Thin elastic plates supported over small areas. I. Korn's inequalities and boundary layers

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Dedicated to the memory of Jean-Jacques Moreau

Abstract

A thin anisotropic elastic plate clamped along its lateral side and also supported at a small area θ_h of one base is considered; the diameter of θ_h is of the same order as the plate relative thickness $h \ll 1$. In addition to the standard Kirchhoff model with the Sobolev point condition, a three-dimensional boundary layer is investigated in the vicinity of the support θ_h, which with the help of the derived weighted inequality of Korn's type, will provide an error estimate with the bound $ch^{1/2}|\ln h|$. Ignoring this boundary layer effect reduces the precision order down to $|\ln h|^{-1/2}$.

Keywords: Kirchhoff plate, small support zones, asymptotic analysis, boundary layers, weighted Korn inequality.

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1 Introduction

1.1 A plate supported over a small area

Let ω be a domain in the plane $\mathbb{R}^2$ with a smooth boundary $\partial\omega$ and a compact closure $\overline{\omega} = \omega \cup \partial\omega$. We introduce the cylindrical plate

$$\Omega_h = \omega \times (-h/2, h/2) \tag{1.1}$$

of a small thickness $h > 0$. By rescaling, the characteristic size of ω is reduced to one so that the Cartesian coordinates $x = (y, z) \in \mathbb{R}^2 \times \mathbb{R}$ and all geometric parameters become dimensionless. The bases of the plate and its lateral side are given by

$$\Sigma_h^{\pm} = \{x : y = (y_1, y_2) \in \omega,\ z = \pm h/2\}, \quad \upsilon_h = \{x : y \in \partial\omega,\ |z| < h/2\} \tag{1.2}$$

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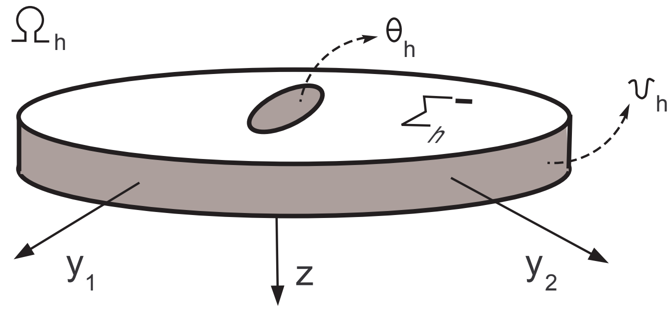

Figure 1: A plate clamped over the lateral side υ_h and a small support area θ_h.

respectively. We fix a point $\mathcal{O}$ inside ω and place the y-coordinate origin at $\mathcal{O}$. Denoting by $\theta \subset \mathbb{R}^2$ an open, not necessarily connected, set with a compact closure $\overline{\theta}$, we assume that the plate (1.1) is clamped over the lateral side υ_h as well as at the small supporting area

$$\theta_h = \left\{x : \eta := h^{-1}y \in \theta,\ z = -h/2\right\} \tag{1.3}$$

on the base Σ_h^-. Characteristic sizes of θ are supposed to be of order one, too. In other words, the diameter of the supporting zone (1.3) is comparable with the plate thickness h. The bound $h_0 > 0$ for the small parameter h is chosen such that $\overline{\theta_h} \subset \Sigma_h^-$ for all $h \in (0, h_0]$, however if necessary, we will reduce h_0 but always keep the notation. Throughout this paper we do not distinguish in the notation for θ and θ_h between two-dimensional sets and their immersions in $\mathbb{R}^3$ along the planes $\{x : z = -1/2\}$ and $\{x : z = -h/2\}$, respectively.

The plate Ω_h is made out of a homogeneous anisotropic elastic material and its deformation is caused by volume forces. It is ideally fixed over the set

$$\Gamma_h = \upsilon_h \cup \theta_h \tag{1.4}$$

while the rest of the plate surface, in particular $\Sigma_h^\bullet = \Sigma_h^- \setminus \overline{\theta_h}$, is traction-free. The clamped area is shaded in Figure 1.

The main goal of this and consequent paper [5] is to examine the influence of the supporting area (1.3) on the stress-strain state of the whole plate. To this end, we construct asymptotics of elastic fields as $h \to +0$, prove error estimates, and create a two-dimensional model which reflects adequately all principal effects of the small support. We emphasize that clamping along the lengthy set $\upsilon_h \subset \partial\Omega_h$ plays a considerable role in technicalities of our study and the case of a traction-free lateral side will be treated in a subsequent paper where, in contrast to the present case, the number of small supporting areas and their location become of a major importance.

1.2 Formulation of the elasticity problem; the Mandel-Voigt notation

Since the Cartesian coordinate system $x = (x_1, x_2, x_3)$ attached to the plate Ω_h is fixed, we can regard the displacement field u as the column $(u_1, u_2, u_3)^\top$ in $\mathbb{R}^3$ where u_j is the projection of u onto the x_j-axis and $\top$ stands for transposition. The strain column

$$\varepsilon = \left(\varepsilon_{11}, \varepsilon_{22}, 2^{1/2}\varepsilon_{12}, 2^{1/2}\varepsilon_{13}, 2^{1/2}\varepsilon_{23}, \varepsilon_{33}\right)^\top \tag{1.5}$$

substitutes for the strain tensor of rank 2 with the Cartesian components

$$\varepsilon_{jk}(u) = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right), \qquad j,k = 1,2,3, \tag{1.6}$$

and can be computed by the formula

$$\varepsilon(u) = D(\nabla)\,u \tag{1.7}$$

where $\nabla = \operatorname{grad}$ and

$$D(\nabla)^{\top} = \begin{pmatrix} \partial_1 & 0 & 2^{-1/2}\partial_2 & 2^{-1/2}\partial_3 & 0 & 0 \\ 0 & \partial_2 & 2^{-1/2}\partial_1 & 0 & 2^{-1/2}\partial_3 & 0 \\ 0 & 0 & 0 & 2^{-1/2}\partial_1 & 2^{-1/2}\partial_2 & \partial_3 \end{pmatrix}, \qquad \partial_j = \frac{\partial}{\partial x_j}. \tag{1.8}$$

Notice that, according to the Mandel-Voigt notation, the factors $2^{1/2}$ and $2^{-1/2}$ are introduced in (1.5) and (1.8) for the purpose of equalizing the intrinsic norms of the tensor and the column of height 6 (cf. [3, 37] and others).

The stress column

$$\sigma = \left(\sigma_{11}, \sigma_{22}, 2^{1/2}\sigma_{12}, 2^{1/2}\sigma_{13}, 2^{1/2}\sigma_{23}, \sigma_{33}\right)^{\top}$$

of the same structure as in (1.5) is to be found through the Hooke's law

$$\sigma(u) = A\varepsilon(u) = AD(\nabla)u \tag{1.9}$$

where A is the stiffness matrix of size 6×6, symmetric and positive definite. This matrix contains elastic moduli, for instance, to an isotropic elastic material there corresponds

$$A = \begin{pmatrix} \lambda+2\mu & \lambda & 0 & 0 & 0 & \lambda \\ \lambda & \lambda+2\mu & 0 & 0 & 0 & \lambda \\ 0 & 0 & 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ \lambda & \lambda & 0 & 0 & 0 & \lambda+2\mu \end{pmatrix} \tag{1.10}$$

where $\lambda \geq 0$ and $\mu > 0$ are the Lamé constants.

In what follows we use matrix, rather than tensor, notation in elasticity known as the Mandel-Voigt notation, cf. [3]. In this way we write the equilibrium equations as follows:

$$L(\nabla)\,u(h,x) := D(-\nabla)^{\top} AD(\nabla)\,u(h,x) = f(h,x), \quad x \in \Omega_h, \tag{1.11}$$

where $f = (f_1, f_2, f_3)^{\top}$ is the vector (column) of the volume (mass) forces. The three-dimensional elasticity system (1.11) is supplied with the traction-free boundary condition

$$\begin{aligned} N^{+}(\nabla)\,u(h,x) &:= D(e_3)^{\top} AD(\nabla)\,u(h,x) = 0, \quad x \in \Sigma_h^{+}, \\ N^{-}(\nabla)\,u(h,x) &:= D(-e_3)^{\top} AD(\nabla)\,u(h,x) = 0, \quad x \in \Sigma_h^{\bullet}, \end{aligned} \tag{1.12}$$

where $e_3 = (0,0,1)^\top$ is the unit vector of the outward normal on the bases $\Sigma_h^\pm$. At the clamped parts of the surfaces $\partial\Omega_h$, we write

$$u(h,x) = 0, \quad x \in \theta_h, \tag{1.13}$$

$$u(h,x) = 0, \quad x \in \upsilon_h. \tag{1.14}$$

We further refer to (1.12) and (1.13), (1.14) as the Neumann and Dirichlet conditions respectively.

The variational statement of problem (1.11)–(1.14) reads: to find a vector function $u \in H_0^1(\Omega_h;\Gamma_h)^3$ such that

$$(AD(\nabla)u, D(\nabla)v)_{\Omega_h} = (f,v)_{\Omega_h}, \quad \forall v \in H_0^1(\Omega_h;\Gamma_h)^3, \tag{1.15}$$

where $(\ ,\ )_{\Omega_h}$ is the natural scalar product in the Lebesgue space $L^2(\Omega_h)$, $H_0^1(\Omega_h;\Gamma_h)$ is a subspace of functions in the Sobolev class $H^1(\Omega_h)$ which vanish at set (1.4), and the last superscript 3 in (1.15) means that test (vector) function v has three components.

In view of the Dirichlet conditions (1.13) and (1.14) the Korn inequality [19]

$$\left\|v; H^1(\Omega_h)\right\| \le K_h \left\|D(\nabla)v; L^2(\Omega_h)\right\|, \quad \forall v \in H_0^1(\Omega_h;\Gamma_h)^3 \tag{1.16}$$

is valid and the unique solvability of the variational problem (1.15) with any $f \in L^2(\Omega_h)^3$ follows from the Riesz representation theorem. Although there exist various approaches, see [13, 14, 18, 45] and many others, to verify inequality (1.16), we still need to clarify the dependence of the Korn constant K_h on the small parameter h. To this end, we exhibit in Section 2 several variants of weighted anisotropic Korn's inequalities.

1.3 Asymptotics and boundary layers

The Kirchhoff theory of thin plates created more than 150 years ago by means of intuitive asymptotic analysis, has got a justification in miscellaneous formulations by various methods and at different level of rigor, we refer only to the mathematical monographs [9, 10, 21, 37, 48] and intensive lists of literature therein, although setting aside a vast volume of publications with important theoretical and practical results. If an external loading is scaled to provide the elastic energy of a plate to gain order $1 = h^0$, the energy norm of asymptotic remainders in Kirchhoff's asymptotic formulas becomes $O(\sqrt{h})$ and this is the best error estimate available within a two-dimensional theory of plates. This limitation of the asymptotic accuracy is prescribed exclusively by boundary layer effect near the plate edge. Namely, in the vicinity of the lateral side υ_h the standard plane-stress state in the (y_1, y_2)-directions, dominant in the midst of the plane, couples with a plane-strain state in the (n, z)-directions, where n is the normal vector to $\partial\omega$. The latter is represented by special elastic fields which slowly vary along the edge, produce strains and stresses of order 1 at the lateral side υ_h but quickly, at the exponential rate, decay at a distance from υ_h. We again refer only to the mathematical papers [33, 50] and [11, 12], where the edge boundary layer phenomenon was investigated in elasticity, and to Chapters 15 and 16 of the monograph [23] where simplest scalar and general elliptic problems were examined and basic principles to construct boundary layers are laid down.

The energy norm

$$(AD(\nabla)u, D(\nabla)v)_{\Omega_h}^{1/2} \tag{1.17}$$

of the boundary layer equals $O(\sqrt{h})$ that just predetermines the error bound in an integral norm. However, the proximity of the plane-stress state in a weighted Hölder norm is much less, cf. [31, 39]. For example, bounds in the pointwise estimates of remainders in two-dimensional asymptotic forms of stresses get the same order in h as the main terms and this is known in mechanics as "edge effect" in a thin plate. Thus, the error $O(\sqrt{h})$ of the Kirchhoff model is mostly due to the intrinsic localization of the edge effect in a ch-neighborhood of the lateral side υ_h.

The two-dimensional structure of boundary layers is kept for a smooth contour $\partial\omega$ only. Plates with angulate edges were considered in [26] where three-dimensional boundary layers with a power-law decay rate were detected. However, their energy norm (1.17) becomes $o(\sqrt{h})$ so that they play a secondary role.

As it can be easily predicted due to the Sobolev embedding theorem, the Dirichlet condition (1.13) leads to the Sobolev condition (3.6) in the limit, that is the overage deflection $w_3 \in H^2(\omega)$ of the plate vanishes at the point $\mathcal{O}$. However, the small support area (1.3) also provokes a fully three-dimensional boundary layer in the vicinity of θ_h and provides a perturbation of order $|\ln h|^{-1/2}$ in the two-dimensional model. In other words, the convergence rate $O(|\ln h|^{-1/2})$ of the rescaled three-dimensional displacements, cf. Theorem 7, to a solution of the limit Dirichlet-Sobolev problem (3.7), (3.8), (3.5), (3.6) is very slow and, therefore, unsatisfactory. At the same time, the asymptotic structures derived in this paper provide the same accuracy as in the Kirchhoff theory.

In [7], the same effect of a crucial reduction of the asymptotic accuracy due to a small Dirichlet area has been observed and discussed in detail for a scalar homogenization problem in a perforated planar domain.

1.4 The elasticity capacity and self-adjoint extensions

To the best knowledge of the authors, this paper is the first mathematically rigorous and complete study of the boundary layer effect near a small support area and its influence on the whole stress-strain state of a plate. The necessity to examine rapid variations of elastic fields in the vicinity of the support area θ_h accounts for the following result obtained in the consequential paper [5]: the convergence rate to the two-dimensional Kirchhoff solution of the true three-dimensional one is extremely slow, of order $|\ln h|^{-1/2}$, but involving the boundary layer reduces the error estimate bound down to $ch^{1/2}|\ln h|$ and makes it acceptable for applications. At the same time, the whole boundary layer solution is not known explicitly and can be determined by solving a complicated elasticity problem (3.23)–(3.25) in the spatial infinite ply $\mathbb{R}^2 \times (-1/2, 1/2)$, for example, numerically. In this way a simplification of asymptotic expansions and further modeling take on special significance.

As usual in the theory of elliptic problems in singularly perturbed domains (see the monographs [16, 23] and others) the alternation of boundary conditions on a small set leads to asymptotic forms engaging singular solutions of limit problems and in our case the Dirichlet condition (1.13) on θ_h affects the limit problem in ω in two ways. First, it requires to impose the point condition $w_3(\mathcal{O}) = 0$, (3.6), for the average deflection $w_3(y)$ of the plate. Second, the far-field, eligible at a distance from the support θ_h, contains the Green matrix, cf. Section 3.4 and see [5], whose entries are solutions in ω with logarithmic singularities at the point $y = 0$. Due to the Sobolev embedding theorem $H^2 \subset C$ in $\mathbb{R}^2$ the point condition is well-posed within the Hilbert theory, cf. [22, Ch.2] but the Green matrix does not belong to the intrinsic energy space for the Kirchhoff plate ω and therefore breaks the usual variational formulation of the model.

To develop a two-dimensional model of the locally supported plate Ω_h, we will turn in [5] to the technique of self-adjoint extensions of differential operators (see the pioneering paper [2] and, e.g., the review [46]). Namely, we get rid of the complicated three-dimensional structure of the boundary layer and simulate its influence on the stress-strain state of the plate by a proper choice of singular solutions. In other words, we select a special self-adjoint extension in $L^2(\omega)^3$ of the matrix $\mathcal{L}(\nabla)$ of differential operators in the conventional Kirchhoff model described in Section 3.2, formula (3.14). Parameters of this extension depend only on the quantity $|\ln h|$ and the algebraic characteristics $C^\sharp(A,\Theta)$ of the small support area, namely the elastic logarithmic capacity matrix introduced heuristically in [25], a 4×4-matrix composed by coefficients in asymptotics of solutions to the elasticity problem in Λ, see Section 3.5, and is quite similar to the logarithmic capacity in harmonic analysis, cf. [20, 47] and Remark 12, and polarization and capacity matrices in elasticity, cf. [1] and [36, 44]. There exist numerical schemes [42, 43] to evaluate such characteristics.

