RELAXATION OF p-GROWTH INTEGRAL FUNCTIONALS UNDER SPACE-DEPENDENT DIFFERENTIAL CONSTRAINTS

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Abstract. A representation formula for the relaxation of integral energies

$$(u,v)\mapsto\int_\Omega f(x,u(x),v(x))\,dx,$$

is obtained, where f satisfies p-growth assumptions, $1<p<+\infty$, and the fields v are subjected to space-dependent first order linear differential constraints in the framework of $\mathscr{A}$-quasiconvexity with variable coefficients.

1. Introduction

The analysis of constrained relaxation problems is a central question in materials science. Many applications in continuum mechanics and, in particular, in magnetoelasticity, rely on the characterization of minimizers of non-convex multiple integrals of the type

$$u\mapsto\int_\Omega f(x,u(x),\nabla u(x),\dots,\nabla^k u(x))\,dx$$

or

$$(u,v)\mapsto\int_\Omega f(x,u(x),v(x))\,dx, \tag{1.1}$$

where Ω is an open, bounded subset of $\mathbb{R}^N$, $u:\Omega\to\mathbb{R}^m$, $m\in\mathbb{N}$, and the fields $v:\Omega\to\mathbb{R}^d$, $d\in\mathbb{N}$, satisfy partial differential constraints of the type "$\mathscr{A}v=0$" other than $\operatorname{curl} v=0$ (see e.g. [5, 9]).

In this paper we provide a representation formula for the relaxation of non-convex integral energies of the form (1.1), in the case in which the energy density f satisfies p-growth assumptions, and the fields v are subjected to linear first-order space-dependent differential constraints.

The natural framework to study this family of relaxation problems is within the theory of $\mathscr{A}$-quasiconvexity with variable coefficients. In order to present this notion, we need to introduce some notation.

For $i=1\cdots,N$, let $A^i\in C^\infty(\mathbb{R}^N;\mathbb{M}^{l\times d})\cap W^{1,\infty}(\mathbb{R}^N;\mathbb{M}^{l\times d})$, let $1<p<+\infty$, and consider the differential operator

$$\mathscr{A}:L^p(\Omega;\mathbb{R}^d)\to W^{-1,p}(\Omega;\mathbb{R}^l),\quad d,l\in\mathbb{N},$$

defined as

$$\mathscr{A}v:=\sum_{i=1}^N A^i(x)\frac{\partial v(x)}{\partial x_i} \tag{1.2}$$

for every $v\in L^p(\Omega;\mathbb{R}^d)$, where (1.2) is to be interpreted in the sense of distributions. Assume that the symbol $\mathbb{A}:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{M}^{l\times d}$,

$$\mathbb{A}(x,w):=\sum_{i=1}^N A^i(x)w_i\quad\text{for }(x,w)\in\mathbb{R}^N\times\mathbb{R}^N,$$

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satisfies the uniform constant rank condition (see [22])

$$\text{rank } \mathbb{A}(x,w) = r \quad \text{for every } x \in \mathbb{R}^N \text{and } w \in \mathbb{S}^{n-1}. \tag{1.3}$$

Let Q be the unit cube in $\mathbb{R}^N$ with sides parallel to the coordinate axis, i.e.,

$$Q := \Big(-\frac{1}{2}, \frac{1}{2}\Big).$$

Denote by $C^\infty_{\rm per}(\mathbb{R}^N;\mathbb{R}^m)$ the set of $\mathbb{R}^m$-valued smooth maps that are Q-periodic in $\mathbb{R}^N$, and for every $x \in \Omega$ consider the set

$$\mathcal{C}_x := \Big\{ w \in C^\infty_{\rm per}(\mathbb{R}^N;\mathbb{R}^m) : \int_Q w(y)\,dy = 0, \text{ and } \sum_{i=1}^N A^i(x)\frac{\partial w(y)}{\partial y_i} = 0 \Big\}.$$

Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0,+\infty)$ be a Carathéodory function. The $\mathscr{A}-$quasiconvex envelope of $f(x,u,\cdot)$ for $x \in \Omega$ and $u \in \mathbb{R}^m$ is defined for $\xi \in \mathbb{R}^d$ as

$$Q_{\mathscr{A}(x)} f(x,u,\xi) := \inf\Big\{ \int_Q f(x,u,\xi + w(y))\,dy : \, w \in \mathcal{C}_x \Big\}.$$

We say that f is $\mathscr{A}-$quasiconvex if $f(x,u,\xi) = Q_{\mathscr{A}(x)} f(x,u,\xi)$ for a.e. $x \in \Omega$, and for all $u \in \mathbb{R}^m$ and $\xi \in \mathbb{R}^d$.

The notion of $\mathscr{A}$-quasiconvexity was first introduced by B. Dacorogna in [8], and extensively characterized in [17] by I. Fonseca and S. Müller for operators $\mathscr{A}$ defined as in (1.2), satisfying the constant rank condition (1.3), and having constant coefficients,

$$A^i(x) \equiv A^i \in \mathbb{M}^{l\times d} \quad \text{for every } x \in \mathbb{R}^N,\ i = 1,\dots,N.$$

In that paper the authors proved (see [17, Theorems 3.6 and 3.7]) that under p-growth assumptions on the energy density f, $\mathscr{A}$-quasiconvexity is necessary and sufficient for the lower-semicontinuity of integral functionals

$$I(u,v) := \int_\Omega f(x,u(x),v(x))\,dx \quad \text{for every } (u,v) \in L^p(\Omega;\mathbb{R}^m) \times L^p(\Omega;\mathbb{R}^d)$$

along sequences (u^n, v^n) satisfying $u^n \to u$ in measure, $v^n \rightharpoonup v$ in $L^p(\Omega;\mathbb{R}^d)$, and $\mathscr{A}v^n \to 0$ in $W^{-1,p}(\Omega)$. We remark that in the framework $\mathscr{A} = \text{curl}$, i.e., when $v^n = \nabla\phi^n$ for some $\phi^n \in W^{1,p}(\Omega;\mathbb{R}^m)$, $d = n \times m$, $\mathscr{A}$-quasiconvexity reduces to Morrey's notion of quasiconvexity.

The analysis of properties of $\mathscr{A}-$quasiconvexity for operators with constant coefficients was extended in the subsequent paper [6], where A. Braides, I. Fonseca and G. Leoni provided an integral representation formula for relaxation problems under p-growth assumptions on the energy density, and presented (via Γ-convergence) homogenization results for periodic integrands evaluated along $\mathscr{A}-$free fields. These homogenization results were later generalized in [13], where I. Fonseca and S. Krömer worked under weaker assumptions on the energy density f. In [19, 20], simultaneous homogenization and dimension reduction was studied in the framework of $\mathscr{A}$-quasiconvexity with constant coefficients. Oscillations and concentrations generated by $\mathscr{A}$-free mappings are the subject of [14]. Very recently an analysis of the case in which the energy density is nonpositive has been carried out in [18], and applications to the theory of compressible Euler systems have been studied in [7]. A parallel analysis for operators with constant coefficients and under linear growth assumptions for the energy density has been developed in [1, 4, 15, 21]. A very general characterization in this setting has been obtained in [2], following the new insight in [12].

The theory of $\mathscr{A}$-quasiconvexity for operators with variable coefficients has been characterized by P. Santos in [23]. Homogenization results in this setting have been obtained in [10] and [11].

This paper is devoted to proving a representation result for the relaxation of integral energies in the framework of $\mathscr{A}$-quasiconvexity with variable coefficients. To be precise, let $1 < p, q < +\infty$, $d, m, l \in \mathbb{N}$, and consider a Carathéodory function $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0, +\infty)$ satisfying

$$\text{(H)} \quad 0 \le f(x, u, v) \le C(1 + |u|^p + |v|^q), \quad 1 < p, q < +\infty,$$

for a.e. $x \in \Omega$, and all $(u, v) \in \mathbb{R}^m \times \mathbb{R}^d$, with $C > 0$.

