VARIATIONAL PROBLEMS WITH LONG-RANGE INTERACTION

NICOLA SOAVE, HUGO TAVARES, SUSANNA TERRACINI, AND ALESSANDRO ZILIO

Abstract. We consider a class of variational problems for densities that repel each other at distance. Typical examples are given by the Dirichlet functional and the Rayleigh functional

$$D(\mathbf{u}) = \sum_{i=1}^k \int_\Omega |\nabla u_i|^2 \quad \text{or} \quad R(\mathbf{u}) = \sum_{i=1}^k \frac{\int_\Omega |\nabla u_i|^2}{\int_\Omega u_i^2}$$

minimized in the class of $H^1(\Omega, \mathbb{R}^k)$ functions attaining some boundary conditions on $\partial\Omega$, and subjected to the constraint

$$\mathrm{dist}(\{u_i > 0\}, \{u_j > 0\}) \geqslant 1 \qquad \forall i \neq j.$$

For these problems, we investigate the optimal regularity of the solutions, prove a free-boundary condition, and derive some preliminary results characterizing the free boundary $\partial\{\sum_{i=1}^k u_i > 0\}$.

1. Introduction

The object of this paper is the study of a class of minimal configurations for variational problems involving arbitrarily many densities related by long-range repulsive interactions. The mathematical setting we consider is described by the following two archetypical situations.

Problem (A) Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geqslant 2$, and let

$$\Omega_1 = \bigcup_{x \in \Omega} B_1(x) = \{x \in \mathbb{R}^N : \ \mathrm{dist}(x, \Omega) < 1\}.$$

Given $k \geqslant 2$ nonnegative nontrivial functions $f_1, \dots, f_k \in H^1(\Omega_1) \cap C(\overline{\Omega}_1)$ satisfying [1]

$$\mathrm{dist}(\mathrm{supp}\, f_i, \mathrm{supp}\, f_j) \geqslant 1 \qquad \forall i \neq j,$$

we consider the minimization problem

$$\inf_{\mathbf{u} \in H_\infty} J_\infty(\mathbf{u}),$$

where the set H_∞ and the functional J_∞ are defined by

$$H_\infty = \left\{ \mathbf{u} = (u_1, \dots, u_k) \in H^1(\Omega_1, \mathbb{R}^k) \;\middle|\; \begin{array}{ll} \mathrm{dist}(\mathrm{supp}\, u_i, \mathrm{supp}\, u_j) \geqslant 1 & \forall i \neq j \\ u_i = f_i \text{ a.e. in } \Omega_1 \setminus \Omega & \end{array} \right\}, \tag{1.1}$$

Date: January 17, 2017.
Key words and phrases. Free boundary condition, Lipschitz regularity, Long-range interaction, Optimal partition problems, Segregation problems, Variational methods.

[1] Here and in the rest of the paper, the distance between two sets A and B is understood as

$$\mathrm{dist}(A, B) := \inf\{|x - y| : \ x \in A, \ y \in B\}.$$

and

$$J_\infty(\mathbf{u}) = \sum_{i=1}^k \int_\Omega |\nabla u_i|^2.$$

The support of each component u_i is taken in the weak sense: it corresponds to the complement in Ω_1 of the largest open set $\omega \subseteq \mathbb{R}^N$ where $u_i = 0$ a.e. on ω (cf. [3, Proposition 4.17]). Notice also that the existence of $f_1, \dots, f_k$ with the above properties imposes some conditions on Ω (for instance, the diameter of Ω cannot be too small), and we suppose that such conditions are satisfied.

We are interested in existence and qualitative properties of minimizers.

Problem (B) Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geqslant 2$, and let $k \geqslant 2$. We consider the *set of open partitions of* Ω *at distance* 1, defined as

$$\mathcal{P}_k(\Omega) = \left\{ (\omega_1, \dots, \omega_k) \,\middle|\, \begin{array}{l} \omega_i \subset \Omega \text{ is open and non-empty for every } i, \\ \text{and } \operatorname{dist}(\omega_i, \omega_j) \geqslant 1 \quad \forall i \neq j \end{array} \right\}.$$

Then, for a cost function $F \in \mathcal{C}^1((\mathbb{R}^+)^k, \mathbb{R})$ satisfying

- $\partial_i F(x) > 0$ for all $x \in (\mathbb{R}^+)^k$ and $i = 1, \dots, k$, which in particular yields that F is component-wise increasing;
- for any given $i = 1, \dots, k$,
$$\lim_{x_i \to +\infty} F(\bar{x}_1, \dots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1} \dots, \bar{x}_k) = +\infty$$
for all $(\bar{x}_1, \dots, \bar{x}_{i-1}, \bar{x}_{i+1} \dots, \bar{x}_k) \in (\mathbb{R}^+)^{k-1}$,

we consider the minimization problem

$$\inf_{(\omega_1, \dots, \omega_k) \in \mathcal{P}_k(\Omega)} F(\lambda_1(\omega_1), \dots, \lambda_1(\omega_k)), \tag{1.2}$$

where $\lambda_1(\omega)$ is the first eigenvalue of the Laplace operator in ω with homogeneous Dirichlet boundary conditions. Problem (1.2) is a particular case of an *optimal partition problem* (cf. [1, 4]). A typical case we have in mind is the cost function $F(\lambda_1(\omega_1), \dots, \lambda_1(\omega_k)) = \sum_{i=1}^k \lambda_1(\omega_i)$.

We are interested in existence and qualitative properties of an optimal partition.

Our main results are, for problem (A):

- the existence of a minimizer;
- the optimal interior regularity of any minimizer;
- the derivation of several properties of the positivity sets $\{u_i > 0\}$;
- the derivation of a free boundary condition involving the normal derivatives of different components of any minimizers on the regular part of the free-boundary $\partial\{u_i > 0\}$.

For problem (B):

- the introduction of a weak formulation in terms of densities, and the existence of weak solutions;
- the global optimal regularity of any weak solution, which leads in particular to the existence of a strong solution for the original problem;
- the derivation of properties of the subsets ω_i, and of a free boundary condition on the regular part of $\partial\omega_i$.

In a forthcoming paper, we will study more in detail the regularity of the free-boundary.

We stress that, both in problems (A) and (B), the interaction among different densities takes place at distance: in problem (A) the positivity sets $\{u_i > 0\}$, and in problem (B) the open subsets ω_i, are indeed forced to stay at a fixed minimal distance from each other.

When the interaction among the densities takes place point-wisely, segregation problems analogue to (A) and (B) have been studied intensively, in connection with optimal partition problems for Laplacian eigenvalues [5, 9, 10, 11, 21, 25, 26], with the regularity theory of harmonic maps into singular manifold [6, 12, 25], and with segregation phenomena for systems of elliptic equations arising in quantum mechanics driven by strong competition [6, 13, 18, 22, 23, 24, 30].

In contrast, the only results available so far regarding segregation problems driven by long-range competition are given in [7], where the authors analyze the spatial segregation for systems of type

$$\begin{cases} -\Delta u_{i,\beta} = -\beta u_{i,\beta} \sum_{j \neq i} (\mathbb{1}_{B_1} \star |u_j|^p) & \text{in } \Omega \\ u_{i,\beta} = f_i \geq 0 & \text{in } \Omega_1 \setminus \Omega, \end{cases} \tag{1.3}$$

with $1 \leqslant p \leqslant +\infty$. In the above equation, $\mathbb{1}_{B_1}$ denotes the characteristic function of B_1, the ball[2] of center 0 and radius 1, and $\star$ stays for the convolution for $p < +\infty$, so that

$$(\mathbb{1}_{B_1} \star |u_j|^p)(x) = \int_{B_1(x)} |u_j(y)|^p \, dy \qquad \forall x \in \Omega, \text{ with } 1 \leqslant p < +\infty;$$

in case $p = +\infty$, we intend that the integral is replaced by the supremum over $B_1(x)$ of $|u_j|$. In [7], the authors prove the equi-continuity of families of viscosity solutions $\{\mathbf{u}_\beta : \beta > 0\}$ to (1.3), the local uniform convergence to a limit configuration $\mathbf{u}$, and then study the free-boundary regularity of the positivity sets $\{u_i > 0\}$ in cases $p = 1$ and $p = +\infty$, mostly in dimension $N = 2$. As we shall see, our problem (A) is strictly related with the asymptotic study of the solutions to (1.3) in case $p = 2$ (see the forthcoming Theorem 2.1); nevertheless, also in such a situation our approach is very different with respect to the one in [7], since we heavily rely on the variational nature of the problem. This gives differenti free boundary conditions which requires different techniques, and allows us to prove new results.

Regarding problem (1.3), we also refer to [2], where the author proves uniqueness results in the cases $p = 1$ and $p = +\infty$.

1.1. **Main results.** We adopt the notation previously introduced. First of all, we have the following existence results for problems (A) and (B).

Theorem 1.1 (Problem (A))**.** *There exists a minimizer* $\mathbf{u} = (u_1, \dots, u_k)$ *for* $\inf_{H_\infty} J_\infty$.

Theorem 1.2 (Problem (B))**.** *There exists a minimizer* $(\omega_1, \dots, \omega_k) \in \mathcal{P}_k$ *for* (1.2).

Observe that, to each optimal partition $(\omega_1, \dots, \omega_k)$, we can associate a vector of signed first eigenfunctions. To fix ideas, from now on we always consider nonnegative eigenfunctions. The second part of our analysis concerns the properties satisfied by any minimizer of problems (A) and (B).

Theorem 1.3. *Let* $\mathbf{u} = (u_1, \dots, u_k)$ *be either any minimizer of* J_∞ *in* H_∞*, or a vector of first eigenfunctions associated to an optimal partition* $(\omega_1, \dots, \omega_k)$ *of* (1.2)*. Then* $\mathbf{u}$ *is a vector of nonnegative functions in* Ω*, and denoting by* S_i *the positivity set* $\{x \in \Omega : \ u_i > 0\}$*, for every* $i = 1, \dots, k$*, we have:*

(1) Subsolution in Ω*: We have that*
$-\Delta u_i \leqslant 0$ *in distributional sense in* Ω*, if* $\mathbf{u}$ *is a solution to problem (A),*
$-\Delta u_i \leqslant \lambda_1(\omega_1) u_i$ *in distributional sense in* Ω*, if* $\mathbf{u}$ *is a solution to problem (B).*

(2) Solution in S_i: *We have that*
$-\Delta u_i = 0$ *in int*(S_i)*, if* $\mathbf{u}$ *is a solution to problem (A),*
$-\Delta u_i = \lambda_1(\omega_i)$ *in int*(S_i)*, if* $\mathbf{u}$ *is a solution to problem (B).*

[2]We denote by $B_r(x)$ the ball of center x and radius r in $\mathbb{R}^N$. In case $x = 0$, we simply write B_r.

(3) Exterior sphere condition for the positivity sets: *S_i satisfies the* 1-uniform exterior sphere condition in Ω*, in the following sense: for every $x_0 \in \partial S_i \cap \Omega$ there exists a ball B with radius* 1 *which is exterior to S_i and tangent to S_i at x_0, i.e.*

$$S_i \cap B = \emptyset \quad \text{and} \quad x_0 \in \overline{S_i} \cap \overline{B}.$$

Moreover, in $B \cap B_1(x_0)$ we have $u_j \equiv 0$ for every $j = 1, \dots, k$ (including $j = i$).

(4) Lipschitz continuity: *u_i is Lipschitz continuous in Ω, and in particular S_i is an open set, for every i.*

(5) Lebesgue measure of the free-boundary: *the free-boundary $\partial\{u_i > 0\}$ has zero Lebesgue measure, and its Hausdorff dimension is strictly smaller than N.*

(6) Exact distance between the supports: *for every $x_0 \in \partial S_i \cap \Omega$ there exists $j \neq i$ such that*

$$\overline{B_1(x_0)} \cap \partial \operatorname{supp} u_j \neq \emptyset.$$

Notice that, if $y_0 \in \partial S_j$ is such that $|x_0 - y_0| = 1$, then $B_1(y_0)$ is an exterior sphere to S_i at x_0. Moreover, by the Hopf lemma, the interior Lipschitz regularity is optimal.

Regarding the regularity of a vector of eigenfunctions $\mathbf{u}$ of problem (B), if we ask that Ω satisfies the exterior sphere condition, then we have actually a stronger statement.