1.5 Architecture of the paper

In Section 2 we derive anisotropic weighted Korn inequality in the plate (1.1) clamped along the lateral surface and the small set (1.3) (Theorem 2) as well as demonstrate that weights introduced in the Sobolev norms are optimal. Then we modify our approach to prove Theorem 4 which serves for a plate with a traction-free lateral surface but several small support areas.

The last section is devoted to three-dimensional boundary layers emerging in the vicinity of small support areas. First of all, we outline the two-dimensional Kirchhoff model of the plate Ω_h and the Sobolev point condition $w_3(\mathcal{O})=0$, (3.6) for the average deflection $w_3(y)$, which imitates the entire Dirichlet condition (1.13) on θ_h. Moreover, we formulate Theorem 7 from [5] on the convergence of the spatial true solution to the Kirchhoff-Sobolev solution in order to indicate the extremely slow convergence rate $O(|\ln h|^{-1/2})$ and to emphasize the constitutive influence of boundary layers described by an elasticity problem in the layer

$$\Lambda = \mathbb{R}^2 \times (-1/2, 1/2) \tag{1.18}$$

clamped along the set θ on its planar base $\mathbb{R}^2\times\{-1/2\}$.

To prove the existence of a unique solution in the unbounded domain Λ but with a finite energy, we need a new anisotropic weighted Korn inequality which, however, can be derived by means of the same tricks as in Section 2. At the same time, a rigorous derivation of asymptotic expansions of the solution at infinity requires both, a formal procedure of dimension reduction scheduled in Section 3.2 and a specific application of the Kondratiev theory [17], cf. [32]. This evident similarity of several approaches in our paper explains its architecture.

In Section 3.4 we strictly determine the elastic logarithmic capacity matrix which was mentioned in Section 1.4 and will be a particular object in [5]. In Theorem 13 we establish its general properties.

2 Weighted anisotropic inequalities of Korn's type

2.1 Some Hardy-type inequalities

Here below we collect some Hardy-type inequalities that will be used subsequently.

Proposition 1 *Let $a(x)$ be a nonnegative function in $(0,1)$; then the following Hardy inequality holds:*

$$\int_0^1 a(x)|u(x)|^2\,dx \le 4\int_0^1 \frac{1}{a(x)}\Big(\int_x^1 a(t)\,dt\Big)^2|u'(x)|^2\,dx \qquad \forall u\in H^1(0,1)\ \textit{with}\ u(0)=0.$$

Proof. Since $u(0)=0$ we have

$$u^2(x) = 2\int_0^x u(t)u'(t)\,dt,$$

so that

$$\int_0^1 a(x)|u(x)|^2\,dx \le 2\int_0^1 a(x)\Big(\int_0^x |u(t)||u'(t)|\,dt\Big)\,dx.$$

Interchanging the order of integration gives for the right-hand side

$$2\int_0^1 |u(t)||u'(t)|\Big(\int_t^1 a(x)\,dx\Big)\,dt,$$

which, by Hölder inequality, is majorized by

$$2\Big[\int_0^1 a(t)|u(t)|^2\,dt\Big]^{1/2}\Big[\int_0^1 \frac{1}{a(t)}\Big(\int_t^1 a(x)\,dx\Big)^2|u'(t)|^2\,dt\Big]^{1/2},$$

and this concludes the proof. ∎

We list below some particular cases of functions $a(x)$, with the corresponding Hardy-type inequalities. By taking

$$a(x) = x^{-2}$$

we obtain for every $u\in H^1(0,1)$ with $u(0)=0$

$$\int_0^1 x^{-2}|u(x)|^2\,dx \le 4\int_0^1 (1-x)^2|u'(x)|^2\,dx,$$

which implies the classical Hardy inequality

$$\int_0^T x^{-2}|u(x)|^2\,dx \le 4\int_0^T |u'(x)|^2\,dx \tag{2.1}$$

for every $u\in H^1(0,T)$ with $u(0)=0$. Taking

$$a(x) = (1-x)^{-1}|\ln(1-x)|^{-2}$$

gives for every $u\in H^1(0,1)$ with $u(0)=0$

$$\int_0^1 (1-x)^{-1}|\ln(1-x)|^{-2}|u(x)|^2\,dx \le 4\int_0^1 (1-x)|u'(x)|^2\,dx,$$

which implies the inequality

$$\int_0^R x^{-1}|\ln(x/R)|^{-2}|u(x)|^2\,dx \le 4\int_0^R x|u'(x)|^2\,dx \tag{2.2}$$

for every $u \in H^1(0,R)$ with $u(R) = 0$. Taking

$$a(x) = x^{-3}|\ln(x/2)|^{-2}$$

gives for every $u \in H^1(0,1)$ with $u(0) = 0$

$$\int_0^1 x^{-3}|\ln(x/2)|^{-2}|u(x)|^2\,dx \le 4\int_0^1 x^{-1}|\ln(x/2)|^{-2}|u'(x)|^2\,dx$$

which implies the inequality

$$\int_0^{R/2} x^{-3}|\ln(x/R)|^{-2}|u(x)|^2\,dx \le 4\int_0^{R/2} x^{-1}|\ln(x/R)|^{-2}|u'(x)|^2\,dx \tag{2.3}$$

for every $u \in H^1(0,R/2)$ with $u(0) = 0$. Taking

$$a(x) = (x+h)^{-4} \qquad \text{with } h > 0$$

gives for every $u \in H^1(0,1)$ with $u(0) = 0$

$$\int_0^1 (x+h)^{-4}|u(x)|^2\,dx \le \frac{4}{9}\int_0^1 (x+h)^{-2}|u'(x)|^2\,dx,$$

which implies the inequality

$$\int_0^T (x+h)^{-4}|u(x)|^2\,dx \le \frac{4}{9}\int_0^T (x+h)^{-2}|u'(x)|^2\,dx \tag{2.4}$$

for every $u \in H^1(0,T)$ with $u(0) = 0$.

2.2 Anisotropic weighted Korn inequality

In the paper [49] devoted to justification of the Kirchhoff theory of plates, see also the pioneering papers [24] and [8] together with the monographs [9, 10, 21, 37, 48] etc., it was proved that a constant $K(\omega)$ in the inequality

$$|||u;\Omega_h||| \le K(\omega)\left\|D(\nabla)\,u; L^2(\Omega_h)\right\|, \qquad \forall u \in H_0^1(\Omega_h, \upsilon_h) \tag{2.5}$$

does not depend on $h \in (0,1]$ and, of course, on u, where $|||u;\Omega_h|||$ is the anisotropic Sobolev norm

$$|||u;\Omega_h|||^2 = \int_{\Omega_h}\left[\sum_{i=1}^{2}\left(|\nabla_y u_i|^2 + h^2\left(\left|\frac{\partial u_i}{\partial z}\right|^2 + \left|\frac{\partial u_3}{\partial y_i}\right|^2\right) + |u_i|^2\right)\right. \\ \left. + |\partial_z u_3|^2 + h^2|u_3|^2\right]dx. \tag{2.6}$$

Notice that the distribution of coefficients h^p in (2.6) is optimal, namely one cannot replace h^2 by h^α, with $\alpha < 2$, without losing the independence property of $K(\omega)$.

In the paper [27] a convenient modification of norm (2.6) was suggested, namely the weighted anisotropic norm

$$|||u;\Omega_h|||_0^2 = \int_{\Omega_h} \left[\sum_{i=1}^{2} \left(|\nabla_y u_i|^2 + \frac{h^2}{s_h^2} \left(\left| \frac{\partial u_i}{\partial z} \right|^2 + \left| \frac{\partial u_3}{\partial y_i} \right|^2 \right) + \frac{1}{s_h^2} |u_i|^2 \right) + |\partial_z u_3|^2 + \frac{h^2}{s_h^4} |u_3|^2 \right] dx \tag{2.7}$$

with the weighting function

$$s_h\left(y\right) = h + \operatorname{dist}\left(y, \partial\omega\right). \tag{2.8}$$

Notice that in the middle of the plate the weights in (2.6) and (2.7) are equivalent but in the vicinity of the lateral side υ_h where the Dirichlet condition (1.14) is imposed, the displacements u_i, u_3 and all their derivatives get multipliers h^{-1} and $h^0 = 1$ respectively.

Our further justification scheme requires to insert into the norm (2.7) some additional weights

$$S_{hq}(y) = (h^2 + |y|^2)^{-q/2} \left(1 + \left|\ln(h^2 + |y|^2)\right|\right)^{-1} \tag{2.9}$$

that become big in a ch-neighborhood of the set $\overline{\theta_h}$ and, therefore, take the Dirichlet condition (1.13) into account. Note that the function S_{hk} is smooth with any $h > 0$ and in the sequel it is proper to put into (2.7) the weight $h + s_0\left(y\right)$, tantamount to (2.8). Here, s_0 stands for a smooth in $\overline{\omega}$ and positive in ω function equivalent to $\operatorname{dist}(\cdot, \partial\omega)$ in the vicinity of the boundary $\partial\omega$.

Then we will prove the anisotropic weighted Korn inequality in the following theorem.

Theorem 2 *Any displacement field $u \in H^1\left(\Omega_h\right)^3$ satisfying the Dirichlet conditions (1.13) and (1.14), meets the weighted anisotropic Korn inequality*

$$|||u;\Omega_h|||_\bullet \leq K_\bullet\left(\omega\right) \left\| D\left(\nabla\right) u; L^2\left(\Omega_h\right) \right\|, \qquad \forall u \in H_0^1\left(\Omega_h, \upsilon_h \cup \theta_h\right) \tag{2.10}$$

where

$$|||u;\Omega_h|||_\bullet^2 = \int_{\Omega_h} \left[\sum_{i=1}^{2} \left(|\nabla_y u_i|^2 + \frac{h^2}{s_h^2} S_{h1}^2 \left(\left| \frac{\partial u_i}{\partial z} \right|^2 + \left| \frac{\partial u_3}{\partial y_i} \right|^2 \right) + \frac{1}{s_h^2} S_{h1}^2 |u_i|^2 \right) + |\partial_z u_3|^2 + \frac{h^2}{s_h^4} S_{h2}^2 |u_3|^2 \right] dx. \tag{2.11}$$

with a constant $K\left(\omega\right)$ independent of $h \in (0, h_0]$ and $u \in H_0^1\left(\Omega_h; \upsilon_h \cup \theta_h\right)^3$.

Proof. Due to a completion argument, it suffices to verify inequality (2.10) for any smooth function u in $\overline{\Omega_h}$ which vanishes near $\overline{\upsilon_h}$ and $\overline{\theta_h}$, see (1.2). We use the stretched coordinates

$$\xi = (\eta, \zeta) = \left(h^{-1} y, h^{-1} z\right) \tag{2.12}$$

and consider the circular cylinder $\mathbf{Q}_{hR} = \mathbb{B}^2_{hR} \times (-h/2, h/2)$ where $\mathbb{B}^2_\rho$ is the disk $\{y : |y| < \rho\}$ and the radius R is fixed such that $\theta \subset \mathbb{B}^2_{R/2}$. We write

$$\begin{aligned}\left\|\nabla_x u; L^2\left(\mathbf{Q}_{hR}\right)\right\|^2 + h^{-2}\left\|u; L^2\left(\mathbf{Q}_{hR}\right)\right\|^2 &= h\left\|\nabla_\xi u; L^2\left(\mathbf{Q}_R\right)\right\|^2 + h^{-1}\left\|u; L^2\left(\mathbf{Q}_R\right)\right\|^2 \\ &= h\left\|\xi \mapsto u(x); H^1\left(\mathbf{Q}_R\right)\right\|^2 \leq c(R,\omega) h\left\|D(\nabla_\xi) u; L^2\left(\mathbf{Q}_R\right)\right\|^2 \\ &= c(R,\omega)\left\|D(\nabla_x) u; L^2\left(\mathbf{Q}_{hR}\right)\right\|^2 \leq c(R,\omega)\left\|D(\nabla) u; L^2(\Omega_h)\right\|^2.\end{aligned} \tag{2.13}$$

Here, we have applied the standard Korn inequality

$$\left\|v; H^1\left(\mathbf{Q}_R\right)\right\|^2 \leq c\left(R,\omega\right)\left\|D\left(\nabla_\xi\right) v; L^2\left(\mathbf{Q}_R\right)\right\|^2, \quad \forall v \in H^1_0\left(\mathbf{Q}_R; \theta\right)^3$$

which holds true due to the Dirichlet condition on θ.

Let $\chi \in C^\infty(\mathbb{R})$ be a reference cut-off function such that

$$\chi\left(r\right) = 1 \text{ for } r < 1/2 \quad \text{and} \quad \chi\left(r\right) = 0 \text{ for } r \geq 1, \quad 0 \leq \chi \leq 1. \tag{2.14}$$

We set $X_h(y) = 1 - \chi\left((hR)^{-1}|y|\right)$ and, in view of $|\nabla_y X_h\left(y\right)| \leq ch^{-1}$, obtain

$$\begin{aligned}\left\|D\left(\nabla\right)\left(X_h u\right); L^2\left(\Omega_h\right)\right\|^2 &\leq c(\left\|D(\nabla) u; L^2(\Omega_h)\right\|^2 \\ &+ h^{-2}\left\|u; L^2\left(\left(\mathbb{B}_{hR} \setminus \mathbb{B}_{hR/2}\right) \times (-h/2, h/2)\right)\right\|^2) \leq c\left\|D(\nabla) u; L^2(\Omega_h)\right\|^2.\end{aligned} \tag{2.15}$$

Since $S_{h1}(y) \leq ch^{-1}$ in $\mathbf{Q}_{hR}$, estimates (2.13) and (2.15) mean that we further can treat the product $X_h u$ only. This product is still denoted by u but we remember its specific property

$$u(y,z) = 0 \quad \text{for } |y| < hR/2. \tag{2.16}$$

Formulas (1.6), (1.14) and integrating by parts yield

$$\begin{aligned}4\int_{\Omega_h} |\varepsilon_{12}(u)|^2\, dx &= \int_{\Omega_h}\left(\left|\frac{\partial u_1}{\partial y_2}\right|^2 + \left|\frac{\partial u_2}{\partial y_1}\right|^2 + 2\frac{\partial u_1}{\partial y_2}\frac{\partial u_2}{\partial y_1}\right) dx \\ &= \int_{\Omega_h}\left(\left|\frac{\partial u_1}{\partial y_2}\right|^2 + \left|\frac{\partial u_2}{\partial y_1}\right|^2 + 2\frac{\partial u_1}{\partial y_2}\frac{\partial u_2}{\partial y_1}\right) dx \\ &\geq \int_{\Omega_h}\left(\left|\frac{\partial u_1}{\partial y_2}\right|^2 + \left|\frac{\partial u_2}{\partial y_1}\right|^2 - \left|\frac{\partial u_1}{\partial y_1}\right|^2 - \left|\frac{\partial u_2}{\partial y_2}\right|^2\right) dx.\end{aligned}$$

Hence, recalling the strains $\varepsilon_{11}\left(u\right)$ and $\varepsilon_{22}\left(u\right)$, we obtain

$$\left\|\nabla_y u_i; L^2\left(\Omega_h\right)\right\|^2 \leq 2\left\|D\left(\nabla\right) u; L^2\left(\Omega_h\right)\right\|^2, \quad i = 1,2. \tag{2.17}$$

We have the Friedrichs inequality in ω integrated in $z \times (-h/2, h/2)$, that is

$$\left\|u_i; L^2\left(\Omega_h\right)\right\|^2 \leq c_\omega\left\|\nabla_y u_i; L^2\left(\Omega_h\right)\right\|^2. \tag{2.18}$$

Let us explain how inequalities (2.17)–(2.18) provide the estimates

$$\left\|s_h^{-1} S_{01} u_i; L^2\left(\Omega_h\right)\right\|^2 \leq c\left\|D\left(\nabla\right) u; L^2\left(\Omega_h\right)\right\|^2, \qquad i = 1,2, \tag{2.19}$$

where $S_{01}(y)$ is given by (2.9) at $h = 0$. To attach the weight s_h^{-1}, we rewrite the function u_i in the natural curvilinear coordinates n, s while n is the oriented distance to $\partial\omega$, $n < 0$ in ω, and s is the arc length along $\partial\omega$. Setting $t = h - n$ and $U(t) = u_i(n, s)$ in (2.1), we observe that the Jacobian of the coordinate change $y \mapsto (n, s)$ and its inverse are bounded in the δ-neighborhood $\mathcal{V}_\delta$ of $\partial\omega$, $\delta > 0$ being fixed small. Since $\left|\partial_{|n|}u_i(x)\right| \le |\nabla_y u_i(x)|$, integrating in $s \in \omega$ and $z \in (-h/2, h/2)$ converts (2.1) into the inequality

$$\left\|s_h^{-1}u_i; L^2\left((\omega \cap \mathcal{V}_\delta) \times (-h/2, h/2)\right)\right\|^2 \le c\left\|\nabla_y u_i; L^2\left((\omega \cap \mathcal{V}_\delta) \times (-h/2, h/2)\right)\right\|^2 \qquad (2.20)$$

because $s_h(y) \sim h + |n|$. Formula (2.18) allows us to replace $\omega \cap \mathcal{V}_\delta$ with ω in (2.20).

