Geometric properties of the heat content

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Abstract

In this paper we study short time behavior of heat semigroup in connection with the geometry of sets with finite perimeter; we assume $C^{1,1}$ regularity and we relate the heat semigroup with curvatures of the initial datum. We also study the behavior when singularities occur; this is the case when the mean curvature is no more a function, but has to be considered as a Radon measure. This work is the natural continuation of [11] and is in the spirit of [4].

1 Introduction

In this paper we investigate the connections between the theory of semigroups and that of perimeters. This link was first established by De Giorgi [5]; he noticed that by taking the convolution with the Gauss–Weierstrass kernel

$$T_t\chi_E(x) = g_t * \chi_E(x) = \frac{1}{(4\pi t)^{n/2}}\int_E e^{-\frac{|x-y|^2}{4t}}dy,$$

then the map

$$t \mapsto \int_{\mathbb{R}^n} |\nabla T_t\chi_E|dx$$

is monotone decreasing, showing the existence of the limit

$$P(E) = \lim_{t\to 0}\int_{\mathbb{R}^n} |\nabla T_t\chi_E|dx$$

for any measurable set $E \subset \mathbb{R}^n$. In addition, De Giorgi proved that the finiteness of the perimeter, $P(E) < +\infty$, is equivalent to the existence of a vector–valued measure μ_E, with finite total variation such that

$$\int_E \mathrm{div}\phi dx = -\int_{\mathbb{R}^n}\phi\cdot d\mu_E, \qquad \forall \phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n).$$

The measure μ_E can be characterized as a Hausdorff measure; in fact, defining the reduced boundary of E as

$$\mathcal{F}E = \left\{ x \in \mathrm{spt}\mu_E : \exists \nu_E(x) = \lim_{r\to 0}\frac{\mu_E(B_r(x))}{|\mu_E|(B_r(x))}, |\nu_E(x)| = 1 \right\},$$

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then $\mathcal{H}^{n-1}(\partial_* E \setminus \mathcal{F}E) = 0$, with $\partial_* E = \mathbb{R}^n \setminus (E_0 \cup E_1)$, E_t the set of points of E with density t; finally, there holds

$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \mathcal{F}E.$$

It is also possible to prove that $\mathcal{F}E$ is an $(n-1)$–countable rectifiable set and at any point $x \in \mathcal{F}E$, the blow-up of the set E converges to a half–space, that is the measures

$$\mathcal{L}^n \llcorner \left(\frac{E - x}{t} \right)$$

are weakly convergent as $t \to 0$ to the measure

$$\mathcal{L}^n \llcorner H_{\nu_E(x)}$$

where

$$H_{\nu_E(x)} = \{ y \in \mathbb{R}^n : \langle y, \nu_E(x) \rangle \geq 0 \}.$$

We also have, at any point $x \in \mathcal{F}E$, the weak convergence of the measures

$$\mathcal{H}^{n-1} \llcorner \left(\frac{\mathcal{F}E - x}{t} \right)$$

as $t \to 0$ to the tangent measure

$$\mathcal{H}^{n-1} \llcorner \nu_E(x)^{\perp}$$

where

$$\nu_E(x)^{\perp} = \{ y \in \mathbb{R}^n : \langle y, \nu_E(x) \rangle = 0 \}.$$

1.1 The heat content

A characterization of the perimeter measure similar to the original definition of De Giorgi can also be given by following an approach suggested by Ledoux in [8], where the author used the diffusion functional defined by

$$K_t(E, F) = \int_F T_t \chi_E(x) dx = \int_{\mathbb{R}^n} T_{t/2} \chi_E(x) T_{t/2} \chi_F(x) dx, \qquad t > 0; \tag{1}$$

his motivation was a semigroup characterization of isoperimetry. In fact, he proved that for any set with finite perimeter $E \subset \mathbb{R}^n$

$$K_t(E, E^c) \leq \sqrt{\frac{t}{\pi}} P(E),$$

whether if $E = B$ is a ball, then

$$\lim_{t \to 0} \sqrt{\frac{\pi}{t}} K_t(B, B^c) = P(B). \tag{2}$$

Ledoux noticed also that, since

$$K_t(E, E^c) = |E| - \|T_{t/2} \chi_E\|^2_{L^2(\mathbb{R}^n)},$$

the isoperimetry of B is equivalent to the inequality

$$\|T_t \chi_E\|_{L^2(\mathbb{R}^n)} \leq \|T_t \chi_B\|_{L^2(\mathbb{R}^n)}$$

for any measurable set with $|E| = |B|$. In [11], it is proved that equation (2) holds for any set with finite perimeter and, moreover, an equivalent characterization of sets with finite perimeter is given, in the sense that

$$\liminf_{t \to 0} \frac{K_t(E, E^c)}{\sqrt{t}} < +\infty$$

if and only if E has finite perimeter. Summarizing, for a set E with finite measure and finite perimeter, the following expansion holds

$$K_t(E, E^c) = \sqrt{\frac{t}{\pi}} P(E) + o(\sqrt{t}),$$

or equivalently

$$\|T_t \chi_E\|^2_{L^2(\mathbb{R}^n)} = |E| - \sqrt{\frac{2t}{\pi}} P(E) + o(\sqrt{t}).$$

More generally, for any two sets E and F with finite perimeter, the following formula holds

$$\int_F T_t \chi_E(x) dx = |E \cap F| - \sqrt{\frac{t}{\pi}} \int_{\mathcal{F}E \cap \mathcal{F}F} \langle \nu_E(x), \nu_F(x) \rangle d\mathcal{H}^{n-1}(x) + o(\sqrt{t}). \tag{3}$$

Due to the presence of $\sqrt{t}$, it is more convenient to introduce the functions $f_{E,F}(t) = K_{t^2}(E, F)$ and $f_E(t) = K_{t^2}(E, E^c)$. A way to prove (3) simply consists in considering the limits

$$\lim_{t \to 0} f_{E,F}(t) = |E \cap F|$$

and (see [11, Theorem 3.1] for a detailed proof)

$$\lim_{t \to 0} f'_{E,F}(t) = \lim_{t \to 0} 2t \int_F \Delta T_{t^2} \chi_E(x) dx = -\frac{1}{\sqrt{\pi}} \int_{\mathcal{F}E \cap \mathcal{F}F} \langle \nu_E(x), \nu_F(x) \rangle d\mathcal{H}^{n-1}(x).$$

In [11], a more accurate description of the function $f_E(t)$ is given under additional regularity of the boundary of E; in fact, assuming $C^{1,1}$ regularity of ∂E, it is possible to prove that, for small time, the heat amount is essentially contained in a neighborhood $E_r \setminus E$, with

$$E_r = \{x \in \mathbb{R}^n : \mathrm{dist}(x, E) < r\},$$

that is $f_E(t) \sim f_{E, E_r \setminus E}(t)$. So, the fact that

$$\lim_{t \to 0} \frac{f_E(t)}{t} = \lim_{t \to 0} \frac{f_{E, E_r \setminus E}(t)}{t} = \frac{P(E)}{\sqrt{\pi}}$$

resembles the characterization of the perimeter measure by the Minkowski content

$$P(E) = \lim_{r \to 0} \frac{|E_r \setminus E|}{r},$$

justifying the terminology of heat content for $f_E(t)$. It is worth noticing that the heat content is much more accurate than the Minkowski content, since it gives the perimeter measure without any further condition on the regularity of the reduced boundary of E.

This result is similar in spirit to that contained in [4], where it is proved that

$$Q_D(t) = |D| - \frac{2\sqrt{t} P(D)}{\sqrt{\pi}} + \frac{t}{2} \int_{\partial D} H_D(x) d\mathcal{H}^{n-1}(x) + o(t)$$

with D a bounded connected domain such that ∂D is of class C^3 and H_D is the mean curvature of ∂D; here $Q_D(t) = \int_D u(t,x)dx$ and u is the solution of the Dirichlet Laplacian on D

$$\begin{cases} \partial_t u = \Delta u & (0,+\infty) \times D \\ u(t,x) = 0 & (0,+\infty) \times \partial D \\ u(0,x) = 1 & D. \end{cases}$$

More recently, the same authors generalized the result in order to consider also Neumann boundary conditions; they also considered mixed, Dirichlet and Neumann, boundary conditions and initial data other then χ_D (see for instance [2], [3] and the references therein). On the other hand, in the recent paper [1], two different characterizations of $P(E,\Omega)$, the perimeter of a set in a domain are given in terms of the short-time behavior of the solution of a parabolic initial boundary value problem in Ω.

In this paper we are interested in the higher order expansion of $f_{E,F}$ as t goes to 0; this is similar in spirit to the Minkowski content, where it is possible to prove that under suitable properties of E, the quantity $|E_r|$ is a polynomial in r for r small enough. A classical result is due to Steiner in the case E convex, for ∂E of class C^2 there is a result of Weyl [18]; both these results were generalized by Federer in [7] to the class of sets with positive reach, by showing the existence of measures $\psi_k(E,\cdot)$, called the curvature measures, for which

$$|E_r \cap B| = \sum_{k=0}^{n} r^k \psi_k(E,B), \qquad \forall B \subset \mathbb{R}^n \text{ Borel.}$$

Recently, these measures have been characterized by Rataj [12], [13] and [14] (see also the survey of Thäle [16]) as $(n-1)$–dimensional Hausdorff measures on the unit normal bundle $\mathrm{nor}(E)$ of E. We stress the fact that the positive reach assumption is not so far from the requirement that ∂E is $C^{1,1}$. In the case of the heat content, a similar result can be expected; in general, it is not true that the heat content is a polynomial. We want also to stress the fact that the heat content is much more precise of the Minkowski content and in some sense it is much more adapted to the geometry of the sets involved; in fact, the heat content gives an equivalent definition of the perimeter, whether the Minkowski content is not always comparable with the perimeter measure (as for instance happens for the enlarged rational numbers).

Our result is also similar in spirit to a recent paper of Wang [17], where the second fundamental form for the boundary of a Riemannian manifold is described by the short time behavior for the gradient of the Neumann semigroup.

In order to obtain an expansion for $f_{E,F}$, we start by taking second derivative, that is given by

$$\begin{aligned} f''_{E,F}(t) =& 2\int_F \Delta T_{t^2}\chi_E(x)dx + 4t^2 \int_F \Delta^2 T_{t^2}\chi_E(x)dx \\ =& -2\int_{\mathcal{F}F} \langle \nabla T_{t^2}\chi_E(x), \nu_F(x)\rangle d\mathcal{H}^{n-1}(x) - 4t^2 \int_{\mathcal{F}F} \langle \nabla\Delta T_{t^2}\chi_E(x), \nu_F(x)\rangle d\mathcal{H}^{n-1}(x). \end{aligned}$$

We notice that

$$\begin{aligned} \nabla T_{t^2}\chi_E(x) =& \frac{1}{(4\pi)^{n/2}t^n} \nabla_x \int_E e^{-\frac{|x-y|^2}{4t^2}} dy \\ =& -\frac{1}{(4\pi)^{n/2}t^n} \int_E \nabla_y e^{-\frac{|x-y|^2}{4t^2}} dy \\ =& \frac{1}{(4\pi)^{n/2}t^n} \int_{\mathcal{F}E} \nu_E(y) e^{-\frac{|x-y|^2}{4t^2}} d\mathcal{H}^{n-1}(y) \end{aligned}$$

and also

$$\begin{aligned}\langle \nabla \Delta T_{t^2}\chi_E(x), \nu_F(x)\rangle =& \sum_{i,j=1}^n \nu_F^i(x) \frac{\partial}{\partial x_i}\frac{\partial^2}{\partial x_j^2} T_{t^2}\chi_E(x)\\ =& -\frac{1}{(4\pi)^{n/2}t^n}\sum_{i,j=1}^n \nu_F^i(x)\frac{\partial^2}{\partial x_i \partial x_j}\int_E \frac{\partial}{\partial y_j} e^{-\frac{|x-y|^2}{4t^2}}dy\\ =&\frac{1}{(4\pi)^{n/2}t^n}\sum_{i,j=1}^n \nu_F^i(x)\frac{\partial^2}{\partial x_i \partial x_j}\int_{\mathcal{F}E}\nu_E^j(y) e^{-\frac{|x-y|^2}{4t^2}}d\mathcal{H}^{n-1}(y)\\ =&\frac{1}{(4\pi)^{n/2}t^n}\sum_{i,j=1}^n \nu_F^i(x)\int_{\mathcal{F}E}\left(-\frac{\delta_{ij}}{2t^2}+\frac{(x_i-y_i)(x_j-y_j)}{4t^4}\right)\nu_E^j(y) e^{-\frac{|x-y|^2}{4t^2}}d\mathcal{H}^{n-1}(y).\end{aligned}$$

In this way we obtain that

$$f''_{E,F}(t) = -\frac{1}{(4\pi)^{n/2}t^{n+2}}\int_{\mathcal{F}F}\int_{\mathcal{F}E}\langle \nu_F(x), y-x\rangle\langle \nu_E(y), y-x\rangle e^{-\frac{|x-y|^2}{4t^2}}d\mathcal{H}^{n-1}(y)d\mathcal{H}^{n-1}(x).$$

If we set, for countably $\mathcal{H}^{n-1}$–rectifiable orientable surfaces Σ, Γ with $\mathcal{H}^{n-1}(\Sigma), \mathcal{H}^{n-1}(\Gamma) < +\infty$,

$$I_t(\Sigma;\Gamma) = \int_\Sigma\int_\Gamma \langle \nu_\Gamma(x), y-x\rangle\langle \nu_\Sigma(y), y-x\rangle e^{-\frac{|x-y|^2}{4t^2}}d\mathcal{H}^{n-1}(y)d\mathcal{H}^{n-1}(x),$$

with ν_Σ and ν_Γ unit vector fields orienting Σ and Γ respectively, we see that the small time behavior of $f_{E,F}$ is strictly related with the small time behavior of $I_t(\mathcal{F}F, \mathcal{F}E)$.

The paper is then organized as follows; in Section 2 we fix the notations we shall use in the paper and we give the standing hypotheses on our surfaces Σ and Γ; in Section 3 we shall study the functional $I_t(\Sigma) = I_t(\Sigma;\Sigma)$ for a suitable regular hypersurface Σ; in Section 4 we study the functional $I_t(\Sigma;\Gamma)$ and finally in Section 5 we study the third order expansion of $f_{E,F}$ under suitable regular assumptions on E and F.

2 Notations

In this section we fix the main definitions we shall use later. By $B_r(x)$ we denote the open ball centered at x and with radius $r > 0$; if $x = 0$, we simply write B_r. We also denote by B_r^+ the set of points $y \in B_r \subset \mathbb{R}^n$ such that $y_n > 0$. Given a set $M \subset \mathbb{R}^n$, we shall use the notation $M_r^x = M \cap B_r(x)$.

We shall use the notations introduced by Federer in [7]; in particular, given a closed set $M \subset \mathbb{R}^n$, we define the tangent cone of M at x by

$$\mathrm{Tan}(M, x) = \left\{\lambda u : u = \lim_{M \ni y \to x}\frac{y-x}{|y-x|}, \lambda \geq 0\right\}.$$

If $\mathrm{Tan}(M, x)$ is a vector space, we shall denote it by T_xM; if in general $\mathrm{Tan}(M, x)$ is only a cone, we shall denote by T_xM its span, that is the smallest vector space containing it. We shall also denote by $\Pi_M^x : \mathbb{R}^n \to T_xM$ the orthogonal projection on T_xM.

