partial\omega}] = 0$ on $\partial_n\omega$ and $b_0(\hat{\sigma}) = b_1(\hat{\sigma}) = 0$ on $\partial_n\omega$.

Proof. Let $\Gamma := \left(\partial_n\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right) \cup \left(\omega \times \left\{\pm\frac{1}{2}\right\}\right)$. From [51, Chapter II, Section 4] it follows that

$$\begin{aligned} \mathcal{H}^*(p) &= \sup\left\{\int_{\Omega\cup\partial_d\Omega} \sigma : dp : \sigma \in C^\infty(\mathbb{R}^3;\mathbb{M}^{2\times 2}_{sym}) \cap \mathcal{K}^*(\Omega), \operatorname{supp}\sigma \cap \Gamma = \emptyset\right\} \\ &\le \sup\{\langle\sigma, p\rangle^* : \sigma \in \Theta(\Omega)\} \\ &\le \sup\{\langle\sigma, p\rangle^* : \sigma \in \Sigma(\Omega) \cap \mathcal{K}^*(\Omega)\}. \end{aligned}$$

The converse inequality can be proved as in [13, Proposition 7.8] by an approximation argument, where the density result is provided in our framework by Lemma 3.2.7. □

Now we are ready to state and prove the main result of this section.

Theorem 3.5.10. *Let $w \in \mathrm{Lip}([0,T]; H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega))$. Let $t \mapsto (u(t), e(t), p(t))$ be a map from $[0,T]$ into $KL(\Omega) \times L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}) \times M_b(\Omega \cup \partial_d\Omega;\mathbb{M}^{2\times 2}_{sym})$. Let $\sigma(t) := \mathbb{C}^* e(t)$. Then the following conditions are equivalent:*

(a) $t \mapsto (u(t), e(t), p(t))$ *is a reduced quasistatic evolution for the boundary datum* w*;*

(b) $t \mapsto (u(t), e(t), p(t))$ *is Lipschitz continuous and*

(b1) for every $t \in [0, T]$ *we have* $(u(t), e(t), p(t)) \in \mathcal{A}_{GKL}(w(t))$*,* $\sigma(t) \in \Theta(\Omega)$*,* $\operatorname{div} \bar{\sigma}(t) = 0$ *in* ω *and* $\frac{1}{12}\operatorname{div} \operatorname{div} \hat{\sigma}(t) + \bar{\sigma}(t) : D^2\theta = 0$ *in* ω*;*

(b2) for a.e. $t \in [0, T]$ *there holds*

$$\mathcal{H}^*(\dot{p}(t)) = \langle \sigma(t), \dot{p}(t) \rangle^*.$$

Proof. Arguing as in [12, Theorem 5.2] one can prove that every reduced quasistatic evolution is Lipschitz continuous.

We first prove the equivalence between (qs1*) and (b1). Let $t \in [0, T]$. By Lemma 3.5.5 we have to show that (b1) is equivalent to the following condition:

$$-\mathcal{H}^*(q) \leq \int_\Omega \sigma(t) : \eta \, dx \quad \text{for every } (v, \eta, q) \in \mathcal{A}_{GKL}(0). \tag{3.5.39}$$

Assume (3.5.39). Let $B \subset \Omega$ be a Borel set and let χ_B be its characteristic function. Let $\xi \in \mathbb{M}^{2\times 2}_{sym}$ and let $\eta := \chi_B \xi$. By choosing $(0, \eta, -\eta) \in \mathcal{A}_{GKL}(0)$ as test function in (3.5.39), we have that

$$\sigma(t, x) : \xi \leq H^*(\xi) \quad \text{for a.e. } x \in B.$$

Since B is arbitrary, we conclude that $\sigma(t) \in \mathcal{K}^*(\Omega)$.

Let now $v \in H^1(\Omega; \mathbb{R}^3) \cap KL(\Omega)$ be such that $v = 0$ on $\partial_d\Omega$. Since $(\pm v, \pm \bar{E}v, 0) \in \mathcal{A}_{GKL}(0)$, equation (3.5.39) implies

$$\int_\Omega \sigma(t) : \bar{E}v \, dx = 0 \tag{3.5.40}$$

for every $v \in H^1(\Omega; \mathbb{R}^3) \cap KL(\Omega)$ with $v = 0$ on $\partial_d\Omega$. By choosing $v = \psi_\alpha e_\alpha$ with $\psi \in H^1(\omega; \mathbb{R}^2)$ and $\psi = 0$ on $\partial_d\omega$ in (3.5.40), we deduce that

$$\int_\omega \bar{\sigma}(t) : \operatorname{sym} D\psi \, dx' = 0$$

for every $\psi \in H^1(\omega; \mathbb{R}^2)$, $\psi = 0$ on $\partial_d\omega$. Since this holds, in particular, for every $\psi \in C^\infty_c(\omega; \mathbb{R}^2)$, we have

$$\operatorname{div} \bar{\sigma}(t) = 0 \quad \text{in } \omega. \tag{3.5.41}$$

Moreover, by [13, Lemma 7.10-(i)] we obtain

$$[\bar{\sigma}(t)\nu_{\partial\omega}] = 0 \quad \text{on } \partial_n\omega. \tag{3.5.42}$$

We now choose v in (3.5.40) of the form $v = \varphi e_3$, with $\varphi \in H^2(\omega)$, $\varphi = 0$ and $\nabla\varphi = 0$ on $\partial_d\omega$. This leads to

$$\int_\omega \bar{\sigma}(t) : (\nabla\theta \odot \nabla\varphi) \, dx' - \frac{1}{12} \int_\omega \hat{\sigma}(t) : D^2\varphi \, dx' = 0.$$

By (3.5.41), (3.5.42), and (3.5.33) we obtain

$$\begin{aligned} \int_\omega \bar{\sigma}(t) : (\nabla\theta \odot \nabla\varphi) \, dx' &= \int_\omega \bar{\sigma}(t) : \nabla(\varphi\nabla\theta) \, dx' - \int_\omega \varphi\bar{\sigma}(t) : D^2\theta \, dx' \\ &= -\int_\omega \varphi\bar{\sigma}(t) : D^2\theta \, dx'. \end{aligned}$$

Thus, we deduce that

$$\int_\omega \varphi\bar\sigma(t) : D^2\theta\, dx' + \frac{1}{12}\int_\omega \hat\sigma(t) : D^2\varphi\, dx' = 0$$

for every $\varphi \in H^2(\omega)$, $\varphi = 0$ and $\nabla\varphi = 0$ on $\partial_d\omega$. Since this holds, in particular, for every $\varphi \in C_c^\infty(\omega)$, we have

$$\bar\sigma(t) : D^2\theta + \frac{1}{12}\mathrm{div}\,\mathrm{div}\,\hat\sigma(t) = 0 \quad \text{in } \omega.$$

Moreover, by [13, Lemma 7.10-(ii)] we obtain that $b_0(\hat\sigma) = b_1(\hat\sigma) = 0$ on $\partial_n\omega$. In particular, $\sigma(t) \in \Theta(\Omega)$ and (b1) holds.

Assume now (b1) and let $(v, \eta, q) \in \mathcal{A}_{GKL}(0)$. Applying Proposition 3.5.8 to (v, η, q) and $\varphi = 1$ yields

$$\langle\sigma(t), q\rangle^* = -\int_\Omega \sigma(t) : \eta\, dx.$$

Since $\sigma \in \Theta(\Omega)$, we deduce (3.5.39) by Lemma 3.5.9.

