INTEGRAL REPRESENTATION RESULTS FOR FUNCTIONALS DEFINED ON SBV$(\Omega;\mathbb{R}^{\mathbf{m}})$

ANDREA BRAIDES [1] AND VALERIA CHIADÒ PIAT [2]

Abstract. We show that lower semicontinuous functionals defined on De Giorgi and Ambrosio's space of *special* functions of bounded variation admit an integral representation with Carathéodory integrands, under some growth and continuity conditions.

1. Introduction

The space $SBV(\Omega;\mathbb{R}^m)$ of $\mathbb{R}^m$-valued special functions of bounded variation on the open set Ω, introduced by Ambrosio and De Giorgi in [28], is the subset of $BV(\Omega;\mathbb{R}^m)$ of all integrable functions whose distributional derivative Du can be expressed through the equality of measures

$$Du = \nabla u \cdot \mathcal{L}_n \llcorner \Omega + (u^+ - u^-) \otimes \nu_u \cdot \mathcal{H}^{n-1} \llcorner S_u,$$

where $\mathcal{L}_n$ is the Lebesgue measure on $\mathbb{R}^n$, $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure, S_u represents the set of *jump points* of u in Ω, ν_u the measure theoretical normal to S_u, and u^-, u^+ the traces of u on both sides of S_u. The introduction of this space allows a weak formulation for many problems involving a "free discontinuity set" such as in image segmentation, fracture mechanics, minimal partitioning, etc. (see [34], [11], [2], [15], [8], [24]), by taking into account functionals of the form

$$F(u,B) = \int_B f(x,u,\nabla u)\,dx + \int_{S_u \cap B} \varphi(x,u^+,u^-,\nu)\,d\mathcal{H}^{n-1}, \tag{1.1}$$

defined on the space $SBV(\Omega;\mathbb{R}^m)$ (here $B \in \mathcal{B}(\Omega)$, the family of Borel subsets of Ω). The application of the so-called direct methods of the calculus of variations to minimum problems involving these functionals can be carried on thanks to a compactness theorem by Ambrosio [5], under some conditions that guarantee the lower semicontinuity of the functional F with respect to a.e. convergence (see [3], [8], [4], [5], [6]).

In order to describe the behaviour of minimizing sequences for non-lower semicontinuous functionals, or to perform an asymptotic analysis for sequences of functionals of the form (1.1) through the methods of relaxation and Γ-convergence, a fundamental step is to give an integral representation theorem, which characterizes functionals that can be written as in (1.1) by a list of abstract properties, easier to verify. The purpose of this work is to generalize previous results on functionals defined on Sobolev spaces to the case of functionals defined on special functions of bounded variation, obtaining the representation above with Carathéodory integrands. Note that, unlike the Sobolev spaces case, this result does not cover all cases, since even homogeneous lower semicontinuous functionals invariant by translations may not admit an integral

[1]Permanent Address: Dipartimento di Elettronica per l'Automazione, Università di Brescia, via Valotti 9, I-25060 Brescia, Italy.

[2]Permanent Address: Dipartimento di Matematica del Politecnico di Torino, C.so Duca degli Abruzzi 24, I-10129 Torino, Italy.

representation with continuous integrands; some steps towards a general integral representation result without Carathéodory conditions are obtained in Section 3. We remark, however, that the study of integrals as in (1.1) seems to be sufficient for all main applications.

The main result of this paper is Theorem 2.4 which shows that functionals $F : SBV(\Omega;\mathbb{R}^m) \times \mathcal{B}(\Omega) \to [0,+\infty]$ satisfying standard locality, measurability and semicontinuity hypotheses (conditions (i)–(iii) and (v) in Theorem 2.4, which were already stated by Buttazzo and Dal Maso in [20] for functionals on Sobolev spaces), the growth hypothesis

$$\alpha\Big(\int_B |\nabla u|^p\,dx + \int_{S_u\cap B}(1+|u^+-u^-|)d\mathcal{H}^{n-1}\Big) \le F(u,B)$$
$$\le \beta\Big(\int_B (a(x)+|\nabla u|^p)\,dx + \int_{S_u\cap B}(1+|u^+-u^-|)d\mathcal{H}^{n-1}\Big)$$

for some α, $\beta > 0$, $a \in \mathrm{L}^1(\Omega)$, and a "continuity condition for the jump energy" (see hypothesis (vi), which is somehow the analogue of Buttazzo and Dal Maso's "weak ω condition" (v)), can be represented as in (1.1) with a Carathéodory integrand. Since all these conditions are also necessary, Theorem 2.4 gives a complete characterization of such integral functionals.

The proof of this result is rather technical, involving blow-up techniques and approximation results for functions of bounded variation. As a first step, in Section 3 we give an integral representation result (Theorem 3.2) for functionals defined on the subspace of $BV(\Omega;\mathbb{R}^m)$ of piecewise constant functions. This has an interest both in itself (since it provides a good model for some problems in image segmentation, see for instance [25]) and for subsequent applications. Results of this kind have already been obtained only under strong continuity assumptions (see, for instance, [7]). The main feature of our method is that it permits to deal with the case when the function φ is discontinuous in the space variable. Note moreover that no continuity of the integrands is required also with respect to the jumps. A first application of this result is given in Section 4 to obtain an integral representation of $F(u, S_u \cap B)$ when u is piecewise constant. The second main step is an approximation lemma: a crucial point in the proof of integral representation results is often the passage from the representation on a set of simpler functions to a wider set by "strong approximation" (*e.g.*, from piecewise affine functions to Sobolev functions by strong approximation in $\mathrm{W}^{1,p}$; see the proof of Theorem 4.3.2 in [18]). In the case of $SBV(\Omega;\mathbb{R}^m)$-functions we introduce, in Section 5, the notion of *strong convergence in SBV*, and we prove a "strong density theorem" of piecewise smooth functions (see Lemma 5.2). This result is crucial to obtain an inequality for the representation (1.1) (see Lemma 6.2). The converse inequality is obtained adapting an argument by Ambrosio [5] which allows to pass from SBV-functions to Lipschitz maps (see the proof of Proposition 6.9), through a "partial locality result" (Lemma 6.8). We conclude the paper with applications to problems in image segmentation and fracture mechanics (Sections 7 and 8).

2. Preliminaries and statement of the main result

Let Ω be a bounded open subset of $\mathbb{R}^n$; we will use standard notation for the Sobolev and Lebesgue spaces $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^m)$ and $\mathrm{L}^p(\Omega;\mathbb{R}^m)$. The L^∞-norm of a function u is denoted simply by $\|u\|_\infty$. We denote by $\mathcal{A}(\Omega)$ and $\mathcal{B}(\Omega)$ the families of the open and Borel subsets of Ω, respectively, and if $x, y \in \mathbb{R}^n$ then $\langle x, y\rangle$ stands for their scalar product. $B_\rho(x)$ is the open ball of center x and radius ρ; $B_\rho = B_\rho(0)$. $M^{n\times m}$ will denote the space of $n \times m$ matrices. If $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ is an affine transformation $\Phi x = b + Lx$ with L linear, then $\|\Phi\| = \|L\| + |b|$. The letter c will denote a strictly positive constant independent from the parameters under consideration, whose value may vary from line to line.

The Lebesgue measure and the Hausdorff $(n-1)$-dimensional measure in $\mathbb{R}^n$ are denoted by $\mathcal{L}_n$ and $\mathcal{H}^{n-1}$, respectively, but we write also $|E|$ in place of $\mathcal{L}_n(E)$. Note that for $n = 1$ we have $\mathcal{H}^0 = \#$ the counting measure. Sometimes we use the shorter notation $\{u < t\}$ for $\{x \in \mathbb{R}^n : u(x) < t\}$ (and similar) when no confusion is possible. If E is a subset of $\mathbb{R}^n$ then χ_E is its *characteristic function*, defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

Given a vector-valued measure μ on Ω, we adopt the notation $|\mu|$ for its total variation (see Federer [31]), and $\mathcal{M}(\Omega)$ is the set of all signed measures on Ω with bounded total variation. We say that $u \in \mathrm{L}^1(\Omega;\mathbb{R}^m)$ is a *function of bounded variation*, and we write $u \in BV(\Omega;\mathbb{R}^m)$, if every its distributional first derivatives $D_i u_j$ belong to $\mathcal{M}(\Omega)$. We denote by Du the $M^{n\times m}$-valued measure whose entries are $D_i u_j$. We write $BV(\Omega) = BV(\Omega;\mathbb{R})$. For the general exposition of the theory of functions of bounded variation we refer to Federer [31], Evans and Gariepy [30], Giusti [32], Vol'pert [35], and Ziemer [36]. We recall some results needed in the sequel.

The space $BV(\Omega;\mathbb{R}^m)$ is a Banach space, if endowed with the BV norm

$$\|u\|_{BV(\Omega;\mathbb{R}^m)} = \|u\|_{\mathrm{L}^1(\Omega;\mathbb{R}^m)} + |Du|(\Omega).$$

We say that $u_h \rightharpoonup u$ in BV-w^* (weakly* in $BV(\Omega;\mathbb{R}^m)$) if $\sup_h |Du_h|(\Omega) < +\infty$ and $u_h \to u$ in $\mathrm{L}^1(\Omega;\mathbb{R}^m)$. Recall that if Ω is bounded and $\partial\Omega$ is Lipschitz, then every bounded sequence in $BV(\Omega;\mathbb{R}^m)$ admits a subsequence converging in BV-w^*.

We denote by ∇u the density of the absolutely continuous part of Du with respect to the Lebesgue measure, and by S_u the complement of the Lebesgue set of u; *i.e.*, $x \notin S_u$ if and only if

$$\lim_{\rho\to 0^+} \rho^{-n} \int_{B_\rho(x)} |u(y) - z|\, dy = 0$$

for some $z \in \mathbb{R}^m$. If z exists then it is unique, and we denote it by $\widetilde{u}(x)$, the *approximate limit* of u at x. For any function $u \in L^1(\Omega;\mathbb{R}^m)$ the set S_u is Lebesgue-negligible and $\widetilde{u}$ is a Borel function equal to u almost everywhere. In the sequel we will tacitly assume $u(x) = \widetilde{u}(x)$ at *every* point of $\Omega \setminus S_u$. If $u \in BV(\Omega;\mathbb{R}^m)$, then the Hausdorff dimension of S_u is at most $(n-1)$; more precisely, S_u is *rectifiable*; i.e., there is a countable sequence of C^1 hypersurfaces Γ_i which covers $\mathcal{H}^{n-1}$-almost all of S_u; i.e., $\mathcal{H}^{n-1}\big(S_u \setminus \bigcup_{i=1}^\infty \Gamma_i\big) = 0$. Moreover, for $\mathcal{H}^{n-1}$-almost every $x \in S_u$ it is possible to find a, $b \in \mathbb{R}^m$ and $\nu \in S^{n-1}$ such that

$$\lim_{\rho\to 0^+} \rho^{-n} \int_{B_\rho^\nu(x)} |u(y) - a|\, dy = 0, \qquad \lim_{\rho\to 0^+} \rho^{-n} \int_{B_\rho^{-\nu}(x)} |u(y) - b|\, dy = 0,$$

where $B_\rho^\nu(x) = \{y \in B_\rho(x) : \langle y - x, \nu\rangle > 0\}$. The triplet (a, b, ν) is uniquely determined up to a change of sign of ν and an interchange between a and b; it will be denoted by $(u^+(x), u^-(x), \nu_u(x))$. We define the relation $(a, b, \nu) \sim (a', b', \nu')$ if $a = a'$, $b = b'$ and $\nu = \nu'$, or $a = b'$, $b = a'$ and $\nu = -\nu'$. If $x \in \Omega \setminus S_u$ we set $u^+(x) = u^-(x) = \widetilde{u}(x)$.

We say that a set E is of *finite perimeter in* Ω, or a Caccioppoli set, if $\chi_E \in BV(\Omega)$. We will set $\partial^* E \cap \Omega = S_{\chi_E} \cap \Omega$ the *reduced boundary* of E in Ω. For $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E$ it is possible to define a measure theoretical *interior normal* to E $\nu_E(x) \in S^{n-1}$ such that

$$D\chi_E(B) = \int_{B\cap\partial^* E} \nu_E(x) d\mathcal{H}^{n-1}(x)$$

for every $B \in \mathcal{B}(\Omega)$. Note that $|D\chi_E|(\Omega) = \mathcal{H}^{n-1}(\partial^* E \cap \Omega)$ for every E of finite perimeter in Ω. Note that if E is a set of finite perimeter then $\partial^* E$ is rectifiable.

We recall the Fleming & Rishel *coarea formula.* Let u be a Lipschitz function. We have that $\{u > t\}$ is a set of finite perimeter for a.e. $t \in \mathbb{R}$, and

$$\int_\Omega v|\nabla u|\, dx = \int_{-\infty}^{+\infty} dt \int_{\partial^*\{u>t\}\cap\Omega} \widetilde{v}\, d\mathcal{H}^{n-1}$$

for every $v \in BV(\Omega)$.

In general, for a function $u \in BV(\Omega; \mathbb{R}^m)$, we have the decomposition

$$Du = \nabla u \cdot \mathcal{L}_n + J_u + C_u,$$

where $\nabla u \cdot \mathcal{L}_n$ is the *Lebesgue part* of Du,

$$J_u = (u^+ - u^-) \otimes \nu_u \cdot \mathcal{H}^{n-1} \llcorner S_u$$

is the *Hausdorff part*, or *jump* part, and C_u the *Cantor part* of Du. We recall that the measure Cu is singular with respect to the Lebesgue measure and it is "diffuse"; *i.e.*, $C_u(S) = 0$ for every set S of Hausdorff dimension $n-1$. We will use the notation $D_s u$ for the singular part of Du with respect to the Lebesgue measure; *i.e.*, $D_s u = J_u + C_u$, so that $C_u = D_s u \llcorner \Omega \setminus S_u$.