Denoting by $\mathcal{O}(\Omega)$ the collection of open subsets of Ω, for every $D \in \mathcal{O}(\Omega)$, $u \in L^p(\Omega; \mathbb{R}^m)$ and $v \in L^q(\Omega; \mathbb{R}^d)$ with $\mathscr{A} v = 0$, we define

$$\begin{aligned} \mathcal{I}((u,v),D) := \inf \Big\{ \liminf_{n\to+\infty} \int_D f(x, u_n(x), v_n(x)) : \; u_n \to u \quad \text{strongly in } L^p(\Omega;\mathbb{R}^m), \\ v_n \rightharpoonup v \quad \text{weakly in } L^q(\Omega;\mathbb{R}^d) \text{ and } \mathscr{A} v_n \to 0 \quad \text{strongly in } W^{-1,q}(\Omega;\mathbb{R}^l) \Big\}. \end{aligned} \tag{1.4}$$

Our main result is the following.

Theorem 1.1. *Let $\mathscr{A}$ be a first order differential operator with variable coefficients, satisfying* (1.3)*. Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0, +\infty)$ be a Carathéodory function satisfying* (H)*. Then,*

$$\int_D Q_{\mathscr{A}(x)} f(x, u(x), v(x))\, dx = \mathcal{I}((u, v), D)$$

for all $D \in \mathcal{O}(\Omega), u \in L^p(\Omega; \mathbb{R}^m)$ and $v \in L^q(\Omega; \mathbb{R}^d)$ with $\mathscr{A} v = 0$.

Adopting the "blow-up" method introduced in [16], the proof of the theorem consists in showing that the functional $\mathcal{I}((u, v), \cdot)$ is the trace of a Radon measure absolutely continuous with respect to the restriction of the Lebesgue measure $\mathcal{L}^N$ to Ω, and proving that for a.e. $x \in \Omega$ the Radon-Nicodym derivative $\frac{d\mathcal{I}((u,v)\cdot)(x)}{d\mathcal{L}^N}$ coincides with the $\mathscr{A}-$quasiconvex envelope of f.

The arguments used are a combination of the ideas from [6, Theorem 1.1] and from [23]. The main difference with [6, Theorem 1.1], which reduces to our setting in the case in which the operator $\mathscr{A}$ has constant coefficients, is in the fact that while defining the operator $\mathcal{I}$ in (1.4) we can not work with exact solutions of the PDE, but instead we need to study sequences of asymptotically $\mathscr{A}-$vanishing fields. As pointed out in [23], in the case of variable coefficients the natural framework is the context of pseudo-differential operators. In this setting, we don't know how to project directly onto the kernel of the differential constraint, but we are able to construct an "approximate" projection operator P such that for every field $v \in L^p$, the $W^{-1,p}$ norm of $\mathscr{A} P v$ is controlled by the $W^{-1,p}$ norm of v itself (we refer to [23, Subsection 2.1] for a detailed explanation of this issue and to the references therein for a treatment of the main properties of pseudo-differential operators). For the same reason, in the proof of the inequality

$$\frac{d\mathcal{I}((u,v)\cdot)(x)}{d\mathcal{L}^N} \le Q_{\mathscr{A}(x)} f(x, u(x), v(x)) \quad \text{for a.e. } x \in \Omega,$$

an equi-integrability argument is needed (see Proposition 3.2). We also point out that the representation formula in Theorem 1.1 was obtained in a simplified setting in [11] as a corollary of the main homogenization result. Here we provide an alternative, direct proof, which does not rely on homogenization techniques.

The paper is organized as follows: in Section 2 we establish the main assumptions on the differential operator $\mathscr{A}$ and we recall some preliminary results on $\mathscr{A}-$quasiconvexity with variable coefficients. Section 3 is devoted to the proof of Theorem 1.1.

Notation

Throughout the paper $\Omega \subset \mathbb{R}^N$ is a bounded open set, $1 < p, q < +\infty$, $\mathcal{O}(\Omega)$ is the set of open subsets of Ω, Q denotes the unit cube in $\mathbb{R}^N$, $Q(x_0, r)$ and $B(x_0, r)$ are, respectively, the open cube and the open ball in $\mathbb{R}^N$, with center x_0 and radius r. Given an exponent $1 < q < +\infty$, we denote by q' its conjugate exponent, i.e., $q' \in (1, +\infty)$ is such that

$$\frac{1}{q} + \frac{1}{q'} = 1.$$

Whenever a map $v \in L^q, C^\infty, \cdots$ is $Q-$periodic, that is

$$v(x + e_i) = v(x) \quad i = 1, \cdots, N,$$

for a.e. $x \in \mathbb{R}^N$, $\{e_1, \cdots, e_N\}$ being the standard basis of $\mathbb{R}^N$, we write $v \in L^q_{\rm per}, C^\infty_{\rm per}, \ldots$ We implicitly identify the spaces $L^q(Q)$ and $L^q_{\rm per}(\mathbb{R}^N)$.

We adopt the convention that C will denote a generic constant, whose value may change from line to line in the same formula.

2. Preliminary results

In this section we introduce the main assumptions on the differential operator $\mathscr{A}$ and we recall some preliminary results about $\mathscr{A}-$quasiconvexity.

For $i = 1, \cdots, N$, $x \in \mathbb{R}^N$, consider the linear operators $A^i(x) \in \mathbb{M}^{l\times d}$, with $A^i \in C^\infty(\mathbb{R}^N; \mathbb{M}^{l\times d}) \cap W^{1,\infty}(\mathbb{R}^N; \mathbb{M}^{l\times d})$. For every $v \in L^q(\Omega; \mathbb{R}^d)$ we set

$$\mathscr{A} v := \sum_{i=1}^N A^i(x) \frac{\partial v(x)}{\partial x_i} \in W^{-1,q}(\Omega; \mathbb{R}^l).$$

The symbol $\mathbb{A} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{0\} \to \mathbb{M}^{l\times d}$ associated to the differential operator $\mathscr{A}$ is

$$\mathbb{A}(x, \lambda) := \sum_{i=1}^N A^i(x)\lambda_i \in \mathbb{M}^{l\times d}$$

for every $x \in \mathbb{R}^N$, $\lambda \in \mathbb{R}^N \setminus \{0\}$. We assume that $\mathscr{A}$ satisfies the following *uniform constant rank condition*:

$$\text{rank}\,\Big(\sum_{i=1}^N A^i(x)\lambda_i\Big) = r \quad \text{for all } x \in \mathbb{R}^N \text{ and } \lambda \in \mathbb{R}^N \setminus \{0\}. \tag{2.1}$$

For every $x \in \mathbb{R}^N$, $\lambda \in \mathbb{R}^N \setminus \{0\}$, let $\mathbb{P}(x, \lambda) : \mathbb{R}^d \to \mathbb{R}^d$ be the linear projection on Ker $\mathbb{A}(x, \lambda)$, and let $\mathbb{Q}(x, \lambda) : \mathbb{R}^l \to \mathbb{R}^d$ be the linear operator given by

$$\begin{aligned} &\mathbb{Q}(x, \lambda)\mathbb{A}(x, \lambda)v := v - \mathbb{P}(x, \lambda)v \quad \text{for all } v \in \mathbb{R}^d, \\ &\mathbb{Q}(x, \lambda)\xi = 0 \quad \text{if } \xi \notin \text{Range } \mathbb{A}(x, \lambda). \end{aligned}$$

The main properties of $\mathbb{P}(\cdot, \cdot)$ and $\mathbb{Q}(\cdot, \cdot)$ are recalled in the following proposition (see *e.g.* [23, Subsection 2.1]).