We now show, that if (b1) holds, then (qs2*) and (b2) are equivalent. Assume (b1). Since p is Lipschitz continuous, [12, Theorem 7.1] guarantees that

$$\mathcal{D}^*(p; 0, t) = \int_0^t \mathcal{H}^*(\dot p(s))\, ds \tag{3.5.43}$$

for every $t \in [0, T]$. Moreover, using Lemma 3.2.2 one can prove that $(\dot u(t), \dot e(t), \dot p(t)) \in \mathcal{A}_{GKL}(\dot w(t))$ for a.e. $t \in [0, T]$. Applying Proposition 3.5.8 to $(\dot u(t), \dot e(t), \dot p(t))$ and $\varphi = 1$ yields

$$\langle\sigma(t), \dot p(t)\rangle^* = \int_\Omega \sigma(t) : (E^*\dot w(t) - \dot e(t))\, dx. \tag{3.5.44}$$

Differentiation of (qs2*) with respect to time, together with (3.5.43) and (3.5.44), yields (b2), and conversely, integration of (b2) with respect to time yields (qs2*). □

Remark 3.5.11. Observe that, in contrast with the plate model deduced in [13], the equilibrium equations $\mathrm{div}\,\bar\sigma(t) = 0$ and $\frac{1}{12}\mathrm{div}\,\mathrm{div}\,\hat\sigma(t) + \bar\sigma(t) : D^2\theta = 0$ are coupled. In particular, in the case of plates one can show that the reduced quasistatic evolution problem can be written in the two-dimensional domain ω, when initial and boundary data are “horizontal”, [13, Proposition 7.6]. This result is in general false for a shallow shell with $\theta \not\equiv 0$. We also underline that, as in the case of plates, the reduced problem is genuinely three-dimensional. Indeed, in general, the stress component $\sigma_\perp(t)$, which has a non trivial dependence on x_3, is different from 0 (for an explicit example see, e.g., [14, Section 5]). From a mechanical point of view, this is due to the plastic response of the material, since the location of the plastic zone (that is, the region where $\sigma(t) \in \partial K^*$) may depend also on the thickness variable x_3.

3.6 Applied loads

In this section we show that Theorem 3.5.3 still holds when the shallow shell is subjected to applied loads. We consider a *body force* of density

$$f_h \in \mathrm{Lip}([0, T]; L^3(\Sigma_h; \mathbb{R}^3))$$

and a *surface force* of density

$$g_h \in \mathrm{Lip}([0,T]; L^\infty(\partial_n\Sigma_h;\mathbb{R}^3)),$$

where

$$\partial_n\Sigma_h := \Psi_h\big(\partial_n\omega \times (-\tfrac{1}{2},\tfrac{1}{2})\big).$$

We also set

$$\partial\Sigma_h^- := \Psi_h\big(\omega\times\{-\tfrac{1}{2}\}\big), \qquad \partial\Sigma_h^+ := \Psi_h\big(\omega\times\{\tfrac{1}{2}\}\big).$$

For every $t\in[0,T]$ we introduce the functional $\mathcal{L}_h(t)\in(BD(\Sigma_h))'$, defined as

$$\langle\mathcal{L}_h(t),v\rangle := \int_{\Sigma_h} f_h(t)\cdot v\,dx + \int_{\partial_n\Sigma_h} g_h(t)\cdot v\,d\mathcal{H}^2$$

for every $v\in BD(\Sigma_h)$. We assume the following *safe-load condition*: there exist a function $\rho_h\in\mathrm{Lip}([0,T];L^2(\Sigma_h;\mathbb{M}^{3\times3}_{sym}))$, with $(\rho_h)_D\in\mathrm{Lip}([0,T];C(\overline{\Sigma}_h;\mathbb{M}^{3\times3}_{sym}))$, and a constant $\alpha>0$ such that

$$\begin{gathered}-\mathrm{div}\,\rho_h(t) = f_h(t) \quad \text{in } \Sigma_h,\\ \rho_h(t)\nu_{\partial\Sigma_h} = g_h(t) \quad \text{on } \partial_n\Sigma_h, \qquad \rho_h(t)\nu_{\partial\Sigma_h}=0 \quad \text{on } \partial\Sigma_h^-\cup\partial\Sigma_h^+,\\ (\rho_h(t))_D+\xi\in K\end{gathered} \tag{3.6.1}$$

for every $\xi\in\mathbb{M}^{3\times3}_D$ with $|\xi|\le\alpha$.

Remark 3.6.1. As proved in [12], conditions (3.6.1) are crucial to guarantee the existence of a quasistatic evolution $t\mapsto(u_h(t),e_h(t),p_h(t))$ in presence of nonzero loads. Note that we assume $(\rho_h(t))_D\in C(\overline{\Sigma}_h;\mathbb{M}^{3\times3}_{sym})$ and not in $L^\infty(\Sigma_h;\mathbb{M}^{3\times3}_{sym})$ as in [12], since we prefer not to rely on the notion of stress-strain duality in this setting.

Condition (3.6.1) lead to the following formula (for a proof see, e.g., [12, Lemma 3.1]):

$$\begin{aligned}\langle\mathcal{L}_h(t),\phi(v-z)\rangle = \int_{\Sigma_h}\phi\rho_h(t):(\eta-\mathrm{sym}\,Dz)\,dx + \int_{\Sigma_h\cup\partial_d\Sigma_h}\phi(\rho_h)_D(t):\,dq\\ +\int_{\Sigma_h}\rho_h(t):(v-z)\odot\nabla\phi\,dx,\end{aligned} \tag{3.6.2}$$

for every $(v,\eta,q)\in\mathcal{A}(\Sigma_h,z)$, $z\in H^1(\Sigma_h;\mathbb{R}^3)$, and for every $\phi\in C^1(\overline{\Sigma}_h)$.

We introduce the following scaling for the forces:

$$f^h(t) := R_h f_h(t)\circ\Psi_h, \quad g^h(t) := R_h g_h(t)\circ\Psi_h \tag{3.6.3}$$

for every $t\in[0,T]$, while we scale $\rho_h(t)$ as

$$\rho^h(t) := \rho_h(t)\circ\Psi_h \tag{3.6.4}$$

for every $t\in[0,T]$. For every $v\in BD(\Omega)$ we denote by $u\in V_h(\Omega)$ the vectorfield defined in (3.2.11). Owing to (3.6.3), we can rewrite $\mathcal{L}_h(t)$ as

$$\begin{aligned}\langle\mathcal{L}_h(t),v\rangle &= h\int_\Omega f^h(t)\cdot u\det F_h dx + h\int_{\partial_n\Omega} g^h(t)\cdot u|(\mathrm{cof}F_h)R_h\nu_{\partial\Omega}|d\mathcal{H}^2\\ &= h\int_\Omega f^h(t)\cdot u\det F_h dx + h\int_{\partial_n\Omega} g^h(t)\cdot u|\mathrm{cof}F_h\nu_{\partial\Omega}|d\mathcal{H}^2,\end{aligned} \tag{3.6.5}$$

where we used that $R_h\nu_{\partial\Omega} = \nu_{\partial\Omega}$ on $\partial_n\Omega$, because $\nu_{\partial\Omega}\cdot e_3 = 0$ on $\partial_n\Omega$. We thus define the functional

$$\langle \mathcal{L}^h(t), u\rangle := \int_\Omega f^h(t)\cdot u \det F_h dx + \int_{\partial_n\Omega} g^h(t)\cdot u|\mathrm{cof} F_h\nu_{\partial\Omega}|d\mathcal{H}^2 \tag{3.6.6}$$

for every $u \in V_h(\Omega)$. In the next proposition we collect some consequences of (3.6.1) in the scaled domain Ω.

