LOCAL MINIMALITY OF THE BALL FOR THE GAUSSIAN PERIMETER

DOMENICO ANGELO LA MANNA
UNIVERSITÀ DEGLI STUDI DI NAPOLI FEDERICO II

1. INTRODUCTION

For a set $E \subset \mathbb{R}^n$ the Gaussian measure is defined as

$$\gamma(E) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_E e^{-\frac{|x|^2}{2}}\, dx.$$

Note that the Gauss space $(\mathbb{R}^n, \gamma)$ is a probability space, since $\gamma(\mathbb{R}^n) = 1$. It plays a central role in various branches of Probability Theory. For a smooth set E, the Gaussian perimeter is defined as

$$P_\gamma(E) = \frac{1}{(2\pi)^{n/2}} \int_{\partial E} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x).$$

This definition can be extended to general sets of locally finite perimeter by replacing ∂E with the so called reduced boundary, see Section 2. The isoperimetric inequality in Gaussian space states that among all subsets of $\mathbb{R}^n$ with prescribed Gaussian measure halfspaces have the least Gaussian perimeter. More precisely, given $s \in \mathbb{R}$ and setting $H_s = \{x \in \mathbb{R}^n : x_n < s\}$, we have

$$P_\gamma(E) \geq P_\gamma(H_s) \tag{1}$$

for all Borel subsets $E \subset \mathbb{R}^n$ such that $\gamma(E) = \gamma(H_s)$, with equality holding if and only if E coincides with H_s up to a rotation around the origin. Inequality (1) was indipendently established in [3], [18], while the equality case was obtained much later in [8].

Let us now set for all $s \in \mathbb{R}$

$$\Phi(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-\frac{t^2}{2}} dt.$$

Note that Φ is an increasing function from $\mathbb{R}$ into $(0, 1)$ and that

$$\gamma(H_s) = \Phi(s), \quad P_\gamma(H_s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}}.$$

With this notation the Gaussian isoperimetric inequality can be restated in the following analytical way

$$P_\gamma(E) \geq \frac{1}{\sqrt{2\pi}} e^{-\frac{[\Phi^{-1}(\gamma(E))]^2}{2}}.$$

In this paper we consider th of sets E symmetric around the origin, i.e., $E = -E$. We show that when restricted to this class balls B_r centered at the origin are local minimizers for the perimeter, at least when r is not too big. More precisely we have the following local minimality result.

Theorem 1. *Let $n \geq 2$ and $\sigma \in (0, 1/2)$. There exist δ and κ such that if $r \in [\sigma, \sqrt{n+1} - \sigma]$, E is a set of locally finite perimeter with $E = -E$, $\gamma(E) = \gamma(B_r)$ and $\gamma(E \Delta B_r) < \delta$, then*

$$P_\gamma(E) - P_\gamma(B_r) \geq \kappa(n, \sigma) \gamma(E \Delta B_r)^2. \tag{2}$$

Let us now comment briefly on this result. First, observe that Theorem 1, beside stating the local minimality of balls B_r when $r \in (0, \sqrt{n+1})$, provides also a quantitative estimate of the isoperimetric gap $P_\gamma(E) - P_\gamma(B_r)$ in terms of the square of the measure of the symmetric difference between E and B_r. In this respect this inequality is close to the recente quantitative isoperimetric inequalities in Gaussian space proved in [4], [13], [16], [17]. Note also that the constant κ in (2) is uniformly bounded from below when r is away from 0 and $\sqrt{n+1}$. In addition, Proposition 1 shows that the result above is sharp in the sense that if $r > \sqrt{n+1}$ then B_r is *never* a local minimizer for the perimeter. Also the power 2 is optimal, as it can be easily checked with an argument similar to the one used for the quantitative inequality in the Euclidean case, see [11, Section 4]. However, if balls B_r are not in general global minimizers among symmetric sets with the same Gaussian measure, at least if r is sufficiently small, see Proposition 2.

Our Theorem 1 is closely related to a well known conjecture, known as *Symmetric Gaussian Problem*, see [14]. Indeed, as observed in [14], if this conjecture were true it would imply the global minimality of B_r or of its complement in $\mathbb{R}^n$. This is precisely what happens in the 1-dimensional case where one can prove that B_r is always a local minimizer of the perimeter among all symmetric sets with the same Gaussian measure, see Section 4. Moreover, balls are the unique global minimizers for $r > r_0$, where r_0 is the unique positive number such that

$$\frac{1}{\sqrt{2\pi}} \int_{-r_0}^{r_0} e^{-\frac{t^2}{2}}\, dt = \frac{1}{2},$$

while $\mathbb{R} \setminus B_r$ is the unique global minimizers when $0 < r < R_0$.

We conclude this introduction with a few words about the proof of Theorem 1, which is achieved following the strategy introduced in this context by Cicalese and Leonardi in [10] and later on modified in [1]. More precisely, we first prove inequality (1) for nearly spherical sets, i.e., sets that are close in C^1 to a ball B_r with the same Gaussian volume and symmetric around the origin. Then we extend it to the general case with a contradiction argument based on the regularity theory for sets of minimal perimeter, see a more detailed account of this strategy in Section 3, before the proof of Theorem 1. Note that in our case the above strategy is more complicated. An obvious difficulty comes from the constraint that the competing sets must be symmetric with respect to the origin. However the main source of problems is represented by possible unbounded competitors of balls.

2. Nearly spherical sets

In the following we shall denote by $B_r(x_0)$ the ball with center at x_0 and radius r. If the center is at the origin we shall write B_r.

We recall the basic definitions of the theory of sets of finite perimeter. If E is any Borel subset of $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ is an open set, the *perimeter of* E *in* Ω is defined by setting

$$P(E;\Omega) = \sup\left\{ \int_E \operatorname{div}\varphi\, dx : \varphi \in C_c^\infty(\Omega;\mathbb{R}^n),\ \|\varphi\|_\infty \le 1 \right\}.$$

The perimeter of E in $\mathbb{R}^n$ is denoted by $P(E)$. We say that E has *locally finite perimeter* if $P(E;\Omega) < \infty$ for all bounded open sets Ω. It is well known, see [2, Ch. 3], that E is a set of locally finite perimeter if and only if its characteristic function χ_E has distributional derivative $D\chi_E$ which is a measure in $\mathbb{R}^n$ with values in $\mathbb{R}^n$. Then, from the above definition we have immediately that $P(E;\Omega) = |D\chi_E|(\Omega)$ for every open set Ω.

From Besicovitch derivation theorem we have that for $|D\chi_E|$-a.e. $x \in \mathbb{R}^n$ there exists

$$\nu^E(x) = -\lim_{r\to 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} \quad \text{and} \quad |\nu^E(x)| = 1. \tag{3}$$

The set $\partial^* E$ where (3) holds is called the *reduced boundary* of E, while the vector $\nu^E(x)$ is the *generalized exterior normal* at x. For all the properties of sets of finite perimeter used herein we refer to the book [2].

As recalled in the Introduction, the *Gauss space* is the space $\mathbb{R}^n$ endowed with the measure γ given for any Lebesgue measurable set $E \subset \mathbb{R}^n$ by

$$\gamma(E) = \frac{1}{(2\pi)^{n/2}} \int_E e^{\frac{-|x|^2}{2}}\, dx.$$

Similarly to the Euclidean case, the *Gaussian perimeter* of a Borel set E in an open set Ω is defined by setting

$$P_\gamma(E;\Omega) = \sup\left\{ \int_E (\operatorname{div}\varphi - \varphi\cdot x)\, d\gamma : \varphi \in C_c^\infty(\Omega;\mathbb{R}^n),\ \|\varphi\|_\infty \le 1 \right\}.$$

The *Gaussian perimeter* of E in the whole $\mathbb{R}^n$ will be denoted by $P_\gamma(E)$. If E has finite Gaussian perimeter then E has locally finite perimeter (in the Euclidean sense) and

$$P_\gamma(E) = \frac{1}{(2\pi)^{n/2}} \int_{\partial^* E} e^{\frac{-|x|^2}{2}}\, d\mathcal{H}^{n-1},$$

where, for $0 \le s \le n$, $\mathcal{H}^s$ stands for the s-dimensional *Hausdorff measure*. In the following we shall denote by $\mathcal{H}^s_\gamma$ the measure defined by setting for every Borel set $E \subset \mathbb{R}^n$

$$\mathcal{H}^s_\gamma(E) = \frac{1}{(2\pi)^{n/2}} \int_E e^{-\frac{|x|^2}{2}}\, d\mathcal{H}^s.$$

Thus, if E is a set of locally finite perimeter in $\mathbb{R}^n$ we may write

$$P_\gamma(E) = \mathcal{H}^{n-1}_\gamma(\partial^* E).$$

A set $E \subset \mathbb{R}^n$ is said to be *nearly spherical* if there exist a ball B_r and a Lipschitz function $u : \mathbb{S}^{n-1} \to (-1/2, 1/2)$ such that

$$E = \{y = trx(1 + u(x)) : x \in \mathbb{S}^{n-1},\ 0 \le t < 1\}. \tag{4}$$

In the following, given any function $u : \mathbb{S}^{n-1} \to \mathbb{R}$ we shall always assume that u is extended to $\mathbb{R}^n \setminus \{0\}$ by setting $u(x) = u\big(\frac{x}{|x|}\big)$.

It is easily checked that if E is defined as in (4) then its Gaussian measure and it Gaussian perimeter are given, respectively, by the two formulas below

$$\gamma(E) = \frac{r^n}{(2\pi)^{n/2}} \int_B (1 + u(x))^n e^{-\frac{r^2|x|^2(1+u(x))^2}{2}}\, dx \tag{5}$$

$$P_\gamma(E) = \frac{r^{n-1}}{(2\pi)^{n/2}} \int_{\mathbb{S}^{n-1}} (1 + u(x))^{n-1} \sqrt{1 + \frac{|D_\tau u(x)|^2}{(1 + u(x))^2}}\, e^{-\frac{r^2(1+u(x))^2}{2}}\, d\mathcal{H}^{n-1}, \tag{6}$$

where $D_\tau u$ stands for the tangential gradient of u on $\mathbb{S}^{n-1}$.