Theorem 1.4. *Let $\mathbf{u}$ be a vector of first eigenfunctions associated to an optimal partition $(\omega_1, \dots, \omega_k)$ of (1.2). Assume that Ω satisfies the exterior sphere condition with radius $r > 0$. Then $\mathbf{u}$ is globally Lipschitz continuous in $\overline{\Omega}$.*

Next, we establish a relation involving the normal derivatives of two "adjacent components" on the regular part of the free boundary.

In what follows, for each i, $\nu_i(x)$ will denote the exterior normal at a point $x \in \partial S_i$ (at points where such a normal vector does exist).

Assumptions. Let $x_0 \in \partial S_i \cap \Omega$, and let us assume that $\Gamma_i^R := \partial S_i \cap B_R(x_0)$ is a smooth hypersurface, for some $R > 0$. By the 1-uniform exterior sphere condition, we know that the principal curvatures of ∂S_i in x_0, denoted by $\chi_h^i(x_0)$, $h = 1, \dots, N-1$, are smaller than or equal to 1 (where we agree that outward is the positive direction). We further suppose that the strict inequality holds, that is there exists $\delta > 0$ such that

$$\chi_1^i(x_0), \dots, \chi_{N-1}^i(x_0) \leqslant 1 - \delta. \tag{1.4}$$

We know that there exists $j \neq i$ and $y_0 \in \partial \operatorname{supp} u_j$ such that $|x_0 - y_0| = 1$.

Theorem 1.5. *Let $\mathbf{u} = (u_1, \dots, u_k)$ be either any minimizer of J_∞ in H_∞, or a vector of first eigenfunctions associated to an optimal partition $(\omega_1, \dots, \omega_k)$ of (1.2). Under the previous assumptions and notations, we have that $y_0 = x_0 + \nu_i(x_0)$ is the unique point in $\bigcup_{k \neq i} \partial \operatorname{supp} u_k$ at distance 1 from x_0. If $y_0 \in \partial \operatorname{supp} u_j \cap \Omega$, then $\partial \operatorname{supp} u_j$ is also smooth around y_0, and*

$$\frac{(\partial_\nu u_i(x_0))^2}{(\partial_\nu u_j(y_0))^2} = \begin{cases} \displaystyle\prod_{\substack{h=1 \\ \chi_h^i(x_0) \neq 0}}^{N-1} \left| \frac{\chi_h^i(x_0)}{\chi_h^j(y_0)} \right| & \text{if } \chi_h^i(x_0) \neq 0 \text{ for some } h, \\ 1 & \text{if } \chi_h^i(x_0) = 0 \text{ for all } h = 1, \dots, N-1. \end{cases} \tag{1.5}$$

We stress that, since the sets S_i and S_j are at distance 1 from each other and (1.4) holds, $\chi_h^i(x_0) \neq 0$ if and only if $\chi_h^j(y_0) \neq 0$, and hence the term on the right hand side is always well defined.

The proof of Theorem 1.5 is based on the introduction of a family of domain variations for the minimizer $\mathbf{u}$. As we shall see, the possibility of producing admissible domain variations, preserving the constraint on the distance of the supports in H_∞, presents major difficulties. At the moment, we can only overcome such obstructions and produce more or less explicit variations supposing that ∂S_i is locally regular. This is the main problem when trying to study the regularity of the free boundary. Regarding this point, we mention that the proofs of all our results (and also of those in [7], in a nonvariational case) are completely different with respect to the analogue counterpart in problems with point-wise interaction. Indeed, all the local techniques, such as blow-up analysis and monotonicity formulae, cannot be straightforwardly adapted when dealing with long-range interaction; the reason is that the interface between different positivity sets $\{u_i > 0\}$ and $\{u_j > 0\}$ with $i \neq j$ is now a strip of width at least 1, and hence with a standard blow-up one cannot catch the interaction on the free-boundary at the limit.

We also mention that the validity of a uniform exterior sphere condition does not directly imply any extra regularity for ∂S_i: if we could show that ∂S_i is a set with positive reach (see [14]), then we could argue as in [7, Corollary 6.3] and prove at least that the Hausdorff dimension of ∂S_i is $N-1$ (see also [8, Theorem 4.2] for a different proof of this fact), but on the other hand sets enjoying the uniform exterior sphere condition are not necessarily of positive reach, as shown in [19, Section 2].

Remark 1.6. A very interesting feature of Theorem 1.5 stays in the fact that it reveals a deep difference between segregation models with point-wise interaction, and with long-range interaction. To explain this difference, let us consider a sequence $\{\mathbf{u}_\beta\}$ of solutions to (1.3), with $p = 1$ and $\beta \to +\infty$. This is the setting studied in [7]. In [7, Theorem 9.1], the authors derive a free-boundary condition analogous to (1.5) for the limit configurations in case $p = 1$, but in their situation, the left hand side is replaced by the ratio between the normal derivatives, $\partial_\nu u_i(x_0)/\partial_\nu u_j(y_0)$. This difference is in contrast with respect to segregation phenomena with point-wise interaction, where, as proved in [25], limit configurations associated with

$$-\Delta u_i = -\beta u_i \sum_{j\neq i} u_j \quad \text{or} \quad -\Delta u_i = -\beta u_i \sum_{j\neq i} u_j^2$$

belong to the same functional class [13, 25], and hence in particular satisfy the same free-boundary condition, that is $|\partial_\nu u_i(x_0)| = |\partial_\nu u_j(x_0)|$ on the regular part of the free boundary. A similar difference has been observed in [27, 28, 29] in the case of fractional operators, that is when the non-locality is in the differential operator.

Finally, in comparison with the free boundary condition derived in [7], it is worthwhile noticing that the analogue of (1.5) there involves the plain quotient of the normal derivatives, while here we find the squared one.

Remark 1.7. The previous result may fail if the right hand side in (1.4) is replaced by the constant 1. Indeed, if $\partial S_i \cap B_R(x_0) = \partial B_1(0) \cap B_R(x_0)$ for some $x_0 \in \partial B_1(0)$ and $R > 0$, and the set S_i is contained in the exterior of $B_1(0)$, then $y_0 = 0$ is a cusp for ∂S_j.

1.2. **Structure of the paper.** We first treat problem (A). In Section 2 we prove Theorem 1.1 for this problem, relating this segregation problem with a variational competition–diffusion of type (1.3). Then some qualitative properties of any possible minimizer of problem (A) are shown in Section 3, where we prove Theorem 1.3 for this problem. Section 4 contains the proof of the free boundary condition contained in the statement of Theorem 1.5 for problem (A).

The analogous statements for problem (B) – existence and properties of minimizers, and free boundary condition – are proved in Section 5.

Finally, in Appendix A we state and prove an Hadamard's type formula which we need along this paper.

2. Existence of a minimizer for Problem (A)

In this section we prove Theorem 1.1. To this purpose, we introduce a competition parameter $\beta > 0$ which allows us to remove the segregation constraint. To be precise, let

$$H = \{\mathbf{u} \in H^1(\Omega_1, \mathbb{R}^k) : \ u_i = f_i \text{ a.e. in } \Omega_1 \setminus \Omega\} \supset H_\infty,$$

and let $\beta > 0$. We consider the minimization of the functional

$$J_\beta(\mathbf{u}) = \sum_{i=1}^k \int_\Omega |\nabla u_i|^2 + \sum_{1 \leqslant i < j \leqslant k} \iint_{\Omega_1 \times \Omega_1} \beta \, \mathbb{1}_{B_1}(x-y) u_i^2(x) u_j^2(y) \, dx \, dy$$

in the set H. With respect to the search of a minimizer for $\inf_{H_\infty} J_\infty$, the advantage stays in the fact that we can get rid of the infinite dimensional constraint $\operatorname{dist}(\operatorname{supp} u_i, \operatorname{supp} u_j) \geqslant 1$ for $i \neq j$, and we can easily show that a minimizer for J_β in H does exists, and satisfies an Euler-Lagrange equation of type (1.3) with $p = 2$. This allows us to obtain Theorem 1.1 as a direct corollary of the following statement:

Theorem 2.1. *For every $\beta > 0$, there exists a minimizer $\mathbf{u}_\beta = (u_{1,\beta}, \dots, u_{k,\beta})$ for $\inf_H J_\beta$, which is a solution of*

$$\begin{cases} -\Delta u_i = -\beta u_i \sum_{j \neq i} (\mathbb{1}_{B_1} \star u_j^2) & \text{in } \Omega \\ u_i > 0 & \text{in } \Omega \\ u_i = f_i & \text{in } \Omega_1 \setminus \Omega. \end{cases} \tag{2.1}$$

The family $\{\mathbf{u}_\beta : \beta > 0\}$ is uniformly bounded in $H^1(\Omega_1, \mathbb{R}^k) \cap L^\infty(\Omega_1)$, and there exists $\mathbf{u} = (u_1, \dots, u_k) \in H$ such that:

1. *$\mathbf{u}_\beta \to \mathbf{u}$ strongly in $H^1(\Omega)$ as $\beta \to +\infty$, up to a subsequence;*
2. *$\operatorname{dist}(\operatorname{supp} u_i, \operatorname{supp} u_j) \geqslant 1$ for every $i \neq j$, so that $\mathbf{u} \in H_\infty$;*
3. *for every $i \neq j$,*
$$\lim_{\beta \to +\infty} \iint_{\Omega_1 \times \Omega_1} \mathbb{1}_{B_1}(x-y) u_{i,\beta}^2(x) u_{j,\beta}^2(y) \, dx \, dy = 0$$
4. *$\mathbf{u}$ is a minimizer for $\inf_{H_\infty} J_\infty$. In particular, $\mathbf{u}$ is a solution to problem (A).*

Remark 2.2. Without any additional complication, we can replace in the previous theorem the indicator function $\mathbb{1}_{B_1}$ with a more general function $V \in L^\infty(\mathbb{R}^N)$ satisfying $V > 0$ a.e. in B_1, $V = 0$ a.e. on $\mathbb{R}^N \setminus \overline{B_1}$.

The proof of Theorem 2.1 is the object of the rest of the section. Before proceeding, we observe that, by the definition of support given in [3, Proposition 4.17], the set H_∞ can be defined in the following equivalent way:

$$H_\infty = \left\{ \mathbf{u} \in H : \iint_{\Omega_1 \times \Omega_1} \mathbb{1}_{B_1}(x-y) u_i^2(x) u_j^2(y) \, dx \, dy = 0 \ \forall i \neq j \right\}$$

(see the proof of Lemma 3.1 below for more details).

Remark 2.3. Here it is worth to stress that we consider the functions u_i as defined in Ω_1, and hence the supports have to be considered in this set (and not only in Ω).

Proof of Theorem 2.1. The existence of a minimizer $\mathbf{u}_\beta$ follows by the direct method of the calculus of variations, and the fact that minimizers solve (2.1) is straightforward. Observe that $f_i \geqslant 0$, hence the minimizers are positive in Ω, by the strong maximum principle.

For the uniform L^∞ estimate, since $u_{i,\beta} > 0$ is subharmonic in Ω for every $i = 1, \dots, k$, by the maximum principle we have $\|u_{i,\beta}\|_{L^\infty(\Omega)} \leqslant \|f_i\|_{L^\infty(\partial\Omega)}$. Let us set

$$c_\beta := \inf_H J_\beta \quad \text{and} \quad c_\infty := \inf_{H_\infty} J_\infty.$$

We observe that, since $J_\beta(\mathbf{v}) = J_\infty(\mathbf{v})$ for every $\mathbf{v} \in H_\infty$, we have $c_\beta \leqslant c_\infty$. Then, by the minimality of $\mathbf{u}_\beta$, for every $\beta > 0$ we have $J_\beta(\mathbf{u}_\beta) \leqslant c_\infty$. Since moreover $u_{i,\beta} \equiv f_i$ in $\Omega_1 \setminus \Omega$, the uniform $H^1(\Omega_1, \mathbb{R}^k)$ boundedness of $\{\mathbf{u}_\beta\}$ follows. Hence, up to a subsequence, $\mathbf{u}_\beta \rightharpoonup \mathbf{u}$ weakly in $H^1(\Omega_1, \mathbb{R}^k)$ and a.e. in Ω. Moreover

$$\lim_{\beta\to+\infty} \iint_{\Omega_1\times\Omega_1} \mathbb{1}_{B_1}(x-y) u_i^2(x) u_j^2(y)\, dx\, dy = 0 \qquad \forall i \neq j$$

and by the Fatou lemma we have

$$0 \leqslant \iint_{\Omega_1\times\Omega_1} \mathbb{1}_{B_1}(x-y) u_i^2(x) u_j^2(y)\, dx\, dy \leqslant \liminf_{\beta\to+\infty} \iint_{\Omega_1\times\Omega_1} \mathbb{1}_{B_1}(x-y) u_{i,\beta}^2(x) u_{j,\beta}^2(y)\, dx\, dy = 0$$

for every $i \neq j$. This in particular proves point (2) in the thesis and implies that $\mathbf{u} \in H_\infty$, defined in (1.1).

