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**EIGENVALUES AND ENTROPIES UNDER THE
HARMONIC-RICCI FLOW**

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In this paper, the author discusses the eigenvalues and entropies under the harmonic-Ricci flow, which is the Ricci flow coupled with the harmonic map flow. We give an alternative proof of results for compact steady and expanding harmonic-Ricci breathers. In the second part, we derive some monotonicity formulas for eigenvalues of the Laplacian under the harmonic-Ricci flow. Finally, we obtain the first variation of the shrinker and expanding entropies of the harmonic-Ricci flow.

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1. Introduction

Since the successful application of the Ricci flow to topological and geometric problems, several analogous flows have been studied, including the harmonic-Ricci flow [List 2006; Müller 2012], connection Ricci flow [Streets 2008], Ricci–Yang–Mills flow [Streets 2007; 2010; Young 2008], and renormalization group flows [He et al. 2008; Li 2012; Oliynyk et al. 2006; Streets 2009]. In this article, we study the

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eigenvalue problems of the harmonic-Ricci flow, which is the following coupled system:

$$(1-1) \quad \frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}_{g(t)} + 4 du(t) \otimes du(t),$$

$$(1-2) \quad \frac{\partial}{\partial t} u(t) = \Delta_{g(t)} u(t).$$

For convenience, we introduce a new symmetric 2-tensor $\mathcal{S}_{g(t), u(t)}$ whose components S_{ij} are defined by

$$S_{ij} := R_{ij} - 2\partial_i u \partial_j u.$$

Its trace is $S_{g(t), u(t)} := g^{ij} S_{ij} = R_{g(t)} - 2|\nabla_{g(t)} u(t)|_{g(t)}^2$.

Suppose that M is a compact Riemannian manifold. For any Riemannian metric g and any smooth functions u, f , we have a number of functionals:

$$\mathcal{F}(g, u, f) = \int_M (R_g + |\nabla_g f|_g^2 - 2|\nabla_g u|_g^2) e^{-f} dV_g,$$

$$\mathcal{E}(g, u, f) = \int_M (R_g - 2|\nabla_g u|_g^2) e^{-f} dV_g,$$

$$\mathcal{F}_k(g, u, f) = \int_M (kR_g + |\nabla_g f|_g^2 - 2k|\nabla_g u|_g^2) e^{-f} dV_g.$$

List [2006] and Müller [2012] showed that, as in the case of Perelman's \mathcal{F} -functional, under the evolution equation

$$(1-3) \quad \begin{aligned} \frac{\partial}{\partial t} g(t) &= -2 \operatorname{Ric}_{g(t)} + 4 du(t) \otimes du(t), \\ \frac{\partial}{\partial t} u(t) &= \Delta_{g(t)} u(t), \\ \frac{\partial}{\partial t} f(t) &= -\Delta_{g(t)} f(t) - R_{g(t)} + |\nabla_{g(t)} f(t)|_{g(t)}^2 + 2|\nabla_{g(t)} u(t)|_{g(t)}^2, \end{aligned}$$

the evolution equation for the \mathcal{F} -functional is

$$(1-4) \quad \begin{aligned} \frac{d}{dt} \mathcal{F}(g(t), u(t), f(t)) &= 2 \int_M |\mathcal{S}_{g(t), u(t)} + \nabla_{g(t)}^2 f(t)|_{g(t)}^2 e^{-f(t)} dV_{g(t)} \\ &\quad + 4 \int_M |\Delta_{g(t)} u(t) - \langle du(t), df(t) \rangle_{g(t)}|_{g(t)}^2 e^{-f(t)} dV_{g(t)}, \end{aligned}$$

which is nonnegative. Based on (1-4), we derive the following.

Theorem 1.1. *Under the evolution equation (1-3), one has*

$$(1-5) \quad \begin{aligned} \frac{d}{dt} \mathcal{E}(g(t), u(t), f(t)) \\ = 2 \int_M |\mathcal{S}_{g(t), u(t)}|_{g(t)}^2 e^{-f(t)} dV_{g(t)} + 4 \int_M |\Delta_{g(t)} u(t)|_{g(t)}^2 e^{-f(t)} dV_{g(t)}, \end{aligned}$$

and

$$\begin{aligned}
 (1-6) \quad & \frac{d}{dt} \mathcal{F}_k(g(t), u(t), f(t)) \\
 & = 2(k-1) \int_M |\mathcal{S}_{g(t), u(t)}|_{g(t)}^2 e^{-f(t)} dV_{g(t)} + 2 \int_M |\mathcal{S}_{g(t), u(t)} \\
 & \quad + \nabla_{g(t)}^2 f(t)|_{g(t)}^2 e^{-f(t)} dV_{g(t)} + 4(k-1) \int_M |\Delta_{g(t)} u(t)|_{g(t)}^2 e^{-f(t)} dV_{g(t)} \\
 & \quad + 4 \int_M |\Delta_{g(t)} u(t) - \langle du(t), df(t) \rangle_{g(t)}|_{g(t)}^2 e^{-f(t)} dV_{g(t)}.
 \end{aligned}$$

As a corollary we give a new proof of the following result.

Corollary 1.2. *There is no compact steady harmonic-Ricci breather unless the manifold $(M, g(t))$ is Ricci-flat and $u(t)$ is a constant.*

To deal with the expanding harmonic-Ricci breather, we need the functionals

$$\begin{aligned}
 \mathcal{L}_+(g, u, \tau, f) & = \tau^2 \int_M \left(R_g + \frac{n}{2\tau} + \Delta_g f - 2|\nabla_g u|_g^2 \right) e^{-f} dV_g, \\
 \mathcal{L}_{+,k}(g, u, \tau, f) & = \tau^2 \int_M \left(k \left(R_g + \frac{n}{2\tau} \right) + \Delta_g f - 2k|\nabla_g u|_g^2 \right) e^{-f} dV_g.
 \end{aligned}$$

Under the evolution equation

$$\begin{aligned}
 \frac{\partial}{\partial t} g(t) & = -2 \operatorname{Ric}_{g(t)} + 4 du(t) \otimes du(t), \\
 \frac{\partial}{\partial t} u(t) & = \Delta_{g(t)} u(t), \\
 \frac{\partial}{\partial t} f(t) & = -\Delta_{g(t)} f(t) + |\nabla_{g(t)} f(t)|_{g(t)}^2 - R_{g(t)} + 2|\nabla_{g(t)} u(t)|_{g(t)}^2, \\
 \frac{d}{dt} \tau(t) & = 1,
 \end{aligned}$$

we have:

Theorem 1.3. *Under the evolution equation, one has*

$$\begin{aligned}
 (1-7) \quad & \frac{d}{dt} \mathcal{L}_+(g(t), u(t), \tau(t), f(t)) \\
 & = 2\tau(t)^2 \int_M \left| \mathcal{S}_{g(t), u(t)} + \nabla_{g(t)}^2 f(t) + \frac{1}{2\tau(t)} g(t) \right|_{g(t)}^2 e^{-f(t)} dV_{g(t)} \\
 & \quad + 4\tau(t)^2 \int_M |\Delta_{g(t)} u(t) - \langle du(t), df(t) \rangle_{g(t)}|_{g(t)}^2 e^{-f(t)} dV_{g(t)}
 \end{aligned}$$

and

$$\begin{aligned}
 (1-8) \quad & \frac{d}{dt} \mathcal{L}_{+,k}(g(t), u(t), \tau(t), f(t)) \\
 & = 2\tau(t)^2 \int_M |\mathcal{S}_{g(t),u(t)} + \nabla_{g(t)}^2 f(t) + \frac{1}{2\tau(t)} g(t)|_{g(t)}^2 e^{-f(t)} dV_{g(t)} \\
 & \quad + 2(k-1)\tau(t)^2 \int_M |\mathcal{S}_{g(t),u(t)} + \frac{1}{2\tau(t)} g(t)|_{g(t)}^2 e^{-f(t)} dV_{g(t)} \\
 & \quad + 4\tau(t)^2 \int_M |\Delta_{g(t)} u(t) - \langle du(t), df(t) \rangle_{g(t)}|_{g(t)}^2 e^{-f(t)} dV_{g(t)} \\
 & \quad + 4(k-1)\tau(t)^2 \int_M |\Delta_{g(t)} u(t)|_{g(t)}^2 e^{-f(t)} dV_{g(t)}.
 \end{aligned}$$

As a corollary, we obtain a new proof of the following.