We also denote the normal space to M at x by

$$\mathrm{Nor}(M, x) = \{v \in \mathbb{R}^n : \langle v, u\rangle \leq 0, \quad \forall u \in \mathrm{Tan}(M, x)\};$$

by N_xM we shall denote the orthogonal complement of T_xM, that is the linear space such that

$$\mathbb{R}^n = T_xM \otimes N_xM.$$

We recall that a map φ is said to be L–bilipschitz, $L \geq 1$, if

$$\frac{1}{L}|y_1 - y_2| \leq |\varphi(y_1) - \varphi(y_2)| \leq L|y_1 - y_2|.$$

Definition 2.1 *Let $M \subset \mathbb{R}^n$ be an m–dimensional closed manifold, with or without boundary ∂M such that $\mathcal{H}^m(\partial M) = 0$. We shall say that:*

- *M is* a piece of an uniform $(C^{1.1}, m, L, r)$–regular manifold*, $r > 0$ and $L \geq 1$, if for any $x \in M$ there exists a $C^{1,1}$ map $\varphi^x_M : B_{Lr} \subset \mathbb{R}^m \to \mathbb{R}^n$ such that*

  1. *$\varphi^x_M(0) = x$ and $\{\partial_i \varphi^x_M(0)\}_{i=1,\dots,m}$ orthonormal;*
  2. *for any $i = 1, \dots, m$, the maps $\partial_i \varphi^x_M : B_{Lr} \to \mathbb{R}^n$ are L–bilipschitz;*
  3. *$M^x_r \subset \varphi^x_M(B_{Lr})$.*

- *M is* $(C^{1,1}, m, L, r)$–regular at $x \in M$ *if;*

  (i) *for $x \in M \setminus \partial M$, in addition to requirements 1. and 2., we have that*

  $$\varphi^x_M(B_{\frac{r}{L}}) \subset M^x_r \subset \varphi^x_M(B_{Lr});$$

  (ii) *for $x \in \partial M$, in addition to requirements 1. and 2., we have that*

  $$\varphi^x_M(B^+_{\frac{r}{L}}) \subset M^x_r \subset \varphi^x_M(B^+_{Lr});$$

We shall simply say a piece of and $C^{1,1}$–regular manifold if the dimension m is clear from the context and there exist $L, r > 0$ such that the manifold is a piece of or $(C^{1,1}, m, L, r)$–regular manifold.

Remark 2.2 In view of the previous definition, if M is $C^{1,1}$–regular and $x \in M \setminus \partial M$, we shall use the same notation M^x_r to mean both, $M \cap B_r(x)$ or the image $\varphi^x_M(B_r)$ (or $\varphi^x_M(B_r) \cap M$ in case M is a piece of $C^{1,1}$–regular manifold). Moreover, if the part of M inside $B_r(x)$ is strictly contained in $\varphi^x_M(B_r)$, we can extend M adding a disjoint set $\widetilde{M}$ to M with $M^x_r \cup \widetilde{M} = \varphi^x_M(B_r)$. The same argument can be repeated also in the case $x \in \partial M$, replacing B_r with B^+_r. We shall call this procedure the tangential completion of M at x.

2.1 Manifolds of codimension one

We consider now the case $M = \Sigma$ a surface of dimension $(n-1)$; with a little abuse of notation, we shall write $I_t(\Sigma; \Sigma^x_r)$ to mean the integral

$$I_t(\Sigma; \Sigma^x_r) = \int_\Sigma d\mathcal{H}^{n-1}(x) \int_{\Sigma^x_r} \langle \nu_\Sigma(x), y - x\rangle \langle \nu_\Sigma(y), y - x\rangle e^{-\frac{|y-x|^2}{4t^2}} d\mathcal{H}^{n-1}(y).$$

The second fundamental form $\vec{\mathrm{II}}^x_\Sigma : T_x\Sigma \times T_x\Sigma \to N_x\Sigma$ for an hypersurface Σ at a point x is a bilinear map and is related to the scalar second fundamental form A^x_Σ by equality

$$\vec{\mathrm{II}}^x_\Sigma(v, w) = A^x_\Sigma(v, w)\nu_\Sigma(x), \qquad \forall v, w \in T_x\Sigma.$$

The principal curvatures $\kappa^x_{\Sigma,i}$, $i = 1, \dots, n-1$, are defined as the eigenvalues of A^x_Σ; moreover, fixed a vector $v \in T_x\Sigma$, the sectional curvature in x of Σ in direction v is defined as

$$\kappa^x_\Sigma[v] = A^x_\Sigma(v, v).$$

In this way, if $\{v_i\}_{i=1,\dots,n-1}$ are the eigenvectors of A^x_Σ associated to the $\kappa^x_{\Sigma,i}$'s, we have

$$\kappa^x_{\Sigma,i} = \kappa^x_\Sigma[v_i].$$

If no confusion may arise, we simply denote by κ_i the principal curvatures.

The mean curvature of Σ in x is defined by

$$H^x_\Sigma = \frac{1}{n-1} \sum_{i=1}^{n-1} \kappa_i;$$

in addition, the square of the length of the second fundamental form is given by

$$c^2_\Sigma(x) = \sum_{i=1}^{n-1} \kappa_i^2.$$

Remark 2.3 With the previous assumptions, we notice that if Σ is a piece of a $C^{1,1}$–regular hypersurface, then if $x \in \Sigma \setminus \partial\Sigma$ and r is small enough, we can parametrize Σ^x_r by

$$\varphi^x_\Sigma : A^x_r \to \Sigma^x_r, \qquad \varphi^x_\Sigma(w) = x + w + u(w)\nu_\Sigma(x), \tag{4}$$

where $A^x_r = \Pi^x_\Sigma(\Sigma^x_r)$ is an open set containing 0 and diameter less then r and $u(0) = 0$; u is a function with the same regularity in 0 as that of the surface Σ in x, then in particular u is differentiable in 0 and $\nabla u(0) = 0$. Using the Lipschitz regularity of ∇u, we also obtain that

$$|\nabla u(w)| \le L|w|, \qquad |u(w)| \le \frac{L}{2}|w|^2. \tag{5}$$

In other terms, Σ^x_r is contained in the graph of u on the tangent space $T_x\Sigma$; if x has been chosen in such a way that u is twice differentiable in 0, then the second fundamental form of Σ in x is then given by

$$\vec{\mathrm{II}}^x_\Sigma(w, z) = \langle Hu(0)w, z\rangle \nu_\Sigma(x) \tag{6}$$

and the eigenvalues of $Hu(0)$ are the principal curvatures of Σ at x. For the normal unit field $\nu_\Sigma : \Sigma \to \mathbb{S}^{n-1}$, we have that for $y \in \Sigma^x_r$

$$|\nu_\Sigma(y) - \nu_\Sigma(x)| \le L|y - x|; \tag{7}$$

moreover, $d_x\nu_\Sigma : T_x\Sigma \to T_{\nu_x}\mathbb{S}^{n-1} = T_x\Sigma$ and, by setting $\gamma^x_z(t) = x + tz + u(tz)\nu_\Sigma(x)$, the following holds

$$d_x\nu_\Sigma[z] = \frac{d}{dt}\nu_\Sigma(\gamma^x_z(t))_{|t=0} = \lim_{t\to 0} \frac{\nu_\Sigma(x + tz + u(tz)\nu_\Sigma(x)) - \nu_\Sigma(x)}{t} = -Hu(0)z. \tag{8}$$

Finally, if $x \in \partial\Sigma$, we can define as before the parametrization $\varphi^x_\Sigma : A^x_r \to \Sigma^x_r$, but in this case $0 \in \partial A^x_r$. By the $C^{1,1}$ assumption, the function u can be extended to 0 in such a way that $u(0) = \nabla u(0) = 0$, and (5)-(8) continue to hold.

2.2 Submanifolds of higher codimension

Let $M \subset \mathbb{R}^n$ be a manifold with codimension $k > 1$; then, for any $x \in M$, T_xM and N_xM have, respectively, dimension $n-k$ and k. The second fundamental form is a bilinear form $\vec{\mathrm{II}}^x_M : T_xM \times T_xM \to N_xM$, so for any $\eta \in N_xM$ we have the scalar second fundamental form in direction η

$$A^x_{M,\eta}(v,w) = \left\langle \vec{\mathrm{II}}^x_M(v,w), \eta \right\rangle, \qquad \forall v, w \in T_xM.$$

The principal curvatures of M in direction η are the eigenvalues of $A^x_{M,\eta}$, and are denoted by $(\kappa^x_{M,\eta,i})_{i=1,\dots,n-k}$. It is then defined the mean curvature of M at x in direction η by the identity

$$H^x_M[\eta] = \frac{1}{n-k} \sum_{i=1}^{n-k} \kappa^x_{M,\eta,i}.$$

Finally, if M is a parametrized manifold and $\varphi : A \to \mathbb{R}^n$, $A \subset \mathbb{R}^{n-k}$ is an open set with $\varphi(0) = x$, then the metric tensor at x is given by

$$g_{i,j}(x) = \langle \partial_i \varphi(0), \partial_j \varphi(0) \rangle;$$

using such a metric, the quadratic form $A^x_{M,\eta}$ is determined by the matrix

$$\sum_{h=1}^{n-k} g^{i,h} \langle \partial^2_{h,j} \varphi(0), \eta \rangle,$$

where g^{ih} are the coefficient of the inverse of the metric g. In particular, if M is a piece of a $C^{1,1}$–regular manifold, $x \in M$ and φ^x_M given in Definition 2.1, then the metric induced by φ^x_M is the identity in x, so that the second fundamental form is determined by the matrix

$$(A^x_{M,\eta})_{i,j} = \langle \partial^2_{i,j} \varphi(0), \eta \rangle;$$

as a consequence, the mean curvature of M at x in direction η is given by

$$H^x_M[\eta] = \frac{1}{n-k} \sum_{i=1}^{n-k} \langle \partial^2_{i,i} \varphi(0), \eta \rangle.$$

3 The functional $I_t(\Sigma)$

In this section we study $I_t(\Sigma)$ for Σ a piece of $C^{1,1}$–regular hypersurface. First of all, due to the exponential map, the part of I_t with $|y-x| \geq r$, $r > 0$ fixed, goes to 0 exponentially as $t \to 0$. With a little abuse of notation, we shall write $I_t(\Sigma, \Sigma^x_r)$ by meaning that $x \in \Sigma$ is a fixed point in the first integral of I_t and $\Sigma^x_r = \{y \in \Sigma : |y-x| < r\}$. With the change of variable $z = \frac{y-x}{t}$, we can write

$$\begin{aligned} I_t(\Sigma, \Sigma^x_r) =& t^{n+1} \int_\Sigma d\mathcal{H}^{n-1}(x) \int_{\frac{\Sigma^x_r - x}{t}} \langle \nu_\Sigma(x), z \rangle \langle \nu_\Sigma(x+tz), z \rangle e^{-\frac{|z|^2}{4}} d\mathcal{H}^{n-1}(z) \\ =& \int_\Sigma d\mathcal{H}^{n-1}(x) \int \langle \nu_\Sigma(x), z \rangle \langle \nu_\Sigma(x+tz), z \rangle e^{-\frac{|z|^2}{4}} d\mu_t(z). \end{aligned}$$

We then need a time expansion of both $\nu_\Sigma(x+tz)$ and the measures

$$\mu_t = \mathcal{H}^{n-1} \llcorner \left(\frac{\Sigma_r^x - x}{t} \right);$$

by the regularity assumptions we have done, we can see Σ locally as a graph around any point $x \in \Sigma$.

Theorem 3.1 *Let Σ be a piece of a $C^{1,1}$–regular hypersurface; then*

$$\lim_{t\to 0} \frac{I_t(\Sigma)}{(4\pi)^{n/2} t^{n+2}} = 0. \tag{9}$$

Proof. We start with the decomposition

$$I_t(\Sigma) = I_t(\Sigma; \Sigma_r^x) + I_t(\Sigma; \Sigma \setminus \Sigma_r^x).$$

A direct computation shows that if $2t < r$

$$|I_t(\Sigma; \Sigma \setminus \Sigma_r^x)| \le r^2 e^{-\frac{r^2}{4t^2}} \left(\mathcal{H}^{n-1}(\Sigma) \right)^2, \tag{10}$$

and then

$$I_t(\Sigma; \Sigma) = I_t(\Sigma; \Sigma_r^x) + o(t^k), \qquad \forall k > 0, \tag{11}$$

so we can restrict our attention on $I_t(\Sigma; \Sigma_r^x)$. We can write

$$I_t(\Sigma; \Sigma_r^x) = \int_\Sigma g_t^r(x) d\mathcal{H}^{n-1}(x);$$

we fix $x \in \Sigma$ and, with the change of variable $y = x + tz$ we obtain

$$\begin{aligned} g_t^r(x) =& t^{n+1} \int_{\frac{\Sigma_r^x - x}{t}} \langle \nu_\Sigma(x), z \rangle^2 e^{-\frac{|z|^2}{4}} d\mathcal{H}^{n-1}(z) + \\ &+ t^{n+1} \int_{\frac{\Sigma_r^x - x}{t}} \langle \nu_\Sigma(x), z \rangle \langle \nu_\Sigma(x+tz) - \nu_\Sigma(x), z \rangle e^{-\frac{|z|^2}{4}} d\mathcal{H}^{n-1}(z). \end{aligned}$$

Using the parametrization φ_Σ^x of Remark 2.3 and a change of variable, we can write

$$\begin{aligned} g_t^r(x) = t^{n+1} \int_{A_r^x/t} & e^{-\frac{|w|^2}{4}} e^{-\frac{u(tw)^2}{4t^2}} \sqrt{1 + |\nabla u(tw)|^2} \left(\frac{u(tw)^2}{t^2} + \right. \\ &+ \left. \frac{u(tw)}{t} \left\langle \nu_\Sigma(x + tw + u(tw)\nu_\Sigma(x)) - \nu_\Sigma(x), w + \frac{u(tw)}{t} \nu_\Sigma(x) \right\rangle \right) dw. \end{aligned} \tag{12}$$

As a consequence, by (5) and (7), we get

$$|g_t^r(x)| \le t^{n+3} \frac{3}{2} L^5 \int_{T_x\Sigma} |w|^4 (1 + |w|^2)^{3/2} e^{-\frac{|w|^2}{4}} dw = ct^{n+3} \tag{13}$$

with $c = c(L,n)$ a constant depending only on L and the integral of $|w|^4(1+|w|^2)^{3/2} e^{-\frac{|w|^2}{4}}$ on $\mathbb{R}^{n-1}$. This implies (9). □

Remark 3.2 In the previous proof the $C^{1,1}$–regularity essentially shows that the dominated convergence theorem can be used; the same argument, each time $C^{1,1}$–regularity holds, can be used in the sequel, also in the case of manifolds with higher codimension, To keep the proofs a little shorter, we shall use the dominated convergence without repeating the check of it.

We can go further in the expansion, in order to obtain the following result.