On the other hand, by the the minimality of $\mathbf{u}_\beta$ and weak convergence,

$$\begin{aligned} c_\infty \leqslant J_\infty(\mathbf{u}) &= \sum_{i=1}^k \int_\Omega |\nabla u_i|^2 \leqslant \liminf_{\beta\to\infty} \sum_{i=1}^k \int_\Omega |\nabla u_{i,\beta}|^2 \\ &\leqslant \limsup_{\beta\to\infty} \sum_{i=1}^k \int_\Omega |\nabla u_{i,\beta}|^2 \leqslant \limsup_{\beta\to\infty} J_\beta(\mathbf{u}_\beta) = \limsup_{\beta\to\infty} c_\beta \leqslant c_\infty. \end{aligned}$$

This means that all the previous inequalities are indeed equalities, and in particular:

- we have convergence $\|\nabla u_{i,\beta}\|_{L^2(\Omega)} \to \|\nabla u_i\|_{L^2(\Omega)}$, which together with the weak convergence ensures that $\mathbf{u}_\beta \to \mathbf{u}$ strongly in $H^1(\Omega, \mathbb{R}^k)$ (recall that Ω is bounded);
- point (3) of the thesis holds;
- we have $c_\infty = J_\infty(\mathbf{u})$, which proves the minimality of $\mathbf{u} \in H_\infty$. □

3. Properties of minimizers for problem (A)

This section is devoted to the proof of Theorem 1.3 for the solutions of problem (A). Let then $\mathbf{u}$ be a minimizer for $\inf_{H_\infty} J_\infty$. Theorem 1.1 (see also Theorem 2.1) does not give any information about the continuity of u_i, and in particular we do not know if the sets $S_i = \{x \in \Omega : u_i(x) > 0\}$ are open. On the other hand it is reasonable to work at a first stage with the functions

$$\Phi_i : \Omega \to \mathbb{R}, \qquad \Phi_i(x) := \int_{B_1(x)} u_i^2(y)\, dy,$$

which are clearly continuous due to the Lebesgue dominated convergence theorem.

Let us consider the open sets

$$C_i = \Omega \cap \left(\bigcup_{y \in \{\Phi_i = 0\}} B_1(y) \right), \qquad D_i := \operatorname{int}(\Omega \setminus C_i),$$

for $i = 1, \dots, k$, so that

$$\Omega = C_i \cup D_i \cup (\partial D_i \cap \Omega), \qquad \text{and} \qquad \partial D_i \cap \Omega = \partial C_i \cap \Omega.$$

Observe that, by the definition of Φ_i, we have $u_i = 0$ a.e. in C_i. Moreover

$$D_i = \{x \in \Omega : \mathrm{dist}(x, \{\Phi_i = 0\}) > 1\} \subset \{\Phi_i > 0\}.$$

The strategy of the proof of Theorem 1.3 can be summarized as follows:

- At first, we prove some simple properties of the set D_i and of the restriction of u on D_i.
- In particular, we show that S_i is the union of connected components of D_i, so that the regularity of u_i in Ω is reduced to the regularity of u_i on ∂D_i.
- Using the basic properties of D_i, we show that u_i is locally Lipschitz continuous across ∂D_i, and hence in Ω. It follows in particular that S_i is open, and directly inherits from D_i properties (3) and (5) in Theorem 1.3. Moreover, points (1) and (2) holds.
- As a last step, we prove point (6) by using the minimality of $\mathbf{u}$.

Lemma 3.1. *The function u_i is harmonic in D_i. In particular, if $\tilde{D}_i$ is any connected component of D_i, then either $u_i \equiv 0$ or $u_i > 0$ in $\tilde{D}_i$.*

Proof. The set D_i is open. If we know that $\mathrm{dist}(D_i, \mathrm{supp}\, u_j) \geqslant 1$, then we can consider any $\phi \in C_c^\infty(D_i)$ and observe that, by the minimality of $\mathbf{u}$ for J_∞ on the set H_∞, the function

$$f(\varepsilon) := J_\infty(u_1, \dots, u_{i-1}, u_i + \varepsilon\phi, u_{i+1}, \dots, u_k)$$

has a minimum at $\varepsilon = 0$. This implies that u_i is harmonic in D_i, and all the other conclusions follow immediately. Therefore, in what follows we have to show that

$$\mathrm{dist}(D_i, \mathrm{supp}\, u_j) \geqslant 1 \qquad \forall j \neq i. \tag{3.1}$$

By definition of H_∞ we have $u_i^2(x) u_j^2(y) \mathbb{1}_{B_1}(x-y) = 0$ for a.e. $x, y \in \Omega_1$, that is

$$u_i^2(x) u_j^2(y) = 0 \quad \text{ for a.e. } x, y \in \Omega_1, \ |x-y| < 1.$$

As a consequence, $u_j(x)\Phi_i(x) = 0$ for a.e. $x \in \Omega$ and every $j \neq i$. In particular, this implies that

$$\{\Phi_i > 0\} \subset (\Omega \setminus \mathrm{supp}\, u_j)\,. \tag{3.2}$$

Let $x_0 \in D_i$. Then by definition of D_i, $\mathrm{dist}(x_0, \{\Phi_i = 0\}) > 1$, and hence $B_1(x_0) \subset \{\Phi_i > 0\}$. But then, due to (3.2), and since x_0 has been arbitrarily chosen, we deduce that (3.1) holds. □

Let A_i be the union of the connected components of D_i on which $u_i > 0$, and let N_i be the union of those on which $u_i \equiv 0$, so that $D_i = A_i \cup N_i$. We know that u_i is positive and harmonic in A_i, while $u_i = 0$ a.e. in $N_i \cup C_i$. Since A_i, N_i and C_i are open, this means that (if necessary replacing u_i with a different representative in its same equivalence class) u_i is continuous in A_i, N_i, and C_i. To discuss the continuity of u_i in Ω, we have to derive some properties of the boundary $\partial D_i \cap \Omega = (\partial A_i \cup \partial N_i) \cap \Omega = \partial C_i \cap \Omega$. In the next lemma we show that D_i satisfies a uniform exterior sphere condition.

Lemma 3.2. *For each i, the set D_i satisfies the 1-uniform exterior sphere condition in Ω, in the following sense: for every $x_0 \in \partial D_i \cap \Omega$ there exists a ball B of radius 1 such that*

$$D_i \cap B = \emptyset \quad \text{and} \quad x_0 \in \overline{D_i} \cap \overline{B}.$$

Moreover, in B we have $u_i \equiv 0$.

Proof. This comes directly from the definitions: we have

$$\partial D_i \cap \Omega = \partial C_i \cap \Omega = \{x : \mathrm{dist}(x, \{\Phi_i = 0\}) = 1\} \cap \Omega.$$

Thus, given $x \in \partial D_i \cap \Omega$, there exists $y \in \partial B_1(x)$ with $\Phi_i(y) = 0$. The ball $B_1(y)$ is the desired exterior tangent ball, since $B_1(y) \subset C_i$, and hence $B_1(y) \cap D_i = \emptyset$. □

The exterior sphere condition permits to deduce that ∂D_i has zero Lebesgue measure.

Lemma 3.3. *The boundary ∂D_i is a porous set, and in particular it has 0 Lebesgue measure and $dim_{\mathfrak{H}}(\partial D_i) < N$.*

For the definition of "porosity", we refer to [20, Section 3.2], while here and in what follows $\dim_{\mathfrak{H}}$ denotes the Hausdorff dimension.

Proof. Since $\partial D_i \subset \Omega$ is bounded, to prove its porosity it is sufficient to show that there exists $\delta > 0$ such that: for every ball $B_r(x_0)$ with $x_0 \in \partial D_i$, there exists $y \in B_r(x_0)$ with $B_{\delta r}(y) \subset B_r(x_0) \setminus \partial D_i$ (see [20, Exercise 3.4]).

The existence of such $\delta = 1/2$ follows immediately by the exterior sphere condition: given $x_0 \in \partial D_i$, there exists $z \in \Omega_1$ such that $B_1(z)$ is exterior to D_i. Let then y be the point on the segment $x_0 z$ at distance $r/2$ from D_i. The ball $B_{r/2}(y)$ is contained both in $\Omega_1 \setminus \partial D_i$ and in $B_r(x_0)$, and this proves that ∂D_i is porous. The rest of the proof follows by [20, Page 62]. □

It is not difficult now to deduce that u_i is continuous at every point of ∂N_i. Indeed, notice that $\partial N_i \subset \partial C_i$, and in both N_i and C_i we have $u_i \equiv 0$. Since $\partial N_i \subset \partial D_i$ has 0 Lebesgue measure, we deduce that $u_i = 0$ a.e. in $\overline{N_i} \cup C_i = \Omega \setminus \overline{A_i}$. That is, up to the choice of a different representative, $u_i \equiv 0$ in $\Omega \setminus \overline{A_i}$, and hence it is real analytic therein. At this stage, it remains to discuss the continuity of u_i on ∂A_i. This is the content of the forthcoming Corollary 3.6, where we show that actually $\mathbf{u}$ is locally Lipschitz continuous in Ω. We postpone the proof, proceeding here with the conclusion of Theorem 1.3. The continuity of u_i implies in particular that $\{u_i > 0\}$ is open for every i, so that $\{u_i > 0\} = A_i$. Thus, Lemmas 3.1-3.3 establish the validity of points (2) and (5) in Theorem 1.3. The subharmonicity of u_i, point (1), follows from (2).

Regarding point (3), the existence of an exterior sphere B of radius 1 for $\{u_i > 0\}$ at any boundary point x_0 comes directly from Lemma 3.2. We also know that $u_i \equiv 0$ in B, and furthermore, by (3.1), $B_1(x_0) \cap \operatorname{supp} u_j = \emptyset$ for every $j \neq i$. This proves the validity of (3).

It remains only to show that also point (6) holds.

Proof of Theorem 1.3-(6). This is a consequence of the minimality. Take $x_0 \in \partial S_i \cap \Omega$ and assume, in view of a contradiction, that $\operatorname{dist}(x_0, \operatorname{supp} u_j) > 1$ for some $x_0 \in \partial S_i \cap \Omega$, for every $j \neq i$. Then there exists $\rho > 0$ such that $B_\rho(x_0) \subset \Omega$ and

$$\operatorname{dist}(B_\rho(x_0), \operatorname{supp} u_j) > 1 \qquad \forall j \neq i. \tag{3.3}$$

Let v be the harmonic extension of u_i in $B_\rho(x_0)$:

$$\begin{cases} \Delta v = 0 & \text{in } B_\rho(x_0) \\ v = u_i & \text{on } \partial B_\rho(x_0). \end{cases}$$

Since $u_i \not\equiv 0$ on $\partial B_\rho(x_0)$, we infer that $v > 0$ in $B_\rho(x_0)$, and in particular $v \not\equiv u_i$ in $B_\rho(x_0)$. Let now $\tilde{\mathbf{u}}$ be defined by

$$\tilde{u}_i = \begin{cases} u_i & \text{in } \Omega \setminus B_\rho(x_0) \\ v & \text{in } B_\rho(x_0) \end{cases}, \qquad \tilde{u}_j = u_j \quad \forall j \neq i.$$

Due to (3.3), it belongs to H_∞, so that by minimality $J_\infty(\mathbf{u}) \leqslant J_\infty(\tilde{\mathbf{u}})$. On the other hand, by the definition of harmonic extension we have also $J_\infty(\tilde{\mathbf{u}}) < J_\infty(\mathbf{u})$ (the strict inequality comes from the fact that $v \not\equiv u_i$ in $B_\rho(x_0)$), a contradiction. □

Remark 3.4. In [7], the authors proved harmonicity, local Lipschitz continuity, and exterior sphere condition for limits of any sequence of solutions to (2.1). Nevertheless, the result here is not contained in [7], since we establish harmonicity, Lipschitz continuity, and exterior sphere condition for any minimizer of $\inf_{H_\infty} J_\infty$, independently on wether it can be approximated with a sequence of solutions to (2.1) or not. Also, it is worth to point out that the approach is completely different: while in [7] the authors proceed with careful uniform estimates for viscosity solution of (1.3), here we use the variational structure of the limit problem.