Proposition 2.1. *Under the constant rank condition* (2.1)*, for every* $x \in \mathbb{R}^N$ *the operators* $\mathbb{P}(x, \cdot)$ *and* $\mathbb{Q}(x, \cdot)$ *are, respectively,* 0*-homogeneous and* (-1)*-homogeneous. In addition,* $\mathbb{P} \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N \setminus \{0\}; \mathbb{M}^{d\times d})$ *and* $\mathbb{Q} \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N \setminus \{0\}; \mathbb{M}^{d\times l})$.

Let $\eta \in C^\infty_c(\Omega; [0, 1])$, $\eta = 1$ in Ω' for some $\Omega' \subset\subset \Omega$. We denote by $\mathbb{A}_\eta$ the symbol

$$\mathbb{A}_\eta(x, \lambda) := \sum_{i=1}^N \eta(x)A^i(x)\lambda_i, \tag{2.2}$$

for every $x \in \mathbb{R}^N$, $\lambda \in \mathbb{R}^N \setminus \{0\}$, and by $\mathscr{A}_\eta$ the corresponding pseudo-differential operator (see [23, Subsection 2.1] for an overview of the main properties of pseudo-differential operators). Let $\chi \in C^\infty(\mathbb{R}^+; \mathbb{R})$ be such that $\chi(|\lambda|) = 0$ for $|\lambda| < 1$ and $\chi(|\lambda|) = 1$ for $|\lambda| > 2$. Let also P_η be the operator associated to the symbol

$$\mathbb{P}_\eta(x, \lambda) := \eta^2(x)\mathbb{P}(x, \lambda)\chi(|\lambda|) \tag{2.3}$$

for every $x \in \mathbb{R}^N$, $\lambda \in \mathbb{R}^N \setminus \{0\}$. The following proposition (see [23, Theorem 2.2 and Subsection 2.1]) collects the main properties of the operators P_η and $\mathscr{A}_\eta$.

Proposition 2.2. *Let $1 < q < +\infty$, and let $\mathscr{A}_\eta$ and P_η be the pseudo-differential operators associated with the symbols (2.2) and (2.3), respectively. Then there exists a constant C such that*

$$\|P_\eta v\|_{L^q(\Omega;\mathbb{R}^d)} \leq C\|v\|_{L^q(\Omega;\mathbb{R}^d)} \tag{2.4}$$

for every $v \in L^q(\Omega;\mathbb{R}^d)$, and

$$\begin{aligned}&\|P_\eta v\|_{W^{-1,q}(\Omega;\mathbb{R}^d)} \leq C\|v\|_{W^{-1,q}(\Omega;\mathbb{R}^d)},\\ &\|v - P_\eta v\|_{L^q(\Omega;\mathbb{R}^d)} \leq C\big(\|\mathscr{A}_\eta v\|_{W^{-1,q}(\Omega;\mathbb{R}^l)} + \|v\|_{W^{-1,q}(\Omega;\mathbb{R}^d)}\big),\\ &\|\mathscr{A}_\eta P_\eta v\|_{W^{-1,q}(\Omega;\mathbb{R}^l)} \leq C\|v\|_{W^{-1,q}(\Omega;\mathbb{R}^d)}\end{aligned}$$

for every $v \in W^{-1,q}(\Omega;\mathbb{R}^d)$.

3. Proof of Theorem 1.1

Before proving Theorem 1.1 we state and prove a decomposition lemma, which generalizes [17, Lemma 2.15] to the case of operators with variable coefficients.

Lemma 3.1. *Let $1 < q < +\infty$. Let $\mathscr{A}$ be a first order differential operator with variable coefficients, satisfying (2.1). Let $v \in L^q(\Omega;\mathbb{R}^d)$, and let $\{v_n\}$ be a bounded sequence in $L^q(\Omega;\mathbb{R}^d)$ such that*

$$\begin{aligned}&v_n \rightharpoonup v \quad \text{weakly in } L^q(\Omega;\mathbb{R}^d),\\ &\mathscr{A} v_n \to 0 \quad \text{strongly in } W^{-1,q}(\Omega;\mathbb{R}^l),\\ &\{v_n\} \text{ generates the Young measure } \nu.\end{aligned}$$

Then, there exists a q-equiintegrable sequence $\{\tilde{v}_n\} \subset L^q(\Omega;\mathbb{R}^d)$ such that

$$\mathscr{A}\tilde{v}_n \to 0 \quad \text{strongly in } W^{-1,s}(\Omega;\mathbb{R}^l) \quad \text{for every } 1 < s < q, \tag{3.1}$$

$$\int_\Omega \tilde{v}_n(x)\,dx = \int_\Omega v(x)\,dx,$$

$$\tilde{v}_n - v_n \to 0 \quad \text{strongly in } L^s(\Omega;\mathbb{R}^d) \quad \text{for every } 1 < s < q, \tag{3.2}$$

$$\tilde{v}_n \rightharpoonup v \quad \text{weakly in } L^q(\Omega;\mathbb{R}^d). \tag{3.3}$$

In addition, if $\Omega \subset Q$ then we can construct the sequence $\{\tilde{v}^n\}$ so that $\tilde{v}_n - v \in L^q_{per}(\mathbb{R}^N;\mathbb{R}^d)$ for every $n \in \mathbb{N}$.

Proof. Arguing as in the first part of [23, Proof of Theorem 1.1], we construct a q-equiintegrable sequence $\{\hat{v}_n\}$ satisfying (3.1), (3.2) and (3.3). The conclusion follows by setting $\tilde{v}_n := \hat{v}_n - \int_\Omega \hat{v}_n(x)\,dx + \int_\Omega v(x)\,dx$.

In the case in which $\Omega \subset Q$, let $\{\varphi^i\}$ be a sequence of cut-off functions in Q with $0 \leq \varphi^i \leq 1$ in Q, such that $\varphi^i = 0$ on $Q \setminus \Omega$ and $\varphi^i \to 1$ pointwise in Ω. Define $w^i_n := \varphi^i(\hat{v}_n - v)$. By (3.3) for every $\psi \in L^{q'}(\Omega;\mathbb{R}^d)$ we have

$$\lim_{i\to+\infty}\lim_{n\to+\infty}\int_\Omega w^i_n(x)\psi(x)\,dx = 0.$$

By (3.1), (3.2), and the compact embedding of $L^q(\Omega;\mathbb{R}^d)$ into $W^{-1,q}(\Omega;\mathbb{R}^d)$, there holds

$$\mathscr{A} w^i_n = \varphi^i \mathscr{A}\hat{v}_n + \left(\sum_{j=1}^N A^j \frac{\partial \varphi^i}{\partial x_j}\right)\hat{v}_n \to 0 \quad \text{strongly in } W^{-1,s}(\Omega;\mathbb{R}^l)$$

as $n \to +\infty$, for every $1 < s < q$. Extending the maps w^i_n outside Q by periodicity, by the metrizability of the weak topology on bounded sets and by Attouch's diagonalization lemma (see [3, Lemma 1.15 and Corollary 1.16]), we obtain a sequence

$$w_n := w^{i(n)}_n,$$

with $\{w_n\} \subset L^q_{\rm per}(\mathbb{R}^N;\mathbb{R}^d)$, and such that $w_n + v$ satisfies (3.1), (3.2) and (3.3). The thesis follows by setting

$$\tilde{v}_n := w_n - \int_\Omega w_n(x)\,dx + v.$$

□

The following proposition will allow us to neglect vanishing perturbations of q-equiintegrable sequences.