Theorem 3.3 *Let Σ be a piece of a $C^{1,1}$–regular hypersurface; then*

$$\begin{aligned}\lim_{t\to 0}\frac{I_t(\Sigma)}{(4\pi)^{n/2}t^{n+3}} =& -\frac{1}{4(4\pi)^{n/2}}\int_\Sigma d\mathcal{H}^{n-1}(x)\int_{T_x\Sigma} A^x_\Sigma(z,z)^2 e^{-\frac{|z|^2}{4}}\,dz \\ =& -\frac{(n-1)^2}{2\sqrt{\pi}}\int_\Sigma \left((H^x_\Sigma)^2+\frac{2}{(n-1)^2}c^2_\Sigma(x)\right)d\mathcal{H}^{n-1}(x).\end{aligned} \tag{14}$$

Proof. Thanks to estimate (13), we can use the dominated convergence and compute the pointwise limit

$$\lim_{t\to 0}\frac{g^r_t(x)}{(4\pi)^{n/2}t^{n+3}}.$$

So we fix $x\in\Sigma$ and denote simply by ν the vector $\nu_\Sigma(x)$; we also use the parametrization φ^x_Σ of Remark 2.3. We can also assume that the point x has been chosen in such a way that $u\in C^{1,1}$ and u twice differentiable in 0. By (12), we can write

$$\begin{aligned}g^r_t(x) =& t^{n+3}\int_{A^x_r/t} e^{-\frac{|w|^2}{4}-\frac{|u(tw)|^2}{4t^2}}\sqrt{1+|\nabla u(tw)|^2}\Big(\frac{u(tw)^2}{t^4}+ \\ &+\frac{u(tw)}{t^2}\left\langle\frac{\nu_\Sigma(x+tw+u(tw)\nu)-\nu}{t}, w+\frac{u(tw)}{t}\nu\right\rangle\Big)dw\end{aligned}$$

where in the last integral we have performed the change of variable $z=tw$. By the dominated convergence, it suffices to consider the pointwise limits as $t\to 0$ of $u(tw)/t$, $\nabla u(tw)$ and $u(tw)/t^2$. We have that

$$\lim_{t\to 0}\frac{u(tw)}{t}=0,\qquad \lim_{t\to 0}\nabla u(tw)=0,$$

and

$$\lim_{t\to 0}\frac{u(tw)}{t^2}=\frac{1}{2}\langle Hu(0)w,w\rangle.$$

We also have that

$$\lim_{t\to 0}\left\langle\frac{\nu_\Sigma(x+tw+u(tw)\nu)-\nu}{t}, w+\frac{u(tw)}{t}\nu\right\rangle=-\langle Hu(0)w,w\rangle,$$

as can be easily proved by assuming $\nu=e_n$, the last element of the canonical base of $\mathbb{R}^n$ and by writing

$$\nu_\Sigma(x+tw+u(tw)\nu)=\frac{(-\nabla u(tw),1)}{\sqrt{1+|\nabla u(tw)|^2}}.$$

Summarizing, we can conclude that

$$\lim_{t\to 0}\frac{g^r_t(x)}{(4\pi)^{n/2}t^{n+3}}=-\frac{1}{4(4\pi)^{n/2}}\int_{T_x\Sigma}\langle Hu(0)w,w\rangle^2 e^{-\frac{|w|^2}{4}}\,dw:=I(x).$$

We take then an orthonormal basis in $T_x\Sigma$ of eigenvector $(\tau_i)_{i=1,\dots,n-1}$ with $Hu(0)\tau_i = \kappa_i\tau_i$, κ_i the principal curvatures of Σ at x; so we can write $w = \sum_{i=1}^{n-1} w_i\tau_i$ so that we have

$$\begin{aligned} I(x) &= -\frac{1}{4(4\pi)^{n/2}}\int_{T_x\Sigma}\langle Hu(0)w,w\rangle^2 e^{-\frac{|w|^2}{4}}\,dw \\ &= -\frac{1}{4(4\pi)^{n/2}}\int_{\mathbb{R}^{n-1}}\Big(\sum_{i=1}^{n-1}\kappa_i w_i^2\Big)^2 e^{-\frac{w_1^2+\ldots+w_{n-1}^2}{4}}\,dw_1\ldots dw_{n-1} \\ &= -\frac{1}{4(4\pi)^{n/2}}\sum_{i,j=1}^{n-1}\kappa_i\kappa_j\int_{\mathbb{R}^{n-1}} w_i^2 w_j^2 e^{-\frac{|w|^2}{4}}\,dw. \end{aligned}$$

Since for $i \neq j$

$$\int_{\mathbb{R}^{n-1}} w_i^2 w_j^2 e^{-\frac{|w|^2}{4}}\,dw = (4\pi)^{\frac{n-3}{2}}\left(\int_{\mathbb{R}} s^2 e^{-s^2/4}ds\right)^2 = 4(4\pi)^{\frac{n-1}{2}}$$

and for $i = j$

$$\int_{\mathbb{R}^{n-1}} w_i^4 e^{-\frac{|w|^2}{4}}\,dw = (4\pi)^{\frac{n-2}{2}}\int_{\mathbb{R}} s^4 e^{-s^2/4}ds = 12(4\pi)^{\frac{n-1}{2}},$$

we then obtain

$$\begin{aligned} I(x) &= -\frac{1}{4(4\pi)^{n/2}}\sum_{i,j=1}^{n-1}\kappa_i\kappa_j\Big(12(4\pi)^{\frac{n-1}{2}}\delta_{ij} + 4(4\pi)^{\frac{n-1}{2}}(1-\delta_{ij})\Big) \\ &= -\frac{1}{2\sqrt{\pi}}\sum_{i,j=1}^{n-1}\kappa_i\kappa_j - \frac{1}{\sqrt{\pi}}\sum_{i=1}^{n-1}\kappa_i^2 = -\frac{(n-1)^2}{2\sqrt{\pi}}(H_\Sigma^x)^2 - \frac{1}{\sqrt{\pi}}c_\Sigma^2(x). \end{aligned}$$

□

4 The functional $I_t(\Sigma;\Gamma)$

In this section we study the quantity $I_t(\Sigma;\Gamma)$ for two oriented pieces of uniform $C^{1,1}$ hypersurfaces. In this computation, a crucial rôle is played by the intersection $S = \overline{\Sigma}\cap\overline{\Gamma}$ of the two manifolds. Since we are interested in the case when Σ and Γ are parts of the boundaries of sets of finite perimeter, we can restrict to the case when for $\mathcal{H}^{n-2}$–almost every $x_0 \in S$, $\mathrm{Tan}(\Sigma, x_0)$ and $\mathrm{Tan}(\Gamma, x_0)$ are both half hyperplanes. On S we shall always assume the following regularity.

Definition 4.1 (Regular skeleton) *We shall say that S has regular skeleton if, denoting by $S_1 = S$ and $S_{k+1} = \partial S_k$ (boundary in the sense of manifolds), S_k, for any $k = 1,\dots,n-1$, is a finite union of closed pieces of $C^{1,1}$–regular $(n-1-k)$–dimensional manifolds.*

Given Σ and Γ, we define the sets

$$\Sigma_r = \{x \in \Sigma : \mathrm{dist}(x,S) \leq r\}, \qquad \Gamma_r = \{y \in \Gamma : \mathrm{dist}(y,S) \leq r\};$$

we point out that if $y \in \Gamma \setminus \Gamma_r$, then

$$\mathrm{dist}(y,\Sigma) \geq cr \tag{15}$$

for some positive constant $c > 0$. This in particular implies that

$$I_t(\Sigma;\Gamma) = I_t(\Sigma_r;\Gamma_r) + o(t^m), \qquad \forall m \in \mathbb{N}. \tag{16}$$

In fact, we can write

$$I_t(\Sigma;\Gamma) = I_t(\Sigma_r;\Gamma_r) + I_t(\Sigma \setminus \Sigma_r;\Gamma_r) + I_t(\Sigma;\Gamma \setminus \Gamma_r),$$

and, as in (10), there holds

$$|I_t(\Sigma;\Gamma \setminus \Gamma_r)| \leq r^2 e^{-\frac{r^2}{4t^2}} \mathcal{H}^{n-1}(\Sigma)\mathcal{H}^{n-1}(\Gamma). \tag{17}$$

Same estimate holds for $I_t(\Sigma \setminus \Sigma_r;\Gamma_r)$.

From now on we consider Σ_r; the same type of considerations and constructions can also be performed for Γ_r. If r is small enough, it is well defined the projection $\pi : \Sigma_r \to S$; for any $k = 0, \dots, n-1$, we define the sets

$$S^k = \{x_0 \in S : \dim \pi^{-1}(x_0) = k\}, \qquad \Sigma_r^{k,x_0} = \pi^{-1}(x_0), x_0 \in S^k.$$

By the fact that S has regular skeleton, we deduce that

$$S_k \setminus S_{k+1} \subset S^k \subset S_k, \qquad k = 1, \dots n-1 \tag{18}$$

and, since $\mathcal{H}^{n-1-k}(S_{k+1}) = 0$, the fact that $\mathcal{H}^{n-1-k}$ a.e. point of S_k belongs to S^k. The sets $\Sigma_r^k = \pi^{-1}(S^k)$ are $(n-1)$–dimensional manifolds for any $k \geq 1$, whether $\mathcal{H}^{n-1}(\Sigma_r^0) = 0$; we then have the following disjoint decomposition

$$\Sigma_r = \bigcup_{k=0}^{n-1} \Sigma_r^k. \tag{19}$$

Moreover, by (18), for $\mathcal{H}^{n-1-k}$ a.e. $x_0 \in S^k$, we have the decomposition

$$\mathrm{Tan}(\Sigma_r^k, x_0) = T_{x_0} S^k \oplus \mathrm{Tan}(\Sigma_r^{k,x_0}, x_0). \tag{20}$$

In the same way we can define the sets Γ_r^k and Γ_r^{k,x_0}. The cones $\mathrm{Tan}(\Sigma_r^{k,x_0}, x_0)$ and $\mathrm{Tan}(\Gamma_r^{k,x_0}, x_0)$ are k–dimensional and we denote by $\{\sigma_j^k(x_0)\}_{j=1,\dots,k}$ and $\{\gamma_j^k(x_0)\}_{j=1,\dots,k}$ the sets of their generators.

We give the following definition.

Definition 4.2 *We say that Σ meets Γ transversally if there exists $c > 0$ such that for almost every $x_0 \in S$, $-1 + c \leq \langle \sigma_1^1(x_0), \gamma_1^1(x_0) \rangle \leq 1 - c$, or equivalently $|\sigma_1^1(x_0) \wedge \gamma_1^1(x_0)| \geq c$.*

Remark 4.3 In what follows, the transversality can aslo be replaced by a weaker condition, requiring the existence of a function $\omega \in L^1(S, \mathcal{H}^{n-2})$ such that

$$\frac{1}{|\sigma_1^1(x_0) \wedge \gamma_1^1(x_0)|} \leq \omega(x_0), \qquad \mathcal{H}^{n-2} - \text{a.e. } x_0 \in S.$$

We also define, for $\mathcal{H}^{n-1-k}$ a.e. $x_0 \in S^k$, the cone

$$V_{x_0}^k = \mathrm{Tan}(\Gamma_r^{k,x_0}, x_0) - \mathrm{Tan}(\Sigma_r^{k,x_0}, x_0);$$

this cone can be parametrized by using the maps $Q\Sigma^k_{x_0} : \mathbb{R}^k_+ \to \mathrm{Tan}(\Sigma^{k,x_0}_r, x_0)$, $Q\Gamma^k_{x_0} : \mathbb{R}^k_+ \to \mathrm{Tan}(\Gamma^{k,x_0}_r, x_0)$

$$Q\Sigma^k_{x_0}(\alpha) = \sum_{i=1}^k \alpha_i \sigma^k_i(x_0), \qquad Q\Gamma^k_{x_0}(\beta) = \sum_{i=1}^k \beta_i \gamma^k_i(x_0) :$$

such a parametrization is given by $Q^k_{x_0} : \mathbb{R}^{2k}_+ \to V^k_{x_0}$,

$$Q^k_{x_0}(\alpha, \beta) = Q\Gamma^k_{x_0}(\beta) - Q\Sigma^k_{x_0}(\alpha)$$

and is determined by tha matrix

$$\begin{pmatrix} -\sigma^k_1(x_0) & \dots & -\sigma^k_k(x_0) & \gamma^k_1(x_0) & \dots & \gamma^k_k(x_0) \end{pmatrix}. \tag{21}$$

The dimension of $V^k_{x_0}$ is given by the rank of (21), that is, due to the transversality condition on Σ and Γ, always equals to $k+1$. We shall then denote by $J_k Q^k_{x_0}$ the factor

$$D_k Q^k_{x_0} = \begin{cases} |\sigma^1_1(x_0) \wedge \gamma^1_1(x_0)| & \text{if } k = 1 \\ \\ \dfrac{J_k dQ\Sigma^k_{x_0} J_k dQ\Gamma^k_{x_0}}{C_k dQ^k_{x_0}} & \text{if } k > 1, \end{cases} \tag{22}$$

where by $J_k dQ\Sigma^k_{x_0}$ and $J_k dQ\Gamma^k_{x_0}$ we denote the area factor of the maps $Q\Sigma^k_{x_0}$ and $Q\Gamma^k_{x_0}$, and by $C_k dQ^k_{x_0}$ the coarea factor associated to $Q^k_{x_0}$.

For the restrictions $\pi_k : \Sigma^k_r \to S^k$ of π, we can consider, for any measurable function g, the coarea formula

$$\int_{\Sigma^k_r} g(x) C_k d_x \pi_k d\mathcal{H}^{n-1}(x) = \int_{S^k} d\mathcal{H}^{n-1-k}(x_0) \int_{\Sigma^{k,x_0}_r} g(x) d\mathcal{H}^k(x),$$

where $C_k d_x \pi_k$ is the coarea factor. Equivalently, since $C_k d_x \pi_k \neq 0$ on Σ^k_r,

$$\int_{\Sigma^k_r} g(x) d\mathcal{H}^{n-1}(x) = \int_{S^k} d\mathcal{H}^{n-1-k}(x_0) \int_{\Sigma^{k,x_0}_r} \frac{g(x)}{C_k d_x \pi_k} d\mathcal{H}^k(x),$$

Remark 4.4 We can write the coarea factor using local parametrizations; we do it explicitly since we shall use it in Lemma 4.5. Using a partition of unity argument, we can assume that Σ^k_r is a parametrized surface in a neighbourhood of a fixed point $x_0 \in S^k$. We shall then assume to have a map $\psi : A^k_{x_0} \times I^k_{x_0} \to \mathbb{R}^n$ with the following properties:

1. $A^k_{x_0}$ is an open subset of $\mathbb{R}^{n-1-k}$ and $I^k_{x_0} \subset \mathbb{R}^k$, both sets containing 0;
2. $\psi(0,0) = x_0$;
3. the map $a \mapsto \phi(a) = \psi(a, 0)$ is a parametrization for S^k;
4. for any $a \in A^k_{x_0}$, $\psi(a, a') \in \mathrm{Nor}(S^k, \phi(a))$;
5. the set $\{\partial_i \psi(0,0)\}_{i=1,\dots,n-1-k}$ is an orthonormal basis of $T_{x_0} S^k$ and

$$\partial_{n-1-k+j} \psi(0,0) = \sigma^k_j(x_0), \qquad j = 1, \dots, k.$$

We shall also write ψ^{-1} and ϕ^{-1} to denote the inverses of ψ and ϕ defined on Σ_r^k and S^k respectively. With the previous assumptions, it is clear that

$$\pi_k(\psi(a,a')) = \phi(a).$$

Moreover, the parametrization ψ induces then metrics g_{n-1}, g_{n-1-k} and g_k on Σ_r^k, S^k and $\Sigma_r^{k,y}$, $y \in S^k$, respectively given by

$$(g_{n-1}(a,a'))_{i,j} = \langle \partial_i \psi(a,a'), \partial_j \psi(a,a') \rangle, \qquad i,j = 1,\dots,n-1,$$

$$(g_{n-1-k}(a,0))_{i,j} = \langle \partial_i \psi(a,0), \partial_j \psi(a,0) \rangle, \qquad i,j = 1,\dots,n-1-k,$$

and

$$(g_k(a,a'))_{i,j} = \langle \partial_i \psi(a,a'), \partial_j \psi(a,a') \rangle, \qquad i,j = n-k,\dots,n-1$$

so that for $x \in \Sigma_r^k$ with $x = \psi(a,a')$, $y \in S^k$ with $y = \psi(a,0)$

$$d\mathcal{H}^{n-1}(x) = \sqrt{\det[g_{n-1}(a,a')]}\,da\,da', \qquad d\mathcal{H}^{n-1-k}(y) = \sqrt{\det[g_{n-1-k}(a)]}\,da,$$

$$d\mathcal{H}^k(x) = \sqrt{\det[g_k(a,a')]}\,da'.$$

We can then write

$$\int_{\Sigma_r^k} g(x)\,d\mathcal{H}^{n-1}(x) = \int_{A_{x_0}^k \times I_{x_0}^k} g(\psi(a,a'))\sqrt{\det[g_{n-1}(a,a')]}\,da\,da'$$

$$= \int_{A_{x_0}^k} \sqrt{\det[g_{n-1-k}(a,0)]}\,da \int_{I_{x_0}^k} g(\psi(a,a'))\sqrt{\frac{\det[g_{n-1}(a,a')]}{\det[g_{n-1-k}(a,0)]\det[g_k(a,a')]}}\sqrt{\det[g_k(a,a')]}\,da',$$

so that

$$C_k d_x \pi_k = \sqrt{\frac{\det[g_{n-1-k}(\phi^{-1}(\pi_k(x)),0)]\det[g_k(\psi^{-1}(x))]}{\det[g_{n-1}(\psi^{-1}(x))]}}. \tag{23}$$

The following lemma contains the main properties of the coarea factor $C_k d_{x_0}\pi_k$ we use in the sequel.

