RECTIFIABLE CURVES IN SIERPIŃSKI CARPETS

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Abstract. We characterize the slopes of nontrivial line segments contained in self-similar and non-self-similar Sierpiński carpets. The set of slopes is related to Farey sequences and the dynamics of punctured square toral billiards. Our results provide a first step towards a description of the rectifiable curves contained in such carpets.

1. **Introduction.** Let $Q = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\} \subset \mathbb{R}^2$ be the unit square. Divide $R_0 := Q$ into nine equal squares of side length $1/3$ and remove the central one. In this way, we obtain a set R_1 which is the union of 8 squares $Q_{1,j}$ of side length $1/3$. Repeating this procedure on each square we get a sequence of sets R_k, where R_k consists of 8^k squares $Q_{k,j}$ of side length 3^{-k}. We define the *Sierpiński carpet* to be

$$S_{\mathbf{3}} = \bigcap_{k \ge 1} R_k.$$

See Figure 1.

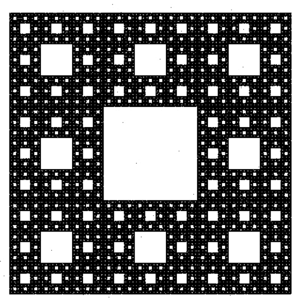

Figure 1. Standard Sierpiński carpet $S_{\mathbf{3}}$

A *carpet* is a metric space which is homeomorphic to $S_{\mathbf{3}}$. The following fundamental problem arises in the study of quasiconformal and bi-Lipschitz maps between carpets.

Characterize the rectifiable curves contained in a given carpet.

For instance, such a characterization could perhaps be used to give a direct proof of the following bi-Lipschitz rigidity property of $S_{\mathbf{3}}$: *every bi-Lipschitz map of $S_{\mathbf{3}}$ onto itself is the restriction of an isometry of the plane which preserves the unit square Q.* The bi-Lipschitz rigidity of $S_{\mathbf{3}}$ is a corollary of the quasisymmetric rigidity, which has been established by Bonk and Merenkov [7] using conformal modulus techniques. As far as we are aware, there is no independent proof of bi-Lipschitz rigidity which does not use conformal methods.[1] Further results on the conformal geometry of carpets can be found in [10], [6], [4], [13], [11]. We remark that the conformal geometry of carpets

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[1]We are grateful to Enrico Le Donne for suggesting this problem.

arises in connection with the Kapovich–Kleiner conjecture on quasisymmetric uniformization of Sierpiński carpet group boundaries. See [5] for additional details.

Let us consider planar carpets, i.e., carpets which are realized as subsets of the plane. Every rectifiable curve contained in such a carpet is, in particular, a rectifiable curve in the plane and hence admits a tangent line at almost every point by the theorem of Rademacher [14], [12, Theorem 7.3]. We thus naturally begin by considering the line segments contained in such carpets. Our starting point for this paper was the following folklore observation: *there exist points in $S_{\mathbf{3}}$ which are joined by straight line segments which lie entirely within $S_{\mathbf{3}}$, yet are not horizontal or vertical.* See Figure 2 for an illustration of some of these line segments.

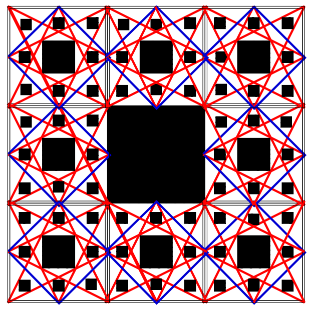

FIGURE 2. Line segments contained in $S_{\mathbf{3}}$

From the figure, we see that the set of slopes of nontrivial line segments contained in $S_{\mathbf{3}}$ is

$$\left\{0, \pm\frac{1}{2}, \pm 1, \pm 2, \infty\right\}.$$

A proof of this fact was given by Bandt and Mubarak [1].

A planar carpet S is called a *square carpet* if the bounded components of $\mathbb{R}^2 \setminus S$ are Euclidean squares. The boundaries of the omitted square domains are called the *peripheral squares* of S. Let us note that according to this definition the boundary of the largest square is not a peripheral square; this disagrees with the terminology used by some other authors.

In this paper, we give a complete description of the slopes of nontrivial line segments contained in the members of a class of square Sierpiński carpets. Our main results are Theorem 4.1 and Theorem 5.3. As a consequence, we deduce conclusions about the collection of everywhere differentiable curves contained in such carpets.

We conclude this introduction with an outline of the paper. In Section 2 we introduce the class of carpets under consideration in this paper. Section 3 describes a coordinate system for points in these carpets. In Section 7, where we present the proofs for our main theorems, we make substantial use of this coordinate system. Sections 4 and 5 contain the statements of our principal results as well as various corollaries. Here we also indicate the relationship between our results and the theory of Farey sequences and billiards. Section 6 contains preliminary material relevant for the proofs in Section 7. In the final Section 8, we conclude with results and examples connected with more general differentiable and rectifiable curves in carpets.

Acknowledgments. We benefited from useful conversations on this subject with Enrico Le Donne, John Mackay and Kevin Wildrick. We also thank the referee, whose careful reading and helpful comments improved the paper greatly. Research for this paper was completed while both authors were guests in the Mathematics Institute of the University of Bern in Fall 2009. The hospitality of the institute is gratefully appreciated. This work would have been impossible to perform without the help of a ruler, a protractor and Swiss chocolate.

2. **Self-similar and non-self-similar Sierpiński carpets.** Let

$$\mathbf{a} = (a_1^{-1}, a_2^{-1}, \dots) \in \left\{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots\right\}^{\mathbb{N}}.$$

Divide $R_0 := Q$ into a_1^2 equal squares of side length a_1^{-1} and remove the central one. We obtain a set R_1 which is the union of $a_1^2 - 1$ squares $Q_{1,j}$ of side length a_1^{-1}. Consider the remaining $a_1^2 - 1$ squares, divide each into a_2^2 squares of side length $a_1^{-1} \cdot a_2^{-1}$ and again remove each open central square. Iterating this procedure yields a sequence of sets R_k, where R_k consists of

$$(a_1^2 - 1) \cdot (a_2^2 - 1) \dots (a_j^2 - 1)$$

squares $Q_{k,j}$ of side length $a_1^{-1} \cdot a_2^{-1} \dots a_k^{-1}$. We define the *generalized Sierpiński carpet* to be

$$S_{\mathbf{a}} = \bigcap_{k \geq 1} R_k.$$

For any sequence $\mathbf{a}$, the carpet $S_{\mathbf{a}}$ is a compact set without interior which is rectifiably connected. Furthermore, $S_{\mathbf{a}}$ has positive area (Lebesgue 2-measure) if and only if $\mathbf{a} \in \ell^{\mathbf{2}}$, i.e., $\sum_j a_j^{-2} < \infty$. The metric measure space $(S_{\mathbf{a}}, d, \mathscr{L}^2)$ (where d denotes the Euclidean metric and $\mathscr{L}^2$ denotes the Lebesgue measure in $\mathbb{R}^2$) admits a $(1,p)$-Poincaré inequality for each $1 < p < \infty$ if $\mathbf{a} \in \ell^{\mathbf{2}}$. We will not need these facts in this paper. See [11] for these and other results.

We will consider the special case when $\mathbf{a} = (\frac{1}{a}, \frac{1}{a}, \frac{1}{a}, \dots)$ is a constant sequence. Note that if $\mathbf{3} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots\right)$, we obtain the standard Sierpiński carpet $S_{\mathbf{3}}$. Similarly, we write $S_{\mathbf{5}}$, $S_{\mathbf{7}}$, and so on, for the self-similar Sierpiński carpets defined via the constant sequences $\mathbf{5} = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \dots\right)$, $\mathbf{7} = \left(\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \dots\right)$, and so on.

See Figure 3 for a picture of the carpet $S_{\mathbf{a}}$ when $\mathbf{a} = (\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots)$.

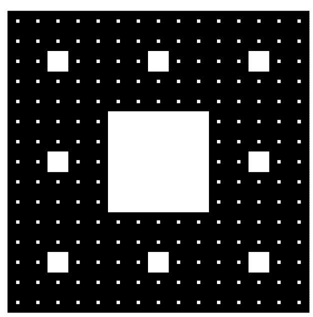

FIGURE 3. Sierpiński carpet $S_{(1/3,1/5,1/7,\dots)}$

3. **Coordinates in the carpet.** The easiest way to characterize points in the usual Cantor set C is via 3-adic expansions. In fact, a point x lies in C if and only if x admits a 3-adic expansion which uses no 1's.