Proposition 3.6.2. *Let $t\in[0,T]$. Then the following hold:*

i) for every $\xi \in \mathbb{M}^{3\times3}_D$ with $|\xi|\le\alpha$ we have

$$\rho^h_D(t) + \xi \in K; \tag{3.6.7}$$

ii) for every $\varphi \in H^1(\Omega;\mathbb{R}^3)$ with $\varphi = 0$ on $\partial_d\Omega$ we have

$$\int_\Omega \rho^h(t) : \mathrm{sym}(R_h D\varphi R_h F_h^{-1}) \det F_h\, dx = \langle \mathcal{L}^h(t), \varphi\rangle; \tag{3.6.8}$$

iii) for every $\varphi \in C^1(\overline{\Omega})$ we have

$$\begin{aligned}
&\langle \mathcal{L}^h(t), \varphi(u-w)\rangle\\
&= \int_\Omega \varphi\rho^h(t) : (e - \mathrm{sym}(R_h Dw R_h F_h^{-1}))\det F_h\, dx\\
&+ \int_{\Omega\cup\partial_d\Omega} \varphi \det F_h \rho^h_D(t) : dp + \int_\Omega R_h\rho^h(t)F_h^{-T}R_h : (u-w)\otimes\nabla\varphi \det F_h\, dx
\end{aligned} \tag{3.6.9}$$

for every $(u,e,p)\in\mathcal{A}_h(\Omega,w)$, and $w\in H^1(\Omega;\mathbb{R}^3)$.

Proof. Condition i) immediately follows from the last equation in (3.6.1) and from (3.6.4).

Now we prove (3.6.8). Let $\varphi \in H^1(\Omega;\mathbb{R}^3)$ with $\varphi = 0$ on $\partial_d\Omega$ and let $\phi := \varphi\circ\Psi_h^{-1}$. Since $\phi = 0$ on $\partial_d\Sigma_h$, the first three conditions in (3.6.1) imply that

$$\int_{\Sigma_h} \rho_h(t) : \mathrm{sym}\, D\phi\, dx = \langle \mathcal{L}_h(t), \phi\rangle.$$

Equation (3.6.8) is now a consequence of (3.6.4), (3.2.14), (3.6.5), and of a change of variable.

Now we show (3.6.9). Let $\varphi\in C^1(\overline{\Omega})$ and let $\phi := \varphi\circ\Psi_h^{-1}$. Let $w \in H^1(\Omega;\mathbb{R}^3)$ and $(u,e,p)\in\mathcal{A}_h(w)$, let $z\in H^1(\Sigma_h;\mathbb{R}^3)$ and $(v,\eta,q)\in\mathcal{A}(\Sigma_h,z)$ be defined as in (3.2.11) and (3.2.15). By a change of variable we rewrite formula (3.6.2) in Ω and we divide by h. All the terms in (3.6.9) are straightforward, except for the last one, which comes from the last integral in (3.6.2). In fact, since $\nabla\phi\circ\Psi_h = (D\Psi_h)^{-T}\nabla\varphi$, owing to (3.2.11), (3.2.15), and (3.6.4) we infer

$$\int_{\Sigma_h} \rho_h(t) : (v-z)\odot\nabla\phi\, dx = h\int_\Omega \rho^h(t) : R_h(u-w)\odot(D\Psi_h^{-T})\nabla\varphi \det F_h\, dx.$$

Since $D\Psi_h^{-T} = F_h^{-T}R_h$, we obtain the last term in (3.6.9) by dividing by h. □

Hypotheses on the forces.

Let $F_\top$ and F_3 be the tangential component and the out-of-plane component of a vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$, respectively. We suppose that there exists a body load f and a surface load g, with

$$f_\top \in \mathrm{Lip}([0,T]; H^1(\Omega;\mathbb{R}^2)), \quad f_3 \in \mathrm{Lip}([0,T]; L^3(\Omega)), \tag{3.6.10}$$

$$g_\top \in \mathrm{Lip}([0,T]; H^1(\partial_n\Omega;\mathbb{R}^2) \cap L^\infty(\partial_n\Omega;\mathbb{R}^2)), \quad g_3 \in \mathrm{Lip}([0,T]; L^\infty(\partial_n\Omega)), \tag{3.6.11}$$

a matrix-valued

$$\rho \in \mathrm{Lip}([0,T]; L^3(\Omega;\mathbb{M}^{3\times3}_{sym}))$$

and a vector-valued

$$\tilde\rho \in \mathrm{Lip}([0,T]; L^3(\Omega;\mathbb{R}^2)),$$

such that for every $t \in [0,T]$

$$f^h(t) \to f(t) \quad \text{strongly in } L^3(\Omega;\mathbb{R}^3), \tag{3.6.12}$$

$$g^h(t) \to g(t) \quad \text{strongly in } L^2(\partial_n\Omega;\mathbb{R}^3), \tag{3.6.13}$$

$$\rho^h(t) \to \rho(t) \quad \text{strongly in } L^3(\Omega;\mathbb{M}^{3\times3}_{sym}), \tag{3.6.14}$$

$$\frac{1}{h}\rho^h_{\alpha3}(t) \to \tilde\rho_\alpha(t) \quad \text{strongly in } L^3(\Omega) \tag{3.6.15}$$

for every $t \in [0,T]$, and

$$\dot f^h(t) \to \dot f(t) \quad \text{strongly in } L^3(\Omega;\mathbb{R}^3), \tag{3.6.16}$$

$$\dot\rho^h(t) \to \dot\rho(t) \quad \text{strongly in } L^3(\Omega;\mathbb{M}^{3\times3}_{sym}), \tag{3.6.17}$$

$$\frac{1}{h}\dot\rho^h_{\alpha3}(t) \to \dot{\tilde\rho}_\alpha(t) \quad \text{strongly in } L^3(\Omega) \tag{3.6.18}$$

for a.e. $t \in [0,T]$. Moreover, we suppose that there exists a constant $C > 0$, independent of h, such that

$$\|\rho^h_D\|_{W^{1,\infty}([0,T];L^\infty)} \le C, \tag{3.6.19}$$

$$\|\rho^h\|_{W^{1,\infty}([0,T];L^2)} \le C \tag{3.6.20}$$

for every $0 < h \ll 1$.

In the next Lemma we deduce some properties of ρ and $\tilde\rho$.