We say that a function $u \in BV(\Omega; \mathbb{R}^m)$ is a *special function of bounded variation* if $C_u \equiv 0$, or equivalently if

$$Du = \nabla u \cdot \mathcal{L}_n + (u^+ - u^-) \otimes \nu_u \cdot \mathcal{H}^{n-1} \llcorner S_u.$$

We denote the space of the special functions of bounded variation by $SBV(\Omega; \mathbb{R}^m)$. The introduction of this space is due to De Giorgi and Ambrosio [28]. For the properties of functions $u \in SBV(\Omega)$ we refer to [3], [4] and [28].

Remark 2.1. Note that a standard use of the coarea formula (see *e.g.* [15] Theorem 2.1) shows that fixed $u \in BV(\Omega; \mathbb{R}^m)$ and $\varepsilon > 0$, it is possible to construct $u_\varepsilon \in SBV(\Omega; \mathbb{R}^m)$ with $\nabla u_\varepsilon \equiv 0$ a.e., such that

$$\|u - u_\varepsilon\|_\infty \le \varepsilon, \qquad \mathcal{H}^{n-1}(S_{u_\varepsilon} \cap \Omega) \le \mathcal{H}^{n-1}(S_u \cap \Omega) + c\frac{1}{\varepsilon}|Du|(\Omega \setminus S_u),$$

with c depending only on n. This construction will be used in Remark 4.2 and in the proof of Lemma 6.2.

Let $p > 1$; the space $SBV^p(\Omega; \mathbb{R}^m)$ is defined as the subspace of $SBV(\Omega; \mathbb{R}^m)$ of functions u such that

$$\mathcal{H}^{n-1}(S_u \cap \Omega) < +\infty \qquad \text{and} \qquad \nabla u \in \mathrm{L}^p(\Omega; M^{n\times m}).$$

We define the *weak convergence* on $SBV^p(\Omega;\mathbb{R}^m)$ as follows. We say that a sequence (u_h) in $SBV^p(\Omega;\mathbb{R}^m)$ converges weakly in $SBV^p(\Omega;\mathbb{R}^m)$ to $u \in SBV^p(\Omega;\mathbb{R}^m)$ if $u_h \to u$ in $\mathrm{L}^1(\Omega;\mathbb{R}^m)$, $\sup_h |Du_h|(\Omega) < +\infty$, and $\nabla u_h \rightharpoonup \nabla u$ weakly in $\mathrm{L}^p(\Omega; M^{n\times m})$.

The introduction of this kind of convergence is motivated by the following compactness theorem (Ambrosio [3]).

Theorem 2.2. *Let (u_h) be a sequence in $SBV^p(\Omega;\mathbb{R}^m)$ such that*

$$\sup_h \|u_h\|_{BV(\Omega;\mathbb{R}^m)} < +\infty,$$

and

$$\sup_h \Big\{ \int_\Omega |\nabla u_h|^p \, dx + \mathcal{H}^{n-1}(S_{u_h} \cap \Omega) \Big\} < +\infty;$$

then there exists a subsequence (u_{h_j}) converging weakly in $SBV^p(\Omega;\mathbb{R}^m)$ to some function u. Moreover $\mathcal{H}^{n-1}(S_u \cap \Omega) \le \liminf_j \mathcal{H}^{n-1}(S_{u_{h_j}} \cap \Omega)$.

From Theorem 2.2 above we deduce the lower semicontinuity with respect to the L^1 convergence of the functional

$$\int_\Omega |\nabla u|^p \, dx + \int_{S_u} |u^+ - u^-| d\mathcal{H}^{n-1} + \mathcal{H}^{n-1}(S_u \cap \Omega)$$

defined on $SBV^p(\Omega;\mathbb{R}^m)$. The lower semicontinuity of the term $\int_{S_u} |u^+ - u^-| \, d\mathcal{H}^{n-1}$ follows from the weak convergence of the jump part of the measures Du_h to the jump part of Du. Similarly, we have the lower semicontinuity of the functional

$$\int_\Omega |\nabla u|^p \, dx + \mathcal{H}^{n-1}(S_u \cap \Omega)$$

defined on $\{u \in SBV^p(\Omega;\mathbb{R}^m) : \|u\|_\infty \le c\}$, with c a positive constant. More general lower semicontinuity theorems have been proven for functionals of the form

$$\int_\Omega f(x,u,\nabla u) \, dx + \int_{S_u\cap\Omega} \varphi(x,u^+,u^-,\nu_u) d\mathcal{H}^{n-1}$$

(see [4], [5], [8]).

In the framework of the direct methods of the calculus of variations it is convenient to give a characterization of integral functionals as above by some abstract properties which are stable by perturbations (as relaxation or Γ-convergence). As for functionals defined on Sobolev spaces we have the following result by Buttazzo and Dal Maso [20].

Theorem 2.3. *Let $F : \mathrm{W}^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{B}(\Omega) \to [0,+\infty)$ be a functional satisfying the following conditions:*

(i) (locality on $\mathcal{A}(\Omega)$) *if $u = v$ a.e. on $A \in \mathcal{A}(\Omega)$ then $F(u,A) = F(v,A)$;*

(ii) (measure property) *for every $u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^m)$ the set function $B \mapsto F(u,B)$ is a Borel measure;*

(iii) (lower semicontinuity) *for all $A \in \mathcal{A}(\Omega)$ the functional $F(\cdot,A)$ is lower semicontinuous on $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^m)$ with respect to the $\mathrm{L}^1(\Omega;\mathbb{R}^m)$ convergence;*

(iv) (growth condition of order p) *there exists $a \in \mathrm{L}^1(\Omega)$ such that*

$$\alpha \int_B |\nabla u|^p \, dx \le F(u,B) \le \beta \int_B (a(x) + |\nabla u|^p) \, dx$$

for all $u \in \mathrm{W}^{1,p}(\Omega; \mathbb{R}^m)$ *and* $B \in \mathcal{B}(\Omega)$*;*

(v) ("weak ω condition") *there exists a sequence* (ω_k) *of integrable moduli of continuity such that*

$$|F(u+s, A) - F(u, A)| \leq \int_A \omega_k(x, |s|)\, dx$$

for every $k \in \mathbb{N}$, $A \in \mathcal{A}(\Omega)$, $s \in \mathbb{R}^m$, $u \in C^1$ *such that* $\|u\|_\infty \leq k$, $\|u+s\|_\infty \leq k$ *and* $\|Du\|_\infty \leq k$.

Then there exists a Carathéodory function $f : \Omega \times \mathbb{R}^m \times M^{n\times m} \to [0, +\infty)$ *such that*

$$F(u, B) = \int_B f(x, u(x), \nabla u(x))\, dx$$

for all $u \in \mathrm{W}^{1,p}(\Omega; \mathbb{R}^m)$ *and* $B \in \mathcal{B}(\Omega)$.

We are going to prove a similar integral representation theorem for functionals defined on $SBV^p(\Omega; \mathbb{R}^m)$. Our main result is the following.

THEOREM 2.4. *Let* $F : SBV^p(\Omega; \mathbb{R}^m) \times \mathcal{B}(\Omega) \to [0, +\infty)$ *be a functional satisfying the following conditions:*

(i) (locality on $\mathcal{A}(\Omega)$) *if* $u = v$ a.e. *on* $A \in \mathcal{A}(\Omega)$ *then* $F(u, A) = F(v, A)$*;*

(ii) (measure property) *for every* $u \in SBV^p(\Omega; \mathbb{R}^m)$ *the set function* $B \mapsto F(u, B)$ *is a Borel measure;*

(iii) (lower semicontinuity) *for all* $A \in \mathcal{A}(\Omega)$ *the functional* $F(\cdot, A)$ *is lower semicontinuous on* $SBV^p(\Omega; \mathbb{R}^m)$ *with respect to the* $\mathrm{L}^1(\Omega; \mathbb{R}^m)$ *convergence;*

(iv) (growth condition of order p) *there exist* α, $\beta > 0$, $a \in \mathrm{L}^1(\Omega)$ *such that*

$$\alpha\Big(\int_B |\nabla u|^p\, dx + \int_{S_u\cap B} (1 + |u^+ - u^-|) d\mathcal{H}^{n-1}\Big) \leq F(u, B)$$
$$\leq \beta\Big(\int_B (a(x) + |\nabla u|^p)\, dx + \int_{S_u\cap B} (1 + |u^+ - u^-|) d\mathcal{H}^{n-1}\Big)$$

for all $u \in SBV^p(\Omega; \mathbb{R}^m)$ *and* $B \in \mathcal{B}(\Omega)$*;*

(v) ("weak ω condition") *there exists a sequence* (ω_k) *of integrable moduli of continuity such that*

$$|F(u+s, A) - F(u, A)| \leq \int_A \omega_k(x, |s|)\, dx$$

for every $k \in \mathbb{N}$, $A \in \mathcal{A}(\Omega)$, $s \in \mathbb{R}^m$, $u \in C^1$ *such that* $\|u\|_\infty \leq k$, $\|u+s\|_\infty \leq k$ *and* $\|Du\|_\infty \leq k$*;*

(vi) (continuity of the jump energy) *there exists a modulus of continuity* ω *such that*

$$|F(u, S) - F(v, S)| \leq \int_S \omega(|u^+ - v^+| + |u^- - v^-|) d\mathcal{H}^{n-1},$$

for all $u, v \in SBV^p(\Omega; \mathbb{R}^m)$ *and* $S \subset S_u \cap S_v$ *(we choose the orientation* $\nu_v = \nu_u$ $\mathcal{H}^{n-1}$-a.e. *on* $S_u \cap S_v$*).*

Then there exist Carathéodory functions $f : \Omega \times \mathbb{R}^m \times M^{n\times m} \to [0, +\infty)$ *and* $\varphi : \Omega \times \mathbb{R}^m \times \mathbb{R}^m \times S^{n-1} \to [0, +\infty)$ *such that*

$$F(u, B) = \int_B f(x, u(x), \nabla u(x))\, dx + \int_{S_u\cap B} \varphi(x, u^+, u^-, \nu_u) d\mathcal{H}^{n-1} \tag{2.1}$$

for all $u \in SBV^p(\Omega;\mathbb{R}^m)$ *and* $B \in \mathcal{B}(\Omega)$.

Remark 2.5. Sometimes condition (vi) may not be easy to verify. An alternative condition, which is suitable for many applications and much easier to handle, is given in Lemma 6.2. Condition (vi) can be completely removed if we want to obtain the integral representation only on "Caccioppoli partitions" (see the next section). Note that by Theorem 2.2 we can replace L^1-convergence with weak SBV^p-convergence in condition (iii).

Remark 2.6. The function f can be defined simply by

$$f(x_0,u_0,\xi_0) = \limsup_{\rho\to 0+} \frac{F(u_0+\xi_0(x-x_0), B_\rho(x_0))}{\mathcal{L}_n(B_\rho)}.$$

Such a simple description for φ, substituting somehow $\mathcal{L}_n$ by $\mathcal{H}^{n-1}$, is not possible, and in general false. However a more complex derivation formula for φ can be given, and is described in Section 3 (see Theorems 3.2 and 4.1).

Remark 2.7. The value of φ on the set $\Omega\times\Delta\times S^{n-1}$, where $\Delta = \{(a,a):\ a\in\mathbb{R}^m\}$ is the "diagonal" of $\mathbb{R}^m\times\mathbb{R}^m$, will never be taken into account. Hence, the Carathéodory condition for φ means that $\varphi(\cdot,a,b,\nu)$ is measurable for all $(a,b,\nu)\in\mathbb{R}^m\times\mathbb{R}^m\times S^{n-1}$, and $\varphi(x,\cdot,\cdot,\cdot)$ is continuous on $(\mathbb{R}^m\times\mathbb{R}^m\setminus\Delta)\times S^{n-1}$ for all $x\in\Omega$.

We can easily deduce from Theorem 2.4 a further integral representation result for functionals satisfying a different kind of growth conditions.

COROLLARY 2.8. *Let* $F : SBV^p(\Omega;\mathbb{R}^m)\times\mathcal{B}(\Omega)\to[0,+\infty)$ *be a functional satisfying conditions* (i)–(iii), (v) *and* (vi) *of Theorem* 2.4, *and the growth condition*

$$0 \le F(u,B) \le \beta\Big(\int_B (a(x)+|\nabla u|^p)\,dx + \mathcal{H}^{n-1}(S_u\cap B) + \int_{S_u\cap B} |u^+-u^-|d\mathcal{H}^{n-1}\Big)$$

for all $u\in SBV^p(\Omega;\mathbb{R}^m)$ *and* $B\in\mathcal{B}(\Omega)$. *Then there exist Carathéodory functions* $f:\Omega\times\mathbb{R}^m\times M^{n\times m}\to[0,+\infty)$ *and* $\varphi:\Omega\times\mathbb{R}^m\times\mathbb{R}^m\times S^{n-1}\to[0,+\infty)$ *such that* (2.1) *holds for all* $u\in SBV^p(\Omega;\mathbb{R}^m)$ *and* $B\in\mathcal{B}(\Omega)$.

Proof. It suffices to apply Theorem 2,4 to the functional

$$F(u,B) + \int_B |\nabla u|^p\,dx + \mathcal{H}^{n-1}(S_u\cap B) + \int_{S_u\cap B} |u^+-u^-|d\mathcal{H}^{n-1},$$

which satisfies all conditions (i)–(vi) thanks to Theorem 2.2. □

3. Integral representation of functionals defined on Caccioppoli partitions

In this section we introduce some notation and properties of piecewise constant BV-functions, and give a first integral representation result. In order to simplify the statements, in the following we use the symbol $SBV_0(\Omega;\mathbb{R}^m)$ to denote the space

$$SBV_0(\Omega;\mathbb{R}^m) = \{u\in BV(\Omega;\mathbb{R}^m):\ \mathcal{H}^{n-1}(S_u\cap\Omega)<+\infty,\ \nabla u = 0 \text{ a.e. in } \Omega\}$$

of SBV^p-functions whose approximate gradient vanishes almost everywhere in Ω. Moreover, we say that a sequence (E_i) is a *Borel partition* of a given set $B \in \mathcal{B}(\mathbb{R}^n)$ if and only if

$$E_i \in \mathcal{B}(\mathbb{R}^n), \qquad \forall i \in \mathbb{N}; \qquad E_i \cap E_j = \emptyset \quad \text{when } i \neq j; \qquad \bigcup_{i=1}^{\infty} E_i = B.$$

More in general, we can weaken the above conditions by requiring that $|E_i \cap E_j| = 0$ when $i \neq j$ and $|B \triangle (\bigcup_{i=1}^{\infty} E_i)| = 0$. We say that (E_i) is a *Caccioppoli partition* if each E_i is a set of finite perimeter. The relation between Caccioppoli partitions and functions in $SBV_0(\Omega; \mathbb{R}^m)$ is expressed in the following lemma, whose proof can be found in [25] (see Lemmas 1.4, 1.10 and Remark 1.5 therein).