We use the same idea to represent points in the self-similar carpet $S_{\mathbf{a}}$. Let us consider the following a-adic expansion for points $x \in \mathbb{R}$:

$$x = x_0 + \sum_{k=1}^{\infty} \frac{x_k}{a^k} \qquad x_0 \in \mathbb{Z}, x_k \in \{0, 1, \dots a-1\}. \tag{3.1}$$

In the remainder of this paper, we will use the notation

$$x = (x_0.x_1|x_2|x_3|\cdots)_a \tag{3.2}$$

to denote such an expansion. In several places, we will abuse notation and express points x in the form (3.2) for positive integers x_k, $k \geq 1$, which are not necessarily in the set $\{0, 1, \ldots, a-1\}$. This has the obvious interpretation as in (3.1).

We now state the desired characterization of the carpet $S_{\mathbf{a}}$.

Proposition 3.1. *Let $\mathbf{a} = (a^{-1}, a^{-1}, \ldots)$ for some $a \in \{3, 5, 7, \ldots\}$ and let (x, y) be a point in Q. Then $(x, y) \in S_{\mathbf{a}}$ if and only if $x = (0.x_1|x_2|x_3|\cdots)_a$ and $y = (0.y_1|y_2|y_3|\cdots)_a$ where, for each $k \in \mathbb{N}$, either $x_k \neq (a-1)/2$ or $y_k \neq (a-1)/2$.*

The proof is elementary.

Proposition 3.1 extends to cover the general (not necessarily self-similar) carpets. Let $\mathbf{a} = (a_1^{-1}, a_2^{-1}, \ldots)$ and consider the following $\mathbf{a}$-adic expansion for points $x \in \mathbb{R}$:

$$x = (x_0.x_1|x_2|x_3|\cdots)_{\mathbf{a}} = x_0 + \sum_{k=1}^{\infty} \frac{x_k}{a_1 \cdot a_2 \cdots a_k} \qquad x_0 \in \mathbb{Z}, x_k \in \{0, 1, \ldots a_k - 1\}. \tag{3.3}$$

Proposition 3.2. *Let (x, y) be a point in Q. Then $(x, y) \in S_{\mathbf{a}}$ if and only if $x = (0.x_1|x_2|x_3|\cdots)_{\mathbf{a}}$ and $y = (0.y_1|y_2|y_3|\cdots)_{\mathbf{a}}$ where, for each $k \in \mathbb{N}$, either $x_k \neq (a_k-1)/2$ or $y_k \neq (a_k-1)/2$.*

4. **Slopes of nontrivial line segments in Sierpiński carpets.** Since the carpet $S_{\mathbf{a}}$ admits all of the symmetries of the unit square $\{(x, y) : 0 \leq x, y \leq 1\}$ (i.e., the dihedral group D_4), we observe that a value α occurs as a slope if and only if each of the quantities $-\alpha$, $\frac{1}{\alpha}$, and $-\frac{1}{\alpha}$ occurs as a slope (with the usual interpretation regarding 0 and ∞). Thus it suffices to characterize the slopes which lie between 0 and 1. We denote by

$$\mathrm{Slopes}(S_{\mathbf{a}})$$

the set of slopes, in the interval $[0, 1]$, of nontrivial line segments contained in the carpet $S_{\mathbf{a}}$.

The following theorem is the main result of this paper. It characterizes self-similar carpets in terms of their slope sets, in the sense that it gives a one-to-one correspondence between self-similar carpets and the set of slopes of nontrivial line segments contained in such carpets.

Theorem 4.1. *Let $\mathbf{a} = (\frac{1}{a}, \frac{1}{a}, \frac{1}{a}, \ldots)$ be a constant sequence. Then the set of slopes* $\mathrm{Slopes}(S_{\mathbf{a}})$ *is the union of the following two sets:*

$$A = \left\{ \frac{p}{q} \,:\, p+q \leq a, \quad 0 \leq p < q \leq a-1, \quad p, q \in \mathbb{N} \cup \{0\}, \quad p+q \text{ odd} \right\}$$

and

$$B = \left\{ \frac{p}{q} \,:\, p+q \leq a-1, \quad 1 \leq p \leq q \leq a-2, \quad p, q \in \mathbb{N}, \quad p, q \text{ odd} \right\}.$$

Moreover, if $\alpha \in A$, then each nontrivial line segment in $S_{\mathbf{a}}$ with slope α touches vertices of peripheral squares, while if $\alpha \in B$, then each nontrivial line segment in $S_{\mathbf{a}}$ with slope α is disjoint from all peripheral squares. For each $\alpha \in A \cup B$, there exist maximal line segments in $S_{\mathbf{a}}$ with slope α. Finally, if $b < a$, then any maximal nontrivial line segment in $S_{\mathbf{b}}$ is also contained in $S_{\mathbf{a}}$. In particular, $\mathrm{Slopes}(S_{\mathbf{b}}) \subset \mathrm{Slopes}(S_{\mathbf{a}})$.

We say that a line segment in $S_{\mathbf{a}}$ is *maximal* if it connects two points on the boundary of the initial square $Q = \{(x, y) : 0 \leq x, y \leq 1\}$.

We list the set of slopes of the first few carpets $S_{\mathbf{a}}$. Observe that the slopes appear in strictly increasing order:

$$\mathrm{Slopes}(S_{\mathbf{3}}) = \{0, \frac{1}{2}, 1\},$$

$$\mathrm{Slopes}(S_{\mathbf{5}}) = \{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\},$$

$$\mathrm{Slopes}(S_{\mathbf{7}}) = \{0, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1\},$$

and

$$\text{Slopes}(S_{\mathbf{9}}) = \{0, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, 1\}.$$

Remark 4.2. If $S_{\mathbf{a}}$ contains a nontrivial line segment of some slope α, then $S_{\mathbf{a}}$ contains a nontrivial line segment of slope α which intersects the x-axis. Indeed, any line segment contained in $S_{\mathbf{a}}$ must intersect the boundary of one of the defining squares $Q_{k,j}$. Since for fixed k, all of the sets $Q_{k,j} \cap S_{\mathbf{a}}$ are isometric, there is a corresponding line segment of the same slope which intersects the boundary of the original square Q. Applying an isometry of Q if necessary, and using the invariance of the set of slopes under the operations $\alpha \mapsto -\alpha$, $\alpha \mapsto \frac{1}{\alpha}$ and $\alpha \mapsto -\frac{1}{\alpha}$, we conclude the desired fact. Figure 4 shows nontrivial line segments of each allowed slope in the Sierpiński carpets $S_{\mathbf{3}}$ and $S_{\mathbf{5}}$.

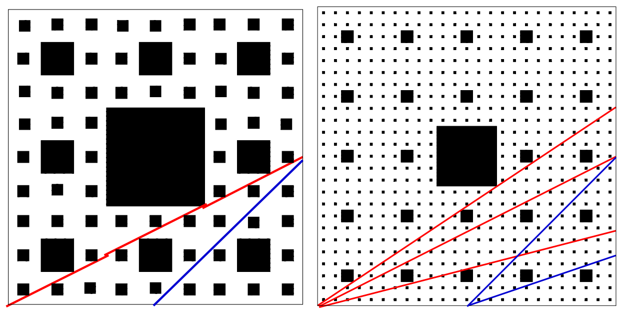

FIGURE 4. Nontrivial line segments of various slopes in the carpets $S_{\mathbf{3}}$ and $S_{\mathbf{5}}$

Figure 4 suggests the following refinement of Remark 4.2, which is in fact correct and will be confirmed in the proof of Theorem 4.1.

Remark 4.3. Fix $\mathbf{a}$, write $\text{Slopes}(S_{\mathbf{a}}) = A \cup B$ as in the statement of Theorem 4.1, and fix $\alpha \in A \cup B$. If $\alpha \in A$, then there exists a line segment of slope α passing through the origin $(0,0)$. On the other hand, if $\alpha \in B$, then there exists a line segment of slope α passing through the midpoint $(\frac{1}{2}, 0)$. Other line segments of this slope are obtained by applying Euclidean translations.

Using Remark 4.2 we can give a quick proof that no irrational slopes can occur in any of the carpets $S_{\mathbf{a}}$.