3.1. **Lipschitz continuity of the minimizers.** In this subsection we show that the solutions of problem (A) are Lipschitz continuous inside Ω, which is the highest regularity one can expect for the minimizers of J_∞ (by the Hopf lemma). This is a consequence of the following general statement.

Theorem 3.5. *Let Λ be a domain of $\mathbb{R}^N$, and let $A \subset \Lambda$ be an open subset, satisfying the r-uniform exterior sphere condition in Λ: for any $x_0 \in \partial A \cap \Lambda$ there exists a ball B with radius r which is exterior to A and tangent to ∂A at x_0, i.e.*

$$A \cap B = \emptyset \quad \text{and} \quad x_0 \in \overline{A} \cap \overline{B}.$$

Let $f \in L^\infty(\Lambda)$, and let $u \in H^1(\Lambda) \cap L^\infty(\Lambda)$ satisfy

$$\begin{cases} -\Delta u = f & \text{in } A \\ u = 0 & \text{a.e. in } \Lambda \setminus A \end{cases}$$

Then u is locally Lipschitz continuous in Λ, and for every compact set $K \Subset \Lambda$ there exists a constant $C = C(r, N, K) > 0$ such that

$$\|\nabla u\|_{L^\infty(K)} \leqslant C \left(\|u\|_{L^\infty(\Lambda)} + \|f\|_{L^\infty(\Lambda)}\right).$$

For the sake of generality, we required no sign condition on the function u, even though we will apply the result only to nonnegative solutions.

Corollary 3.6. *Let $\mathbf{u}$ be any minimizer of J_∞ in H_∞. Then $\mathbf{u}$ is locally Lipschitz continuous in Ω.*

Proof. We apply Theorem 3.5 to the harmonic functions u_i in $A := A_i$, with $\Lambda := \Omega$ and $r = 1$. □

The proof of Theorem 3.5 is based upon a simple barrier argument. For any $R > 0$, let us define

$$w_R(x) := \frac{1}{2N}(R^2 - |x|^2)^+ \quad \Longrightarrow \quad \begin{cases} -\Delta w_R = 1 & \text{in } B_R \\ w_R = 0 & \text{in } \mathbb{R}^N \setminus B_R, \end{cases}$$

and let

$$w_R^*(x) := \left(\frac{R}{|x|}\right)^{N-2} w_R\left(\frac{R^2}{|x|^2}x\right) = \frac{R^N}{2N|x|^N}\left(|x|^2 - R^2\right)^+ \tag{3.4}$$

be its Kelvin transform with respect to the sphere of radius R. It is not difficult to check that

$$-\Delta w_R^*(x) = -\left(\frac{R}{|x|}\right)^{N+2} \Delta w_R\left(\frac{R^2}{|x|^2}x\right) = \left(\frac{R}{|x|}\right)^{N+2}. \tag{3.5}$$

With this preliminary observation, we can easily prove the following estimate:

Lemma 3.7. *Let $x_0 \in \partial A \cap \Lambda$, and let $\rho > 0$ be such that $B_\rho(x_0) \Subset \Lambda$. Under the assumptions of Theorem 3.5, there exists a constant $C > 0$ depending on the dimension N, on r and on ρ, such that*

$$|u(x)| \leqslant C\left(\|u\|_{L^\infty(\Lambda)} + \|f\|_{L^\infty(\Lambda)}\right)|x - x_0| \qquad \forall x \in B_\rho(x_0).$$

Proof. Let $y_0 \in \mathbb{R}^N$ be the center of the exterior sphere in x_0:

$$A \cap B_r(y_0) = \emptyset \quad \text{and} \quad x_0 \in \overline{A} \cap \overline{B_r(y_0)}$$

Let z_0 be the medium point on the segment $x_0\, y_0$. Up to a rigid motion, we can suppose that $z_0 = 0$ and that $x_0 = (0', r/2)$, where $0'$ denotes the 0 vector in $\mathbb{R}^{N-1}$. In this setting, we aim at proving that $u \leqslant w^*_{r/2}$ in $B_\rho(x_0) \cap A$, with $w^*_{r/2}$ defined by (3.4). Since $u = 0$ a.e. in $\Omega \setminus A$, we have (in the sense of traces) that $u = 0$ on $\partial A \cap \overline{B_\rho(x_0)}$. Moreover, since $B_{r/2}(z_0) \subset B_r(y_0)$, and $\partial B_{r/2}(z_0) \cap \partial B_r(y_0) = \{x_0\}$, there exists a value $\delta = \delta(r, \rho, N)$ (independent on the point x_0) such that $\operatorname{dist}(z_0, A \cap \partial B_\rho(x_0)) \geqslant \operatorname{dist}(z_0, \partial B_r(y_0) \cap \partial B_\rho(x_0)) > r/2 + \delta$. Hence

$$\inf_{A \cap \partial B_\rho(x_0)} w^*_{r/2} \geqslant m(r, \rho, N) > 0,$$

and we can define

$$\varphi(x) := \left(\frac{\|u\|_{L^\infty(\Omega)}}{m(r,\rho,N)} + \left(\frac{2\rho}{r}\right)^{N+2} \|f\|_{L^\infty(\Lambda)} \right) w^*_{r/2}(x).$$

It is now not difficult to check that

$$\begin{cases} -\Delta(\varphi - u) \geqslant 0 & \text{in } A \cap B_\rho(x_0) \\ (\varphi - u) \geqslant 0 & \text{on } \partial(A \cap B_\rho(x_0)) \end{cases}$$

Indeed, in $A \cap B_\rho(x_0)$, by recalling (3.5),

$$-\Delta u = f \leq \|f\|_{L^\infty(\Lambda)} \qquad \text{and} \qquad -\Delta w^*_{r/2} = \left(\frac{r}{2|x|}\right)^{N+2} \geq \left(\frac{r}{2\rho}\right)^{N+2}$$

The boundary $\partial(A \cap B_\rho(x_0))$ splits into two parts. On the first part $\partial A \cap \overline{B_\rho(x_0)}$ we know that $u = 0$ in the sense of traces, and since $\varphi \geqslant 0$ there, we have $\varphi - u \geqslant 0$ on $\partial A \cap \overline{B_\rho(x_0)}$ in the sense of traces. On the remaining part $A \cap \partial B_{\rho(x_0)}$, the function u can be evaluated point-wisely, since in the interior of A the function u is of class $\mathcal{C}^{1,\alpha}$; therefore, it makes sense to write that $u(x) \leqslant \|u\|_{L^\infty(\Omega)} \leqslant \varphi$ for any $x \in A \cap \partial B_{\rho(x_0)}$. All together, we obtain that $u \leqslant \varphi$ on $\partial(A \cap B_\rho(x_0))$ in the sense of traces.

In conclusion, we have $u \leqslant \varphi$ in $A \cap B_\rho(x_0)$ by the maximum principle. Observing that

$$\begin{aligned} \frac{r^N}{2^{N+1} N |x|^N} \left(|x|^2 - \left(\frac{r}{2}\right)^2 \right) &= \frac{r^N}{2^{N+1} N |x|^N} \left(|x| + \frac{r}{2}\right)\left(|x| - \frac{r}{2}\right) \\ &\leqslant \frac{r^N}{2^{N+1} N (r/2)^N} \left(\rho + \frac{r}{2}\right)(|x| - |x_0|) \leqslant \frac{2^{N-1}(2\rho + r)}{N} |x - x_0|. \end{aligned}$$

for every $x \in B_\rho(x_0)$, we obtain the desired upper estimate for u. Arguing in the same way on $-u$, we obtain also the lower estimate, and the proof is complete. □

As an immediate consequence:

Corollary 3.8. *For every compact set $K \Subset \Lambda$ there exists $C = C(K, r, N) > 0$ such that*

$$|u(x)| \leqslant C \left(\|u\|_{L^\infty(\Lambda)} + \|f\|_{L^\infty(\Lambda)} \right) \operatorname{dist}(x, \partial A)$$

whenever $x \in K$ with $\operatorname{dist}(x, \partial A) < \operatorname{dist}(K, \partial\Lambda)$.

Proof. Let $x \in A \cap K$ such that $\operatorname{dist}(x, \partial A) < \operatorname{dist}(K, \partial\Lambda)$. Then take $x_0 \in \partial A$ such that $|x - x_0| = \operatorname{dist}(x, \partial A)$. Then we can apply the previous theorem to $B_{\operatorname{dist}(K, \partial\Lambda)/2}(x_0)$. □

We are ready to proceed with the:

Proof of Theorem 3.5. Recall that $-\Delta u = f$ in A, hence there the function u is of class $\mathcal{C}^{1,\alpha}$. Since moreover $u \in H^1(\Lambda)$ and $\nabla u = 0$ a.e. in $\Lambda \setminus A$, it is sufficient to obtain a uniform estimate for ∇u in a neighborhood of ∂A (and actually only in A). Notice that in A it makes sense to consider point-wise values of the gradient of u.

We use the notation $d_x := \operatorname{dist}(x, \partial A)$, for every $x \in \Omega$. Take $x_0 \in \partial A \cap \Lambda$ and let $\delta > 0$ be small enough such that, considering the compact set

$$K := \overline{\bigcup_{x \in B_\delta(x_0)} B_{d_x}(x)}$$

then

$$\operatorname{dist}(x, \partial A) < \operatorname{dist}(K, \partial \Lambda) \qquad \forall x \in K.$$

By Corollary 3.8, there exists $C = C(K, N, r) > 0$ such that

$$|u(x)| \leqslant C \left(\|u\|_{L^\infty(\Lambda)} + \|f\|_{L^\infty(\Lambda)} \right) d_x \qquad \forall x \in K.$$

In particular, for every $x \in A \cap B_\delta(x_0)$, since $B_{d_x}(x) \subset K$, then

$$\|u\|_{L^\infty(B_{d_x}(x))} \leq 2C \left(\|u\|_{L^\infty(\Lambda)} + \|f\|_{L^\infty(\Lambda)} \right) d_x. \tag{3.6}$$

Now, let

$$Q_x := \left\{ y \in \mathbb{R}^N : |y_i - x_i| < \frac{d_x}{m}, \ i = 1, \dots, N \right\},$$

where $m > 0$ is chosen so large that the cube Q_x is contained in the ball $B_{d_x}(x)$ ($m > 0$ is a universal constant, depending only on the dimension N). Since $B_{d_x}(x) \subset A$, then $-\Delta u = f$ in $B_{d_x}(x)$ and we can combine (3.6) with interior gradient estimates for the Poisson equation (see [15, Formula 3.15)]), deducing that

$$|\nabla u(x)| \leqslant \frac{Nm}{d_x} \sup_{\partial Q_x} |u| + \frac{d_x}{2m} \sup_{Q_x} |f| \leqslant C' \left(\|u\|_{L^\infty(\Lambda)} + \|f\|_{L^\infty(\Lambda)} \right) \qquad \forall x \in A \cap B_\delta(x_0). \qquad \square$$

4. Free-boundary condition for problem (A)

In this section we prove Theorem 1.5. We briefly recall the setting.

Let $x_0 \in \partial S_i \cap \Omega$, and let us assume that $\Gamma_i^R := \partial S_i \cap B_R(x_0)$ is a smooth hypersurface, for some $R > 0$. We suppose that, for a positive δ, condition (1.4) holds on Γ_i^R:

$$\chi_1^i(x), \dots, \chi_{N-1}^i(x) \leqslant 1 - \delta \qquad \forall x \in \Gamma_i^R,$$

where $\chi_1^i, \dots, \chi_{N-1}^i$ denote the principal curvatures of ∂S_i. Without loss of generality, we can suppose that Γ_i^R is a graph:

$$\Gamma_i^R = \{(x', \psi(x')) : \ x' \in B_R^{N-1}(x_0')\}, \qquad \text{and} \qquad S_i \cap B_R(x_0) = \{(x', z) \in B_R(x_0) : \ z \leqslant \psi(x')\}$$

for a function $\psi : B_R^{N-1}(x_0') \to \mathbb{R}$, where $B_R^{N-1}(x_0')$ denotes the ball of radius R in $\mathbb{R}^{N-1}$ centered at $x_0' = (x_0^1, \dots, x_0^{N-1})$. We know from Theorem 1.3-(6) that there exists $j \neq i$ and $y_0 \in \partial \operatorname{supp} u_j$ such that $|x_0 - y_0| = 1$.