Corollary 1.4. *There is no expanding harmonic-Ricci breather on compact Riemannian manifolds unless the manifold M is an Einstein manifold and $u(t)$ a constant.*

The second part of this paper focuses on the eigenvalue of the Laplacian operator under the harmonic-Ricci flow.

Theorem 1.5. *If $(g(t), u(t))$ is a solution of the harmonic-Ricci flow on a compact Riemannian manifold M and $\lambda(t)$ denotes the eigenvalue of the Laplacian $\Delta_{g(t)}$ with eigenfunction $f(t)$,*

$$\begin{aligned}
 (1-9) \quad & \frac{d}{dt} \lambda(t) \cdot \int_M f(t)^2 dV_{g(t)} \\
 & = \lambda(t) \int_M S_{g(t),u(t)} f(t)^2 dV_{g(t)} - \int_M S_{g(t),u(t)} |\nabla_{g(t)} f|_{g(t)}^2 dV_{g(t)} \\
 & \quad + 2 \int_M \langle \mathcal{S}_{g(t),u(t)}, df(t) \otimes df(t) \rangle_{g(t)} dV_{g(t)}.
 \end{aligned}$$

Equation (1-9) is a general formula to describe the evolution of $\lambda(t)$ under the harmonic-Ricci flow. Under a curvature assumption, we can derive some monotonicity formulas for the eigenvalue $\lambda(t)$. Set

$$(1-10) \quad S_{\min}(0) := \min_{x \in M} S_{g(0),u(0)}(x),$$

the minimum of $S_{g(t),u(t)}$ over M at the time 0.

Theorem 1.6. *Let $(g(t), u(t))_{t \in [0, T]}$ be a solution of the harmonic-Ricci flow on a compact Riemannian manifold M and let $\lambda(t)$ denote the eigenvalue of the Laplacian $\Delta_{g(t)}$. Suppose that $\mathcal{S}_{g(t),u(t)} - \alpha S_{g(t),u(t)} g(t) \geq 0$ along the harmonic-Ricci flow for some $\alpha \geq \frac{1}{2}$.*

(1) If $S_{\min}(0) \geq 0$, $\lambda(t)$ is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.

(2) If $S_{\min}(0) > 0$, the quantity

$$\left(1 - \frac{2}{n} S_{\min}(0)t\right)^{n\alpha} \lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for $T \leq n/(2S_{\min}(0))$.

(3) If $S_{\min}(0) < 0$, the quantity

$$\left(1 - \frac{2}{n} S_{\min}(0)t\right)^{n\alpha} \lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.

Corollary 1.7. Let $(g(t), u(t))_{t \in [0, T]}$ be a solution of the harmonic-Ricci flow on a compact Riemannian surface Σ and let $\lambda(t)$ denote the eigenvalue of the Laplacian $\Delta_{g(t)}$.

(1) Suppose that $\text{Ric}_{g(t)} \leq \epsilon du(t) \otimes du(t)$ where

$$\epsilon \leq 4 \frac{1 - \alpha}{1 - 2\alpha}, \quad \alpha > \frac{1}{2}.$$

(i) If $S_{\min}(0) \geq 0$, $\lambda(t)$ is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.

(ii) If $S_{\min}(0) > 0$, the quantity

$$(1 - S_{\min}(0)t)^{2\alpha} \lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for $T \leq 1/S_{\min}(0)$.

(iii) If $S_{\min}(0) < 0$, the quantity

$$(1 - S_{\min}(0)t)^{2\alpha} \lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.

(2) Suppose that

$$|\nabla_{g(t)} u(t)|_{g(t)}^2 g(t) \geq 2du(t) \otimes du(t).$$

(i) If $S_{\min}(0) \geq 0$, $\lambda(t)$ is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.

(ii) If $S_{\min}(0) > 0$, the quantity

$$(1 - S_{\min}(0)t)\lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for $T \leq 1/S_{\min}(0)$.

(iii) If $S_{\min}(0) < 0$, the quantity

$$(1 - S_{\min}(0)t)\lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.

When we restrict to the Ricci flow, we obtain:

Corollary 1.8. *Let $(g(t))_{t \in [0, T]}$ be a solution of the Ricci flow on a compact Riemannian surface Σ and let $\lambda(t)$ denote the eigenvalue of the Laplacian $\Delta_{g(t)}$.*

- (1) *If $R_{\min}(0) \geq 0$, $\lambda(t)$ is nondecreasing along the Ricci flow for any $t \in [0, T]$.*
- (2) *If $R_{\min}(0) > 0$, the quantity $(1 - R_{\min}(0)t)\lambda(t)$ is nondecreasing along the Ricci flow for $T \leq 1/R_{\min}(0)$.*
- (3) *If $R_{\min}(0) < 0$, the quantity $(1 - R_{\min}(0)t)\lambda(t)$ is nondecreasing along the Ricci flow for any $t \in [0, T]$.*

Remark 1.9. Let $(g(t))_{t \in [0, T]}$ be a solution of the Ricci flow on a compact Riemannian surface Σ with nonnegative scalar curvature and let $\lambda(t)$ denote the eigenvalue of the Laplacian $\Delta_{g(t)}$. Then $\lambda(t)$ is nondecreasing along the Ricci flow for any $t \in [0, T]$.

Since

$$(1-11) \quad \mu(g, u) := \inf \left\{ \mathcal{F}(g, u, f) \mid f \in C^\infty(M), \int_M e^{-f} dV_g = 1 \right\}$$

is the smallest eigenvalue of the operator $\Delta_{g,u} := -4\Delta_g + R_g - 2|\nabla_g u|_g^2$, we can consider the evolution equation for this eigenvalue under the harmonic-Ricci flow. To the operator $\Delta_{g,u}$ we associate a functional

$$(1-12) \quad \lambda_{g,u}(f) := \int_M f \Delta_{g,u} f dV_g.$$

When f is an eigenfunction of the operator $\Delta_{g,u}$ with the eigenvalue λ and normalized by $\int_X f^2 dV_g = 1$, we obtain $\lambda_{g,u}(f) = \lambda$. Hence it suffices to study the evolution equation for $(d/dy)\lambda_{g,u}(f)$ under the harmonic-Ricci flow.