Lemma 4.5 *Let $C_k d\pi_k$ be the k–th coarea factor; then, if $x_0 \in S^k$,*

$$C_k d_{x_0}\pi_k = 1, \tag{24}$$

whether for $z \in \mathrm{Tan}(\Sigma_r^{k,x_0}, x_0)$,

$$d_{x_0} C_k d_{x_0}\pi_k[z] = (n-1-k) H_{S^k}^{x_0}[z], \tag{25}$$

where $H_{S^k}^{x_0}[z]$ is the mean curvature of S^k at x_0 in direction $z \in N_{x_0}S^k$.

Proof. We use the parametrized characterization of the coarea factor given in (23); if we fix $x_0 \in S_k$, then (24) is a direct consequence of point 5. in Remark 4.4.

To prove (25), by linearity, it suffices to consider $z = \sigma_j^k(x_0)$, $j = 1,\dots,k$. The curve $\gamma(t) = \psi(c(t)) = \psi(te_{n-1-k+j})$ has the property that $\gamma(0) = x_0$ and $\gamma'(0) = \sigma_j^k(x_0)$, so

$$d_{x_0} C_k d_{x_0}\pi_k[\sigma_j^k(x_0)] = \frac{d}{dt} C_k d_{\gamma(t)}\pi_k(\gamma(t))_{|t=0}.$$

Since $\pi_k(\gamma(t)) = x_0$, again by point 5. of Remark 4.4 we get

$$d_{x_0} C_k d_{x_0} \pi_k[\sigma_j^k(x_0)] = \frac{1}{2}\left(\frac{d}{dt}\det[g_k(c(t)]_{|t=0} - \frac{d}{dt}\det[g_{n-1}(c(t))]_{|t=0}\right)$$

For the derivative of the determinant of an invertible matrix $a = a(t)$, we can use formula

$$\frac{d}{dt}\det a(t) = \det a(t) \sum_{\alpha,\beta} a^{\alpha\beta}(t) \frac{d}{dt} a_{\alpha\beta}(t),$$

where $a_{\alpha\beta}$ are the entries of a and $a^{\alpha\beta}$ the entries of a^{-1}. Since $g_k(0)$ and $g_{n-1}(0)$ are the identity matrix, we get that

$$\begin{aligned}\frac{d}{dt}\det[g_k(c(t)]_{|t=0} =&2\sum_{\alpha=1}^{k} \big\langle \partial_{n-1-k+\alpha}\psi(0), \partial^2_{n-1-k+\alpha,n-1-k+j}\psi(0)\big\rangle \\ =&2\sum_{\alpha=n-k}^{n-1} \big\langle \partial_{\alpha}\psi(0), \partial^2_{\alpha,n-1-k+j}\psi(0)\big\rangle,\end{aligned}$$

and

$$\frac{d}{dt}\det[g_{n-1}(c(t)]_{|t=0} = 2\sum_{\alpha=1}^{n-1} \big\langle \partial_{\alpha}\psi(0), \partial^2_{\alpha,n-1-k+j}\psi(0)\big\rangle.$$

We now use the fact that $\langle \partial_\alpha\psi(a,0), \partial_{n-1-k+j}\psi(a,0)\rangle = 0$ for all $a \in A^k_{x_0}$ and $\alpha = 1,\dots,n-1-k$. Then for all $\alpha,\beta = 1,\dots,n-1-k$

$$0 = \big\langle \partial^2_{\alpha,\beta}\psi(0), \partial_{n-1-k+j}\psi(0)\big\rangle + \big\langle \partial_{\alpha}\psi(0), \partial^2_{\beta,n-1-k+j}\psi(0)\big\rangle. \tag{26}$$

By using (26) with $\beta = \alpha$, we obtain that

$$\begin{aligned}d_{x_0} C_k d_{x_0} \pi_k[\sigma_j^k(x_0)] =& -\sum_{\alpha=1}^{n-1-k} \big\langle \partial_{\alpha}\psi(0), \partial^2_{\alpha,n-1-k+j}\psi(0)\big\rangle \\ =& \sum_{\alpha=1}^{n-1-k} \big\langle \partial^2_{\alpha,\alpha}\psi(0), \sigma_j^k(x_0)\big\rangle = (n-1-k) H^{x_0}_{S^k}[\sigma_j^k(x_0)].\end{aligned}$$

□

Using the decomposition (19), we can write

$$I_t(\Sigma_r;\Gamma) = \sum_{k=1}^{n-1} I_t(\Sigma_r^k;\Gamma); \tag{27}$$

the following result holds.

Lemma 4.6 *The quantity $I_t(\Sigma^k_r;\Gamma)$ is asymptotic to t^{n+k+1} and there holds*

$$I_0^k(\Sigma;\Gamma) := \lim_{t\to 0} \frac{I_t(\Sigma_r^k;\Gamma)}{(4\pi)^{n/2} t^{n+1+k}} = \int_{S^k} \Theta_k(x_0) d\mathcal{H}^{n-1-k}(x_0),$$

where, if $k = 1$,

$$\Theta_1(x_0) = \frac{1}{4\pi|\sigma^1(x_0) \wedge \gamma_1^1(x_0))|} \int_{V_{x_0}^1} \langle \nu_\Sigma(x_0), v\rangle \langle \nu_\Gamma(x_0), v\rangle \exp\left(-\frac{|v|^2}{4}\right) dv$$

and if $k > 1$,

$$\Theta_k(x_0) = \frac{D_k Q_{x_0}^k}{(4\pi)^{\frac{k+1}{2}}} \int_{V_{x_0}^k} \langle \nu_\Sigma(x_0), v\rangle \langle \nu_\Gamma(x_0), v\rangle e^{-\frac{|v|^2}{4}} \mathcal{H}^{k-1}((Q_{x_0}^k)^{-1}(v)) dv$$

with $D_k Q_{x_0}^k$ given by (22).

Proof. First of all we note that, using the changes of variables $x = x_0 + tz$ and $y = x_0 + tw$,

$$\begin{aligned} I_t(\Sigma_r^k;\Gamma) =& \int_{\Sigma_r^k} d\mathcal{H}^{n-1}(x) \int_\Gamma \langle \nu_\Sigma(x), y - x\rangle \langle \nu_\Gamma(x), y - x\rangle e^{-\frac{|y-x|^2}{4t^2}} d\mathcal{H}^{n-1}(y) \\ =& \int_{S^k} d\mathcal{H}^{n-1-k}(x_0) \int_{\Sigma_r^{k,x_0}} d\mathcal{H}^k(x) \int_\Gamma \frac{\langle \nu_\Sigma(x), y - x\rangle \langle \nu_\Gamma(x), y - x\rangle}{C_k d_x \pi_k} e^{-\frac{|y-x|^2}{4t^2}} d\mathcal{H}^{n-1}(y) \\ =& t^{n+1+k} \int_{S^k} d\mathcal{H}^{n-1-k}(x_0) \int_{\frac{\Sigma_r^{k,x_0} - x_0}{t}} d\mathcal{H}^k(z) \int_{\frac{\Gamma - x_0}{t}} F_{x_0}^k(t,z,w) d\mathcal{H}^{n-1}(w), \end{aligned}$$

with

$$F_{x_0}^k(t,z,w) = \frac{\langle \nu_\Sigma(x_0 + tz), w - z\rangle \langle \nu_\Gamma(x_0 + tw), w - z\rangle}{C_k d_{x_0+tz} \pi_k} e^{-\frac{|w-z|^2}{4}}.$$

Using the convergences

$$\lim_{t\to 0} F_{x_0}^k(t,z,w) = \langle \nu_\Sigma(x_0), w - z\rangle \langle \nu_\Gamma(x_0), w - z\rangle e^{-\frac{|w-z|^2}{4}},$$

$$\mathcal{H}^k \llcorner \left(\frac{\Sigma_r^{k,x_0} - x_0}{t}\right) \rightharpoonup \mathcal{L}^k \llcorner \operatorname{Tan}(\Sigma_r^{k,x_0}, x_0), \qquad \mathcal{H}^{n-1} \llcorner \left(\frac{\Gamma - x_0}{t}\right) \rightharpoonup \mathcal{L}^{n-1} \llcorner \operatorname{Tan}(\Gamma, x_0)$$

and the decomposition $\operatorname{Tan}(\Gamma, x_0) = T_{x_0} S^k \oplus \operatorname{Tan}(\Gamma_r^{k,x_0}, x_0)$, we get

$$\lim_{t\to 0} \frac{I_t(\Sigma_r^k;\Gamma)}{(4\pi)^{n/2} t^{n+1+k}} = \int_{S^k} \Theta_k(x_0) d\mathcal{H}^{n-1-k}(x_0),$$

with

$$\Theta_k(x_0) = \frac{1}{(4\pi)^{\frac{k+1}{2}}} \int_{\operatorname{Tan}(\Sigma_r^{k,x_0}, x_0)} dz \int_{\operatorname{Tan}(\Gamma_r^{k,x_0}, x_0)} \langle \nu_\Sigma(x_0), w - z\rangle \langle \nu_\Gamma(x_0), w - z\rangle e^{-\frac{|w-z|^2}{4}} dw.$$

To compute the last integral we distinguish the cases $k = 1$ and $k > 1$; in the first case we parametrize $\operatorname{Tan}(\Sigma^{1,x_0}, x_0)$ and $\operatorname{Tan}(\Gamma^{1,x_0}, x_0)$ using the maps

$$Q\Sigma_{x_0}^1(\alpha) = \alpha \sigma_1^1(x_0), \qquad Q\Gamma_{x_0}^1(\beta) = \beta \gamma_1^1(x_0),$$

whose area factor is 1, so that we obtained

$$\begin{aligned} \Theta_1(x_0) =& \frac{1}{4\pi} \int_{\mathbb{R}_+^2} \langle \nu_\Sigma(x_0), Q_{x_0}^1(\alpha,\beta)\rangle \langle \nu_\Gamma(x_0), Q_{x_0}^1(\alpha,\beta)\rangle e^{-\frac{|Q_{x_0}^1(\alpha,\beta)|^2}{4}} d(\alpha,\beta) \\ =& \frac{1}{4\pi|\sigma_1^1(x_0) \wedge \gamma_1^1(x_0)|} \int_{V_{x_0}^1} \langle \nu_\Sigma(x_0), v\rangle \langle \nu_\Gamma(x_0), v\rangle e^{-\frac{|v|^2}{4}} dv \end{aligned}$$

The case $k > 1$ is similar, but since $Q^k_{x_0}$ maps $\mathbb{R}^{2k}$ onto the $(k+1)$–dimensional cone $V^k_{x_0}$, we have to use the coarea formula, that is

$$\begin{aligned}\Theta_k(x_0) =&\frac{J_k dQ\Sigma^k_{x_0} J_k dQ\Gamma^k_{x_0}}{(4\pi)^{\frac{k+1}{2}}} \int_{\mathbb{R}^{2k}_+} \big\langle \nu_\Sigma(x_0), Q^k_{x_0}(\alpha,\beta)\big\rangle \big\langle \nu_\Gamma(x_0), Q^k_{x_0}(\alpha,\beta)\big\rangle e^{-\frac{|Q^k_{x_0}(\alpha,\beta)|^2}{4}} d(\alpha,\beta)\\ =&\frac{D_k Q^k_{x_0}}{(4\pi)^{\frac{k+1}{2}}} \int_{V^k_{x_0}} \langle \nu_\Sigma(x_0), v\rangle \langle \nu_\Gamma(x_0), v\rangle e^{-\frac{|v|^2}{4}} \mathcal{H}^{k-1}((Q^k_{x_0})^{-1}(v)) dv.\end{aligned}$$

□

Remark 4.7 We point out some consequence and possible generalization of the previous result; it states that the quantity $I_t(\Sigma;\Gamma)$ is infinitesimal of order strictly related to the dimension of the intersection $\Sigma \cap \Gamma$. In general, if M_1 and M_2 are two oriented manifolds of dimensions m_1 and m_2 respectively, $S_{12} = M_1 \cap M_2$ has dimension $m < \min\{m_1, m_2\}$ and ξ_1, ξ_2 are two vector fields in $\mathbb{R}^n$, then the quantity

$$I_t(M_1,\xi_1; M_2,\xi_2) := \int_{M_1} d\mathcal{H}^{m_1}(x) \int_{M_2} \langle \xi_1(x), y-x\rangle \langle \xi_2(y), y-x\rangle e^{-\frac{|y-x|^2}{4t^2}} d\mathcal{H}^{m_2}(y)$$

is infinitesimal of order $t^{m_1+m_2-m+2}$. The proof of this essentially uses the projection on the intersection S_{12}; this fact has been exploited in the previous Lemma by the quantity $I_t(\Sigma^k;\Gamma)$, since in this case the intersection between Σ^k and Γ, having dimension $n-1$, is contained in S^k, which has dimension $n-1-k$.

Since we are interested in the cases $k = 1$, we compute Θ_1 explicitly. In order to do this, we define the angle ϑ_0 as the unique $\vartheta_0 \in [0,\pi)$ such that

$$\big\langle \sigma^1_1(x_0), \gamma^1_1(x_0)\big\rangle = \cos\vartheta_0.$$

With this choice, we also have that

$$\big\langle \nu_\Sigma(x_0), \gamma^1_1(x_0)\big\rangle = \pm\sin\vartheta_0, \qquad \big\langle \nu_\Gamma(x_0), \sigma^1_1(x_0)\big\rangle = \pm\sin\vartheta_0$$

where the signs depend on the orientations of Σ and Γ.