Proposition 3.2. *For every $n \in \mathbb{N}$, let $f_n : Q \times \mathbb{R}^d \to [0,+\infty)$ be a continuous function. Assume that there exists a constant $C > 0$ such that, for $q > 1$,*

$$\sup_{n\in\mathbb{N}} f_n(y,\xi) \le C(1+|\xi|^q) \quad \textit{for every } y \in Q \textit{ and } \xi \in \mathbb{R}^d, \tag{3.4}$$

and that the sequence $\{f_n(y,\cdot)\}$ is equicontinuous in $\mathbb{R}^d$, uniformly in y. Let $\{w_n\}$ be a q-equiintegrable sequence in $L^q(Q;\mathbb{R}^d)$, and let $\{v_n\} \subset L^q(Q;\mathbb{R}^d)$ be such that

$$v_n \to 0 \quad \textit{strongly in } L^q(Q;\mathbb{R}^d). \tag{3.5}$$

Then

$$\lim_{n\to+\infty} \Big| \int_Q f_n\big(y, w_n(y)\big)\,dy - \int_Q f_n\big(y, v_n(y) + w_n(y)\big)\,dy \Big| = 0.$$

Proof. Fix $\eta > 0$. In view of (3.5), the sequence $\{C(1+|v_n|^q + |w_n|^q)\}$ is equiintegrable in Q, thus there exists $0 < \varepsilon < \frac{\eta}{3}$ such that

$$\sup_{n\in\mathbb{N}} \int_A C\big(1+|v_n(y)|^q + |w_n(y)|^q\big)\,dy < \frac{\eta}{3} \tag{3.6}$$

for every $A \subset Q$ with $|A| < \varepsilon$. By the q-equiintegrability of $\{w_n\}$ and $\{v_n\}$, and by Chebyshev's inequality there holds

$$\big|Q \cap \big(\{|w_n| > M\} \cup \{|v_n| > M\}\big)\big| \le \frac{1}{M^q} \int_Q (|w_n(y)|^q + |v_n(y)|^q)\,dy \le \frac{C}{M^q}$$

for every $n \in \mathbb{N}$. Therefore, there exists M_0 satisfying

$$\sup_{n\in\mathbb{N}} \big|Q \cap \big(\{|w_n| > M_0\} \cup \{|v_n| > M_0\}\big)\big| \le \frac{\varepsilon}{2}. \tag{3.7}$$

By the uniform equicontinuity of the sequence $\{f_n(y,\cdot)\}$, there exists $\delta > 0$ such that, for every $\xi_1, \xi_2 \in \overline{B(0,M_0)}$, with $|\xi_1 - \xi_2| < \delta$, we have

$$\sup_{y\in Q} |f_n(y,\xi_1) - f_n(y,\xi_2)| < \varepsilon \tag{3.8}$$

for every $n \in \mathbb{N}$. By (3.5) and Egoroff's theorem, there exists a set $E_\varepsilon \subset Q$, $|E_\varepsilon| < \frac{\varepsilon}{2}$, such that

$$v_n \to 0 \quad \text{uniformly in } Q \setminus E_\varepsilon,$$

and, in particular,

$$|v_n(x)| < \delta \quad \text{for a.e. } x \in Q \setminus E_\varepsilon, \tag{3.9}$$

for every $n \ge n_0$, for some $n_0 \in \mathbb{N}$.

We observe that

$$\int_Q f_n(y, v_n(y) + w_n(y))\,dy = \int_{Q\cap\{|w_n|\le M_0\}\cap\{|v_n|\le M_0\}} f_n(y, v_n(y) + w_n(y))\,dy \tag{3.10}$$
$$+ \int_{Q\cap(\{|w_n|>M_0\}\cup\{|v_n|>M_0\})} f_n(y, v_n(y) + w_n(y))\,dy.$$

The first term in the right-hand side of (3.10) can be further decomposed as

$$\begin{aligned}
&\int_{Q\cap\{|w_n|\le M_0\}\cap\{|v_n|\le M_0\}} f_n(y,v_n(y)+w_n(y))\,dy\\
&=\int_{(Q\setminus E_\varepsilon)\cap\{|w_n|\le M_0\}\cap\{|v_n|\le M_0\}} f_n(y,v_n(y)+w_n(y))\,dy\\
&\quad+\int_{E_\varepsilon\cap\{|w_n|\le M_0\}\cap\{|v_n|\le M_0\}} f_n(y,v_n(y)+w_n(y))\,dy\\
&=\int_{(Q\setminus E_\varepsilon)\cap\{|w_n|\le M_0\}\cap\{|v_n|\le M_0\}} f_n(y,w_n(y))\,dy\\
&\quad+\int_{(Q\setminus E_\varepsilon)\cap\{|w_n|\le M_0\}\cap\{|v_n|\le M_0\}} \big(f_n(y,v_n(y)+w_n(y))-f_n(y,w_n(y))\big)\,dy\\
&\quad+\int_{E_\varepsilon\cap\{|w_n|\le M_0\}\cap\{|v_n|\le M_0\}} f_n(y,v_n(y)+w_n(y))\,dy\\
&=\int_Q f_n(y,w_n(y))\,dy-\int_{E_\varepsilon\cap\{|w_n|\le M_0\}\cap\{|v_n|\le M_0\}} f_n(y,w_n(y))\,dy\\
&\quad-\int_{Q\cap(\{|w_n|>M_0\}\cup\{|v_n|>M_0\})} f_n(y,w_n(y))\,dy\\
&\quad+\int_{(Q\setminus E_\varepsilon)\cap\{|w_n|\le M_0\}\cap\{|v_n|\le M_0\}} \big(f_n(y,v_n(y)+w_n(y))-f_n(y,w_n(y))\big)\,dy\\
&\quad+\int_{E_\varepsilon\cap\{|w_n|\le M_0\}\cap\{|v_n|\le M_0\}} f_n(y,v_n(y)+w_n(y))\,dy.
\end{aligned}$$

We observe that by (3.7)

$$|E_\varepsilon\cup(\{|w_n|>M_0\}\cup\{|v_n|>M_0\})|<\varepsilon.$$

Hence, for $n\ge n_0$, by (3.4), (3.6), (3.8), and (3.9) we deduce the estimate

$$\begin{aligned}
&\Big|\int_Q f_n(y,w_n(y))\,dy-\int_Q f_n(y,v_n(y)+w_n(y))\,dy\Big| \\
&\le \varepsilon+\int_{E_\varepsilon\cup(\{|w_n|>M_0\}\cup\{|v_n|>M_0\})} 2C(1+|w_n(y)|^p+|v_n(y)|^p)\,dy\le\varepsilon+\frac{2\eta}{3}.
\end{aligned}\tag{3.11}$$

The thesis follows by the arbitrariness of η. □

We now prove our main result.

Proof of Theorem 1.1. The proof is subdivided into 4 steps. Steps 1 and 2 follow along the lines of [6, Proof of Theorem 1.1]. Step 3 is obtained by modifying [6, Lemma 3.5], whereas Step 4 follows by adapting an argument in [23, Proof of Theorem 1.2]. We only outline the main ideas of Steps 1 and 2 for convenience of the reader, whilst we provide more details for Steps 3 and 4.

Step 1:

The first step consists in showing that

$$\begin{aligned}
\mathcal{I}((u,v),D)=\inf\Big\{\liminf_{n\to+\infty}\int_D f(x,u(x),v_n(x))\,dx:\ &\{v_n\}\text{ is } q-\text{equiintegrable },\\
&\mathscr{A}v_n\to 0\text{ strongly in } W^{-1,s}(D;\mathbb{R}^l)\text{ for every } 1<s<q\\
&\text{and } v_n\rightharpoonup v\text{ weakly in } L^q(D;\mathbb{R}^d)\Big\}.
\end{aligned}$$

This identification is proved by adapting [6, Proof of Lemma 3.1]. The only difference is the application of Lemma 3.1 instead of [6, Proposition 2.3 (i)].

Step 2:

The second step is the proof that $\mathcal{I}((u,v),\cdot)$ is the trace of a Radon measure absolutely continuous with respect to $\mathcal{L}^N\lfloor\Omega$. This follows as a straightforward adaptation of [6, Lemma 3.4]. The only modifications are due to the fact that [6, Proposition 2.3 (i)] and [6, Lemma 3.1] are now replaced by Lemma 3.1 and Step 1.