Theorem 1.10. *Suppose that $(g(t), u(t))$ is a solution of the harmonic-Ricci flow on a compact Riemannian manifold M and $f(t)$ is an eigenfunction of $\Delta_{g(t),u(t)}$, that is, $\Delta_{g(t),u(t)} f(t) = \lambda(t) f(t)$ (where $\lambda(t)$ is only a function of time t), with the normalized condition $\int_M f(t)^2 dV_{g(t)} = 1$. Then we have*

$$(1-13) \quad \begin{aligned} \frac{d}{dt} \lambda(t) &= \frac{d}{dt} \lambda_{g,u}(f(t)) = \int_M 2 \langle \mathcal{S}_{g(t),u(t)}, df(t) \otimes df(t) \rangle_{g(t)} dV_{g(t)} \\ &\quad + \int_M f(t)^2 (|\mathcal{S}_{g(t),u(t)}|_{g(t)}^2 + 2|\Delta_{g(t)} u(t)|_{g(t)}^2) dV_{g(t)}. \end{aligned}$$

List [2006] proved the nonnegativity of the operator $\mathcal{S}_{g(t),u(t)}$ is preserved by the harmonic-Ricci flow. Hence we get the following.

Corollary 1.11. *If $\text{Ric}_{g(0)} - 2du(0) \otimes du(0) \geq 0$, the eigenvalues of the operator $\Delta_{g(t),u(t)}$ are nondecreasing under the harmonic-Ricci flow.*

Remark 1.12. If we choose $u(t) \equiv 0$, we obtain X. Cao's result [2007].

There is another expression for $d\lambda(t)/dt$.

Theorem 1.13. *Suppose that $(g(t), u(t))$ is a solution of the harmonic-Ricci flow on a compact Riemannian manifold M and $f(t)$ is an eigenfunction of $\Delta_{g(t),u(t)}$, that is, $\Delta_{g(t),u(t)}f(t) = \lambda(t)f(t)$ (where $\lambda(t)$ is only a function of time t), with the normalized condition $\int_M f(t)^2 dV_{g(t)} = 1$. Then we have*

$$(1-14) \quad \begin{aligned} \frac{d}{dt}\lambda(t) &= \frac{d}{dt}\lambda_{g,u}(f(t)) = \frac{1}{2} \int_M |\mathcal{S}_{g(t),u(t)} + \nabla_{g(t)}^2 \varphi(t)|_{g(t)}^2 e^{-\varphi(t)} dV_{g(t)} \\ &+ \frac{1}{4} \int_M |\mathcal{S}_{g(t),u(t)}|_{g(t)}^2 e^{-\varphi(t)} dV_{g(t)} + \int_M |\langle du(t), d\varphi(t) \rangle_{g(t)}|^2 e^{-\varphi(t)} dV_{g(t)} \\ &+ 2 \int_M |\nabla_{g(t)}^2 u(t)|_{g(t)}^2 e^{-\varphi(t)} dV_{g(t)} - \int_M \Delta_{g(t)}(|\nabla_{g(t)} u(t)|_{g(t)}^2) e^{-\varphi(t)} dV_{g(t)} \\ &+ \frac{1}{4} \int_M |\mathcal{S}_{g(t),u(t)} + 4du(t) \otimes du(t)|_{g(t)}^2 e^{-\varphi(t)} dV_{g(t)}, \end{aligned}$$

where $f(t)^2 = e^{-\varphi(t)}$.

Remark 1.14. When $u \equiv 0$, (1-14) reduces to J. Li's formula [2007].

Suppose that M is a compact manifold of dimension n . For any Riemannian metric g , any smooth functions u, f , and any positive number τ , we define

$$(1-15) \quad \mathcal{W}_{\pm}(g, u, f, \tau) := \int_M [\tau(S_g + |\nabla_g f|_g^2) \mp f \pm n] \frac{e^{-f}}{(4\pi\tau)^{n/2}} dV_g.$$

Set

$$\begin{aligned} \mu_{\pm}(g, u, \tau) &:= \inf \left\{ \mathcal{W}_{\pm}(g, u, f, \tau) \mid f \in C^{\infty}(M), \int_M \frac{e^{-f}}{(4\pi\tau)^{n/2}} dV_g = 1 \right\}, \\ v_{-}(g, u) &:= \inf \{ \mu_{-}(g, u, \tau) \mid \tau > 0 \}, \quad v_{+}(g, u) := \sup \{ \mu_{+}(g, u, \tau) \mid \tau > 0 \}. \end{aligned}$$

The first variation of $v_{\pm}(g(s), u(s))$ is the following.

Theorem 1.15. *Suppose that (M, g) is a compact Riemannian manifold and u a smooth function on M . Let h be any symmetric covariant 2-tensor on M and set $g(s) := g + sh$. Let v be any smooth function on M and $u(s) := u + sv$. If*

$v_{\pm}(g(s), u(s)) = \mathcal{W}_{\pm}(g(s), u(s), f_{\pm}(s), \tau_{\pm}(s))$ for some smooth functions $f_{\pm}(s)$ with $\int_M e^{-f_{\pm}(s)} dV_g / (4\pi \tau_{\pm}(s))^{n/2} = 1$ and constants $\tau_{\pm}(s) > 0$,

$$(1-16) \quad \frac{d}{ds} \Big|_{s=0} v_{\pm}(g(s), u(s)) = 4\tau_{\pm} \int_M v(\Delta_g u - \langle du, df_{\pm} \rangle_g) \frac{e^{-f_{\pm}}}{(4\pi \tau_{\pm})^{n/2}} dV_g \\ - \tau_{\pm} \int_M \left(\langle h, \mathcal{G}_{g,u} \rangle_g + \langle h, \nabla_g^2 f \rangle_g \pm \frac{\text{tr}_g h}{2\tau_{\pm}} \right) \frac{e^{-f_{\pm}} dV_g}{(4\pi \tau_{\pm})^{n/2}},$$

where $f_{\pm} := f_{\pm}(0)$ and $\tau_{\pm} := \tau_{\pm}(0)$. In particular, the critical points of $v_{\pm}(\cdot, \cdot)$ satisfy

$$\mathcal{G}_{g,u} + \nabla_g^2 f \pm \frac{1}{2\tau_{\pm}} g = 0, \quad \Delta_g u = \langle du, df_{\pm} \rangle_g.$$

Consequently, if $\mathcal{W}_{\pm}(g, u, f, \tau)$ and $v_{\pm}(g, u)$ achieve their extremum, (M, g) is a gradient expanding and shrinker harmonic-Ricci soliton according to the sign.

Corollary 1.16. *Suppose that (M, g) is a compact Riemannian manifold and u a smooth function on M . Let h be any symmetric covariant 2-tensor on M and set $g(s) := g + sh$. Let v be any smooth function on M and $u(s) := u + sv$. If $v_{\pm}(g(s), u(s)) = \mathcal{W}_{\pm}(g(s), u(s), f_{\pm}(s), \tau_{\pm}(s))$ for some smooth function $f_{\pm}(s)$ with $\int_M e^{-f_{\pm}(s)} dV / (4\pi \tau_{\pm}(s))^{n/2} = 1$ and a constant $\tau_{\pm}(s) > 0$, and (g, u) is a critical point of $v_{\pm}(\cdot, \cdot)$, then*

$$\text{Ric}_g = \mp \frac{1}{2\tau_{\pm}} g, \quad f_{\pm} \equiv \text{constant}, \quad u \equiv \text{constant}.$$

Thus, if $\mathcal{W}_{\pm}(g, u, \cdot, \cdot)$ achieve their minimum and (g, u) is a critical point of $v_{\pm}(\cdot, \cdot)$, (M, g) is an Einstein manifold and u is a constant function.