Lemma 3.6.3. *Assume* (3.6.10)–(3.6.20)*. Then for every* $t \in [0,T]$

$$\rho_{i3}(t) = 0 \quad \text{in } \Omega. \tag{3.6.21}$$

Moreover, $\rho(t)$ *satisfies this uniform safe-load condition: for every* $t \in [0,T]$

$$-\operatorname{div}\bar\rho(t) = \bar f_\top(t) \quad \text{in } \omega, \tag{3.6.22}$$

$$[\bar\rho(t)\nu_{\partial\omega}] = \bar g_\top(t) \quad \text{on } \partial_n\omega, \tag{3.6.23}$$

$$-\frac{1}{12}\operatorname{div}\operatorname{div}\hat\rho(t) - \bar\rho(t) : D^2\theta = \bar f_3(t) + \frac{1}{12}\operatorname{div}\hat f_\top(t) - \bar f_\top(t)\cdot\nabla\theta \quad \text{in } \omega, \tag{3.6.24}$$

$$b_0(\hat\rho(t)) = m_0(t), \quad b_1(\hat\rho(t)) = m_1(t) \quad \text{on } \partial_n\omega, \tag{3.6.25}$$

$$\rho(t) + \xi \in K^* \tag{3.6.26}$$

for every $\xi \in \mathbb{M}^{2\times 2}_{sym}$ *such that* $|\xi| \le \alpha$. *Here*

$$\begin{aligned} m_0(t) &= 12\bar{g}_3(t) + \frac{\partial}{\partial\tau}(\hat{g}_\top(t)\cdot\tau_{\partial\omega}) - \hat{f}_\top(t)\cdot\nu_{\partial\omega} - 12\bar{g}_\top(t)\cdot\nabla\theta, \\ m_1(t) &= \hat{g}_\top(t)\cdot\nu_{\partial\omega}. \end{aligned} \tag{3.6.27}$$

Finally, the following equation holds for every $t \in [0,T]$*:*

$$\frac{1}{12}\operatorname{div}\hat{\rho}(t) + \bar{\bar{\rho}}(t) + \bar{\rho}(t)\nabla\theta = \frac{1}{12}\hat{f}_\top(t) \qquad \text{in } \omega. \tag{3.6.28}$$

Proof. It is a consequence of (3.6.14) and (3.6.15) that $\rho_{\alpha 3}(t) = 0$ for every $t \in [0,T]$. Furthermore, if in (3.6.8) we choose a variation $\varphi = \phi_n^h$, where ϕ_n^h is defined in (3.5.30), the same argument as in Step 4 of Theorem 3.5.3 implies that $\rho_{33}(t) = 0$ for every $t \in [0,T]$. This proves (3.6.21). Because of this property, we will identify $\rho(t)$ with a two-dimensional matrix.

Assume that $\varphi \in KL(\Omega)$ in formula (3.6.8). By applying the expansions in (3.4.16) we deduce that

$$\operatorname{sym}(R_h D\varphi R_h F_h^{-1}) \to E^*\varphi \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{sym}).$$

Moreover, by Lemma 3.2.1 we have that $\det F_h \to 1$ and $\operatorname{cof} F_h \to I_{3\times 3}$ uniformly, as h tends to 0. These facts, together with (3.6.12), (3.6.13), and (3.6.14), allow us to pass to the limit in (3.6.8) and obtain that

$$\int_\Omega \rho(t) : E^*\varphi\,dx = \int_\Omega f(t)\cdot\varphi\,dx + \int_{\partial_n\Omega} g(t)\cdot\varphi\,d\mathcal{H}^2. \tag{3.6.29}$$

Choosing φ in (3.6.29) of the form $\varphi = (\bar{\varphi}, 0)$, with $\bar{\varphi} \in H^1(\omega;\mathbb{R}^2)$ and $\bar{\varphi} = 0$ on $\partial_d\omega$, yields

$$\int_\omega \bar{\rho}(t) : \operatorname{sym} D\bar{\varphi}\,dx' = \int_\omega \bar{f}_\top(t)\cdot\bar{\varphi}\,dx' + \int_{\partial_n\omega} \bar{g}_\top(t)\cdot\bar{\varphi}\,d\mathcal{H}^1.$$

Therefore, (3.6.22) and (3.6.23) hold.

We now choose φ in (3.6.29) of the form $\varphi = (-x_3\nabla\psi, \psi)$, where $\psi \in H^2(\omega)$, $\psi = 0$ on $\partial_d\omega$, and $\nabla\psi = 0$ on $\partial_d\omega$. This leads to

$$\begin{aligned} &-\frac{1}{12}\int_\omega \hat{\rho}(t) : D^2\psi\,dx' + \int_\omega \bar{\rho}(t) : \nabla\theta \odot \nabla\psi\,dx' = \int_\omega \left(\bar{f}_3(t)\psi - \frac{1}{12}\hat{f}_T(t)\cdot\nabla\psi\right)dx' \\ &+ \int_{\partial_n\omega} \left(\bar{g}_3(t)\psi - \frac{1}{12}\hat{g}_T(t)\cdot\nabla\psi\right)d\mathcal{H}^1. \end{aligned}$$

Therefore, (3.5.34), (3.6.10), (3.6.11), (3.6.22), and integration by parts imply (3.6.24) and (3.6.25).

Now we prove (3.6.26). We recall that $K = \partial H(0)$. Hence, the last condition in (3.6.1), together with (3.6.4), leads to

$$\int_{U\times(a,b)} (\rho_D^h(t) + \xi) : \zeta\,dx \le \int_{U\times(a,b)} H(\zeta)\,dx$$

for every $\zeta \in \mathbb{M}^{3\times 3}_D$, for every open set $U \subseteq \omega$ and for every $(a,b) \subseteq \left(-\frac{1}{2},\frac{1}{2}\right)$. Owing to convergence (3.6.14), we can pass to the limit in the previous inequality, as h tends to 0. Since U and (a,b) are arbitrary, we deduce that

$$(\rho(t) + \xi)_D \in K$$

for a.e. $x \in \Omega$. This is equivalent to (3.6.26), by (3.6.21) and (3.2.26).

To conclude we have to prove (3.6.28). To this aim, we consider in (3.6.8) variations of the form $\varphi = (x_3\phi, 0)$, with $\phi \in H^1_0(\omega;\mathbb{R}^2)$. It follows from (3.4.16) that

$$\rho^h(t) : \mathrm{sym}(R_h D\varphi(t) R_h F_h^{-1}) = \rho^h_{\alpha\beta}(t)(x_3\,\mathrm{sym}\,D\phi(t) - \phi(t)\odot\nabla\theta)_{\alpha\beta}$$
$$+ O(h^2)\rho^h_{\alpha\beta}(t)\|\phi(t)\|_{H^1} + \frac{1}{h}\rho^h_{\alpha3}(t)\phi_\alpha(t) + O(h)\rho^h_{\alpha3}(t)\|\phi(t)\|_{H^1} + O(h^2)\rho^h_{33}(t)\|\phi(t)\|_{H^1}.$$

Hence (3.6.12), (3.6.14), and (3.6.15) yield

$$\int_\Omega \Big(\rho(t) : (x_3\,\mathrm{sym}\,D\phi - \phi\odot\nabla\theta) + \tilde\rho(t)\cdot\phi\Big)\,dx = \int_\Omega f(t)\cdot(x_3\phi, 0)\,dx. \tag{3.6.30}$$

An easy computation shows that

$$\int_\Omega \Big(\rho(t) : (x_3\,\mathrm{sym}\,D\phi - \phi\odot\nabla\theta) - \tilde\rho(t)\cdot\phi\Big)\,dx = \frac{1}{12}\int_\omega \hat\rho(t) : D\phi\,dx'$$
$$- \int_\omega \left(\bar\rho(t)\nabla\theta\cdot\phi - \bar{\tilde\rho}(t)\cdot\phi\right)\,dx',$$

while

$$\int_\omega f(t)\cdot(x_3\phi, 0)\,dx = \frac{1}{12}\int_\omega \hat f_\top(t)\cdot\phi\,dx'.$$

These two equalities, together with (3.6.30), yield (3.6.28). □

Now we give the definitions of h-quasistatic and reduced quasistatic evolution for a shallow shell subjected to nonzero applied loads.