Step 3:

We claim that

$$\frac{d\mathcal{I}((u,v),\cdot)}{d\mathcal{L}^N}(x_0) \geq Q_{\mathscr{A}(x_0)}f(x_0,u(x_0),v(x_0)) \quad \text{for a.e. } x_0 \in \Omega. \tag{3.12}$$

Indeed, since $g(x,\xi) := f(x,u(x),\xi)$ is a Carathéodory function, by Scorza-Dragoni Theorem there exists a sequence of compact sets $K_j \subset \Omega$ such that

$$|\Omega \setminus K_j| \leq \tfrac{1}{j}$$

and the restriction of g to $K_j \times \mathbb{R}^d$ is continuous. Hence, the set

$$\omega := \bigcup_{j=1}^{+\infty}(K_j \cap K_j^*) \cap \mathcal{L}(u,v), \tag{3.13}$$

where K_j^* is the set of Lebesgue point for the characteristic function of K_j and $\mathcal{L}(u,v)$ is the set of Lebesgue points of u and v, is such that

$$|\Omega \setminus \omega| \leq |\Omega \setminus K_j| \leq \frac{1}{j} \quad \text{for every } j,$$

and so $|\Omega \setminus \omega| = 0$. Let $x_0 \in \omega$ be such that

$$\lim_{r\to 0^+}\frac{1}{r^N}\int_{Q(x_0,r)}|u(x)-u(x_0)|^p\,dx = \lim_{r\to 0^+}\frac{1}{r^N}\int_{Q(x_0,r)}|v(x)-v(x_0)|^q\,dx = 0, \tag{3.14}$$

and

$$\frac{d\mathcal{I}((u,v),\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{r\to 0^+}\frac{\mathcal{I}((u,v),Q(x_0,r))}{r^N} < +\infty, \tag{3.15}$$

where the sequence of radii r is such that $\mathcal{I}((u,v),\partial Q(x_0,r)) = 0$ for every r. (Such a choice of the sequence is possible due to Step 2).

By Step 1, for every r there exists a $q-$equiintegrable sequence $\{v_{n,r}\}$ such that

$$\begin{aligned} &v_{n,r} \rightharpoonup v \quad \text{weakly in } L^q(Q(x_0,r);\mathbb{R}^d),\\ &\mathscr{A}v_{n,r} \to 0 \quad \text{strongly in } W^{-1,s}(Q(x_0,r);\mathbb{R}^l) \text{ for every } 1<s<q \end{aligned} \tag{3.16}$$

as $n \to +\infty$, and

$$\lim_{n\to+\infty}\int_{Q(x_0,r)} g(x,v_{n,r}(x))\,dx \leq \mathcal{I}((u,v),Q(x_0,r)) + r^{N+1}.$$

A change of variables yields

$$\frac{d\mathcal{I}((u,v),\cdot)}{d\mathcal{L}^N}(x_0) \geq \liminf_{r\to 0^+}\lim_{n\to+\infty}\int_Q g(x_0+ry, v(x_0)+w_{n,r}(y))\,dy,$$

where

$$w_{n,r}(y) := v_{n,r}(x_0+ry) - v(x_0) \quad \text{for a.e. } y \in Q.$$

Arguing as in [6, Proof of Lemma 3.5], Hölder's inequality and a change of variables imply

$$w_{n,r} \rightharpoonup 0 \quad \text{weakly in } L^q(Q;\mathbb{R}^d) \tag{3.17}$$

as $n \to +\infty$ and $r \to 0^+$, in this order. We claim that

$$\mathscr{A}(x_0 + r\cdot)w_{n,r} \to 0 \quad \text{strongly in } W^{-1,s}(Q;\mathbb{R}^l), \tag{3.18}$$

as $n \to +\infty$, for every r and every $1 < s < q$.

Indeed, let $\varphi \in W^{1,s'}_0(Q;\mathbb{R}^d)$. There holds

$$\langle \mathscr{A}(x_0+r\cdot)w_{n,r},\,\varphi\rangle_{W^{-1,s}(Q;\mathbb{R}^l),W^{1,s'}_0(Q;\mathbb{R}^l)} = -\sum_{i=1}^N \Bigg\{ r\int_Q \frac{\partial A^i(x_0+ry)}{\partial x_i} v_{n,r}(x_0+ry)\cdot\varphi(y)\,dy + \int_Q A^i(x_0+ry)v_{n,r}(x_0+ry)\cdot\,\frac{\partial\varphi(y)}{\partial y_i}\,dy\Bigg\}$$

$$= -\sum_{i=1}^N \Bigg\{ \frac{1}{r^{N-1}}\int_{Q(x_0,r)} \frac{\partial A^i(x)}{\partial x_i} v_{n,r}(x)\cdot\psi_r(x)\,dx + \frac{1}{r^{N-1}}\int_{Q(x_0,r)} A^i(x)v_{n,r}(x)\cdot\frac{\partial\psi_r(x)}{\partial x_i}\,dx\Bigg\}$$

$$= \frac{1}{r^{N-1}}\langle \mathscr{A}v_{n,r},\,\psi_r\rangle_{W^{-1,s}(Q(x_0,r);\mathbb{R}^l),W^{1,s'}_0(Q(x_0,r);\mathbb{R}^l)},$$

where $\psi_r(x) := \varphi\big(\frac{x-x_0}{r}\big)$ for a.e. $x\in Q(x_0,r)$. Since $\psi_r\in W^{1,s'}_0(Q(x_0,r);\mathbb{R}^d)$ and

$$\|\psi_r\|_{W^{1,s'}_0(Q(x_0,r);\mathbb{R}^d)} \le C(r)\|\varphi\|_{W^{1,s'}_0(Q;\mathbb{R}^d)},$$

we obtain the estimate

$$\|\mathscr{A}(x_0+r\cdot)w_{n,r}\|_{W^{-1,s}(Q;\mathbb{R}^l)} \le C(r)\|\mathscr{A}v_{n,r}\|_{W^{-1,s}(Q(x_0,r);\mathbb{R}^l)}.$$

Claim (3.18) follows by (3.16).

In view of (3.17) and (3.18), a diagonalization procedure yields a $q-$equiintegrable sequence $\{\hat{w}_k\}\subset L^q(Q;\mathbb{R}^d)$ satisfying

$$\hat{w}_k \rightharpoonup 0 \quad \text{weakly in } L^q(Q;\mathbb{R}^d), \tag{3.19}$$

$$\mathscr{A}(x_0+r_k\cdot)\hat{w}_k \to 0 \quad \text{strongly in } W^{-1,s}(Q;\mathbb{R}^l) \quad \text{for every } 1<s<q, \tag{3.20}$$

and

$$\frac{d\mathcal{I}((u,v),\cdot)}{d\mathcal{L}^N}(x_0) \ge \liminf_{k\to+\infty}\int_Q g(x_0+r_ky, v(x_0)+\hat{w}_k(y))\,dy. \tag{3.21}$$

For every $\varphi\in W^{1,s'}_0(Q;\mathbb{R}^l)$, $1<s<q$, there holds

$$\langle(\mathscr{A}(x_0+r_k\cdot)-\mathscr{A}(x_0))\hat{w}_k,\,\varphi\rangle_{W^{-1,s}(Q;\mathbb{R}^l),W^{1,s'}_0(Q;\mathbb{R}^l)}$$
$$= -\sum_{i=1}^N\Bigg[r_k\int_Q\frac{\partial A^i(x_0+r_ky)}{\partial x_i}\hat{w}_k(y)\cdot\varphi(y)\,dy + \int_Q (A^i(x_0+r_ky)-A^i(x_0))\hat{w}_k(y)\cdot\frac{\partial\varphi(y)}{\partial y_i}\,dy\Bigg].$$

Thus,

$$\|(\mathscr{A}(x_0+r_k\cdot)-\mathscr{A}(x_0))\hat{w}_k\|_{W^{-1,s}(Q;\mathbb{R}^l)} \le r_k\sum_{i=1}^N\|A^i\|_{W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^{l\times d})}\|\hat{w}_k\|_{L^q(Q;\mathbb{R}^d)}$$

for every $1<s<q$. By (3.19) and (3.20) we conclude that

$$\mathscr{A}(x_0)\hat{w}_k \to 0 \quad \text{strongly in } W^{-1,s}(Q;\mathbb{R}^l) \quad \text{for every } 1<s<q. \tag{3.22}$$