The proof of Theorem 1.5 is divided into several steps. We start with the uniqueness and characterization of y_0.

Lemma 4.1. *If $x \in \partial S_i \cap \Omega$ and ∂S_i is smooth in a neighbourhood of x, then $y = x + \nu_i(x)$ is the unique point in $\bigcup_{l \neq i} \partial \operatorname{supp} u_l$ at distance 1 from x.*

Proof. By Theorem 1.3 (points (3) and (6)), we know that there exists a point $y \in \bigcup_{l\neq i} \partial \operatorname{supp} u_l$ such that

$$|x-y| = 1, \quad \text{and} \quad |x-z| \geqslant 1 \quad \text{for all } z \in \bigcup_{l\neq i} \partial \operatorname{supp} u_l.$$

This means that $y - x \in Q := \{v : \operatorname{dist}(x+v, S_i) = |v|\}$. By [14, Theorem 4.8-(2)], Q is a subset of the normal cone to S_i in x, and since ∂S_i is smooth in x, we deduce that $y - x = \nu_i(x)$. □

The previous lemma implies that there exists a unique j and a unique $y_0 \in \partial \operatorname{supp} u_j$ at distance 1 from x_0. In order to simplify the notation, let $i = 1$ and $j = 1$, and so $x_0 \in \partial S_1 \cap \Omega$, $y_0 \in \partial \operatorname{supp} u_2$. Assume from now on that $y_0 \in \Omega$, so that $y_0 \in \partial S_2 \cap \Omega$. We denote $\Gamma_1^R := \partial S_1 \cap B_R(x_0)$ and $\Gamma_2^R := \{x + \nu_1(x) : x \in \Gamma_1^R\}$. Notice that by Lemma 4.1 and by continuity, we have that $y_0 \in \Gamma_2^R \subset \partial S_2 \cap \Omega$, where the last inclusion holds for sufficiently small $R > 0$.

Lemma 4.2. *The set Γ_2^R is a smooth hypersurface.*

Proof. The set Γ_2^R can be parametrized by $\Phi : B_R^{N-1}(x_0') \to \mathbb{R}^N$,

$$\Phi(x') = (x', \psi(x')) + \nu_1(x', \psi(x')) = \left(x' - \frac{\nabla\psi(x')}{\sqrt{1+|\nabla\psi(x')|^2}}, \psi(x') + \frac{1}{\sqrt{1+|\nabla\psi(x')|^2}} \right),$$

and hence we need to prove that $D\Phi(x')$ has maximum rank. We have

$$D\left(x' - \frac{\nabla\psi(x')}{\sqrt{1+|\nabla\psi(x')|^2}} \right) = \mathrm{Id}_{N-1} - D\left(\frac{\nabla\psi(x')}{\sqrt{1+|\nabla\psi(x')|^2}} \right), \tag{4.1}$$

where Id_{N-1} denotes the identity in $\mathbb{R}^{N-1}$. Observe that $D\left(\nabla\psi(x')/\sqrt{1+|\nabla\psi(x')|^2}\right)$ is the curvature tensor of Γ_1^R at $(x', \psi(x'))$ (see for instance [15, p.356]). Assumption (1.4) implies that all its eigenvalues are strictly smaller than one. Then the determinant of (4.1) does not vanish, and the result follows. □

Observe that, with the previous notations,

$$\nu_1(x) = -\nu_2(x + \nu_1(x)) \ \forall x \in \Gamma_1^R \quad \text{and} \quad \nu_2(x) = -\nu_1(x + \nu_2(x)) \ \forall x \in \Gamma_2^R. \tag{4.2}$$

Let $\eta \in \mathcal{C}_c^\infty(B_R(x_0))$ be a nonnegative test function. We define two deformations, one acting on S_1, and the other on S_2. The first one, which deforms S_1, is a function denoted by $F_{1,\varepsilon} : \mathbb{R}^N \to \mathbb{R}^N$, $\varepsilon \in [0, \bar\varepsilon)$, such that,

$$F_{1,\varepsilon}(x) = \begin{cases} x & \text{if } x \notin B_R(x_0) \\ x + \varepsilon\eta(x)\nu_1(x) & \text{if } x \in \Gamma_1^R, \end{cases}$$

extended to the whole $\mathbb{R}^N$ in such a way that $(\varepsilon, x) \in [0, \bar\varepsilon) \times \mathbb{R}^N \mapsto F_{1,\varepsilon}(x)$ is of class $\mathcal{C}^1$, and $F_{1,0}(\cdot) = \mathrm{Id}$. We denote

$$S_{1,\varepsilon} := F_{1,\varepsilon}(S_1) := S_1 \cup \{x + s\eta(x)\nu_1(x) : \ x \in \Gamma_1^R, \ 0 \leqslant s < \varepsilon\}$$

and

$$\Gamma_{1,\varepsilon}^R := F_{1,\varepsilon}(\Gamma_1^R) = \{x + \varepsilon\eta(x)\nu_1(x) : \ x \in \Gamma_1^R\}.$$

Lemma 4.3. *The set $\Gamma_{1,\varepsilon}^R$ is a smooth hypersurface. Moreover, if we denote its exterior normal at a point $x + \varepsilon\eta(x)\nu_1(x)$ (for $x \in \Gamma_1^R$) by $\nu^\varepsilon(x)$, then $\varepsilon \mapsto \nu^\varepsilon(x)$ is differentiable at $\varepsilon = 0$ and*

$$\left.\frac{d}{d\varepsilon}\nu^\varepsilon(x)\right|_{\varepsilon=0} \ \textit{is orthogonal to } \nu_1(x), \ \textit{for every } x \in \Gamma_1^R. \tag{4.3}$$

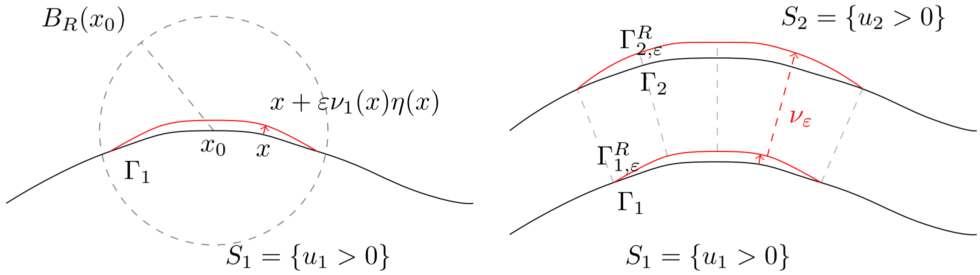

FIGURE 1. The picture on the left represents the deformation acting on S_1. The picture on the right represents the deformation acting on S_2.

Proof. By the smoothness of Γ_1^R and of the perturbation η, it follows that ν^ε is differentiable in ε for ε small. By deriving the identity $|\nu^\varepsilon(x)|^2=1$ in ε for each $x\in\Gamma_1^R$, we have $\frac{d}{d\varepsilon}\nu^\varepsilon(x)\cdot\nu^\varepsilon(x)=0$. Since $\nu^0(x)=\nu_1(x)$, the statement (4.3) follows. □

Now we consider an open neighbourhood $B^R_{y_0}$ of y_0 such that $B^R_{y_0}\cap\partial S_2=\Gamma_2^R$ and $\operatorname{dist}(B^R_{y_0},\partial\Omega)>0$. In order to deform S_2, we take $F_{2,\varepsilon}:\mathbb{R}^N\to\mathbb{R}^N$, $\varepsilon\in[0,\bar\varepsilon)$, such that

$$F_{2,\varepsilon}(y)=\begin{cases} y & \text{if } y\notin B^R_{y_0},\\ x+\varepsilon\eta(x)\nu_1(x)+\nu^\varepsilon(x), & \text{if } x=y+\nu_2(y),\ y\in\Gamma_2^R\end{cases}$$

extended to the whole $\mathbb{R}^N$ in such a way that $(\varepsilon,x)\in[0,\bar\varepsilon)\times\mathbb{R}^N\mapsto F_{2,\varepsilon}(x)$ is of class $\mathcal{C}^1$, and $F_{2,0}(\cdot)=\mathrm{Id}$. Define

$$S_{2,\varepsilon}:=F_{2,\varepsilon}(S_2):=S_2\setminus\{x+s\eta(x)\nu_1(x)+\nu^\varepsilon(x):\ x\in\Gamma_1^R,\ 0\leqslant s<\varepsilon\}$$

and

$$\Gamma^R_{2,\varepsilon}:=F_{2,\varepsilon}(\Gamma_2^R)=\{x+\varepsilon\eta(x)\nu_1(x)+\nu^s(x):\ x\in\Gamma_1^R\}.$$

Notice that, since $\eta\geqslant0$, we have $\Gamma^R_{2,\varepsilon}\subset\overline{S_2}$ for every $\varepsilon>0$.

Remark 4.4. We observe that the map $x\in\Gamma_1^R\mapsto x_\varepsilon:=x+\varepsilon\eta(x)\nu_1(x)\in\Gamma^R_{1,\varepsilon}$ is a diffeomorphism for $\varepsilon>0$ small enough. For this reason, we can see the normal ν^ε as defined on $\Gamma^R_{1,\varepsilon}$, and use the notation

$$\nu^\varepsilon(x_\varepsilon):=\nu^\varepsilon(x)\qquad\Longleftrightarrow\quad x_\varepsilon=x+\varepsilon\eta(x)\nu(x).$$

The crucial point in our argument is the following:

Lemma 4.5. *We have* $\operatorname{dist}(S_{1,\varepsilon},S_{2,\varepsilon})\geqslant1$. *Moreover,* $\operatorname{dist}(S_{i,\varepsilon},S_j)\geqslant1$ *for every* $i\in\{1,2\}$, $j\neq1,2$.

For the proof we will need the following elementary fact.

Lemma 4.6. *Let* (x_1,y_1), (x_2,y_2), *two points on the lower semi-circle* $\partial B_1^-:=\{x^2+y^2=1,y<0\}$ *in* $\mathbb{R}^2$. *Let* γ *be the graph of a* $\mathcal{C}^2$ *function* $f:[x_1,x_2]\to\mathbb{R}$, *and let us suppose that:*

- *the curvature of* γ *is strictly smaller than* 1;
- $f(x_1)=y_1$, *i.e.* (x_1,y_1) *is the initial point of* γ;
- *there exists* $\rho>0$ *such that* $f(t)\leqslant-\sqrt{1-t^2}$ *for* $t\in(x_1,x_1+\rho)$.

Then $f(x_2)<y_2$, *i.e.* γ *cannot contain any other point on* ∂B_1.