Lemma 4.4. *Let $S_{\mathbf{a}}$ be a carpet (possibly non-self-similar) of the type defined in section 2. Let $\alpha \in [0,1]$ and for each point $x \in [0,1]$ consider the set*

$$A_x^\alpha = \{x + \alpha n \pmod 1 : n \in \mathbb{N}\}.$$

If each of the sets A_x^α, $x \in [0,1]$, is dense in $[0,1]$, then there is no nontrivial line segment in $S_{\mathbf{a}}$ with slope α.

Proof. By Remark 4.2, it suffices to consider line segments meeting the x-axis.

If each of the sets A_x^α is dense in $[0,1]$, then the union of the lines with slope α through the points of A_x^α meets $[0,1]^2$ in a dense set. It follows that every nontrivial line segment through any point of the x-axis must meet complementary squares arbitrarily close to the x-axis. □

Corollary 4.5. *There are no nontrivial line segments of irrational slope in any of the carpets $S_{\mathbf{a}}$.*

Proof. If α is irrational, then A_x^α is dense in $[0,1]$. □

Remark 4.6. A similar argument can be used to prove that if $\mathbf{a} = (\frac{1}{a}, \frac{1}{a}, \ldots)$ for some odd integer $a \geq 3$, and if each of the sets A_x^α, $x \in [0,1]$, has no gaps of length greater than or equal than $1/a$, then there is no nontrivial line segment in $S_{\mathbf{a}}$ with slope α. However, our proof of Theorem 4.1 will proceed along different lines.

A full proof of Theorem 4.1 will be given in section 7. In particular, we will reprove the nonexistence of line segments with irrational slope in the carpets.

5. **The set of slopes and Farey sequences.** In this section, we discuss the connection between the set of slopes for a self-similar carpet $S_{\mathbf{a}}$ and Farey sequences. Our starting point is the following corollary of Theorem 4.1.

Corollary 5.1. *The set* $\mathrm{Slopes}(S_{\mathbf{a}})$ *contains all Farey fractions of order* $(a+1)/2$*, and is contained in the set of all Farey fractions of order* $a-1$*.*

We recall that the *Farey fractions (or Farey sequence) of order* n consist of those rational numbers in $[0,1]$ which, in lowest terms, have denominator no more than n. Farey fractions arise ubiquitously in problems at the intersection of number theory, combinatorics and geometry. Their appearance here stems from one of their well known geometric properties [15, p. 87]: *the nth Farey sequence corresponds to the integer lattice points in the triangle* $\{(x,y) : 0 \le y \le x \le n\}$ *which are directly visible from the origin.* See Remark 5.6. For a previous use of Farey sequences in fractal geometry (enumeration of the components of the Mandelbrot set), see Devaney [9].

Proof of Corollary 5.1. The inclusion of $\mathrm{Slopes}(S_{\mathbf{a}})$ in F_{a-1} is clear from Theorem 4.1.

We prove the inclusion $F_{(a+1)/2} \subset \mathrm{Slopes}(S_{\mathbf{a}})$. Suppose that $\frac{p}{q}$, in lowest terms, is in $F_{(a+1)/2}$. Then $0 \le p \le q \le \frac{a+1}{2}$.

If both p and q are odd, then either $p = q = 1$ or $p < q$. In the latter case, $p+q \le \frac{a+1}{2} + \frac{a-3}{2} = a-1$. Hence $\frac{p}{q} \in B$.

Suppose instead that either p or q is even. Then $0 \le p < q \le \frac{a+1}{2} \le a-1$ (since $a \ge 3$). Furthermore, $p+q \le \frac{a+1}{2} + \frac{a-1}{2} = a$. Hence $\frac{p}{q} \in A$. □

Corollary 5.2. $\mathrm{Slopes}(S_{\mathbf{3}}) \subsetneq \mathrm{Slopes}(S_{\mathbf{5}}) \subsetneq \mathrm{Slopes}(S_{\mathbf{7}}) \subsetneq \dots$ *and*

$$\bigcup \mathrm{Slopes}(S_{\mathbf{a}}) = [0,1] \cap \mathbb{Q}. \tag{5.1}$$

The identity in (5.1) follows from the inclusion of $F_{(a+1)/2}$ in $\mathrm{Slopes}(S_{\mathbf{a}})$. The monotonicity of the sets $\mathrm{Slopes}(S_{\mathbf{a}})$ with respect to a follows from the characterization in Theorem 4.1.

As a consequence of Lemma 4.4 and Corollary 5.1 we draw the following interesting conclusion for Sierpiński carpets $S_{\mathbf{a}}$, when $\mathbf{a}$ is not necessarily a constant sequence.

Theorem 5.3. *Let* $\mathbf{a} = (a_1^{-1}, a_2^{-1}, \dots) \in \left\{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots\right\}^{\mathbb{N}}$.

(a) *If* $\limsup \mathbf{a} = 0$ *(i.e., if* $\mathbf{a} \in c_0$*), then* $S_{\mathbf{a}}$ *contains nontrivial line segments of every rational slope, and contains no nontrivial line segments of any irrational slope.*

(b) *If* $\limsup \mathbf{a} > 0$*, then* $\limsup \mathbf{a} = \frac{1}{a_0}$ *for some* $a_0 \in \{3,5,7,\cdots\}$*. In this case,* $\mathrm{Slopes}(S_{\mathbf{a}})$ *coincides with* $\mathrm{Slopes}(S_{\mathbf{a_0}})$*.*

Proof. If $\limsup \mathbf{a} = 0$, then $\lim \mathbf{a} = 0$ and $\lim_{k\to\infty} a_k = \infty$. By Corollary 5.2 and Theorem 4.1, if $a_k \ge b$ for all sufficiently large k, then all corresponding subsquares $Q_{k,j} \cap S_{\mathbf{a}}$ contain nontrivial line segments of all slopes α in $\mathrm{Slopes}(S_{\mathbf{b}})$. Since b may be chosen arbitrarily large and every positive rational is a Farey fraction of some order, the statement in part (a) follows. The second statement follows from Lemma 4.4.

For the proof of part (b), we note that if $\limsup \mathbf{a} > 0$ and $a_0 = \min\{\frac{1}{b} : b \in \mathbf{a}\}$, then $a_k = a_0$ for infinitely many values of k. From the fact that $a_k = a_0$, we easily deduce that there are no line segments with slope not in $\mathrm{Slopes}(S_{a_0})$ whose length exceeds some quantity ϵ_k, where $\epsilon_k \to 0$ as $k \to \infty$. Hence there are no nontrivial line segments in $S_{\mathbf{a}}$ with slopes which are not in $\mathrm{Slopes}(S_{a_0})$. We postpone discussion of the remaining claim (there exist nontrivial line segments in $S_{\mathbf{a}}$ with each slope in $\mathrm{Slopes}(S_{a_0})$) to Remark 7.2. □

Remark 5.4. Lemma 4.4 can also be explained by the aid of the theory of square billiards [8]. Consider a square billiard table Q and a particle moving inside Q. When the moving particle reaches the boundary ∂Q, the angle of incidence is equal to the angle of reflection. However, instead of reflecting the trajectory of the particle in a side of ∂Q, let us reflect the square Q across that side and allow the particle to move straight into the mirror image of Q. If we repeat this procedure at every collision, the particle will move along a straight line through multiple copies of Q obtained by successive reflections. This construction is called *unfolding* the billiard trajectory. To recover the original trajectory in Q, one *folds* the resulting string of adjacent copies of Q back onto Q. If we consider the 2×2 square

$$Q_2 = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 2\},$$

the standard projection of $\mathbb{R}^2$ onto Q_2 transforms unfolded trajectories into directed straight lines on the 2×2 torus (the latter is obtained by identifying opposite sides of the square Q_2). Billiards in the square thus reduces to simple linear flow on a torus. The linear flow on a flat torus is one of the standard examples in ergodic theory. Its main properties are:

- if a trajectory has rational slope, then it is periodic (it runs along a closed geodesic),
- if a trajectory has irrational slope, then it is dense (its closure is the whole torus).

The theory of square billiards can be applied to study line segments contained in the carpets $S_{\mathbf{a}}$. Instead of considering a square, we consider a "punctured" square, and so a punctured torus. Here by "punctured" we mean a closed square with a square hole in the center of the corresponding size $\frac{1}{a}$. According to the above results, trajectories with irrational slope can not occur in the punctured torus either. However, since we now have a hole, not all rational slopes will occur, since eventually the trajectory will hit the hole. In this way, Theorem 4.1 can be interpreted as a game of "punctured" squared billiards.

The relationship between line segments in the carpet and the dynamics of the corresponding square billiards is made somewhat more precise in Proposition 6.1, which gives a criterion for membership in $\mathrm{Slopes}(S_{\mathbf{a}})$.