The weight S_{01} ought to be inserted in a similar manner. However, we go over to the polar coordinate system (r, φ) in the y-plane and multiply u_i with the cut-off function $\chi_\omega(y) = \chi(y/R_\omega)$ where R_ω is chosen such that $\mathbb{B}^2_{R_\omega} \subset \omega$. It remains to mention that $dy = rdrd\varphi$ and

$$\left\|\partial_r(\chi_\omega u_i); L^2\left(\mathbb{B}^2_{R_\omega}\right)\right\| \le c_\omega\left\|u_i; H^1\left(\mathbb{B}^2_{R_\omega}\right)\right\|,$$

and to apply inequalities (2.2) integrated over $(\varphi, z) \in (0, 2\pi) \times (-h/2, h/2)$ and (2.18), (2.17).

To get an accurate information on the derivatives $\partial_z u_i$ and $\nabla_y u_3$ is a much more complicated issue and we apply a trick from [27]. To this end, we introduce the cut-off function $\chi_h(z) = \chi(2z/h)$ which is null on the bases $\Sigma_h^\pm$ and observe that, according to definition (2.8), (2.9) of the weights, there holds

$$0 < h^2 s_h(y)^{-1} S_{01}(y) \le c_S, \quad y \in \overline{\omega} \setminus \mathbb{B}^2_{hR/2},\ h \in (0, h_0]. \qquad (2.21)$$

Notice that we have reduced our analysis to the case $u = 0$ in $\mathbb{Q}_{hR/2} = \mathbb{B}^2_{hR/2} \times (-h/2, h/2)$. Then we proceed as follows:

$$c_S^2 \int_{\Omega_h} |\varepsilon_{i3}(u)|^2 dx \ge h^2 \int_{\Omega_h} \chi_h^2 s_h^{-2} S_{01}^2 \left(\left|\frac{\partial u_i}{\partial z}\right|^2 + \left|\frac{\partial u_3}{\partial y_i}\right|^2\right) dx$$

$$+2h^2 \int_{\Omega_h} \chi_h^2 s_h^{-2} S_{01}^2 \frac{\partial u_i}{\partial z}\frac{\partial u_3}{\partial y_i} dx =: I_1^h + 2I_2^h,$$

$$I_2^h = -h^2 \int_{\Omega_h} \chi_h^2 s_h^{-2} S_{01}^2 u_i \frac{\partial^2 u_3}{\partial z \partial y_i} dx - 2h^2 \int_{\Omega_h} \chi_h \partial_z \chi_h s_h^{-2} S_{01}^2 u_i \frac{\partial u_3}{\partial y_i} dx =: I_3^h + 2I_4^h.$$

Since $|\partial_z \chi_h(z)| \le c_\chi h^{-1}$, we have

$$\left|I_4^h\right| \le c_\chi \left(h^2 \int_{\Omega_h} \chi_h^2 s_h^{-2} S_{01}^2 \left|\frac{\partial u_3}{\partial y_i}\right|^2 dx\right)^{1/2} \left(\int_{\Omega_h} s_h^{-2} S_{01}^2 |u_i|^2 dx\right)^{1/2}$$

$$\le \delta I_1^h + C\delta^-1 \left\|s_h^- 1 S_0 1 u_i; L^2(\Omega_h)\right\|^2$$

where $\delta > 0$ is arbitrary and the last norm has been estimated in (2.19). Furthermore,

$$I_3^h = h^2 \int_{\Omega_h} \chi_h^2 s_h^{-2} S_{01}^2 \frac{\partial u_i}{\partial y_i}\frac{\partial u_3}{\partial z} dx + h^2 \int_{\Omega_h} \chi_h^2 u_i \frac{\partial u_3}{\partial z}\frac{\partial}{\partial y_i}\left(s_h^{-2} S_{01}^2\right) dx =: I_5^h + I_6^h,$$

both the integrals getting appropriate bounds with simplicity. First, in view of (2.21),

$$\left|I_5^h\right| \le \frac{1}{2} c_S^2 \left(\left\|\varepsilon_{ii}(u); L^2(\Omega_h)\right\|^2 + \left\|\varepsilon_{33}(u); L^2(\Omega_h)\right\|^2\right) \le C\left\|D(\nabla)u; L^2(\Omega_h)\right\|^2.$$

Second, performing differentiation of weight in y we see that, for $y \in \overline{\omega} \setminus \mathbb{B}^2_{hR/2}$ and $h \in (0, h_0]$

$$h^2 \left| \frac{\partial}{\partial y_i} \left(s_h(y)^{-2} S_{01}(y)^2 \right) \right| \leq c h^2 s_h(y)^{-2} S_{01}(y)^2 \left(\frac{1}{h + s_0(y)} + \frac{1}{|y|} \right) \leq C s_h(y)^{-1} S_{01}(y),$$

and, thus,

$$\left| I_6^h \right| \leq c \left(\left\| s_h^{-1} S_{01} u_i; L^2(\Omega_h) \right\|^2 + \left\| \partial_z u_3; L^2(\Omega_h) \right\|^2 \right) \leq C \left\| D(\nabla) u; L^2(\Omega_h) \right\|^2.$$

Collecting relations listed above provides the formula

$$I_1^h \leq c \left(1 + \delta^{-1} \right) \left\| D(\nabla) u; L^2(\Omega_h) \right\|^2 - 2\delta I_1^h$$

so that putting $\delta = 1/4$ leads to the following estimate of the derivatives in question:

$$\begin{aligned} h^2 \left\| s_h^{-1} S_{01} \partial_z u_i; L^2\left(\Omega_{h/2}\right) \right\|^2 + h^2 \left\| s_h^{-1} S_{01} \partial u_3 / \partial y_i; L^2\left(\Omega_{h/2}\right) \right\|^2 \\ \leq C \left\| D(\nabla) u; L^2(\Omega_h) \right\|^2, \end{aligned} \tag{2.22}$$

however only inside the thinner plate $\Omega_{h/2} = \omega \times (-h/4, h/4)$ where the cut-off function equals one.

We postpone spreading of estimate (2.22) onto the entire plate and conclude with the displacement u_3 itself.

Operating with (2.4) and (2.3) in the same way as with (2.1) and (2.2), we derive from estimates (2.22) of $\nabla_y u_3$ the formula

$$h^2 \left\| s_h^{-1} S_{02} u_3; L^2(\Omega_{h/2}) \right\|^2 \leq c \left\| D(\nabla) u; L^2(\Omega_h) \right\|^2 \tag{2.23}$$

with a weight required in (2.11) but again in a thinner plate. By the way, we become in position to lighten weights in (2.22), (2.23) by the replacement $S_{0q} \mapsto S_{hq}$ and write

$$|||u; \Omega_{h/2}|||_\bullet^2 \leq c \left\| D(\nabla) u; L^2(\Omega_h) \right\|^2. \tag{2.24}$$

Moreover, we now may forget about the artificial property (2.16) of the displacement field u.

To improve the obtained estimates, we apply a method of passing anisotropic Korn inequalities from one part of a body to another part which was proposed in [27] and elaborated in [40]. Let $\mathbb{Q}_h$ be the cube $\left\{ x : \left| y_i - y_i^0 \right| < h/2, \ |z| < h/2 \right\}$ with some center $(y^0, 0)$. Extending u by zero from Ω_h onto the layer $\{x : |z| < h/2\}$, we assume that $y^0 \in \omega$ and set $U(\eta, \zeta) = u(y^0 + h\eta, h\zeta)$ where $\xi = (\eta, \zeta) = \left(h^{-1}\left(y - y^0 \right), h^{-1} z \right) \in \mathbb{Q}_1$ are stretched coordinates, cf. (2.12). We introduce the rigid motion matrix of size 3×6

$$d(\xi) = \begin{pmatrix} 1 & 0 & 0 & 0 & \xi_3 & -\xi_2 \\ 0 & 1 & 0 & -\xi_3 & 0 & \xi_1 \\ 0 & 0 & 1 & \xi_2 & -\xi_1 & 0 \end{pmatrix} \tag{2.25}$$

and make the following decomposition in the unit cube $\mathbb{Q}_1$:

$$\mathcal{U}(\xi) = \mathcal{U}^\perp(\xi) + d(\xi) \mathcal{U}^0. \tag{2.26}$$

Here, the last term implies a rigid motion generated by the column

$$\mathcal{U}^0 = \mathbf{d}\left(\mathbb{Q}_1'\right)^{-1} \int_{\mathbb{Q}_1'} d\left(\xi\right)^\top \mathcal{U}\left(\xi\right) d\xi \in \mathbb{R}^6, \tag{2.27}$$

where $\mathbb{Q}_1' = \{\xi \in \mathbb{Q}_1 : |\zeta| < 1/4\}$ is a half of the cube and $\mathbf{d}\left(\mathbb{Q}_1'\right)$ is a Gram matrix of size 6×6, symmetric and positive definite,

$$\mathbf{d}\left(\mathbb{Q}_1'\right) = \int_{\mathbb{Q}_1'} d\left(\xi\right)^\top d\left(\xi\right) d\xi. \tag{2.28}$$

Owing to definition (2.27), (2.28), the component $\mathcal{U}^\perp$ meets the orthogonality conditions

$$\int_{\mathbb{Q}_1'} d\left(\xi\right)^\top \mathcal{U}^\perp\left(\xi\right) d\xi = 0 \in \mathbb{R}^6 \tag{2.29}$$

which, as known (see the proof of Theorem 3.3.3 [13] or Theorem 2.3.3 [37]), assure the Korn inequality on the intact cube

$$\left\|\mathcal{U}^\perp; H^1\left(\mathbb{Q}_1\right)\right\|^2 \le K \left\|D\left(\nabla_\xi\right)\mathcal{U}^\perp; L^2\left(\mathbb{Q}_1\right)\right\|^2. \tag{2.30}$$

Due to the central symmetry of the integration domain in (2.28) the matrix $\mathbf{d}\left(\mathbb{Q}_1'\right)$ is diagonal. Hence, formulas (2.27) and (2.25) immediately show that

$$\left|\mathcal{U}_i^0\right| \le c\left\|\mathcal{U}_i; L^2\left(\mathbb{Q}_1'\right)\right\|, \quad i = 1, 2, \qquad \left|\mathcal{U}_3^0\right| \le c\left\|\mathcal{U}_3; L^2\left(\mathbb{Q}_1'\right)\right\|. \tag{2.31}$$

The component $\mathcal{U}_6^0$ is estimated in the following way:

$$\begin{aligned}
\left|\mathcal{U}_6^0\right| &= \mathbf{d}_{66}^{-1}\left|\int_{\mathbb{Q}_1'} \left(\eta_1\mathcal{U}_2\left(\xi\right) - \eta_2\mathcal{U}_1\left(\xi\right)\right) d\xi\right| \\
&= \mathbf{d}_{66}^{-1}\left|\int_{\mathbb{Q}_1'} \left(\mathcal{U}_2\left(\xi\right)\frac{\partial}{\partial\eta_1}\left(\frac{\eta_1^2}{2} - \frac{1}{8}\right) - \mathcal{U}_1\left(\xi\right)\frac{\partial}{\partial\eta_2}\left(\frac{\eta_2^2}{2} - \frac{1}{8}\right)\right) d\xi\right| \\
&= \mathbf{d}_{66}^{-1}\left|\int_{\mathbb{Q}_1'} \left(\left(\frac{\eta_1^2}{2} - \frac{1}{8}\right)\frac{\partial\mathcal{U}_2}{\partial\eta_1}\left(\xi\right) - \left(\frac{\eta_2^2}{2} - \frac{1}{8}\right)\frac{\partial\mathcal{U}_1}{\partial\eta_2}\left(\xi\right)\right) d\xi\right| \\
&\le c\left(\left\|\frac{\partial\mathcal{U}_1}{\partial\eta_2}; L^2(\mathbb{Q}_1')\right\| + \left\|\frac{\partial\mathcal{U}_2}{\partial\eta_1}; L^2\left(\mathbb{Q}_1'\right)\right\|\right).
\end{aligned}$$

Note that integration by parts did not bring a surface integral because $\left(\eta_i^2/2 - 1/8\right)_{\eta_i = \pm 1/2} = 0$. Referring to the formulas

$$\zeta = \frac{\partial}{\partial\zeta}\left(\frac{\zeta^2}{2} - \frac{1}{32}\right), \qquad \left.\left(\frac{\zeta^2}{2} - \frac{1}{32}\right)\right|_{\zeta = \pm 1/4} = 0,$$

we finally obtain in a similar manner that

$$\left|\mathcal{U}_{6-i}^0\right| \le c\left(\left\|\frac{\partial\mathcal{U}_i}{\partial\zeta}; L^2\left(\mathbb{Q}_1'\right)\right\| + \left\|\frac{\partial\mathcal{U}_3}{\partial\eta_i}; L^2\left(\mathbb{Q}_1'\right)\right\|\right), \qquad i = 1, 2.$$

Now we return to the x-coordinates and the displacement field u. The decomposition (2.26) determines the component $u^{\perp}$ which, according to (2.30), gets the estimate, cf. (2.13),

$$\left\|\nabla u^{\perp}; L^2\left(\mathbb{Q}_h\right)\right\|^2 + h^{-2}\left\|u^{\perp}; L^2\left(\mathbb{Q}_h\right)\right\|^2 \leq c\left\|D\left(\nabla\right)u; L^2\left(\mathbb{Q}_h\right)\right\|^2. \tag{2.32}$$

Then we calculate

$$\begin{aligned}
&\left\|s_h^{-1}S_{1h}u_i; L^2\left(\mathbb{Q}_h\right)\right\|^2 \\
&\quad\leq c\left(h^{-2}\left\|u_i^{\perp}; L^2\left(\mathbb{Q}_h\right)\right\|^2 + s_h\left(y^0\right)^{-1}S_{1h}\left(y^0\right)\operatorname{mes}_3(\mathbb{Q}_h)\left(\left|\mathcal{U}_i^0\right|^2 + \left|\mathcal{U}_{6-i}^0\right|^2 + \left|\mathcal{U}_0^0\right|^2\right)\right) \\
&\quad\leq c\Bigg(\left\|D\left(\nabla\right)u; L^2(\mathbb{Q}_h)\right\|^2 + \left\|s_h^{-1}S_{1h}u_i; L^2(\mathbb{Q}_h')\right\|^2 + h^2\left\|s_h^{-1}S_{1h}\frac{\partial u_i}{\partial z}; L^2(\mathbb{Q}_h')\right\|^2 \\
&\qquad\quad + h^2\left\|s_h^{-1}S_{1h}\frac{\partial u_3}{\partial y_i}; L^2(\mathbb{Q}_h')\right\|^2 + \left\|\nabla_y u'; L^2(\mathbb{Q}_h')\right\|^2\Bigg) \\
&\quad\leq c\left(\left\|D\left(\nabla\right)u; L^2\left(\mathbb{Q}_h\right)\right\|^2 + |||u; \mathbb{Q}_h'|||_{\bullet}^2\right).
\end{aligned} \tag{2.33}$$

This calculation needs a detailed commentary. We here and further take into account the following formulas for weights:

$$\begin{aligned}
&s_h\left(y^0\right)^{-1}S_{hq}\left(y^0\right) \leq \sup_{x\in\mathbb{Q}_h} s_h\left(y\right)^{-1}S_{hq}\left(y\right) \leq cs_h\left(y^0\right)^{-1}S_{hq}\left(y^0\right), \\
&h^{2q}s_h\left(y\right)^{-1}S_{hq}\left(y\right) \leq c_q, \qquad q = 0, 1.
\end{aligned} \tag{2.34}$$

The first inequality in (2.33) was obtained by using decomposition (2.26) and a direct computation of the norm in $L^2(\mathbb{Q}_h)$. The factor $\operatorname{mes}_3\mathbb{Q}_h = h^3$ was compensated due to the relations

$$\left\|u_j; L^2(\mathbb{Q}_h')\right\|^2 = h^{-3}\left\|\mathcal{U}_j; L^2(\mathbb{Q}_1')\right\|^2, \qquad \left\|\frac{\partial u_j}{\partial x_k}; L^2(\mathbb{Q}_h')\right\|^2 = h^{-1}\left\|\frac{\partial \mathcal{U}_j}{\partial \xi_k}; L^2(\mathbb{Q}_1')\right\|^2$$

but the last one still leaves the coefficient h^2 on the square of the $L^2(\mathbb{Q}_h')$-norm of a derivative. Then we applied estimates (2.31)–(2.32). Finally we recalled definition (2.11) of a weighted anisotropic norm.