Proof. In terms of f, the curvature of γ is defined by

$$k(t) := \frac{f''(t)}{(1+(f'(t))^2)^{3/2}}.$$

Thus, by assumption:

$$f''(t) < \big(1+(f'(t))^2\big)^{3/2} \text{ in } [x_1,x_2], \qquad f'(x_1) \leqslant \frac{x_1}{\sqrt{1-x_1^2}} \text{ and } f(x_1) = y_1.$$

Recalling that $v(t) = -\sqrt{1-t^2}$ solves $v'' = (1+(v')^2)^{3/2}$, the thesis follows by a comparison argument for solutions to ODEs. □

Proof of Lemma 4.5. The second statement of the lemma comes from the fact that $\mathrm{dist}(S_i, S_j) \geq 1$ and $\mathrm{dist}(\Gamma_i^R, S_j) > 1$ for $i = 1,2$ and $j > 2$. As for the first statement, observe that it is enough to show that

$$\mathrm{dist}(\partial S_{1,\varepsilon} \cap \Omega, \partial S_{2,\varepsilon} \cap \Omega) \geqslant 1.$$

By construction, $\partial S_{i,\varepsilon} \setminus \Gamma_{i,\varepsilon}^R = \partial S_i \setminus \Gamma_i^R$ for $i = 1,2$, and since $\mathrm{dist}(\partial \operatorname{supp} u_1, \partial \operatorname{supp} u_2) = 1$, then

$$\mathrm{dist}(\partial S_{1,\varepsilon} \setminus \Gamma_{1,\varepsilon}^R, \partial S_{2,\varepsilon} \setminus \Gamma_{2,\varepsilon}^R) \geqslant 1.$$

Since every point in Γ_i^R admits a unique point on ∂S_j at distance exactly one, we have that $\mathrm{dist}(\Gamma_i^R, \partial S_j \setminus \Gamma_j^R) > 1$ for every $i \neq j$, $i,j \in \{1,2\}$. Thus, by the continuity of the deformations $F_{1,\varepsilon}, F_{2,\varepsilon}$,

$$\mathrm{dist}(\Gamma_{i,\varepsilon}^R, \partial S_{j,\varepsilon} \setminus \Gamma_{j,\varepsilon}^R) > 1 \qquad \forall i \neq j,\ i,j \in \{1,2\}.$$

It remains to show that

$$\mathrm{dist}(\Gamma_{1,\varepsilon}^R, \Gamma_{2,\varepsilon}^R) = 1.$$

This follows from the following property (we use the notation introduced in Remark 4.4):

(C) there exists $\varepsilon > 0$ small enough such that any point in $y \in S_{1,\varepsilon}^c$ such that $\mathrm{dist}(y, \Gamma_{1,\varepsilon}^R) = 1$ has unique projection at minimal distance onto $S_{1,\varepsilon}$, this projection lies in $\Gamma_{1,\varepsilon}^R$, and moreover $y = x_\varepsilon + \nu^\varepsilon(x_\varepsilon)$ for some $x_\varepsilon \in \Gamma_{1,\varepsilon}^R$.

Indeed, (C) implies, by definition of $\Gamma_{2,\varepsilon}^R$, that

$$\begin{aligned}\{y \in S_{1,\varepsilon}^c : \mathrm{dist}(y, \Gamma_{1,\varepsilon}^R) = 1\} &= \{y \in \overline{S_2} : \mathrm{dist}(y, \Gamma_{1,\varepsilon}^R) = 1\} \\ &= \{y \in \overline{S_2} : y = x_\varepsilon + \nu^\varepsilon(x_\varepsilon),\ x_\varepsilon \in \Gamma_{1,\varepsilon}^R\} = \Gamma_{2,\varepsilon}^R,\end{aligned}$$

and completes the proof.

Let us now prove property (C). That any point at minimal distance from $S_{1,\varepsilon}$ stays on $\Gamma_{1,\varepsilon}^R$ is a consequence of Lemma 4.1 for $\varepsilon = 0$; the case $\varepsilon > 0$ small follows by continuity of $F_{1,\varepsilon}$, and recalling that η has compact support. Take $y \in \overline{S_2} \cap \{\mathrm{dist}(z, \Gamma_{1,\varepsilon}^R) = 1\}$. To prove the uniqueness of the projection, suppose by contradiction that there exist two points x_1 and x_2 in $\Gamma_{1,\varepsilon}^R$ such that $|x_1 - y| = |x_2 - y| = 1$. Since our argument is local in nature, it is not restrictive to suppose that we chose $R < 1/2$ from the beginning, and hence in particular $|x_1 - x_2| < 1$.

Let Π be the plane containing x_1, x_2 and y, and let γ be the arc of the curve $\Gamma_{1,\varepsilon}^R \cap \Pi$ connecting x_1 and x_2. The basic idea which we develop in what follows is that the existence of both x_1 and x_2 is forbidden by the fact that, thanks to (1.4), the curvature at every point of γ is smaller than 1.

Since $\Gamma_{1,\varepsilon}^R$ is a graph of a function of x_N,

(4.4) also γ can be seen as the graph of a function of x_N for ε small enough.

Also, since the principal curvatures of ∂S_1 are all smaller than $1-\delta$ on Γ_1^R, for ε small enough the principal curvatures of $\partial S_{1,\varepsilon}$ are all smaller than $1-\delta/2)$ on $\Gamma_{1,\varepsilon}^R$. Combining this with the fact that x_1 is a projection of y onto $S_{1,\varepsilon}$, it follows the existence of $r>0$ small (possibly depending on ε) such that

$$B_r(x_1)\cap\Gamma_{1,\varepsilon}^R\cap B_1(y)=\{x_1\}. \tag{4.5}$$

Moreover,

(4.6) the (planar) curvature of γ is also smaller than $1-\delta/2$.

Collecting together (4.4), (4.5), (4.6), we are in position to apply [3] Lemma 4.6 to the curve Γ on the plane Π, deducing that Γ cannot meet $B_1(y)$ in any other point than x_1, in contradiction with the existence of x_2.

It remains to show that $y=x_\varepsilon+\nu^\varepsilon(x_\varepsilon)$ for some $x_\varepsilon\in\Gamma_{1,\varepsilon}^R$. Having proved the uniqueness of the projection, this follows directly from [14, Theorem 4.8-(2)] and the smoothness of $\Gamma_{1,\varepsilon}^R$. □

Lemma 4.5 is crucial since it allows us to produce a family of admissible variations of the minimizer $\mathbf{u}$ in the following way. For $i\in\{1,2\}$, let $u_{i,\varepsilon}\in H^1(S_{i,\varepsilon})$ be such that

$$\begin{cases}\Delta u_{i,\varepsilon}=0 & \text{in } S_{i,\varepsilon}\\ u_{i,\varepsilon}=u_i & \text{on } \partial S_{i,\varepsilon}\backslash\Gamma_{i,\varepsilon}^R=\partial S_i\backslash\Gamma_i^R\\ u_{i,\varepsilon}=0 & \text{on } \Gamma_{i,\varepsilon}^R\end{cases}$$

extended by zero to $\Omega\backslash S_{i,\varepsilon}$. Observe that $S_{i,\varepsilon}=\{x\in\Omega:\ u_{i,\varepsilon}(x)>0\}$, and that for $\varepsilon\geqslant 0$ small the vector $(u_{1,\varepsilon},u_{2,\varepsilon},u_3,\dots,u_k)$ belongs to the set H_∞ — defined in (1.1)— by Lemma 4.5.

Proposition 4.7. *We have*

$$\frac{d}{d\varepsilon}\int_\Omega|\nabla u_{1,\varepsilon}|^2\bigg|_{\varepsilon=0^+}=-\int_{\Gamma_1^R}\eta(x)(\partial_{\nu_1}u_1)^2, \tag{4.7}$$

$$\frac{d}{d\varepsilon}\int_\Omega|\nabla u_{2,\varepsilon}|^2\bigg|_{\varepsilon=0^+}=\int_{\Gamma_2^R}\eta(x+\nu_2(x))(\partial_{\nu_2}u_2)^2. \tag{4.8}$$

Proof. The identity (4.7) is a direct consequence of Lemma A.2 in the appendix, with $S:=S_1$ and $\omega=B_R(x_0)$, since

$$Y_1:=\frac{d}{d\varepsilon}F_{1,\varepsilon}(x)\bigg|_{\varepsilon=0}=\eta(x)\nu_1(x).$$

As for (4.8), we apply the same lemma with $S=S_2$ and $\omega=B_{y_0}^R$. We have

$$Y_2(y):=\frac{d}{d\varepsilon}F_{2,\varepsilon}(y)\bigg|_{\varepsilon=0}=\eta(y+\nu_2(y))\nu_1(y+\nu_2(y))+\frac{d}{d\varepsilon}\nu^\varepsilon(y+\nu_2(y))\bigg|_{\varepsilon=0}$$

for every $y\in\Gamma_2^R$. Recalling (4.2) and taking into account (4.3), we have

$$\left\langle\frac{d}{d\varepsilon}\nu^\varepsilon(y+\nu_2(y))\bigg|_{\varepsilon=0},\nu_2(y)\right\rangle=\left\langle\frac{d}{d\varepsilon}\nu^\varepsilon(y+\nu_2(y))\bigg|_{\varepsilon=0},-\nu_1(y+\nu_2(y))\right\rangle=0.$$

Therefore, using (4.2) once again, $\langle Y_2(y),\nu_2(y)\rangle=\eta(y+\nu_2(y))$, and (4.8) follows by Lemma A.2. □

[3]after a translation and a possible rotation

Proof of Theorem 1.5. Without loss of generality we work in the case $i = 1$ and $j = 2$, and use the notations previously introduced. Take, for $\varepsilon \geqslant 0$ small, the vector $(u_{1,\varepsilon}, u_{2,\varepsilon}, u_3, \dots, u_k)$, which by Lemma 4.5 belongs to the set H_∞. Since $u_{1,0} = u_1$ and $u_{2,0} = u_2$, then by the minimality of $\mathbf{u}$ we have that

$$\frac{d}{d\varepsilon} J_\infty(u_{1,\varepsilon}, u_{2,\varepsilon}, u_3, \dots, u_k)\bigg|_{\varepsilon=0^+} \geqslant 0.$$

By Proposition 4.7, this is equivalent to

$$\int_{\Gamma_1^R} \eta(x)(\partial_{\nu_1} u_1)^2 \leqslant \int_{\Gamma_2^R} \eta(x + \nu_2(x))(\partial_{\nu_2} u_2)^2.$$

This identity holds true for every nonnegative $\eta \in \mathcal{C}^\infty_c(B_R(x_0))$. In particular, by taking $\eta = \eta_\delta$ such that $\eta_\delta(x) = 1$ for $x \in B_{R-2\delta}(x_0)$ and $\eta_\delta(x) = 0$ in $B_R(x_0) \backslash B_{R-\delta}(x_0)$, and by making $\delta \to 0$, we can easily conclude that

$$\int_{\Gamma_1^R} (\partial_{\nu_1} u_1)^2 \leqslant \int_{\Gamma_2^R} (\partial_{\nu_2} u_2)^2.$$

Arguing exactly in the same way, but deforming first $\Gamma_{2,R}$, and afterwards $\Gamma_{1,R}$, we can prove that also the opposite inequality holds, and hence

$$\int_{\Gamma_1^R} (\partial_{\nu_1} u_1)^2 = \int_{\Gamma_2^R} (\partial_{\nu_2} u_2)^2.$$

Therefore

$$\frac{⨍_{\Gamma_1^R} (\partial_{\nu_1} u_1)^2}{⨍_{\Gamma_2^R} (\partial_{\nu_2} u_2)^2} = \frac{|\Gamma_2^R|}{|\Gamma_1^R|}, \tag{4.9}$$

and we can thus end the proof by applying [7, Lemma 9.3], which states that the right-hand-side of (4.9) tends to the right-hand-side of (1.5) as $R \to 0$. We point out that, with respect to [7], the modulus is present in our formula (1.5). This is only a consequence of the different convention that we adopted regarding the sign of the curvatures. □

5. Existence and properties of solutions to problem (B)

We focus now on problem (B). It is convenient to restate the problem as follows. Letting, for all $\mathbf{u} \in H^1_0(\Omega; \mathbb{R}^k)$,

$$J(\mathbf{u}) = F\left(\int_\Omega |\nabla u_1|^2, \dots, \int_\Omega |\nabla u_k|^2\right),$$

we define

$$c := \inf_{\mathbf{u} \in H_\infty} J(\mathbf{u}) \tag{5.1}$$

where

$$H_\infty = \left\{ \mathbf{u} = (u_1, \dots, u_k) \in H^1(\Omega, \mathbb{R}^k) \;\middle|\; \begin{array}{c} \operatorname{dist}(\operatorname{supp} u_i, \operatorname{supp} u_j) \geqslant 1 \quad \forall i \neq j \\ \int_\Omega u_i^2 = 1 \ \forall i \end{array} \right\}.$$

Clearly, since to each set ω_i of an element in $\mathcal{P}_k$ we can associate an eigenvalue $u_i \in H^1_0(\omega_i)$, we have

$$c \leqslant \inf_{(\omega_1, \dots, \omega_k) \in \mathcal{P}_k(\Omega)} F(\lambda_1(\omega_1), \dots, \lambda_1(\omega_k)).$$

We show below that these levels coincide.

5.1. **Existence of a minimizer and its first properties.** We first address the problem of existence of optimal partitions, and derive some preliminary properties of the sets composing the minimal solutions. This part is close the results in Section 2 and for this reason we shall only give a brief sketch of the methodology.