LEMMA 3.1. *If $u \in SBV_0(\Omega; \mathbb{R}^m)$ then there exist a Borel partition (E_i) of Ω, and a sequence (u_i) in $\mathbb{R}^m$ with $u_i \neq u_j$ for $i \neq j$, such that*

$$u = \sum_{i=1}^{\infty} u_i \chi_{E_i} \quad \textit{a.e. in } \Omega,$$

$$\mathcal{H}^{n-1}(S_u \cap \Omega) = \frac{1}{2} \sum_{i=1}^{\infty} \mathcal{H}^{n-1}(\partial^* E_i \cap \Omega) = \frac{1}{2} \sum_{i \neq j}^{\infty} \mathcal{H}^{n-1}(\partial^* E_i \cap \partial^* E_j \cap \Omega)$$

$$(u^+, u^-, \nu_u) \sim (u_i, u_j, \nu_i) \quad \mathcal{H}^{n-1}\text{-a.e. } \textit{on } \partial^* E_i \cap \partial^* E_j \cap \Omega$$

where ν_i is the inner normal to E_i.

From now on we deal with functionals $G : SBV_0(\Omega; \mathbb{R}^m) \times \mathcal{B}(\Omega) \to \mathbb{R}^+$ which are measures on $\mathcal{B}(\Omega)$, and satisfy some locality and lower-semicontinuity condition in the first variable. More precisely, we assume that:

$$G(u, \cdot) \text{ is a measure for every } u \in SBV_0(\Omega; \mathbb{R}^m) \tag{3.1}$$

$$G \text{ is } \textit{local} \text{ on } \mathcal{A}(\Omega) \tag{3.2}$$

(*i.e.*, for all $A \in \mathcal{A}(\Omega)$ $G(u, A) = G(v, A)$ whenever $u = v$ a.e. in A)

$$G(\cdot, A) \qquad \text{is L}^1\text{-lower semicontinuous for all } A \in \mathcal{A}(\Omega). \tag{3.3}$$

In order to state the main result of this section we have to introduce some notation. Let $x \in \mathbb{R}^n$, $\rho > 0$, $\nu \in S^{n-1}$. We denote by $Q_\rho^\nu(x)$ an open cube centered in x, of side length ρ and one face orthogonal to ν. We will suppose that fixed x and ν for each ρ and $\sigma > 0$ the cube $Q_\sigma^\nu(x)$ is obtained from $Q_\rho^\nu(x)$ by an homothety of center x. We also define the function $u^{\nu,x}$ by

$$u^{\nu,x}(y) = \begin{cases} 1 & \text{if } \langle y - x, \nu \rangle > 0 \\ 0 & \text{if } \langle y - x, \nu \rangle \leq 0; \end{cases}$$

i.e., the characteristic function of the half space $\{y \in \mathbb{R}^n : \langle y - x, \nu \rangle > 0\}$. Moreover, given a, $b \in \mathbb{R}^m$, we set

$$u_{a,b}^{\nu,x}(y) = b + (a - b) u^{\nu,x}(y) = \begin{cases} a & \text{if } \langle y - x, \nu \rangle > 0 \\ b & \text{if } \langle y - x, \nu \rangle \leq 0. \end{cases}$$

THEOREM 3.2. *Let $G : SBV_0(\Omega;\mathbb{R}^m) \times \mathcal{B}(\Omega) \to [0,+\infty)$ be a functional satisfying (3.1)–(3.3), such that*

$$\mathcal{H}^{n-1}(B \cap S_u) + |Du|(B) \le G(u,B) \le c(\mathcal{H}^{n-1}(B \cap S_u) + |Du|(B)) \tag{3.4}$$

for a positive constant c. Then for every $u \in SBV_0(\Omega;\mathbb{R}^m)$, and for every set $B \in \mathcal{B}(\Omega)$ we have

$$G(u,B) = \int_{S_u \cap B} \varphi(x,u^+,u^-,\nu)\, d\mathcal{H}^{n-1} \tag{3.5}$$

where $\varphi(x,a,b,\nu)$ is given by

$$\begin{aligned} \varphi(x,a,b,\nu) = \limsup_{\rho\to 0+} \frac{1}{\rho^{n-1}} \min\Big\{ & G(w,\overline{Q^\nu_\rho(x)}) : \\ & : w \in SBV_0(\Omega;\mathbb{R}^m), w = u^{\nu,x}_{a,b} \text{ on } \Omega \setminus Q^\nu_\rho(x)\Big\} \end{aligned} \tag{3.6}$$

for all $x \in \Omega$, a, $b \in \mathbb{R}^m$, $\nu \in S^{n-1}$.

In order to split the proof of our integral representation result into simpler steps, we state some preliminary lemmas. The first one says that our functionals enjoy also a sort of locality property on Borel sets (cf. assumption (3.2)).

LEMMA 3.3. *Let $G : SBV_0(\Omega;\mathbb{R}^m) \times \mathcal{B}(\Omega) \to [0,+\infty)$ be a functional satisfying (3.1)–(3.3) and assume that there exists a positive constant c such that*

$$0 \le G(u,B) \le c(\mathcal{H}^{n-1}(B \cap S_u) + |Du|(B)) \tag{3.7}$$

for all $u \in SBV_0(\Omega;\mathbb{R}^m)$ and $B \in \mathcal{B}(\Omega)$. Then

$$G(u,B) = 0 \text{ for every } B \in \mathcal{B}(\Omega) \text{ such that } \mathcal{H}^{n-1}(B \cap S_u) = 0 \tag{3.8}$$

$$\begin{gathered} G(u,B) = G(v,B) \text{ for every } B \in \mathcal{B}(\Omega), \text{ for every } u,v \text{ such that} \\ \mathcal{H}^{n-1}((S_u \triangle S_v) \cap B) = 0, \text{ with } (u^+,u^-,\nu_u) \sim (v^+,v^-,\nu_v) \quad \mathcal{H}^{n-1}\text{-a.e. in } S_u \cap S_v. \end{gathered} \tag{3.9}$$

Proof. The proof of (3.8) is trivial by (3.7), while the one of (3.9) can be obtained as in [9], Proposition 4.4, Step 1 (see also [7], Lemma 4.1). □

Under the same assumptions, the following lemma states that a functional can be identified on all the characteristic functions of sets of finite perimeter, just by the knowledge of its values on sets having a smooth reduced boundary.

LEMMA 3.4. *Let $G : SBV_0(\Omega) \times \mathcal{B}(\Omega) \to [0,+\infty)$ be a functional satisfying the assumptions of Lemma 3.3, with $m = 1$. If there exists a Borel function $\varphi : \Omega \times S^{n-1} \to [0,+\infty)$ such that*

$$G(\chi_E, A) = \int_{\partial^* E \cap A} \varphi(x,\nu_E)\, d\mathcal{H}^{n-1}$$

for every pair (E,A) such that $A \cap \partial^* E$ is a C^1-hypersurface, then the same integral representation holds for every set E of finite perimeter and every $A \in \mathcal{A}(\Omega)$.

Proof. Let $E \subseteq \Omega$ be a set of finite perimeter, and let $A \in \mathcal{A}(\Omega)$; it is not restrictive to assume that $A \subset\subset \Omega$. Since $\partial^* E$ is $\mathcal{H}^{n-1}$-rectifiable, there exist an increasing sequence of compact sets $K_h \subseteq \partial^* E$ and a sequence of functions $f_h \in \mathcal{C}^1_0(\Omega)$

such that $K_h \subseteq \{f_h = 0\}$, $\mathcal{H}^{n-1}(A \cap \partial^* E \setminus K_h) \to 0$ as $h \to +\infty$ and there exists the inner normal to $\partial^* E$ $\nu_E(x) = \nabla f_h(x)$ for every $x \in K_h$. By our assumptions

$$\begin{aligned} G(\chi_E, A \cap K_h) =& G(\{f_h > 0\}, A \cap K_h) \\ &= \int_{\partial^*\{f_h>0\}\cap A\cap K_h} \varphi(x, \nu_h)\, d\mathcal{H}^{n-1} = \int_{\partial^* E\cap A\cap K_h} \varphi(x, \nu_E)\, d\mathcal{H}^{n-1}, \end{aligned}$$

for every h, and by taking the supremum over h we obtain

$$G(\chi_E, A \cap \partial^* E) = \int_{\partial^* E\cap A} \varphi(x, \nu_E)\, d\mathcal{H}^{n-1}.$$

Hence the proof is completed noticing that, by (3.7), $G(\chi_E, A \cap \partial^* E) = G(\chi_E, A)$. □

The following lemma is crucial in the proof of the main result, and gives an explicit formula to compute the integrand. The proof generalizes the method used in [14] Theorem 2.1.

LEMMA 3.5. *Let $G : SBV_0(\Omega; \mathbb{R}^m) \times \mathcal{B}(\Omega) \to [0, +\infty)$ be a functional satisfying (3.1)–(3.4). Then for every a, $b \in \mathbb{R}^m$, for every set $E \subseteq \Omega$ of finite perimeter, and for every $A \in \mathcal{A}(\Omega)$ we have*

$$G(a\chi_E + b\chi_{\Omega\setminus E}, A) = \int_{\partial^* E\cap A} \varphi(x, a, b, \nu_E)\, d\mathcal{H}^{n-1} \tag{3.10}$$

where φ is given by (3.6).

Proof. Given a, $b \in \mathbb{R}^m$, a set E of finite perimeter, and a set $A \in \mathcal{A}(\Omega)$, denote by $G_{a,b} : SBV_0(\Omega) \times \mathcal{B}(\Omega) \to [0, +\infty)$ the functional

$$G_{a,b}(u, A) = G(au + b(1 - u), A).$$

We want to prove that

$$G_{a,b}(\chi_E, A) = \int_{\partial^* E\cap A} \varphi(x, a, b, \nu_E)\, d\mathcal{H}^{n-1},$$

which is the same as (3.10). By Lemma 3.4, it is not restrictive to assume that $A \cap \partial^* E$ is a C^1-hypersurface. Note that if $x \in \partial^* E$, $\nu = \nu_E(x)$, and

$$w = \begin{cases} \chi_E & \text{in } Q^\nu_\rho(x) \\ u^{\nu,x} & \text{in } \mathbb{R}^n \setminus Q^\nu_\rho(x) \end{cases} \tag{3.11}$$

then we have

$$\begin{aligned} G_{a,b}(w, \overline{Q^\nu_\rho(x)}) &= G_{a,b}(w, Q^\nu_\rho(x)) + G_{a,b}(w, \partial Q^\nu_\rho(x)) \\ &\le G_{a,b}(w, Q^\nu_\rho(x)) + \rho^{n-2} o(\rho) \\ &= G_{a,b}(\chi_E), Q^\nu_\rho(x)) + \rho^{n-2} o(\rho), \end{aligned} \tag{3.12}$$

so that by (3.6)

$$\varphi(x, a, b, \nu_E(x)) \le \limsup_{\rho\to 0+} \frac{1}{\rho^{n-1}} G_{a,b}(\chi_E, Q^\nu_\rho(x)). \tag{3.13}$$

By the Lebesgue derivation theorem applied to the measure

$$\mu(A) = G_{a,b}(\chi_E, A)$$

(note that $\lim_{\rho\to 0+} \rho^{1-n}\mathcal{H}^{n-1}(Q^\nu_\rho(x) \cap \partial^* E) = 1$) we obtain that (3.13) holds for $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E$. Hence we deduce the inequality

$$\int_{A\cap\partial^* E} \varphi(x, a, b, \nu_E(x))\, d\mathcal{H}^{n-1} \le G_{a,b}(\chi_E, A). \tag{3.14}$$

The converse inequality will be proven with the aid of formula (3.6) and of the lower semicontinuity of G. We will exhibit a sequence $u_h \in SBV_0(\Omega; \mathbb{R}^m)$ converging to $a\chi_E + b(1 - \chi_E)$ in $\mathrm{L}^1(A, \mathbb{R}^m)$ such that

$$\begin{aligned} G(a\chi_E + b(1 - \chi_E), A) &\le \liminf_{h\to+\infty} G(u_h, A) \\ &\le \int_{A\cap\partial^* E} \varphi(x, a, b, \nu_E(x))\, d\mathcal{H}^{n-1}. \end{aligned} \tag{3.15}$$

The construction of such u_h will be obtained via a proper combination of solutions of the minimum problems in (3.6).