Definition 3.6.4. Let $0 < h \ll 1$ and let $w^h \in \mathrm{Lip}([0,T]; H^1(\Omega;\mathbb{R}^3))$. An *h-quasistatic evolution* for the boundary datum w^h is a function $t \mapsto (u^h(t), e^h(t), p^h(t))$ from $[0,T]$ into $V_h(\Omega)\times L^2(\Omega;\mathbb{M}^{3\times3}_{sym})\times M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{3\times3}_D)$ that satisfies the following conditions:

(qs1) *global stability:* for every $t\in[0,T]$ we have that $(u^h(t), e^h(t), p^h(t)) \in \mathcal{A}_h(\Omega, w^h(t))$ and

$$\int_\Omega Q(e^h(t))\det F_h\,dx - \langle\mathcal{L}^h(t), u^h(t)\rangle$$
$$\leq \int_\Omega Q(\eta)\det F_h\,dx + \mathcal{H}_h(q - p^h(t)) - \langle\mathcal{L}^h(t), v\rangle \tag{3.6.31}$$

for every $(v,\eta,q)\in\mathcal{A}_h(\Omega, w^h(t))$;

(qs2) *energy balance:* $p^h \in BV([0,T]; M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{3\times3}_{sym}))$ and for every $t\in[0,T]$

$$\int_\Omega Q(e^h(t))\det F_h\,dx + \mathcal{D}_h(p^h; 0, t) - \langle\mathcal{L}^h(t), u^h(t)\rangle$$
$$= \int_\Omega Q(e^h(0))\det F_h\,dx - \langle\mathcal{L}^h(0), u^h(0)\rangle - \int_0^t (\langle\dot{\mathcal{L}}^h(s), u^h(s)\rangle + \langle\mathcal{L}^h(s), \dot w^h(s)\rangle)\,ds$$
$$+ \int_0^t\int_\Omega \mathbb{C}e^h(s) : \mathrm{sym}(R_h D\dot w^h(s) R_h F_h^{-1})\det F_h\,dx\,ds, \tag{3.6.32}$$

where

$$\langle\dot{\mathcal{L}}^h(t), u\rangle := \int_\Omega \dot f^h(t)\cdot u\det F_h\,dx + \int_{\partial_n\Omega}\dot g^h(t)\cdot u|\mathrm{cof}F_h\nu_{\partial\Omega}|\,d\mathcal{H}^2$$

for every $u\in V_h(\Omega)$.

Definition 3.6.5. Let $w \in \mathrm{Lip}([0,T]; H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega))$. A *reduced quasistatic evolution* for the boundary datum w is a function $t \mapsto (u(t), e(t), p(t))$ from $[0,T]$ into $BD(\Omega) \times L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}) \times M_b(\Omega \cup \partial_d\Omega;\mathbb{M}^{2\times 2}_{sym})$ that satisfies the following conditions:

(qs1)* *reduced global stability:* for every $t \in [0,T]$ we have that $(u(t), e(t), p(t)) \in \mathcal{A}_{GKL}(w(t))$ and

$$\mathcal{Q}^*(e(t)) - \langle \mathcal{L}(t), u(t)\rangle \leq \mathcal{Q}^*(\eta) + \mathcal{H}^*(q - p(t)) - \langle \mathcal{L}(t), v\rangle \tag{3.6.33}$$

for every $(v, \eta, q) \in \mathcal{A}_{GKL}(w(t))$, where

$$\langle \mathcal{L}(t), u\rangle := \int_\Omega f(t) \cdot u \, dx + \int_{\partial_n\Omega} g(t) \cdot u \, d\mathcal{H}^2$$

for every $u \in BD(\Omega)$;

(qs2)* *reduced energy balance:* $p \in BV([0,T]; M_b(\Omega \cup \partial_d\Omega;\mathbb{M}^{2\times 2}_{sym}))$ and for every $t \in [0,T]$

$$\begin{aligned}
&\mathcal{Q}^*(e(t)) + \mathcal{D}^*(p; 0, t) - \langle \mathcal{L}(t), u(t)\rangle \\
&= \mathcal{Q}^*(e(0)) - \langle \mathcal{L}(0), u(0)\rangle - \int_0^t (\langle \dot{\mathcal{L}}(s), u(s)\rangle + \langle \mathcal{L}(s), \dot{w}(s)\rangle) \, ds \\
&+ \int_0^t \int_\Omega \mathbb{C}^* e(s) : E^* w(s) \, dx \, ds,
\end{aligned} \tag{3.6.34}$$

where

$$\langle \dot{\mathcal{L}}(t), u\rangle := \int_\Omega \dot{f}(t) \cdot u \, dx + \int_{\partial_n\Omega} \dot{g}(t) \cdot u \, d\mathcal{H}^2$$

for every $u \in BD(\Omega)$.

It follows from (3.6.9) (where we choose $\varphi = 1$) that conditions (3.6.31) and (3.6.32) are equivalent to

(qs1') for every $t \in [0,T]$ we have that $(u^h(t), e^h(t), p^h(t)) \in \mathcal{A}_h(\Omega, w^h(t))$ and

$$\begin{aligned}
&\int_\Omega Q(e^h(t)) \det F_h \, dx - \int_\Omega \rho^h(t) : e^h(t) \det F_h \, dx \\
&\leq \int_\Omega Q(\eta) \det F_h \, dx - \int_\Omega \rho^h(t) : \eta \det F_h \, dx + \mathcal{H}_h(q - p^h(t)) \\
&- \int_{\Omega \cup \partial_d\Omega} \det F_h \rho^h_D(t) : \, d(q - p^h(t))
\end{aligned}$$

for every $(v, \eta, q) \in \mathcal{A}_h(\Omega, w^h(t))$;

(qs2') $p^h \in BV([0,T]; M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times 3}_{sym}))$ and for every $t \in [0,T]$

$$\begin{aligned}
&\int_\Omega Q(e^h(t)) \det F_h \, dx + \mathcal{D}_h(p^h; 0, t) - \int_{\Omega\cup\partial_d\Omega} \det F_h \rho^h_D(t) : dp^h(t) \\
&- \int_\Omega \rho^h(t) : (e^h(t) - \mathrm{sym}(R_h Dw^h(t) R_h F_h^{-1}))\, dx \\
&= \int_\Omega Q(e^h(0)) \det F_h \, dx - \int_\Omega \rho^h(0) : (e^h(0) - \mathrm{sym}(R_h Dw^h(0) R_h F_h^{-1}))\, dx \\
&- \int_{\Omega\cup\partial_d\Omega} \det F_h \rho^h_D(0) : dp^h(0) + \int_0^t \int_\Omega \mathbb{C} e^h(s) : \mathrm{sym}(R_h D\dot{w}^h(s) R_h F_h^{-1}) \det F_h \, dx\, ds \\
&- \int_0^t \int_\Omega \dot{\rho}^h(s) : (e^h(s) - \mathrm{sym}(R_h Dw^h(s) R_h F_h^{-1}))\, dx\, ds \\
&- \int_0^t \int_{\Omega\cup\partial_d\Omega} \det F_h \dot{\rho}^h_D(s) : dp^h(s)\, ds.
\end{aligned}$$

Owing to (3.6.22)–(3.6.25), and Proposition 3.5.8, we have that

$$\langle \mathcal{L}(t), u - w\rangle = \int_\Omega \rho(t) : (e - E^* w)\, dx + \langle \rho(t), p\rangle^*$$

for every $(u, e, p) \in \mathcal{A}_{GKL}(w)$.