Augmenting (2.34) with the relation $S_{h1}\left(y\right)^{-1}S_{h2}\left(y\right) \leq h^{-1}$ in $\mathbb{Q}_h$, we continue as follows:

$$\begin{aligned}
&h^2\left\|s_h^{-1}S_{2h}u_3; L^2(\mathbb{Q}_h)\right\|^2 \\
&\quad\leq c\left(h^{-2}\left\|u_3^{\perp}; L^2(\mathbb{Q}_h)\right\|^2 + h^2 s_h\left(y^0\right)^{-1}S_{2h}\left(y^0\right)\operatorname{mes}_3\mathbb{Q}_h\left(\left|\mathcal{U}_3^0\right|^2 + \left|\mathcal{U}_4^0\right|^2 + \left|\mathcal{U}_5^0\right|^2\right)\right) \\
&\quad\leq c\Bigg(\left\|D\left(\nabla\right)u^{\perp}; L^2(\mathbb{Q}_h)\right\|^2 + h^2\left\|s_h^{-1}S_{2h}u_3; L^2(\mathbb{Q}_h')\right\|^2 + h^2\left\|s_h^{-1}S_{1h}\nabla_y u_3; L^2(\mathbb{Q}_h')\right\|^2 \\
&\qquad\quad + h^2\left\|s_h^{-1}S_{1h}\partial_z u'; L^2(\mathbb{Q}_h')\right\|^2\Bigg) \\
&\quad\leq c\left(\left\|D\left(\nabla\right)u; L^2(\mathbb{Q}_h)\right\|^2 + |||u; \mathbb{Q}_h'|||_{\bullet}^2\right),
\end{aligned}$$

where as usual $u' = (u_1, u_2)^\top$. Since the left 3×3-block of the rigid motion matrix (2.25) is annulled by differentiation, treating derivatives of u_j becomes much simpler:

$$\begin{aligned}
&\left\| \frac{\partial u_1}{\partial y_2}; L^2(\mathbb{Q}_h) \right\|^2 + \left\| \frac{\partial u_2}{\partial y_1}; L^2(\mathbb{Q}_h) \right\|^2 \leq c \left(\left\| \nabla u^\perp; L^2(\mathbb{Q}_h) \right\|^2 + h^3 \left| \mathcal{U}_6^0 \right|^2 \right) \\
&\quad \leq c \Bigg(\left\| D(\nabla) u; L^2(\mathbb{Q}_h) \right\|^2 + \left\| \frac{\partial u_1}{\partial y_2}; L^2(\mathbb{Q}_h') \right\|^2 + \left\| \frac{\partial u_2}{\partial y_1}; L^2(\mathbb{Q}_h') \right\|^2 \Bigg), \\
&h^2 \left\| s_h^{-1} S_{h1} \frac{\partial u'}{\partial z}; L^2(\mathbb{Q}_h) \right\|^2 + h^2 \left\| s_h^{-1} S_{h1} \nabla_y u_3; L^2(\mathbb{Q}_h) \right\|^2 \qquad (2.35) \\
&\quad \leq c \left(\left\| \nabla u^\perp; L^2(\mathbb{Q}_h) \right\|^2 + h^2 s_h \left(y^0 \right)^{-2} S_{h1} \left(y^0 \right)^2 h^3 \left(\left| \mathcal{U}_4^0 \right|^2 + \left| \mathcal{U}_5^0 \right|^2 \right) \right) \\
&\quad \leq c \Bigg(\left\| D(\nabla) u; L^2(\mathbb{Q}_h) \right\|^2 + h^2 \left\| s_h^{-1} S_{h1} \frac{\partial u'}{\partial z}; L^2(\mathbb{Q}_h') \right\|^2 + h^2 \left\| s_h^{-1} S_{h1} \nabla_y u_3; L^2(\mathbb{Q}_h') \right\|^2 \Bigg).
\end{aligned}$$

By the way, one may avoid to present (2.35) down because the $L^2(\Omega_h)$-norms of $\partial u_1/\partial y_2$ and $\partial u_2/\partial y_1$ had been estimated in (2.17). However, we observe that the derivatives $\partial u_i/\partial y_i = \varepsilon_{ii}(u)$ and $\partial_z u_3 = \varepsilon_{33}(u)$ figure in the stress column (1.5)–(1.7) and collect estimates obtained above to arrive at the equality

$$|||u; \mathbb{Q}_h|||_\bullet^2 \leq c \left(\left\| D(\nabla) u; L^2(\mathbb{Q}_h) \right\|^2 + |||u; \mathbb{Q}_h'|||_\bullet^2 \right).$$

Summing these inequalities up over all cubes which are erected from cells of the quadratic net of size h in the plane $\mathbb{R}^2$ and have nonempty intersection with the plate Ω_h, yields the estimate

$$|||u; \Omega_h|||_\bullet^2 \leq c \left(\left\| D(\nabla) u; L^2(\Omega_h) \right\|^2 + |||u; \Omega_{h/2}|||_\bullet^2 \right).$$

We combine it with (2.24) getting the result. ■

Remark 3 *As mentioned above, the optimality of distribution of the weights h and $s_h(y)$ in norms (2.6) and (2.7) is quite known. However, relation (2.13) shows that the norms $\left\| u_j; L^2(\mathbf{Q}_{hR}) \right\|$ in the small cylinder $\mathbf{Q}_{hR} = \mathbb{B}_{hR}^2 \times (-h/2, h/2)$ can be endowed with the big factor h^{-1} but weights in norm (2.11) give the following estimate only:*

$$|\ln h|^{-1} h^{-1} \left\| u_j; L^2(\mathbf{Q}_{hR}) \right\| \leq c |||u; \mathbb{Q}_h|||_\bullet.$$

In other words, it is worth to confirm impossibility of the change $S_{hq}(y) \mapsto \left(h^2 + |y|^2 \right)^{-q/2}$ in (2.11) with the simultaneous preservation of estimate (2.10).

Let ψ_i be smooth nontrivial functions such that $\psi_i(t) = 0$ for $t \notin (1/2, 1)$. We set $u_i(x) = \psi_i(|\ln r| / |\ln h|)$, $i = 1, 2$, $u_3(x) = 0$ and obtain

$$\varepsilon_{il}(u; x) = \frac{1}{r} \frac{1}{|\ln h|} \Psi_{il} \left(\frac{|\ln r|}{|\ln h|} \right), \quad i, l = 1, 2, \quad \varepsilon_{13}(u) = \varepsilon_{23}(u) = \varepsilon_{33}(u) = 0,$$

where again $\Psi_{il}(t) = 0$ for $t \notin (1/2, 1)$, that is $\Psi_{il}(|\ln r| / |\ln h|) = 0$ for $r \notin (h, \sqrt{h})$. We thus have

$$\left\| D(\nabla) u; L^2(\Omega_h) \right\|^2 \leq \frac{c_\Psi h}{|\ln h|^2} \int_h^{\sqrt{h}} \frac{r dr}{r^2} = \frac{c_\Psi h}{2 |\ln h|}. \qquad (2.36)$$

At the same time,

$$\left\|(h^2+|y|^2)^{-1/2}u_i; L^2(\Omega_h)\right\|^2 = 2\pi h\int_h^{\sqrt{h}}\left|\psi_i\left(\frac{|\ln r|}{|\ln h}\right)\right|^2\frac{rdr}{h^2+r^2}$$

$$\geq 2\pi h|\ln h|\int_{1/2}^1\left|\psi_i\left(\frac{|\ln r|}{|\ln h|}\right)\right|^2 d\frac{|\ln r|}{|\ln h|} = 2\pi h|\ln h|\left\|\psi_i; L^2\left(\tfrac{1}{2},1\right)\right\|^2,$$

$$\left\|(h^2+|y|^2)^{-1/2}(1+|\ln(h^2+|y|^2)|)^{-1}u_i; L^2(\Omega_h)\right\|^2$$

$$= 2\pi h\int_h^{\sqrt{h}}\left|\psi_i\left(\frac{|\ln r|}{|\ln h|}\right)\right|^2\left(1+\left|\ln(h^2+|y|^2)\right|\right)^{-2}\frac{rdr}{h^2+r^2}\leq\frac{c_\Psi h}{|\ln h|}.$$

Glancing over these formulas convinces that logarithms cannot be eliminated in the weighted norm (2.11).

2.3 Traction-free edge of the plate with several support areas

The approach applied above and described at length in the review paper [40] helps to derive asymptotically exact weighted anisotropic inequalities of Korn's type without requiring the lateral side υ_h of the plate to be clamped. Let us outline derivation of such inequalities.

To determine small clamped area on the lower base Σ_h^-, we fix some points $y^1,\ldots,y^J$ inside ω, $y^j\neq y^k$ for $j\neq k$, and put

$$\Theta_h = \vartheta_h^1\cup\cdots\cup\vartheta_h^J,\quad \vartheta_h^j = \left\{x: r_j := \left|y-y^j\right| < Rh,\ z=-h/2\right\}. \tag{2.37}$$

Real supporting sets, of course, may be bigger, for instance $\theta_h^j\supset\vartheta_h^j$ like in (1.3), but the Dirichlet condition

$$u(x) = 0,\quad x\in\Theta_h, \tag{2.38}$$

is sufficient for our purpose.

Since the resultant inequality is sensitive to the number of supporting sets θ_h^j, we focus on the case

$$J\geq 2. \tag{2.39}$$

Notice that the Korn inequality remains the same for $J=2$ and the most realistic case $J=3$. This and the peculiar case $J=1$ will be commented in Remark 6.

Theorem 4 *If $J\geq 2$ in (2.37), then any displacement field $u\in H_0^1(\Omega_h;\Theta_h)^3$ satisfies the weighted anisotropic Korn inequality*

$$|||u;\Omega_h|||_\odot\leq K_\odot(\omega)(1+|\ln h|)\left\|D(\nabla)u; L^2(\Omega_h)\right\| \tag{2.40}$$

with a constant $K_\odot(\omega)$ independent of $h\in(0,h_0]$ and the weighted Sobolev norm

$$|||u;\Omega_h|||_\odot^2 = \int_{\Omega_h}\Bigg[\sum_{i=1}^2\left(|\nabla_y u_i|^2 + h^2\mathbf{S}_{h1}^2\left(\left|\frac{\partial u_i}{\partial z}\right|^2+\left|\frac{\partial u_3}{\partial y_i}\right|^2\right)+\mathbf{S}_{h1}^2|u_i|^2\right)|\partial_z u_3|^2$$

$$+ h^2\mathbf{S}_{h2}^2|u_3|^2\Bigg]dx,$$

where

$$\mathbf{S}_{hq}(y) = \max\left\{S_{hq}\left(y - y^i\right) : j = 1, \ldots, J\right\} \tag{2.41}$$

and S_{hq} *are given in (2.9).*

Proof. We take a smooth displacement field u satisfying (2.38) and by the same means as in Section 2.2, see, e.g., (2.13), impose the subsidiary conditions, cf. (2.16),

$$u(y,z) = 0 \quad \text{for} \quad |y - y^j| < hR/2, \quad j = 1, \ldots, J. \tag{2.42}$$

First of all, we employ an elegant device from [49] and define the vector function $\mathcal{U}$ with components

$$\mathcal{U}_i(y,\zeta) = u_i(y, h\zeta), \quad i = 1,2, \quad \mathcal{U}_3(y,\zeta) = hu_3(y, h\zeta) \tag{2.43}$$

in the vertically inflated plate $\Omega_1 = \{(y,\zeta) : y \in \omega,\ |\zeta| < 1/2\}$. The crucial property of (2.43) is expressed by the relation

$$\begin{aligned}
&\left\|D(\nabla)u; L^2(\Omega_h)\right\|^2 \\
&= \int_{\Omega_h} \left[\sum_{i=1}^{2}\left(\left|\frac{\partial u_i}{\partial y_i}\right|^2 + \frac{1}{2}\left|\frac{\partial u_i}{\partial z} + \frac{\partial u_3}{\partial y_i}\right|^2\right) + \frac{1}{2}\left|\frac{\partial u_1}{\partial y_2} + \frac{\partial u_2}{\partial y_1}\right|^2 + \left|\frac{\partial u_3}{\partial z}\right|^2\right] dydz \\
&= h\int_{\Omega_1} \left[\sum_{i=1}^{2}\left(\left|\frac{\partial \mathcal{U}_i}{\partial y_i}\right|^2 + \frac{1}{2}h^{-2}\left|\frac{\partial \mathcal{U}_i}{\partial \zeta} + \frac{\partial \mathcal{U}_3}{\partial y_i}\right|^2\right) + \frac{1}{2}\left|\frac{\partial \mathcal{U}_1}{\partial y_2} + \frac{\partial \mathcal{U}_2}{\partial y_1}\right|^2 + h^{-4}\left|\frac{\partial \mathcal{U}_3}{\partial z}\right|^2\right] dyd\zeta \quad (2.44) \\
&\geq h\int_{\Omega_1} \left[\sum_{i=1}^{2}\left(\left|\frac{\partial \mathcal{U}_i}{\partial y_i}\right|^2 + \frac{1}{2}\left|\frac{\partial \mathcal{U}_i}{\partial \zeta} + \frac{\partial \mathcal{U}_3}{\partial y_i}\right|^2\right) + \frac{1}{2}\left|\frac{\partial \mathcal{U}_1}{\partial y_2} + \frac{\partial \mathcal{U}_2}{\partial y_1}\right|^2 + \left|\frac{\partial \mathcal{U}_3}{\partial z}\right|^2\right] dyd\zeta \\
&= h\left\|D\left(\nabla_y, \partial_\zeta\right)\mathcal{U}; L^2(\Omega_1)\right\|^2.
\end{aligned}$$

The term $\mathcal{U}^\perp$ in the decomposition, cf. (2.26),

$$\mathcal{U}(y,\zeta) = \mathcal{U}^\perp(y,\zeta) + d(y,\zeta)\,\mathcal{U}^0 \tag{2.45}$$

with the column

$$\mathcal{U}^0 = \mathbf{d}(\Omega_1)^{-1}\int_{\Omega_1} d(y,\zeta)^\top\,\mathcal{U}(y,\zeta)\,dyd\zeta \in \mathbb{R}^6$$

meets the orthogonality conditions (2.29) under the replacement $\mathbb{Q}_1' \mapsto \Omega_1$ and, therefore, using the Hardy and Korn inequalities (2.2) and (2.30) yields

$$\begin{aligned}
\left\|r_j^{-1}\left(1 + |\ln r_j|\right)^{-1}\mathcal{U}^\perp; L^2(\Omega_1)\right\|^2 &\leq c\left\|\mathcal{U}^\perp; H^1(\Omega_1)\right\|^2 \\
&\leq c\left\|D\left(\nabla_y, \partial_\zeta\right)\mathcal{U}^\perp; L^2(\Omega_1)\right\|^2 \\
&= c\left\|D\left(\nabla_y, \partial_\zeta\right)u_j; L^2(\Omega_1)\right\|^2.
\end{aligned} \tag{2.46}$$