Denote by $u(x, \rho, \nu) \in SBV_0(\Omega; \mathbb{R}^m)$ the solution to the minimum problem

$$\min\Big\{G(w, \overline{Q^\nu_\rho(x)}) : w \in SBV_0(\Omega; \mathbb{R}^m), w = u^{\nu,x}_{a,b} \text{ on } \Omega \setminus Q^\nu_\rho(x)\Big\},$$

and $\Gamma = \Big\{x \in \partial^* E : \lim\limits_{\rho\to 0+} \dfrac{1}{\rho^{n-1}} \displaystyle\int_{Q^\nu_\rho(x)\cap\partial^* E} \varphi(y, a, b, \nu_E(y)) d\mathcal{H}^{n-1} = \varphi(x, a, b, \nu_E(x))\Big\}$.
By the Lebesgue derivation theorem we have $\mathcal{H}^{n-1}(\partial^* E \setminus \Gamma) = 0$. Fix $h \in \mathbb{N}$. We choose, for each h, the family $\mathcal{Q}_h$ of all closed cubes $\overline{Q^\nu_\rho(x)}$ such that $\rho \le 1/h$, $x \in \Gamma$, $\nu = \nu_E(x)$, $Q^\nu_\rho(x) \subset A$,

$$\rho^{n-1} \le \mathcal{H}^{n-1}(A \cap \partial^* E),$$

$$\varphi(x, a, b, \nu_E(x)) \le \frac{1}{\rho^{n-1}} \int_{Q^\nu_\rho(x)\cap\partial^* E} \varphi(y, a, b, \nu_E(y)) d\mathcal{H}^{n-1} + \frac{1}{h},$$

and

$$\frac{1}{\rho^{n-1}} G(u(x, \rho, \nu), \overline{Q^\nu_\rho(x)}) \le \varphi(x, a, b, \nu_E(x)) + \frac{1}{h}.$$

The family $\mathcal{Q}_h$ covers finely $\partial^* E \cap A$, hence by the (generalized) Besicovitch covering theorem (see Morse [33] Theorem 5.13) there exists a countable sub-family of disjoint cubes $\{\overline{Q^{\nu_i}_{\rho_i}(x_i)} : i \in \mathbb{N}\}$ still covering $\partial^* E \cap A$. We define then

$$u_h(y) = \begin{cases} u(x_i, \rho_i, \nu_i)(y) & \text{if } y \in Q^{\nu_i}_{\rho_i}(x_i), \\ a & \text{if } y \in E \setminus \bigcup_i Q^{\nu_i}_{\rho_i}(x_i) \\ b & \text{if } y \in (A \setminus E) \setminus \bigcup_i Q^{\nu_i}_{\rho_i}(x_i). \end{cases}$$

We have

$$\begin{aligned} G(u_h, A) &= \sum_i G(u(x_i, \rho_i, \nu_i), \overline{Q^{\nu_i}_{\rho_i}(x_i)}) \\ &\le \sum_i {\rho_i}^{n-1}(\varphi(x_i, a, b, \nu_E(x_i)) + \frac{1}{h}) \\ &\le \sum_i \Big(\int_{Q^{\nu_i}_{\rho_i}(x_i)\cap\partial^* E} \varphi(y, a, b, \nu_E(y)) d\mathcal{H}^{n-1} + 2\rho_i^{n-1}\frac{1}{h}\Big) \\ &\le \int_{A\cap\partial^* E} \varphi(y, a, b, \nu_E(y)) d\mathcal{H}^{n-1} + \frac{2}{h}\mathcal{H}^{n-1}(A \cap \partial^* E). \end{aligned}$$

Letting $h \to +\infty$ we obtain then

$$\liminf_{h\to+\infty} G(u_h, A) \le \int_{A\cap\partial^* E} \varphi(y, a, b, \nu_E(y)) d\mathcal{H}^{n-1}.$$

Since it is clear that $u_h \to \chi_E$ in $\mathrm{L}^1(A, \mathbb{R}^m)$, we have proven (3.15). □

Proof of Theorem 3.2. First of all we remark that, since by (3.1) $G(u, \cdot)$ is a finite measure on $\mathcal{B}(\Omega)$, it is enough to prove (3.10) for all pairs (u, A) with $A \in \mathcal{A}(\Omega)$. Given $u \in SBV_0(\Omega; \mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$, by Lemma 3.1 there exist a Borel partition (E_i) of A, and a sequence (u_i) in $\mathbb{R}^m$ with $u_i \ne u_j$ for $i \ne j$, such that

$$\sum_{i=1}^{\infty} \mathcal{H}^{n-1}(\partial^* E_i \cap A) < +\infty, \qquad u = \sum_{i=1}^{\infty} u_i \chi_{E_i} \qquad \textit{a.e. in } A.$$

Moreover, since by Lemma 3.1 $\mathcal{H}^{n-1}(S_u \cap A) = \mathcal{H}^{n-1}(\bigcup_{i\ne j}(A \cap \partial^* E_i \cap \partial^* E_j))$, taking into account estimate (3.4), we have

$$\begin{aligned} G(u, A) &= G(u, A \cap S_u) + G(u, A \setminus S_u) = G(u, \bigcup_{i\ne j}(A \cap \partial^* E_i \cap \partial^* E_j)) \\ &= \frac{1}{2}\sum_{i\ne j} G(u, A \cap \partial^* E_i \cap \partial^* E_j). \end{aligned}$$

For each pair of indices i, j, define now the function $u_{ij} \in SBV_0(\Omega; \mathbb{R}^m)$ as

$$u_{ij}(x) = \begin{cases} u_i & \text{if } x \in E_i \\ u_j & \text{if } x \notin E_i. \end{cases}$$

If we denote by ν_i the inner normal to E_i, by Lemma 3.1 we have $(u_i, u_j, \nu_i) = (u_{ij}^+, u_{ij}^-, \nu_i) \sim (u^+, u^-, \nu_u)$ $\mathcal{H}^{n-1}$-a.e. in $\partial^* E_i \cap \partial^* E_j$. Here we can apply the locality property (3.9) and conclude that

$$G(u, A) = \frac{1}{2}\sum_{i\ne j} G(u, A \cap \partial^* E_i \cap \partial^* E_j) = \frac{1}{2}\sum_{i\ne j} G(u_{ij}, A \cap \partial^* E_i \cap \partial^* E_j).$$

By Lemma 3.5

$$G(u_{ij}, A) = \int_{\partial^* E_i \cap A} \varphi(x, u_i, u_j, \nu_i)\, d\mathcal{H}^{n-1},$$

and since $G(u_{ij}, \cdot)$ is a measure on $\mathcal{B}(\Omega)$, also

$$\begin{aligned} G(u_{ij}, A \cap \partial^* E_i \cap \partial^* E_j) &= \int_{\partial^* E_i \cap \partial^* E_j \cap A} \varphi(x, u_i, u_j, \nu_i)\, d\mathcal{H}^{n-1} \\ &= \int_{\partial^* E_i \cap \partial^* E_j \cap A} \varphi(x, u^+, u^-, \nu_u)\, d\mathcal{H}^{n-1}. \end{aligned}$$

By taking the sum for $i \ne j$ we finally conclude that

$$G(u, A) = \int_{S_u \cap A} \varphi(x, u^+, u^-, \nu_u)\, d\mathcal{H}^{n-1},$$

that completes the proof of the theorem. □

Remark 3.6. If there exists a modulus of continuity ω such that G satisfies

$$|G(u,A)-G(\Phi u,A)|\le \omega(\|\Phi-\mathrm{Id}\|)\int_{S_u\cap A}(1+|u^+|+|u^-|)d\mathcal{H}^{n-1}$$

for every $u\in SBV_0(\Omega;\mathbb{R}^m)$, $A\in\mathcal{A}(\Omega)$, and Φ affine transformation on $\mathbb{R}^m$, then by (3.6) it is easy to see that the function $\varphi(x,\cdot,\cdot,\nu)$ is continuous on $(\mathbb{R}^m\times\mathbb{R}^m)\setminus\{(a,a): a\in\mathbb{R}^m\}$.

Remark 3.7. Theorem 3.2 can be stated also with $SBV_0(\Omega;\mathbb{R}^m)\cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ in place of $SBV_0(\Omega;\mathbb{R}^m)$.

4. Integral representation of the surface energy

We apply the results of Section 3 to obtain a representation of $F(u,A\cap S_u)$ when u is piecewise constant.

THEOREM 4.1. *If F satisfies hypotheses* (i)–(iv) *of Theorem* 2.4 *then there exists a Borel function $\varphi:\Omega\times\mathbb{R}^m\times\mathbb{R}^m\times S^{n-1}\to[0,+\infty)$ such that*

$$F(u,B\cap S_u)=\int_{S_u\cap B}\varphi(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{n-1} \tag{4.1}$$

for all $u\in SBV_0(\Omega;\mathbb{R}^m)$ and $B\in\mathcal{B}(\Omega)$.

Proof. Consider the functional $G: SBV_0(\Omega;\mathbb{R}^m)\times\mathcal{B}(\Omega)\to[0,+\infty)$ defined by

$$G(u,B)=F(u,B\cap S_u).$$

In order to apply Theorem 3.2 we have to show only that $G(\cdot,A)$ is L^1-lower semicontinuous for every $A\in\mathcal{A}(\Omega)$, since properties (3.1), (3.2) and (3.4) follow trivially from the corresponding properties of F. Consider $u_h\to u$ in $\mathrm{L}^1(\Omega;\mathbb{R}^m)$ with $u_h,u\in SBV_0(\Omega;\mathbb{R}^m)$, and let K be a compact subset of $S_u\cap A$. If $K\subset A'\subset\subset A$ then we have

$$F(u,A')\le\liminf_{h\to+\infty}F(u_h,A')\le\liminf_{h\to+\infty}G(u_h,A)+\beta|A'|.$$

By letting $A'\to K$ we obtain

$$F(u,K)\le\liminf_{h\to+\infty}G(u_h,A),$$

and then, letting $K\to S_u\cap A$, we have the lower semicontinuity of G. Hence there exists a Carathéodory function φ such that (4.1) holds for all $u\in SBV_0(\Omega;\mathbb{R}^m)$ and $B\in\mathcal{A}(\Omega)$. Moreover, since $F(u,\cdot)$ is a finite measure we have (4.1) for all $B\in\mathcal{B}(\Omega)$. □

PROPOSITION 4.2. *If F satisfies, in addition to hypotheses* (i)–(iv), *also condition* (vi) *of Theorem 2.4 then φ is Carathéodory and* (4.1) *holds for all $u\in SBV^p(\Omega;\mathbb{R}^m)$.*

Proof. By condition (vi) and (3.6), the function φ satisfies

$$|\varphi(x,a,b,\nu)-\varphi(x,a',b',\nu)|\le c\,\omega(|a-a'|+|b-b'|). \tag{4.2}$$

Then, fix $u\in SBV^p(\Omega;\mathbb{R}^m)$, and, for all $j\in\mathbb{N}$, let $u_j\in SBV_0(\Omega;\mathbb{R}^m)$ be such that $\|u-u_j\|_\infty\le\frac{1}{j}$, and $|u_j^+-u_j^-|\ge\frac{1}{j}$ on S_{u_j}. Let K be a compact subset of S_u, and let $K_j=\{x\in K:\ |u^+-u^-|>\frac{3}{j}\}$, which converge increasingly to K as $j\to\infty$. Note that $K_j\subset S_{u_j}$, and we have

$$\begin{aligned}|F(u,K)-&\int_K\varphi(x,u^+,u^-,\nu_u)d\mathcal{H}^{n-1}|\le|F(u,K)-F(u,K_j)|\\&+|F(u,K_j)-F(u_j,K_j)|\\&+|F(u_j,K_j)-\int_{K_j}\varphi(x,u^+,u^-,\nu_u)d\mathcal{H}^{n-1}|\\&+|\int_{K_j}\varphi(x,u^+,u^-,\nu_u)d\mathcal{H}^{n-1}-\int_K\varphi(x,u^+,u^-,\nu_u)d\mathcal{H}^{n-1}|.\end{aligned}$$

Since $\mathcal{H}^{n-1}(K\setminus K_j)\to 0$ we obtain

$$\begin{aligned}&\lim_j|F(u,K)-F(u,K_j)|\\&=\lim_j|\int_{K_j}\varphi(x,u^+,u^-,\nu_u)d\mathcal{H}^{n-1}-\int_K\varphi(x,u^+,u^-,\nu_u)d\mathcal{H}^{n-1}|=0,\end{aligned}$$

while from (4.2) and the integral representation of $F(u_j,K_j)$ as above we easily get

$$\begin{aligned}0&\le\lim_j|F(u_j,K_j)-\int_{K_j}\varphi(x,u^+,u^-,\nu_u)d\mathcal{H}^{n-1}|\\&\le\int_{K_j}|\varphi(x,u_j^+,u_j^-,\nu_u)-\varphi(x,u^+,u^-,\nu_u)|d\mathcal{H}^{n-1}\le\lim_j c\,\omega(\frac{1}{j})=0.\end{aligned}$$

Finally, from property (vi)

$$0\le\lim_j|F(u,K_j)-F(u_j,K_j)|\le\lim_j c\,\omega(\frac{1}{j})=0.$$

Hence $F(u,K)=\int_K\varphi(x,u^+,u^-,\nu_u)d\mathcal{H}^{n-1}$; taking the supremum on compact subsets of S_u we complete the proof. □

5. Strong approximation in $SBV^p(\Omega;\mathbb{R}^m)$

We introduce the following notion of strong convergence in $SBV^p(\Omega;\mathbb{R}^m)$, and subsequently prove the "strong density of piecewise smooth functions".