Remark 1.17. In the situation of [Corollary 1.16](#), by normalization, we may choose $f_{\pm} = n/2$ and $u = 0$.

2. Notation and commuting identities

Let M be a compact Riemannian manifold of dimension n . For any vector bundle E over M , we denote by $\Gamma(M, E)$ the space of smooth sections of E . Set

$$\odot^2(M) := \{v = (v_{ij}) \in \Gamma(M, T^*M \otimes T^*M) \mid v_{ij} = v_{ji}\}, \\ \odot_+^2(M) := \{g = (g_{ij}) \in \odot^2(M) \mid g_{ij} > 0\}.$$

Thus $\odot^2(M)$ is the space of all symmetric covariant 2-tensors on M , while $\odot_+^2(M)$ is the space of all Riemannian metrics on M . The space of all smooth functions on M is denoted by $C^\infty(M)$.

For a given Riemannian metric $g \in \odot_+^2(M)$, the corresponding Levi-Civita connection $\Gamma_g = (\Gamma_{ij}^k)$ is given by

$$(2-1) \quad \Gamma_{ij}^k = \frac{1}{2} g^{k\ell} (\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij})$$

where $\partial_i := \partial/\partial x^i$ for a local coordinate system $\{x^1, \dots, x^n\}$. The Riemann tensor $\text{Rm}_g = (R_{ijl}^k)$ is determined by

$$(2-2) \quad R_{ijl}^k = \partial_i \Gamma_{jl}^k - \partial_j \Gamma_{il}^k + \Gamma_{ip}^k \Gamma_{jl}^p - \Gamma_{jp}^k \Gamma_{il}^p.$$

The Ricci curvature $\text{Ric}_g = (R_{ij})$ is

$$(2-3) \quad R_{ij} = g^{k\ell} R_{kij}^\ell.$$

The scalar curvature R_g of the metric g now is given by

$$(2-4) \quad R_g = g^{ij} R_{ij}.$$

For any tensor $A = (A_{j_1 \dots j_p}^{k_1 \dots k_q})$ the covariant derivative of A is

$$\nabla_i A_{j_1 \dots j_p}^{k_1 \dots k_q} = \partial_i A_{j_1 \dots j_p}^{k_1 \dots k_q} - \sum_{r=1}^p \Gamma_{ij_r}^m A_{j_1 \dots m \dots j_p}^{k_1 \dots k_q} + \sum_{s=1}^q \Gamma_{im}^{k_s} A_{j_1 \dots j_p}^{k_1 \dots m \dots k_q}.$$

Next we recall the Ricci identity:

$$\nabla_i \nabla_j A_{k_1 \dots k_p}^{\ell_1 \dots \ell_q} - \nabla_j \nabla_i A_{k_1 \dots k_p}^{\ell_1 \dots \ell_q} = \sum_{r=1}^q R_{ijm}^{\ell_r} A_{k_1 \dots k_p}^{\ell_1 \dots m \dots \ell_q} - \sum_{s=1}^p R_{ijk_s}^m A_{k_1 \dots m \dots k_p}^{\ell_1 \dots \ell_q}.$$

In particular, for any smooth function $f \in C^\infty(M)$, we have

$$\nabla_i \nabla_j f = \nabla_j \nabla_i f.$$

The Bianchi identities are

$$(2-5) \quad 0 = R_{ijk\ell} + R_{iklj} + R_{i\ell jk},$$

$$(2-6) \quad 0 = \nabla_q R_{ijk\ell} + \nabla_i R_{jqk\ell} + \nabla_j R_{qik\ell},$$

and the contracted Bianchi identities are

$$(2-7) \quad 0 = 2\nabla^j R_{ij} - \nabla_i R_g,$$

$$(2-8) \quad 0 = \nabla_i R_{jk} - \nabla_j R_{ik} + \nabla^\ell R_{lkij}.$$

3. Harmonic-Ricci flow and the evolution equations

Motivated by the static Einstein vacuum equation, List [2006] introduced the harmonic-Ricci flow (originally called the Ricci flow coupled with the harmonic map flow). This flow is similar to the Ricci flow and is given by the coupled system

$$(3-1) \quad \frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}_{g(t)} + 4 du(t) \otimes du(t),$$

$$(3-2) \quad \frac{\partial}{\partial t} u(t) = \Delta_{g(t)} u(t)$$

for a family of Riemannian metrics $g(t)$ and a family of smooth functions $u(t)$. Locally, we have

$$(3-3) \quad \frac{\partial}{\partial t} g_{ij} = -2R_{ij} + 4\partial_i u \cdot \partial_j u, \quad \frac{\partial}{\partial t} u = \Delta_{g(t)} u(t).$$

Introduce a new symmetric tensor field $\mathcal{S}_{g(t),u(t)} = (S_{ij}) \in \odot^2(M)$,

$$(3-4) \quad S_{ij} := R_{ij} - 2\partial_i u \cdot \partial_j u.$$

Then its trace $S_{g(t),u(t)}$ is equal to

$$(3-5) \quad S_{g(t),u(t)} = g^{ij} S_{ij} = R_{g(t)} - 2|\nabla_{g(t)} u(t)|_{g(t)}^2.$$

The evolution equation for $R_{g(t)}$ is

$$(3-6) \quad \begin{aligned} \frac{\partial}{\partial t} R_{g(t)} = & \Delta_{g(t)} R_{g(t)} + 2|\operatorname{Ric}_{g(t)}|_{g(t)}^2 + 4|\Delta_{g(t)} u(t)|_{g(t)}^2 \\ & - 4|\nabla_{g(t)}^2 u(t)|_{g(t)}^2 - 8\langle \operatorname{Ric}_{g(t)}, du(t) \otimes du(t) \rangle_{g(t)}. \end{aligned}$$

Also, we have the evolution equation for $|\nabla_{g(t)} u|_{g(t)}^2$,

$$(3-7) \quad \frac{\partial}{\partial t} |\nabla_{g(t)} u(t)|_{g(t)}^2 = \Delta_{g(t)} |\nabla_{g(t)} u(t)|_{g(t)}^2 - 2|\nabla_{g(t)}^2 u(t)|_{g(t)}^2 - 4|\nabla_{g(t)} u(t)|_{g(t)}^4,$$

and the evolution equation for $S_{g(t),u(t)}$,

$$(3-8) \quad \frac{\partial}{\partial t} S_{g(t),u(t)} = \Delta_{g(t)} S_{g(t),u(t)} + 2|\mathcal{S}_{g(t),u(t)}|_{g(t)}^2 + 4|\Delta_{g(t)} u(t)|_{g(t)}^2.$$

4. Entropies for harmonic-Ricci flow

Motivated by Perelman's entropy, List [2006] introduced a similar functional for the harmonic-Ricci flow:

$$\odot_+^2(M) \times C^\infty(M) \times C^\infty(M) \rightarrow \mathbb{R}, \quad (g, u, f) \mapsto \mathcal{F}(g, u, f)$$

where

$$(4-1) \quad \mathcal{F}(g, u, f) := \int_M (R_g + |\nabla_g f|_g^2 - 2|\nabla_g u|_g^2) e^{-f} dV_g.$$