Lemma 4.8 *Let us fix $x_0 \in S^1$; for $k = 1$, we have that*

$$\Theta_1(x_0) = -\frac{\big\langle \nu_\Sigma(x_0), \gamma^1_1(x_0)\big\rangle \big\langle \nu_\Gamma(x_0), \sigma^1_1(x_0)\big\rangle}{\pi \sin^2\vartheta_0} \Big(1 + (\pi - \vartheta_0)\mathrm{c\,tg}\vartheta_0\Big).$$

Proof. For $\eta \in V^1_{x_0}$, we write $\eta = \eta_1 v_1 + \eta_2 v_2$ where $\{v_1, v_2\}$ is the orthogonal system determined by

$$v_1 = \sigma^1_1(x_0), \qquad v_2 = \frac{1}{\sqrt{1 - \langle \sigma^1_1(x_0), \gamma^1_1(x_0)\rangle^2}} (\big\langle \sigma^1_1(x_0), \gamma^1_1(x_0)\big\rangle \sigma^1_1(x_0) - \gamma^1_1(x_0)).$$

With this choice, we obtain

$$4\pi\Big(1-\big\langle\sigma_1^1(x_0),\gamma_1^1(x_0)\big\rangle^2\Big)\Theta_1(x_0)=-\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle\int_{V^1_{x_0}}\eta_1\eta_2 e^{-\frac{|\eta|^2}{4}}d\eta+$$

$$-\frac{\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle\big\langle\sigma_1^1(x_0),\gamma_1^1(x_0)\big\rangle}{\sqrt{1-\langle\sigma_1^1(x_0),\gamma_1^1(x_0)\rangle^2}}\int_{V^1_{x_0}}\eta_2^2 e^{-\frac{|\eta|^2}{4}}d\eta$$

$$=-\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle\int_0^{+\infty}\varrho^3 e^{-\frac{\varrho^2}{4}}d\varrho\int_0^\alpha\sin\vartheta\cos\vartheta d\vartheta+$$

$$-\frac{\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle\big\langle\sigma_1^1(x_0),\gamma_1^1(x_0)\big\rangle}{\sqrt{1-\langle\sigma_1^1(x_0),\gamma_1^1(x_0)\rangle^2}}\int_0^{+\infty}\varrho^3 e^{-\frac{\varrho^2}{4}}d\varrho\int_0^\alpha\sin^2\vartheta d\vartheta,$$

with $\alpha=\pi-\vartheta_0$ since $V^1_{x_0}$ is the positive cone generated by $\sigma_1^1(x_0)$ and $-\gamma_1^1(x_0)$. □

Proposition 4.9 *Let Σ and Γ two pieces of $C^{1,1}$–regular surfaces such that $S=\overline{\Sigma}\cap\overline{\Gamma}$ has regular skeleton; then*

$$\lim_{t\to 0}\frac{I_t(\Sigma;\Gamma)}{(4\pi)^{n/2}t^{n+2}}=I_0^1(\Sigma;\Gamma)=\int_{S^1}\Theta_1(x_0)d\mathcal{H}^{n-2}(x_0).$$

Proof. The proposition follows by using the decomposition (27), the estimate (17) and applying Lemma 4.6. □

Remark 4.10 In the proof of Lemma 4.6, we have essentially used the weak convergence of

$$\mu_t^{\Sigma_r^{k,x_0}}=\mathcal{H}^k\llcorner\left(\frac{\Sigma_r^{k,x_0}-x_0}{t}\right)$$

to the measure

$$\mu_0^{\Sigma_r^{k,x_0}}=\mathcal{L}^k\llcorner\operatorname{Tan}(\Sigma_r^{k,x_0},x_0)$$

and the weak convergence of

$$\mu_t^{\Gamma_r}=\mathcal{H}^{n-1}\llcorner\left(\frac{\Gamma_r-x_0}{t}\right)$$

to the measure

$$\mu_0^{\Gamma_r}=\mathcal{L}^{n-1}\llcorner\operatorname{Tan}(\Gamma,x_0).$$

In Lemma 4.11 we shall investigate the behavior of the distributions defined by

$$\delta_1\mu_0^{\Sigma_r^{k,x_0}}=\lim_{t\to 0}\frac{\mu_t^{\Sigma_r^{k,x_0}}-\mu_0^{\Sigma_r^{k,x_0}}}{t},\qquad \delta_1\mu_0^{\Gamma_r^{k,x_0}}=\lim_{t\to 0}\frac{\mu_t^{\Gamma_r^{k,x_0}}-\mu_0^{\Gamma_r^{k,x_0}}}{t}.$$

Lemma 4.11 *If $\phi\in C_c^1(\mathbb{R}^n)$; if $x_0\in S$ and Σ_r^{1,x_0} is $C^{1,1}$–regular at x_0, then there holds*

$$\Big\langle\phi,\delta_1\mu_0^{\Sigma_r^{1,x_0}}\Big\rangle=\frac{1}{2}\int_{\operatorname{Tan}(\Sigma_r^{1,x_0},x_0)}\langle\nabla\phi(z),\nu_\Sigma(x_0)\rangle\kappa_\Sigma^{x_0}[z]dz. \tag{28}$$

Moreover, if $x_0\in S$ and Γ is $C^{1,1}$–regular at x_0,

$$\Big\langle\phi,\delta_1\mu_0^{\Gamma_r^{x_0}}\Big\rangle=\frac{1}{2}\int_{\operatorname{Tan}(\Gamma,x_0)}\langle\nabla\phi(w),\nu_\Gamma(x_0)\rangle A_\Gamma^{x_0}(w,w)dw-\frac{1}{2}\int_{T_{x_0}S}\phi(s)A^{x_0}_{S,\gamma_1^1(x_0)}(s,s)ds. \tag{29}$$

PROOF. We start by proving (29); we use the parametrization of $\frac{\Gamma_r^{x_0}-x_0}{t}$ obtained given by Remark 2.3; this parametrization is given by $\varphi_t : B^+_{r/t} \to \mathbb{R}^n$

$$\begin{aligned}\varphi_t(w,b) =& \frac{\varphi(tw,tb)-x_0}{t} = \sum_{h=1}^{n-2} \partial_h\varphi(0)w_h + \partial_{n-1}\varphi(0)b+ \\ &+ \frac{t}{2}\left(\sum_{h,k=1}^{n-2}\partial^2_{h,k}\varphi(0)w_hw_k + 2\sum_{h=1}^{n-2}\partial^2_{h,n-1}\varphi(0)w_hb + \partial^2_{n-1,n-1}\varphi(0)b^2\right) + o(t).\end{aligned}$$

We may also assume that $\partial_i\varphi(0)$ coincide with the elements e_i of the standard basis for $i = 1,\dots,n-1$. Moreover, for the metric we have

$$\begin{aligned}\frac{d}{dt}\sqrt{\det g(tw,tb)}_{t=0} = &\sum_{h,k=1}^{n-2}\partial^2_{h,k}\varphi^k(0)w_h + \sum_{h=1}^{n-2}\partial^2_{n-1,h}\varphi^h(0)b+ \\ &+\sum_{h=1}^{n-2}\partial^2_{h,n-1}\varphi^{n-1}(0)w_h + \partial^2_{n-1,n-1}\varphi^{n-1}(0)b.\end{aligned}$$

We now use the fact that for $\phi \in C^1_c(\mathbb{R}^n)$ and t small enough, we have that

$$\mathrm{spt}(\phi)\cap\frac{\Gamma_r - x_0}{t} \subset \varphi_t(B^+_{r/t}).$$

We also have that for $i \in \{1,\dots,n-2\}$ there hold

$$\int_{\mathbb{R}^{n-1}_+}\partial_i\phi(w,b,0)w_hw_k dwdb = -\int_{\mathbb{R}^{n-1}_+}\phi(w,b,0)(\delta_{ih}w_k + w_h\delta_{ik})dwdb,$$

$$\int_{\mathbb{R}^{n-1}_+}\partial_i\phi(w,b,0)w_hbdwdb = -\int_{\mathbb{R}^{n-1}_+}\phi(w,b,0)\delta_{ih}bdwdb,$$

and

$$\int_{\mathbb{R}^{n-1}_+}\partial_i\phi(w,b,0)b^2dwdb = 0,$$

whether

$$\int_{\mathbb{R}^{n-1}_+}\partial_{n-1}\phi(w,b,0)w_hw_kdwdb = -\int_{\mathbb{R}^{n-2}}\phi(w,0,0)w_hw_kdw,$$

$$\int_{\mathbb{R}^{n-1}_+}\partial_{n-1}\phi(w,b,0)w_hbdwdb = -\int_{\mathbb{R}^{n-1}_+}\phi(w,b,0)w_hdwdb,$$

and

$$\int_{\mathbb{R}^{n-1}_+}\partial_{n-1}\phi(w,b,0)b^2dwdb = -2\int_{\mathbb{R}^{n-1}_+}\phi(w,b,0)bdwdb.$$

In this way we obtain that

$$\int \phi d\mu_t^{\Gamma_r^{x_0}} = \int_{\mathbb{R}_+^{n-1}} \phi(w,b,0)dwdb + \frac{t}{2}\Big(\sum_{h,k=1}^{n-2} \partial^2_{h,k}\varphi^n(0) \int_{\mathbb{R}_+^{n-1}} \partial_n\phi(w,b,0) w_h w_k dwdb +$$

$$+ 2\sum_{h=1}^{n-2} \partial^2_{h,n-1}\varphi^n(0) \int_{\mathbb{R}_+^{n-1}} \partial_n\phi(w,b,0)w_h b dwdb + \partial^2_{n-1,n-1}\varphi^n(0) \int_{\mathbb{R}_+^{n-1}} \partial_n\phi(w,b,0)b^2 dwdb\Big) +$$

$$+ -\frac{t}{2} \sum_{h,k=1}^{n-2} \partial^2_{h,k}\varphi^{n-1}(0) \int_{\mathbb{R}^{n-2}} \phi(w,0,0) w_h w_k dw + o(t)$$

$$= \int \phi d\mu_0^{\Gamma_r^{x_0}} + \frac{t}{2}\int_{\mathrm{Tan}(\Gamma,x_0)} \langle \nabla\phi(w), \nu_\Gamma(x_0)\rangle A_\Gamma^{x_0}(w,w)dw - \frac{t}{2}\int_{T_{x_0}S} \phi(s) A^{x_0}_{S,\gamma_1^1(x_0)}(s,s)ds + o(t)$$

and this proves (29). The proof of (28) is similar. □

Remark 4.12 Lemma 4.11 applies also to function in the Schwarz space and in particular, as in our case, to a polynomial times the Gaussian.

Remark 4.13 In the proof of next result we need to compute integrals of the type

$$F_{hk} = \int_0^{+\infty} d\alpha \int_0^{+\infty} d\beta \alpha^h \beta^k \exp\left(-\frac{\alpha^2+\beta^2-2\alpha\beta\cos\vartheta_0}{4}\right).$$

The integrals we are interested in are F_{10}, F_{21}, F_{30} and F_{32}. With standard computation, we get

$$F_{10} = \frac{2\sqrt{\pi}}{1-\cos\vartheta_0}, \qquad F_{30} = \frac{4\sqrt{\pi}(2-\cos\vartheta_0)}{(1-\cos\vartheta_0)^2},$$

and

$$F_{21} = \frac{4\sqrt{\pi}}{(1-\cos\vartheta_0)^2}, \qquad F_{32} = \frac{16\sqrt{\pi}}{(1-\cos\vartheta_0)^3}.$$

In next lemma we shall also deal with point x_0 belonging to ∂S; we then have for $\mathcal{H}^{n-3}$–almost every $x_0 \in S^2 \subset \partial S$, we have the $(n-3)$–dimensional tangent space $T_{x_0}S^2$ and a vector $s_1^1(x_0)$, pointing inside S, in such a way that the $(n-2)$–dimensional cone $\mathrm{Tan}(S,x_0)$ is given by

$$\mathrm{Tan}(S,x_0) = T_{x_0}S^2 \otimes \mathbb{R}_+\langle s_1^1(x_0)\rangle.$$

Moreover, for such $x_0 \in S^2$, we have that $\mathrm{Tan}(\Sigma,x_0)$ is the product of $T_{x_0}S^2$ and a positive cone genereted by two vectors, $s_1^1(x_0)$ and a second vector $\widetilde{\sigma}_1^1(x_0)$, belonging to the plane generated by $\sigma_1^1(x_0)$ (a vector orthogonal to $s_1^1(x_0)$) and $s_1^1(x_0)$. If $\big\langle \widetilde{\sigma}_1^1(x_0), s_1^1(x_0)\big\rangle > 0$, then $\Sigma_r^{2,x_0} = \emptyset$ and then Σ has a defect of orthogonality around x_0, that is for points $x \in S^1 \cap B_r(x_0)$ close to x_0, $\Sigma_r^{1,x}$ is not $C^{1,1}$–regular at x. We have then to complete $\Sigma_r^{1,x}$ using the sets $\widetilde{\Sigma}_r^{1,x}$ of Remark 2.2; we then set

$$\widetilde{\Sigma}_r = \bigcup_{x_0\in S_2} \bigcup_{x\in S\cap B_r(x_0)} \widetilde{\Sigma}_r^{1,x}.$$

In case $\big\langle \widetilde{\sigma}_1^1(x_0), s_1^1(x_0)\big\rangle < 0$, then Σ has an excess of orthogonality and then $\Sigma_r^{2,x_0} \neq \emptyset$ and as sets of generators of $\mathrm{Tan}(\Sigma_r^{2,x_0}, x_0)$ we can choose $\sigma_1^2(x_0) = \sigma_1^1(x_0)$, $\sigma_2^2(x_0) = \widetilde{\sigma}_1^1(x_0)$. In case $\big\langle \widetilde{\sigma}_1^1(x_0), s_1^1(x_0)\big\rangle = 0$, then both Σ_r^{2,x_0} and $\widetilde{\Sigma}_r \cap B_r(x_0)$ are empty. In the same way, for Γ, we have the vectors $\gamma_1^1(x_0)$ and $\widetilde{\gamma}_1^1(x_0)$ such that $\mathrm{Tan}(\Gamma, x_0)$ is given by the product of $T_{x_0}S^2$ and the cone generated by $s_1^1(x_0)$ and $\widetilde{\gamma}_1^1(x_0)$. The fact that Γ can be not $C^{1,1}$–regular at $x_0 \in S^2$ means that Γ has a defect of tangentiality at x_0; in this case, $\mathrm{Tan}(\Gamma_r^{2,x_0}, x_0)$ is the positive cone generated by $s_1^1(x_0)$ and $\widetilde{\gamma}_1^1(x_0)$, that is we can set $\gamma_1^2(x_0) = s_1^1(x_0)$ and $\gamma_2^2(x_0) = \widetilde{\gamma}_1^1(x_0)$. We also use the definition of

$$\widetilde{\Gamma}_r = \bigcup_{x_0 \in S_2} \bigcup_{x \in B_r(x_0)} \widetilde{\Gamma}_r^x,$$

given by Remark 2.2 is such a way that $\Gamma \cup \widetilde{\Gamma}_r$ is $C^{1,1}$–regular at any point of S; finally, for almost any point $x_0 \in S^2$, we notice that $\mathrm{Tan}(\widetilde{\Gamma}_r, x_0)$ is given by $T_{x_0}S^2$ and the positive cone generated by $\widetilde{\gamma}_1^2(x_0) = -s_1^1(x_0)$ and $\widetilde{\gamma}_2^2(x_0) = \widetilde{\gamma}_1^1(x_0)$.