Definition 5.1. Let (u_h) be a sequence of functions in $SBV^p(\Omega;\mathbb{R}^m)$. We say that u_h *converge strongly* to u in $SBV^p(\Omega;\mathbb{R}^m)$ if

$$u_h\to u \text{ in } \mathrm{L}^1(\Omega;\mathbb{R}^m), \tag{5.1}$$

$$\nabla u_h\to\nabla u \text{ strongly in } \mathrm{L}^p(\Omega;M^{n\times m}), \tag{5.2}$$

$$\mathcal{H}^{n-1}(S_{u_h}\triangle S_u)\to 0, \tag{5.3}$$

$$\int_{S_{u_h}\cup S_u}(|u_h^+-u^+|+|u_h^--u^-|)d\mathcal{H}^{n-1}\to 0 \tag{5.4}$$

(in (5.4) we choose the orientation $\nu_{u_h} = \nu_u$ $\mathcal{H}^{n-1}$-a.e. on $S_{u_h} \cap S_u$; recall that if $v \in BV(\Omega;\mathbb{R}^m)$ then we set $v^+ = v^- = \widetilde{v}$ on $\Omega \setminus S_v$)

LEMMA 5.2. *If $u \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ then there exists a sequence (u_h) in $SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ with $\|u_h\|_\infty \le \|u\|_\infty$, strongly converging to u in $SBV^p(\Omega;\mathbb{R}^m)$, such that for each $h \in \mathbb{N}$ there exists a closed rectifiable set R_h such that $u_h \in C^1(\Omega \setminus R_h;\mathbb{R}^m)$.*

Proof. We suppose $m = 1$; the general case been dealt with by arguing componentwise. For $h \in \mathbb{N}$, let K_h be a compact subset of S_u such that

$$\mathcal{H}^{n-1}(S_u \setminus K_h) \le \frac{1}{h}.$$

We consider the minimum problem

$$\min\Big\{ \int_\Omega |\nabla v|^p\,dx + \mathcal{H}^{n-1}(S_v \setminus K_h) + \mathcal{H}^{n-1}(K_h) + h\int_\Omega |u-v|^p\,dx + \int_{K_h} (|v^+ - u^+| + |v^- - u^-|) \wedge 1\,d\mathcal{H}^{n-1} :\ v \in SBV^p(\Omega;\mathbb{R}^m)\Big\} \tag{5.5}$$

(we choose $\nu_v = \nu_u$ on $S_u \cap S_v \cap K_h$). By a truncation argument it is easy to see that we can limit out analysis in (5.5) to v satisfying $\|v\|_\infty \le \|u\|_\infty$. Moreover the functional in (5.5) is clearly coercive with respect to the weak convergence in $SBV^p(\Omega;\mathbb{R}^m)$ on the set $\{v \in SBV^p(\Omega;\mathbb{R}^m) :\ \|v\|_\infty \le \|u\|_\infty\}$. Hence, to prove the existence of a minimum point it suffices to show that this functional is lower semicontinuous with respect to weak convergence in $SBV^p(\Omega;\mathbb{R}^m)$. This can be obtained following word for word the slicing argument in [3] (see also [12] Section 3); the only thing to prove is the lower semicontinuity of 1-dimensional functionals, which represent the "1-dimensional sections" of the functional in (5.5), of the form

$$F(v,I) = \int_I |v'|^p\,dt + \#((S_v \setminus K) \cap I) + \sum_{j=1}^N (|v(t_j+) - a_j| + |v(t_j-) - b_j|) \wedge 1,$$

where I is an open subset of $\mathbb{R}^n$, $v \in SBV^p(I)$, $K = \{t_1,\dots,t_N\} \subset I$, and $\{a_1,\dots,a_N\}, \{b_1,\dots,b_N\} \subset \mathbb{R}$; $v(t+)$ and $v(t-)$ denote the right hand side and left hand side approximate limits of v at t, respectively. Since the set K is finite, it suffices to show the lower semicontinuity of F with respect to the weak $SBV^p(I)$ convergence in the case $K = \{0\}$ and $I = (-1,1)$, so that

$$F(v,I) = \int_{(-1,1)} |v'|^p\,dt + \#((S_v \setminus \{0\}) \cap (-1,1)) + (|v(0+) - a| + |v(0-) - b|) \wedge 1.$$

Let now $v_h \to v$ weakly in $SBV^p(I)$. We can suppose that $S_{u_h} = \{t_0^h,\dots,t_N^h\}$ with $t_j^h < t_{j+1}^h$, and $t_j^h \to t_j \in [0,1]$ for $j = 0,\dots,N$. If $0 \notin \{t_j :\ j = 1,\dots,N\}$ then $v_h(0+) = v_h(0-)$ for h large enough, and $v_h(0\pm) \to v(0)$ so that the last term of F is continuous, and $F(v,I) \le \liminf_{h\to+\infty} F(v_h,I)$ by Theorem 2.2. Suppose that $0 \in \{t_j :\ j = 1,\dots,N\}$, so that there exist $k, k+1,\dots,l$ such that $t_k = t_{k+1} = \ldots = t_l$. Let $\varepsilon > 0$ be such that $t_{k-1} < -\varepsilon$ and $t_{l+1} > \varepsilon$. Again by Theorem 2.2 we have

$$F(v,(-1,-\varepsilon)) \le \liminf_{h\to+\infty} F(v_h,(-1,-\varepsilon)), \quad F(v,(\varepsilon,1)) \le \liminf_{h\to+\infty} F(v_h,(\varepsilon,1)). \tag{5.6}$$

If $\#\{h \in \mathbb{N} : \exists j \in \{l, \dots, m\},\ t_j^h \neq 0\} = +\infty$ then trivially

$$(|v(0+) - a| + |v(0-) - b|) \wedge 1 \leq 1 \leq \lim_{h\to+\infty} \#((S_{v_h} \setminus \{0\}) \cap (-1,1));$$

if not, then we can suppose $S_{v_h} \cap [-\varepsilon, \varepsilon] = \{0\}$, and $v_h(0\pm) \to v(0\pm)$, so that

$$|v_h(0+) - a| + |v_h(0-) - b| \to |v(0+) - a| + |v(0-) - b|.$$

In all cases, taking into account also (5.6),

$$F(v, (-1, -\varepsilon)) + F(v, (\varepsilon, 1)) + (|v(0+) - a| + |v(0-) - b|) \wedge 1 \leq \liminf_{h\to+\infty} F(v_h, I).$$

Letting $\varepsilon \to 0$ we obtain $F(v, I) \leq \liminf_{h\to+\infty} F(v_h, I)$. Hence, the functional $F(\cdot, I)$ is lower semicontinuous, and there exists a solution v_h to (5.5) with $\|v_h\|_\infty \leq \|u\|_\infty$.

If we insert $v = u$ in (5.5) then we have immediately the estimate

$$\begin{aligned}
&\int_\Omega |\nabla v_h|^p\,dx + \mathcal{H}^{n-1}(S_{v_h} \setminus K_h) + \mathcal{H}^{n-1}(K_h) \\
&\qquad + h\int_\Omega |u - v_h|^p\,dx + \int_{K_h} (|v_h^+ - u^+| + |v_h^- - u^-|) \wedge 1\, d\mathcal{H}^{n-1} \\
&\leq \int_\Omega |\nabla u|^p\,dx + \mathcal{H}^{n-1}(S_u),
\end{aligned} \tag{5.7}$$

in particular

$$\int_\Omega |u - v_h|^p\,dx \leq \frac{c}{h}, \tag{5.8}$$

so that $v_h \to u$ in $\mathrm{L}^p(\Omega)$. Since from (5.7) also

$$\begin{aligned}
|Dv_h|(\Omega) &\leq c\|\nabla v_h\|_{\mathrm{L}^p(\Omega;\mathbb{R}^n)} + 2\|v_h\|_\infty \mathcal{H}^{n-1}(S_{v_h}) \\
&\leq c\|\nabla v_h\|_{\mathrm{L}^p(\Omega;\mathbb{R}^n)} + 2\|v_h\|_\infty \mathcal{H}^{n-1}(K_h \cup S_{v_h}) \leq c
\end{aligned}$$

we can apply Theorem 2.2 and obtain that $v_h \to u$ weakly in $SBV^p(\Omega;\mathbb{R}^m)$. Passing possibly to a subsequence we can suppose that

$$\lim_{h\to+\infty} \int_\Omega |\nabla v_h|^p\,dx \geq \int_\Omega |\nabla u|^p\,dx, \quad \text{and} \quad \lim_{h\to+\infty} \mathcal{H}^{n-1}(S_{v_h}) \geq \mathcal{H}^{n-1}(S_u), \tag{5.9}$$

so that

$$\begin{aligned}
\int_\Omega |\nabla u|^p\,dx + \mathcal{H}^{n-1}(S_u) &\leq \lim_{h\to+\infty} \Big(\int_\Omega |\nabla v_h|^p\,dx + \mathcal{H}^{n-1}(S_{v_h})\Big) \\
&\leq \lim_{h\to+\infty} \int_\Omega |\nabla v_h|^p\,dx + \lim_{h\to+\infty} \mathcal{H}^{n-1}(K_h) \\
&\quad + \limsup_{h\to+\infty} \Big(\mathcal{H}^{n-1}(S_{v_h} \setminus K_h) + h\int_\Omega |u - v_h|^p\,dx \\
&\quad + \int_{K_h} (|v_h^+ - u^+| + |v_h^- - u^-|) \wedge 1\, d\mathcal{H}^{n-1}\Big) \\
&\leq \int_\Omega |\nabla u|^p\,dx + \mathcal{H}^{n-1}(S_u).
\end{aligned}$$

Hence, we obtain

$$\lim_{h\to+\infty}\int_\Omega |\nabla v_h|^p\,dx = \int_\Omega |\nabla u|^p\,dx; \tag{5.10}$$

$$\lim_{h\to+\infty}\mathcal{H}^{n-1}(S_{v_h}) = \mathcal{H}^{n-1}(S_u); \tag{5.11}$$

$$\lim_{h\to+\infty}\mathcal{H}^{n-1}(S_{v_h}\setminus K_h) = 0; \tag{5.12}$$

$$\lim_{h\to+\infty}\int_{K_h}(|v_h^+ - u^+| + |v_h^- - u^-|)\wedge 1\,d\mathcal{H}^{n-1} = 0. \tag{5.13}$$

From (5.10), (5.9) we have $\nabla v_h \to \nabla u$ strongly in $\mathrm{L}^p(\Omega; M^{n\times m})$. From (5.13) we have also

$$\lim_{h\to+\infty}\int_{K_h}(|v_h^+ - u^+| + |v_h^- - u^-|)\,d\mathcal{H}^{n-1} = 0,$$

so that

$$\begin{aligned}0 &\le \lim_{h\to+\infty}\int_{S_u\cup S_{v_h}}(|v_h^+ - u^+| + |v_h^- - u^-|)\,d\mathcal{H}^{n-1}\\ &\le \lim_{h\to+\infty}\Big(4\|u\|_\infty(\mathcal{H}^{n-1}(S_{v_h}\setminus K_h) + \mathcal{H}^{n-1}(S_u\setminus K_h))\\ &\quad + \int_{K_h}(|v_h^+ - u^+| + |v_h^- - u^-|)\,d\mathcal{H}^{n-1}\Big) = 0\end{aligned}$$

and we conclude that $v_h \to u$ strongly in $SBV^p(\Omega; \mathbb{R}^m)$.

Note that v_h is a local minimum point for the functional

$$\int_\Omega |\nabla v|^p\,dx + \mathcal{H}^{n-1}(S_v) + h\int_\Omega |u - v|^p\,dx$$

on $\Omega\setminus K_h$. By the regularity results for such minimum points (obtained by De Giorgi, Carriero and Leaci in [29] for $p = 2$, and generalized to the case $p > 1$ by Carriero and Leaci [23]) we have

$$\mathcal{H}^{n-1}\big((\overline{S_{v_h}}\setminus S_{v_h})\cap(\Omega\setminus K_h)\big) = 0,$$

and $v_h \in C^1(\Omega\setminus(\overline{S_{v_h}}\cup K_h))$. We conclude the proof taking $R_h = K_h \cup \overline{S_{v_h}}$. □

Remark 5.3. From (5.11), (5.12) and (5.15) we have also that

$$\lim_{k\to+\infty}\mathcal{H}^{n-1}(\overline{S_{v_k}}\setminus S_{v_k}) = 0.$$

Moreover, by the properties of minima of (5.14) we have that each S_{v_h} has locally finite Minkowsky content, which means that for all $\Omega' \subset\subset \Omega$ there exists $c > 0$ such that

$$|\{x \in \Omega' :\ \mathrm{dist}(x, S_{v_h}) < \rho\}| \le c\rho$$

for all $\rho > 0$ (see [11] Proposition 5.3(i)). Lemma 5.2 improves the strong/weak density Theorem 4.2 in [6].

6. Proof of the main result

We begin by defining the volume energy density f in Theorem 2.4.

Remark 6.1. The restriction of the functional F in Theorem 2.4 to $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^m)\times\mathcal{A}(\Omega)\to[0,+\infty)$ satisfies the hypotheses of the integral representation Theorem 2.3. Hence, there exists a Carathéodory function $f:\Omega\times\mathbb{R}^m\times M^{n\times m}\to[0,+\infty)$ satisfying the growth condition

$$\alpha|\xi|^p\le f(x,u,\xi)\le\beta(1+|\xi|^p) \tag{6.1}$$

for all $x\in\Omega$, $u\in\mathbb{R}^m$ and $\xi\in M^{n\times m}$, such that

$$F(u,B)=\int_B f(x,u,\nabla u)\,dx \tag{6.2}$$

for all $B\in\mathcal{B}(\Omega)$ and $u\in\mathrm{W}^{1,p}(\Omega;\mathbb{R}^m)$. If $u\in SBV^p(\Omega;\mathbb{R}^m)$ and $u\in\mathrm{W}^{1,p}(A;\mathbb{R}^m)$ for some $A\in\mathcal{A}(\Omega)$ then the same equality holds by the locality and inner regularity properties of F for $B\subset A$.

From now on f will be given by Remark 6.1.

Lemma 6.2. *Let F satisfy hypotheses* (i)–(v) *in Theorem* 2.4, *and*

(vi)$'$ *for all $u\in SBV^p(\Omega;\mathbb{R}^m)$ and Φ affine transformation on $\mathbb{R}^m$*

$$|F(u,S_u\cap A)-F(\Phi u,S_u\cap A)|\le\omega(\|\Phi-\mathrm{Id}\|)\int_{S_u\cap A}(1+|u^+|+|u^-|)d\mathcal{H}^{n-1}$$

for all $A\in\mathcal{A}(\Omega)$.

If f, φ are defined by Remark 6.1 *and Theorem* 4.1, *respectively, then*

$$F(u,A)\le\int_A f(x,u,\nabla u)\,dx+\int_{S_u\cap A}\varphi(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{n-1} \tag{6.3}$$

for all $u\in SBV^p(\Omega;\mathbb{R}^m)$ and $A\in\mathcal{A}(\Omega)$.