We consider the auxiliary problem: for any $\mathbf{u} \in H_0^1(\Omega, \mathbb{R}^k)$ we let

$$J_\beta(\mathbf{u}) = F\left(\int_\Omega |\nabla u_1|^2, \dots, \int_\Omega |\nabla u_k|^2\right) + \sum_{1 \leqslant i < j \leqslant k} \iint_{\Omega \times \Omega} \beta \mathbb{1}_{B_1}(x-y) u_i^2(x) u_j^2(y)\, dx\, dy.$$

We have, similarly to Theorem 2.1:

Theorem 5.1. *For every $\beta > 0$, there exists a nonnegative minimizer $\mathbf{u}_\beta = (u_{1,\beta}, \dots, u_{k,\beta})$ of J_β in the set*

$$H := \left\{\mathbf{u} = (u_1, \dots, u_k) \in H_0^1(\Omega, \mathbb{R}^k) : \int_\Omega u_i^2 = 1 \qquad \forall i = 1, \dots, k\right\}.$$

There exist $\mu_{1,\beta}, \dots, \mu_{k,\beta} > 0$ such that $\mathbf{u}_\beta$ is a nonnegative solution of

$$-\partial_i F\left(\int_\Omega |\nabla u_1|^2, \dots, \int_\Omega |\nabla u_k|^2\right) \Delta u_i = \mu_{i,\beta} u_i - \beta u_i \sum_{j \neq i} \left(\mathbb{1}_{B_r} \star u_j^2\right). \tag{5.2}$$

Moreover, the family $\{\mathbf{u}_\beta : \beta > 0\}$ is uniformly bounded in $H_0^1 \cap L^\infty(\Omega, \mathbb{R}^k)$, and there exists $\mathbf{u} = (u_1, \dots, u_k) \in H$ such that:

1. *$\mathbf{u}_\beta \to \mathbf{u}$ strongly in $H^1(\Omega, \mathbb{R}^k)$ as $\beta \to +\infty$, up to a subsequence;*
2. *$\operatorname{dist}(\operatorname{supp} u_i, \operatorname{supp} u_j) \geqslant 1$, for every $i \neq j$, so that $\mathbf{u} \in H_\infty$;*
3. *for every $i \neq j$,*
$$\lim_{\beta \to +\infty} \iint_{\Omega \times \Omega} \mathbb{1}_{B_1}(x-y) u_{i,\beta}^2(x) u_{j,\beta}^2(y)\, dx\, dy = 0;$$
4. *$\mathbf{u}$ is a minimizer for c, defined in (5.1).*

Proof. All the listed properties can be shown by very similar arguments of Theorem 2.1, we shall only consider here those that are new. In particular, we focus on the uniform bounds on $\{\mathbf{u}_\beta\}$.

The existence of a nonnegative minimizer $\mathbf{u}_\beta$ for J_β on H is given by the direct method of the calculus of variations (J is lower-semicontinuous because F is component-wise increasing). Since H_∞ is not empty, it contains a smooth function $\mathbf{v} = (v_1, \dots, v_k)$. Thus, $J_\beta(\mathbf{u}_\beta) \leqslant c \leqslant J(\mathbf{v}) < +\infty$ for every $\beta > 0$, and this implies that $\{\mathbf{u}_\beta, \beta > 0\}$ is bounded in H_0^1. Notice also that, by definition,

$$\int_\Omega |\nabla u_{i,\beta}|^2 \geqslant \lambda_1(\Omega) \qquad \text{for any } i = 1, \dots, k \text{ and } \beta > 0.$$

Therefore, by the assumptions on F, there exists $a > 0$ such that

$$a < \partial_i F\left(\int_\Omega |\nabla u_{1,\beta}|^2, \dots, \int_\Omega |\nabla u_{k,\beta}|^2\right) < \frac{1}{a} \qquad \text{for any } i = 1, \dots, k \text{ and } \beta > 0.$$

It follows, by the method of the Lagrange multipliers, that any minimizer $\mathbf{u}_\beta$ is a weak solution to (5.2). Testing such equations by $\mathbf{u}_\beta$ itself and using the uniform bound on $J_\beta(\mathbf{u}_\beta)$, we obtain that the exists $\mu > 0$ such that

$$0 < \mu_{i,\beta} < \mu \quad \text{for any } i = 1, \dots, k \text{ and } \beta > 0.$$

The proof of the uniform L^∞ bounds is then a rather standard consequence of the Brezis-Kato iteration technique, since $-\Delta u_{i,\beta} \leq \mu u_{i,\beta}$. The remaining properties can be shown reasoning exactly as in the proof of Theorem 2.1. □

The previous result shows the existence of minimizers for problem c, in connection with an elliptic system with long-range competition. Since both H_∞ and J are invariant under the transformation $(u_1, \ldots, u_k) \mapsto (|u_1|, \ldots, |u_k|)$, we can work from now on, without loss of generality, with nonnegative functions. In what follows, we will show that all the minimizers for c are continuous (actually, we will show that they are Lipschitz continuous in Ω), and this will imply that (1.2) and (5.1) coincide, and there is a one-to-one correspondence between (open) optimal partitions $(\omega_1, \ldots, \omega_k)$ of (1.2) and minimizers $\mathbf{u}$ of (5.1): for every $\mathbf{u}$ minimizer of c, the sets $\omega_i = \{u_i > 0\}$ constitute an optimal partition at distance 1 of Ω.

5.2. **Proof of Theorems 1.3 and 1.4 for problem (B).** By following exactly the same lines of the proof of Theorem 1.3, (1)–(2)–(3), (5)–(6) for problem (A), we can show the exact same properties for any minimizer $\mathbf{u}$ of the level c.

Regarding the regularity of the eigenfunctions, using the notations of Section 3, we observe that $\mathbf{u} = 0$ on $\partial\Omega$, and that Ω satisfies the r-uniform exterior sphere condition for some $r > 0$. Then the Lipschitz continuity in $\overline{\Omega}$ is a direct application of Theorem 3.5 with $f = \lambda_1(\omega_i)u_i$, $\Lambda = \mathbb{R}^N$ and $A := A_i$ (this shows Theorem 1.4).

Observe that the continuity of $\mathbf{u}$ implies that then $\omega_i = \{u_i > 0\}$, $i = 1, \ldots, k$ are minimizers for problem (B). Thus c and (1.2) coincide, and given any optimal partition of (1.2), then the conclusions of Theorem 1.3 hold also for the associated eigenvalues $\mathbf{u}$.

5.3. **Proof of Theorem 1.5 for problem (B).** The proof of this result for problem (B) follows word by word the lines of the proof for problem (A), replacing only Lemma A.2 by the classical Hadamard's variational formula [16, Theorem 2.5.1].

Appendix A. Shape Derivatives

In this appendix we establish a formula which relates the change of the energy of the harmonic extension of a function φ, defined on a boundary ∂S and vanishing on a portion $\partial S \cap \omega$ of ∂S. The domain variation is localized on $\partial S \cap \omega$. Although similar results are by now well known, and excellent references are available (we refer for instance to [17, Chapter 5]), we could not find exactly the result we needed, and therefore we provide here a short discussion for the sake of completeness.

Let $S \subset \mathbb{R}^N$ be a open set, and let $\omega \subset \mathbb{R}^N$ be a bounded smooth domain such that $\partial S \cap \operatorname{int}(\omega) \neq \emptyset$. For a function $\varphi : \partial S \to \mathbb{R}$ such that $\varphi \in Lip(\partial S)$ and $\varphi(x) = 0$ if $x \in \partial S \cap \overline{\omega}$, we consider its harmonic extension in S, that is the function $u \in H^1(S)$ solution to

$$\begin{cases} \Delta u = 0 & \text{in } S \\ u = \varphi & \text{on } \partial S \end{cases} \quad \text{or, equivalently,} \quad \int_S |\nabla u|^2 = \min\left\{ \int_S |\nabla v|^2 : \begin{array}{l} v \in H^1(S), \\ v = \varphi \text{ on } \partial S \end{array} \right\}.$$

The question we want to address is how a smooth deformation of a regular part of ∂S where $u = 0$ impacts the energy of the corresponding harmonic extension. We start by analyzing the derivative with respect to a global homotopy $F : [0, T) \times \mathbb{R}^N \to \mathbb{R}^N$, for some $T > 0$, satisfying:

(H1) $t \in [0, T) \mapsto F(t, \cdot) \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ is differentiable at 0;
(H2) $F(0, \cdot) = \mathrm{Id}$;
(H3) $F(t, x) = x$ for every $t \in [0, T)$, $x \in \partial S \setminus \omega$.

For notation convenience, we let $F_t(x) = F(t,x)$, while $DF_t(x) := D_xF(t,x)$. We can assume that $T > 0$ is sufficiently small so that $D_xF(t,x)$ is an invertible matrix for $(t,x) \in [0,T[\times\mathbb{R}^N$. Moreover, we define

$$Y = F_0' := \frac{d}{dt}F_t(\cdot)\Big|_{t=0} \in W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N),$$

so that, by (H1), $F_t(x) = x + tY(x) + \mathrm{o}(t)$ in $W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N)$, as $t \to 0$.

For every $t \in [0,T)$ we let $S_t = F_t(S)$ and $\Gamma_t = F_t(\partial S \cap \omega)$. Let $u_t \in H^1(S_t)$ be such that

$$\begin{cases} \Delta u_t = 0 & \text{in } S_t \\ u = \varphi & \text{on } \partial S\backslash\omega \\ u = 0 & \text{on } \Gamma_t \end{cases} \text{ that is } I_t := \int_{S_t} |\nabla u_t|^2 = \min\left\{ \int_{S_t} |\nabla v|^2 : \begin{array}{c} v \in H^1(S_t), \\ v = \varphi \text{ on } \partial S\backslash\omega, \\ v = 0 \text{ on } \Gamma_t \end{array} \right\}$$

Lemma A.1. *Under the previous assumptions, the function I_t is differentiable at $t = 0$, with*

$$\frac{d}{dt}I_t\Big|_{t=0} = \int_S \langle(\operatorname{div} Y\,\mathrm{Id} - 2DY)\nabla u, \nabla u\rangle$$

Proof. **Step 1:** Fixing the domain through a change of variables. For any $t \in [0,T[$, let $v_t \in H^1(S)$ be defined as $v_t := u_t \circ F_t$. Observe that for every $v \in H^1(S_t)$ one has

$$\int_{S_t} |\nabla v(y)|^2\,dy = \int_{F_t(S)} |\nabla v(y)|^2\,dy = \int_S |[(DF_t(x))^{-1}]^T \nabla(v(F_t(x))|^2 \det(DF(x))\,dx.$$

Thus v_t is the minimizer of

$$I_t = \min\left\{ \int_S \det(DF_t)|[(DF_t)^{-1}]^T\nabla w|^2 : \begin{array}{c} w \in H^1(S), \\ w = \varphi \text{ on } \partial S \end{array} \right\}$$

(recall that $\varphi = 0$ on $\partial S \cap \omega$) and a solution to the problem

$$\begin{cases} -\operatorname{div}(A_t\nabla v_t) = 0 & \text{in } S \\ v_t = \varphi & \text{on } \partial S \end{cases}$$

with $A_t(x) = \det(DF_t(x))(DF_t(x))^{-1}[(DF_t(x))^{-1}]^T$. Observe that $A_t(x)$ is symmetric and there exist $0 < \lambda < \Lambda$ such that

$$\lambda|\xi|^2 \leqslant \langle A_t(x)\xi,\xi\rangle \leqslant \Lambda|\xi|^2 \qquad \text{for all } x \in \mathbb{R}^N, t \in [0,T), \xi \in \mathbb{R}^N;$$

the map $t \in [0,T) \mapsto A_t \in L^\infty(\mathbb{R}^N)$ is differentiable at $t = 0$, and $\lim_{t\to0} A_t = A_0 = \mathrm{Id}$ uniformly in $\mathbb{R}^N$; and by recalling that $Y := F_0'$, we have by Jacobi's formula

$$\frac{d}{dt}A_t(x)\Big|_{t=0} = \operatorname{div} Y\,\mathrm{Id} - (DY + DY^T) \qquad \text{uniformly in } \mathbb{R}^N.$$

Step 2: Differentiability of the map $t \in [0,T) \mapsto v_t \in H^1(S)$ at $t = 0$. We introduce the incremental quotients

$$w_{t,0} := \frac{v_t - v_0}{t - 0} = \frac{v_t - u}{t} \in H^1_0(S), \qquad t \in]0,T[.$$

Each $w_{t,0}$ is a solution to

$$\text{(A.1)} \qquad \begin{cases} -\operatorname{div}(A_t\nabla w_{t,0}) = \operatorname{div}\left(\frac{A_t - \mathrm{Id}}{t}\nabla u\right) & \text{in } S \\ w_{t,0} = 0 & \text{on } \partial S. \end{cases}$$

We introduce the function $w_0 \in H^1_0(S)$ solution to

$$\begin{cases} -\Delta w_0 = \operatorname{div}(A_0' \nabla u) & \text{in } S \\ w_0 = 0 & \text{on } \partial S. \end{cases} \tag{A.2}$$

and show that indeed $w_{t,0} \to w_0$ as $t \to 0$, strongly in $H^1_0(S)$, so that $t \mapsto v_t$ is differentiable at $t = 0$, with $v_0' = w_0$. To do this, we subtract (A.2) from (A.1) and obtain the identity

$$-\operatorname{div}(A_t \nabla (w_{t,0} - w_0)) = \operatorname{div}((A_t - A_0)\nabla w_0) + \operatorname{div}\left(\left(\frac{A_t - A_0}{t} - A_0'\right)\nabla v_0\right)$$

Testing this equation by $w_{t,0} - w_0 \in H^1_0(S)$, we can conclude that

$$\left(\int_S |\nabla(w_{t,0} - w_0)|^2\right)^{\frac{1}{2}} \leqslant \frac{1}{\lambda}\left(\|A_t - A_0\|_\infty \|w_0\|_{H^1} + \left\|\frac{A_t - A_0}{t} - A_0'\right\|_\infty \|v_0\|_{H^1}\right)$$

and the claim follows recalling the properties of the functions A_t.