Now in view of (2.45) and (2.42), we write

$$\mathbf{d}(\mathbf{Q}^j_{hR/2})\mathcal{U}^0 = -\int_{\mathbf{Q}^j_{hR/2}} d(y,\zeta)^\top\,\mathcal{U}^\perp(y,\zeta)\,dyd\zeta =: \mathcal{F}^j \in \mathbb{R}^6 \tag{2.47}$$

where $\mathbf{Q}^j_{hR/2} = \{(y,\zeta) : |y-y^j| < hR/2,\ |\zeta| < 1/2\}$ is a circular cylinder and, according to (2.46), the right-hand side admits the estimate

$$\begin{aligned} |\mathcal{F}^j| &\le c(\mathrm{mes}_3\,\mathbf{Q}^j_{hR/2})^{1/2} h\,(1+|\ln h|)\left\| r_j^{-1}\,(1+|\ln r_j|)^{-1}\,U^{\perp}; L^2\big(\mathbf{Q}^j_{hR/2}\big)\right\|^2 \\ &\le ch^2\,(1+|\ln h|)\,\|D\,(\nabla_y,\partial_z)\,U; L^2\,(\Omega_1)\|^2 \end{aligned} \tag{2.48}$$

Following [29], see also [40, §3.4], we sum up equations (2.47), $j=1,\dots,J$, and obtain the linear algebraic system

$$\mathcal{M}(h)U^0 = \mathcal{F} := \mathcal{F}^1 + \cdots + \mathcal{F}^J \in \mathbb{R}^6 \tag{2.49}$$

with the 6×6-matrix

$$\mathcal{M}\,(h) = \mathbf{d}\big(\mathbf{Q}^1_{hR/2}\big) + \cdots + \mathbf{d}\big(\mathbf{Q}^J_{hR/2}\big). \tag{2.50}$$

Each of summands in (2.50) is a Gram matrix, symmetric and positive definite. Moreover, by virtue of (2.25) and (2.28), a simple calculation shows that

$$\mathbf{d}\big(\mathbf{Q}^j_{hR/2}\big) = \frac{\pi}{4}h^2R^2\left(d\,(y^j,0)^{\top}\,d\,(y^j,0) + \frac{1}{12}\mathsf{t}^{\top}\mathsf{t} + O\,(h)\right) \tag{2.51}$$

where $O\,(h)$ stands for a 6×6-matrix of size 6×6 with the natural norm of order h and

$$\mathsf{t} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \frac{1}{12} = \int_{-1/2}^{1/2} \zeta^2 d\zeta, \quad \frac{\pi}{4}h^2R^2 = \mathrm{mes}_3\,\mathbf{Q}^j_{hR/2}. \tag{2.52}$$

We observe that under requirement (2.39) the inverse of matrix (2.50) satisfies the estimate

$$\left\|\mathcal{M}\,(h)^{-1}; \mathbb{R}^{6\times 6}\right\| \le ch^{-2}. \tag{2.53}$$

In fact, the 6×6-matrix $M^j = d\,(y^j,0)^{\top}\,d\,(y^j,0) + \frac{1}{12}\mathsf{t}^{\top}\mathsf{t}$ on the right of (2.51), is symmetric but only positive. However, in view of (2.50), estimate (2.53) is a direct consequence of the following fact:

$$b \in \mathbb{R}^6 \quad \text{and} \quad b^{\top}M^j b = 0,\ j=1,\dots,J \quad \Longrightarrow \quad b=0. \tag{2.54}$$

Moreover, the premise in (2.54) is equivalent to

$$\mathsf{t}b = 0 \ \text{ and } \ d\,(y^j,0)\,b = 0, \quad j=1,\dots,J. \tag{2.55}$$

We put the coordinate origin $y=0$ at the point y^1 and direct the y_1-axis through y^2 so that $y^2_1 > 0$ and $y^2_2 = 0$. Owing to (2.52) and (2.25) the equalities $\mathsf{t}b = 0$ and $d\,(y^1,0)\,b = 0$ in (2.55) assure that $b_4 = b_5 = 0$ and $b_1 = b_2 = b_3 = 0$. Furthermore,

$$d\,(y^2,0) - d\,(y^1,0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -y^2_1 \\ 0 & 0 & 0 & 0 & y^2_2 & 0 \end{pmatrix}$$

and, therefore, $b_6 = 0$ due to (2.55) with $j=1$ and $j=2$. So (2.53) is proved.

From (2.49) together with (2.53) and (2.48) we derive that

$$|\mathcal{U}^0| \le c\,(1+|\ln h|)\,\|D\,(\nabla_y,\partial_z)\,\mathcal{U}; L^2\,(\Omega_1)\|\,. \tag{2.56}$$

Thus, representation (2.45), estimates (2.46), (2.56) and relations (2.44), (2.43) between $\mathcal{U}$ and u give the following Korn inequality

$$|||u;\Omega_h|||^2 \leq c\,(1+|\ln h|)^2 \left\|D\,(\nabla)\,u;L^2\,(\Omega_h)\right\|^2, \tag{2.57}$$

with the only exception: after returning to x and u the right-hand side of (2.44) gains the integral $h^4\int|\partial_z u_3|^2\,dx$ instead of $\int|\partial_z u_3|^2\,dx$ as in (2.6). This gap is filled easily because, according to (1.5), (1.8), the expression $\left\|D\,(\nabla)\,u;L^2\,(\Omega_h)\right\|^2$ on the right of (2.57) includes $\int|\varepsilon_{33}\,(u)|^2\,dx=\int|\partial_z u_3|^2\,dx$.

A repetition of calculations in Section 2.2 (word by word but with $s_h=1$, the basic relations (2.21), (2.34) and (2.42) being preserved) allows us to introduce the weights $\mathbf{S}_{hq}$ given in (2.41) into the norm. Note that the real reason to put $s_h=1$ is the absence of the Dirichlet condition (1.14) on the lateral side υ_h and the consequent impossibility to apply the Hardy inequalities (2.1), (2.4).

A completion argument again completes the proof. ■

Remark 5 *The Dirichlet condition (2.38) at small support zones, see (2.37) and (2.39), cannot maintain the Korn inequality without the factor* $1+|\ln h|$ *as in (2.40). To corroborate this statement, we use the test function* $u_i\,(x)=\prod_{j=1}^{J}\chi\,(|\ln(r_j/R)|\,/\,|\ln h|)$ *where* χ *is taken from (2.14) so that* $u_i\,(x)=1$ *if all* $r_j=|y-y_j|>\sqrt{h}R$, *and* $u_i\,(x)=0$ *if some* $r_j<hR$. *Clearly, for* $u=e_iu_i$, *we obtain*

$$\left\|u;L^2\,(\Omega_h)\right\|^2=h^{1/2}(|\omega|^{1/2}+O\,(h)),\quad C\geq\left\|\mathbf{S}_{h1}u';L^2\,(\Omega_h)\right\|^2\geq c>0,$$

and, similarly to (2.36),

$$\left\|D\,(\nabla)\,u;L^2\,(\Omega_h)\right\|^2\leq\frac{c_\chi}{R^2\,|\ln h|^2}\int_h^{\sqrt{h}}\frac{rdr}{r^2}\leq\frac{c}{|\ln h|}.$$

The desired inference follows.

Remark 6 *If* $J=1$ *and* $y^1=0$, *the matrix* $\mathcal{M}\,(h)=\mathbf{d}(\mathbf{Q}^1_{hR/2})$ *in (2.50) becomes*

$$\pi^2h^2R^2\,\mathrm{diag}\left\{1,1,1,\tfrac14\left(\tfrac14+h^2R^2\right),\tfrac14\left(\tfrac14+h^2R^2\right),\tfrac12h^2R^2\right\}$$

and gets the right-hand bottom entry $O\left(h^4\right)$. *In this way (2.53) loses validity and estimate (2.56) alters crucially for* U^0_6. *Thus, the resultant inequality requires a serious modification, too (cf. [38] and [40, §3.4 and §5.2]). We mention the displacement field* $u\,(x)=\left(1-\chi\left(\frac{r}{2hR}\right)\right)(-y_2,y_1,0)^\top$ *which satisfies the relations*

$$h^{-1/2}\left\|u;L^2\,(\Omega_h)\right\|\geq c>0,\quad h^{-3/2}\left\|D\,(\nabla)\,u;L^2\,(\Omega_h)\right\|\leq C$$

and indicates a power-law growth of the Korn constant as $h\to+0$.

3 The convergence theorem and the three-dimensional boundary layer

3.1 The Kirchhoff model with the Sobolev point condition

In [5] we will present a detailed procedure of dimension reduction with turns the elasticity problem (1.11)–(1.14) into the two-dimensional Kirchhoff model of an anisotropic plate (1.1) clamped over the lateral side υ_h and the small area θ_h, cf. (1.13) and (1.14). We however start with a classical and simple result on convergence which can be achieved by any of known methods, cf. monographs [9, 10, 21, 37, 48] and other literature.

We assume the following representation[1] for the right-hand side in (1.11) :

$$f_i\left(h,y,z\right) = h^{-1/2}g_i(y),\ i=1,2, \qquad f_3\left(h,y,z\right) = h^{1/2}g_3(y) \tag{3.1}$$

with $g=(g_1,g_2,g_3)\in L^2\left(\omega\right)^3$.

If the rigidity matrix A has the form (1.10) in the isotropic Hooke's law (1.9), the average longitudinal displacements $w'=(w_1,w_2)^\top$ is solution of the two-dimensional elasticity system

$$-\mu\Delta_y w'(y)-(\lambda'+\mu)\nabla_y\nabla_y^\top w'(y)=g'(y), \qquad y\in\omega, \tag{3.2}$$

with $g'=(g_1,g_2)^\top$ and the deflexion w_3 is solution of the bi-harmonic equation

$$\frac{\mu}{3}\frac{\lambda+\mu}{\lambda+2\mu}\Delta_y^2 w_3\left(y\right)=g_3\left(y\right), \qquad y\in\omega. \tag{3.3}$$

Here, $\nabla_y=(\partial/\partial y_1,\partial/\partial y_2)^\top$, $\Delta_y=\nabla_y^\top\nabla_y$ is the Laplace operator in the y-variables, and the coefficient

$$\lambda'=\frac{2\lambda\mu}{\lambda+2\mu} \tag{3.4}$$

is computed through the Lamé constants $\lambda\geq0$ and $\mu\geq0$.

The condition (1.13) on the lateral side υ_h, see (1.2), requires for the Dirichlet condition on the contour $\partial\omega$

$$w_i(y)=0,\ i=1,2, \qquad w_3(y)=0, \qquad \partial_n w_3(y)=0, \qquad y\in\partial\omega, \tag{3.5}$$

where $\partial_n=n^\top\nabla_y$ and $n=(n_1,n_2)^\top$ is the unit vector of the outward normal at $\partial\omega$. Finally, the support area θ_h, see (1.3) and (1.13), is reflected by the Sobolev point condition

$$w_3\left(\mathcal{O}\right)=0. \tag{3.6}$$

It is well known, see, e.g., [6], that for any $g\in L^2\left(\omega\right)^3$ the problem (3.2), (3.3), (3.5), (3.6) has a unique generalized solution $w\in H_0^1\left(\omega\right)^2\times H_0^2\left(\omega\right)$ while $w_1,w_2\in H^2\left(\omega\right)$ and $w_3\in H^4_{loc}\left(\overline{\omega}\setminus\mathcal{O}\right)$ but in general $w_3\notin H^3\left(\omega\right)$. The next convergence theorem holds and it can be proved by an impalpable modification of the standard approach for a plate without the small support area θ_h.

Theorem 7 *The rescaled displacements $h^{3/2}u_3(h,y,h\zeta)$ and $h^{1/2}u_i(h,y,h\zeta)$, $i=1,2$, in the three-dimensional problem (1.11)–(1.14) with the right-hand side (3.1) converge in $L^2(\omega\times(-1/2,1/2))$ as $h\to+0$ to the functions $w_3(y)$ and $w_i(y)-\zeta\frac{\partial w_3}{\partial y_i}(y)$, $i=1,2$, respectively, where $\zeta=h^{-1}z$ is the stretched coordinate and $w=(w_1,w_2,w_3)^\top\in H_0^1\left(\omega\right)^2\times H_0^2\left(\omega\right)$ is a solution of the two-dimensional Dirichlet-Sobolev problem (3.2), (3.3), (3.5), (3.6).*

[1] In [5] this assumption will be weakned quite much.

The main unfavorable result of [5] reads: the convergence rate in Theorem 7 is unacceptably low, namely $O(|\ln h|^{-1/2})$ and thereafter we prepare for an elaboration of the asymptotic structures of elastic fields in Ω_h near θ_h.

3.2 Sketch of asymptotic expansions in a thin plate.

Theorem 7 remains valid for an anisotropic plate, i.e., with arbitrary rigidity matrix A in the Hooke's law, where the differential equations (3.2) and (3.3) are replaced with

$$\mathcal{L}'(\nabla_y)\, w'(y) = g'(y), \quad y \in \omega, \tag{3.7}$$

$$\mathcal{L}_3(\nabla_y)\, w_3(y) = g_3(y), \quad y \in \omega, \tag{3.8}$$

where the 2×2-matrix $\mathcal{L}'$ of second-order operators and the scalar fourth-order operator $\mathcal{L}_3$ are given by

$$\mathcal{L}'(\nabla_y) = \mathcal{D}'(-\nabla_y)^\top \mathcal{A}^0 \mathcal{D}'(\nabla_y), \tag{3.9}$$

$$\mathcal{L}_3(\nabla_y) = \frac{1}{6}\mathcal{D}_3(\nabla_y)^\top \mathcal{A}^0 \mathcal{D}_3(\nabla_y). \tag{3.10}$$

Here, the symmetric and positive definite 3×3-matrix A^0 is computed as follows:

$$A^0 = A_{(yy)} - A_{(yz)} A_{(zz)}^{-1} A_{(zy)}, \qquad A = \begin{pmatrix} A_{(yy)} & A_{(yz)} \\ A_{(zy)} & A_{(zz)} \end{pmatrix}$$

and

$$\mathcal{D}'(\nabla_y) = \begin{pmatrix} \partial_1 & 0 & 2^{-1/2}\partial_2 \\ 0 & \partial_2 & 2^{-1/2}\partial_1 \end{pmatrix}^\top, \quad \mathcal{D}_3(\nabla_y) = \left(2^{-1/2}\partial_1^2, 2^{-1/2}\partial_2^2, \partial_1\partial_2\right)^\top. \tag{3.11}$$

Calculations of asymptotic expansions and the differential operators (3.9), which are entirely adapted to the Mandel-Voigt notation, will be presented in [5] but also can be derived by any asymptotic procedure for an asymptotic analysis of thin plates, for example, [35] and [37, §4]. In Section 3 we will apply the ordinary asymptotic expansion of the solution to the problem (1.11)–(1.14) with the right-hand side (3.1)

$$u(h, x) \sim h^{-3/2} \sum_{p=0}^{3} h^p W^p(\zeta, \nabla_y)\, w(y) + \dots \tag{3.12}$$

written in an unusual form, i.e. with the help of the following 3×3-matrix differential operators

$$\begin{gathered} W^0(\zeta, \nabla_y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad W^1(\zeta, \nabla_y) = \begin{pmatrix} 1 & 0 & -\zeta\partial_1 \\ 0 & 1 & -\zeta\partial_2 \\ 0 & 0 & 0 \end{pmatrix}, \\ W^2(\zeta, \nabla_y) = \mathbb{J}^{-1} A_{(zz)}^{-1} A_{(zy)} \left(-\zeta \mathbb{I}_3, 2^{1/2}\left(\frac{\zeta^2}{2} - \frac{1}{24}\right)\mathbb{I}_3\right) \mathcal{D}(\nabla_y) \end{gathered} \tag{3.13}$$

where $\mathbb{I}_3 = \mathrm{diag}\{1, 1, 1\}$ and $\mathbb{J} = \mathrm{diag}\{2^{-1/2}, 2^{-1/2}, 1\}$ are diagonal matrices and $\mathcal{D}(\nabla_y)$ is the 6×3-matrix composed of the blocks (3.11)

$$\mathcal{D}(\nabla_y) = \begin{pmatrix} \mathcal{D}'(\nabla_y) & \mathbb{O}_{3\times1} \\ \mathbb{O}_{3\times2} & \mathcal{D}_3(\nabla_y) \end{pmatrix}$$

and the null matrices $\mathbb{O}_{p\times q}$ of size $p\times q$. Coherently with (3.14), the matrix differential operator $\mathcal{L}(\nabla_y)$ of the affiliated system (3.7), (3.8) involves the following block-diagonal matrix $\mathcal{A}$ of size 6×6,

$$\mathcal{A}=\mathrm{diag}\{A^0,\frac{1}{6}A^0\},\quad \mathcal{L}\left(\nabla\right)=\mathrm{diag}\{\mathcal{L}'\left(\nabla_y\right),\mathcal{L}_3\left(\nabla_y\right)\}. \tag{3.14}$$