Remark 5.5. Boca, Gologan and Zaharescu [2], [3] already used the Farey sequences to study the statistics of the first exit time and collision number for punctured toral billiards with circular punctures (which in turn can be used to model the periodic 2D Lorentz gas).

Remark 5.6. We indicate a more geometric way to look at the set of slopes which illuminates the connection to Farey sequences. First, let us introduce a bijection between

$$Z = \{(q, p) \in \mathbb{N}^2 : p \text{ and } q \text{ are coprime}\}$$

and the positive rationals by the rule $\varphi : (q, p) \mapsto \frac{p}{q}$. Consider the set Z' consisting of all elements (q, p) of Z satisfying $p + q \leq a$ and $p \leq q$. Then $\mathrm{Slopes}(S_{\mathbf{a}}) = \{0\} \cup \varphi(Z')$. This follows directly from Theorem 4.1. See Figure 5.

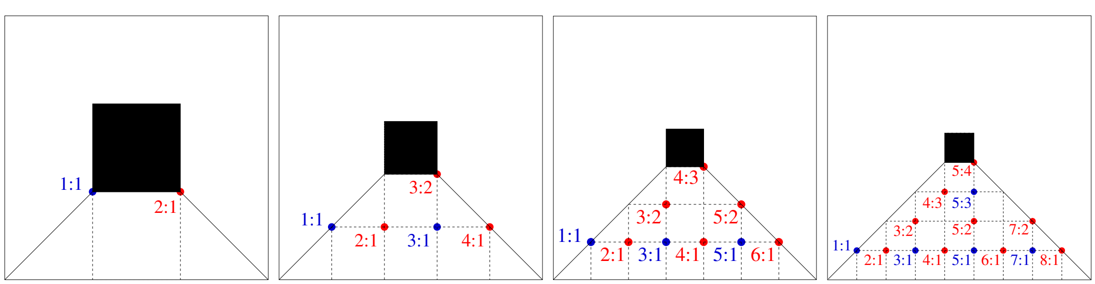

FIGURE 5. Pictorial representation of the slope set for the carpets $S_{\mathbf{a}}$, $\mathbf{a} \in \{\mathbf{3}, \mathbf{5}, \mathbf{7}, \mathbf{9}\}$

Remark 5.7. The inclusion of Slopes($S_{\mathbf{a}}$) in F_{a-1} will not be directly useful for us since Slopes($S_{\mathbf{a}}$) does not appear in F_{a-1} as a consecutive block of elements, that is, there exist elements in $F_{a-1} \setminus$ Slopes($S_{\mathbf{a}}$) which lie between two elements of Slopes($S_{\mathbf{a}}$). In order to take advantage of properties of Farey sequences we will give another description of Slopes($S_{\mathbf{a}}$).

For each odd a, consider the finite set of fractions

$$f_n = \frac{n}{a-1-n}, \qquad n = 0, \dots, \frac{a-1}{2}.$$

Observe that $\{f_n\}_n \in \mathrm{Slopes}(S_{\mathbf{a}})$. Under the bijection φ from Remark 5.6, this set corresponds to lattice points which appear just below the "main diagonal", that is, points which lie on the segment which connects $(a-1,0)$ to $(\frac{a-1}{2}, \frac{a-1}{2})$.

Next, consider the following inclusion between ordered sets:

$$\psi : (\{f_n\}_n, \leq) \longrightarrow (\mathrm{Slopes}(S_{\mathbf{a}}), \leq).$$

Let $\mathrm{Slopes}(S_{\mathbf{a}}) = \{0 = s_0, \cdots, s_r = 1\}$ (in increasing order) and define

$$\phi : \{0, \dots, \frac{a-1}{2}\} \longrightarrow \{0, \cdots, r\}$$

by setting $\phi(n) = j$ if and only if $\psi(f_n) = s_j$. The set of slopes Slopes($S_{\mathbf{a}}$) can be written as the union of

$$[s_0 = s_{\phi(0)}, s_{\phi(1)}] \cap \mathrm{Slopes}(S_{\mathbf{a}}),$$
$$[s_{\phi(1)}, s_{\phi(2)}] \cap \mathrm{Slopes}(S_{\mathbf{a}}),$$

and so on, through

$$[s_{\phi(\frac{a-3}{2})}, s_{\phi(\frac{a-1}{2})} = s_r] \cap \mathrm{Slopes}(S_{\mathbf{a}}).$$

Proposition 5.8. *For each $n = 1, \dots, \frac{a-1}{2}$, the set*

$$[s_{\phi(n-1)}, s_{\phi(n)}] \cap \mathrm{Slopes}(S_{\mathbf{a}})$$

is a sequence of consecutive elements in F_{a-n}.

Proof. Let $n = 1, \dots, \frac{a-1}{2}$. It suffices to prove that

$$[s_{\phi(n-1)}, s_{\phi(n)}] \cap \mathrm{Slopes}(S_{\mathbf{a}}) = [s_{\phi(n-1)}, s_{\phi(n)}] \cap F_{a-n}. \tag{5.2}$$

Let $\frac{p}{q}$ be a rational number expressed in lowest terms and satisfying

$$\frac{n-1}{a-n} \leq \frac{p}{q} \leq \frac{n}{a-1-n}. \tag{5.3}$$

The identity in (5.2) asserts that under these hypotheses,

$$0 \leq p \leq q \leq a-1 \quad \text{and} \quad p+q \leq a \qquad \text{if and only if} \qquad 0 \leq p \leq q \leq a-n.$$

First, we prove the "only if" statement. Assume that $0 \leq p \leq q \leq a-1$ and $p+q \leq a$. The conclusion being obvious if $n=1$, we also assume that $n \geq 2$. Then from (5.3) we obtain

$$(a-1)q = (a-n+n-1)q \leq (a-n)(p+q) \leq a(a-n)$$

so

$$q \leq \frac{a(a-n)}{a-1} < a-n+1$$

(since $n \geq 2$). Since q is an integer, we must have $q \leq a-n$.

Next, we prove the "if" statement. Assume that $0 \leq p \leq q \leq a-n$. Then from (5.3) we obtain

$$(a-1-n)(p+q) \leq (a-1)q \leq (a-n)(a-1)$$

so

$$p+q \leq \frac{(a-n)(a-1)}{a-1-n} \leq a+1. \tag{5.4}$$

If strict inequality holds in either place in (5.4), then $p+q \leq a$, since $p+q$ is an integer. Otherwise, $n = \frac{a-1}{2}$ and $a+1 = p+q \leq 2q \leq 2(a-n) = a+1$ which yields $p = q = \frac{a+1}{2}$. This contradicts the initial assumption that $\frac{p}{q}$ is in lowest terms (recall that $a \geq 3$). □

Proposition 5.8 asserts that $\mathrm{Slopes}(S_{\mathbf{a}})$ can be written as a union of "intervals", each of which consists of a consecutive block within a particular Farey sequence. This observation allows us to use many of the properties of the Farey sequences in the proof of Theorem 4.1. We enumerate here some of the more remarkable properties of Farey sequences. See [8, Ch. 3].

Proposition 5.9. *Farey sequences enjoy the following properties:*

(1) $F_n \subset F_{n+1}$. *If $p_1/q_1 < p_2/q_2$ are consecutive in F_n and separated in F_{n+1}, then the fraction $\frac{p_1+p_2}{q_1+q_2}$ lies in between p_1/q_1 and p_2/q_2 and no other elements of F_{n+1} lies between p_1/q_1 and p_2/q_2. The fraction $\frac{p_1+p_2}{q_1+q_2}$ is called the* mediant *of p_1/q_1 and p_2/q_2.*

(2) *If p_1/q_1 and p_2/q_2 are consecutive in any F_n, then they satisfy the unimodular relation $p_1 \cdot q_2 = p_2 \cdot q_1 - 1$.*

Observe that the mediant of consecutive Farey fractions is already in reduced form. Indeed, suppose that p/q is the mediant of p_1/q_1 and p_2/q_2, and that p_1/q_1 and p_2/q_2 are consecutive Farey fractions of some order. Then $p_2q - q_2p \geq 1$ and $pq_1 - qp_1 \geq 1$. Furthermore, by Proposition 5.9(2), we have

$$q_1 + q_2 = q = q_1(p_2q - q_2p) + q_2(pq_1 - qp_1) \geq q_1 + q_2$$

which shows that $p_2q - q_2p = pq_1 - qp_1 = 1$. By Euclid's algorithm, p and q are coprime.