When E is a measurable set such that $\gamma(E) = \gamma(B_r)$ we shall often use the following notation

$$D_\gamma(E) = (2\pi)^{n/2}\big[P_\gamma(E) - P_\gamma(B_r)\big] \tag{7}$$

to denote its *Gaussian isoperimetric deficit* with respect to the ball B_r.

Next theorem states that if r is smaller than a critical radius depending on the dimension, the Gaussian isoperimetric deficit of a nearly spherical set symmetric with respect to the origin is strictly positive and the following Fuglede type estimate holds.

Theorem 2. *Let $n \ge 2$ and $r \in (0, \sqrt{n+1})$. There exist $\varepsilon \in (0, 1/2)$, depending on n and r, and κ_0, depending only on n, with the following property. If E is a nearly spherical set as in* (4) *with*

$\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} < \varepsilon$, *symmetric with respect to the origin and such that* $\gamma(E) = \gamma(B_r)$, *then*

$$P_\gamma(E) - P_\gamma(B_r) \geq \kappa_0 r^{n-1}(n+1-r^2)\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}. \tag{8}$$

Proof. **Step 1.** Fix $r \in (0, \sqrt{n+1})$. Using the expression of $P_\gamma(E)$ provided in (6) we may split

$$\begin{aligned} D_\gamma(E) = (2\pi)^{n/2}\big[P_\gamma(E) - P_\gamma(B_r)\big] &= r^{n-1}\int_{\mathbb{S}^{n-1}}(1+u)^{n-1}e^{-\frac{r^2(1+u)^2}{2}}\left(\sqrt{1+\frac{|D_\tau u|^2}{(1+u)^2}}-1\right)d\mathcal{H}^{n-1} \\ &\quad + r^{n-1}\int_{\mathbb{S}^{n-1}}\Big[(1+u)^{n-1}e^{-\frac{r^2(1+u)^2}{2}} - e^{-\frac{r^2}{2}}\Big]d\mathcal{H}^{n-1} \\ &= r^{n-1}e^{-\frac{r^2}{2}}I_1 + r^{n-1}e^{-\frac{r^2}{2}}I_2. \end{aligned} \tag{9}$$

Observe that $\sqrt{1+t} \geq 1 + \frac{t}{2} - \frac{t^2}{8}$ for all $t > 0$. Therefore, from the smallness assumption $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} < \varepsilon \leq \frac{1}{2}$, we get

$$\begin{aligned} I_1 &= \int_{\mathbb{S}^{n-1}}(1+u)^{n-1}e^{-r^2(u+u^2/2)}\left(\sqrt{1+\frac{|D_\tau u|^2}{(1+u)^2}}-1\right)d\mathcal{H}^{n-1} \\ &\geq \int_{\mathbb{S}^{n-1}}(1+u)^{n-1}e^{-r^2(u+u^2/2)}\Big(\frac{1}{2}\frac{|D_\tau u|^2}{(1+u)^2} - \frac{1}{8}\frac{|D_\tau u|^4}{(1+u)^4}\Big)d\mathcal{H}^{n-1} \\ &\geq \Big(\frac{1}{2} - C\varepsilon\Big)\int_{\mathbb{S}^{n-1}}(1+u)^{n-1}e^{-r^2(u+u^2/2)}|D_\tau u|^2\,d\mathcal{H}^{n-1} \geq \Big(\frac{1}{2} - C\varepsilon\Big)\int_{\mathbb{S}^{n-1}}|D_\tau u|^2\,d\mathcal{H}^{n-1}, \end{aligned} \tag{10}$$

for some constant C depending only on n, but not on r. Concerning the integral term I_2 we have, by Taylor expansion,

$$\begin{aligned} I_2 &= \int_{\mathbb{S}^{n-1}}\big[(1+u)^{n-1}e^{-r^2(u+u^2/2)} - 1\big]d\mathcal{H}^{n-1} \\ &= (n-1-r^2)\int_{\mathbb{S}^{n-1}}u\,d\mathcal{H}^{n-1} + \Big[\frac{(n-1)(n-2)}{2} - \Big(n-\frac{1}{2}\Big)r^2 + \frac{r^4}{2}\Big]\int_{\mathbb{S}^{n-1}}u^2\,d\mathcal{H}^{n-1} + R_1, \end{aligned}$$

where the remainder term R_1 can be again estimated by $C\varepsilon\|u\|_2^2$, for some constant C depending only on n. Therefore, recalling the previous estimate (10) and the equality in (9) we have

$$\begin{aligned} r^{1-n}e^{\frac{r^2}{2}}D_\gamma(E) &\geq \frac{1}{2}\int_{\mathbb{S}^{n-1}}|D_\tau u|^2\,d\mathcal{H}^{n-1} + (n-1-r^2)\int_{\mathbb{S}^{n-1}}u\,d\mathcal{H}^{n-1} \\ &\quad + \Big[\frac{(n-1)(n-2)}{2} - \Big(n-\frac{1}{2}\Big)r^2 + \frac{r^4}{2}\Big]\int_{\mathbb{S}^{n-1}}u^2\,d\mathcal{H}^{n-1} - C\varepsilon\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}. \end{aligned} \tag{11}$$

To estimate the integral of u in the previous inequality we are going to use the assumption that the Gaussian measures of E and B_r are equal. This equality, using (5), can be written as

$$\int_0^1 t^{n-1}\,dt\int_{\mathbb{S}^{n-1}}\Big[(1+u)^n e^{-\frac{r^2t^2(1+u)^2}{2}} - e^{-\frac{r^2t^2}{2}}\Big]d\mathcal{H}^{n-1} = 0$$

Using again Taylor expansion, we then easily get

$$\begin{aligned} 0 &= \int_0^1 t^{n-1}e^{-\frac{r^2t^2}{2}}\,dt\int_{\mathbb{S}^{n-1}}\big[(1+u)^n e^{-r^2t^2(u+u^2/2)} - 1\big]d\mathcal{H}^{n-1} \\ &= \int_0^1 t^{n-1}e^{-\frac{r^2t^2}{2}}\,dt\int_{\mathbb{S}^{n-1}}\Big[(n-r^2t^2)u + \Big(\frac{n(n-1)}{2} - \frac{(2n+1)r^2t^2}{2} + \frac{r^4t^4}{2}\Big)u^2\Big]d\mathcal{H}^{n-1} + R_2 \\ &= \int_{\mathbb{S}^{n-1}}\Big[(na_n - r^2b_n)u + \Big(\frac{n(n-1)a_n}{2} - \frac{(2n+1)r^2b_n}{2} + \frac{r^4c_n}{2}\Big)u^2\Big]d\mathcal{H}^{n-1} + R_2, \end{aligned} \tag{12}$$

where we have set

$$a_n = \int_0^1 t^{n-1} e^{-\frac{r^2t^2}{2}}\, dt, \qquad b_n = \int_0^1 t^{n+1} e^{-\frac{r^2t^2}{2}}\, dt, \qquad c_n = \int_0^1 t^{n+3} e^{-\frac{r^2t^2}{2}}\, dt$$

and where the remainder term R_2 is estimated as before

$$|R_2| \le C\varepsilon \|u\|^2_{L^2(\mathbb{S}^{n-1})}. \tag{13}$$

A simple integration by parts gives that

$$b_n = \frac{na_n}{r^2} - \frac{e^{-\frac{r^2}{2}}}{r^2}, \qquad c_n = \frac{n(n+2)a_n}{r^4} - \frac{(n+2)e^{-\frac{r^2}{2}}}{r^4} - \frac{e^{-\frac{r^2}{2}}}{r^2}.$$

Thus, inserting the above values of b_n and c_n into (12) we immediately get that

$$\int_{\mathbb{S}^{n-1}} u\, d\mathcal{H}^{n-1} = -\frac{n-1-r^2}{2}\int_{\mathbb{S}^{n-1}} u^2\, d\mathcal{H}^{n-1} - e^{\frac{r^2}{2}} R_2. \tag{14}$$

Then, substituting in (11) the integral of u on $\mathbb{S}^{n-1}$ by the right hand side of the above equality, we obtain the following estimate

$$r^{1-n} e^{\frac{r^2}{2}} D_\gamma(E) \ge \frac{1}{2}\int_{\mathbb{S}^{n-1}} |D_\tau u|^2\, d\mathcal{H}^{n-1} - \frac{n-1+r^2}{2}\int_{\mathbb{S}^{n-1}} u^2\, d\mathcal{H}^{n-1} - C\varepsilon\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}. \tag{15}$$

Step 2. For any integer $k \ge 0$, let us denote by $y_{k,i}$, $i = 1, \dots, G(n,k)$, the spherical harmonics of order k, i.e., the restrictions to $\mathbb{S}^{n-1}$ of the homogeneous harmonic polynomials of degree k, normalized so that $||y_{k,i}||_{L^2(\mathbb{S}^{n-1})} = 1$, for all $k \ge 0$ and $i \in \{1, \dots, G(n,k)\}$. The functions $y_{k,i}$ are eigenfunctions of the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$ and for all k and i

$$-\Delta_{\mathbb{S}^{n-1}} y_{k,i} = k(k+n-2) y_{k,i}\, .$$

Therefore if we write

$$u = \sum_{k=0}^{\infty} \sum_{i=1}^{G(n,k)} a_{k,i} y_{k,i}, \quad \text{where} \quad a_{k,i} = \int_{\mathbb{S}^{n-1}} u y_{k,i} d\mathcal{H}^{n-1},$$

we have

$$||u||^2_{L^2(\mathbb{S}^{n-1})} = \sum_{k=0}^{\infty}\sum_{i=1}^{G(n,k)} a^2_{k,i}, \quad ||D_\tau u||^2_{L^2(\mathbb{S}^{n-1})} = \sum_{k=1}^{\infty} k(k+n-2) \sum_{i=1}^{G(n,k)} a^2_{k,i}\, . \tag{16}$$

Note that since E is symmetric with respect to the origin, we have that u is an even function, hence in the harmonic decomposition only the terms with k even will appear. In particular $a_{1,i} = 0$ for all $i = 1, \dots, n$. Note also that from (14) and (13) we have

$$|a_{0,1}| \le C\varepsilon \|u\|_{L^2(\mathbb{S}^{n-1})}. \tag{17}$$

Thus, from (15),(16) and (17) we have

$$\begin{aligned}
r^{1-n} e^{\frac{r^2}{2}} D_\gamma(E) &\ge \frac{1}{2}\sum_{k=2}^{\infty} k(k+n-2) \sum_{i=1}^{G(n,k)} a^2_{k,i} - \frac{n-1+r^2}{2}\sum_{k=2}^{\infty}\sum_{i=1}^{G(n,k)} a^2_{k,i} - C_0\varepsilon\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})} \\
&= \frac{n+1-r^2}{2}\sum_{i=1}^{G(n,2)} a^2_{2,i} + \frac{1}{2}\sum_{k=4}^{\infty}[k(k+n-2) - (n-1-r^2)] \sum_{i=1}^{G(n,k)} a^2_{k,i} - C_0\varepsilon\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}, \\
&\ge c_0(n+1-r^2)\sum_{k=4}^{\infty} k(k+n-2) \sum_{i=1}^{G(n,k)} a^2_{k,i} - C_0\varepsilon\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})},
\end{aligned}$$

for some positive constants c_0, C_0 depending only on n. Using again (17) and the fact that $a_{1,i} = 0$ for $i = 1, \ldots, n$, from the previous inequality we deduce that there exist two constants $c_1, C_1 > 0$ depending only on n such that

$$r^{1-n} e^{\frac{r^2}{2}} D_\gamma(E) \geq c_1(n+1-r^2)\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})} - C_1\varepsilon\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}.$$

From this inequality (8) immediately follows provided that we choose

$$0 < \varepsilon \leq \min\left\{\frac{1}{2}, \frac{c_1(n+1-r^2)}{2C_1}\right\}. \tag{18}$$

□

The following uniform estimate is a straightforward consequence of the previous theorem.