Let us hint at the choice of operators (3.13) in (3.12) even if a detailed description of the dimension reduction procedure will be given in [5]. The differential operators $L\left(\nabla\right)$ and $N^{\pm}\left(\nabla\right)$ on the left in (1.11) and (1.12) admit the decompositions

$$\begin{aligned} L(\nabla)&=h^{-2}L^0\left(\partial_\zeta\right)+h^{-1}L^1\left(\nabla_y,\partial_\zeta\right)+h^0L^2\left(\nabla_y\right),\\ N^{\pm}(\nabla)&=h^{-1}N^{0\pm}\left(\partial_\zeta\right)+h^0N^{1\pm}\left(\nabla_y\right),\end{aligned} \tag{3.15}$$

where $\zeta=h^{-1}z\in(-1/2,1/2)$ is the stretched coordinate, $\partial_\zeta=\partial/\partial\zeta$, and

$$\begin{aligned} L^0\left(\partial_\zeta\right)&=D(0,0,-\partial_\zeta)^\top AD(0,0,\partial_\zeta),\qquad L^2\left(\nabla_y\right)=D(-\nabla_y,0)^\top AD(\nabla_y,0),\\ L^1\left(\nabla_y,\partial_\zeta\right)&=D(0,0,-\partial_\zeta)^\top AD(\nabla_y,0)+D(-\nabla_y,0)^\top AD(0,0,\partial_\zeta),\\ N^{0\pm}\left(\partial_\zeta\right)&=D(\pm e_3)^\top AD(0,0,\partial_z),\qquad N^{1\pm}\left(\nabla_y\right)=D(\pm e_3)^\top AD(\nabla_y,0).\end{aligned} \tag{3.16}$$

Then we have

$$\begin{aligned} L\sum_{p=0}^{3}h^pW^p&=h^{-2}L^0W^0+h^{-1}\left(L^0W^1+L^1W^0\right)\\ &\quad+h^0\left(L^0W^2+L^1W^1+L^2W^0\right)+h^1\left(L^0W^3+L^1W^2+L^2W^1\right)\\ &\quad+h^2\left(L^1W^3+L^2W^2\right)+h^3L^2W^3=:\sum_{q=0}^{5}h^{q-2}F^q,\\ N^{\pm}\sum_{p=0}^{3}h^pW^p&=h^{-1}N^{0\pm}W^0+h^0\left(N^{0\pm}W^1+N^{1\pm}W^0\right)\\ &\quad+h^1\left(N^{0\pm}W^2+N^{1\pm}W^1\right)+h^2\left(N^{0\pm}W^3+N^{1\pm}W^2\right)\\ &\quad+h^3+N^{1\pm}W^3=:\sum_{q=0}^{4}h^{q-1}G^{q\pm}.\end{aligned} \tag{3.17}$$

The operators (3.13) are selected such that

$$F^q=0,\ G^{q\pm}=0\text{ with }q=0,1,2. \tag{3.18}$$

It is not possible to annul both F^3 and $G^{3\pm}$ but $W^3\left(\zeta,\nabla_y\right)$ is fixed such that

$$\left(F_1^3,F_2^3\right)^\top=\mathcal{L}'\left(\nabla_y\right)\quad\text{and}\quad F_3^3=0,\ G^{3\pm}=0. \tag{3.19}$$

Moreover,

$$\int_{-1/2}^{1/2}F_3^4d\zeta+G_3^{4\pm}+G_3^{4-}=\mathcal{L}_3\left(\nabla_y\right). \tag{3.20}$$

The differential operators $W^p\left(\zeta,\nabla_y\right)$ in (3.12), (3.13) as well as (3.9) in (3.19), (3.20) are defined uniquely through the matrices $D(\nabla)$ and A in (1.8) and (1.9). For general elliptic problem, an algebraic procedure to construct asymptotics type (3.12) and the resultant operator $\mathcal{L}(\nabla_y)$ in (3.14) is developed in [28], see also [37, §2 Ch.5] and the review [34]. This procedure serves for constructing asymptotic expansions in thin domains and in unbounded domains with cylindrical and layer-shaped outlets to infinity, cf. [5, §2] and Section 3.4 below.

Finally it should be mentioned that explicit formulas for $W^3(\zeta, \nabla_y) w(y)$ and higher order terms in (3.12) are available but are of no further use.

3.3 Boundary layer effects

Although Theorem 7 singles out the solution w of the limit problem (3.7), (3.8), (3.5), (3.6) in ω as a limit of the solution u of the original problem in the thin plate Ω_h, the Dirichlet conditions (1.13) at the small support zone θ_h maintain only the point condition (3.6) for the deflection w_3 and are not reflected in the problem for the longitudinal displacements w'. At the same time, the components w_1 and w_2 of w' leave small discrepancies in conditions (1.13) unless accidentally. As usual, a variation of boundary conditions on a set with a small diameter brings boundary layer effects, see [23, Ch.5] for general apprehension. However, due to the assumed comparability of the plate thickness h and diameters of ω_h^j the boundary layer in problem (1.11)–(1.14) exhibits a very specific and intricate structure. To describe it, we introduce the stretched coordinates (2.12) and observe that changing $x \mapsto \xi$ and putting $h = 0$ transform the domains Ω_h and θ_h into the layer (1.18) between the planes $\Sigma_\pm = \mathbb{R}^2 \times \{\pm 1/2\}$ and the set $\theta \subset \Sigma_-$ of unit size respectively. We also denote $\Sigma_\bullet = \Sigma_- \setminus \overline{\theta}$.

Let us recall the following weighted anisotropic Korn inequality which is proved in [27], see also [32] and [37], and looks quite similar to (2.40) and (2.10).

Lemma 8 *For any smooth and compactly supported vector function v satisfying the Dirichlet condition on θ the inequality*

$$\left\| u; V_0^1(\Lambda; \theta) \right\| \leq c \left\| D(\nabla_\xi) v; L^2(\Lambda) \right\| \tag{3.21}$$

is valid, where

$$\left\| u; V_0^1(\Lambda; \theta) \right\|^2 = \int_\Lambda \left[\sum_{i=1}^{2} \left(|\nabla_\xi v_i|^2 + S_1^2 \left(\left| \frac{\partial v_i}{\partial \zeta} \right|^2 + \left| \frac{\partial v_3}{\partial \eta_i} \right|^2 + |v_i|^2 \right) \right) + |\partial_\zeta v_3|^2 + S_2^2 |v_3|^2 \right] d\xi, \tag{3.22}$$

the weighted space $V_0^1(\Lambda; \theta)$ is a completion of $C_c^\infty \left(\overline{\Lambda} \setminus \overline{\theta} \right)^3$ with respect to the norm (3.22) and

$$S_k(\eta) = \left(1 + \rho^2\right)^{-k/2} \left(1 + \ln\left(1 + \rho^2\right)\right)^{-1}, \qquad \rho = |\eta|.$$

Our aim is to investigate the elasticity problem

$$-D(-\nabla_\xi)^\top A D(\nabla_\xi) v(\xi) = F(\xi), \qquad \xi \in \Lambda, \tag{3.23}$$

$$\begin{cases} D(e_3)^\top A D(\nabla_\xi) v(\xi) = G^+(\xi), & \xi \in \Sigma^+, \\ D(-e_3)^\top A D(\nabla_\xi) v(\xi) = G^\bullet(\xi), & \xi \in \Sigma^\bullet, \end{cases} \tag{3.24}$$

$$v(\xi) = 0, \qquad \xi \in \theta, \tag{3.25}$$

in layer (1.18) clamped along the area θ; we assume that

$$\begin{aligned} & S_1^{-1}F_i \in L^2(\Lambda), \qquad S_1^{-1}G_i^{\pm} \in L^2(\Sigma^{\pm}), \qquad i=1,2, \\ & F_3(\eta,\zeta) = F_3^0(\eta,\zeta) + F_3^1(\eta), \qquad \int_{-1/2}^{1/2} F_3^0(\eta,\zeta)d\zeta + \sum_{\pm} G_3^{\pm}(\eta) = 0, \\ & S_2^{-1}F_3^0 \in L^2(\mathbb{R}^2), \qquad F_3^0 \in L^2(\Lambda), \qquad G_3^{\pm} \in L^2(\Sigma^{\pm}), \end{aligned} \tag{3.26}$$

where G^- is an extension of $G^\bullet$ over θ (observe that $u=0$ on θ).

By Lemma 8 we can give a variational formulation to problem (3.23)-(3.25).

Multiplying (3.23) scalarly with a test function $u \in C_c^\infty(\overline{\Lambda} \setminus \overline{\theta})^3$, smooth and compactly supported, and integrating by parts with the help of the Neumann boundary conditions (3.24) lead to the integral identity

$$(AD(\nabla_\xi)v, D(\nabla_\xi)u)_\Lambda = (F,u)_\Lambda + \sum_{\pm} (G^{\pm},u)_{\Sigma^{\pm}}. \tag{3.27}$$

Then the following result holds.

Proposition 9 *Under conditions (3.26), problem (3.23)–(3.25) has a unique weak solution* $v \in V_0^1(\Lambda;\theta)^3$ *verifying the integral identity (3.27) and the norm* $\|u; V_0^1(\Lambda;\theta)\|$*, see (3.22), does not exceed the sum of norms of functions in (3.26) multiplied with a constant.*

Proof. By virtue of the orthogonality condition in (3.26), the right-hand side $\mathcal{F}(u)$ of (3.27) can be rewritten as follows:

$$\begin{aligned} \mathcal{F}(u) = & \sum_{i=1}^{2} \left((F_i,u_i)_\Lambda + \sum_{\pm} (G_i^{\pm},u_i)_{\Sigma^{\pm}} \right) + (F_3^0,\overline{u}_3)_{\mathbb{R}^2} \\ & + (F_3^1, u_3-\overline{u}_3)_\Lambda + \sum_{\pm} (G_3^{\pm}, u_3-\overline{u}_3)_{\Sigma^{\pm}} \end{aligned} \tag{3.28}$$

where $\overline{u}_3(\eta) = \int_{-1/2}^{1/2} u_3(\eta,\zeta)\,d\zeta$ and $\|S_2\overline{u}_3; L^2(\mathbb{R}^2)\| \le \|S_2u_3; L^2(\Lambda)\|$.

Since $\partial_\zeta u_3 = \varepsilon_{33}(u) \in L^2(\Lambda)$, the one-dimensional Poincaré and trace inequalities in the interval $(-1/2,1/2) \ni \zeta$ integrated over the plane $\mathbb{R}^2 \ni \eta$, provide that

$$\begin{aligned} & \|u_3-\overline{u}_3; L^2(\Lambda)\| + \|u_3-\overline{u}_3; L^2(\Sigma^{\pm})\| \\ & \le c\|\partial_z(u_3-\overline{u}_3); L^2(\Lambda)\| = c\|\partial_z u_3; L^2(\Lambda)\| \le c\|u; V_0^1(\Lambda;\theta)\|. \end{aligned} \tag{3.29}$$

These together with the weighted trace inequality

$$\|S_1u_i; L^2(\Sigma^{\pm})\| \le c\left(\|S_1\partial_\zeta u_i; L^2(\Lambda)\| + \|S_1u_i; L^2(\Lambda)\|\right) \le c\|u; V_0^1(\Lambda;\theta)\|$$

demonstrate that, under conditions (3.26), we have the continuous functional (3.28) on the right in (3.27) if test functions are taken from the space $V_0^1(\Lambda;\theta)^3$. Since inequality (3.21) serves the left-hand side of (3.27) to be a scalar product in $V_0^1(\Lambda;\theta)^3$, the Riesz representation theorem proves the thesis. ■

All elements of $V^1(\Lambda)$ (no condition on θ is imposed) belong to $H^1_{loc}(\overline{\Lambda})$. Any rigid motion, except for the rotation $\xi_1 e_2 - \xi_2 e_1$ which makes the integral in (3.22) divergent, falls into $V^1(\Lambda)^3$ because it can be approximated in norm (3.22) by smooth compactly supported vector functions (see [30, 32] for details). We introduce the submatrix

$$d^\sharp(\eta,\zeta) = \begin{pmatrix} 1 & 0 & 0 & \zeta \\ 0 & 1 & -\zeta & 0 \\ 0 & 0 & \eta_2 & -\eta_1 \end{pmatrix} \tag{3.30}$$

of rigid motion and notice that the third and sixth columns are excluded from the original matrix $d(\eta,\zeta)$ in (2.25). Exhibiting a general result in [32], the next assertion detect

$$v^\sharp(\eta,\zeta) = d^\sharp(\eta,\zeta)c^\sharp, \quad c^\sharp \in \mathbb{R}^4, \tag{3.31}$$

as the main asymptotic term of the solution $v \in V^1_0(\Lambda;\theta)^3$ under certain conditions on the right-hand sides.

Proposition 10 *Let the right-hand sides of problem (3.23)–(3.25) satisfy the following smoothness and decay requirements*

$$\begin{aligned} \left|\nabla^p_\eta \partial^q_\zeta F_i(\xi)\right| + \left|\nabla^p_\eta G^+_i(\eta)\right| + \left|\nabla^p_\eta G^\bullet_i(\eta)\right| &\le c_{pq}\rho^{-2+\epsilon}, \qquad i=1,2, \\ \left|\nabla^p_\eta \partial^q_\zeta F^0_3(\xi)\right| + \left|\nabla^p_\eta G^+_3(\eta)\right| + \left|\nabla^p_\eta G^\bullet_3(\eta)\right| &\le c_{pq}\rho^{-2+\epsilon}, \\ \left|\nabla^p_\eta \partial^q_\zeta F^1_3(\xi)\right| &\le c_{pq}\rho^{-3+\epsilon}, \qquad \textit{for } \rho \ge R_\theta, \end{aligned} \tag{3.32}$$

where $p,q = 0,1,2,\dots$, $\epsilon \in (0,1)$ and R_θ is such that $\overline{\theta} \subset \mathbb{B}^2_{R_\theta}$. Then a solution $v \in V^1_0(\Lambda;\theta)^3$ of problem (3.23)–(3.25) given in Proposition 9, becomes smooth for $\rho \ge R^0_\theta$ with any $R^0_\theta > R_\theta$ and verifies the inequalities

$$\left|\nabla^p_\eta \partial^q_\zeta \left(v_i(\xi) - v^0_i(\xi)\right)\right| \le c_{pq}\rho^{\epsilon}, \quad \left|\nabla^p_\eta \partial^q_\zeta \left(v_3(\xi) - v^0_3(\xi)\right)\right| \le c_{pq}\rho^{1+\epsilon} \quad \textit{for } \rho \ge R^0_\theta, \tag{3.33}$$

where v^0 is the rigid motion (3.31) with a coefficient column $c^\sharp \in \mathbb{R}^4$ depending on F, G^+, $G^\bullet$ and other notation is the same as in (3.32).