He also showed that if $(g(t), u(t), f(t))$ satisfies the system

$$(4-2) \quad \begin{aligned} \frac{\partial}{\partial t} g(t) &= -2 \operatorname{Ric}_{g(t)} + 4 du(t) \otimes du(t) - 2 \nabla_{g(t)}^2 f(t), \\ \frac{\partial}{\partial t} u(t) &= \Delta_{g(t)} u(t) - \langle du(t), df(t) \rangle_{g(t)}, \\ \frac{\partial}{\partial t} f(t) &= -\Delta_{g(t)} f(t) - R_{g(t)} + 2|\nabla_{g(t)} u(t)|_{g(t)}^2, \end{aligned}$$

the evolution of the entropy is given by

$$(4-3) \quad \begin{aligned} \frac{d}{dt} \mathcal{F}(g(t), u(t), f(t)) \\ &= 2 \int_M \left(|\mathcal{G}_{g(t), u(t)} + \nabla_{g(t)}^2 f(t)|_{g(t)}^2 \right. \\ &\quad \left. + 2|\Delta_{g(t)} u(t) - \langle du(t), df(t) \rangle_{g(t)}|_{g(t)}^2 \right) e^{-f(t)} dV_{g(t)} \\ &\geq 0. \end{aligned}$$

Remark 4.1. The system (4-2) is equivalent to

$$(4-4) \quad \begin{aligned} \frac{\partial}{\partial t} g(t) &= -2 \operatorname{Ric}_{g(t)} + 4 du(t) \otimes du(t), \\ \frac{\partial}{\partial t} u(t) &= \Delta_{g(t)} u(t), \\ \frac{\partial}{\partial t} f(t) &= -\Delta_{g(t)} f(t) - R_{g(t)} + |\nabla_{g(t)} f(t)|_{g(t)}^2 + 2|\nabla_{g(t)} u(t)|_{g(t)}^2. \end{aligned}$$

The same evolution of the entropy holds for system (4-4).

In particular, the entropy is nondecreasing and the equality holds if and only if $(g(t), u(t), f(t))$ satisfies

$$(4-5) \quad \begin{aligned} \mathcal{G}_{g(t), u(t)} + \nabla_{g(t)}^2 f(t) &= 0, \\ \Delta_{g(t)} u(t) - \langle du(t), df(t) \rangle_{g(t)} &= 0. \end{aligned}$$

Definition 4.2. The \mathcal{E} -functional is defined as

$$\mathcal{O}_+^2(M) \times C^\infty(M) \times C^\infty(M) \rightarrow \mathbb{R}, \quad (g, u, f) \mapsto \mathcal{E}(g, u, f),$$

where

$$(4-6) \quad \mathcal{E}(g, u, f) := \int_M (R_g - 2|\nabla_g u|_g^2) e^{-f} dV_g.$$

Proposition 4.3. *Under the evolution equation (4-4), one has*

$$(4-7) \quad \begin{aligned} & \frac{d}{dt} \mathcal{E}(g(t), u(t), f(t)) \\ &= 2 \int_M |\mathcal{J}_{g(t), u(t)}|_{g(t)}^2 e^{-f(t)} dV_{g(t)} + 4 \int_M |\Delta_{g(t)} u(t)|_{g(t)}^2 e^{-f(t)} dV_{g(t)}. \end{aligned}$$

Proof. Since $S_{g(t), u(t)} = R_{g(t)} - 2|\nabla_{g(t)} u(t)|_{g(t)}^2$ and

$$\begin{aligned} \frac{\partial}{\partial t} S_{g(t), u(t)} &= \Delta_{g(t)} S_{g(t), u(t)} + 2|\mathcal{J}_{g(t), u(t)}|_{g(t)}^2 + 4|\Delta_{g(t)} u(t)|_{g(t)}^2, \\ \frac{\partial}{\partial t} dV_{g(t)} &= -S_{g(t), u(t)} dV_{g(t)}, \end{aligned}$$

we have

$$\begin{aligned} & \frac{d}{dt} \mathcal{E}(g(t), u(t), f(t)) \\ &= \int_M \left(\frac{\partial}{\partial t} S_{g(t), u(t)} \right) e^{-f(t)} dV_{g(t)} + \int_M S_{g(t), u(t)} \frac{\partial}{\partial t} (e^{-f(t)} dV_{g(t)}) \\ &= \int_M (\Delta_{g(t)} S_{g(t), u(t)} + 2|\mathcal{J}_{g(t), u(t)}|_{g(t)}^2 + 4|\Delta_{g(t)} u(t)|_{g(t)}^2) e^{-f(t)} dV_{g(t)} \\ & \quad + \int_M S_{g(t), u(t)} \left(-\frac{\partial}{\partial t} f(t) - S_{g(t), u(t)} \right) e^{-f(t)} dV_{g(t)} \\ &= 2 \int_M |\mathcal{J}_{g(t), u(t)}|_{g(t)}^2 e^{-f(t)} dV_{g(t)} + 4 \int_M |\Delta_{g(t)} u(t)|_{g(t)}^2 e^{-f(t)} dV_{g(t)} \\ & \quad - \int_M S_{g(t), u(t)} \left(\Delta_{g(t)} f(t) - |\nabla_{g(t)} f(t)|_{g(t)}^2 + \frac{\partial}{\partial t} f(t) + S_{g(t), u(t)} \right) e^{-f(t)} dV_{g(t)}, \end{aligned}$$

which implies (4-7). □

Definition 4.4. For any $k \geq 1$ we define

$$(4-8) \quad \mathcal{F}_k(g, u, f) := \int_M (kR_g + |\nabla_g f|_g^2 - 2k|\nabla_g u|_g^2) e^{-f} dV_g.$$

Using the definition, it is easy to show that

$$(4-9) \quad \mathcal{F}_k(g, u, f) = (k-1)\mathcal{E}(g, u, f) + \mathcal{F}(g, u, f).$$

When $k = 1$, this is the \mathcal{F} -functional.

Theorem 4.5. *Under the evolution equation (4-4), one has*

$$(4-10) \quad \begin{aligned} & \frac{d}{dt} \mathfrak{F}_k(g(t), u(t), f(t)) \\ &= 2(k-1) \int_M |\mathcal{G}_{g(t), u(t)}|_{g(t)}^2 e^{-f(t)} dV_{g(t)} + 2 \int_M |\mathcal{G}_{g(t), u(t)} \\ & \quad + \nabla_{g(t)}^2 f(t)|_{g(t)}^2 e^{-f(t)} dV_{g(t)} + 4(k-1) \int_M |\Delta_{g(t)} u(t)|_{g(t)}^2 e^{-f(t)} dV_{g(t)} \\ & \quad + 4 \int_M |\Delta_{g(t)} u(t) - \langle du(t), df(t) \rangle_{g(t)}|_{g(t)}^2 e^{-f(t)} dV_{g(t)}. \end{aligned}$$

Furthermore, the monotonicity is strict unless $g(t)$ is Ricci-flat, $u(t)$ is constant, and $f(t)$ is constant.

Proof. It immediately follows from (4-3) and (4-7). \square

Set

$$(4-11) \quad \mu_k(g, u) := \inf \left\{ \mathfrak{F}_k(g, u, f) \mid f \in C^\infty(M), \int_M e^{-f} dV_g = 1 \right\}.$$

Then $\mu_k(g, u)$ is the lowest eigenvalue of $-4\Delta_g + k(R_g - 2|\nabla_g u|_g^2)$.