The following lemma provides us with a recursive way to construct the set of slopes.[2]

Lemma 5.10. *Suppose p_1/q_1 and p_2/q_2 are consecutive fractions in* $\mathrm{Slopes}(S_{\mathbf{a}})$, *both in reduced form. Then p_1/q_1 and p_2/q_2 are separated in* $\mathrm{Slopes}(S_{\mathbf{a+2}})$ *if and only if $p_1 + q_1 + p_2 + q_2 \leq a + 2$.*

Proof. Recall (see Remark 5.6) that

$$\mathrm{Slopes}(S_{\mathbf{a}}) = \{(q,p) \in \mathbb{Z}^2 : p+q \leq a,\ 0 \leq p \leq q\}. \tag{5.5}$$

Since p_1/q_1 and p_2/q_2 are consecutive in $\mathrm{Slopes}(S_{\mathbf{a}})$, they are consecutive in some F_n and hence their mediant is in reduced form.

First let us assume that p_1/q_1 and p_2/q_2 are separated in $\mathrm{Slopes}(S_{\mathbf{a+2}})$. Then the mediant

$$\frac{p}{q} = \frac{p_1+p_2}{q_1+q_2}$$

appears in $\mathrm{Slopes}(S_{\mathbf{a+2}})$, and so $p+q = p_1+q_1+p_2+q_2 \leq a+2$. On the other hand, if p_1/q_1 and p_2/q_2 are consecutive fractions in $\mathrm{Slopes}(S_{\mathbf{a+2}})$, then the fraction $\frac{p}{q} = \frac{p_1+p_2}{q_1+q_2}$ is not in $\mathrm{Slopes}(S_{\mathbf{a+2}})$ and so, by (5.5), we conclude that $p_1+q_1+p_2+q_2 = p+q > a+2$. □

Observe that one can generate $\mathrm{Slopes}(S_{\mathbf{a+2}})$ from $\mathrm{Slopes}(S_{\mathbf{a}})$ just by adding the mediants of those consecutive fractions p_1/q_1 and p_2/q_2 in $\mathrm{Slopes}(S_{\mathbf{a}})$ for which $p_1+q_1+p_2+q_2 \leq a+2$. Notice also that between $0/1$ and $1/(a-1)$ there always appear two fractions in $\mathrm{Slopes}(S_{\mathbf{a+2}})$. The reason is simple. In this case, the mediant of $0/1$ and $1/(a-1)$ is $1/a$. However, there is still space between $p_1/q_1 = 0/1$ and $p_2/q_2 = 1/a$, since $p_1+q_1+p_2+q_2 = a+1 < a+2$. Thus, the fractions $1/a$ and $1/(a+1)$ appear between $0/1$ and $1/(a-1)$.

[2] We would like to thank Javier Gómez Gil for some useful comments regarding this lemma.

6. **A necessary condition for a line segment to lie in the carpet $S_{\mathbf{a}}$.** To show that the values in Slopes($S_{\mathbf{a}}$) are the only slopes which occur, we will need the following useful criterion. The idea of this criterion is that *going deeper into the carpet* corresponds to *tiling the plane with squares.* This is closely related to the interpretation of line segments in the carpet in terms of square billiards, as in Remark 5.4.

Proposition 6.1. *If there exists a nontrivial line segment of a certain slope α emanating from a point $(c,0)$, $c \in [0,1]$, and contained in the carpet $S_{\mathbf{a}}$, that is, if the set*

$$L_{c,\alpha}^{S_{\mathbf{a}}} = \{(x,y) \in S_{\mathbf{a}} : y = \alpha(x-c)\}$$

contains a line segment containing $(c,0)$, then the line

$$L_{c,\alpha} = \{(x,y) \in \mathbb{R}^2 : y = \alpha(x-c)\}$$

does not intersect any member of the collection

$$\mathbb{Z}^2 + Q_{\mathbf{a}} := \left\{(k,\ell) + Q_{\mathbf{a}} : (k,\ell) \in \mathbb{Z}^2\right\},$$

where $Q_{\mathbf{a}} = \{(x,y) \in \mathbb{R}^2 : \frac{a-1}{2a} < x < \frac{a+1}{2a},\ \frac{a-1}{2a} < y < \frac{a+1}{2a}\}$.

We sketch the proof of Proposition 6.1. Suppose that we are at the mth level of the construction of $S_{\mathbf{a}}$. Replace each square that we have removed in all the previous steps by a concentric square of side length a^{-m} and call the resulting set A_m. Observe that $A_m \supset S_{\mathbf{a}}$. If $L_{c,\alpha}^{S_{\mathbf{a}}}$ contains a line segment containing $(c,0)$, then that line segment also lies in the sets A_m for each $m \in \mathbb{N}$. The conclusion now follows by rescaling and passing to the limit as m tends to infinity.

Corollary 6.2. *If there exists a nontrivial line segment of a certain slope α emanating from a point $(c,0)$, $c \in [0,1]$, and contained in the carpet $S_{\mathbf{a}}$, then the line $L_{0,\alpha}$ emanating from the origin of slope α does not intersect any member of the collection*

$$a\mathbb{Z}^2 + Q' := \left\{(ak,a\ell) + Q' : (k,\ell) \in \mathbb{Z}^2\right\}, \tag{6.1}$$

where $Q' = \{(x,y) : -1 < x < 0,\ 0 < y < 1\}$.

We indicate how Corollary 6.2 follows from Proposition 6.1. First, apply the homothety $(x,y) \mapsto (ax - \frac{a+1}{2}, ay - \frac{a-1}{2})$. Then some line $L_{c',\alpha}$ of slope α does not intersect the collection $a\mathbb{Z}^2 + Q'$. If $L_{c',\alpha}$ passes through the inferior right vertex of any of the squares in $a\mathbb{Z}^2 + Q'$, then applying another translation shows that the line through the origin of slope α also does not intersect the collection $a\mathbb{Z}^2 + Q'$. If $L_{c',\alpha}$ does not pass through the inferior right vertex of any square in $a\mathbb{Z}^2 + Q'$, we distinguish two cases:

(i) The distance from $L_{c',\alpha}$ to the set S of all inferior right vertices of squares in $a\mathbb{Z}^2 + Q'$ is positive. In this case, identify a vertex v in S whose distance to $L_{c',\alpha}$ is minimal. Translate $L_{c',\alpha}$ to pass through v; such translation does not affect the fact that this line does not intersect $a\mathbb{Z}^2 + Q'$. Finally, translating v to the origin completes the proof.

(ii) The distance from $L_{c',\alpha}$ to S is equal to zero, but is not achieved. Choose a sequence of vertices (v_n) in S such that $\mathrm{dist}(L_{c',\alpha}, v_n) \to 0$. Applying the corresponding sequence of translations (which take these points successively to the origin) yields a sequence of lines, all of slope α, which do not intersect the collection $a\mathbb{Z}^2 + Q'$ and whose distance to the origin tends to zero. The limiting line also has slope α, passes through the origin, and does not intersect the collection $a\mathbb{Z}^2 + Q'$.

Remark 6.3. We emphasize a subtle point in the preceding argument. Consider the decomposition Slopes($S_{\mathbf{a}}$) $= A \cup B$ associated to a specific self-similar carpet $S_{\mathbf{a}}$. For α in B, as already mentioned, there are no lines of slope α which meet any of the vertices of the peripheral squares associated to $S_{\mathbf{a}}$. However, there **do** exist such lines passing through vertices of squares associated to the corresponding collection given in Corollary 6.2. The reason is that this collection is not a self-similar fractal construction but rather has a definite lower scale; all of the constituent squares in the collection have mutual distance at least one.

7. **Proof of the main theorem.** We are now in a position to prove Theorem 4.1. We divide the proof into two parts. In the first part, we show that nontrivial line segments exist whenever the slope α is chosen from the set $A \cup B$. In the second part, we show that no other slopes occur.

Part 1. Let $\alpha \in A \cup B$. The strategy of this part of the proof is to use the carpet coordinates introduced in section 3 to see that the lines $y = \alpha(x-c)$ do not intersect the omitted open squares. It is important to note here that if the line segment L has slope $\alpha \in A$, then we can assume that $c = 0$, that is, L emanates from a vertex of the unit square. On the other hand, if L has slope $\alpha \in B$, then we can assume $c = \frac{1}{2}$, that is, L emanates from a midpoint of an edge of the unit square.