Remark 3.6.6. Note that $\rho(t) \in \Sigma(\Omega)$ by Lemma 3.6.3, so that the duality $\langle \rho(t), p(t)\rangle^*$ is well defined for every $t \in [0,T]$.

It follows that (3.6.33) and (3.6.34) are equivalent to

(qs1')* *reduced global stability:* for every $t \in [0,T]$ we have that $(u(t), e(t), p(t)) \in \mathcal{A}_{GKL}(w(t))$ and

$$\mathcal{Q}^*(e(t)) - \int_\Omega \rho(t) : e(t)\, dx \le \mathcal{Q}^*(\eta) - \int_\Omega \rho(t) : \eta\, dx + \mathcal{H}^*(q - p(t)) - \langle \rho(t), q - p(t)\rangle^*$$

for every $(v, \eta, q) \in \mathcal{A}_{GKL}(w(t))$;

(qs2')* *reduced energy balance:* $p \in BV([0,T]; M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{2\times 2}_{sym}))$ and for every $t \in [0,T]$

$$\begin{aligned}
&\mathcal{Q}^*(e(t)) + \mathcal{D}^*(p; 0, t) - \int_\Omega \rho(t) : (e(t) - E^* w(t))\, dx - \langle \rho(t), p(t)\rangle^* \\
&= \mathcal{Q}^*(e(0)) - \int_\Omega \rho(0) : (e(0) - E^* w(0))\, dx - \langle \rho(0), p(0)\rangle^* \\
&+ \int_0^t \int_\Omega \mathbb{C}^* e(s) : E^* \dot{w}(s)\, dx\, ds - \int_0^t \int_\Omega \dot{\rho}(s) : (e(s) - E^* w(s))\, dx\, ds \\
&- \int_0^t \langle \dot{\rho}(s), p(s)\rangle^*\, ds.
\end{aligned}$$

We are ready to state the main result of this section.

Theorem 3.6.7. *Assume* (3.5.5)–(3.5.8), (3.6.7)–(3.6.9), *and* (3.6.10)–(3.6.20). *Furthermore, assume that* $(u^h_0, e^h_0, p^h_0) \in \mathcal{A}_h(\Omega, w^h(0))$ *satisfies* (3.5.10), (3.5.11), *and*

$$\int_\Omega Q(e^h_0) \det F_h \, dx - \langle \mathcal{L}^h(0), u^h_0\rangle \le \int_\Omega Q(\eta) \det F_h \, dx + \mathcal{H}_h(q - p^h_0) - \langle \mathcal{L}^h(0), v\rangle \quad (3.6.35)$$

for every $(v,\eta,q)\in\mathcal{A}_h(\Omega,w^h(0))$. *For every* $0<h\ll 1$ *let* $t\mapsto(u^h(t),e^h(t),p^h(t))$ *be an h-quasistatic evolution for the boundary datum* w^h, *according to Definition 3.6.4, such that* $(u^h(0),e^h(0),p^h(0))=(u_0^h,e_0^h,p_0^h)$. *Then there exist* $w\in\mathrm{Lip}([0,T];H^1(\Omega;\mathbb{R}^3)\cap KL(\Omega))$ *and a reduced quasistatic evolution*

$$(u,e,p)\in\mathrm{Lip}\left([0,T];BD(\Omega)\times L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})\times M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{2\times 2}_{sym})\right)$$

for the boundary datum w, *according to Definition 3.6.5, such that, up to subsequences,*

$$w^h(t)\to w(t)\quad\text{strongly in } H^1(\Omega;\mathbb{R}^3),\tag{3.6.36}$$

$$u^h(t)\to u(t)\quad\text{strongly in } L^1(\Omega;\mathbb{R}^3),\tag{3.6.37}$$

$$\mathrm{sym}(R_hDu^h(t)R_hF_h^{-1})_{\alpha\beta}\rightharpoonup(\bar{E}u(t))_{\alpha\beta}\quad\text{weakly* in } M_b(\Omega),\tag{3.6.38}$$

$$e^h(t)\to\mathbb{M}e(t)\quad\text{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{sym}),\tag{3.6.39}$$

$$p^h_{\alpha\beta}(t)\rightharpoonup p_{\alpha\beta}(t)\quad\text{weakly* in } M_b(\Omega\cup\partial_d\Omega),\tag{3.6.40}$$

as $h\to 0$, *for every* $t\in[0,T]$.

In the remaining of this section we discuss how to modify the proof of Theorem 3.5.3, in order to establish Theorem 3.6.7.

The proof of Step 0 of Theorem 3.5.3 is exactly the same. To prove the remaining steps, it is convenient to start from conditions (qs1') and (qs2'), and deduce (qs1')* and (qs2')*. Now we focus on the proof of Step 1. It follows from (3.6.7) and Lemma 3.2.1 that

$$\mathcal{H}_h(q)-\int_{\Omega\cup\partial_d\Omega}\det F_h\rho_D^h : dq\geq\left(\alpha+O(h^2)\right)\|q\|_{M_b}\tag{3.6.41}$$

for every $q\in M_b(\Omega\cup\partial_d\Omega;\mathbb{M}^{3\times 3}_D)$. Owing to (3.6.41), we can argue as in [12, Theorem 5.2] and infer that there exists a constant $C>0$, independent of h, such that

$$\|e^h(t_2)-e^h(t_1)\|_{L^2}$$
$$\leq C|t_2-t_1|\left(\|\mathrm{sym}(R_hD\dot{w}^hR_hF_h^{-1})\|_{L^\infty([0,T];L^2)}+\|\dot{\rho}_D^h\|_{L^\infty([0,T];L^\infty)}+\|\dot{\rho}^h\|_{L^\infty([0,T];L^2)}\right),\tag{3.6.42}$$

$$\|p^h(t_2)-p^h(t_1)\|_{M_b}$$
$$\leq C|t_2-t_1|\left(\|\mathrm{sym}(R_hD\dot{w}^hR_hF_h^{-1})\|_{L^\infty([0,T];L^2)}+\|\dot{\rho}_D^h\|_{L^\infty([0,T];L^\infty)}+\|\dot{\rho}^h\|_{L^\infty([0,T];L^2)}\right)\tag{3.6.43}$$

for every $t_1,t_2\in[0,T]$ and for every $0<h\ll 1$. In particular, it follows from (3.5.6), (3.6.19), and (3.6.20), that the right-hand side of (3.6.42) and (3.6.43) is uniformly bounded with respect to h. Therefore, Step 1 is proved.