In view of (3.19) and (3.22), an adaptation of [6, Corollary 3.3] yields a $q-$equiintegrable sequence $\{w_k\}$ such that

$$w_k \rightharpoonup 0 \quad \text{weakly in } L^q(Q;\mathbb{R}^d),$$
$$\int_Q w_k(y)\,dy = 0 \quad \text{for every } k,$$
$$\mathscr{A}(x_0)w_k = 0 \quad \text{for every } k, \tag{3.23}$$

and

$$\liminf_{k\to+\infty}\int_Q g(x_0, v(x_0)+w_k(y))\,dy \le \liminf_{k\to+\infty}\int_Q g(x_0+r_ky, v(x_0)+\hat{w}_k(y))\,dy. \tag{3.24}$$

Finally, by combining (3.21), (3.23), and (3.24), and by the definition of $\mathscr{A}$-quasiconvex envelope for operators with constant coefficients, we obtain

$$\begin{aligned}\frac{d\mathcal{I}((u,v),\cdot)}{d\mathcal{L}^N}(x_0) &\geq \liminf_{k\to+\infty}\int_Q g(x_0, v(x_0)+w_k(y))\,dy\\ &= \liminf_{k\to+\infty}\int_Q f(x_0,u(x_0),v(x_0)+w_k(y))\,dy \geq Q_{\mathscr{A}(x_0)}f(x_0,u(x_0),v(x_0))\end{aligned}$$

for a.e. $x_0\in\Omega$. This concludes the proof of Claim (3.12).
Step 4:
To complete the proof of the theorem we need to show that

$$\frac{d\mathcal{I}((u,v),\cdot)}{d\mathcal{L}^N}(x_0) \leq Q_{\mathscr{A}(x_0)}f(x_0,u(x_0),v(x_0)) \quad \text{for a.e. } x_0\in\Omega. \tag{3.25}$$

To this aim, let $\mu>0$, and $x_0\in\omega$ be such that (3.14) and (3.15) hold. Let $w\in C^\infty_{\rm per}(\mathbb{R}^N;\mathbb{R}^d)$ be such that

$$\int_Q w(y)\,dy = 0, \quad \mathscr{A}(x_0)w=0, \tag{3.26}$$

and

$$\int_Q f(x_0,u(x_0),v(x_0)+w(y))\,dy \leq Q_{\mathscr{A}(x_0)}f(x_0,u(x_0),v(x_0))+\mu. \tag{3.27}$$

Let $\eta\in C^\infty_c(\Omega;[0,1])$ be such that $\eta\equiv 1$ in a neighborhood of x_0 and let r be small enough so that

$$Q(x_0,r)\subset\{x:\eta(x)=1\} \quad \text{and } Q(x_0,2r)\subset\subset\Omega. \tag{3.28}$$

Consider a map $\varphi\in C^\infty_c(Q(x_0,r);[0,1])$ satisfying

$$\mathcal{L}^N(Q(x_0,r)\cap\{\varphi\neq 1\})<\mu r^N, \tag{3.29}$$

and define

$$z^r_m(x) := \varphi(x)w\Big(\frac{m(x-x_0)}{r}\Big) \quad \text{for } x\in\mathbb{R}^N. \tag{3.30}$$

We observe that $z^r_m\in L^q(\Omega;\mathbb{R}^d)$, and for $\psi\in L^{q'}(\Omega;\mathbb{R}^d)$ we have

$$\begin{aligned}\int_\Omega z^r_m(x)\cdot\psi(x)\,dx &= \int_\Omega \varphi(x)w\Big(\frac{m(x-x_0)}{r}\Big)\cdot\psi(x)\,dx\\ &= r^N\int_Q \varphi(x_0+ry)w(my)\cdot\psi(x_0+ry)\,dy.\end{aligned}$$

By (3.26) and by the Riemann-Lebesgue lemma we have

$$z^r_m \rightharpoonup 0 \quad \text{weakly in } L^q(\Omega;\mathbb{R}^d) \tag{3.31}$$

as $m\to+\infty$. We claim that

$$\limsup_{m\to+\infty}\|\mathscr{A}_\eta z^r_m\|_{W^{-1,q}(\Omega;\mathbb{R}^l)} \leq Cr^{\frac{N}{q}+1}, \tag{3.32}$$

where $\mathscr{A}_\eta$ is the pseudo-differential operator defined in (2.2). Indeed, by (3.28) we obtain

$$\begin{aligned}\mathscr{A}_\eta z^r_m &= \mathscr{A}z^r_m - \mathscr{A}(x_0)z^r_m + \mathscr{A}(x_0)z^r_m\\ &= \sum_{i=1}^N \frac{\partial((A^i(x)-A^i(x_0))z^r_m(x))}{\partial x_i} + \sum_{i=1}^N A^i(x_0)\frac{\partial z^r_m(x)}{\partial x_i} - \sum_{i=1}^N \frac{\partial A^i(x)}{\partial x_i}z^r_m(x).\end{aligned} \tag{3.33}$$

By the regularity of the operators A^i and by a change of variables, the first term in the right-hand side of (3.33) is estimated as

$$\begin{aligned}&\Big\|\sum_{i=1}^N \frac{\partial((A^i(x)-A^i(x_0))z_m^r(x))}{\partial x_i}\Big\|_{W^{-1,q}(\Omega;\mathbb{R}^l)} \\ &\leq \sum_{i=1}^N \Big\|(A^i(x)-A^i(x_0))\varphi(x)w\Big(\frac{m(x-x_0)}{r}\Big)\Big\|_{L^q(Q(x_0,r);\mathbb{R}^l)} \\ &\leq \sum_{i=1}^N \|A^i\|_{W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^{l\times d})}\|\varphi\|_{L^\infty(Q(x_0,r))}\|w(m\cdot)\|_{L^q(Q;\mathbb{R}^d)}r^{\frac{N}{q}+1} \leq Cr^{\frac{N}{q}+1}.\end{aligned} \tag{3.34}$$

In view of (3.26) the second term in the right-hand side of (3.33) becomes

$$\sum_{i=1}^N A^i(x_0)\frac{\partial z_m^r(x)}{\partial x_i} = \sum_{i=1}^N A^i(x_0)\frac{\partial \varphi(x)}{\partial x_i}w\Big(\frac{m(x-x_0)}{r}\Big),$$

and thus converges to zero weakly in $L^q(\Omega;\mathbb{R}^l)$, as $m\to+\infty$, due to (3.26) and by the Riemann-Lebesgue lemma. Hence,

$$\Big\|\sum_{i=1}^N A^i(x_0)\frac{\partial z_m^r(x)}{\partial x_i}\Big\|_{W^{-1,q}(\Omega;\mathbb{R}^l)}\to 0 \quad \text{as } m\to+\infty \tag{3.35}$$

by the compact embedding of $L^q(\Omega;\mathbb{R}^l)$ into $W^{-1,q}(\Omega;\mathbb{R}^l)$. Finally, the third term in the right-hand side of (3.33) satisfies

$$\sum_{i=1}^N \frac{\partial A^i(x)}{\partial x_i}z_m^r(x) = \sum_{i=1}^N \frac{\partial A^i(x)}{\partial x_i}\varphi(x)w\Big(\frac{m(x-x_0)}{r}\Big),$$

which again converges to zero weakly in $L^q(\Omega;\mathbb{R}^l)$, as $m\to+\infty$, owing again to (3.26) and the Riemann-Lebesgue lemma. Therefore,

$$\Big\|\sum_{i=1}^N \frac{\partial A^i(x)}{\partial x_i}z_m^r(x)\Big\|_{W^{-1,q}(\Omega;\mathbb{R}^l)}\to 0 \quad \text{as } m\to+\infty. \tag{3.36}$$

Claim (3.32) follows by combining (3.34)–(3.36).