With the introduction of the sets $\tilde{\Sigma}_r$ and $\tilde{\Gamma}_r$ we shall consider in the next lemma the quantities $I_t(\tilde{\Sigma}_r; \Gamma_r \cup \tilde{\Gamma}_r)$ and $I_t(\tilde{\Gamma}_r; \Sigma_r)$; as pointed out in Remark 4.7, they are infinitesimal with order depending on the dimension of the intersections $\tilde{\Sigma}_r \cap \Gamma_r \cup \tilde{\Gamma}_r$ and $\tilde{\Gamma}_r \cap \Sigma_r$. Such intersections are contained in S_2 and have dimensions $n-3$; we shall also use the dimensional decompositions of $\tilde{\Sigma}_r$, $\tilde{\Gamma}_r$ and $\Gamma_r \cup \tilde{\Gamma}_r$ induced by the projections on $S_2 = \partial S$. With an abuse of notation, we shall keep the notation $\tilde{\Sigma}_r^{k,x_0}$, $(\tilde{\Gamma}_r)^{k,x_0}$ and $(\Gamma_r \cup \tilde{\Gamma}_r)^{k,x_0}$ to denote such slicings. We shall use in particular the spaces

$$\widetilde{V}_{x_0}^2 = \mathrm{Tan}((\Gamma \cup \widetilde{\Gamma}_r)_r^{2,x_0}, x_0) - \mathrm{Tan}(\widetilde{\Sigma}_r^{2,x_0}, x_0), \qquad \widetilde{W}_{x_0}^2 = \mathrm{Tan}(\widetilde{\Gamma}_r^{2,x_0}, x_0) - \mathrm{Tan}(\Sigma_r^{2,x_0}, x_0)$$

and the linear maps $\widetilde{Q}_{x_0}^2 : \mathbb{R}_+^3 \times \mathbb{R} \to \widetilde{V}_{x_0}^2$ and $\widehat{Q}_{x_0}^2 : \mathbb{R}_+^4 \to \widetilde{W}_{x_0}^2$, defined by

$$\widetilde{Q}_{x_0}^2(\alpha, \beta) = -\alpha_1 \sigma_1^1(x_0) - \alpha_2 \widetilde{\sigma}_1^1(x_0) + \beta_1 \gamma_1^1(x_0) + \beta_2 s_1^1(x_0).$$

and

$$\widehat{Q}_{x_0}^2(\alpha, \beta) = -\alpha_1 \tilde{\sigma}_1^1(x_0) - \alpha_2 s_1^1(x_0) + \beta_1 \tilde{\gamma}_1^1(x_0) - \beta_2 s_1^1(x_0).$$

Lemma 4.14 *Let Σ and Γ as before; then*

$$\delta_1 I_0^1(\Sigma;\Gamma) := \lim_{t\to 0} \frac{1}{t}\left(\frac{I_t(\Sigma_r^1;\Gamma)}{(4\pi)^{n/2} t^{n+2}} - I_0^1(\Sigma;\Gamma)\right) = \int_S T_2(x_0) d\mathcal{H}^{n-2} - \int_{S^2} \widetilde{\Theta}_2(x_0) d\mathcal{H}^{n-3}(x_0),$$

where

$$\begin{aligned} T_2(x_0) =& \frac{1}{(1-\cos\vartheta_0)^2\sqrt{\pi}} \Big[(\cos\vartheta_0 - 2)\big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle \kappa_\Sigma^{x_0}[\sigma_1^1(x_0)] + \\ & + (\cos\vartheta_0 - 2)\big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\big\rangle \kappa_\Gamma^{x_0}[\gamma_1^1(x_0)] + \\ & + (n-2)\cos\vartheta_0 \big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\big\rangle H_S^{x_0}[\nu_\Gamma(x_0)] + \\ & + (n-2)\cos\vartheta_0 \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle H_S^{x_0}[\nu_\Sigma(x_0)]\Big] \end{aligned}$$

and

$$\begin{aligned} \widetilde{\Theta}_2(x_0) =& \frac{D_2 \widetilde{Q}_{x_0}^2}{(4\pi)^{3/2}} \int_{\widetilde{V}_{x_0}^2} \langle \nu_\Sigma(x_0), v\rangle \langle \nu_\Gamma(x_0), v\rangle e^{-\frac{|v|^2}{4}} \mathcal{H}^1((\widetilde{Q}_{x_0}^2)^{-1}(v)) dv + \\ & + \frac{D_2 \widehat{Q}_{x_0}^2}{(4\pi)^{3/2}} \int_{\widetilde{W}_{x_0}^2} \langle \nu_\Sigma(x_0), v\rangle \langle \nu_\Gamma(x_0), v\rangle e^{-\frac{|v|^2}{4}} \mathcal{H}^1((\widehat{Q}_{x_0}^2)^{-1}(v)) dv. \end{aligned}$$

Proof. By (17), we have to compute the derivative of

$$\frac{I_t(\Sigma^1_r;\Gamma_r)}{(4\pi)^{n/2}t^{n+2}} = \int_S d\mathcal{H}^{n-2}(x_0)\int d\mu_t^{\Sigma_r^{1,x_0}}(z)\int F_{x_0}(t,z,w)d\mu_t^{\Gamma_r}(w)$$

where

$$F_{x_0}(t,z,w) = \frac{1}{(4\pi)^{n/2}}\frac{\langle \nu_\Sigma(x_0+tz), w-z\rangle\langle \nu_\Gamma(x_0+tw), w-z\rangle}{J\pi_1(x_0+tz)}e^{-\frac{|w-z|^2}{4}}.$$

First of all, we have that

$$\begin{aligned}
\partial_t F_{x_0}(0,z,w) =& \frac{e^{-\frac{|w-z|^2}{4}}}{(4\pi)^{n/2}}\Big(\langle d_{x_0}\nu_\Sigma[z], w-z\rangle\langle \nu_\Gamma(x_0), w-z\rangle + \\
&+\langle \nu_\Sigma(x_0), w-z\rangle\langle d_{x_0}\nu_\Gamma[w], w-z\rangle - (n-2)\langle \nu_\Sigma(x_0), w-z\rangle\langle \nu_\Gamma(x_0), w-z\rangle H_S^{x_0}[z]\Big)\\
=&\frac{e^{-\frac{|w-z|^2}{4}}}{(4\pi)^{n/2}}\Big(\langle \nu_\Gamma(x_0), z\rangle A_\Sigma^{x_0}(z, \Pi_\Sigma^{x_0}(w)-z) + \\
&-\langle \nu_\Sigma(x_0), w\rangle A_\Gamma^{x_0}(w, w-\Pi_\Gamma^{x_0}(z)) + (n-2)\langle \nu_\Sigma(x_0), w\rangle\langle \nu_\Gamma(x_0), z\rangle H_S^{x_0}[z]\Big), \qquad (30)
\end{aligned}$$

where in the last line we have used the fact that $z\in \mathrm{Tan}(\Sigma^{1,x_0}_r, x_0)$ and $w\in\mathrm{Tan}(\Gamma,x_0)$. If we write $w = w_\tau+\zeta$, with $w_\tau\in T_{x_0}S$ and $\zeta\in\mathrm{Tan}(\Gamma^{1,x_0}_r,x_0)$ and fix an orthonormal basis $\{e_1,\dots,e_{n-2}\}$ of $T_{x_0}S$. Using the fact that $T_{x_0}S$ is a vector space, we can discard the summand of (30) that are odd in the variable w_τ; in addition, we use the fact that $\langle\nu_\Sigma(x_0),w\rangle = \langle\nu_\Sigma(x_0),\zeta\rangle$ and the decompositions $w_\tau = \sum_{i=1}^{n-2} w_\tau^i e_i$ and

$$\zeta = \big\langle\zeta, \sigma^1_1(x_0)\big\rangle\sigma^1_1(x_0) + \langle\zeta,\nu_\Sigma(x_0)\rangle\nu_\Sigma(x_0), \qquad z = \big\langle z, \gamma^1_1(x_0)\big\rangle\gamma^1_1(x_0) + \langle z,\nu_\Gamma(x_0)\rangle\nu_\Gamma(x_0),$$

whence the fact that

$$\Pi_\Sigma^{x_0}(\zeta) = \big\langle\zeta,\sigma_1^1(x_0)\big\rangle\sigma^1_1(x_0), \qquad \Pi_\Gamma^{x_0}(z) = \big\langle z,\gamma_1^1(x_0)\big\rangle\gamma^1_1(x_0).$$

In this way we obtain

$$\begin{aligned}
&\int_{\mathrm{Tan}(\Sigma_r^{1,x_0},x_0)} dz\int_{\mathrm{Tan}(\Gamma,x_0)}\partial_t F_{x_0}(0,z,w)dw = \\
=&\int_{T_{x_0}S} dw_\tau \frac{e^{-\frac{|w_\tau|^2}{4}}}{(4\pi)^{n/2}}\int_{\mathrm{Tan}(\Sigma_r^{1,x_0},x_0)}dz\int_{\mathrm{Tan}(\Gamma_r^{1,x_0},x_0)}\Big(\langle\nu_\Gamma(x_0),z\rangle\big\langle\zeta,\sigma^1_1(x_0)\big\rangle A_\Sigma^{x_0}(z,\sigma^1_1(x_0)) + \\
&-\langle\nu_\Gamma(x_0),z\rangle A_\Sigma^{x_0}(z,z) - \langle\nu_\Sigma(x_0),\zeta\rangle A_\Gamma^{x_0}(w_\tau,w_\tau) - \langle\nu_\Sigma(x_0),\zeta\rangle A_\Gamma^{x_0}(\zeta,\zeta) + \\
&+\langle\nu_\Sigma(x_0),\zeta\rangle\big\langle z,\gamma^1_1(x_0)\big\rangle A_\Gamma^{x_0}(\zeta,\gamma^1_1(x_0)) + (n-2)\langle\nu_\Sigma(x_0),\zeta\rangle\langle\nu_\Gamma(x_0),z\rangle H_S^{x_0}[z]\Big)e^{-\frac{|\zeta-z|^2}{4}}d\zeta.
\end{aligned}$$

We also need the fact that

$$\begin{aligned}
\int_{T_{x_0}S} A_\Gamma^{x_0}(w_\tau,w_\tau)e^{-\frac{|w_\tau|^2}{4}}dw_\tau =& 2(4\pi)^{\frac{n-2}{2}}\sum_{i=1}^{n-2}A_\Gamma^{x_0}(e_i,e_i)\\
=&2(4\pi)^{\frac{n-2}{2}}(n-2)H_S^{x_0}[\nu_\Gamma(x_0)],
\end{aligned}$$

so we get

$$\begin{aligned}
&\int_{\mathrm{Tan}(\Sigma_r^{1,x_0},x_0)} dz \int_{\mathrm{Tan}(\Gamma,x_0)} \partial_t F_{x_0}(0,z,w)dw = \\
=&\frac{1}{4\pi}\Big[\kappa_\Sigma^{x_0}[\sigma_1^1(x_0)]\left(\big\langle\gamma_1^1(x_0),\sigma_1^1(x_0)\big\rangle\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle F_{21}-\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle F_{30}\right)+\\
&+\kappa_\Gamma^{x_0}[\gamma_1^1(x_0)]\left(\big\langle\gamma_1^1(x_0),\sigma_1^1(x_0)\big\rangle\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle F_{12}-\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle F_{03}\right)+\\
&+(n-2)\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle H_S^{x_0}[\sigma_1^1(x_0)]F_{21}+\\
&-2(n-2)\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle H_S^{x_0}[\nu_\Gamma(x_0)]F_{01}\Big]; \qquad (31)
\end{aligned}$$

here the F_{hk} are the coefficients defined in Remark 4.13. We now have to consider the derivatives of the measures $\mu_t^{\Sigma_r^{1,x_0}}$ and $\mu_t^{\Gamma_r}$; in order to apply Lemma 4.11, we need that for any $x_0 \in S^1$, Σ_r^{1,x_0} and Γ have to be $C^{1,1}$–regular at x_0. So, in case, we can consider the completions $\widetilde{\Sigma}_r$ of Σ_r and $\widetilde{\Gamma}_r$ of Γ. We also notice that

$$I_t(\Sigma_r;\Gamma_r) = I_t(\Sigma_r\cup\widetilde{\Sigma}_r;\Gamma_r\cup\widetilde{\Gamma}_r) - I_t(\widetilde{\Sigma}_r;\Gamma_r\cup\widetilde{\Gamma}_r) - I_t(\Sigma_r;\widetilde{\Gamma}_r).$$

To deal with $I_t(\widetilde{\Sigma}_r;\Gamma_r\cup\widetilde{\Gamma}_r)$, we consider the projection $\widetilde{\pi}_\Sigma : \widetilde{\Sigma}_r \to \partial S$ and define the sets

$$(\partial S)^k = \{x_0 \in \partial S : \dim\widetilde{\pi}^{-1}(x_0) = k\}$$

and

$$\widetilde{\Sigma}_r^k = \widetilde{\pi}^{-1}((\partial S)^k).$$

Since ∂S is an $(n-3)$–dimensional manifold, we deduce that

$$\mathcal{H}^{n-1}(\widetilde{\Sigma}_r^0) = \mathcal{H}^{n-1}(\widetilde{\Sigma}_r^1) = 0,$$

so that

$$I_t(\widetilde{\Sigma}_r;\Gamma_r\cup\widetilde{\Gamma}_r) = \sum_{k=2}^{n-1} I_t(\widetilde{\Sigma}_r^k;\Gamma_r\cup\widetilde{\Gamma}_r)$$

and, arguing as in Lemma 4.6, the term $I_t(\widetilde{\Sigma}_r^k;\Gamma_r\cup\widetilde{\Gamma}_r)$ is asymptotic to t^{n+1+k}. In the same way, by considering the projection $\widetilde{\pi}_\Gamma : \widetilde{\Gamma}_r \to \partial S$, we can write

$$I_t(\widetilde{\Gamma}_r;\Sigma_r) = \sum_{k=2}^{n-1} I_t(\widetilde{\Gamma}_r^k;\Sigma_r).$$

We shall still denote by $\widetilde{\Sigma}_r^{2,x_0}$ and by $(\Gamma\cup\widetilde{\Gamma_r})_r^{2,x_0}$ the sections induced by the projection's $\widetilde{\pi}$.

We are interested in the term asymptotic to t^{n+3}; we have that

$$\begin{aligned}
&\lim_{t\to 0}\frac{I_t(\widetilde{\Sigma}_r^2;\Gamma_r\cup\widetilde{\Gamma}_r)}{(4\pi)^{n/2}t^{n+3}} := \widetilde{I}_0(\Sigma;\Gamma) \qquad (32)\\
&=\int_{\partial S} d\mathcal{H}^{n-3}(x_0)\int_{\mathrm{Tan}(\widetilde{\Sigma}_r^{2,x_0},x_0)} dz\int_{\mathrm{Tan}(\Gamma\cup\widetilde{\Gamma}_r,x_0)}\frac{\langle\nu_\Sigma(x_0),w-z\rangle\langle\nu_\Gamma(x_0),w-z\rangle}{(4\pi)^{n/2}}e^{-\frac{|w-z|^2}{4}}dw\\
&=\int_{\partial S} d\mathcal{H}^{n-3}(x_0)\int_{\mathrm{Tan}(\widetilde{\Sigma}_r^{2,x_0},x_0)} dz\int_{\mathrm{Tan}((\Gamma\cup\widetilde{\Gamma}_r^{2,x_0},x_0)}\frac{\langle\nu_\Sigma(x_0),w-z\rangle\langle\nu_\Gamma(x_0),w-z\rangle}{(4\pi)^{3/2}}e^{-\frac{|w-z|^2}{4}}dw\\
&=\int_{\partial S} d\mathcal{H}^{n-3}(x_0)\frac{D_2\widetilde{Q}_{x_0}^2}{(4\pi)^{3/2}}\int_{\widetilde{V}_{x_0}^2}\langle\nu_\Sigma(x_0),v\rangle\langle\nu_\Gamma(x_0),v\rangle e^{-\frac{|v|^2}{4}}\mathcal{H}^1((\widetilde{Q}_{x_0}^2)^{-1}(v))dv;
\end{aligned}$$

moreover we get that

$$
\begin{aligned}
\lim_{t\to 0}\frac{I_t(\widetilde{\Gamma}^2_r;\Sigma_r)}{(4\pi)^{n/2}t^{n+3}} &:= \widetilde{I}_1(\Sigma;\Gamma) \\
&=\frac{D_2\widehat{Q}^2_{x_0}}{(4\pi)^{3/2}}\int_{\partial S} d\mathcal{H}^{n-3}(x_0)\int_{\widetilde{W}^2_{x_0}}\langle\nu_\Sigma(x_0),v\rangle\langle\nu_\Gamma(x_0),v\rangle e^{-\frac{|v|^2}{4}}\mathcal{H}^1((\widehat{Q}^2_{x_0})^{-1}(v))dv,
\end{aligned}
\tag{33}
$$