Step 3: Differentiability of the map $t \in [0,T) \mapsto I_t \in \mathbb{R}$ at $t = 0$. As a result of the previous step, the derivative of I_t at $t = 0$ is equal to

$$\lim_{t\to 0} \int_S \left(\left\langle \frac{A_t - A_0}{t}\nabla v_t + A_0 \nabla w_{t,0}, \nabla v_t\right\rangle + \langle A_0 \nabla v_0, \nabla w_{t,0}\rangle\right) = \int_S \langle A_0' \nabla u, \nabla u\rangle + 2\int_S \langle \nabla w_0, \nabla u\rangle$$

By testing the equation of u by $w_0 \in H^1_0(S)$, we see that the last term in the previous expression is zero, and by exploiting the symmetry of the scalar product we obtain

$$\left.\frac{d}{dt} I_t\right|_{t=0} = \int_S \langle A_0' \nabla u, \nabla u\rangle = \int_S \langle (\operatorname{div} Y \mathrm{Id} - 2DY)\nabla u, \nabla u\rangle \qquad \square$$

We now show that, if F_t leaves invariant a neighborhood of $\partial S \backslash \omega$, then the derivatives in Lemma A.1 can be expressed only in terms of the value of the first order behavior of F around $\partial S \cap \omega$.

Lemma A.2. *Assume* $(H1)$,$(H2)$, *and instead of* $(H3)$ *assume the stronger condition*

(H3') $F(t,x) = x$ *for every* $t \in [0,T)$, $x \in S\backslash\omega'$, *for some* $\omega' \Subset \omega$;

and assume also that $\partial S \cap \omega$ *is a smooth hypersurface Then we have*

$$\left.\frac{d}{dt} I_t\right|_{t=0} = -\int_{\omega\cap\partial S} (Y\cdot\nu)(\partial_\nu u)^2$$

In particular, the first derivative of the energy at 0, I_0', *depends on* F_t *only through the value of* $Y = F_0'$ *over* $\omega \cap \partial S$.

Proof. Observe that the assumptions imply that $Y \in W^{1,\infty}(\mathbb{R}^N)$ satisfies $Y = 0$ in $S\backslash\omega$. Moreover, since u is harmonic in S, $u \in H^2(O)$, for every $O \Subset \omega \cap S$. Thus we can test the equation of u with $Y \cdot \nabla u \in H^1(S)$, obtaining

$$\begin{aligned} 0 &= \int_S \nabla u \cdot \nabla(Y\cdot\nabla u) - \int_{\omega\cap\partial S}(Y\cdot\nabla u)(\nu\cdot\nabla u) \\ &= \int_{\omega\cap S}\left(\langle\nabla u, DY\nabla u\rangle + \langle \nabla u, D^2 u Y\rangle\right) - \int_{\omega\cap\partial S}(Y\cdot\nabla u)(\nu\cdot\nabla u) \\ &= \int_{\omega\cap S}\left(\langle\nabla u, DY\nabla u\rangle + \frac{1}{2}\langle\nabla|\nabla u|^2, Y\rangle\right) - \int_{\omega\cap\partial S}(Y\cdot\nabla u)(\nu\cdot\nabla u) \end{aligned}$$

(the boundary term is well defined since $\omega \cap \partial S$ is a smooth hypersurface). A further integration by parts and the observation that, since $u = 0$ on $\omega \cap \partial S$, we have $|\nabla u| = |\partial_\nu u|$ and $\nabla u = (\nu \cdot \nabla u)\nu$ on $\omega \cap \partial S$, yields the identities

$$\int_{\omega\cap S} \langle (\operatorname{div} Y\,\mathrm{Id} - 2DY)\nabla u, \nabla u\rangle = \int_{\omega\cap\partial S} \big((Y\cdot\nu)|\nabla u|^2 - 2(Y\cdot\nabla u)(\nu\cdot\nabla u)\big) = -\int_{\omega\cap\partial S} (Y\cdot\nu)(\partial_\nu u)^2.$$

□

Acknowledgments. The authors are partially supported by the ERC Advanced Grant 2013 n. 339958 "Complex Patterns for Strongly Interacting Dynamical Systems - COMPAT".

N. Soave is partially supported by the PRIN-2015KB9WPT_010 Grant: "Variational methods, with applications to problems in mathematical physics and geometry".

H. Tavares is partially supported by FCT - Portugal through the project PEst-OE/EEI/LA0009/2013.

A. Zilio is partially supported by the ERC Advanced Grant 2013 n. 321186 "ReaDi – Reaction-Diffusion Equations, Propagation and Modelling".

References

[1] B. Bourdin, D. Bucur, and É. Oudet. Optimal partitions for eigenvalues. *SIAM J. Sci. Comput.*, 31(6):4100–4114, 2009/10.

[2] F. Bozorgnia. Uniqueness result for long range spatially segregation elliptic system. 2016.

[3] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.

[4] D. Bucur, G. Buttazzo, and A. Henrot. Existence results for some optimal partition problems. *Adv. Math. Sci. Appl.*, 8(2):571–579, 1998.

[5] L. A. Caffarelli and F. H. Lin. An optimal partition problem for eigenvalues. *J. Sci. Comput.*, 31(1-2):5–18, 2007.

[6] L. A. Caffarelli and F.-H. Lin. Singularly perturbed elliptic systems and multi-valued harmonic functions with free boundaries. *J. Amer. Math. Soc.*, 21(3):847–862, 2008.

[7] L. A. Caffarelli, V. Quitalo, and S. Patrizi. On a long range segregation model. 2015, arXiv:1505.05433.

[8] G. Colombo and A. Marigonda. Differentiability properties for a class of non-convex functions. *Calc. Var. Partial Differential Equations*, 25(1):1–31, 2006.

[9] M. Conti, S. Terracini, and G. Verzini. Nehari's problem and competing species systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 19(6):871–888, 2002.

[10] M. Conti, S. Terracini, and G. Verzini. An optimal partition problem related to nonlinear eigenvalues. *J. Funct. Anal.*, 198(1):160–196, 2003.

[11] M. Conti, S. Terracini, and G. Verzini. On a class of optimal partition problems related to the Fučík spectrum and to the monotonicity formulae. *Calc. Var. Partial Differential Equations*, 22(1):45–72, 2005.

[12] M. Conti, S. Terracini, and G. Verzini. A variational problem for the spatial segregation of reaction-diffusion systems. *Indiana Univ. Math. J.*, 54(3):779–815, 2005.

[13] E. N. Dancer, K. Wang, and Z. Zhang. The limit equation for the Gross-Pitaevskii equations and S. Terracini's conjecture. *J. Funct. Anal.*, 262(3):1087–1131, 2012.

[14] H. Federer. Curvature measures. *Trans. Amer. Math. Soc.*, 93:418–491, 1959.

[15] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[16] A. Henrot. *Extremum problems for eigenvalues of elliptic operators.* Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.

[17] A. Henrot and M. Pierre. *Variation et optimisation de formes*, volume 48 of *Mathématiques & Applications (Berlin) [Mathematics & Applications].* Springer, Berlin, 2005. Une analyse géométrique. [A geometric analysis].

[18] B. Noris, H. Tavares, S. Terracini, and G. Verzini. Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition. *Comm. Pure Appl. Math.*, 63(3):267–302, 2010.

[19] C. Nour, R. J. Stern, and J. Takche. Proximal smoothness and the exterior sphere condition. *J. Convex Anal.*, 16(2):501–514, 2009.

[20] A. Petrosyan, H. Shahgholian, and N. Uraltseva. *Regularity of free boundaries in obstacle-type problems*, volume 136 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2012.

[21] M. Ramos, H. Tavares, and S. Terracini. Extremality conditions and regularity of solutions to optimal partition problems involving Laplacian eigenvalues. *Arch. Ration. Mech. Anal.*, 220(1):363–443, 2016.
[22] N. Soave, H. Tavares, S. Terracini, and A. Zilio. Hölder bounds and regularity of emerging free boundaries for strongly competing Schrödinger equations with nontrivial grouping. *Nonlinear Anal.*, 138:388–427, 2016.
[23] N. Soave and A. Zilio. Uniform bounds for strongly competing systems: the optimal Lipschitz case. *Arch. Ration. Mech. Anal.*, 218(2):647–697, 2015.
[24] N. Soave and A. Zilio. On phase separation in systems of coupled elliptic equations: Asymptotic analysis and geometric aspects. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, pages –, 2016.
[25] H. Tavares and S. Terracini. Regularity of the nodal set of segregated critical configurations under a weak reflection law. *Calc. Var. Partial Differential Equations*, 45(3-4):273–317, 2012.
[26] H. Tavares and S. Terracini. Sign-changing solutions of competition-diffusion elliptic systems and optimal partition problems. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 29(2):279–300, 2012.
[27] S. Terracini, G. Verzini, and A. Zilio. Uniform Hölder regularity with small exponent in competition-fractional diffusion systems. *Discrete Contin. Dyn. Syst.*, 34(6):2669–2691, 2014.
[28] S. Terracini, G. Verzini, and A. Zilio. Uniform Hölder bounds for strongly competing systems involving the square root of the laplacian. *J. Eur. Math. Soc. (JEMS)*, 18(12):2865–2924, 2016.
[29] G. Verzini and A. Zilio. Strong competition versus fractional diffusion: the case of Lotka-Volterra interaction. *Comm. Partial Differential Equations*, 39(12):2284–2313, 2014.
[30] J. Wei and T. Weth. Asymptotic behaviour of solutions of planar elliptic systems with strong competition. *Nonlinearity*, 21(2):305–317, 2008.

Nicola Soave
Dipartimento di Matematica, Politecnico di Milano,
Via Edoardo Bonardi 9, 20133 Milano, Italy
E-mail address: nicola.soave@gmail.com; nicola.soave@polimi.it

Hugo Tavares
CAMGSD (Center for Mathematical Analysis, Geometry and Dynamical Systems)
Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa
Av. Rovisco Pais, 1049-001 Lisboa, Portugal
and
Departamento de Matemática, Faculdade de Ciências da Universidade de Lisboa
Edifício C6, Piso 1, Campo Grande 1749-016 Lisboa, Portugal
E-mail address: htavares@math.ist.utl.pt; hrtavares@ciencias.ulisboa.pt

Susanna Terracini
Dipartimento di Matematica "Giuseppe Peano", Università di Torino,
Via Carlo Alberto, 10, 10123 Torino, Italy
E-mail address: susanna.terracini@unito.it

Alessandro Zilio
École des Hautes Études en Sciences Sociales, PSL Research University Paris, Centre d'analyse et de mathématique sociales (CAMS), CNRS,
190-198 Avenue de France, 75244, Paris CEDEX 13, France
E-mail address: azilio@ehess.fr, alessandro.zilio@polimi.it