5. Compact steady harmonic-Ricci breathers

In this section we give an alternative proof on some results on compact steady harmonic-Ricci breathers that were proved in [List 2006; Müller 2012].

Definition 5.1. A solution $(g(t), u(t))$ of the harmonic-Ricci flow (1-1)–(1-2) is called a *harmonic-Ricci breather* if there exist $t_1 < t_2$, a diffeomorphism $\psi : M \rightarrow M$, and a constant $\alpha > 0$ such that

$$g(t_2) = \alpha \psi^* g(t_1), \quad u(t_2) = \psi^* u(t_1).$$

The cases $\alpha < 1$, $\alpha = 1$, and $\alpha > 1$, correspond to *shrinking*, *steady*, and *expanding harmonic-Ricci breathers*.

Theorem 5.2. *If $(g(t), u(t))$ is a solution of the harmonic-Ricci flow on a compact Riemannian manifold M , the lowest eigenvalue $\mu_k(g(t), u(t))$ of the operator $-4\Delta_{g(t)} + k(R_{g(t)} - 2|\nabla_{g(t)} u(t)|_{g(t)}^2)$ is nondecreasing under the harmonic-Ricci flow. The monotonicity is strict unless $g(t)$ is Ricci-flat and $u(t)$ is constant.*

Proof. The proof is similar to that given in [Li 2007]. For any $t_1 < t_2$, suppose that

$$\mu_k(g(t_2), u(t_2)) = \mathfrak{F}_k(g(t_2), u(t_2), f_k(t_2))$$

for some smooth function $f_k(x)$. Being an initial value, $f_k(x) = f_k(x, t_2)$ for some smooth function $f_k(x, t)$ satisfying the evolution equation (4-4). The monotonicity

formula (4-10) implies $\mu_k(g(t_2), u(t_2)) \geq \mathcal{F}_k(g(t_1), u(t_1), f_k(t_1)) \geq \mu_k(g(t_1), u(t_1))$. This completes the proof. \square

Corollary 5.3. *On a compact Riemannian manifold, the lowest eigenvalues of $-\Delta_{g(t)} + (1/2)(R_{g(t)} - 2|\nabla_{g(t)}u(t)|_{g(t)}^2)$ are nondecreasing under the harmonic-Ricci flow.*

Proof. Since $\mu_2(g(t), u(t))/4$ is the lowest eigenvalue of this operator, the result immediately follows from Theorem 5.2. \square

Corollary 5.4. *There is no compact steady harmonic-Ricci breather unless the manifold $(M, g(t))$ is Ricci-flat and u is a constant.*

Proof. If $(g(t), u(t))$ is a steady harmonic-Ricci breather, then, for $t_1 < t_2$ given in the definition, we have

$$\mu_k(g(t_1), u(t_1)) = \mu_k(g(t_2), u(t_2)).$$

Hence, using Theorem 5.2, for any $t \in [t_1, t_2]$, we must have

$$\frac{d}{dt}\mu_k(g(t), u(t)) \equiv 0.$$

Thus $(M, g(t))$ is Ricci-flat and $u(t)$ is constant. \square

6. Compact expanding harmonic-Ricci breathers

Inspired by [Li 2007], we define a new functional

$$\mathcal{O}_+^2(M) \times C^\infty(M) \times C^\infty(\mathbb{R}) \times C^\infty(M) \rightarrow \mathbb{R}, \quad (g, u, \tau, f) \mapsto \mathcal{W}_+(g, u, \tau, f),$$

where $(\tau = \tau(t), t \in \mathbb{R})$.

$$(6-1) \quad \mathcal{W}_+(g, u, \tau, f) := \tau^2 \int_M \left(R_g + \frac{n}{2\tau} + \Delta_g f - 2|\nabla_g u|_g^2 \right) e^{-f} dV_g.$$

Similarly, we define a family of functionals

$$(6-2) \quad \mathcal{W}_{+,k}(g, u, \tau, f) := \tau^2 \int_M \left(k \left(R_g + \frac{n}{2\tau} \right) + \Delta_g f - 2k|\nabla_g u|_g^2 \right) e^{-f} dV_g.$$

It's clear that $\mathcal{W}_{+,1}(g, u, \tau, f) = \mathcal{W}_+(g, u, \tau, f)$.

Lemma 6.1. *One has*

$$\begin{aligned} \mathcal{W}_+(g, u, \tau, f) &= \tau^2 \mathcal{F}(g, u, f) + \frac{n}{2} \tau \int_M e^{-f} dV_g, \\ \mathcal{W}_{+,k}(g, u, \tau, f) &= \tau^2 \mathcal{F}_k(g, u, f) + \frac{kn}{2} \tau \int_M e^{-f} dV_g, \\ \mathcal{W}_{+,k}(g, u, \tau, f) &= \mathcal{W}_+(g, u, \tau, f) + (k-1) \left(\tau^2 \mathcal{E}(g, u, f) + \frac{n}{2} \tau \int_M e^{-f} dV_g \right). \end{aligned}$$

Proof. Since $\Delta(e^{-f}) = (-\Delta f + |\nabla f|^2)e^{-f}$, it follows that

$$\begin{aligned} \mathfrak{W}_+(g, u, \tau, f) - \tau^2 \mathfrak{F}(g, u, f) &= \frac{n}{2} \tau \int_M e^{-f} dV_g + \tau^2 \int_M (\Delta_g f - |\nabla_g f|_g^2) e^{-f} dV_g \\ &= \frac{n}{2} \tau \int_M e^{-f} dV_g + \tau^2 \int_M \Delta_g(e^{-f}) dV_g = \frac{n}{2} \tau \int_M e^{-f} dV_g. \end{aligned}$$

We can similarly prove the remaining two relations. \square

Theorem 6.2. *Under the coupled system*

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= -2 \operatorname{Ric}_{g(t)} + 4 du(t) \otimes du(t) - 2 \nabla_{g(t)}^2 f(t), \\ \frac{\partial}{\partial t} u(t) &= \Delta_{g(t)} u(t) - \langle du(t), df(t) \rangle_{g(t)}, \\ \frac{\partial}{\partial t} f(t) &= -\Delta_{g(t)} f(t) - R_{g(t)} + 2 |\nabla_{g(t)} u(t)|_{g(t)}^2, \\ \frac{d}{dt} \tau(t) &= 1, \end{aligned}$$

the first variation formula for $\mathfrak{W}_+(g(t), u(t), \tau(t), f(t))$ is

$$\begin{aligned} (6-3) \quad \frac{d}{dt} \mathfrak{W}_+(g(t), u(t), \tau(t), f(t)) &= 2\tau(t)^2 \int_M |\mathcal{S}_{g(t), u(t)} + \nabla_{g(t)}^2 f(t) + \frac{1}{2\tau(t)} g(t)|_{g(t)}^2 e^{-f(t)} dV_{g(t)} \\ &\quad + 4\tau(t)^2 \int_M |\Delta_{g(t)} u(t) - \langle du(t), df(t) \rangle_{g(t)}|_{g(t)}^2 e^{-f(t)} dV_{g(t)}, \end{aligned}$$

and the first variation formula for $\mathfrak{W}_{+,k}(g(t), u(t), \tau(t), f(t))$ is