Consider the maps

$$v_m^r := P_\eta z_m^r,$$

where P_η is the projection operator introduced in (2.3). By Proposition 2.2 we have

$$\|v_m^r\|_{L^q(Q(x_0,r);\mathbb{R}^d)} \leq C\|z_m^r\|_{L^q(\Omega;\mathbb{R}^d)}, \tag{3.37}$$

$$\|v_m^r\|_{W^{-1,q}(Q(x_0,r);\mathbb{R}^d)} \leq C\|z_m^r\|_{W^{-1,q}(\Omega;\mathbb{R}^d)}, \tag{3.38}$$

$$\|\mathscr{A}_\eta v_m^r\|_{W^{-1,q}(Q(x_0,r);\mathbb{R}^l)} \leq C\|z_m^r\|_{W^{-1,q}(\Omega;\mathbb{R}^d)}, \tag{3.39}$$

$$\|v_m^r - z_m^r\|_{L^q(Q(x_0,r);\mathbb{R}^d)} \leq C(\|\mathscr{A}_\eta z_m^r\|_{W^{-1,q}(\Omega;\mathbb{R}^l)} + \|z_m^r\|_{W^{-1,q}(\Omega;\mathbb{R}^d)}). \tag{3.40}$$

By (3.31) and (3.37), the sequence $\{v_m^r\}$ is uniformly bounded in $L^q(Q(x_0,r);\mathbb{R}^d)$. Thus, there exists a map $v^r\in L^q(Q(x_0,r);\mathbb{R}^d)$ such that, up to the extraction of a (not relabelled) subsequence,

$$v_m^r \rightharpoonup v^r \quad \text{weakly in } L^q(Q(x_0,r);\mathbb{R}^d) \tag{3.41}$$

as $m\to+\infty$. Again by (3.31), and by the compact embedding of L^q into $W^{-1,q}$, we deduce that

$$z_m^r \to 0 \quad \text{strongy in } W^{-1,q}(\Omega;\mathbb{R}^d) \tag{3.42}$$

as $m\to+\infty$. Therefore, by combining (3.38) and (3.41), we conclude that

$$v_m^r \rightharpoonup 0 \quad \text{weakly in } L^q(Q(x_0,r);\mathbb{R}^d)$$

as $m \to +\infty$, and the convergence holds for the entire sequence. Additionally, by (3.28), (3.39), and (3.42), we obtain

$$\mathscr{A} v_m^r = \mathscr{A}_\eta v_m^r \to 0 \quad \text{strongly in } W^{-1,q}(Q(x_0,r);\mathbb{R}^l)$$

as $m \to +\infty$. Finally, by (3.32), (3.40), and (3.42), there holds

$$\lim_{r\to 0} \lim_{m\to+\infty} r^{-\frac{N}{q}} \|v_m^r - z_m^r\|_{L^q(Q(x_0,r);\mathbb{R}^d)} = 0. \tag{3.43}$$

We recall that, since x_0 satisfies (3.15), Step 1 yields

$$\frac{d\mathcal{I}(u,v)}{d\mathcal{L}^N}(x_0) = \lim_{r\to0^+} \frac{\mathcal{I}((u,v);Q(x_0,r))}{r^N} \leq \liminf_{r\to0^+} \liminf_{m\to+\infty} \frac{1}{r^N} \int_{Q(x_0,r)} f(x,u(x),v(x)+v_m^r(x))\,dx. \tag{3.44}$$

We claim that

$$\frac{d\mathcal{I}(u,v)}{d\mathcal{L}^N}(x_0) = \lim_{r\to0^+} \frac{\mathcal{I}((u,v);Q(x_0,r))}{r^N} \leq \liminf_{r\to0^+} \liminf_{m\to+\infty} \frac{1}{r^N} \int_{Q(x_0,r)} g(x,v(x)+z_m^r(x))\,dx, \tag{3.45}$$

where g is the function introduced in Step 3. Indeed, for every $r \in \mathbb{R}$, consider the function $g^r : Q \times \mathbb{R}^d \to [0,+\infty)$ defined as

$$g^r(y,\xi) := g(x_0 + ry, \xi) \quad \text{for every } y \in Q, \xi \in \mathbb{R}^d.$$

Since $x_0 \in \omega$, by (3.13) there exists K_j such that $x_0 \in K_j$. In particular, this yields the existence of $r_0 > 0$ such that for $r \leq r_0$, the maps g^r are continuous on $Q \times \mathbb{R}^d$, and the family $\{g^r(y,\cdot)\}$ is equicontinuous in $\mathbb{R}^d$, uniformly with respect to y. A change of variables yields

$$\begin{aligned}&\frac{1}{r^N}\Big| \int_{Q(x_0,r)} f(x,u(x),v(x)+v_m^r(x))\,dx - \int_{Q(x_0,r)} f(x,u(x),v(x)+z_m^r(x))\,dx\Big|\\ &= \Big| \int_Q g^r(y, v(x_0+ry) + v_m^r(x_0+ry))\,dy - \int_Q g^r(y, v(x_0+ry)+z_m^r(x_0+ry))\,dy\Big|.\end{aligned}$$

On the other hand, by (3.43) we have

$$\lim_{r\to0}\lim_{m\to+\infty} \|z_m^r(x_0+r\cdot) - v_m^r(x_0+r\cdot)\|_{L^q(Q;\mathbb{R}^d)} = \lim_{r\to0}\lim_{m\to+\infty} r^{-\frac{N}{q}} \|z_m^r - v_m^r\|_{L^q(Q(x_0,r);\mathbb{R}^d)} = 0.$$

Therefore, by a diagonal procedure we extract a subsequence $\{m_r\}$ such that

$$\begin{aligned}&\limsup_{r\to0}\limsup_{m\to+\infty} \Big| \int_Q g^r(y, v(x_0+ry)+v_m^r(x_0+ry))\,dy - \int_Q g^r(y, v(x_0+ry)+z_m^r(x_0+ry))\,dy\Big| \\ &= \lim_{r\to0}\Big| \int_Q g^r(y, v(x_0+ry)+v_{m_r}^r(x_0+ry))\,dy - \int_Q g^r(y, v(x_0+ry)+z_{m_r}^r(x_0+ry))\,dy\Big|,\end{aligned} \tag{3.46}$$

and

$$z_{m_r}^r(x_0+r\cdot) - v_{m_r}^r(x_0+r\cdot) \to 0 \quad \text{strongly in } L^q(Q;\mathbb{R}^d).$$

In view of (3.14), (3.30) and the Riemann-Lebesgue lemma, the sequence $\{v(x_0+r\cdot) + z_{m_r}^r(x_0+r\cdot)\}$ is q-equiintegrable in Q. Hence, by (H) we are under the assumptions of Proposition 3.2, and we conclude that

$$\lim_{r\to0}\Big| \int_Q g^r(y, v(x_0+ry)+v_{m_r}^r(x_0+ry))\,dy - \int_Q g^r(y, v(x_0+ry)+z_{m_r}^r(x_0+ry))\,dy\Big| = 0. \tag{3.47}$$

Claim (3.45) follows by combining (3.46) with (3.47).