Proof. The proof will be achieved by successive approximations. The first step is to observe that if $u\in SBV_0(\Omega;\mathbb{R}^m)$ and $|\overline{S_u}|=0$ then by the measure property of F, by Remark 6.1 and by Theorem 4.1

$$\begin{aligned}F(u,A)&=F(u,A\setminus S_u)+F(u,A\cap S_u)\\&=F(u,A\setminus\overline{S_u})+F(u,A\cap S_u)\\&=\int_{A\setminus\overline{S_u}}f(x,u,0)\,dx+\int_{A\cap S_u}\varphi(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{n-1}.\end{aligned}$$

Next, we suppose that $u\in SBV^p(\Omega;\mathbb{R}^m)\cap\mathrm{L}^\infty(\Omega;\mathbb{R}^m)$, and that the sets

$$B_h=\{x\in A:\ \mathrm{dist}\,(x,S_u)<\frac{1}{h}\}$$

satisfy

$$|B_h|\le c\frac{1}{h}, \tag{6.4}$$

in particular

$$|\overline{S_u} \setminus S_u| = 0 \tag{6.5}$$

In this case fixed a sequence (ρ_h) decreasing to 0 we can find by Remark 2.1 a sequence $v_h \in SBV_0(B_h; \mathbb{R}^m)$ such that

$$\|u - v_h\|_\infty \le \rho_h \qquad \mathcal{H}^{n-1}((S_{v_h} \cap B_h) \setminus S_u) \le \frac{1}{\rho_h} \int_{B_h} |\nabla u|\, dx.$$

We define $u_h = \phi_h u + (1 - \phi_h) v_h$, where ϕ_h is a cut-off function between B_{2h} and B_h, *i.e.* $\phi_h \in C_0^\infty(B_h)$ with $\phi_h = 1$ on B_{2h}, such that $|D\phi_h| \le ch$. Since $u_h \to u$, by lower semicontinuity we obtain

$$\begin{aligned} F(u, A) \le F(u_h, A) + o(1) &\le F(u, A \setminus \overline{S_u}) + F(v_h, B_h) \\ &\quad + c \int_{B_h \cap S_{u_h}} (1 + |v_h^+ - v_h^-|) d\mathcal{H}^{n-1} + c \int_{B_h} |u - u_h|^p |D\phi_h|^p\, dx + o(1) \\ &\le \int_A f(x, u, \nabla u)\, dx + \int_{B_h \cap S_u} \varphi(x, v_h^+, v_h^-, \nu_u) d\mathcal{H}^{n-1} \\ &\quad + c \frac{1}{\rho_h} \int_{B_h} |\nabla u|\, dx + c\rho_h^p h^p |B_h| + o(1). \end{aligned}$$

We can choose $(\frac{1}{p} + \frac{1}{p'} = 1)$

$$\rho_h = h^{-1/p'} \Big(\int_{B_h} |\nabla u|^p\, dx \Big)^{1/2p}$$

to get

$$\frac{1}{\rho_h} \int_{B_h} |\nabla u|\, dx + c\rho_h^p h^p |B_h| \to 0 \text{ as } h \to +\infty,$$

so that

$$F(u, A) \le \int_A f(x, u, \nabla u)\, dx + \int_{A \cap S_u} \varphi(x, v_h^+, v_h^-, \nu_u) d\mathcal{H}^{n-1} + o(1).$$

From the continuity of $\varphi(x, \cdot, \cdot, \nu)$, which follows easily from (vi)$'$ (see Remark 3.6), we get

$$F(u, A) \le \int_A f(x, u, \nabla u)\, dx + \int_{A \cap S_u} \varphi(x, u^+, u^-, \nu_u) d\mathcal{H}^{n-1}.$$

Suppose now only that $u \in SBV^p(\Omega; \mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega; \mathbb{R}^m)$. By Lemma 5.2 there exists a sequence of functions u_h converging strongly to u in $SBV^p(\Omega; \mathbb{R}^m)$ such that

$$|\overline{S_{u_h}} \setminus S_{u_h}| = 0 \tag{6.6}$$

and by Remark 5.3 condition (6.4) holds locally in Ω. If $A \subset\subset \Omega$ then we have, by the lower semicontinuity of F and the previous step,

$$\begin{aligned} F(u, A) &\le \liminf_{h \to +\infty} F(u_h, A) \\ &= \lim_{h \to +\infty} \Big(\int_A f(x, u_h, \nabla u_h)\, dx + \int_{S_{u_h} \cap A} \varphi(x, u_h^+, u_h^-, \nu_{u_h})\, d\mathcal{H}^{n-1} \Big) \\ &= \int_A f(x, u, \nabla u) dx + \int_{S_u \cap A} \varphi(x, u^+, u^-, \nu_u)\, d\mathcal{H}^{n-1}. \end{aligned} \tag{6.7}$$

The last equality is obtained by the Lebesgue Dominated Convergence Theorem, taking into account the strong convergence of the sequence (u_h). Inequality (6.7) holds for arbitrary $A \in \mathcal{A}(\Omega)$ since both its sides are measures.

The final step is obtained by a truncation argument. We define u_h componentwise, by $(u_h)_j = (-h \vee u_j) \wedge h, \quad h \in \mathbb{N}$. If $u \in SBV^p(\Omega; \mathbb{R}^m)$ then we have $u_h \to u$ in $L^1(\Omega; \mathbb{R}^m)$, and, again by the Lebesgue Dominated Convergence Theorem,

$$\begin{aligned} &\lim_{h\to+\infty} \Big(\int_A f(x, u_h, \nabla u_h)\, dx + \int_{S_{u_h}\cap A} \varphi(x, u_h^+, u_h^-, \nu_{u_h})\, d\mathcal{H}^{n-1} \Big) \\ &= \int_A f(x, u, \nabla u)\, dx + \int_{S_u \cap A} \varphi(x, u^+, u^-, \nu_u)\, d\mathcal{H}^{n-1}. \end{aligned} \tag{6.8}$$

Hence the thesis follows as in (6.7) by the lower semicontinuity of $F(\cdot, A)$. □

In order to prove the converse inequality of (6.3) we show now that

$$F(u, A \setminus S_u) \geq \int_A f(x, u, \nabla u)\, dx. \tag{6.9}$$

We will make use of a derivation technique, and an approximation argument by Lipschitz functions following the ideas of Ambrosio [5].

Definition 6.3. If μ is a Borel measure on the ball $B_r = B_r(0)$ then we define the *maximal function of μ on B_r* as

$$M_r(\mu)(x) = \sup_{0<\rho<(r-|x|)} \frac{\mu(B_\rho(x))}{|B_\rho|}, \tag{6.10}$$

for $x \in B_r$.

Remark 6.4. The maximal function $M_r(|Du|)$, where $u \in BV(B_r; \mathbb{R}^m)$ remains unchanged by scaling; more precisely, if we define

$$u_t(y) = \frac{1}{t} u(ty) \tag{6.11}$$

then we have

$$M_r(|Du|)\Big(\frac{ry}{s}\Big) = M_s(|Du_{\frac{r}{s}}|)(y) \tag{6.12}$$

for $y \in B_s$.

The following lemma has been proven in [5] using the maximal function of $|Du|$.

Lemma 6.5. *Let $u \in BV(B_r; \mathbb{R}^m)$ and*

$$E^r_\lambda = E^r_\lambda(u) = \{x \in B_r : \ M_r(|Du|)(x) \geq \lambda\}; \tag{6.13}$$

then there exists a Lipschitz function $v^r_\lambda : B_{r/4} \to \mathbb{R}^m$ such that

$$\|\nabla v^r_\lambda\|_\infty \leq c(n,m)\lambda \tag{6.14}$$

such that $u = v^r_\lambda$ a.e. on $B_{r/4} \setminus E^r_\lambda$.

Proof. The proof follows from the first part of the proof of [5] Theorem 2.3, where an estimate of $\|\nabla v_\lambda^r\|_\infty$ is given on the whole B_r. □

Remark 6.6. As in [5] Remark 2.4, we have the estimate

$$|C \cap E_{2\lambda}^r| \le \lambda^{-p} \int_{C \cap E_\lambda^r} (M_r(|\nabla u|))^p \, dx + \frac{1}{\lambda} c(n) |D_s u|(B_r).$$

for all Borel subset C of B_r. Note also that if u_t is defined by (6.11) then by (6.13), (6.12) we have $E_\lambda^s(u_{\frac{r}{s}}) = \frac{s}{r} E_\lambda^r(u)$.

Lemma 6.5 will allow us to pass from SBV functions to Lipschitz maps, using the following "partial locality result".

LEMMA 6.7. *Let $u \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$, and let $v \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ be such that $u = v$ a.e. on a Borel set $B \subset A \subset\subset \Omega$; then $F(u,A) \ge F(v,B)$.*

Proof. We can suppose $m = 1$, otherwise we argue componentwise. For every $h \in \mathbb{N}$ we consider the set

$$B_h = \Big\{ x \in \Omega : \; v(x) - \frac{1}{h} < u^+(x) \wedge u^-(x) \le u^+(x) \vee u^-(x) < v(x) + \frac{1}{h} \Big\}.$$

The set B_h is quasi-open with respect to the 1-capacity, defined as

$$Cap_1(E,\Omega) = \inf\{\mathcal{H}^{n-1}(\partial^* C) : \; E \subset C \subset\subset \Omega\};$$

this means that for every $\varepsilon > 0$ there exist a set E_ε with $Cap_1(E_\varepsilon,\Omega) < \varepsilon$ such that $B_h \cup E_\varepsilon$ is open (see [21] Theorem 2.5). Let C_h satisfy $E_{\frac{1}{h}} \subset C_h \subset\subset \Omega$, let $\mathcal{H}^{n-1}(\partial^* C_h) < \frac{2}{h}$, and let $C_h \cup B_h$ be open. Set $\gamma = \|u\|_\infty + \|v\|_\infty$,

$$\begin{aligned} U_h = \Big\{ x \in \Omega : \; & v(x) - \frac{1}{h} - \gamma\,\chi_{C_h}(x) < u^+(x) \wedge u^-(x), \\ & u^+(x) \vee u^-(x) < v(x) + \frac{1}{h} + \gamma \chi_{C_h}(x) \Big\}. \end{aligned}$$

and

$$v_h(x) = \begin{cases} v(x) - \frac{1}{h} - \gamma\,\chi_{C_h}(x) & \text{if } u(x) \le v(x) - \frac{1}{h} - \gamma\,\chi_{C_h} \\ u(x) & \text{if } x \in U_h \\ v(x) + \frac{1}{h} + \gamma\,\chi_{C_h}(x) & \text{if } u(x) \le v(x) - \frac{1}{h} - \gamma \chi_{C_h}. \end{cases}$$

We have $v_h \to u$ in $L^1(A)$, and $C_h \cup B_h \subset U_h$. Let now $A' \in \mathcal{A}(\Omega)$ be such that $B \subset A' \subset A$. By the lower semicontinuity, locality and measure properties of F we have

$$\begin{aligned} F(v,B) \le F(v,A') &\le \liminf_{h\to+\infty} F(v_h,A') \\ &= \liminf_{h\to+\infty} \Big(F(v_h,(C_h \cup B_h) \cap A') + F(v_h, A' \setminus (C_h \cup B_h)) \Big) \\ &= \liminf_{h\to+\infty} \Big(F(u,(C_h \cup B_h) \cap A') + F(v_h, A' \setminus (C_h \cup B_h)) \Big) \\ &\le \liminf_{h\to+\infty} \Big(F(u,U_h \cap A') + F(v_h, A' \setminus (C_h \cup B_h)) \Big). \end{aligned}$$

The last term can be estimated by

$$\begin{aligned}F(v_h, A'\setminus(C_h\cup B_h)) &\le F(v_h, A'\setminus B)\\ &\le \beta\Big(\int_{A'\setminus B}|\nabla v_h|^p\,dx+\int_{S_{v_h}\cap A'\setminus B}(1+|v_h^+-v_h^-|)d\mathcal{H}^{n-1}\Big)\\ &\le \beta\Big(\int_{A'\setminus B}(|\nabla v|^p+|\nabla u|^p)\,dx\\ &\qquad+\int_{S_u\cap A'\setminus B}(1+|u^+-u^-|)d\mathcal{H}^{n-1}+\gamma\,\mathcal{H}^{n-1}(\partial^*C_h\cap A')\Big),\end{aligned}$$

so that it tends to 0 as $A'\to B$, uniformly with h. We have then

$$F(v,B)\le\liminf_{h\to+\infty}F(u,U_h\cap A')\le F(u,A),$$

and the proof is concluded. □

The proof of (6.9) will be achieved by the following proposition.

Proposition 6.8. *We have*

$$\liminf_{\rho\to0+}\frac{F(u,B_\rho(x_0))}{|B_\rho|}\ge f(x_0,u(x_0),\nabla u(x_0)) \tag{6.15}$$

for almost all $x_0\in\Omega$.