Observe that if $(x, y) \notin R_n$ for some $n \in \mathbb{N}$, i.e., if (x, y) is contained in some omitted square, then

$$\left(0.x_1\middle|\cdots\middle|x_{n-1}\middle|\frac{a-1}{2}\middle|0\middle|0\middle|\cdots\right)_a < x < \left(0.x_1\middle|\cdots\middle|x_{n-1}\middle|\frac{a+1}{2}\middle|0\middle|0\middle|\cdots\right)_a \tag{7.1}$$

and

$$\left(0.y_1|\cdots\middle|y_{n-1}\middle|\frac{a-1}{2}\middle|0\middle|0\middle|\cdots\right)_a < y < \left(0.y_1|\cdots\middle|y_{n-1}\middle|\frac{a+1}{2}\middle|0\middle|0\middle|\cdots\right)_a. \tag{7.2}$$

The proof will involve detailed computations and estimates of the coordinates of points in base a, comparing the condition for membership in one of the omitted squares with membership in the line L.

Case 1a: $\alpha \in A$. We claim that the line L given by the equation

$$y = \alpha x$$

does not meet any of the omitted squares from the construction of $S_{\mathbf{a}}$.

Suppose that (x, y) is a point contained in some omitted square and also contained in L. Since $\alpha \in A$, there exist $p, q \in \mathbb{N} \cup \{0\}$ with $p+q$ odd, $p+q \le a$, $0 \le p < q \le a-1$, and

$$qy = px.$$

If we multiply by p in (7.1) and by q in (7.2), we obtain

$$\left(\widetilde{x_0}.\widetilde{x_1}\middle|\cdots\middle|\widetilde{x_{n-1}}\middle|\frac{(a-1)p}{2}\middle|0\middle|\cdots\right)_a < px < \left(\widetilde{x_0}.\widetilde{x_1}\middle|\cdots\middle|\widetilde{x_{n-1}}\middle|\frac{(a+1)p}{2}\middle|0\middle|\cdots\right)_a \tag{7.3}$$

and

$$\left(\widetilde{y_0}.\widetilde{y_1}\middle|\cdots\middle|\widetilde{y_{n-1}}\middle|\frac{(a-1)q}{2}\middle|0\middle|\cdots\right)_a < qy < \left(\widetilde{y_0}.\widetilde{y_1}\middle|\cdots\middle|\widetilde{y_{n-1}}\middle|\frac{(a+1)q}{2}\middle|0\middle|\cdots\right)_a \tag{7.4}$$

respectively. Observe that coordinates are written modulo a, and we employ the previously mentioned abuse of notation (the coefficients need not be integers in the range $\{0, 1, \ldots, a-1\}$). Moreover, it follows from (7.3) that $p \neq 0$.

If p is even we make a simple arithmetic calculation to recast (7.3) and (7.4) as follows:

$$\left(\widetilde{x_0}.\widetilde{x_1}\middle|\cdots\middle|\frac{p}{2}-1+\widetilde{x_{n-1}}\middle|a-\frac{p}{2}\middle|0\middle|\cdots\right)_a < px < \left(\widetilde{x_0}.\widetilde{x_1}\middle|\cdots\middle|\frac{p}{2}+\widetilde{x_{n-1}}\middle|\frac{p}{2}\middle|0\middle|\cdots\right)_a$$

and

$$\left(\widetilde{y_0}.\widetilde{y_1}\middle|\cdots\middle|\frac{q-1}{2}+\widetilde{y_{n-1}}\middle|\frac{a-q}{2}\middle|0\middle|\cdots\right)_a < qy < \left(\widetilde{y_0}.\widetilde{y_1}\middle|\cdots\middle|\frac{q-1}{2}+\widetilde{y_{n-1}}\middle|\frac{a+q}{2}\middle|0\middle|\cdots\right)_a.$$

Since $p \geq 2$ and $q \leq a-2$ (note that q is odd), we observe that we have reduced the nth coefficients to the range $\{0, 1, \ldots, a-1\}$. We next observe that $\frac{p}{2} \leq \frac{a-q}{2}$ and $\frac{a+q}{2} \leq a - \frac{p}{2}$ by the conditions on p and q. Since the $(n-1)$st coefficients in the bounds for qy are equal, while the $(n-1)$st coefficients in the bounds for px disagree by one, we conclude that no such point (x, y) can exist.

Similarly, if p is odd, we recast (7.3) and (7.4) as follows:

$$\left(\widetilde{x_0}.\widetilde{x_1}\middle|\cdots\middle|\frac{p-1}{2}+\widetilde{x_{n-1}}\middle|\frac{a-p}{2}\middle|0\middle|\cdots\right)_a < px < \left(\widetilde{x_0}.\widetilde{x_1}\middle|\cdots\middle|\frac{p-1}{2}+\widetilde{x_{n-1}}\middle|\frac{a+p}{2}\middle|0\middle|\cdots\right)_a$$

and

$$\left(\widetilde{y_0}.\widetilde{y_1}\middle|\cdots\middle|\frac{q-2}{2}+\widetilde{y_{n-1}}\middle|a-\frac{q}{2}\middle|0\middle|\cdots\right)_a < qy < \left(\widetilde{y_0}.\widetilde{y_1}\middle|\cdots\middle|\frac{q}{2}+\widetilde{y_{n-1}}\middle|\frac{q}{2}\middle|0\middle|\cdots\right)_a.$$

Since $q \geq 2$ and $p \leq a-2$ (note that p is odd), we observe that we have reduced the nth coefficients to the range $\{0, 1, \ldots, a-1\}$. We next observe that $\frac{a+p}{2} \leq a - \frac{q}{2}$ and $\frac{q}{2} \leq \frac{a-p}{2}$ by the conditions on p and q. Since the $(n-1)$st coefficients in the bounds for px are equal, while the $(n-1)$st coefficients in the bounds for qy disagree by one, we conclude that no such point (x, y) can exist.

Case 1b: $\alpha \in B$. We claim that the line L given by the equation

$$y = \alpha(x - \frac{1}{2})$$

does not meet any of the omitted squares from the construction of $S_{\mathbf{a}}$.

Suppose that (x, y) is a point contained in some omitted square and also contained in L. Since $\alpha \in B$, there exist odd integers $p, q \in \mathbb{N}$ with $p + q \leq a - 1$, $1 \leq p \leq q \leq a-2$, and

$$\frac{p}{2} + qy = px.$$

Note that

$$\frac{p}{2} = \frac{p-1}{2} + \frac{1}{2} = \left(\frac{p-1}{2}.\frac{a-1}{2}\middle|\frac{a-1}{2}\middle|\frac{a-1}{2}\middle|\cdots\right)_a. \tag{7.5}$$

If we multiply by p in (7.1), by q in (7.2) and add $\frac{p}{2}$ (written in the form (7.5)) to the latter, we obtain

$$\left(\widetilde{x_0}.\widetilde{x_1}\middle|\cdots\middle|\widetilde{x_{n-1}}\middle|\frac{(a-1)p}{2}\middle|0\middle|\cdots\right)_a < px < \left(\widetilde{x_0}.\widetilde{x_1}\middle|\cdots\middle|\widetilde{x_{n-1}}\middle|\frac{(a+1)p}{2}\middle|0\middle|\cdots\right)_a \tag{7.6}$$

and

$$\begin{aligned}&\left(\frac{p-1}{2}+\widetilde{y_0}.\frac{a-1}{2}+\widetilde{y_1}\middle|\cdots\middle|\frac{a-1}{2}+\widetilde{y_{n-1}}\middle|\frac{(a-1)(q+1)}{2}\middle|\frac{a-1}{2}\middle|\cdots\right)_a\\ &\qquad < \frac{p}{2}+qy <\\ &\left(\frac{p-1}{2}+\widetilde{y_0}.\frac{a-1}{2}+\widetilde{y_1}\middle|\cdots\middle|\frac{a-1}{2}+\widetilde{y_{n-1}}\middle|\frac{(a-1)+(a+1)q}{2}\middle|\frac{a-1}{2}\middle|\cdots\right)_a\end{aligned} \tag{7.7}$$

respectively. Note that (7.6) coincides with (7.3), while (7.7) is the sum of (7.6) and (7.5).