For the term $I_t(\Sigma_r\cup\widetilde{\Sigma}_r;\Gamma\cup\widetilde{\Gamma}_r)$ we can apply Lemma 4.11 and the fact that $\mathrm{Tan}(\Sigma^{1,x_0}_r,x_0)=\mathrm{Tan}((\Sigma\cup\widetilde{\Sigma}_r)^{1,x_0}_r,x_0)$, so that

$$
\begin{aligned}
\delta_1 I^1_0(\Sigma;\Gamma) =& \int_S d\mathcal{H}^{n-2}(x_0)\int_{\mathrm{Tan}(\Sigma^{1,x_0}_r,x_0)} dz\int_{\mathrm{Tan}(\Gamma,x_0)}\partial_t F(0,z,w)dw+ \\
&+\int_S\left(\left\langle\phi_\Gamma,\delta_1\mu_0^{(\Sigma\cup\widetilde{\Sigma}_r)^{1,x_0}_r}\right\rangle+\left\langle\phi_\Sigma,\delta_1\mu_0^{\Gamma\cup\widetilde{\Gamma}_r}\right\rangle\right)d\mathcal{H}^{n-2}(x_0)-\widetilde{I}_0(\Sigma;\Gamma)-\widetilde{I}_1(\Sigma;\Gamma),
\end{aligned}
$$

where

$$
\phi_\Gamma(z)=\int_{\mathrm{Tan}(\Gamma,x_0)}F_{x_0}(0,z,w)dw,\qquad \phi_\Sigma(w)=\int_{\mathrm{Tan}(\Sigma^{1,x_0}_r,x_0)}F_{x_0}(0,z,w)dz.
$$

We start with ϕ_Γ; using the decomposition $\mathrm{Tan}(\Gamma,x_0)=T_{x_0}S\oplus\mathrm{Tan}(\Gamma^{1,x_0}_r,x_0)$, we deduce that

$$
\phi_\Gamma(z)=-\frac{\langle\nu_\Gamma(x_0),z\rangle}{4\pi}\int_{\mathrm{Tan}(\Gamma^{1,x_0}_r,x_0)}\langle\nu_\Sigma(x_0),w-z\rangle e^{-\frac{|w-z|^2}{4}}dw,
$$

so that

$$
\begin{aligned}
\nabla\phi_\Gamma(z)=&-\frac{1}{4\pi}\int_{\mathrm{Tan}(\Gamma^{1,x_0}_r,x_0)}\Big(\langle\nu_\Sigma(x_0),w-z\rangle\nu_\Gamma(x_0)-\langle\nu_\Gamma(x_0),z\rangle\nu_\Sigma(x_0)+ \\
&+\frac{1}{2}\langle\nu_\Sigma(x_0),w-z\rangle\langle\nu_\Gamma(x_0),z\rangle(w-z)\Big)e^{-\frac{|w-z|^2}{4}}dw.
\end{aligned}
$$

We then obtain that

$$
\begin{aligned}
\left\langle\phi_\Gamma,\delta_1\mu_0^{\Sigma^{1,x_0}_r}\right\rangle=&-\frac{1}{8\pi}\int_{\mathrm{Tan}(\Sigma^{1,x_0}_r,x_0)}dz\int_{\mathrm{Tan}(\Gamma^{1,x_0}_r,x_0)}\Big(\langle\nu_\Sigma(x_0),w\rangle\langle\nu_\Sigma(x_0),\nu_\Gamma(x_0)\rangle+ \\
&-\langle\nu_\Gamma(x_0),z\rangle+\frac{1}{2}\langle\nu_\Sigma(x_0),w\rangle^2\langle\nu_\Gamma(x_0),z\rangle\Big)\kappa^{x_0}_\Sigma[z]e^{-\frac{|w-z|^2}{4}}dw \\
=&-\frac{1}{8\pi}\kappa^{x_0}_\Sigma[\sigma^1_1(x_0)]\Big(\langle\nu_\Sigma(x_0),\nu_\Gamma(x_0)\rangle\big\langle\nu_\Sigma(x_0),\gamma^1_1(x_0)\big\rangle F_{21}+ \\
&-\big\langle\nu_\Gamma(x_0),\sigma^1_1(x_0)\big\rangle F_{30}+\frac{1}{2}\big\langle\nu_\Sigma(x_0),\gamma^1_1(x_0)\big\rangle^2\big\langle\nu_\Gamma(x_0),\sigma^1_1(x_0)\big\rangle F_{32}\Big).
\end{aligned}
\tag{34}
$$

Concerning $\nabla\phi_\Sigma$, we have

$$
\begin{aligned}
\nabla\phi_\Sigma(w)=&\frac{1}{(4\pi)^{\frac{n}{2}}}\int_{\mathrm{Tan}(\Sigma^{1,x_0}_r,x_0)}(\langle\nu_\Gamma(x_0),w-z\rangle\nu_\Sigma(x_0)+\langle\nu_\Sigma(x_0),w\rangle\nu_\Gamma(x_0)+ \\
&-\frac{1}{2}\langle\nu_\Sigma(x_0),w\rangle\langle\nu_\Gamma(x_0),w-z\rangle(w-z)\Big)e^{-\frac{|w-z|^2}{4}}dz;
\end{aligned}
$$

we also get

$$\left\langle \phi_\Sigma, \delta_1 \mu_0^{\Gamma_r^{x_0}} \right\rangle = \frac{1}{8\pi}\kappa_\Gamma^{x_0}[\gamma_1^1(x_0)]\Big(-\langle \nu_\Sigma(x_0), \nu_\Gamma(x_0)\rangle \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle F_{12} +$$
$$+ \big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\big\rangle F_{03} - \frac{1}{2}\big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\big\rangle \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle^2 F_{23}\Big) +$$
$$+ \frac{(n-2)}{4\pi} H_S^{x_0}[\nu_\Gamma(x_0)]\Big(-\big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle \langle \nu_\Sigma(x_0), \nu_\Gamma(x_0)\rangle F_{10} +$$
$$+ \big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\big\rangle F_{01} - \frac{1}{2}\big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\big\rangle \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle^2 F_{21}\Big) \tag{35}$$

Summing the three terms (31), (34) and (35), we obtain the term

$$T_2(x_0) = \frac{1}{4\pi}\Big\{ \kappa_\Sigma^{x_0}[\sigma_1^1(x_0)]\Big(-\frac{1}{2}\langle \nu_\Sigma(x_0), \nu_\Gamma(x_0)\rangle \big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\big\rangle F_{21} - \frac{1}{2}\big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle F_{30} +$$
$$- \frac{1}{4}\big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\big\rangle^2 \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle F_{32} + \big\langle \gamma_1^1(x_0), \sigma_1^1(x_0)\big\rangle \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle F_{21}\Big) +$$
$$+ \kappa_\Gamma^{x_0}[\gamma_1^1(x_0)]\Big(-\frac{1}{2}\langle \nu_\Sigma(x_0), \nu_\Gamma(x_0)\rangle \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle F_{12} - \frac{1}{2}\big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\big\rangle F_{03} +$$
$$- \frac{1}{4}\big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle^2 \big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\big\rangle F_{23} + \big\langle \gamma_1^1(x_0), \sigma_1^1(x_0)\big\rangle \big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\big\rangle F_{12}\Big) +$$
$$+ (n-2)H_S^{x_0}[\nu_\Gamma]\Big(-\langle \nu_\Sigma(x_0), \nu_\Gamma(x_0)\rangle \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle F_{10} - \big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\big\rangle F_{01} +$$
$$- \frac{1}{2}\big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\big\rangle \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle^2 F_{21}\Big) +$$
$$+ (n-2)H_S^{x_0}[\sigma_1^1(x_0)]\big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\big\rangle \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle F_{21}\Big\}.$$

We write $\sigma_1^1(x_0)$ as a combination of $\nu_\Sigma(x_0)$ and ν_Γ

$$\sigma_1^1(x_0) = \alpha \nu_\Sigma(x_0) + \beta \nu_\Gamma(x_0),$$

where, since $\big\langle \nu_\Sigma(x_0), \sigma_1^1(x_0)\big\rangle = 0$ and $1 = \big\langle \sigma_1^1(x_0), \sigma_1^1(x_0)\big\rangle$,

$$\alpha = \frac{\cos\vartheta_0}{\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\rangle}, \qquad \beta = \frac{1}{\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\rangle}.$$

By the fact that $\cos\vartheta_0 = \big\langle \sigma_1^1(x_0), \gamma_1^1(x_0)\big\rangle$, we also obtain that

$$\langle \nu_\Sigma(x_0), \nu_\Gamma(x_0)\rangle = -\frac{\cos\vartheta_0 \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle}{\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\rangle}.$$

$$T_2(x_0) = \frac{1}{(1-\cos\vartheta_0)^2\sqrt{\pi}}\Big[(\cos\vartheta_0 - 2)\big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle \kappa_\Sigma^{x_0}[\sigma_1^1(x_0)] +$$
$$+ (\cos\vartheta_0 - 2)\big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\big\rangle \kappa_\Gamma^{x_0}[\gamma_1^1(x_0)] +$$
$$+ (n-2)\cos\vartheta_0 \big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\big\rangle H_S^{x_0}[\nu_\Gamma(x_0)] +$$
$$+ (n-2)\cos\vartheta_0 \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle H_S^{x_0}[\nu_\Sigma(x_0)]\Big]$$

□

Proposition 4.15 *Let Σ and Γ as before, then there holds*

$$\frac{I_t(\Sigma;\Gamma)}{(4\pi)^{n/2}t^{n+2}} = I_0^1(\Sigma;\Gamma) + t\left(I_0^2(\Sigma;\Gamma) + \delta_1 I_0^1(\Sigma;\Gamma)\right) + o(t)$$

PROOF. The proof simply follows by the previous results. □

Remark 4.16 We end this section by pointing out the following.

1. The hypotheses on S and ∂S can be weakened by requiring that S has positive reach, following the definition of Federer in [7], in both surfaces Σ and Γ.
2. It is also clear that the previous expansion can also continue for higher powers of t; this expansion will contains higher derivatives of the objects involved so far, but also the higher codimensional part of the skeleton of S.

5 Expansion for sets of finite perimeter

As direct corollaries of Theorems 3.1 and 3.3, we can state and prove the following properties. The assumptions is that E is a set with ∂E union of $C^{1,1}$–regular surfaces and such that $\partial E \setminus \mathcal{F}E$ with regular skeleton. We write

$$\partial E = \Sigma = \bigcup_{i=i}^{m} \Sigma_i$$

and

$$A_i = \{j \neq i : S_{i,j} = \Sigma_i \cap \Sigma_j \neq \emptyset\}.$$

Theorem 5.1 *Under the conditions that ∂E is a finite union of regular surfaces and $\partial E \setminus \mathcal{F}E$ has regular skeleton, we have that*

$$\|T_t\chi_E\|^2_{L^2(\mathbb{R}^n)} = |E| - \sqrt{\frac{2t}{\pi}}P(E) + t\sum_{i=1}^{m}\sum_{j\in A_i} I_0^1(\Sigma_i;\Sigma_j) +$$
$$-\frac{\sqrt{2t^3}}{3}\left[\frac{(n-1)^2}{2\sqrt{\pi}}\int_\Sigma\left((H^x_\Sigma)^2 + \frac{2}{(n-1)^2}c^2_\Sigma\right)d\mathcal{H}^{n-1} - \sum_{i=1}^{m}\sum_{j\in A_i}\left(I_0^2(\Sigma_i;\Sigma_j) + \delta_1 I_0^1(\Sigma_i;\Sigma_j)\right)\right]$$
$$+\, o(t^{3/2}).$$

PROOF. It suffices to notice that, if K_t is the function defined in (1), then

$$\|T_t\chi_E\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} T_{2t}\chi_E dx = |E| - \int_{E^c} T_{2t}\chi_E dx = |E| - K_{2t}(E, E^c).$$

The result then follows by the properties described in Section 1.1. □

Theorem 5.2 *Under the same conditions of Theorem 3.1, we have that as $t \to 0^+$,*

$$\|T_t\chi_E - \chi_E\|_{L^1(\mathbb{R}^n)} = 2\sqrt{\frac{t}{\pi}}P(E) - t\sum_{i=1}^{m}\sum_{j\in A_i} I_0^1(\Sigma_i;\Sigma_j) +$$
$$+\frac{\sqrt{t^3}}{3}\left[\frac{(n-1)^2}{2\sqrt{\pi}}\int_\Sigma\left((H^x_\Sigma)^2 + \frac{2}{(n-1)^2}c^2_\Sigma\right)d\mathcal{H}^{n-1} - \sum_{i=1}^{m}\sum_{j\in A_i}\left(I_0^2(\Sigma_i;\Sigma_j) + \delta_1 I_0^1(\Sigma_i;\Sigma_j)\right)\right]$$
$$+\, o(t^{3/2}).$$

Proof. Setting $f_t(x) = T_t\chi_E(x) - \chi_E(x)$, we have

$$\int_{\mathbb{R}^n} f_t(x)dx = 0, \qquad \forall t \geq 0,$$

hence

$$\int_{\mathbb{R}^n} f_t^+(x)dx = \int_{\mathbb{R}^n} f_t^-(x)dx.$$

Since $f_t^+(x) = (T_t\chi_E(x) - \chi_E(x))\chi_{E^c}(x)$, it suffices to notice that

$$\|T_t\chi_E - \chi_E\|_{L^1(\mathbb{R}^n)} = 2\int_{E^c} T_t\chi_E(x)dx. \tag{36}$$

□

The previous result can also be extended to the case of two sets $E, F \subset \mathbb{R}^n$ with piecewice $C^{1,1}$–regularity; by assuming that both E and F are open sets, one has to consider the splittings

$$\partial E = (\partial E \cap F) \cup (\partial E \cap \bar{F}^c) \cup (\partial E \cap \partial F)$$

and

$$\partial F = (\partial F \cap E) \cup (\partial F \cap \bar{E}^c) \cup (\partial F \cap \partial E)$$

and the decomposition of these three parts into finite unions of $C^{1,1}$–surfaces.

Remark 5.3 In view of Remark 4.16, we can also go further in the expansion of $\|T_t\chi_E\|^2_{L^2(\mathbb{R}^n)}$, also requiring less regularity (positive reach) of the singularities of ∂E. It is clear from the argument used that, in general, there is no reason why this expansion has to be a polynomial in $\sqrt{t}$. This happens, up to a term that is infinitesimal of exponential type, if for instance E is a polyhedral set, that is a finite intersection of halfspaces. It would be interesting to investigate the reverse implication of this, that is if it is true that knowing that the expansion is a polynomial of degree n in $\sqrt{t}$, then E has to be a polyhedral. The fact that only integer powers of $\sqrt{t}$ appear is a consequence of the regularity of the skeleton of ∂E. In general, any power on t can be involved; take for instance u to be the Cantor–Vitali function and by U its primitive. Then the epigraph of U is a convex set and, direct computations, show that in the expansion the term $t^{1+\alpha}$ appears, with $\alpha = \log_3 2$ the Hausdorff dimension of the Cantor set.

This fact shows a crucial difference between the heat and the Minkowski content, also in the class of convex sets.

6 Examples

6.1 Two-dimensional region

Example 6.1 Let $\gamma : [0, L] \to \mathbb{R}^2$ be a $C^{1,1}$ planar simple curve parametrized by arclength; then

$$\lim_{t\to 0} \frac{I_t(\gamma)}{4\pi t^5} = -\frac{3}{2\sqrt{\pi}}\int_0^L \kappa_\gamma^2(\alpha)d\alpha,$$

with κ_γ the curvature of γ. In particular, if E is a bounded set with ∂E parametrized by γ, then

$$\|T_t\chi_E\|^2_{L^2(\mathbb{R}^2)} = |E| - \sqrt{\frac{2t}{\pi}}L - \sqrt{\frac{t^3}{2\pi}}\int_0^L \kappa_\gamma^2(\alpha)d\alpha + o(t^{3/2}).$$

A first example is given by the circle $E = B_r(0)$ in the plane; in this case ∂E is parametrized by $\gamma(\alpha) = (r\cos\frac{\alpha}{r}, r\sin\frac{\alpha}{r})$, and $\kappa_\gamma(\alpha) = r^{-1}$ for every $\alpha \in [0, 2\pi r)$. Then

$$\|T_t\chi_E\|^2_{L^2(\mathbb{R}^2)} = \pi r^2 - 2r\sqrt{2\pi t} - \frac{\sqrt{2\pi t^3}}{r} + o(t^{3/2}).$$

The following example shows how the second order expansion of the heat content of a set E with finite perimeter takes into account the behavior of the boundary ∂E along the 0-singular set of ∂E.