Proof. Without loss of generality we suppose that $x_0=0$,

$$\lim_{\rho\to0+}\frac{|D_s(u)|(B_\rho(0))}{|B_\rho|}=0, \tag{6.16}$$

and that 0 is a Lebesgue point for u and ∇u. Let $B_r\subset\Omega$, and $\rho\le r/4$. Let $v_\lambda^{4\rho}$ be as in Lemma 6.5 (with 4ρ in the place of r), and define

$$v_{\lambda,\rho}(y)=\frac{1}{\rho}v_\lambda^{4\rho}(\rho y),\qquad u_\rho(y)=\frac{1}{\rho}u(\rho y). \tag{6.17}$$

By Lemma 6.7 above applied to $v=v_\lambda^{4\rho}$, and $B=B_\rho\setminus E_\lambda^{4\rho}$ ($E_\lambda^{4\rho}$ defined by (6.13)), and by Remark 6.1, we have

$$\begin{aligned}\frac{F(u,B_\rho)}{\rho^n}&\ge\frac{F(v_\lambda^{4\rho},B_\rho\setminus E_\lambda^{4\rho})}{\rho^n}\\ &=\frac{1}{\rho^n}\int_{B_\rho\setminus E_\lambda^{4\rho}}f(x,v_\lambda^{4\rho}(x),\nabla v_\lambda^{4\rho}(x))\,dx\\ &=\frac{1}{\rho^n}\int_{B_\rho\setminus E_\lambda^{4\rho}}f(x,u(x),\nabla v_\lambda^{4\rho}(x))\,dx\\ &=\int_{B_1\setminus E_\lambda^4}f(\rho y,u(\rho y),\nabla v_{\lambda,\rho}(y))\,dy,\end{aligned} \tag{6.18}$$

where $E_\lambda^4=E_\lambda^4(u_\rho)=\frac{1}{\rho}E_\lambda^{4\rho}$ by Remark 6.6. Note that

$$\begin{aligned}\int_{B_4}M_4(|\nabla u_\rho|)^p\,dx&=\frac{1}{\rho^n}\int_{B_{4\rho}}M_{4\rho}(|\nabla u|)^p\,dx\\ &\le\frac{c}{\rho^n}\int_{B_{4\rho}}|\nabla u|^p\,dx\le c.\end{aligned} \tag{6.19}$$

Let $\varepsilon > 0$; we can suppose by [1] Lemma I.7 that there exist $C_\varepsilon \subset B_4$, with $|C_\varepsilon| \le \varepsilon$, and $\lambda_\varepsilon \ge 1$ such that if $\lambda \ge \lambda_\varepsilon$ then

$$\int_{\Big(B_1\setminus\{M_4(|\nabla u_\rho|)>\lambda/2\}\Big)\setminus C_\varepsilon} M_4(|\nabla u_\rho|)^p\,dx \le \varepsilon, \tag{6.20}$$

for all $\rho \in (0,1)$. Hence, by Remark 6.6

$$\lambda^p|(E^4_\lambda \cap B_1)\setminus C_\varepsilon| \le \varepsilon + \Big(\frac{\lambda}{2}\Big)^{p-1} c(n)|D_s u|(B_{4\rho}), \tag{6.21}$$

and

$$\begin{aligned}\frac{F(u,B_\rho)}{\rho^n} &\ge \int_{(B_1\setminus E^4_\lambda)\setminus C_\varepsilon} f(\rho y, u(\rho y), \nabla v_{\lambda,\rho}(y))\,dy\\ &\ge \int_{B_1\setminus C_\varepsilon} f(\rho y, u(\rho y), \nabla v_{\lambda,\rho}(y))\,dy - \beta 2\lambda^p|(B_1\cap E^4_\lambda)\setminus C_\varepsilon|.\end{aligned} \tag{6.22}$$

Let $v_{\lambda,0}$ be the weak* limit in $\mathrm{W}^{1,\infty}(B_1;\mathbb{R}^m)$ of $(v_{\lambda,\rho})$ as $\rho \to 0$, which we can suppose exists up to passage to a subsequence. Since $u(\rho y) \to 0$ pointwise, we have

$$\liminf_{\rho\to 0+}\int_{B_1\setminus C_\varepsilon} f(\rho y, u(\rho y), \nabla v_{\lambda,\rho}(y))\,dy \ge \int_{B_1\setminus C_\varepsilon} f(0,u(0),\nabla v_{\lambda,0}(y))\,dy. \tag{6.23}$$

This inequality follows by the lower semicontinuity of $u \mapsto \int f(\nabla u)\,dx$ if $f = f(s)$, and is a consequence of standard Γ-convergence results in the general case (see *e.g.* [19], or [27]). We have then

$$\liminf_{\rho\to 0+}\frac{F(u,B_\rho)}{|B_\rho|} \ge \frac{1}{|B_1|}\int_{B_1\setminus C_\varepsilon} f(0,u(0),\nabla v_{\lambda,0}(y))\,dy - \frac{\varepsilon}{|B_1|}.$$

Since $|\{x\in B_1:\ \nabla u(0)\ne\nabla v_{\lambda,0}\}| \le \frac{c}{\lambda}$, by the lower semicontinuity of the functional $w \mapsto |\{x \in B_1:\ w(x)\ne 0\}|$ with respect to convergence in measure, we have

$$\liminf_{\rho\to 0+}\frac{F(u,B_\rho)}{|B_\rho|} \ge f(0,u(0),\nabla u(0))\Big(1-\frac{\varepsilon}{|B_1|}-\frac{1}{|B_1|}\frac{c}{\lambda}\Big).$$

Letting $\lambda \to +\infty$, and $\varepsilon \to 0$ we obtain the thesis. □

Proof of Theorem 2.4. If $u \in SBV^p(\Omega;\mathbb{R}^m)$ then by the hypotheses on F the set function $B \mapsto F(u, B\setminus S_u)$ is a Borel measure absolutely continuous with respect to the Lebesgue measure. By Proposition 6.8 we have

$$F(u, A\setminus S_u) \ge \int_A f(x,u(x),\nabla u(x))\,dx$$

for any $A \in \mathcal{A}(\Omega)$; hence, taking into account Proposition 4.2 we have

$$\begin{aligned}F(u,A) &= F(u,A\setminus S_u) + F(u, A\cap S_u)\\ &\ge \int_A f(x,u(x),\nabla u(x))\,dx + \int_{S_u\cap A}\varphi(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{n-1}.\end{aligned}$$

Lemma 6.2 concludes the proof. □

Remark 6.10. If the hypotheses of Theorem 2.4 are satisfied only for $u \in SBV^p(\Omega;\mathbb{R}^m)\cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ then its thesis holds for u in this space. The same remark applies to Lemma 6.2 (taking into account Remark 3.7). Similarly, we can see that in Corollary 2.8 it is sufficient that the hypotheses be satisfied on $u \in SBV^p(\Omega;\mathbb{R}^m)\cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$.

7. Relaxation of image segmentation problems

As an example of application of Theorem 2.4 we give a relaxation result for a special class of integrands. The functionals we treat are modeled on the Mumford and Shah image segmentation functional (see [34])

$$F(u)=\int_\Omega |\nabla u|^2\,dx+c_1\int_\Omega |u-g|^2\,dx+c_2\mathcal{H}^{n-1}(S_u\cap\Omega),\qquad u\in SBV^2(\Omega). \tag{7.1}$$

In this case $m=1$, the "grey function" g, with $0\le g\le 1$, represents the input picture, and S_u is expected to detect the relevant contours of the objects in the picture. The existence of minima for the functional F can be obtained using the compactness theorem and lower semicontinuity results by Ambrosio [3]. More in general we can consider functionals of the form

$$G(u)=\int_\Omega h(\nabla u)\,dx+c_1\int_\Omega |u-g|^p\,dx+\int_{S_u\cap\Omega}\psi(u^+,u^-)d\mathcal{H}^{n-1},\ u\in SBV^p(\Omega). \tag{7.2}$$

If the functional G is not lower semicontinuous, then the behaviour of minimizing sequences is described by the study of the lower semicontinuous envelope, or relaxation, $\overline{G}$ of G. Since the proof of a general relaxation result lies beyond the scopes of this paper, we will be content to limit our analysis to integrands satisfying some technical assumptions. These will be of two types. First, we suppose that

$$h(0)=0, \tag{7.3}$$

and that ψ is increasing, which means that

$$\psi(a,b)\le\psi(a',b')\quad\text{if}\quad a'\le a<b\le b', \tag{7.4}$$

These hypotheses allow a simple truncation argument. If (7.3) or (7.4) do not hold, or if $m>1$, a more complex truncation procedure can be applied, following [22] (see also [17] Section 3). Secondly, we will need some Lipschitz continuity for h and ψ, that permits a short proof of property (vi)$'$ of Lemma 6.2. Namely, we suppose that

$$|h(\xi)-h(\eta)|\le c(1+|\xi|^{p-1}+|\eta|^{p-1})|\xi-\eta| \tag{7.5}$$

for all ξ, η, and that for all $R>0$ there exists c_R such that

$$|\psi(u,v)-\psi(u',v')|\le c_R(|u-u'|+|v-v'|) \tag{7.6}$$

if $|u|$, $|u'|$, $|v|$, $|v'|\le R$.

Theorem 2.4 allows the integral representation and a simple description of $\overline{G}$, that leads to the following result.

PROPOSITION 7.1. *Let $h:\mathbb{R}^n\to\mathbb{R}$ and $\psi:\mathbb{R}^2\to[0,+\infty)$ be functions satisfying (7.3)–(7.6); we suppose that $\psi(a,b)=\psi(b,a)$ for all $a,b\in\mathbb{R}$, and the growth conditions*

$$\alpha|\xi|^p\le h(\xi)\le\beta(1+|\xi|^p), \tag{7.7}$$

for some $p > 1$, and

$$\alpha \le \psi(u,v) \le \beta(1+|v-u|). \tag{7.8}$$

for all $u, v \in \mathbb{R}$. Then, if $g \in L^\infty(\Omega)$, we have

$$\begin{aligned}
&\inf\Big\{\int_\Omega h(\nabla u)\,dx + c_1\int_\Omega |u-g|^p\,dx + \int_{S_u\cap\Omega}\psi(u^+,u^-)d\mathcal{H}^{n-1} :\ u \in SBV^p(\Omega)\Big\}\\
&= \min\Big\{\int_\Omega h^{**}(\nabla u)\,dx + c_1\int_\Omega |u-g|^p\,dx + \int_{S_u\cap\Omega}(\operatorname{sub}\psi)(u^+,u^-)d\mathcal{H}^{n-1}\\
&\qquad\qquad :\ u \in SBV^p(\Omega)\Big\},
\end{aligned}$$

where h^{**} is the convex envelope of h (i.e., the greatest convex function not greater than h), and $\operatorname{sub}\psi$ is the subadditive envelope of ψ (i.e., the greatest function ϕ not greater than ψ satisfying

$$\phi(a,b) \le \phi(a,c) + \phi(c,b)$$

for all $a, b, c \in \mathbb{R}$).

Proof. Note that if we substitute u by $(u \vee (-\|g\|_\infty)) \wedge \|g\|_\infty$ then the value of all integrals decreases. Hence, our minimum problems will be carried on without loss of generality on $SBV^p(\Omega) \cap \{\|u\|_\infty \le \|g\|_\infty\}$.

Let $H : SBV^p(\Omega) \times \mathcal{A}(\Omega) \to [0,+\infty)$ be defined by

$$H(u,A) = \int_A h(\nabla u)dx + \int_{S_u\cap A}\psi(u^+,u^-)d\mathcal{H}^{n-1} + \int_{S_u\cap A}(|u^+-u^-|-2\|g\|_\infty)\vee 0\, d\mathcal{H}^{n-1}.$$

Note that the last integral is 0 if $\|u\|_\infty \le \|g\|_\infty$. Let $\overline{H}$ denote the lower semicontinuous envelope of H in the L^1-topology:

$$\overline{H}(u,A) = \inf\Big\{\liminf_{h\to+\infty} H(u_h,A),\ u_h \to u \text{ in } L^1(A)\Big\}.$$

By a truncation argument as above, if $u \in L^\infty(A)$ we have

$$\begin{aligned}
\overline{H}(u,A) &= \inf\Big\{\liminf_{h\to+\infty} H(u_h,A),\ u_h \to u \text{ in } L^1(A),\ \|u_h\|_\infty \le \|u\|_\infty\Big\}\\
&= \inf\Big\{\liminf_{h\to+\infty} H(u_h,A),\ u_h \to u \text{ in } L^p(A)\Big\}.
\end{aligned}$$

It is not difficult to see, following the standard techniques of relaxation (see [27], [18]) that the extension to $\mathcal{B}(\Omega)$ of $\overline{H}(u,\cdot)$ is a measure for all $u \in SBV^p(\Omega) \cap L^\infty(\Omega)$ (a complete proof can be found in [17] Section 3 in a far more general setting). Hence, the functional $F(u,B) = \overline{H}(u,B)$ satisfies conditions (i)–(v) of Theorem 2.4. Moreover, it also satisfies condition (vi)′ of Lemma 6.2. In fact, let A' be any open subset of A with $S_u \cap A \subset A'$, and let $u_h \to u$ with $\|u_h\|_\infty \le \|u\|_\infty$ be such that $F(u,A') = \lim_{h\to+\infty} H(u_h,A')$. Since $\lim_{h\to+\infty} H(u_h,A') \le H(u,A')$, we have

$$\int_{A'} |\nabla u_h|^p\,dx + \mathcal{H}^{n-1}(S_{u_h}\cap A') \le c\Big(\int_{A'}(1+|\nabla u|^p)\,dx + \mathcal{H}^{n-1}(S_u\cap A')\Big). \tag{7.9}$$

Let $v_h = \Phi u_h$. We have $v_h \to \Phi u$, and, by (7.5)–(7.8),

$$\begin{aligned} F(\Phi u, A') &\le \liminf_{j\to+\infty} H(v_j, A') \\ &= \liminf_{j\to+\infty}\Big(\int_{A'} h(\Phi\nabla u_j)\,dx + \int_{S_{u_j}\cap A'} \psi(\Phi u_j^+, \Phi u_j^-)\,d\mathcal{H}^{n-1} \\ &\qquad + \int_{S_{u_j}\cap A'} (|\Phi u_j^+ - \Phi u_j^-| - 2\|g\|_\infty)\vee 0\,d\mathcal{H}^{n-1}\Big) \\ &\le \liminf_{j\to+\infty}\Bigg(H(u_j, A') + c\|\Phi - Id\|\Big(\big(\int_{A'} |\nabla u_j|^p\,dx\big)^{1/p} \\ &\qquad + \int_{A'\cap S_{u_j}} (1+|u_j^+|+|u_j^-|)d\mathcal{H}^{n-1}\Big)\Bigg), \end{aligned}$$

so that, by (7.9),

$$F(\Phi u, A') - F(u, A') \le c\|\Phi - Id\|\Big(\big(\int_{A'} |\nabla u|^p\,dx\big)^{1/p} + \int_{A\cap S_u} (1+|u^+|+|u^-|)d\mathcal{H}^{n-1}\Big).$$

Letting $A' \to S_u \cap A$ and using a symmetry argument we get (vi)′.