Corollary 1. *Let $n \geq 2$ and $r_0 \in (0, \sqrt{n+1})$. There exist $\varepsilon \in (0, 1/2)$, $\kappa_1 > 0$, depending only on n and r_0, such that if $r \in (0, r_0]$ and E is a nearly spherical set as in (4) with $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} < \varepsilon$, then*

$$P_\gamma(E) - P_\gamma(B_r) \geq \kappa_1 r^{-1-n}\gamma(E\Delta B_r)^2. \tag{19}$$

Proof. Fix r_0 and a nearly spherical set E as in the statement. Then, arguing as in the proof of (12), we get

$$\begin{aligned}\gamma(E\Delta B_r) &= \frac{r^n}{(2\pi)^{\frac{n}{2}}}\int_B \left|(1+u)^n e^{-\frac{r^2|x|^2(1+u)^2}{2}} - e^{-\frac{r^2|x|^2}{2}}\right| dx \\ &= \frac{r^n}{(2\pi)^{\frac{n}{2}}}\int_0^1 t^{n-1}e^{-\frac{r^2t^2}{2}}\,dt\int_{\mathbb{S}^{n-1}}\left|(1+u)^n e^{-r^2t^2(u+u^2/2)} - 1\right| d\mathcal{H}^{n-1} \\ &\leq C(n)r^n\int_{\mathbb{S}^{n-1}}|u|\,d\mathcal{H}^{n-1},\end{aligned} \tag{20}$$

where in the last inequality we have used the assumption that $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} \leq 1/2$. Then, choosing

$$\varepsilon = \min\left\{\frac{1}{2}, \frac{c_1(n+1-r_0^2)}{2C_1}\right\},$$

where c_1 and C_1 are as in (18), from (20) and (8) we get at once

$$\gamma(E\Delta B_r)^2 \leq \frac{C(n)r^{n+1}}{n+1-r^2}\big[P_\gamma(E) - P_\gamma(B_r)\big],$$

for a suitable constant $C(n)$. Hence (19) follows. □

Consider the isoperimetric problem in the Gaussian space

$$\min\{P_\gamma(E) : \gamma(E) = m\} \tag{21}$$

for some fixed $m > 0$. The Euler-Lagrange equation associated with the minimum problem (21)

$$H_{\partial E} - x\cdot\nu^E = \lambda \qquad \text{on } \partial E, \tag{22}$$

where $H_{\partial E}$ is the sum of the the principal curvatures of the boundary of E and λ is a suitable Lagrange multiplier. Observe that B_r is a solution of (22), hence a critical point of the isoperimetric problem (21) for all $r > 0$. Theorem 2 shows that if $0 < r < \sqrt{n+1}$ then B_r is also a local minimizer for the isoperimetric problem with respect to small variations, close to B_r in C^1 and symmetric with respect to the origin. In this respect the above theorem is sharp since if $r > \sqrt{n+1}$ then B_r is *never* a local minimizer for the Gaussian perimeter under the constraints $\gamma(E) = m$ and $E = -E$, as it can be shown by a simple second variation argument.

Proposition 1. *Let $n \geq 2$, $r > \sqrt{n+1}$ and k a positive integer. For every $\varepsilon > 0$ there exists a function $u \in C^\infty(\mathbb{S}^{n-1})$, with $\|u\|_{C^k(\mathbb{S}^{n-1})} < \varepsilon$ such that the corresponding nearly spherical set*

$$E = \{y = trx(1 + u(x)) : x \in \mathbb{S}^{n-1}, 0 \leq t < 1\} \tag{23}$$

is symmetric with respect to the origin, $\gamma(E) = \gamma(B_r)$ and $P_\gamma(E) < P_\gamma(B_r)$.

Proof. Fix $r > \sqrt{n+1}$, a positive integer k and $\varepsilon > 0$. Given an even function $\varphi \in C^\infty(\mathbb{S}^{n-1})$ such that

$$\int_{\mathbb{S}^{n-1}} \varphi(x)\, d\mathcal{H}^{n-1} = 0, \tag{24}$$

let $X \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ be a vector field such that $X(-x) = -X(x)$ and

$$X(x) = \frac{e^{\frac{|x|^2}{2}}}{|x|^n}\varphi\Big(\frac{x}{|x|}\Big)x \qquad \text{for } x \in B_{2r} \setminus \overline{B}_{\frac{r}{2}}. \tag{25}$$

Let Φ be the flow associated to X, that is the unique C^∞ map $\Phi : \mathbb{R}^n \times (-1, 1) \to \mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$ and $t \in (-1, 1)$

$$\frac{\partial \Phi}{\partial t}(x, t) = X(\Phi(x, t)), \quad \Phi(x, 0) = x. \tag{26}$$

Set $E_t = \Phi(\cdot, t)(B_r)$ for all $t \in (-1, 1)$. Since $\Psi(x, t) = -\Phi(-x, t)$ is also a solution to (26), by uniqueness we have that $\Phi(-x, t) = -\Phi(x, t)$, hence each E_t is symmetric with respect to the origin. Moreover, there exists $\delta > 0$ such that for $|t| < \delta$ the set E_t is a nearly spherical set as in (23) and the corresponding function u satisfies $\|u\|_{C^k(\mathbb{S}^{n-1})} < \varepsilon$. We claim that we can choose δ so that $\gamma(E_t) = \gamma(B_r)$ for $|t| < \delta$. To see this, let us choose $\delta > 0$ so that $B_{\frac{r}{2}} \subset\subset E_t \subset\subset B_r$. Then, see [15, Prop. 17.8], for all $t \in (-\delta, \delta)$

$$\frac{d}{dt}\gamma(E_t) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\partial E_t} X \cdot \nu^{E_t} e^{-\frac{|x|^2}{2}}\, d\mathcal{H}^{n-1}.$$

The equality $\gamma(E_t) = \gamma(B_r)$ will follow by observing that the integral on the right hand side of the above formula vanishes for all $t \in (-\delta, \delta)$. Indeed, if $\varrho > r/2$ is such that $B_\varrho \subset\subset E_t$, from the divergence theorem, recalling (24) and (25), we have

$$\begin{aligned}\int_{\partial E_t} X \cdot \nu^{E_t} e^{-\frac{|x|^2}{2}}\, d\mathcal{H}^{n-1} &= \int_{\partial B_\varrho} X \cdot \frac{x}{|x|} e^{-\frac{|x|^2}{2}}\, d\mathcal{H}^{n-1} + \int_{E_t \setminus B_\varrho} \operatorname{div}\big(X e^{-\frac{|x|^2}{2}}\big)\, dx \\ &= \frac{1}{\varrho^{n-1}}\int_{\partial B_\varrho} \varphi\Big(\frac{x}{|x|}\Big)\, d\mathcal{H}^{n-1} + \int_{E_t \setminus B_\varrho} \operatorname{div}\Big(\frac{x}{|x|^n}\varphi\Big(\frac{x}{|x|}\Big)\Big)\, dx = 0.\end{aligned}$$

Set now $p(t) = (2\pi)^{\frac{n}{2}} P_\gamma(E_t)$ for $t \in (-\delta, \delta)$. Using the formula for the first variation of the perimeter, see [15, Th. 17.5], the divergence theorem on manifolds and (24), we have

$$\begin{aligned}p'(0) &= \int_{\partial B_r} (\operatorname{div}_\tau X - X \cdot x) e^{-\frac{|x|^2}{2}}\, d\mathcal{H}^{n-1} = \int_{\partial B_r} \Big(X \cdot \frac{x}{|x|} H_{\partial B_r} - X \cdot x\Big) e^{-\frac{|x|^2}{2}}\, d\mathcal{H}^{n-1} \\ &= \int_{\partial B_r} \Big(\frac{n-1}{r^{n+1}} - \frac{1}{r^{n-2}}\Big)\varphi\Big(\frac{x}{|x|}\Big)\, d\mathcal{H}^{n-1} = 0\end{aligned}$$

Thus, in order to conclude the proof it will be enough to show that we may always choose φ satisfying (24) and such that $p''(0) < 0$.