Example 6.2 Let E be a simple oriented polygonal region in the plane with angles $\alpha_i \in (0,\pi), i = 1,\ldots,m$, then in the expansion of its heat content the coefficients of t is not zero and depends on the not $C^{1,1}$ contact of pair of consecutive segments; indeed if Σ and Γ are two segments that have a common endpoint x_0 and generate an angle $\alpha \in (0,\pi)$, by Lemma 4.8 we get that

$$I^1_0(\Sigma;\Gamma) = \Theta_1(x_0) = -\frac{1}{\pi}\big(1 + (\pi-\alpha)\mathrm{c\,tg}\,\alpha\big).$$

In the case of E as before we have to consider m segments and for each of them two contacts with the adjacent segments; moreover, since the curvature of a line is zero, the heat content of E is, up to an exponential infinitesimal term, a quadratic polynomial in $\sqrt{t}$

$$\|T_t\chi_E\|^2_{L^2(\mathbb{R}^2)} = |E| - \sqrt{\frac{2t}{\pi}}P(E) - 2t\Big(\frac{m}{\pi} + \sum_{i=1}^m \frac{\mathrm{c\,tg}\,\alpha_i}{\pi}(\pi-\alpha_i)\Big) + o(t^h) \quad \forall\, h > 1.$$

In the case of the square $E = [0,1]^2 \subset \mathbb{R}^2$, ∂E is the union of four orthogonal segments and

$$\lim_{t\to 0}\frac{\|T_t\chi_E\|^2_{L^2(\mathbb{R}^2)} - |E| + \sqrt{\frac{2t}{\pi}}P(E)}{t} = -\frac{8}{\pi}$$

being $\alpha_i = \frac{\pi}{2}$, $i = 1,..,4$. Hence,

$$\|T_t\chi_E\|^2_{L^2(\mathbb{R}^2)} = 1 - 4\sqrt{\frac{2t}{\pi}} - \frac{8}{\pi}t + o(t^h) \quad \forall\, h > 1.$$

6.2 Three-dimensional region

Example 6.3 Let $E = B_r(0) \subset \mathbb{R}^3$, then ∂E is $C^{1,1}$- regular and being $\kappa^x_{\partial E,i} = r^{-1}$ for $i = 1,2$ and for every $x \in \partial E$, as immediate consequence of Theorem 5.1 we get that

$$\|T_t\chi_{B_r(0)}\|^2_{L^2(\mathbb{R}^3)} = \frac{4}{3}\pi r^3 - 4\sqrt{2\pi t}r^2 - \frac{16}{3}\sqrt{2\pi t^3} + o(t^{3/2})$$

Example 6.4 We consider now the set $E = B^+_r(0) = B_r(0) \cap \{(x,y,z) : z > 0\}$; we divide $\partial E = \Sigma_1 \cup \Sigma_2$, where $\Sigma_1 = \{(x,y,z) : x^2+y^2 < r^2, z = 0\}$ and $\Sigma_2 = \{(x,y,z) : x^2+y^2+z^2 = r^2, z > 0\}$; both Σ_1 and Σ_2 can be parametrized by uniformly Lipschitz functions. The presence of the 1-singular set $S = \{x^2+y^2 = r^2, z = 0\}$ gives a nontrivial coefficient of t in the asymptotic expansion of the heat content of E. We have that

$$-(4\pi)^{3/2}t^5 f''_E(t) = I_t(\partial E) = \sum_{i,j=1}^2 I_t(\Sigma_i;\Sigma_j)$$

By Theorem 3.3, it holds that

$$I_t(\Sigma_i) = I_t(\Sigma_i;\Sigma_i) = -16\pi t^6 \int_{\Sigma_i} \Big((H^x_{\Sigma_i})^2 + \frac{1}{2}c^2_{\Sigma_i}(x)\Big)\Big) d\mathcal{H}^2(x) + o(t^6) \quad i = 1, 2.$$

Since $\kappa^x_{\Sigma_1,i} = 0$ for $i = 1, 2$, $x \in \Sigma_1$ and $\kappa^x_{\Sigma_2,i} = \frac{1}{r}$ for $i = 1, 2$, $x \in \Sigma_2$ then $I_t(\Sigma_1) = 0$ and

$$\lim_{t\to 0} \frac{I_t(\Sigma_2)}{(4\pi)^{3/2}t^6} = -8\sqrt{\pi}$$

Whereas for $i, j = 1, 2$ and $i \neq j$ we have that

$$I_t(\Sigma_i;\Sigma_j) = I_t(\Sigma_j;\Sigma_i) = (4\pi)^{3/2}t^5 \int_S \Theta_1(x)d\mathcal{H}^1(x) + o(t^5)$$

and

$$I^1_0(\Sigma_1;\Sigma_2) = I^1_0(\Sigma_2;\Sigma_1) = \lim_{t\to 0} \frac{I_t(\Sigma_2;\Sigma_1)}{(4\pi)^{3/2}t^5} = \int_S \Theta_1(x)d\mathcal{H}^1(x) = -2r$$

being $\theta_0 = \frac{\pi}{2}$ and $\Theta_1(x_0) = -\frac{1}{\pi}$ for every $x_0 \in S$. In addition, since $\kappa^{x_0}_{\Sigma_2}[e_3] = \frac{1}{r}$ for any $x_0 \in S$, we also deduce that $T_2(x_0) = -\frac{2}{r\sqrt{\pi}}$ for any $x_0 \in S$. Since $S^2 = \emptyset$, we can conclude that

$$\|T_t\chi_{B^+_r(0)}\|^2_{L^2(\mathbb{R}^3)} = \frac{2}{3}\pi r^3 - 3\sqrt{2\pi t}r^2 - 4rt - \frac{16}{3}\sqrt{2\pi t^3} + o(t^{3/2}).$$

Example 6.5 The example of the square in $\mathbb{R}^2$ can be easily extended to the case $E = [0, 1]^3 \subset \mathbb{R}^3$; in this case

$$\partial E = \sum_{1=1}^{6} \Sigma_i,$$

where the Σ_i's are 1 side-squares and $|A_i| = |\{j \neq i : S_{i,j} = \Sigma_i \cap \Sigma_j \neq \emptyset\}| = 4$ for every $i = 1, \ldots, 6$. Moreover, there holds that

$$I^1_0(\Sigma_i;\Sigma_j) = \int_{S_{i,j}} \Theta_1(x)d\mathcal{H}^1(x) = -\frac{1}{\pi}, \quad i = 1, \ldots, 6,\ j \in A_i.$$

Since $\kappa^x_{\Sigma_i} = 0$ for every $x \in \Sigma_i$ and $S^2_{i,j} = \emptyset$ the coefficient of $\sqrt{t^3}$ in the asymptotic expansion of the heat content of E reduces to $\sum_{i=1}^{6}\sum_{j\in A_i} \delta_1 I^1_0(\Sigma_i;\Sigma_j)$. In order to describe a generic term of the form $\delta_1 I^1_0(\Sigma_i;\Sigma_j)$ we consider an orthogonal coordinate system, we fix $\Sigma = \Sigma_i$ and $\Gamma = \Sigma_j$ with $S = \Sigma \cap \Gamma \neq \emptyset$ and, without loss of generality, we assume that $\Sigma = \{0\} \times [0, 1]^2$ and $\Gamma = [0, 1]^2 \times \{0\}$, then the origin of the axis belongs to ∂S. Using the notation introduced in Section 4 we have that

$$\nu_\Sigma(0) = e_1,\ \nu_\Gamma(0) = e_3,\ \sigma^1_1(0) = e_3,\ \gamma^1_1(0) = e_1,\ s^1_1(0) = e_2,$$

where $\{e_i\}_{i=1,2,3}$ denote the canonical basis in $\mathbb{R}^3$. It is obvious that, being $\kappa^x_{\Sigma_i} = 0$, $\kappa^x_{S_{i,j}} = 0$ respectively for every $x \in \Sigma_i$ and $x \in S_{i,j}$ and $\tilde{\Sigma}^{2,0}_r = \emptyset$, there holds that $\delta_1 I^1_0(\Sigma;\Gamma)$ reduces to $-\tilde{I}_1(\Sigma;\Gamma)$ which depends on the defect of tangentiality of Γ in 0; in this case we have to consider $\tilde{\Gamma}^{2,0}_r$ in such a way that $\Gamma \cup \tilde{\Gamma}^{2,0}_r$ is $C^{1,1}$-regular at 0, then $\mathrm{Tan}(\tilde{\Gamma}^{2,0}_r, 0)$ is generated by $-e_2$ and e_1.

We have to consider $\bar{\pi}:\Sigma_r \to (\Gamma\cup\tilde{\Gamma}_r^{2,0})\cap\Sigma_r$ and $\Sigma_{r,\bar{\pi}}^{2,0}=\bar{\pi}^{-1}(0)=\Sigma_r$; in this case $\mathrm{Tan}(\Sigma_{r,\bar{\pi}}^{2,0},0)$ is generated by e_2 and e_3. In this case we get that

$$\widetilde{W}_0^2=\mathrm{Tan}(\tilde{\Gamma}_r^{2,0},0)-\mathrm{Tan}(\Sigma_{r,\bar{\pi}}^{2,0},0)=(0,+\infty)\times(-\infty,0)^2$$

and

$$\widehat{Q}_0^2:\mathbb{R}_+^4\to\widetilde{W}_0^2,\ (\alpha,\beta)\mapsto(\beta_1,-\alpha_2-\beta_2,-\alpha_1),$$

then $D_2\widehat{Q}_0^2=\frac{1}{\sqrt{2}}$

$$\tilde{I}_1(\Sigma;\Gamma)=\frac{1}{8\pi\sqrt{2\pi}}\int_{\widetilde{W}_0^2}v_2\langle\nu_\Sigma(0),v\rangle\langle\nu_\Gamma(0),v\rangle e^{-\frac{|v|^2}{4}}\mathcal{H}^1((\hat{Q}_0^2)^{-1}(v))dv=-\frac{1}{\sqrt{\pi^3}}.$$

Hence, we get

$$\|T_t\chi_E\|_{L^2(\mathbb{R}^3)}^2=1-6\sqrt{\frac{2t}{\pi}}-\frac{24}{\pi}t-8\sqrt{\frac{2t^3}{\pi^3}}+o(t^h)\quad\forall\,h>3/2.$$

Example 6.6 Let E be an oriented regular tetrahedron, that is a polyhedron whose four faces are triangles equilateral T_i (with side equal to a), three of which meet at each vertex. In this case $|E|=\frac{\sqrt{2}}{12}a^3$, $P(E)=\sqrt{3}a^2$; moreover $\partial E=\sum_{i=1}^4 T_i$ and $|A_i|=|\{j\neq i: T_{i,j}=T_i\cap T_j\neq\emptyset\}|=3$ for every $i=1,..,4$. We notice that $\vartheta_0=\frac{\pi}{3}$ and for every $x_0\in T_{i,j}$, $\Theta_1(x_0)=-(\frac{1}{\pi}+\frac{2}{\sqrt{3}})$, hence

$$I_0^1(T_i;T_j)=-a\left(\frac{1}{\pi}+\frac{2}{\sqrt{3}}\right),\quad i=1,\dots,4,\ j\in A_i.$$

It is easy to see that in the expansion of the heat content of E all the terms which depend on the curvatures of T_i and $T_{i,j}$ will be zero. However, in order to go further in the expansion we first observe that, since $(T_i)_r^2=\emptyset$ then $I_0^2(T_i;T_j)=0$ for every $i=1,..,4$ and $j\in A_i$; moreover, fixed T_i, T_j (with $i=1,\dots,4$, $j\in A_i$), $\delta_1 I_0^1(T_i;T_j)$ reduces to $\widetilde{I}_0(T_i;T_j)+\widetilde{I}_1(T_i;T_j)=2\widetilde{\Theta}_2(x_0)$ with $x_0\in\partial T_{i,j}$. Without loss of generality, we fix an orthogonal coordinate system, we assume that x_0 coincides with the origin of the axis and that Γ and Σ are two faces of the tetrahedron whose common side coincides with the segment of endpoints the origin O and $A(a,0,0)$. Assume that Γ belongs to $z=0$ whereas Σ is contained in the plane $z-\sqrt{3}y=0$. In order to compute $\widetilde{\Theta}_2(0)$ we have to complete $\Sigma_r^{1,0}$ which has a defect of orthogonality around 0 using the set $\widetilde{\Sigma}_r^{1,0}$, and also Γ, which has a defect of tangentiality around 0 using the set $\widetilde{\Gamma}_r$ both introduced in Remark 4.13. In this case $\nu_\Gamma=(0,0,1)$, $\gamma=(0,1,0)$, $\tilde{\gamma}=(\frac{1}{2},\frac{\sqrt{3}}{2},0)$, $\nu_\Sigma=(0,\frac{\sqrt{3}}{2},-\frac{1}{2})$, $\sigma=(0,\frac{1}{2},\frac{\sqrt{3}}{2})$, $\tilde{\sigma}=\left(\frac{1}{2},\frac{\sqrt{3}}{4},\frac{3}{4}\right)$. By (32), we get that

$$\widetilde{I}_0=\widetilde{I}_0(\Sigma;\Gamma)=\frac{2D_2\widetilde{Q}_0^2}{(4\pi)^{3/2}}\int_{\widetilde{V}_0^2}\langle\nu_\Sigma,v\rangle\langle\nu_\Gamma,v\rangle e^{-\frac{|v|^2}{4}}\mathcal{H}^1(\widetilde{Q}_0^2)^{-1}(v)dv$$

where

$$\widetilde{Q}_0^2:\mathbb{R}_+^3\times\mathbb{R}\to\widetilde{V}_0^2,\ (\alpha,\beta)\mapsto\left(-\frac{1}{2}\alpha_2+\beta_1,-\frac{1}{2}\alpha_1-\frac{\sqrt{3}}{4}\alpha_2+\beta_2,-\frac{\sqrt{3}}{2}\alpha_1-\frac{3}{4}\alpha_2\right),$$

and $\widetilde{V}_0^2=\{v_3\leq 0\}\cap\{v_3-\sqrt{3}v_2\leq 0\}$. Analogously, by (33), we get

$$\widetilde{I}_1=\widetilde{I}_1(\Sigma;\Gamma)=\frac{2D_2\widehat{Q}_0^2}{(4\pi)^{3/2}}\int_{\widetilde{W}_0^2}\langle\nu_\Sigma,v\rangle\langle\nu_\Gamma,v\rangle e^{-\frac{|v|^2}{4}}\mathcal{H}^1(\widehat{Q}_0^2)^{-1}(v)dv,$$

where

$$\widehat{Q}_0^2 : \mathbb{R}^4_+ \to \widetilde{W}_0^2,\ (\alpha,\beta) \mapsto \left(-\alpha_1 - \frac{1}{2}\alpha_2 - \beta_1 + \frac{1}{2}\beta_2, -\frac{\sqrt{3}}{4}\alpha_2 + \frac{\sqrt{3}}{2}\beta_2, -\frac{3}{4}\alpha_2 \right),$$

and $\widetilde{W}_0^2 = \{v_2 \geq \max\{0, v_1\sqrt{3}\}, 3v_1 - v_2\sqrt{3} \leq v_3 \leq 0\} \cup \{v_1\frac{\sqrt{3}}{2} \leq v_2 \leq 0, 3v_1 - v_2\sqrt{3} \leq v_3 \leq \sqrt{3}v_2\}$. Finally, summing all the terms obtained, we get

$$\|T_t\chi_E\|^2_{L^2(\mathbb{R}^3)} = \frac{\sqrt{2}}{12}a^3 - a^2\sqrt{\frac{6t}{\pi}} - 12a\left(\frac{1}{\pi} + \frac{2}{\sqrt{3}}\right)t + 4\sqrt{2t^3}(\widetilde{I}_0 + \widetilde{I}_1) + o(t^h) \quad \forall\, h > 3/2.$$

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