$$\begin{aligned} (6-4) \quad \frac{d}{dt} \mathfrak{W}_{+,k}(g(t), u(t), \tau(t), f(t)) &= 2\tau(t)^2 \int_M |\mathcal{S}_{g(t), u(t)} + \nabla_{g(t)}^2 f(t) + \frac{1}{2\tau(t)} g(t)|_{g(t)}^2 e^{-f(t)} dV_{g(t)} \\ &\quad + 2(k-1)\tau(t)^2 \int_M |\mathcal{S}_{g(t), u(t)} + \frac{1}{2\tau(t)} g(t)|_{g(t)}^2 e^{-f(t)} dV_{g(t)} \\ &\quad + 4\tau(t)^2 \int_M |\Delta_{g(t)} u(t) - \langle du(t), df(t) \rangle_{g(t)}|_{g(t)}^2 e^{-f(t)} dV_{g(t)} \\ &\quad + 4(k-1)\tau(t)^2 \int_M |\Delta_{g(t)} u(t)|_{g(t)}^2 e^{-f(t)} dV_{g(t)}. \end{aligned}$$

Proof. Under this coupled system, we first observe that

$$(6-5) \quad \frac{d}{dt} \left(\int_M e^{-f(t)} dV_{g(t)} \right) = 0.$$

In fact, from $\frac{\partial}{\partial t} dV_{g(t)} = -S_{g(t),u(t)} - \Delta_{g(t)} f(t) dV_{g(t)}$ we obtain

$$\begin{aligned} \frac{d}{dt} \left(\int_M e^{-f(t)} dV_{g(t)} \right) &= \int_M \left(-\frac{\partial}{\partial t} f(t) \cdot dV_{g(t)} + \frac{\partial}{\partial t} dV_{g(t)} \right) e^{-f(t)} \\ &= \int_M [\Delta_{g(t)} f(t) + S_{g(t),u(t)} - S_{g(t),u(t)} - \Delta_{g(t)} f(t)] e^{-f(t)} dV_{g(t)} \\ &= 0. \end{aligned}$$

Lemma 6.1 and the identity (6-5) imply

$$\begin{aligned} \frac{d}{dt} \mathfrak{W}_+(g(t), u(t), \tau(t), f(t)) &= \tau(t)^2 \frac{d}{dt} \mathfrak{F}(g(t), u(t), f(t)) + 2\tau(t) \mathfrak{F}(g(t), u(t), f(t)) + \frac{n}{2} \int_M e^{-f(t)} dV_{g(t)} \\ &= 2\tau(t)^2 \int_M |\mathcal{S}_{g(t),u(t)} + \nabla_{g(t)}^2 f(t)|_{g(t)}^2 e^{-f(t)} dV_{g(t)} \\ &\quad + 4\tau(t)^2 \int_M |\Delta_{g(t)} u(t) - \langle du(t), df(t) \rangle_{g(t)}|^2 e^{-f(t)} dV_{g(t)} \\ &\quad + 2\tau(t) \int_M (S_{g(t),u(t)} + |\nabla_{g(t)} f(t)|_{g(t)}^2) e^{-f(t)} dV_{g(t)} + \frac{n}{2} \int_M e^{-f(t)} dV_{g(t)}, \end{aligned}$$

which is (6-3). Using **Lemma 6.1** and the same method, we can prove (6-4). \square

Remark 6.3. Under the coupled system

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= -2 \operatorname{Ric}_{g(t)} + 4 du(t) \otimes du(t), \\ \frac{\partial}{\partial t} u(t) &= \Delta_{g(t)} u(t), \\ \frac{\partial}{\partial t} f(t) &= -\Delta_{g(t)} f(t) + |\nabla_{g(t)} f(t)|_{g(t)}^2 - R_{g(t)} + 2|\nabla_{g(t)} u(t)|_{g(t)}^2, \\ \frac{d}{dt} \tau(t) &= 1, \end{aligned}$$

the same formulas (6-3) and (6-4) hold for \mathfrak{W}_+ and $\mathfrak{W}_{+,k}$.

Define

$$(6-6) \quad \mu_+(g, u, \tau) := \inf \left\{ \mathfrak{W}_+(g, u, \tau, f) \mid f \in C^\infty(M), \int_M e^{-f} dV_g = 1 \right\}.$$

Lemma 6.4. For any $\alpha > 0$, one has

$$(6-7) \quad \mu_+(\alpha g, u, \alpha \tau) = \alpha \mu_+(g, u, \tau).$$

Proof. Set $\bar{g} := \alpha g$; then $R_{\bar{g}} = \alpha^{-1} R_g$, $\Delta_{\bar{g}} f = \alpha^{-1} \Delta_g f$, and $|\nabla_{\bar{g}} u|_{\bar{g}}^2 = \alpha^{-1} |\nabla_{g(t)} u|_g^2$. Hence

$$\begin{aligned} \mathcal{W}_+(\bar{g}, u, \alpha\tau, f) &= \alpha^2\tau^2 \int_M \left(R_{\bar{g}} + \frac{n}{2\alpha\tau} + \Delta_{\bar{g}}f - 2|\nabla_{\bar{g}}u|_{\bar{g}}^2 \right) e^{-f} dV_{\bar{g}} \\ &= \alpha\tau^2 \int_M \left(R_g + \frac{n}{2\tau} + \Delta_g f - 2|\nabla_{g(t)}u|_g^2 \right) \alpha^{n/2} e^{-f} dV_g. \end{aligned}$$

Since $f \mapsto f - (n/2) \ln \alpha$ is one-to-one and onto, by taking the infimum, we derive $\mu_+(\alpha g, u, \alpha\tau) = \alpha\mu_+(g, u, \tau)$. □

Definition 6.5. A solution $(g(t), u(t))$ of the harmonic-Ricci flow is called a *harmonic-Ricci soliton* if there exists a one-parameter family of diffeomorphisms $\psi_t : M \rightarrow M$, satisfying $\psi_0 = \text{id}_M$, and a positive scaling function $\alpha(t)$ such that

$$g(t) = \alpha(t)\psi_t^*g(0), \quad u(t) = \psi_t^*u(0).$$

The cases $(\partial/\partial t)\alpha(t) = \dot{\alpha} < 0$, $\dot{\alpha} = 0$, and $\dot{\alpha} > 0$ correspond to *shrinking*, *steady*, and *expanding harmonic-Ricci solitons*, respectively. If the diffeomorphisms ψ_t are generated by a (possibly time-dependent) vector field $X(t)$ that is the gradient of some function $f(t)$ on M , the soliton is called a *gradient harmonic-Ricci soliton* and f is called the *potential of the harmonic-Ricci soliton*.

Müller [2012] showed that if $(g(t), u(t))$ is a gradient harmonic-Ricci soliton with potential f ,

$$\begin{aligned} 0 &= \text{Ric}_{g(t)} - 2du(t) \otimes du(t) + \nabla_{g(t)}^2 f(t) + cg(t), \\ 0 &= \Delta_{g(t)}u(t) - \langle \nabla_{g(t)}u(t), \nabla_{g(t)}f(t) \rangle_{g(t)} \end{aligned}$$

for some constant c .