By Remark 6.10 we obtain the inequality

$$\overline{H}(u, A) \le \int_A f(x, u, \nabla u)\,dx + \int_{A\cap S_u} \varphi(x, u^+, u^-, \nu_u)d\mathcal{H}^{n-1}, \tag{7.10}$$

where f and φ are given by Remark 6.1 and Theorem 4.1. It is easy to see by formula (3.6) and Remark 2.6 that indeed we may suppose that

$$f(x, u, \xi) = f(\xi), \qquad \text{and} \qquad \varphi(x, u, v, \nu) = \varphi(u, v).$$

It is well-known that convexity is a necessary condition for the weak lower semicontinuity on $W^{1,p}(\Omega)$ of the functional $u \mapsto \int_A f(\nabla u)\,dx$ (see *e.g.* [26]). Hence, the function f is convex and not greater than h, so that

$$f \le h^{**}. \tag{7.11}$$

On the other hand, subadditivity is a necessary condition for the lower semicontinuity of the functional $u \mapsto \int_{A\cap S_u} \varphi(u^+, u^-)\,dx$ on spaces of Caccioppoli partitions $BV(A; T)$ where T is any finite subset of $\mathbb{R}$ (see [8]). Hence, φ is subadditive and not greater than $\widetilde{\psi}$, where $\widetilde{\psi}(u, v) = \psi(u, v) + (|v - u| - 2\|g\|_\infty) \vee 0$ so that

$$\varphi \le \operatorname{sub}\widetilde{\psi}. \tag{7.12}$$

From (7.10)–(7.12) we obtain

$$\overline{H}(u, A) \le \int_A h^{**}(\nabla u)\,dx + \int_{A\cap S_u} \operatorname{sub}\widetilde{\psi}(u^+, u^-)d\mathcal{H}^{n-1}. \tag{7.13}$$

The functional on the right hand side of (7.13) is lower semicontinuous with respect to L^1-convergence (see [4]), and not greater than $H(\cdot, A)$. Hence,

$$\int_A h^{**}(\nabla u)\,dx + \int_{A\cap S_u} \operatorname{sub}\widetilde{\psi}(u^+, u^-)d\mathcal{H}^{n-1} \le \overline{H}(u, A), \tag{7.14}$$

and by (7.13) we have the equality.

If $|a|, |b| \le \|g\|_\infty$, then, by the truncation argument as above applied to the test functions in formula (3.6) defining φ, we have

$$\operatorname{sub}\widetilde{\psi}(a,b) = \operatorname{sub}\psi(a,b),$$

so that

$$\overline{H}(u,A) = \int_A h^{**}(\nabla u)\,dx + \int_{A\cap S_u} \operatorname{sub}\psi(u^+,u^-)d\mathcal{H}^{n-1}$$

if $\|u\|_\infty \le \|g\|_\infty$. By the continuity of $u \mapsto \int_\Omega |u-g|^p\,dx$, and the compactness of the sets

$$\Big\{u \in SBV^p(\Omega) : \|u\|_\infty \le \|g\|_\infty,\ \int_\Omega |\nabla u|^p\,dx + \mathcal{H}^{n-1}(S_u \cap A) \le c\Big\}$$

with respect to the L^1-topology, we obtain

$$\begin{aligned}
&\inf\Big\{H(u,\Omega) + \int_\Omega |u-g|^p\,dx :\ u \in SBV^p(\Omega)\Big\}\\
&\quad= \inf\Big\{H(u,\Omega) + \int_\Omega |u-g|^p\,dx :\ u \in SBV^p(\Omega),\ \|u\|_\infty \le \|g\|_\infty\Big\}\\
&\quad= \min\Big\{\overline{H}(u,\Omega) + \int_\Omega |u-g|^p\,dx :\ u \in SBV^p(\Omega),\ \|u\|_\infty \le \|g\|_\infty\Big\}\\
&\quad= \min\Big\{\overline{H}(u,\Omega) + \int_\Omega |u-g|^p\,dx :\ u \in SBV^p(\Omega)\Big\}.
\end{aligned}$$

The last equality follows again by the truncation argument, and the fact that $\operatorname{sub}\widetilde{\psi}$ is increasing. □

Remark 7.2. Condition $\psi(a,b) = \psi(b,a)$ is necessary to have a good definition of the second integral in (7.2). It can be avoided, *e.g.* by assuming $u^+ > u^-$ by definition. Theorem 7.1 holds without changes in the proof if $h = h(x,\xi)$ and $\varphi = \varphi(x,u,v)$ are continuous, while simple examples show that it does not hold if we suppose only a Carathéodory condition for h and φ.

8. Relaxation of nonlinear elasticity energies for brittle fracture problems

The Griffith theory of brittle fracture can be included in our framework by a suitable choice of the surface energy. In the case of isotropic and homogeneous response to fracture the energy for crack initiation is proportional to the crack surface. For an hyperelastic material occupying the reference configuration Ω, we can write then the total energy

$$E(u) = \int_\Omega W(x,\nabla u)\,dx + \lambda\mathcal{H}^{n-1}(S_u),$$

where W is the elastic energy density of the uncracked portion of the body, and the value of the constant λ is given by Griffith's criterion. In this notation ∇u represents the deformation gradient and S_u the crack surface. If the functional E is not lower semicontinuous on $SBV^p(\Omega;\mathbb{R}^m)$ then we can give a relaxation results by applying Corollary 2.8.

THEOREM 8.1. *Let $W : \Omega \times M^{n\times m} \to [0,+\infty)$ be a Borel function satisfying the growth condition*

$$\alpha|\xi|^p \le W(x,\xi) \le \beta(1+|\xi|^p)$$

for all $x \in \Omega$, $\xi \in M^{n\times m}$ (β, $\alpha > 0$). Then the lower semicontinuous envelope of E in the $\mathrm{L}^1(\Omega;\mathbb{R}^m)$-topology is given by

$$\overline{E}(u) = \int_\Omega QW(x,\nabla u)\,dx + \lambda\mathcal{H}^{n-1}(S_u),$$

for all $u \in SBV^p(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$, where

$$QW(x,\xi) = \inf\Big\{\int_\Omega W(x, Du(y)+\xi)\,dy : \ u \in \mathcal{C}_0^\infty(\Omega;\mathbb{R}^m)\Big\}$$

is the quasiconvex envelope of W (see [26]).

Proof. We localize the functional E, by setting

$$E(u,A) = \int_A W(x,\nabla u)\,dx + \lambda\mathcal{H}^{n-1}(S_u\cap A)$$

for all $A \in \mathcal{A}(\Omega)$. As in the proof of Proposition 7.1 it can be shown that for all $u \in SBV^p(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ the set function $\overline{E}(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure. We denote by $F(u,\cdot)$ its extension to $\mathcal{B}(\Omega)$. The functional F satisfies hypotheses (i)–(iii) and (v) of Theorem 2.4 on $SBV^p(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$, and the growth condition

$$\alpha\int_B |\nabla u|^p\,dx+\lambda\mathcal{H}^{n-1}(S_u\cap B) \le F(u,B) \le \beta\int_B (1+|\nabla u|^p)\,dx+\lambda\mathcal{H}^{n-1}(S_u\cap B) \tag{8.1}$$

for all $B \in \mathcal{B}(\Omega)$ and $u \in SBV^p(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$. In particular

$$F(u,S) = \lambda\mathcal{H}^{n-1}(S) \tag{8.2}$$

for all $S \subset S_u$, so that hypothesis (vi) of Theorem 2.4 is also satisfied. By (8.1) we see that the hypotheses of Corollary 2.8 are all satisfied for $u \in SBV^p(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$. Recalling Remark 6.10 and (8.2) we have then the integral representation

$$F(u,B) = \int_B f(x,u,\nabla u)\,dx + \lambda\mathcal{H}^{n-1}(S_u\cap B) \tag{8.3}$$

for $u \in SBV^p(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ and $B \in \mathcal{B}(\Omega)$, with f given by Remark 6.1. By Remark 2.6 it is easy to see that there is no loss of generality in supposing $f(x,u,\xi) = f(x,\xi)$. It is well-known that the quasiconvexity of f is a necessary condition for the lower semicontinuity of $u \mapsto F(u,A)$ on $\mathrm{W}^{1,p}(A;\mathbb{R}^m)$ for all $A \in \mathcal{A}(\Omega)$. Hence, since $f \le W$, we have

$$f(x,\xi) \le QW(x,\xi) \qquad \text{for all } x \in \Omega,\ \xi \in M^{n\times m}. \tag{8.4}$$

On the other hand, the functional

$$G(u,A) = \int_A QW(x,\nabla u)\,dx + \lambda\mathcal{H}^{n-1}(S_u\cap A) \tag{8.5}$$

is lower semicontinuous with respect to the L^1-convergence on $SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ (see [5] and [17] Lemma 3.5), so that, since $G(u,A) \le E(u,A)$, we have

$$(8.6) \qquad G(u,A) \le \overline{E}(u,A) \qquad \text{for all } u \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m),\ A \in \mathcal{A}(\Omega).$$

taking into account (8.3)–(8.6) we have $G = \overline{E}$, and the proof is concluded. □

Remark 8.2. The thesis of Theorem 8.1 can be proven on the whole space $SBV(\Omega;\mathbb{R}^m)$ by a truncation procedure introduced in [22]. In the case of W independent of x this result can be seen as a particular case of the homogenization theorem in [17].

REFERENCES

[1] E. Acerbi and N. Fusco Semicontinuity problems in the calculus of variations *Arch. Rational Mech. Anal.* **86** (1984), 125-145.

[2] F. J. Almgren Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints *Mem. Amer. Mat. Soc.* **4** (1976), 165

[3] L. Ambrosio, A compactness theorem for a new class of functions of bounded variation, *Boll. Un. Mat. Ital.* **3-B** (1989), 857-881.

[4] L. Ambrosio, Existence theory for a new class of variational problems, *Arch. Rational Mech. Anal.* **111** (1990), 291-322.

[5] L. Ambrosio, On the lower semicontinuity of quasi-convex integrals in $SBV(\Omega;\mathbb{R}^k)$, *Nonlinear. Anal.* **23** (1994), 405-425.

[6] L. Ambrosio, The space SBV(Ω) and free discontinuity problems, in *Variational and Free Boundary Problems* (A. Friedman and J. Spruch eds.), Springer-Verlag, New York, 1994, 29-45.

[7] L. Ambrosio and A. Braides, Functionals defined on partitions of sets of finite perimeter, I: integral representation and Γ-convergence, *J. Math. Pures. Appl.* **69** (1990), 285-305.

[8] L. Ambrosio and A. Braides, Functionals defined on partitions of sets of finite perimeter, II: semicontinuity, relaxation and homogenization, *J. Math. Pures. Appl.* **69** (1990), 307-333.

[9] L. Ambrosio, S. Mortola and V. M. Tortorelli, Functionals with linear growth defined on vector-valued BV functions, *J. Math. Pures. Appl.* **70** (1991) 269-323.

[10] L. Ambrosio and V. M. Tortorelli, Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence, *Comm. Pure Appl. Math.* **43** (1990), 999-1036.

[11] L. Ambrosio and V. M. Tortorelli, On the approximation of free discontinuity problems, *Boll. Un. Mat. Ital.* **6-B** (1992), 105-123.

[12] G. Bouchitté, A. Braides and G. Buttazzo, Relaxation results for some free discontinuity problems, *J. Reine Angew. Math.* (to appear).

[13] G. Bouchitté and G. Dal Maso, Integral representation and relaxation of convex local functionals on $BV(\Omega)$, *Ann. Scuola Norm. Sup. Pisa* **20** (1993), 483-533 .

[14] A. Braides and V. Chiadò Piat, A derivation formula for integral functionals defined on $BV(\Omega)$, Preprint SISSA, 1993.

[15] A. Braides and A. Coscia, The interaction between bulk energy and surface energy in multiple integrals, *Proc. Roy. Soc. Edinburgh Sect. A* **124** (1994), 737-756.

[16] A. Braides, A. Defranceschi and E. Vitali, A relaxation approach to Hencky's plasticity. Preprint SISSA, Trieste, 1993.

[17] A. Braides, A. Defranceschi and E. Vitali, Homogenization of free discontinuity problems. Preprint SISSA, Trieste, 1994.

[18] G. Buttazzo, *Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations*, Pitman Res. Notes Math. Ser. **207**, Longman, Harlow 1989.

[19] G. Buttazzo and G. Dal Maso, Γ-limits of integral functionals, *J. Analyse Math.* **37** (1980), 145-185.

[20] G. Buttazzo and G. Dal Maso, A characterization of nonlinear functionals on Sobolev spaces which admit an integral representation with a Carathéodory integrand, *J. Math. Pures Appl.*, **64** (1985), 337-361.

[21] G. Carriero, G. Dal Maso, A. Leaci and E. Pascali, Relaxation of the non-parametric Plateau problem with an obstacle, *J. Math. Pures Appl.*, **67** (1988), 359-396.

[22] P. Celada and G. Dal Maso, Further lower semicontinuity results for polyconvex integrals *Ann. Inst. H. Poincaré Analyse Non Linéaire*, (to appear)
[23] G. Carriero and A. Leaci, S^k-valued maps minimizing the L^p-norm of the gradient, *Ann. Scuola Norm. Sup. Pisa ser. IV*, **18** (1991), 321-.
[24] G. Carriero, A. Leaci and Tomarelli, Plastic free discontinuities and special bounded hessian, *C. R. Acad. Sci. Paris*, **314** (1992), 595-.
[25] G. Congedo and I. Tamanini, On the existence of solutions to a problem in multidimensional segmentation, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **2** (1991), 175-195.
[26] B. Dacorogna, Direct Methods in the Calculus of Variations, Springer-Verlag, Berlin, 1989
[27] G. Dal Maso, *An Introduction to* Γ*-convergence*, Birkhäuser, Boston, 1993.
[28] E. De Giorgi and L.Ambrosio, Un nuovo funzionale del calcolo delle variazioni, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **82** (1988), 199-
[29] E. De Giorgi, G. Carriero, and A. Leaci, Existence theorem for a minimum problem with free discontinuity set, *Arch. Rational Mech. Anal.*, **108** (1989), 195-218.
[30] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, Boca Raton 1992.
[31] H. Federer, *Geometric Measure Theory.* Springer-Verlag, New York, 1969.
[32] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, Basel 1983.
[33] A. P. Morse, Perfect blankets, *Trans. Amer. Math. Soc.* **6** (1947), 418-442.
[34] D. Mumford and J. Shah, Optimal approximation by piecewise smooth functions and associated variational problems, *Comm. Pure Appl. Math.* **17** (1989), 577-685.
[35] A. I. Vol'pert, Spaces BV and quasilinear equations, *Math. USSR Sb.* **17** (1967), 225-267.
[36] W. P. Ziemer, *Weakly Differentiable Functions*, Springer-Verlag, Berlin 1989.