Note that the imposed orthogonality condition in (3.26) allowed us in Proposition 9 to reduce the decay requirement on F^0_3 and $G^\pm_3$ because in the examination of functional (3.28) we used inequality (3.29) without referring to the weighted norm (3.22). Furthermore, these components can be compensated by a solution of a problem on the interval $(-1/2,1/2) \ni \zeta$ which inherits the decay properties from F^0_3 and G^+_3, $G^\bullet_3$ in (3.32) and thus does not influence main asymptotic terms indicated in (3.33).

The asymptotic form $v = v^0 + \widetilde{v}$ is detected in [32] by means of the dimension reduction, cf. Section 3.1, and an application of the Kondratiev theory [17] (see also [41, Ch.3 and Ch.6]) together with various weighted forms of Korn's inequality. The paper [32] furnishes complete multi-scale decompositions of elastic fields in a layer and we will precise the asymptotic forms of Proposition 10 in Section 3.6.

3.4 The fundamental matrix of the differential operator $\mathcal{L}(\nabla_y)$

The most easily understood way to compensate for, e.g., the main part of the discrepancy

$$h^{-1/2}w_i(y)\,e_i = h^{-1/2}w_i(\mathcal{O})\,e_i + O(h^{1/2}) \quad \text{for } y \in \theta_h$$

in condition (1.13) is just to solve problem (3.23)–(3.25) with $F = 0$ and $G^+ = 0$, $G^\bullet = 0$ but with the right-hand side $-w_i(\mathcal{O})\,e_i$ in the Dirichlet condition (3.25) on θ. Since by an appropriate extension the inhomogeneity can be passed over to the right-hand sides in equations (3.23), (3.24), Proposition 9 and our further comments on the rigid motion (3.31) demonstrate that the unique solution $v \in V^1(\Lambda)^3$ of the problem is nothing but the constant vector $-w_i(\mathcal{O})\,e_i$ which does not decay as $\rho \to +\infty$ and by no means can be accepted as a boundary layer. Thereupon, instead of solutions offered by Propositions 9 and 10 we prefer to employ a solution of the homogeneous problem (3.23)–(3.25) which, of course, must live outside $V_0^1(\Lambda;\theta)^3$. Asymptotics at infinity of elastic fields with power-logarithmic growth in a layer has been prepared in the paper [32] and to achieve the goal we need a certain notation only.

General results in [15] supply us with the fundamental matrix of size 2×2

$$\Phi'(y) = \Psi' \ln r + \psi'(\varphi) \tag{3.34}$$

of the elliptic 2×2-matrix (3.9) of the second-order differential operators. Here, (r, φ) is the polar coordinate system on the plane $\mathbb{R}^2 \ni y$, Ψ' is a numeral nondegenerate symmetric 2×2-matrix and ψ' is a smooth matrix function on the unit circle $\mathbb{S}$. By its meaning, the fundamental matrix (3.34) satisfies the relation

$$-\int_\gamma \mathcal{N}'(y, \nabla_y)\,\Phi'(y)\,ds_y = \mathbb{I}_2 \in \mathbb{R}^{2\times 2}$$

where γ is any smooth closed simple contour enveloping the y-coordinates origin $\mathcal{O}$ and

$$\mathcal{N}'(y, \nabla_y) = \mathcal{D}'(n(y))^\top \mathcal{A}'\mathcal{D}'(\nabla_y) \tag{3.35}$$

is the Neumann condition operator for (3.9) with the unit vector $n = (n_1, n_2)^\top$ of the outward normal at γ. Since the matrix ψ' in (3.34) is defined up to a constant summand, it can be fixed such that

$$\int_\gamma' \Phi(y)^\top \mathcal{N}'(y, \nabla_y)\,\Phi'(y)\,ds_y = \mathbb{O}_2 \in \mathbb{R}^{2\times 2}. \tag{3.36}$$

The scalar forth-order elliptic operator (3.10) possesses the fundamental solution $\Phi_3(y)$ which we write down in the convenient form

$$\Phi_3(y) = r^2\left(-\frac{1}{2}\Psi_3 \ln r + \psi_3(\varphi)\right) \tag{3.37}$$

where $\Psi_3 \neq 0$ is a number and ψ_3 is a function on $\mathbb{S}$. Notice that $\Phi(y) = \operatorname{diag}\{\Phi'(y), \Phi_3(y)\}$ implies the fundamental matrix of the operator $\mathcal{L}(\nabla_y)$ in (3.9).

In the isotropic case (1.10) we have

$$\Phi'(y) = \frac{1}{4\pi}\frac{\lambda' + 3\mu}{\mu(\lambda' + 2\mu)}\begin{pmatrix} -\ln r + \beta y_1^2 r^{-2} & \beta y_1 y_2 r^{-2} \\ \beta y_2 y_1 r^{-2} & -\ln r + \beta y_2^2 r^{-2} \end{pmatrix}, \quad \Phi_3(y) = \frac{3}{8\pi}\frac{\lambda + 2\mu}{\mu(\lambda + \mu)} r^2 \ln r$$

where $\beta = (\lambda' + 3\mu)^{-1} (\lambda' + \mu)$, λ and μ are the Lamé constants and λ' is defined in (3.4).

We now introduce the 3×4-matrix

$$\Phi^{\sharp}(y) = \left(d^{\sharp}(-\nabla_y, 0)^{\top} \Phi(y)^{\top} \right)^{\top} = \begin{pmatrix} \Phi'(y) & & 0 & 0 \\ & & 0 & 0 \\ 0 & 0 & \Phi_3^2(y) & \Phi_3^1(y) \end{pmatrix} \tag{3.38}$$

where $\Phi'(y)$ from (3.34) has size 2×2 and

$$\Phi_3^i(y) := \frac{\partial \Phi_3}{\partial y_i}(y) = \Psi_3 y_i \ln r + r \psi_3^i(\varphi), \quad i = 1, 2. \tag{3.39}$$

To clarify properties of (3.37) and (3.39), we observe that $\mathcal{D}_3(\nabla_y) = 2^{-1/2} \mathcal{D}'(\nabla_y) \nabla_y$ in (3.11) and integrate by parts as follows:

$$(\mathcal{L}_3 w_3, v_3)_{\Gamma} + (\mathcal{N}_3 w_3, (1, \nabla_y) v_3)_{\gamma} = (w_3, \mathcal{L}_3 v_3)_{\Gamma} + ((1, \nabla_y) w_3, \mathcal{N}_3 v_3)_{\gamma}. \tag{3.40}$$

Here, Γ is a domain bounded by the contour γ (actually, we need $\Gamma = \omega$ or $\Gamma = \mathbb{B}_R^2$) and $\mathcal{N}_3 = (\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2)$,

$$\begin{aligned} \mathcal{N}_0(y, \nabla_y) &= 2^{-1/2} n(y)^{\top} \mathcal{D}'(-\nabla_y)^{\top} \mathcal{A}_3 \mathcal{D}_3(\nabla_y), \\ \mathcal{N}_i(y, \nabla_y) &= -2^{-1/2} e_i^{\top} \mathcal{D}'(n(y))^{\top} \mathcal{A}_3 \mathcal{D}_3(\nabla_y). \end{aligned} \tag{3.41}$$

The fundamental solution (3.37) and its derivatives (3.39) satisfy the relations, with $i, k = 1, 2$

$$\begin{aligned} -(1, \mathcal{N}_0 \Phi_3)_{\gamma} &= 1, & -((1, \nabla_y) y_k, \mathcal{N}_3 \Phi_3)_{\gamma} &= 0, \\ -\left(1, \mathcal{N}_3 \Phi_3^i\right)_{\gamma} &= 0, & \left((1, \nabla_y) y_k, \mathcal{N}_3 \Phi_3^i\right)_{\gamma} &= \delta_{i,k}. \end{aligned} \tag{3.42}$$

Remark 11 *The simplest way to check up (3.42) requires to use the Dirac mass $\delta(y)$ in the framework of the theory of distributions. Since $\mathcal{L}_3(\nabla_y) \Phi_3(y) = \delta(y)$ by definition, we apply formula (3.40) with either $w_3 = 1$, $v_3 = \Phi_3$ or $w_3 = y_k$, $v_3 = \Phi_3^i$ and obtain*

$$-((1, \nabla_y) 1, \mathcal{N}_3 \Phi_3)_{\gamma} = (1, \delta)_{\Gamma} = 1, \quad -\left((1, \nabla_y) y_k, \mathcal{N}_3 \Phi_3^i\right)_{\gamma} = (y_k, \partial \delta / \partial y_i)_{\Gamma} = -\delta_{i,k}.$$

Other relations in (3.42) as well as (3.35), (3.36) are verified in the same way.

3.5 The elastic logarithmic capacity

According to a general procedure in [32] we search for a matrix solution of the homogeneous problem (3.23)–(3.25) in the form

$$\mathcal{P}(\xi) = \left(1 - \chi\left(\frac{\rho}{2R_\theta}\right)\right) \sum_{p=0}^{3} W^p(\zeta, \nabla_\eta) \Phi^{\sharp}(\eta) + \widehat{\mathcal{P}}(\xi) \tag{3.43}$$

where the notation from (3.12) is used and $\widehat{\mathcal{P}} \in V_0^1(\Lambda; \omega_1)^{3 \times 4}$ is a remainder to be determined. Note that R_θ is fixed such that $\overline{\theta} \subset \mathbb{B}_{R_\theta}^2$.

We insert (3.43) into the differential equations (3.23) with $F=0$ and take decomposition (3.15) into account. If $\rho > R_\theta$ and $1-\chi(\rho/2R_\theta)=1$, we obtain for the sum $\Xi(\eta,\zeta)$ of the detached terms in (3.43) that

$$\begin{aligned} D(-\nabla_\eta)^\top AD(\nabla_\eta)\Xi &= L^0W^0\Phi^\sharp + \left(L^0W^1+L^1W^0\right)\Phi^\sharp \\ &+ \left(L^0W^2+L^1W^1+L^2W^0\right)\Phi^\sharp + \left(L^0W^3+L^1W^2+L^2W^1\right)\Phi^\sharp \\ &+ \left(L^1W^3+L^2W^2\right)\Phi^\sharp + L^2W^3\Phi^\sharp. \end{aligned} \tag{3.44}$$

According to the content of Section 3.2, first four items on the right in (3.44) vanish; recall that $\mathcal{D}\left(-\nabla_\eta\right)^\top \mathcal{AD}\left(\nabla_\eta\right)\Phi^\sharp(\eta)=0$ in the punctured plane $\mathbb{R}^2\setminus 0$. Owing to (3.16), (3.13) and (3.34), (3.39), the last two terms in (3.44) are of order ρ^{-3} and ρ^{-4} respectively.

In the Neumann boundary conditions (3.24) we have

$$\begin{aligned} D\left(\pm e_3\right)^\top AD\left(\nabla_\eta\right)\Xi &= N^{0\pm}W^0\Phi^\sharp + \left(N^{0\pm}W^1+N^{1\pm}W^0\right)\Phi^\sharp \\ &+ \left(N^{0\pm}W^2+N^{1\pm}W^1\right)\Phi^\sharp + \left(N^{0\pm}W^3+N^{1\pm}W^2\right)\Phi^\sharp + N^{1\pm}W^3\Phi^\sharp. \end{aligned} \tag{3.45}$$

First four items on the right vanish again. Recalling that the equation (3.20) appeared as a result of calculation (3.17), we detect the following relation which is crucial for our further consideration: for $\eta\in\mathbb{R}^2\setminus 0$

$$\begin{aligned} e_3^\top \Bigg(&\int_{-1/2}^{1/2} L^1\left(\nabla_\eta,\partial_\zeta\right)W^3\left(\zeta,\nabla_\eta\right) + L^2(\nabla_\eta)W^2\left(\zeta,\nabla_\eta\right)d\zeta \\ &+ \sum_{\pm} N^{1\pm}\left(\nabla_\eta\right)W^3\left(\pm\frac{1}{2},\nabla_\eta\right)\Phi^\sharp\left(\eta\right)\Bigg) = 0. \end{aligned} \tag{3.46}$$

Let us compose a problem of type (3.23)–(3.25) for the remainder $\widehat{\mathcal{P}}$ in (3.43). Since the cut-off function $1-\chi(2\rho/R_\theta)$ is null near the η-coordinates origin where $\Phi^\sharp$ becomes singular, the right-hand sides $\widehat{F}$ and $\widehat{G}^+$, $\widehat{G}^\bullet$ in the problem are smooth and, outside a big ball, coincide with (3.44) and (3.45) respectively. Formula (3.46) furnishes the representation $\widehat{F}_3\left(\xi\right)=\widehat{F}_3^0\left(\xi\right)+\widehat{F}_3^1\left(\eta\right)$ for $\rho>2R_\theta$ where

$$\begin{aligned} \widehat{F}_3^1\left(\eta\right) &= e_3^\top\int_{-1/2}^{1/2} L^2\left(\nabla_\eta\right)W^3\left(\zeta,\nabla_\eta\right)d\zeta\Phi^\sharp\left(\eta\right) = O(\rho^{-4}), \\ \widehat{F}_3^0\left(\xi\right) &= e_3^\top\left(L^1\left(\nabla_\eta,\partial_\zeta\right)W^3\left(\zeta,\nabla_\eta\right)+L^2\left(\nabla_\eta\right)W^2\left(\zeta,\nabla_\eta\right)\right)\Phi^\sharp\left(\eta\right) + \\ &+ e_3^\top L^2\left(\nabla_\eta\right)W^3\left(\zeta,\nabla_\eta\right)\Phi^\sharp\left(\eta\right) - \widehat{F}_3^1\left(\eta\right) = O(\rho^{-3}). \end{aligned} \tag{3.47}$$

Note that

$$G^\pm\left(\eta\right) = N^{1\pm}\left(\nabla_\eta\right)W^3\left(\pm 1/2,\nabla_\eta\right)\Phi^\sharp\left(\eta\right) = O(\rho^{-3}), \quad \rho>2R_\theta. \tag{3.48}$$

Thus, all hypotheses in Proposition 9 are fulfilled and then remainder $\widehat{\mathcal{P}}\in V_0^1\left(\Lambda;\omega_1\right)^{3\times 4}$ exists.

Formulas (3.47) and (3.48) also allow us to derive from Proposition 10 the representation

$$\widehat{\mathcal{P}}\left(\xi\right) = d^\sharp\left(\eta,\zeta\right)C^\sharp + \widetilde{\mathcal{P}}\left(\xi\right) \tag{3.49}$$

where the rows $\widetilde{\mathcal{P}}_i$, $i=1,2$, and $\widetilde{\mathcal{P}}_3$ of the 3×4-matrix $\widetilde{\mathcal{P}}$ meet the inequalities

$$\left|\nabla_\eta^p \partial_\zeta^q \widetilde{\mathcal{P}}_i\left(\xi\right)\right| \le c_{pq}\rho^\epsilon, \quad \left|\nabla_\eta^p \partial_\zeta^q \widetilde{\mathcal{P}}_3\left(\xi\right)\right| \le c_{pq}\rho^{1+\epsilon} \quad \text{for } \rho \ge R_\theta^0, \tag{3.50}$$

with any $\epsilon \in (0,1)$.

We call the numerical matrix $C^\sharp = C^\sharp\left(A,\theta\right)$ of size 4×4 the *elastic logarithmic capacity* (matrix), cf. [25] for the isotropic case. It depends on the stifness matrix A in the Hooke law (1.9) and the shape of the clamped zone θ and is defined uniquely through decompositions (3.43), (3.49) of the *elastic logarithmic potential* matrix $\mathcal{P}\left(\xi\right)$ of size 3×4.