To this aim, let us evaluate $p''(0)$. Note that the general formula for the second variation of the Gaussian perimeter is quite complicate, see for instance [4, Eq. (17)]. However in our case, since B_r satisfies the Euler-Lagrange equation (22), it simplifies a lot. Indeed, see [4, Prop. 3] we have

$$p''(0) = \int_{\partial B_r} \big[|D_\tau(X \cdot \nu^{\partial B_r})|^2 - |II_{B_r}|^2 (X \cdot \nu^{\partial B_r})^2 - (X \cdot \nu^{\partial B_r})^2\big] e^{-\frac{|x|^2}{2}}\, d\mathcal{H}^{n-1},$$

where $|II_{B_r}|^2$ is the sum of the squares of the principal curvatures of ∂B_r. Hence,

$$p''(0) = \frac{e^{\frac{r^2}{2}}}{r^{2n-2}} \int_{\partial B_r} \Big[\Big|D_\tau\Big(\varphi\Big(\frac{x}{|x|}\Big)\Big)\Big|^2 - \frac{n-1}{r^2}\varphi\Big(\frac{x}{|x|}\Big)^2 - \varphi\Big(\frac{x}{|x|}\Big)^2\Big]\, d\mathcal{H}^{n-1}$$
$$= \frac{e^{\frac{r^2}{2}}}{r^{n-1}} \int_{\mathbb{S}^{n-1}} \Big(\frac{1}{r^2}|D_\tau\varphi(x)|^2 - \frac{n-1}{r^2}\varphi(x)^2 - \varphi(x)^2\Big)\, d\mathcal{H}^{n-1}.$$

Then, choosing $\varphi = y_2$, where y_2 is any homogeneous harmonic polynomial of degree 2, normalized so that $\|y_2\|_{L^2(\mathbb{S}^{n-1})} = 1$, (24) is obviously satisfied and from the above formula we get

$$p''(0) = \frac{e^{\frac{r^2}{2}}}{r^{n-1}}\Big(\frac{2n}{r^2} - \frac{n-1}{r^2} - 1\Big) = \frac{(n+1-r^2)e^{\frac{r^2}{2}}}{r^{n+1}} < 0,$$

thus concluding the proof. □

3. L^1-local minimality

In this section we show how to derive from Theorem 2 the L^1-local minimality of balls centered at the origin with sufficiently small radii. Our proof follows the strategy devised in [1] with a few difficulties due to the fact that in the Gauss space the presence of a density in the measure γ does not allow us to reduce the proof to the case of bounded sets as it happens in the Euclidean case.

We now introduce a functional that will be used in the proof of the L^1-local minimality of the ball. Given $r > 0$ for every set E of locally finite perimeter we define

$$J(E) = P_\gamma(E) + \Lambda_1\gamma(E\Delta B_r) + \Lambda_2|\gamma(E) - \gamma(B_r)|, \tag{27}$$

where $\Lambda_1, \Lambda_2 \geq 0$. Next lemma, which is the counterpart in our setting of [1, Lemma 4.1], shows that if Λ_1 is sufficiently large, then the unique minimizer of J among all sets of locally finite perimeter E is the ball B_r.

Lemma 1. *Let $n \geq 2$. There exists $C_0(n) > 0$ such that if $r > 0$, $\Lambda_1 > C_0\big(r + \frac{1}{r}\big)$ and $\Lambda_2 \geq 0$, then B_r is the unique minimizer of J among all sets $E \subset \mathbb{R}^n$ of locally finite perimeter.*

Proof. Let $\eta : \mathbb{R} \to [0, 1]$ be a smooth function such that $\eta \equiv 0$ outside the interval $(1/3, 3)$, $\eta \equiv 1$ on the interval $(1/2, 2)$. Fix $r > 0$ and denote by X_r the vector field

$$X_r(x) = \eta\Big(\frac{|x|}{r}\Big)\frac{x}{|x|} \qquad \text{for all } x \in \mathbb{R}^n.$$

Then

$$J(E) - J(B_r) \geq \int_{\partial^* E} X_r \cdot \nu^E\, d\mathcal{H}^{n-1}_\gamma - \int_{\partial B_r} X_r \cdot \nu^{B_r}\, d\mathcal{H}^{n-1}_\gamma + \Lambda_1\gamma(E\Delta B_r)$$
$$= \int_E (\operatorname{div} X_r - X_r \cdot\ x)\, d\gamma - \int_{B_r} (\operatorname{div} X_r - X_r \cdot\ x)\, d\gamma + \Lambda_1\gamma(E\Delta B_r)$$
$$= \int_{E\Delta B_r} (\operatorname{div} X_r - X_r \cdot\ x)\, d\gamma + \Lambda_1\gamma(E\Delta B_r).$$

Since by construction $||\operatorname{div} X_r - X_r \cdot x||_\infty \leq C_0\big(\frac{1}{r} + r\big)$ for some constant C_0 depending only on n, the result immediately follows. □

One difficulty in the proof of L^1-local minimality of the balls B_r for small radii is that the Gaussian measure is not scaling invariant. The following simple lemma is a helpful tool to deal with this issue.

Lemma 2. *Let $n \geq 2$, $\sigma \in (0, 1/2)$. For every $\varepsilon > 0$ there exists $\delta < \sigma/2$ depending only on ε, n, σ, such that for all $r \in [\sigma, \sqrt{n+1}]$ and $\tau \in (0, \delta)$*

$$P_\gamma(B_r) - P_\gamma(B_{r-\tau}) \leq \varepsilon P_\gamma(H_{s(r,\tau)}), \tag{28}$$

where the half space $H_{s(r,\tau)}$ is such that $\gamma(H_{s(r,\tau)}) = \gamma(B_r) - \gamma(B_{r-\tau})$.

Proof. For $r \in [\sigma, \sqrt{n+1}]$, we set

$$f(r,\tau) = e^{-\frac{s(r,\tau)^2}{2}} \quad \text{for } 0 < \tau \leq r, \qquad f(r,0) = 0.$$

Then, for $0 < \tau \leq \sigma/2$ we have

$$\frac{P_\gamma(B_r) - P(_\gamma(B_{r-\tau})}{P_\gamma(H_{s(r,\tau)})} = \frac{n\omega_n}{(2\pi)^{\frac{n-1}{2}}} \frac{r^{n-1}e^{-\frac{r^2}{2}} - (r-\tau)^{n-1}e^{-\frac{(r-\tau)^2}{2}}}{f(r,\tau) - f(r,0)}.$$

Therefore, by the Cauchy's mean value theorem there exists $\vartheta \in (0, \tau)$, such that

$$\frac{P_\gamma(B_r) - P_\gamma(B_{r-\tau})}{P_\gamma(H_{s(r,\tau)})} = \frac{n\omega_n}{(2\pi)^{\frac{n-1}{2}}} \frac{-(n-1)(r-\vartheta)^{n-2}e^{-\frac{(r-\vartheta)^2}{2}} + (r-\vartheta)^n e^{-\frac{(r-\vartheta)^2}{2}}}{-s(r,\vartheta)e^{-\frac{s(r,\vartheta)^2}{2}} \dfrac{\partial s}{\partial \tau}(r,\vartheta)}. \tag{29}$$

On the other hand, since by definition

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s(r,\tau)} e^{-\frac{t^2}{2}}\, dt = \frac{n\omega_n}{(2\pi)^{\frac{n}{2}}} \int_{r-\tau}^{r} t^{n-1}e^{-\frac{t^2}{2}}\, dt,$$

differentiating this equation with respect to τ we have that

$$\frac{\partial s}{\partial \tau}(r,\vartheta) = \frac{n\omega_n}{(2\pi)^{\frac{n-1}{2}}} n\omega_n (r-\vartheta)^{n-1} e^{\frac{s(r,\vartheta)^2 - (r-\vartheta)^2}{2}}.$$

Thus, inserting this value in (29) we have

$$\begin{aligned}
\frac{P_\gamma(B_r) - P(_\gamma(B_{r-\tau})}{P_\gamma(H_{s(r,\tau)})} &= \frac{-(n-1) + (r-\vartheta)^2}{-s(r,\vartheta)(r-\vartheta)} \\
&\leq \frac{2}{-\Phi^{-1}\big(\gamma(B_r) - \gamma(B_{r-\vartheta})\big)(r-\vartheta)} \\
&\leq \frac{2}{-\Phi^{-1}(C(n)\vartheta)\sigma} \leq \frac{4}{-\Phi^{-1}(C(n)\tau)\sigma},
\end{aligned}$$

for a suitable constant depepnding only on n. Then the conclusion follows since $\Phi^{-1}(\tau) \to -\infty$ as $\tau \to 0^+$. □

As in [1] the proof of Theorem 1 uses heavily the regularity theory for area minimizing sets. For the reader's convenience we recall the relevant definitions and the main results that we need in the sequel.

Definition 1. Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter, $\omega, r_0 > 0$ and let Ω be an open subset of $\mathbb{R}^n$. We say that E is a (ω, r_0)-*quasiminimizer of the (Euclidean) perimeter* in Ω if for every ball $B_\varrho(x) \subset \Omega$ with $\varrho < r_0$ and any set F of locally finite perimeter such that $E\Delta F \subset\subset B_\varrho(x)$

$$P(E; B_\varrho(x)) \leq P(E; B_\varrho(x)) + \omega\varrho^n. \tag{30}$$

Note that this notion of minimality is slightly weaker than the so called *almost minimality* where on the right hand side of (30) the term $\omega\varrho^n$ is replaced by $\Lambda|E\Delta F|$, where Λ is a fixed positive constant. Nevertheless, the regularity theory for perimeter minimizers or almost minimizers carries also to quasiminizers. In particular we have the following two results. For the first one we refer to [19, Th. 1.9], see also [15, Th. 21.14]. For a proof of Theorem 4 the reader may see [10, Prop.

2.2] or [15, Th. 26.6]. Note that when dealing with a set of finite perimeter E we always tacitly assume that E is a Borel set such that its topological boundary ∂E coincides with the support of the perimeter measure, i.e.,

$$\partial E = \{x \in \mathbb{R}^n : 0 < |E \cap B_r(x)| < \omega_n r^n \text{ for every } r > 0\},$$

see for instance [15, Prop. 12.19]. Finally, we say that a sequence of measurable sets E_h *converges to* E*, or converges locally to* E*,* in an open set Ω if the characteristic functions χ_{E_h} converge in $L^1(\Omega)$, respectively in $L^1_{loc}(\Omega)$, to χ_E. Observe that

$$E_h \text{ converge locally in } \mathbb{R}^n \text{ to } E \quad \Longrightarrow \quad \gamma(E_h) \to \gamma(E) \text{ as } h \to \infty. \tag{31}$$

Theorem 3. *Let E_h be a sequence of (ω, r_0)-quasiminimizers in Ω converging locally to a set of locally finite perimeter E. Then the two following properties hold:*

(i) if $x_h \in \partial E_h \cap \Omega$ and $x_h \to x \in \Omega$, then $x \in \partial E$;

(ii) if $x \in \partial E \cap \Omega$, then there exists a sequence x_h such that $x_h \in \partial E_h \cap \Omega$ for all h and $x_h \to x$.