Arguing as in [6, Proof of Lemma 3.5], for every $x_0 \in \omega$ (where ω is the set defined in (3.13)) we have

$$\begin{aligned}
&\liminf_{r\to 0^+}\liminf_{m\to+\infty}\frac{1}{r^N}\int_{Q(x_0,r)} f(x,u(x),v(x)+z^r_m(x))\,dx\\
&\quad\le \liminf_{r\to 0^+}\liminf_{m\to+\infty}\frac{1}{r^N}\int_{Q(x_0,r)} f(x_0,u(x_0),v(x_0)+z^r_m(x))\,dx,
\end{aligned}$$

hence by (3.45) we deduce that

$$\frac{d\mathcal{I}(u,v)}{d\mathcal{L}^N}(x_0)\le \liminf_{r\to 0^+}\liminf_{m\to+\infty}\frac{1}{r^N}\int_{Q(x_0,r)} f(x_0,u(x_0),v(x_0)+z^r_m(x))\,dx.$$

By (3.30) we obtain

$$\begin{aligned}
\frac{d\mathcal{I}(u,v)}{d\mathcal{L}^N}(x_0)&\le \liminf_{r\to 0^+}\liminf_{m\to+\infty}\frac{1}{r^N}\int_{Q(x_0,r)} f(x_0,u(x_0),v(x_0)+z^r_m(x))\,dx\\
&\le \liminf_{r\to 0^+}\liminf_{m\to+\infty}\frac{1}{r^N}\bigg\{\int_{Q(x_0,r)} f\Big(x_0,u(x_0),v(x_0)+w\Big(\frac{m(x-x_0)}{r}\Big)\Big)\,dx\\
&\qquad+\int_{Q(x_0,r)\cap\{\varphi\neq 1\}} f\Big(x_0,u(x_0),v(x_0)+\varphi(x)w\Big(\frac{m(x-x_0)}{r}\Big)\Big)\,dx\bigg\}.
\end{aligned}$$

The growth assumption (H) and estimate (3.29) yield

$$\begin{aligned}
&\int_{Q(x_0,r)\cap\{\varphi\neq 1\}} f\Big(x_0,u(x_0),v(x_0)+\varphi(x)w\Big(\frac{m(x-x_0)}{r}\Big)\Big)\,dx\\
&\quad\le C\int_{Q(x_0,r)\cap\{\varphi\neq 1\}}\Big(1+\Big|w\Big(\frac{m(x-x_0)}{r}\Big)\Big|^q\Big)\,dx\\
&\quad\le C(1+\|w\|^q_{L^\infty(\mathbb{R}^N;\mathbb{R}^d)})\mathcal{L}^N(Q(x_0,r)\cap\{\varphi\neq 1\})\le C\mu r^N.
\end{aligned}\tag{3.48}$$

Thus, by (3.48), the periodicity of w, and Riemann-Lebesgue lemma, we deduce

$$\begin{aligned}
\frac{d\mathcal{I}(u,v)}{d\mathcal{L}^N}(x_0)&\le C\mu+\liminf_{r\to 0^+}\liminf_{m\to+\infty}\frac{1}{r^N}\int_{Q(x_0,r)} f\Big(x_0,u(x_0),v(x_0)+w\Big(\frac{m(x-x_0)}{r}\Big)\Big)\,dx\\
&=C\mu+\liminf_{m\to+\infty}\int_Q f(x_0,u(x_0),v(x_0)+w(my))\,dy\\
&=C\mu+\int_Q f(x_0,u(x_0),v(x_0)+w(y))\,dy\\
&\le C\mu+Q_{\mathscr{A}(x_0)}f(x_0,u(x_0),v(x_0)),
\end{aligned}$$

where the last inequality is due to (3.27). Letting $\mu\to 0^+$ we conclude (3.25). □

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References

[1] Adolfo Arroyo-Rabasa. Relaxation and optimization for linear-growth convex integral functionals under PDE constraints. *arXiv:1603.01310*, 2016.

[2] Adolfo Arroyo-Rabasa, Guido De Philippis, and Filip Rindler. Lower semicontinuity and relaxation of linear-growth integral functionals under pde constraints. *arXiv:1701.02230*, 2017.

[3] Hedy Attouch. *Variational convergence for functions and operators*. Applicable Mathematics Series. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[4] Margarida Baía, Milena Chermisi, José Matias, and Pedro M. Santos. Lower semicontinuity and relaxation of signed functionals with linear growth in the context of $\mathscr{A}$-quasiconvexity. *Calc. Var. Partial Differential Equations*, 47(3-4):465–498, 2013.

[5] Barbora Benešová and Martin Kružík. *Weak lower semicontinuity of integral functionals and applications*. arXiv:1601.00390v4, 2016.

[6] Andrea Braides, Irene Fonseca, and Giovanni Leoni. $\mathscr{A}$-quasiconvexity: relaxation and homogenization. *ESAIM Control Optim. Calc. Var.*, 5:539–577 (electronic), 2000.

[7] Elisabetta Chiodaroli, Eduard Feireisl, Ondřej Kreml, and Emil Wiedemann. $\mathscr{A}$-free rigidity and applications to the compressible Euler system. *Annali di Matematica Pura ed Applicata (1923 -)*, pages 1–16, 2017.

[8] Bernard Dacorogna. *Weak continuity and weak lower semicontinuity of nonlinear functionals*, volume 922 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1982.

[9] Bernard Dacorogna and Irene Fonseca. A-B quasiconvexity and implicit partial differential equations. *Calc. Var. Partial Differential Equations*, 14(2):115–149, 2002.

[10] Elisa Davoli and Irene Fonseca. Homogenization of integral energies under periodically oscillating differential constraints. *Calc. Var. Partial Differential Equations*, 55(3):1–60, 2016.

[11] Elisa Davoli and Irene Fonseca. Periodic homogenization of integral energies under space-dependent differential constraints. *Port. Math.*, 73(4):279–317, 2016.

[12] Guido De Philippis and Filip Rindler. On the structure of $\mathcal{A}$-free measures and applications. *Ann. of Math. (2)*, 184(3):1017–1039, 2016.

[13] Irene Fonseca and Stefan Krömer. Multiple integrals under differential constraints: two-scale convergence and homogenization. *Indiana Univ. Math. J.*, 59(2):427–457, 2010.

[14] Irene Fonseca and Martin Kružík. Oscillations and concentrations generated by $\mathscr{A}$-free mappings and weak lower semicontinuity of integral functionals. *ESAIM Control Optim. Calc. Var.*, 16(2):472–502, 2010.

[15] Irene Fonseca, Giovanni Leoni, and Stefan Müller. $\mathscr{A}$-quasiconvexity: weak-star convergence and the gap. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 21(2):209–236, 2004.

[16] Irene Fonseca and Stefan Müller. Relaxation of quasiconvex functionals in $\mathrm{BV}(\Omega, \mathbf{R}^p)$ for integrands $f(x, u, \nabla u)$. *Arch. Rational Mech. Anal.*, 123(1):1–49, 1993.

[17] Irene Fonseca and Stefan Müller. $\mathscr{A}$-quasiconvexity, lower semicontinuity, and Young measures. *SIAM J. Math. Anal.*, 30(6):1355–1390 (electronic), 1999.

[18] Jan Krämer, Stefan Krömer, Martin Kružík, and Gabriel Pathó. $\mathscr{A}$-quasiconvexity at the boundary and weak lower semicontinuity of integral functionals. *Adv. Calc. Var.*, 10(1):49–67, 2017.

[19] Carolin Kreisbeck and Stefan Krömer. Heterogeneous thin films: combining homogenization and dimension reduction with directors. *SIAM J. Math. Anal.*, 48(2):785–820, 2016.

[20] Carolin Kreisbeck and Filip Rindler. Thin-film limits of functionals on $\mathscr{A}$-free vector fields. *Indiana Univ. Math. J.*, 64(5):1383–1423, 2015.

[21] José Matias, Marco Morandotti, and Pedro M. Santos. Homogenization of functionals with linear growth in the context of $\mathcal{A}$-quasiconvexity. *Appl. Math. Optim.*, 72(3):523–547, 2015.

[22] François Murat. Compacité par compensation: condition nécessaire et suffisante de continuité faible sous une hypothèse de rang constant. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 8(1):69–102, 1981.

[23] Pedro M. Santos. $\mathscr{A}$-quasi-convexity with variable coefficients. *Proc. Roy. Soc. Edinburgh Sect. A*, 134(6):1219–1237, 2004.

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