The proof of Step 2 of Theorem 3.5.3 is unchanged.

To conclude the proof of Theorem 3.6.7, we establish a semicontinuity property for the plastic dissipation and the duality $\langle\cdot,\cdot\rangle^*$.

Proposition 3.6.8. *For every* $t\in[0,T]$ *we have*

$$\mathcal{D}^*(p;0,t)+\int_0^t\langle\dot{\rho}(s),p(s)\rangle^*ds-\langle\rho(t),p(t)\rangle^*+\langle\rho(0),p(0)\rangle^*$$
$$\leq\liminf_{h\to 0}\Big\{\mathcal{D}_h(p^h;0,t)+\int_0^t\int_{\Omega\cup\partial_d\Omega}\det F_h\dot{\rho}_D^h(s):dp^h(s)ds$$
$$-\int_{\Omega\cup\partial_d\Omega}\det F_h\rho_D^h(t):dp^h(t)+\int_{\Omega\cup\partial_d\Omega}\det F_h\rho_D^h(0):dp^h(0)\Big\}.$$

Proof. Let $\delta > 0$, $\phi \in C^\infty(\mathbb{R})$ be such that $\phi(s) = 0$ if $s \leq 1$ and $\phi(s) = 1$ if $s \geq 2$ and assume that $0 \leq \phi \leq 1$. Let $\psi_\delta : \Omega \to \mathbb{R}$ given by

$$\psi_\delta(x) := \phi\left(\frac{1}{\delta}\mathrm{dist}(x', \partial_n\omega)\right)$$

for every $x \in \Omega$. It follows from the definition of H and (3.6.7) that the measure

$$\det F_h H(q) - \det F_h \rho^h : q$$

is nonnegative for every $q \in \mathcal{M}_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times 3}_D)$. Thus,

$$\begin{aligned}&\mathcal{H}_h(\psi_\delta \dot{p}^h(t)) - \int_{\Omega\cup\partial_d\Omega} \psi_\delta \det F_h \rho^h_D(t) : d\dot{p}^h(t)\\ &\leq \mathcal{H}_h(\dot{p}^h(t)) - \int_{\Omega\cup\partial_d\Omega} \det F_h \rho^h_D(t) : d\dot{p}^h(t).\end{aligned} \tag{3.6.44}$$

As a consequence of [12, Theorem 7.1] we have

$$\mathcal{D}_h(q; 0, t) = \int_0^t \mathcal{H}_h(\dot{q}(s))\, ds$$

for every $q \in AC([0,T]; \mathcal{M}_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times 3}_{sym}))$. Applying this identity to $\psi_\delta \dot{p}^h(t)$ and to $\dot{p}^h(t)$, integrating (3.6.44) with respect to time, and using integration by parts we obtain

$$\begin{aligned}&\mathcal{D}_h(\psi_\delta p^h; 0, t) + \int_0^t \int_{\Omega\cup\partial_d\Omega} \psi_\delta \det F_h \dot{\rho}^h_D(s) : dp^h(s)\, ds\\ &- \int_{\Omega\cup\partial_d\Omega} \psi_\delta \det F_h \rho^h_D(t) : dp^h(t) + \int_{\Omega\cup\partial_d\Omega} \psi_\delta \det F_h \rho^h_D(0) : dp^h(0)\\ &\leq \mathcal{D}_h(p^h; 0, t) + \int_0^t \int_{\Omega\cup\partial_d\Omega} \det F_h \dot{\rho}^h_D(s) : dp^h(s)\, ds\\ &- \int_{\Omega\cup\partial_d\Omega} \det F_h \rho^h_D(t) : dp^h(t) + \int_{\Omega\cup\partial_d\Omega} \det F_h \rho^h_D(0) : dp^h(0).\end{aligned} \tag{3.6.45}$$

We know by (3.5.23) that

$$p^h(t) \rightharpoonup \tilde{p}(t) \quad \text{weakly* in } M_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{3\times 3}_D).$$

This convergence, together with the lower semicontinuity of $\mathcal{D}^*$, gives

$$\mathcal{D}^*(\psi_\delta p; 0, t) \leq \liminf_{h\to 0} \mathcal{D}_h(\psi_\delta p^h; 0, t). \tag{3.6.46}$$

It follows from (3.6.9) that for a.e. $t \in [0,T]$

$$\begin{aligned}&\int_{\Omega\cup\partial_d\Omega} \psi_\delta \det F_h \dot{\rho}^h_D(t) : dp^h(t)\\ &= \int_\Omega \psi_\delta \dot{\rho}^h(t) : (\mathrm{sym}(R_h Dw^h(t) R_h F_h^{-1}) - e^h(t)) \det F_h\, dx\\ &+ \int_\Omega \psi_\delta \dot{f}^h(t) \cdot (u^h(t) - w^h(t)) \det F_h\, dx\end{aligned}$$

$$+\int_\Omega R_h\dot\rho^h(t)F_h^{-T}R_h : (w^h(t)-u^h(t))\otimes\nabla\psi_\delta \det F_h\,dx,$$

where we also used that $\psi_\delta = 0$ on $\partial_n\Omega$. We already know, owing to Remark 3.4.2 and to (3.5.22), that for every $t\in[0,T]$

$$u^h(t) \rightharpoonup u(t) \quad \text{weakly in } L^{3/2}(\Omega;\mathbb{R}^3), \tag{3.6.47}$$
$$e^h(t) \rightharpoonup \tilde e(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}_{sym}), \tag{3.6.48}$$
$$w^h(t) \to w(t) \quad \text{strongly in } H^1(\Omega;\mathbb{R}^3). \tag{3.6.49}$$

By Lemma 3.2.1 we have that $\det F_h\to 1$ uniformly in Ω as h tends to 0, and the following expansions hold:

$$(R_h\dot\rho^h(t)F_h^{-T}R_h)_{\alpha\beta} = \dot\rho^h_{\alpha\gamma}(t)(\delta_{\gamma\beta}+O(h^2)) + \dot\rho^h_{\alpha3}(t)O(h), \tag{3.6.50}$$
$$(R_h\dot\rho^h(t)F_h^{-T}R_h)_{3\beta} = \frac{1}{h}\dot\rho^h_{3\beta}(t) + (\dot\rho^h_{31}(t)+\dot\rho^h_{32}(t))O(h) + \dot\rho^h_{33}(t)(\partial_\beta\theta + O(h^2)). \tag{3.6.51}$$