Another simple arithmetic calculation recasts (7.6) and (7.7) as follows:

$$\left(\widetilde{x_0}.\widetilde{x_1}\middle|\cdots\middle|\frac{p-1}{2}+\widetilde{x_{n-1}}\middle|\frac{a-p}{2}\middle|0\middle|\cdots\right)_a < px < \left(\widetilde{x_0}.\widetilde{x_1}\middle|\cdots\middle|\frac{p-1}{2}+\widetilde{x_{n-1}}\middle|\frac{a+p}{2}\middle|0\middle|\cdots\right)_a$$

and

$$\left(\frac{p-1}{2}+\widetilde{y_0}.\frac{a-1}{2}+\widetilde{y_1}\middle|\cdots\middle|\frac{a+q-2}{2}+\widetilde{y_{n-1}}\middle|a-\frac{q+1}{2}\middle|\frac{a-1}{2}\middle|\cdots\right)_a$$
$$<\frac{p}{2}+qy<$$
$$\left(\frac{p-1}{2}+\widetilde{y_0}.\frac{a-1}{2}+\widetilde{y_1}\middle|\cdots\middle|\frac{a+q}{2}+\widetilde{y_{n-1}}\middle|\frac{q-1}{2}\middle|\frac{a-1}{2}\middle|\cdots\right)_a.$$

Again, we have reduced the nth coefficients to the range $\{0,1,\dots,a-1\}$. We now observe that $\frac{a+p}{2}\le a-\frac{q+1}{2}$ and $\frac{q-1}{2}\le\frac{a-p}{2}$ by the conditions on p and q. Since the $(n-1)$st coefficients in the bounds for px are equal, while the $(n-1)$st coefficients in the bounds for $\frac{p}{2}+qy$ disagree by one, we conclude that no such point (x,y) can exist.

Note that in Case 1b there is a definite gap between the ranges of possible values for px and $\frac{p}{2}+qy$. This gap corresponds to the fact that lines with slope in B avoid all of the peripheral squares in the construction of the carpet.

This completes the proof of Part 1.

Remark 7.1. An analysis of the preceding proof confirms the previous assertion that every maximal line segment contained in a carpet $S_{\mathbf{a}}$ is also contained in carpets $S_{\mathbf{b}}$ for $b\ge a$. Suppose that α is a slope in either of the sets A or B, associated to Slopes($S_{\mathbf{a}}$). If $b\ge a$, we may repeat the arithmetic calculations of the preceding proofs, working modulo b instead of modulo a. The conclusions remain the same. We conclude that the appropriate line segments $\{(x,y)\in Q: y=\alpha x\}$ or $\{(x,y)\in Q: y=\alpha(x-\frac{1}{2})\}$ persist as subsets of $S_{\mathbf{b}}$.

Remark 7.2. A straightforward variation on the above proof shows that $S_{\mathbf{a}}$ contains nontrivial line segments of each slope in Slopes(S_{a_0}) whenever $\limsup \mathbf{a}=\frac{1}{a_0}>0$.

Part 2. Now let $\alpha\notin A\cup B$. We claim that there is no nontrivial line segment of slope α contained in $S_{\mathbf{a}}$. By Corollary 6.2, it suffices to show that the line L_α of slope α passing through the origin intersects the planar tiling $a\mathbb{Z}^2+Q'$ given in (6.1).

Observe that lines through the origin which pass through the inferior right vertex of any square in the tiling have slope $a\ell/ak=\ell/k$ for some $k,\ell\in\mathbb{Z}$. On the other hand, the slope of any line through the origin which passes through the superior left vertex of any such square has slope $(a\ell+1)/(ak-1)$ for some $k,\ell\in\mathbb{Z}$. Consequently, the line L_β of slope β passing through the origin intersects a square from the tiling if and only if

$$\frac{\ell}{k}<\beta<\frac{a\ell+1}{ak-1}$$

for some relatively prime integers $0\le\ell<k$.

We may choose consecutive slopes p_1/q_1 and p_2/q_2 in Slopes($S_{\mathbf{a}}$) so that

$$\frac{p_1}{q_1}<\alpha<\frac{p_2}{q_2}. \tag{7.8}$$

Now, for each $n\ge 0$, define the *iterated mediants*

$$\alpha_n=\frac{p_1+np_2}{q_1+nq_2}.$$

Note that α_{n+1} is the mediant of α_n and p_2/q_2. All of these rational numbers are in reduced form. We claim that the union of the intervals

$$\alpha_n=\frac{p_1+np_2}{q_1+nq_2}<\beta<\frac{a(p_1+np_2)+1}{a(q_1+nq_2)-1},\qquad n\ge 0, \tag{7.9}$$

covers the interval (7.8). Thus α is contained in one of the intervals (7.9), and hence the line L_α must intersect one of the squares from the tiling. See Figure 6.

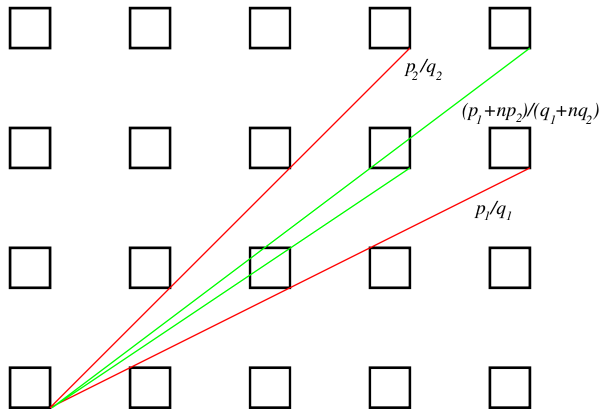

FIGURE 6. Lines with iterated mediant slopes

Since the iterated mediants α_n converge to p_2/q_2 as $n \to \infty$, it suffices to prove that

$$\frac{p_1 + (n-1)p_2}{q_1 + (n-1)q_2} < \frac{p_1 + np_2}{q_1 + nq_2} < \frac{a(p_1 + (n-1)p_2) + 1}{a(q_1 + (n-1)q_2) - 1}$$

for each n. The left hand inequality follows immediately from (7.8). After some computations, the right hand inequality is equivalent to

$$a(q_1p_2 - p_1q_2) < n(p_2 + q_2) + p_1 + q_1,$$

for each $n \in \mathbb{N}$. By property (2) in Proposition 5.9 we know that $q_1p_2 - p_1q_2 = 1$, so we only have to prove that

$$a < p_2 + q_2 + p_1 + q_1. \tag{7.10}$$

Since p_1/q_1 and p_2/q_2 are consecutive fractions in $\mathrm{Slopes}(S_{\mathbf{a}})$, inequality (7.10) can be deduced from Lemma 5.10 and the result follows.

8. **Differentiable and rectifiable curves in the carpets.** Characterizing the slopes of line segments which occur in the carpet permits us to draw conclusions regarding the set of differentiable curves in the carpet. For instance, since the set of slopes has no interior, we easily see that there are no C^1 curves contained in any of the carpets $S_{\mathbf{a}}$ except for the line segments. We now extend this statement to cover all differentiable curves.

Proposition 8.1. *Let* $\mathbf{a}$ *be any sequence in* $\{\frac{1}{3}, \frac{1}{5}, \ldots\}^{\mathbb{N}}$. *Every curve* $\gamma \subset S_{\mathbf{a}}$ *which is differentiable with nonzero derivative everywhere is a line segment.*

The partial derivatives of such a curve satisfy the Darboux property.

Definition 8.2. A real-valued function f defined on an interval I satisfies the *Darboux property* if f takes every connected set to a connected set.

Let $\gamma = (x, y)$ be a curve as in Proposition 8.1. Without loss of generality, we may assume that the curve is parameterized to have speed one everywhere: $x'(t)^2 + y'(t)^2 \equiv 1$. A simple argument using the Darboux property shows that the range of $\gamma' = (x', y')$ is a connected subset of $\mathbb{S}^1$. Since the slope of the tangent vector at time t is given by

$$\alpha(t) = \frac{y'(t)}{x'(t)},$$

we conclude that the range of α is connected. Since the set of slopes has no interior, we conclude the proof of Proposition 8.1 modulo the following lemma.