We will also need a regularity theorem stating that if F is a perimeter quasiminimizer, sufficiently close in L^1 to a smooth open set E, then F is indeed $C^{1,\alpha}$ close to E.

Theorem 4. *Let E_h be a sequence of equibounded (ω, r_0)-quasiminimizers in $\mathbb{R}^n$, converging in $\mathbb{R}^n$ to a bounded open set E of class C^2. Then, for h large enough E_h is of class $C^{1,\frac{1}{2}}$ and*

$$\partial E_h = \{x + \psi_h(x)\nu^E(x) \; : \; x \in \partial E\}$$

with $\psi_h \to 0$ in $C^{1,\alpha}$ for all $\alpha \in (0, \frac{1}{2})$.

We are now ready to prove our main result. Roughly speaking it states that if B_r is a ball whose radius is below the critical value $\sqrt{n+1}$, then it is a local minimizer of the Gaussian perimeter among all sets with the same Gaussian measure and symmetric with respect to the origin. Moreover this local minimality property holds with a uniform quantitative estimate, provided r is away from 0 and from $\sqrt{n+1}$.

Before giving the proof of this theorem, let us briefly describe its strategy. We argue by contradiction assuming that there exists a sequence of symmetric sets E_h, with $\gamma(E_h) = \gamma(B_{r_h})$, such that $\varepsilon_h = \gamma(E_h \Delta B_{r_h}) \to 0$ as $h \to \infty$, for which the inequality (2) is violated. At this point, as first observed in this context by Cicalese and Leonardi in [10], one may replace the sets E_h with a new sequence F_h, still violating inequality (2), but converging in $C^{1,\alpha}$ to a ball $B_{\overline{r}}$. This leads to a contradiction with the local minimality property of $B_{\overline{r}}$, provided the constant κ is sufficiently small. The new sequence F_h is obtained by minimizing the functionals

$$J_h(F) = P_\gamma(F) + \Lambda_1|\gamma(F \Delta B_{r_h}) - \varepsilon_h| + \Lambda_2|\gamma(F) - \gamma(B_{r_h})|, \tag{32}$$

for suitable $\Lambda_1, \Lambda_2 > 0$, among all subsets of $\mathbb{R}^n$ symmetric with respect to the origin. The choice of this particular functional is inspired by a similar one first introduced in [1] and later on successfully modified in [5], [6], [7], [12]. To get the $C^{1,\alpha}$ convergence of the minimizers F_h we prove that they are also (ω, r_0)-quasiminimizers of the Euclidean perimeter in every ball B_R, a fact that in our case is not completely trivial, since each F_h minimizes the functional J_h only among sets which are symmetric with respect to the origin. At this point, if we knew that the sets F_h were equibounded, the $C^{1,\alpha}$ convergence would follow immediately from Theorem 4. However, there is no reason why this should be true and to overcome this difficulty we have to show that even if the F_h may be unbounded they all split into two regions, a bounded one which converge in $C^{1,\alpha}$ to the ball $B_{\overline{r}}$ and another one of small mass which disappears at infinity.

Proof of Theorem 1. Throughout this proof we are going to use the following notation. Given a measurable set E we denote by $r(E)$ the radius of the ball centered at the origin such that

$$\gamma(E) = \gamma(B_{r(E)}). \tag{33}$$

Step 1. We argue by contradiction assuming that there exists a sequence E_h of sets symmetric with respect to the origin, with $\gamma(E_h) = \gamma(B_{r_h})$, $r_h \in [\sigma, \sqrt{n+1} - \sigma]$, such that

$$\varepsilon_h = \gamma(E_h \Delta B_{r_h}) \to 0, \qquad P_\gamma(E_h) - P_\gamma(B_{r_h}) \leq \kappa \varepsilon_h^2, \tag{34}$$

for a suitable κ that will be fixed later in the proof. Let us fix

$$\Lambda_1 > C_0\Big(\sqrt{n+1} + \frac{1}{\sigma}\Big), \qquad \Lambda_2 \geq \max\{3\Lambda_1, \widetilde{C}\}, \tag{35}$$

where C_0 is the constant provided in Lemma 1 and $\widetilde{C}$ is a constant, depending only on n and σ, that will be fixed later.
For every h we consider the following minimum problem

$$\min\big\{J_h(F) : F = -F,\ F \text{ has locally finite perimeter}\big\}, \tag{36}$$

where J_h is the functional defined in (32).
The existence of a minimizer for the the problem in (36) is readily proved by observing that any minimizing sequence is compact with respect to the local convergence in $\mathbb{R}^n$ and recalling the lower semicontinuity of the perimeter and the continuity of the Gaussian measure, see (31), with respect to the local convergence in $\mathbb{R}^n$.
Let us now assume, without loss of generality, that $r_h \to \overline{r}$ for $h \to \infty$ and observe that the minimizers F_h converge locally in L^1 to $B_{\overline{r}}$. In fact, since by the minimality of F_h

$$P_\gamma(F_h) \leq P_\gamma(E_h) \leq C(n), \qquad \text{for all } h,$$

we have, see [2, Th. 3.39], that up to a (not relabelled) subsequence, they converge locally in $\mathbb{R}^n$ to some set of locally finite perimeter $\widetilde{F}$. We claim that $\widetilde{F} = B_{\overline{r}}$. To see this let us take a set of locally finite perimeter E, symmetric with respect to the origin. By the minimality of F_h we have that

$$P_\gamma(F_h) + \Lambda_1|\gamma(F_h \Delta B_{r_h}) - \varepsilon_h| + \Lambda_2|\gamma(F_h) - \gamma(B_{r_h})| \leq J_h(E).$$

Recalling (31), from the previous inequality we get immediately that

$$P_\gamma(\widetilde{F}) + \Lambda_1\gamma(\widetilde{F}\Delta B_{\overline{r}}) + \Lambda_2|\gamma(\widetilde{F}) - \gamma(B_{\overline{r}})| \leq P_\gamma(E) + \Lambda_1\gamma(E\Delta B_{\overline{r}}) + \Lambda_2|\gamma(E) - \gamma(B_{\overline{r}})|.$$

Hence, recalling the first inequality in (35), Lemma 1 yields $\widetilde{F} = B_{\overline{r}}$.
Note that for every B_R there exist $\omega > 0, r_0 \in (0,1)$ such that, for h large, the sets F_h are all (ω, r_0)-quasiminimizers in B_R. The proof of this latter property is given in Lemma 3 below.
Step 2 We claim that for h large

$$\gamma(F_h \Delta B_{r_h}) \geq \frac{\varepsilon_h}{4}. \tag{37}$$

To this end observe that by the minimality of F_h, the inequality in (34) and Lemma 1 again, we have

$$\begin{aligned} P_\gamma(F_h) + \Lambda_1|\gamma(F_h\Delta B_{r_h}) - \varepsilon_h| + \Lambda_2|\gamma(F_h) - \gamma(B_{r_h})| &\leq P_\gamma(E_h) \\ \leq P_\gamma(B_{r_h}) + \kappa\varepsilon_h^2 &\leq P_\gamma(F_h) + \Lambda_1\gamma(F_h\Delta B_{r_h}) + \kappa\varepsilon_h^2. \end{aligned} \tag{38}$$

If $\gamma(F_h\Delta B_{r_h}) \geq \varepsilon_h/2$, inequality (37) is trivially satisfied. Otherwise, if $\gamma(F_h\Delta B_{r_h}) \leq \varepsilon_h/2$, from (38) we deduce

$$\varepsilon_h - \gamma(F_h\Delta B_{r_h})) \leq \gamma(F_h\Delta B_{r_h}) + \frac{\kappa}{\Lambda_1}\varepsilon_h^2.$$

Hence, the claim (37) follows for h sufficiently large, since $\varepsilon_h \to 0$ by (34).

Since $\gamma(F_h)$ may be different from $\gamma(B_{r_h})$, it is convenient to consider the balls $B_{r(F_h)}$ defined as in (33). From (37) and (38), recalling the second inequality in (35), we have for h large

$$|\gamma(F_h) - \gamma(B_{r_h})| \leq \frac{\Lambda_1}{\Lambda_2}\gamma(F_h \Delta B_{r_h}) + \frac{\kappa}{\Lambda_2}\varepsilon_h^2 \leq \frac{\gamma(F_h \Delta B_{r_h})}{2}.$$

Thus, we may estimate, for h large

$$\begin{aligned}\gamma(F_h \Delta B_{r_h}) &\leq \gamma(F_h \Delta B_{r(F_h)}) + \gamma(B_{r(F_h)} \Delta B_{r_h}) \\ &= \gamma(F_h \Delta B_{r(F_h)}) + |\gamma(F_h) - \gamma(B_{r_h})| \leq \gamma(F_h \Delta B_{r(F_h)}) + \frac{\gamma(F_h \Delta B_{r_h})}{2}.\end{aligned}$$

Therefore, recalling (37), we have

$$\gamma(F_h \Delta B_{r_h}) \leq 2\gamma(F_h \Delta B_{r(F_h)}), \text{ hence } \gamma(F_h \Delta B_{r(F_h)}) \geq \frac{\varepsilon_h}{8}. \tag{39}$$

From (38) and the second inequality in (39) we have with some easy computations

$$\begin{aligned}P_\gamma(F_h) + \Lambda_2|\gamma(F_h) - \gamma(B_{r_h})| &\leq P_\gamma(B_{r_h}) + 64\kappa\gamma(F_h \Delta B_{r(F_h)})^2 \\ &= P_\gamma(B_{r(F_h)})) + [P_\gamma(B_{r_h}) - P_\gamma(B_{r(F_h)})] + 64\kappa\gamma(F_h \Delta B_{r(F_h)})^2 \\ &\leq P_\gamma(B_{r(F_h)})) + C(n)|r_h - r(F_h)| + 64\kappa\gamma(F_h \Delta B_{r(F_h)})^2 \\ &\leq P_\gamma(B_{r(F_h)})) + \widetilde{C}(n,\sigma)|\gamma(B_{r(F_h)}) - \gamma(B_{r_h})| + 64\kappa\gamma(F_h \Delta B_{r(F_h)})^2,\end{aligned}$$

for a suitable constant $\widetilde{C}$ depending only on n and σ. Therefore, recalling that $\Lambda_2 \geq \widetilde{C}$ by (34), we end up by proving that also the sets F_h satisfy a 'reverse' quantitative inequality as the one in (34), with a possibly bigger constant

$$P_\gamma(F_h) \leq P_\gamma(B_{r(F_h)})) + 64\kappa\gamma(F_h \Delta B_{r(F_h)})^2. \tag{40}$$