Remark 12 *The names introduced above come by analogy with the logarithmic capacity potential $P\left(\eta\right)$ in harmonic analysis, cf. [20, 47]. The function P is harmonic in $\mathbb{R}^2\setminus\overline{\vartheta}$, vanishes at the boundary $\partial\vartheta$ of a compact set ϑ and admits the representation*

$$P\left(\eta\right) = \left(2\pi\right)^{-1}\left(\ln\rho^{-1} + C_{\log}\left(\vartheta\right)\right) + O(\rho^{-1}), \quad \rho \to +\infty,$$

while $-\left(2\pi\right)^{-1}\ln\left|\eta\right|$ is the fundamental solution of the Laplacian in the plane $\mathbb{R}^2$ and $C_{\log}\left(\vartheta\right)$ is called the logarithmic capacity. Clearly, P also solves the mixed boundary value problem in the perforated layer $\Theta = \left\{\xi : \eta\in\mathbb{R}^2\setminus\overline{\vartheta},\ |\zeta| < 1/2\right\}$, namely

$$-\Delta_\xi P\left(\xi\right) = 0,\ \xi\in\Theta, \quad P\left(\xi\right) = 0,\ \xi\in\partial\vartheta\times\left(-1/2,1/2\right), \quad \pm\partial_\zeta P\left(\eta,\pm 1/2\right) = 0,\ \eta\in\mathbb{R}^2\setminus\overline{\vartheta}$$

which really looks quite similar to a scalar version of the elasticity problem (3.23)–(3.25) under consideration. Note that, as distinct from the standard harmonic capacity in dimension $n\ge 3$ which always is positive, the logarithmic capacity $C_{\log}\left(\vartheta\right)$ can be positive (the obstacle ϑ is big) or negative (ϑ is small).

The elastic logarithmic capacity matrix $C^\sharp$, in general, is neither positive, nor negative but still symmetric. The latter is proved in [25] for an isotropic layer Λ with a defect and we here serve for a much more complicated anisotropic case.

Theorem 13 *The elastic logarithmic capacity 4×4-matrix $C^\sharp = C^\sharp\left(A,\theta\right)$ is symmetric.*

Proof. We insert on both positions in the Green formula for the operator $L\left(\nabla_\xi\right)$ in the truncated layer $\Lambda\left(T\right) = \left\{\xi\in\Lambda : \rho < T\right\}$ and let $T\to+\infty$. Since $\mathcal{P}$ verifies the homogeneous problem (3.23)–(3.25), we are left with an integral over the truncation surface $\mathbb{S}_T\times(-1/2,1/2)$, the lateral surface of the circular cylinder $\Lambda\left(T\right)$. We have

$$0 = \int_{\mathbb{S}_T}\int_{-1/2}^{1/2}\left(\mathcal{P}(\xi)^\top D(\rho^{-1}\eta,0)^\top AD\left(\nabla_\xi\right)\mathcal{P}(\xi) - \left(D(n(\eta),0)^\top AD\left(\nabla_\xi\right)\mathcal{P}(\xi)\right)^\top\mathcal{P}(\xi)\right)d\zeta ds_\eta \tag{3.51}$$

where $n\left(\eta\right) = \rho^{-1}\eta$ is the unit vector of the outward normal and $ds_\eta = \rho d\varphi$ is the arc element on the circle $\mathbb{S}_T$.

We now extract from the integrand in (3.51) all terms of order ρ^{-1} which contribute to the limit. Infinitesimal terms $o\left(\rho^{-1}\right)$, in particular ones generated by the remainder $\widetilde{\mathcal{P}}$ in (3.49) can be removed from the integrand with a clear reason while terms growing as $\rho\to+\infty$ vanish all together after integration in $(\varphi,\zeta)\in(0,2\pi)\times(-1/2,1/2)$ because the limit does exist. In this way, the integral in ζ is approximated by the expression $J(\eta) - J(\eta)^\top$ where, in view of the obvious relation $D(\nabla_\xi) d^\sharp(\xi) = 0$, we have

$$J(\eta) = \int_{-1/2}^{1/2} \left(\sum_{q=0}^{3} W^q(\zeta, \nabla_\eta) \Phi^\sharp(\eta) + d^\sharp(\eta,\zeta) C^\sharp \right)^\top D\left(\frac{\eta}{\rho}, 0\right)^\top AD(\nabla_\xi) \sum_{p=0}^{3} W^p(\zeta, \nabla_\eta) \Phi^\sharp(\eta) d\zeta. \tag{3.52}$$

Similarly to (3.44) and (3.45), in the formula

$$D(\nabla_\xi) \sum_{p=0}^{3} W^p \Phi^\sharp = D_\zeta W^0 \Phi^\sharp + \left(D_\zeta W^1 + D_\eta W^0\right) \Phi^\sharp + \left(D_\zeta W^2 + D_\eta W^1\right) \Phi^\sharp + \\ + \left(D_\zeta W^3 + D_\eta W^2\right) \Phi^\sharp + D_\eta W^3 \Phi^\sharp,$$

where $D_\zeta = D(0,0,\partial_\zeta)$ and $D_\eta = D(\nabla_\eta, 0)$, cf. (3.16), the first two terms on the right vanish.

According to (3.25) and (3.34), (3.39), we have

$$(D_\zeta W^{1+p}(\zeta, \nabla_\eta) + D_\eta W^p(\zeta, \nabla_\eta)) \Phi^\sharp(\eta) = O(\rho^{-p}), \ p = 1, 2 \quad D_\eta W^3(\zeta, \nabla_\eta) \Phi^\sharp(\eta) = O(\rho^{-3}).$$

Taking (3.30) and (3.13) into account, we write the relations

$$\left(W^0 + W^1(\zeta, \nabla_\eta)\right) \Phi^\sharp + d^\sharp(\eta,\zeta) C^\sharp = d^\flat(\zeta, \nabla_\eta) \left(\Phi^\sharp(\eta) + d^\sharp(\eta, 0) C^\sharp\right)$$

and

$$D(n(\eta), 0) d^\flat(\zeta, \nabla_\eta) = D^{1\flat}(n(\eta), \zeta, \nabla_\eta) + D^{0\flat}(n(\eta))$$

with the matrices $D^{0\flat}(n) = (D_\zeta \zeta)\left(n^\top, 0\right)^\top e_3$ of size 6×3 and

$$D^{1\flat}(n, \zeta, \nabla_\eta) = \begin{pmatrix} \mathcal{D}'(n), & -\zeta 2^{1/2} \mathcal{D}'(n) \nabla_\eta \\ & \mathbb{O}_3 \end{pmatrix}, \quad d^\flat(\zeta, \nabla_\eta) = \begin{pmatrix} 1 & 0 & -\zeta \partial/\partial\eta_1 \\ 0 & 1 & -\zeta\partial/\partial\eta_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Comparing decay rates of remaining multipliers shows that we need to keep in (3.52) the following two terms of order $\rho^{-1}(1 + |\ln\rho|)$:

$$\int_{-1/2}^{1/2} D^{1\flat}(n(\eta), \zeta, \nabla_\eta) \left(\Phi^\sharp(\eta) + d^\sharp(\eta, 0) C^\sharp\right)^\top A \left(D_\zeta W^2(\zeta, \nabla_\eta) + D_\eta W^1(\zeta, \nabla_\eta)\right) d\zeta \Phi^\sharp(\eta) \tag{3.53}$$

and

$$\int_{-1/2}^{1/2} D^{0\flat}(n(\eta)) \left(\Phi^\sharp(\eta) + d^\sharp(\eta, 0) C^\sharp\right)^\top A \left(D_\zeta W^3(\zeta, \nabla_\eta) + D_\eta W^2(\zeta, \nabla_\eta)\right) d\zeta \Phi^\sharp(\eta). \tag{3.54}$$

By formulas (3.19), (3.20) for $\mathcal{L}'(\nabla_y)$, $\mathcal{L}_3(\nabla_y)$ and (3.35), (3.41) for $\mathcal{N}'$, $\mathcal{N}_j$ we recall the calculations (3.17), (3.18) and conclude that (3.53) is equal to the sum

$$\left(\left(\Phi^\sharp(\eta)\right)' + \left(d^\sharp(\eta, 0)\right)' C^\sharp \right)^\top \mathcal{N}'(\eta, \nabla_\eta) \left(\Phi^\sharp(\eta)\right)' \\ + \left(\nabla_\eta \left(\Phi_3^\sharp(\eta)\right)' + d_3^\sharp(\eta, 0) C^\sharp\right) \begin{pmatrix} \mathcal{N}_1(\eta, \nabla_\eta) \\ \mathcal{N}_2(\eta, \nabla_\eta) \end{pmatrix} \Phi_3^\sharp(\eta) \tag{3.55}$$

where $\left(\Phi^{\sharp}\right)'$ and $\left(d^{\sharp}\right)'$ are submatrices with eliminated third lines. A similar argument converts (3.54) into

$$\left(\Phi_3^{\sharp}(\eta) + d_3^{\sharp}(\eta,0)\, C^{\sharp}\right)^{\top} \mathcal{N}_0(\eta,\nabla_\eta)\,\Phi_3^{\sharp}(\eta)\,. \tag{3.56}$$

Now we see that

$$0 = \lim_{T\to+\infty} \int_{\mathbb{S}} \left(J(\eta) - J(\eta)^{\top}\right) ds_\eta, \tag{3.57}$$

i.e. the limit of the integral on the right-hand side of (3.51), is a linear combination of scalar products, listed in (3.36) and (3.42), and their conjugates. Thus, a part of the integral (3.57) which involves columns of the matrix $\Phi^{\sharp}$, vanishes due to "orthogonality" conditions in (3.36) and (3.42) with 0 on the right-hand side. The other part of the integral which involves columns of the matrices $d^{\sharp}$ and $\Phi^{\sharp}$, converts into the difference $\left(C^{\sharp}\right)^{\top} - C^{\sharp}$. Indeed, we had reduced expression (3.52) to the sum of (3.55) and (3.56) so that after integration in $\varphi \in \mathbb{S}$ it is sufficient to apply the "bi-orthogonality" conditions (3.36) and (3.42) with 1 on the right-hand side.

Since according to (3.51) the integral in (3.57) is null, the theorem is proved. ■

3.6 Asymptotics at infinity of elastic fields in layer-shaped domains

If the right-hand sides F and G^{+}, $G^{\bullet}$ in problem (3.23)–(3.25) are smooth and decay exponentially as $\rho \to +\infty$, for example have compact supports, then results in [32] serve for an asymptotic expansion of the solution v with a remainder of any given power-law decay $O\left(\rho^{-N}\right)$. Hence, we can make the decomposition of the elastic logarithmic potential $\mathcal{P}$ a bit more precise, namely

$$\mathcal{P}(\xi) = (1-\chi(2\rho/R_\theta)) \left(\sum_{p=0}^{3} W^p(\zeta,\nabla_\eta)\,\Phi^{\sharp}(\eta) + d^{\sharp}(\eta,0)\,C^{\sharp} + \Upsilon^{\sharp}(\eta)\right) + \widetilde{\widetilde{\mathcal{P}}}(\xi)\,. \tag{3.58}$$

The first two terms, of course, are the same as in (3.43) and (3.49), but the new remainder gets the faster decay properties, cf. (3.50),

$$\left|\nabla_\eta^p \partial_\zeta^q \widetilde{\widetilde{\mathcal{P}}}_i(\xi)\right| \le c_{pq}\rho^{-1+\epsilon}, \quad \left|\nabla_\eta^p \partial_\zeta^q \widetilde{\widetilde{\mathcal{P}}}_3(\xi)\right| \le c_{pq}\rho^{\epsilon}, \quad \rho \ge 2R_\theta.$$

The latter is caused by the additional term $\Upsilon^{\sharp}(\eta)$ in (3.58), a 3×4-matrix with the rows

$$\begin{aligned} \Upsilon_i^{\sharp}(\eta) &= \rho^{-1}\Upsilon_i^{0\sharp}(\varphi) + \rho^{-1}\ln\rho\,\Upsilon_i^{1\sharp}(\varphi)\,, \qquad i=1,2,\\ \Upsilon_3^{\sharp}(\eta) &= \Upsilon_1^{0\sharp}(\varphi) + \ln\rho\ \Upsilon_3^{1\sharp}(\varphi) + (\ln\rho)^2\,\Upsilon_3^{2\sharp}(\varphi)\,. \end{aligned} \tag{3.59}$$

Let us explain where the lower-order asymptotic terms (3.59) appear from.

The dimension reduction procedure, quite similar to Section 3.1, leads to the following system for the third term $\Upsilon^{\sharp}$ in the asymptotic ansatz (3.58):

$$\mathcal{D}(-\nabla_\eta)^{\top} \mathcal{A}\mathcal{D}(\nabla_\eta)\,\Upsilon^{\sharp}(\eta) = \mathcal{F}(\eta)\,, \quad \eta \in \mathbb{R}^2 \setminus \mathcal{O}. \tag{3.60}$$

The right-hand side $\mathcal{F} = (\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)^{\top}$ is constructed in the same way as (3.17), (3.18) and is necessary to compensate for discrepancies of the expression Ξ, see a comment to (3.43), in the homogeneous equations (3.23) and (3.24), that are the fifth and sixth terms in (3.44) and the fifth term in (3.45) respectively. In this way, representations (3.13), (3.38) and integration by parts in the variable ζ, cf. calculation in Section 3.5, yield

$$
\begin{aligned}
\mathcal{F}_i(\eta) &= e_i^\top D_\eta \int_{-1/2}^{1/2} A\left(D_\zeta W^3(\zeta,\nabla_\eta) + D_\eta W^2(\zeta,\nabla_\eta)\right) d\zeta \Phi^\sharp(\eta) = \rho^{-3}\mathcal{F}_i^0(\varphi),\\
\mathcal{F}_3(\eta) &= e_i^\top D_\eta \int_{-1/2}^{1/2} A D_\eta W^3(\zeta,\nabla_\eta)\, d\zeta \Phi^\sharp(\eta)\\
&\quad + \sum_{i=1}^{2}\frac{\partial}{\partial\eta_i} D_\eta \int_{-1/2}^{1/2} \zeta A\left(D_\zeta W^3(\zeta,\nabla_\eta) + D_\eta W^2(\zeta,\nabla_\eta)\right) d\zeta \Phi^\sharp(\eta) = \rho^{-4}\mathcal{F}_3^0(\varphi).
\end{aligned}
\tag{3.61}
$$

It is important that logarithms figuring in the matrix $\Phi^\sharp(\eta)$ due to (3.34), (3.39) and (3.38) as coefficients on linear functions in η_i, are eliminated in (3.61) by differentiating sufficiently many times so that functions (3.61) become positive homogeneous in η of degree -3 and -4 respectively. However, the logarithm $\ln\rho$ returns into components (3.59) of the solution $\Upsilon^\sharp(\eta)$ by virtue of the Kondratiev theorem on asymptotics (see [17] and, e.g., [41, Thm. 3.1.4]). To confirm this, we observe system (3.60) with $\mathcal{F}=0$ and find out its vector solutions in the form of derivatives of columns in matrix (3.39), namely

$$
\sum_{i=1}^{2}\begin{pmatrix} \left(a_1^i\partial_1 + a_2^i\partial_2\right)\Phi_i'(\eta) \\ \left(a_1^{2+i}\partial_1 + a_2^{2+i}\partial_2\right)\Phi_3^i(\eta) \end{pmatrix}. \tag{3.62}
$$

As shown in [41, §5.4], any solution with the same positive homogeneity degrees -1 and 0 as in (3.59) takes form (3.62). Furthermore, in the cases $a_2^3 \neq 0$ and $a_1^3 \neq 0$ the third component of (3.62) stays linearly dependent on $\ln\rho$. Hence, the above-mentioned theorem prescribes to search for a solution of system (3.60) with the right-hand sides (3.61) in the form (3.59) while $\Upsilon_1^{1\sharp}$, $\Upsilon_2^{1\sharp}$ and $\Upsilon_3^{2\sharp}$ may vanish only under some orthogonality conditions in $L^2(\mathbb{S})^3$ for the angular part $\mathcal{F}^0(\varphi) = \left(\mathcal{F}_1^0(\varphi), \mathcal{F}_2^0(\varphi), \mathcal{F}_3^0(\varphi)\right)^\top$ of $\mathcal{F}(\eta)$. We do not examine these conditions and specify (3.59) because the component Υ is indicated in (3.58) with an auxiliary technical reason only and will be excluded from the final asymptotic formulas for the solution u of problem (1.11)–(1.14) in the paper [5].

We emphasize that a solution $\Upsilon^\sharp$ of system (3.60) is defined up to a summand of type (3.62). A unique $\Upsilon^\sharp$ in decomposition (3.58) of the elastic logarithmic potential $\mathcal{P}$ depends on the whole data of the problem, in particular on the clamped area θ, and is specified by means of theorem on asymptotics in layer-shaped domains, the most challenging assertion in the paper [32].

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