Lemma 8.3. *Let γ be a differentiable curve in $S_{\mathbf{a}}$. Then $\gamma'(t) \in \mathrm{Slopes}(S_{\mathbf{a}})$ for all t.*

Lemma 8.3 is proved by Bandt and Mubarak in [1] in the case $\mathbf{a} = (\frac{1}{3}, \frac{1}{3}, \ldots)$ and the general case is similar. Here we provide only a sketch. The proof uses the following quantitative version of the fact that $S_{\mathbf{a}}$ contains no nontrivial segments with slopes which are not in $\mathrm{Slopes}(S_{\mathbf{a}})$:

> *If $\alpha \notin \mathrm{Slopes}(S_{\mathbf{a}})$ and L denotes a nontrivial line segment of slope α through a point $(x,y) \in S_{\mathbf{a}}$, then for all sufficiently small ϵ there exists a point (x',y') in $B((x,y),\epsilon) \cap L$ whose distance to $S_{\mathbf{a}}$ is at least $c\epsilon$, where $c > 0$ depends only on $\mathrm{dist}(\alpha, \mathrm{Slopes}(S_{\mathbf{a}}))$.*

Suppose that there exists a differentiable curve γ contained entirely in $S_{\mathbf{a}}$, and $\gamma'(t) \notin \mathrm{Slopes}(S_{\mathbf{a}})$ for some time t. Then $\gamma(s)$ is well approximated by $\gamma(t) + (s-t)\gamma'(t)$ for s near t and hence the line segment $s \mapsto \gamma(t) + (s-t)\gamma'(t)$ remains close to the carpet $S_{\mathbf{a}}$ for s near t. This can be used eventually to contradict the preceding quantitative statement.

Ultimately, we are interested in classifying the rectifiable curves in the carpets $S_{\mathbf{a}}$. The results of this paper form a first step towards this goal. However, the classification of the rectifiable curves will necessarily be more subtle. Any arc length parameterized rectifiable curve γ contained in the carpet $S_{\mathbf{a}}$ is differentiable at $\mathcal{H}^1$-a.e. point of the parameterizing interval. At such times, γ has a tangent line. We might be led to conjecture that γ must contain a (possibly one-sided) nontrivial line segment through $\gamma(t)$ in the direction of $\gamma'(t)$. The following example, due to Enrico Le Donne, shows that this conjecture is false. This illustrates the difficulty in understanding the structure of general rectifiable curves. We are grateful to Enrico for allowing us to include his example.

Example 8.4. There exists a rectifiable curve $\gamma : [0,T] \to S_{\mathbf{3}}$, parameterized by arc length, so that $\gamma'(T)$ exists, but there is no nontrivial line segment through $\gamma(T)$ contained in $S_{\mathbf{3}}$ in the direction of $\gamma'(T)$.

We begin with some observations. Let C denote the usual Cantor set. Construct a set $\widetilde{C} \subset [0,1]$ as follows: start with C and add new (similar) copies of C into all of the omitted intervals, continuing recursively until no omitted intervals remain. The set $\widetilde{C}$ is a dense F_σ subset of $[0,1]$ of Hausdorff dimension $\frac{\log 2}{\log 3}$. For a given $y \in [0,1]$, the following are equivalent:

(i) the line segment $[0,1] \times \{y\}$ is contained in $S_{\mathbf{3}}$,
(ii) $y \in C$, and
(iii) y admits a 3-adic representation containing no copies of the digit 1.

Similarly, for a given $y \in [0,1]$, the following are equivalent:

(i) there exists a line segment $[0,\epsilon] \times \{y\}$ contained in $S_{\mathbf{3}}$, for some $\epsilon > 0$,
(ii) $y \in \widetilde{C}$, and
(iii) y admits a 3-adic representation containing at most finitely many copies of the digit 1.

We will construct a rectifiable curve $\gamma : [0,T] \to S_{\mathbf{3}}$ with the property that $\gamma(T) = (0,y)$ for some $y \notin \widetilde{C}$ and $\gamma'(T) = (-1,0)$. Thus γ is differentiable at time T with horizontal tangent vector, but there is no nontrivial horizontal line segment through $\gamma(T)$ contained in $S_{\mathbf{3}}$.

Let $(m_j)_{j\geq 1}$ be any increasing sequence of positive integers, e.g., $m_j = j$. Set $M_j = \sum_{i=1}^{j} m_i$ with the usual interpretation $M_0 = 0$. Note that $m_j \geq j$ for all j. Set

$$T := \sum_{j \geq 0} 3^{-M_j}.$$

Define sequences (T_k) and (S_k) satisfying

$$0 = S_0 < T_0 < S_1 < T_1 < \cdots < T$$

with $\lim S_k = \lim T_k = T$, as follows: $T_k := \sum_{j=0}^{k} 3^{-M_j}$ for $k \geq 0$ and $S_k := T_{k-1} + 3^{-M_{k+1}} = T_{k+1} - 3^{-M_k}$ for $k \geq 1$.

We define γ by assigning the values of $\gamma(S_k)$ and $\gamma(T_k)$ for all k and extending in a piecewise linear fashion. For $k \geq 0$, let $\gamma(S_k) = (3^{-M_k}, T_{k+1} - 1)$ and $\gamma(T_k) = (3^{-M_{k+1}}, T_{k+1} - 1)$. Then $\gamma|_{[S_k,T_k]}$ is horizontal for all k, while $\gamma|_{[T_k,S_{k+1}]}$ is vertical for all k. From the construction it is clear that the image of $\gamma|_{[0,T)}$ is contained in $S_{\mathbf{3}}$. Extending γ to $t = T$ by continuity gives $\gamma(T) = (0, T-1)$; we observe that the range of γ is contained in $S_{\mathbf{3}}$. Since $T - 1 = \sum_{j\geq 1} 3^{-M_j}$ has a 3-adic representation with infinitely many 1's, $T-1$ is not in $\widetilde{C}$ and there is no nontrivial horizontal line segment contained in $S_{\mathbf{3}}$ passing through $\gamma(T)$.

It remains to show that $\gamma'(T)$ exists and to compute its value. First, we note that when $S_k \leq t \leq T_k$, resp. when $T_k \leq t \leq S_{k+1}$, the difference quotient

$$\frac{\gamma(T) - \gamma(t)}{T - t}$$

is a convex combination of

$$\frac{\gamma(T) - \gamma(S_k)}{T - S_k} \quad \text{and} \quad \frac{\gamma(T) - \gamma(T_k)}{T - T_k},$$

resp., of

$$\frac{\gamma(T) - \gamma(T_k)}{T - T_k} \quad \text{and} \quad \frac{\gamma(T) - \gamma(S_{k+1})}{T - S_{k+1}}.$$

Thus it suffices to show that $\lim_{k\to\infty} \frac{\gamma(T)-\gamma(T_k)}{T-T_k}$ and $\lim_{k\to\infty} \frac{\gamma(T)-\gamma(S_k)}{T-S_k}$ exist and are equal. But

$$\frac{\gamma(T) - \gamma(T_k)}{T - T_k} = (-1, 0) + \left(\frac{T - T_{k+1}}{T - T_k}\right)(1, 1)$$

while

$$\frac{\gamma(T) - \gamma(S_k)}{T - S_k} = (-1, 0) + \left(\frac{T - T_{k+1}}{T - S_k}\right)(1, 1).$$

Lemma 8.5. *For all positive N and $j \geq 0$, $\sum_{i=N}^{N+j} m_i \geq N(j+1)$.*

Proof. Use the fact that $m_i \geq i$ for all i. ◻

Using this lemma, we now conclude the proof with the estimate

$$\left|\frac{T - T_{k+1}}{T - T_k}\right| \leq \frac{\sum_{j=k+2}^{\infty} 3^{-M_j}}{3^{-M_{k+1}}} = \sum_{j=k+2}^{\infty} 3^{-\sum_{i=k+2}^{j} m_i} \leq \sum_{j=0}^{\infty} 3^{-(k+2)(j+1)} = \frac{1}{3^{k+2} - 1} \to 0. \tag{8.1}$$

To see that $\frac{T-T_{k+1}}{T-S_k} \to 0$, repeat (8.1) with the factor $3^{-M_{k+1}}$ replaced by 3^{-M_k} and note that this only increases the rate of convergence.

Remark 8.6. In the above example, the limit point $\gamma(T) = (0, T-1)$ is not contained in any horizontal line segment in $S_{\mathbf{3}}$, but it is contained in a vertical line segment in $S_{\mathbf{3}}$. It is easy to modify the construction to obtain a rectifiable curve $\gamma : [0, T] \to S_{\mathbf{3}}$ so that $\gamma'(T)$ exists, but there are no nontrivial line segments of any slope passing through $\gamma(T)$ and contained in $S_{\mathbf{3}}$. Note that the typical point of $S_{\mathbf{3}}$ lies in no nontrivial line segment contained in $S_{\mathbf{3}}$. Indeed, the union of all nontrivial line segments contained in $S_{\mathbf{3}}$ has Hausdorff dimension

$$1 + \frac{\log 2}{\log 3}$$

which is strictly less than

$$\frac{\log 8}{\log 3},$$

the Hausdorff dimension of $S_{\mathbf{3}}$.

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