Note that if we knew that the F_h were equibounded we would have by Theorem 4 that they were converging in $C^{1,\alpha}$ to the ball $B_{\overline{r}}$ and, taking κ sufficiently small, from (40) we would get a contradiction to (19), thus concluding the proof. Instead, since it may happen that the sets F_h are unbounded or that they are not equibounded, we split them as follows

$$G_h = F_h \cap B_n, \qquad L_h = F_h \setminus B_n.$$

Clearly, the G_h converge in L^1 to $B_{\overline{r}}$, while $\gamma(L_h) \to 0$ as $h \to \infty$. Moreover, since

$$\begin{aligned}\gamma(F_h \Delta B_{r(F_h)}) &\leq \gamma(F_h \Delta G_h) + \gamma(G_h \Delta B_{r(G_h)}) + \gamma(B_{r(G_h)} \Delta B_{r(F_h)}) \\ &= \gamma(G_h \Delta B_{r(G_h)}) + 2\gamma(L_h),\end{aligned}$$

from (40) we conclude that

$$P_\gamma(F_h) \leq P_\gamma(B_{r(F_h)})) + C_2\kappa\big[\gamma(G_h \Delta B_{r(G_h)})^2 + \gamma(L_h)^2\big], \tag{41}$$

for some universal constant C_2 not even depending on n.

Step 3. We claim now that for h large

$$F_h \cap (\overline{B}_{n+1} \setminus B_n) = \emptyset. \tag{42}$$

To prove this we argue by contradiction assuming that for infinitely many h the intersection $F_h \cap (\overline{B}_{n+1} \setminus B_n)$ is not empty. On the other hand, since $F_h \cap \overline{B}_n$ is converging in $\mathbb{R}^n$ to $B_{\overline{r}}$, we have that for h large also $(\overline{B}_{n+1} \setminus B_n) \setminus F_h$ is not empty. Therefore, there exists an increasing sequence $h_k \to \infty$ such that for any k there exists $x_k \in \partial F_{h_k} \cap (\overline{B}_{n+1} \setminus B_n)$ (note that since the sets F_h are quasiminimizers of the perimeter in every ball B_R, they are of class C^1 and thus ∂F_h coincides with the topological boundary). Passing possibly to another, and not relabelled,

subsequence we may assume that $x_k \to x$ for some $x \in \overline{B}_{n+1} \setminus B_n$. But this is impossible since by Theorem 3 the point x should belong to $\partial B_{\overline{r}}$.
Note that (42) yields in particular that

$$G_h \subset B_n, \qquad L_h \subset \mathbb{R}^n \setminus B_{n+1}. \tag{43}$$

As an immediate consequence of the above inclusions we have that the sets G_h are quasiminimizers of the Euclidean perimeter in $\mathbb{R}^n$.
Another consequence of (43) is that for h large $P_\gamma(F_h) = P_\gamma(G_h) + P_\gamma(L_h)$. Thus, from (41) we get that for h large

$$P_\gamma(G_h) \le P_\gamma(B_{r(F_h)}) - P_\gamma(L_h) + C_2\kappa\big[\gamma(G_h \Delta B_{r(G_h)})^2 + \gamma(L_h)^2\big].$$

Now, let $s_h \in \mathbb{R}$ be such that $\gamma(H_{s_h}) = \gamma(L_h)$. From the inequality above and the Gaussian isoperimetric inequality we have

$$P_\gamma(G_h) \le P_\gamma(B_{r(F_h)}) - P_\gamma(H_{s_h}) + C_2\kappa\big[\gamma(G_h \Delta B_{r(G_h)})^2 + \gamma(H_{s_h})^2\big].$$

In turn, using (28) with $\varepsilon = 1/2$ to estimate $P_\gamma(B_{r(F_h)})$, we have that for h sufficiently large

$$P_\gamma(G_h) \le P_\gamma(B_{r(G_h)}) - \frac{1}{2}P_\gamma(H_{s_h}) + C_2\kappa\big[\gamma(G_h \Delta B_{r(G_h)})^2 + \gamma(H_{s_h})^2\big]. \tag{44}$$

Finally, observe that

$$\lim_{s\to-\infty} \frac{\gamma(H_s)}{P_\gamma(H_s)} = 0.$$

Therefore, from (44) we may conclude that for h sufficiently large

$$P_\gamma(G_h) \le P_\gamma(B_{r(G_h)}) + C_2\kappa\gamma(G_h \Delta B_{r(G_h)})^2. \tag{45}$$

Now, since the sets G_h are converging to $B_{\overline{r}}$ in $\mathbb{R}^n$, by Lemma 4 they also converge in $C^{1,\alpha}$ to $B_{\overline{r}}$, i.e.

$$\partial G_h = \{x(1 + u_h(x)) : x \in \partial B_{\overline{r}}\}$$

where $u_h \to 0$ in $C^{1,\alpha}(\partial B_{\overline{r}})$. Thus, still denoting by u_h the 0-homogeneous extension of the above functions u_h, we conclude that

$$G_h = \Big\{y = tr(G_h)x\Big(1 + \frac{\overline{r}(1 + u_h(x)) - r(G_h)}{r(G_h)}\Big) :\, x \in \mathbb{S}^{n-1}, 0 \le t < 1\Big\},$$

where, since $r(G_h) \to \overline{r}$,

$$\frac{\overline{r}(1 + u_h(x)) - r(G_h)}{r(G_h)} \to 0 \qquad \text{in } C^{1,\alpha}(\mathbb{S}^{n1}).$$

Thus, by (19) we conclude that for h sufficiently large

$$P_\gamma(G_h) - P_\gamma(B_{r(G_h)}) \ge \kappa_1 r(G_h)^{-1-n}\gamma(G_h \Delta B_{r(G_h)})^2 \ge \frac{\kappa_1}{(n+1)^{\frac{n+1}{2}}}\gamma(G_h \Delta B_{r(G_h)})^2,$$

which contradicts (45) if we choose

$$\kappa < \frac{\kappa_1}{C_2(n+1)^{\frac{n+1}{2}}}.$$

Hence the conclusion follows by this contradiction. □

The arguments used in the proof of next lemma are similar to the ones used for the standard perimeter. However, in our case the proof is more involved due presence of the constraint $F = -F$ in the minimum problems (36).

Lemma 3. *Let $n \geq 2$ and $\sigma \in (0, 1/2)$ as in Theorem 1. Moreover, let F_h be a sequence of minimizers of the* (36)*, with F_h converging locally in $\mathbb{R}^n$ to a ball $B_{\overline{r}}$, with $\overline{r} \in [\sigma, \sqrt{n+1}-\sigma]$. There exists h_0 such that for very ball B_R, there exist $\omega, r_0 > 0$, such that F_h is a (ω, r_0)-quasiminimizer in B_R for all $h \geq h_0$.*

Proof. **Step 1.** Let us fix $R \geq 1$. We start proving that there exist $r_1, \vartheta > 0$, depending on R, such that if $x \in \partial F_h \cap B_R$, $\varrho < r_1$, then

$$|F_h \cap B_\varrho(x)| \leq (\omega_n - \vartheta)\varrho^n. \tag{46}$$

To this end, let us observe that if $x \in \mathbb{R}^n$, $\varrho > 0$ and G is a set of locally finite perimeter with $G = -G$, such that $F_h \Delta G \subset\subset B_\varrho(x) \cup B_\varrho(-x)$, then from the minimality inequality $J_h(F_h) \leq J_h(G)$ we get

$$P_\gamma(F_h; B_\varrho(x) \cup B_\varrho(-x)) \leq P_\gamma(G; B_\varrho(x) \cup B_\varrho(-x)) + (\Lambda_1 + \Lambda_2)\gamma(F_h \Delta G).$$

From this inequality, setting

$$m(x,\varrho) = \frac{1}{(2\pi)^{\frac{n}{2}}} \min_{y \in B_\varrho(x)} e^{-\frac{|y|^2}{2}}, \qquad M(x,\varrho) = \frac{1}{(2\pi)^{\frac{n}{2}}} \max_{y \in B_\varrho(x)} e^{-\frac{|y|^2}{2}},$$

we immediately get the following inequality for the Euclidean perimeter

$$m(x,\varrho)P(F_h; B_\varrho(x) \cup B_\varrho(-x)) \leq M(x,\varrho)P(G; B_\varrho(x) \cup B_\varrho(-x)) + (\Lambda_1 + \Lambda_2)M(x,\varrho)|F_h \Delta G|.$$

Thus, dividing both sides of this inequality by $m(x, \varrho)$ and observing that if $0 < \varrho < 1$ we have $(M(x, \varrho) - m(x, \varrho))/m(x, \varrho) < C\varrho$, for some constant C depending on R, we get that

$$P(F_h; B_\varrho(x) \cup B_\varrho(-x)) \leq (1 + C\varrho)P(G; B_\varrho(x) \cup B_\varrho(-x)) + C'|F_h \Delta G|. \tag{47}$$

Recalling that $F_h \Delta G \subset\subset B_\varrho(x) \cup B_\varrho(-x)$ from the standard isoperimetric inequality we get

$$\begin{aligned} |F_h \Delta G| &\leq |B_\varrho(x) \cup B_\varrho(-x)|^{\frac{1}{n}} |F_h \Delta G|^{\frac{n-1}{n}} \leq n\omega_n \varrho P(F_h \Delta G; B_\varrho(x) \cup B_\varrho(-x)) \\ &\leq n\omega_n \varrho[P(F_h; B_\varrho(x) \cup B_\varrho(-x)) + P(G; B_\varrho(x) \cup B_\varrho(-x))], \end{aligned}$$

where the last inequality follows by using the precise expression of the reduced boundary of the symmetric difference of two sets of finite perimeter, see [15, Th. 16.3]. Inserting this inequality in (47) we conclude that there exists $\chi > 1$ depending only on n, Λ_1 and Λ_2 and R such that for all $0 < \varrho < 1$

$$(1 - \chi\varrho)P(F_h; B_\varrho(x) \cup B_\varrho(-x)) \leq (1 + \chi\varrho)P(G; B_\varrho(x) \cup B_\varrho(-x)). \tag{48}$$