Moreover, since $\partial_3\psi_\delta = 0$, we have that $((w^h(t)-u^h(t))\otimes\nabla\psi_\delta)_{i3} = 0$. This fact, together with (3.6.47)–(3.6.51), (3.5.18), assumptions (3.5.7), (3.6.16)–(3.6.18), and (3.6.21) give

$$\begin{aligned}
&\lim_{h\to0}\int_{\Omega\cup\partial_d\Omega}\psi_\delta\det F_h\dot\rho^h_D(t) : dp^h(t)\\
&= \int_\Omega \psi_\delta\dot\rho(t) : (E^*w(t)-e(t))\,dx + \int_\Omega\psi_\delta\dot f(t)\cdot(u(t)-w(t))\,dx\\
&+ \int_\Omega\dot\rho(t) : \nabla\psi_\delta\odot(w(t)-u(t))\,dx + \int_\Omega\dot{\underline{\rho}}(t)\cdot\nabla\psi_\delta(w_3(t)-u_3(t))\,dx\\
&= \int_\Omega \psi_\delta\dot\rho(t) : (E^*w(t)-e(t))\,dx + \int_\omega\psi_\delta\dot{\bar f}_{\rm T}(t)\cdot(\bar u(t)-\bar w(t))\,dx'\\
&- \frac{1}{12}\int_\omega\psi_\delta\dot{\hat f}_{\rm T}(t)\cdot(\nabla u_3(t)-\nabla w_3(t))\,dx' + \int_\omega\psi_\delta\dot{\bar f}_3(t)(u_3(t)-w_3(t))\,dx'\\
&+ \int_\omega\dot{\bar\rho}(t) : \nabla\psi_\delta\odot(\bar w(t)-\bar u(t))\,dx' - \frac{1}{12}\int_\omega\dot{\hat\rho}(t) : \nabla\psi_\delta\odot(\nabla w_3(t)-\nabla u_3(t))\,dx'\\
&+ \int_\omega\dot{\bar{\underline{\rho}}}(t)\cdot\nabla\psi_\delta(w_3(t)-u_3(t))\,dx',
\end{aligned}$$

where we also used that $u(t), w(t)\in KL(\Omega)$. An integration by parts, the fact that $u_3(t) = w_3(t)$ on $\partial_d\omega$ and (3.6.22) yield

$$\begin{aligned}
&\lim_{h\to0}\int_{\Omega\cup\partial_d\Omega}\psi_\delta\det F_h\dot\rho^h_D(t) : dp^h(t) =\\
&= \int_\Omega \psi_\delta\dot\rho(t) : (E^*w(t)-e(t))\,dx - \int_\omega\psi_\delta{\rm div}\,\dot{\bar\rho}(t)\cdot(\bar u(t)-\bar w(t))\,dx'\\
&+ \int_\omega\dot{\bar\rho}(t) : \nabla\psi_\delta\odot(\bar w(t)-\bar u(t))\,dx' - \frac{1}{12}\int_\omega\dot{\hat\rho}(t) : \nabla\psi_\delta\odot(\nabla w_3(t)-\nabla u_3(t))\,dx'\\
&+ \int_\omega\psi_\delta\left(\dot{\bar f}_3(t) + \frac{1}{12}{\rm div}\,\dot{\hat f}_{\rm T}(t)\right)(u_3(t)-w_3(t))\,dx'\\
&+ \int_\omega\left(\dot{\bar{\underline{\rho}}}(t) - \frac{1}{12}\dot{\hat f}_{\rm T}(t)\right)\cdot(w_3(t)-u_3(t))\nabla\psi_\delta\,dx'.
\end{aligned}$$

Integrating by parts and using (3.6.24), (3.6.28) we get

$$\begin{aligned}
&\lim_{h\to 0}\int_{\Omega\cup\partial_d\Omega}\psi_\delta \det F_h \dot\rho_D^h(t) : dp^h(t) = \\
&= \int_\Omega \psi_\delta \dot\rho(t) : (E^* w(t) - e(t))\,dx - \int_\omega \psi_\delta \operatorname{div}\dot{\bar\rho}(t)\cdot(\bar u(t) - \bar w(t))\,dx' \\
&+ \int_\omega \dot{\bar\rho}(t) : \nabla\psi_\delta \odot (\bar w(t) - \bar u(t))\,dx' - \frac{1}{6}\int_\omega \dot{\hat\rho}(t) : \nabla\psi_\delta \odot (\nabla w_3(t) - \nabla u_3(t))\,dx' \\
&- \int_\omega \psi_\delta\left(\frac{1}{12}\operatorname{div}\operatorname{div}\dot{\hat\rho}(t) + \dot{\bar\rho}(t) : D^2\theta + \operatorname{div}\dot{\bar\rho}(t)\cdot\nabla\theta\right)(u_3(t) - w_3(t))\,dx' \\
&- \frac{1}{12}\int_\omega (w_3(t) - u_3(t))\dot{\hat\rho}(t) : D^2\psi_\delta\,dx' + \int_\omega \dot{\bar\rho}(t)\nabla\theta\cdot(w_3(t) - u_3(t))\nabla\psi_\delta\,dx',
\end{aligned}$$

It follows now from Proposition 3.5.8 that for every $t \in [0,T]$

$$\lim_{h\to 0}\int_{\Omega\cup\partial_d\Omega}\psi_\delta \det F_h \dot\rho_D^h(t) : dp^h(t) = \langle[\dot\rho(t) : p(t)]^*, \psi_\delta\rangle. \tag{3.6.52}$$

Moreover, owing to (3.6.19) and to the estimate

$$\|p^h\|_{L^\infty([0,T];M_b)} \le C$$

for every $0 < h \ll 1$ (where C is positive and independent of h), we can apply the Dominated Convergence Theorem and infer that

$$\lim_{h\to 0}\int_0^t\int_{\Omega\cup\partial_d\Omega}\psi_\delta \det F_h \dot\rho_D^h(s) :\, dp^h(s)\,ds = \int_0^t \langle[\dot\rho(s) : p(s)]^*, \psi_\delta\rangle\,ds. \tag{3.6.53}$$

Hence (3.6.45), (3.6.46), (3.6.52), and (3.6.53) lead to

$$\begin{aligned}
&\mathcal{D}^*(\psi_\delta p; 0, t) + \int_0^t \langle[\dot\rho(s), p(s)]^*, \psi_\delta\rangle ds - \langle[\rho(t), p(t)]^*, \psi_\delta\rangle + \langle[\rho(0), p(0)]^*, \psi_\delta\rangle \\
&\le \liminf_{h\to 0}\Big\{\mathcal{D}_h(p^h; 0, t) + \int_0^t\int_{\Omega\cup\partial_d\Omega}\det F_h\dot\rho_D^h(s) : dp^h(s)ds \\
&- \int_{\Omega\cup\partial_d\Omega}\det F_h \rho_D^h(t) : dp^h(t) + \int_{\Omega\cup\partial_d\Omega}\det F_h\rho_D^h(0) : dp^h(0)\Big\}.
\end{aligned}$$

We can pass to the limit as δ tends to 0 in the previous inequality and deduce the thesis. □

Now we conclude the proof of Theorem 3.6.7. To prove Step 3, that is, $(u(t), e(t), p(t))$ satisfies the reduced stability, we note that, arguing as in Lemma 3.5.5, this is equivalent to require that

$$-\mathcal{H}^*(q) \le \int_\Omega (\mathbb{C}^* e(t) - \rho(t)) : \eta\,dx - \langle\rho(t), q\rangle^*$$

for every $(v, \eta, q) \in \mathcal{A}_{GKL}(0)$. We can derive this inequality arguing as in Step 3 of Theorem 3.5.3, using (3.6.52) (with $\dot\rho_D^h(t)$ replaced by $\rho_D^h(t)$) and sending δ to 0.

The proof of Step 4 does not present additional difficulties with respect to that of Theorem 3.5.3.

To conclude it remains to prove the energy balance (Step 5). Proposition 3.6.8 provide the lower energy inequality, while the converse inequality follows, as usual, from the reduced stability.

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