Let us now fix $x \in \mathbb{R}^n$ and $0 < \varrho < 1/\chi$ and set $G = F_h \cup (B_{\varrho'}(x) \cup B_{\varrho'}(-x))$ for some $0 < \varrho' < \varrho$. Note that G is an admissible comparison set since $G = -G$. With this choice of G, using again the precise expression of the reduced boundary of the difference between two sets of finite perimeter, see again [15, Th. 16.3], from (48) we easily obtain that

$$\begin{aligned} (1 - \chi\varrho)P(F_h; B_\varrho(x) \cup B_\varrho(-x)) \leq (1 + \chi\varrho)\big[&\mathcal{H}^{n-1}\big(F_h^{(0)} \cap \partial(B_{\varrho'}(x) \cup B_{\varrho'}(-x))\big) \\ &+ P(F_h; (B_\varrho(x) \cup B_\varrho(-x)) \setminus (\overline{B}_{\varrho'}(x) \cup \overline{B}_{\varrho'}(-x))), \end{aligned}$$

where $F_h^{(0)}$ denotes the sets of points in $\mathbb{R}^n$ where F_h has density 0. From this inequality, letting $\varrho' \to \varrho$, we deduce that if $0 < \varrho < 1/\chi$, then

$$P(F_h, B_\varrho(x)) \leq \frac{2(1+\chi)}{1-\chi}\mathcal{H}^{n-1}\big(F_h^{(0)} \cap \partial B_\varrho(x)\big). \tag{49}$$

Let us now fix $x \in \partial F_h$. In this way, setting $m(\varrho) = |B_\varrho(x) \setminus F_h|$, we have that $m(\varrho) > 0$ for all $\varrho > 0$. Since $m'(\varrho) = \mathcal{H}^{n-1}\big(F_h^{(0)} \cap \partial B_\varrho(x)\big)$ for a.e. $\varrho > 0$, from (49) we get that for all $\varrho \in (0, 1/\chi)$

$$m(\varrho)^{\frac{n-1}{n}} \leq \frac{2\kappa_n(1+\chi)}{1-\chi}m'(\varrho),$$

where κ_n is the constant of the Euclidean relative isoperimetric in balls, see for instance [2, Eq. 3.43]. Integrating this inequality we then get that for all $0 < \varrho < 1/\chi$

$$|B_\varrho(x) \setminus F_h| \geq \vartheta \varrho^{n-1},$$

hence (46) follows.

Step 2. Let us now prove that there exists an integer h_0 such that

$$|B_{\frac{\sigma}{2}} \setminus F_h| = 0 \qquad \text{for all } h \geq h_0. \tag{50}$$

To prove this inclusion we argue by contradiction assuming that there exists a strictly increasing sequence h_k of integers such that $|B_{\frac{\sigma}{2}} \setminus F_{h_k}| > 0$ for all k. On the other hand, since the sets $F_{h_k} \cap B_{\frac{\sigma}{2}}$ are converging in $\mathbb{R}^n$ to $B_{\frac{\sigma}{2}}$, we have also that $|B_{\frac{\sigma}{2}} \cap F_{h_k}| > 0$ for all k sufficiently large. Thus, from the relative isoperimetric inequality on balls we have that for all k large $P(F_{h_k}; B_{\frac{\sigma}{2}}) > 0$, hence there exists a point $x_k \in \partial F_{h_k}$. Without loss of generality we may assume that the sequence x_{h_k} converges to a point $x \in \overline{B}_{\frac{\sigma}{2}}$. We now apply (46) with R=1 and $0 < \varrho < \min\{r_1, \sigma/2\}$. From the local convergence of F_h in $\mathbb{R}^n$ and we then have

$$|B_\varrho(x)| = \lim_k |F_{h_k} \cap B_\varrho(x_k)| \leq (\omega_n - \vartheta)\varrho^n.$$

From this contradiction (50) immediately follows.

Step 3. Let us now fix $R \geq 1$ and set $r_0 = \min\{\sigma/4, 1/\chi\}$, where χ is the constant in (49) (note that this constant depends on R but not on h). Let us consider a ball $B_\varrho(x) \subset B_R$, with $0 < \varrho < r_0$, and a set G of locally finite perimeter such that $F_h \Delta G \subset\subset B_\varrho(x)$, for a given $h \geq h_0$. Assume first that $|x| \geq \sigma/4$ and observe that in this case $B_\varrho(x) \cap B_\varrho(-x) = \emptyset$. Then, define

$$G' = [F \setminus (B_\varrho(x) \cup B_\varrho(-x))] \cup (G \cap B_\varrho(x)) \cup (-G \cap B_\varrho(-x)).$$

By construction, the set $-G' = G'$ and, inserting it in (47) we immediately get that

$$P(F_h; B_\varrho(x)) \leq (1 + C\varrho)P(G; B_\varrho(x)) + C'|F_h \Delta G|.$$

Adding $C\varrho$ to both sides of this inequality and recalling (49) we have

$$(1 + C\varrho)P(F_h; B_\varrho(x)) \leq (1 + C\varrho)P(G; B_\varrho(x)) + \frac{2C\varrho(1 + \chi)}{1 - \chi}\mathcal{H}^{n-1}\big(F_h^{(0)} \cap \partial B_\varrho(x)\big) + C'|F_h \Delta G|.$$

Dividing this inequality by $1 + C\varrho$ we immediately get that

$$P(F_h; B_\varrho(x)) \leq P(G; B_\varrho(x)) + \omega\varrho^n, \tag{51}$$

for a suitable ω depending only on $n, \chi, \Lambda_1, \Lambda_2$ and R.

If $|x| < \sigma/4$, recalling the inclusion (50), we have that $P(F_h; B_\varrho(x)) = 0$ and thus (51) holds trivially. This concludes the proof of the lemma. □

Now we want to show that B_r is not a global minimizer among symmetric sets with prescribed Gaussian measure, at least if r is small. To this end, we set $C_s = \mathbb{R}^n \setminus B_s$, and for every $r > 0$ we denote by $s(r)$ the unique number such that

$$\int_0^r t^{n-1} e^{-\frac{t^2}{2}}\, dt = \int_{s(r)}^\infty t^{n-1} e^{-\frac{t^2}{2}}\, dt \tag{52}$$

In other words, $s(r)$ is such that $\gamma(B_r) = \gamma(C_{s(r)})$

Proposition 2. *For every $n > 2$ there exists $r_0 > 0$ such that*

$$P_\gamma(C_{s(r)}) < P_\gamma(B_r) \tag{53}$$

for every $r < r_0$.
If $n = 2$, then B_r is never a global minimizer.

Proof. Differentiating (52) with respect to r, we have

$$r^{n-1}e^{-\frac{r^2}{2}} = -s^{n-1}(r)e^{-\frac{s^2(r)}{2}}s'(r) \tag{54}$$

In order to show that $P_\gamma(B_r) > P_\gamma(C_{s(r)})$ for r small enough, using (54) we evaluate the quotient as follows

$$\frac{P_\gamma(B_r)}{P_\gamma(C_{s(r)})} = \frac{r^{n-1}e^{-\frac{r^2}{2}}}{s^{n-1}(r)e^{-\frac{s^2(r)}{2}}} = -s'(r) \tag{55}$$

Since $\lim_{r\to 0^+} s(r) = +\infty$, $\lim_{r\to 0^+} s'(r) = -\infty$. Then there exists $r_0 > 0$ such that if $r < r_0$ we have $s'(r) < -1$. Therefore

$$\frac{P_\gamma(B_r)}{P_\gamma(C_{s(r)})} = -s'(r) > 1,$$

hence (53) follows.

For $n = 2$ we can give the explicit expression of $s(r)$. In fact integrating (52) we have

$$1 - e^{-\frac{r^2}{2}} = e^{-\frac{s^2(r)}{2}}$$

and then

$$s(r) = \sqrt{-2\log(1 - e^{-\frac{r^2}{2}})}, \quad s'(r) = -\frac{re^{-\frac{r^2}{2}}}{\left(1 - e^{-\frac{r^2}{2}}\right)\sqrt{-2\log(1 - e^{-\frac{r^2}{2}})}}.$$

A numerical plot of $s''(r)$ shows that this function is strictly positive for $r \in [0, 4]$. Therefore we have that for all $r \in [0, \sqrt{3}]$

$$s'(r) < s'(\sqrt{3}) \simeq -1.063$$

Since B_r is a local minimizer only for $0 < r < \sqrt{3}$, the above estimate implies, together with (55), that B_r is never a global minimizer. □

4. The 1-dimensional case

In this section we shall briefly discuss the 1-dimensional case. Beside being much simpler, this case exhibits quite different features. Before stating the local minimality result we recall that in one dimension a set of locally finite perimeter is locally the union of a finitely many intervals.

Proposition 3. *Let $n = 1$. For every $r > 0$, there exists $\delta = \delta(r)$ such that if $E \subset \mathbb{R}$ is a set of finite perimeter, $0 < \gamma(B_r \Delta E) < \delta$, $E = -E$ and $\gamma(E) = \gamma(B_r)$, then*

$$P_\gamma(E) > P_\gamma(B_r) \tag{56}$$

Moreover, there exists r_0 such that:

(a) *if $r > r_0$ then B_r is the unique global minimizer of the perimeter among all the sets E such that $E = -E$ and $\gamma(B_r) = \gamma(E)$.*
(b) *if $r < r_0$ then $C_s = (-\infty, -s) \cup (s, +\infty)$ is the unique global minimizer of the perimeter among all the sets E such that $E = -E$ and $\gamma(B_r) = \gamma(C_s) = \gamma(E)$.*
(c) *if $r = r_0$, then both B_{r_0} and C_{r_0} are global minimizers.*

Proof. The proof is quite easy, and it is based on the minimality property of the halfline. Fix any $r > 0$ and E such that $\gamma(E) = \gamma(B_r)$. Since a set of locally finite perimeter is locally the union of a finite number of intervalls, the generic symmetric set E will be of the type

$$E = \bigcup_{i=1}^{M}(-b_i, -a_i) \cup \bigcup_{i=1}^{M}(a_i, b_i) \cup (-a, a)$$

for some $0 \le a < a_1 < b_1 < \cdots < a_i < b_i < \dots$, with $M \in \mathbb{N} \cup \{\infty\}$ and $b_M \in (0, \infty]$.
Take $R > r$ such that $R \neq a_i, R \neq b_i$, $\forall i \in \mathbb{N}$ and such that $b_j < R < a_{j+1}$ for some $j \in \mathbb{N}$. Using the isoperimetric inequality it is easy to check that if H_{-s} is a halfline such that

$$\gamma(H_{-s}) = \frac{1}{2}\gamma(E \setminus B_R),$$

then

$$P_\gamma((E \cap B_R) \cup C_s) \le P_\gamma(E).$$

Therefore we may assume without loss of generality that

$$E = (-\infty, -b) \cup \bigcup_{i=1}^{k}(-b_i, -a_i) \cup \bigcup_{i=1}^{k}(a_i, b_i) \cup (-a, a) \cup (b, +\infty) \tag{57}$$

where $k = \max\{i \in \mathbb{N} : b_i < R\}$ and $\gamma(E) = \gamma(B_r)$.
Observe that if $0 < a < r$ then

$$\frac{\sqrt{2\pi}}{2}P_\gamma(B_r) = e^{-\frac{r^2}{2}} < e^{-\frac{a^2}{2}} \le \frac{\sqrt{2\pi}}{2}P_\gamma(E)$$

and thus (56) follows. On the other hand, since $\gamma(E) = \gamma(B_r)$, $a = r$ if and only if $E = B_r$.
Therefore we are left with the case $a = 0$. In this case we fix $\delta < \frac{\gamma(B_{\frac{r}{2}})}{2}$ and let $\gamma(B_r \Delta E) < \delta$. This last inequality implies that $a_1 < \frac{r}{2}$. In fact, if $a_1 > \frac{r}{2}$, we would have

$$\frac{\sqrt{2\pi}}{2}\gamma(B_{\frac{r}{2}}) = \int_0^{\frac{r}{2}} e^{-\frac{x^2}{2}}\,dx \le \int_0^{a_1} e^{-\frac{x^2}{2}} < \frac{\sqrt{2\pi}}{2}\gamma(E\Delta B_r) < \frac{\sqrt{2\pi}}{4}\gamma(B_{\frac{r}{2}}).$$

This contradiction shows that $a_1 < \frac{r}{2}$, hence $P_\gamma(B_r) < P_\gamma(E)$.
Let us prove (a). Let $r_0 > 0$ be such that

$$\frac{1}{\sqrt{2\pi}}\int_{-r_0}^{r_0} e^{-x^2}\,dx = \frac{1}{2}.$$

Let $r > r_0$ and assume by contradiction that there exists a set E such that $E = -E$, $\gamma(E) = \gamma(B_r)$ and $P_\gamma(E) < P_\gamma(B_r)$. Arguing as before we may assume

$$E = (-\infty, -b) \cup \bigcup_{i=1}^{k}(-b_i, -a_i) \cup \bigcup_{i=1}^{k}(a_i, b_i) \cup (b, +\infty) \tag{58}$$

for some $a_1 > 0$. Let $s > 0$ be such that

$$\frac{1}{2}\gamma(E) = \frac{1}{\sqrt{2\pi}}\int_s^{\infty} e^{-x^2}\,dx$$

and consider $C_s = (-\infty, -s) \cup (s, \infty)$. Using the isoperymetric inequality and the fact that $a_1 > 0$ we have

$$P_\gamma(E) > P_\gamma(C_s).$$

Since $\gamma(C_s) = \gamma(B_r) > \frac{1}{2}$ we have that $s < r$ and thus

$$P_\gamma(B_r) < P_\gamma(H_s) < P_\gamma(E).$$

Assume now $r < r_0$. In this case $P_\gamma(C_s) < P_\gamma(B_r)$ since $r < s$. Hence B_r cannot be a global minimizer.
In case (b) the proof that C_s is a global minimizer among all the symmetric sets follows by the same argument as in (a).
Finally if $r = r_0$, $P_\gamma(B_{r_0}) = P_\gamma(C_{r_0})$ and both minimize the Gaussian perimeter among symmetric sets. □

We want to emphatize that this argument applies only when $n = 1$ because of the rigidity of the structure of the sets of locally finite perimeter and because in one dimension the measure of the perimeter of the ball is a strictly decreasing function of the radius r. On the other hand, for $n > 1$ the measure of the perimeter of the ball is increasing for $r < \sqrt{n-1}$ and decreasing for $r > \sqrt{n-1}$.

The previous minimality result holds indeed also in a quantitative form. The simple proof of this property uses an argument of [9].

Proposition 4. *Let $n = 1$. For every $r > 0$ there exists $\delta(r) > 0$ such that for any $E \subset \mathbb{R}$, $E = -E$, $\gamma(E) = \gamma(B_r)$ and $\gamma(E\Delta B_r) \le \delta(r)$ there exist a positive constant $C(r)$ such that*

$$P_\gamma(E) - P_\gamma(B_r) > C(r)\gamma(E\Delta B_r)\sqrt{\log\left(\frac{1}{\gamma(E\Delta B_r)}\right)} \tag{59}$$

Proof. First, note that it is enough to prove the inequality in the case $P_\gamma(E) - P_\gamma(B_r) < \delta_0(r)$, for some positve δ_0 to be chosen later. Let $E \subset \mathbb{R}$ be a set of locally finite perimeter. As before, we may assume without loss of generality that E is of the form (58). Let $\delta(r)$ be as in Proposition 3. We have 2 cases: $a = 0$ and $r > a > 0$.
Let $a = 0$. Since $\gamma(E\Delta B_r) < \delta$, as before we have that $a_1 < \frac{r}{2}$ and then

$$P_\gamma(E) - P_\gamma(B_r) > e^{-\frac{r^2}{8}} - e^{-\frac{r^2}{2}} = f(r).$$

Then, we set $\delta_0(r) = f(r)$. With such a choice of $\delta_0(r)$ we are immediately reduced to the case $a > 0$.
Let $0 < a < r$. Since $\gamma(E\Delta B_r) < \delta$, arguing as in the proof of Proposition 3, we have that there exists $\varepsilon > 0$ such that $a > r - \varepsilon$. Let K_ε such that

$$\frac{2}{\sqrt{2\pi}}\int_{K_\epsilon}^{\infty} e^{-\frac{t^2}{2}}\,dt = \gamma(E) - \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-\frac{t^2}{2}}\,dt$$

If we set $E' = (-a, a) \cup C_{K_\epsilon}$, we have $P(E') \le P(E)$ and $\gamma(E'\Delta B_r) \ge \gamma(E\Delta B_r)$ and then it is enough to estimate the isoperimetric gap for E'. From this point on, the calculation are exactly as in [9, Theorem 1.2].

□

References

[1] E. Acerbi, N. Fusco & M. Morini, *Minimality via second variation for a nonlocal isoperimetric problem.* Comm. Math. Phys. **322** (2013), 515-557.

[2] L. Ambrosio, N. Fusco & D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford University Press, 2000.

[3] C. Borell, *The Brunn-Minkowski inequality in Gauss space.* Invent. Math. **30** (1975), 207-216.

[4] M. Barchiesi, A. Brancolini & V. Julin, *Sharp dimension free quantitative estimates for the Gaussian isoperimetric inequality.* arXiv:1409.2106v1 (2015), to appear in Ann. Probab.

[5] V. Bögelein, F. Duzaar & N. Fusco, *A sharp quantitative isoperimetric inequality in higher codimension,* Rend. Lincei Mat. Appl. **26** (2015), 309-362.

[6] V.Bögelein, F.Duzaar & N. Fusco, *A quantitative isoperimetric inequality on the sphere,* http://cvgmt.sns.it/paper/2146/ (2013), to appear in Adv. Calc. Var.

[7] L. Brasco, G. De Philippis & B. Velichkov: *Faber-Krahn inequalities in sharp quantitative form,* Duke Math. J. **164** (2015), 1777-1831.

[8] E. A. Carlen & C. Kerce, *On the cases of equality in Bobkov's inequality and Gaussian rearrangement.* Calc. Var. Partial Differential Equations **13** (2001), 1-18.

[9] A. Cianchi, N. Fusco, F. Maggi, & A. Pratelli, *On the isoperimetric deficit in Gauss space.* Amer. J. Math. **133** (2011), 131-186.

[10] M. Cicalese & G.P. Leonardi, *A selection principle for the sharp quantitative isoperimetric inequality.* Arch. Ration. Mech. Anal. **206** (2012), 617-643.

[11] N. Fusco, *The quantitative isoperimetric inequality and related topics*,Bulletin of Mathematical Sciences **5** (2015), 517-607.

[12] N. Fusco & V. Julin, *A strong form of the quantitative isoperimetric inequality,* Calc. Var. Partial Differential Equations **50** (2014), 925-937.

[13] N. Fusco, F. Maggi & A. Pratelli, *The sharp quantitative isoperimetric inequality.* Ann. of Math. **168** 2008), 941-980.

[14] S. Heilman, *Low Correlation Noise Stability of Symmetric Sets.* arXiv:1511.00382v4, 2016.

[15] F. Maggi, *Sets of finite perimeter and geometric variational problems. An introduction to geometric measure theory.* Cambridge Studies in Advanced Mathematics **135**. Cambridge University Press, Cambridge, 2012.

[16] E. Mossel & J. Neeman, *Robust Dimension Free Isoperimetry in Gaussian Space.* Ann. Probab. **43** (2015), 971-991.

[17] E. Mossel, R. O'Donnell & K. Oleszkiewicz, *Noise stability of functions with low influences: invariance and optimality.* Ann. of Math. **171** (2010), 295-341.

[18] V. N. Sudakov, & B. S. Tsirel'son, *Extremal properties of half-spaces for spherically invariant measures* (Russian). Problems in the theory of probability distributions, II. Zap. Naučn. Sem. Leningrad. Otdel. Mat.Inst. Steklov. (LOMI) **41** (1974), 14-24.

[19] I. Tamanini, *Regularity results for almost minimal oriented hypersurfaces,* Quaderni del Dipartimento di Matematica dell'Università di Lecce, Lecce, 1984. See also http://cvgmt.sns.it/paper/1807/