Corollary 6.6. *There is no expanding breather on compact Riemannian manifolds other than expanding gradient harmonic-Ricci solitons.*

Proof. The proof is similar to that given in [Li 2007]. Suppose there is an expanding breather on a compact Riemannian manifold M . Then, by definition, we have

$$g(t_2) = \alpha\Phi^*g(t_1), \quad u(t_2) = \Phi^*u(t_1)$$

for some $t_1 < t_2$, where Φ be a diffeomorphism and the constant $\alpha > 1$. Let $f_+(x)$ be a smooth function where $\mathcal{W}_+(g(t_2), u(t_2), \tau(t_2), f(t_2))$ attains its minimum. Then there exists a smooth function $f_+(x, t) : M \times [t_1, t_2] \rightarrow \mathbb{R}$ with initial value $f_+(x, t_2) = f_+(x)$ that satisfies the coupled system in Remark 6.3. Define a linear function

$$\tau : [t_1, t_2] \rightarrow (0, +\infty), \quad \tau(t_2) = T + t_2$$

where T is a constant. By the monotonicity formula, we have

$$\begin{aligned}\mu_+(g(t_2), u(t_2), \tau(t_2)) &= \mathfrak{W}_+(g(t_2), u(t_2), \tau(t_2), f_+(t_2)) \\ &\geq \mathfrak{W}_+(g(t_1), u(t_1), \tau(t_1), f_+(t_1)) \\ &\geq \mu_+(g(t_1), u(t_1), \tau(t_1)).\end{aligned}$$

Lemma 6.4 and the diffeomorphic invariant property of the functionals shows

$$\mu_+(g(t_1), u(t_1), \tau(t_1)) \leq \alpha \mu_+(g(t_1), u(t_1), \tau(t_1)),$$

which yields

$$\mu_+(g(t_1), u(t_1), \tau(t_1)) \geq 0,$$

since $\alpha > 1$.

If we impose an additional condition $\tau(t_2) = \alpha \tau(t_1)$ and $\tau(t_1) = T + t_1$, we have

$$\tau(t) = \frac{\alpha(t - t_1) - (t - t_2)}{\alpha - 1}, \quad T = \frac{t_2 - \alpha t_1}{\alpha - 1}.$$

Then

$$\frac{\tau(t_2)^{n/2}}{V_{g(t_2)}} = \frac{[\alpha(t_2 - t_1)/(\alpha - 1)]^{n/2}}{\alpha^{n/2} V_{g(t_1)}} = \frac{\tau(t_1)^{n/2}}{V_{g(t_1)}}.$$

The mean value theorem tells us that there exists a time $\bar{t} \in [t_1, t_2]$ with

$$\begin{aligned}0 &= \left. \frac{d}{dt} \log \frac{\tau(t)^{n/2}}{V_{g(t)}} \right|_{t=\bar{t}} \\ &= \frac{V_{g(\bar{t})}}{\tau(\bar{t})^{n/2}} \cdot \frac{(n/2)\tau(\bar{t})^{n/2-1} V_{g(\bar{t})} - \tau(\bar{t})^{n/2} (d/dt)|_{t=\bar{t}} V_{g(t)}}{V_{g(\bar{t})}^2} \\ &= \frac{n}{2\tau(\bar{t})} - \frac{1}{V_{g(\bar{t})}} \left. \frac{\partial}{\partial t} \right|_{t=\bar{t}} V_{g(\bar{t})}.\end{aligned}$$

From the evolution equation for the volume element $dV_{g(t)}$, we have

$$\frac{d}{dt} V_{g(t)} = \int_M \frac{\partial}{\partial t} dV_{g(t)} = \int_M (-S_{g(t), u(t)} - \Delta_{g(t)} f(t)) dV_{g(t)} = - \int_M S_{g(t), u(t)} dV_{g(t)}.$$

Putting these together yields

$$0 = \frac{n}{2\tau(\bar{t})} + \frac{1}{V_{g(\bar{t})}} \int_M S_{g(\bar{t}), u(\bar{t})} dV_{g(\bar{t})} = \frac{1}{V_{g(\bar{t})}} \int_M \left(S_{g(\bar{t}), u(\bar{t})} + \frac{n}{2\tau(\bar{t})} \right) dV_{g(\bar{t})}.$$

If we set $\bar{f} = \log V_{g(\bar{t})}$,

$$0 = \mathfrak{W}_+(g(\bar{t}), u(\bar{t}), \tau(\bar{t}), \bar{f}) \geq \mu_+(g(\bar{t}), u(\bar{t}), \tau(\bar{t})).$$

By the monotonicity of μ_+ we obtain

$$0 \leq \mu_+(g(t_1), u(t_1), \tau(t_1)) \leq \mu_+(g(\bar{t}), u(\bar{t}), \tau(\bar{t})) \leq 0$$

Hence $\mu_+(g(t_1), u(t_1), \tau(t_1)) = \mu_+(g(t_2), u(t_2), \tau(t_2)) = 0$ and $\mathcal{W}_+ = 0$ on the interval $[t_1, t_2]$. This indicates that the first variation of \mathcal{W}_+ must vanish. So the expanding breather is a gradient soliton, that is,

$$\mathcal{G}_{g(t),u(t)} + \nabla_{g(t)}^2 f(t) + \frac{1}{2\tau(t)}g(t) = 0.$$

Moreover, in this case $\Delta_{g(t)}u(t) = \langle du(t), df(t) \rangle_{g(t)}$. □

Because of (6-7), we define

$$(6-8) \quad \mu_{+,k}(g, u, \tau) := \inf \left\{ \mathcal{W}_{+,k}(g, u, \tau, f) \mid f \in C^{+\infty}(M), \int_M e^{-f} dV_g = 1 \right\}.$$

Due to Lemma 6.4, we still have

$$(6-9) \quad \mu_{+,k}(\alpha g, u, \alpha \tau) = \alpha \mu_{+,k}(g, u, \tau).$$

Corollary 6.7. *If $(g(t), u(t))$ is an expanding harmonic-Ricci breather on compact Riemannian manifolds, M is an Einstein manifold and $u(t)$ is constant.*

Proof. Using the same method as in Corollary 6.6 and $\mu_{+,k}$, we can show that the first variation of $\mathcal{W}_{+,k}$ must vanish. Hence, from (6-4), one has

$$\begin{aligned} \mathcal{G}_{g(t),u(t)} + \nabla_{g(t)}^2 f(t) + \frac{1}{2\tau(t)}g(t) &= 0, \\ \mathcal{G}_{g(t),u(t)} + \frac{1}{2\tau(t)}g(t) &= 0, \\ \Delta_{g(t)}u(t) &= \langle du(t), df(t) \rangle_{g(t)}, \\ \Delta_{g(t)}u(t) &= 0. \end{aligned}$$

The above four equations can be reduced to the coupled equation

$$\mathcal{G}_{g(t),u(t)} + \frac{1}{2\tau(t)}g(t) = 0 = \Delta_{g(t)}u(t),$$

which indicates that $u(t)$ is a constant and $\text{Ric}_{g(t)} = -(1/(2\tau(t)))g(t)$. □

7. Eigenvalues of the Laplacian under the harmonic-Ricci flow

In this section we consider the eigenvalues of the Laplacian $\Delta_{g(t)}$ under the harmonic-Ricci flow

$$(7-1) \quad \frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}_{g(t)} + 4 du(t) \otimes du(t),$$

$$(7-2) \quad \frac{\partial}{\partial t} u(t) = \Delta_{g(t)} u(t).$$

Suppose that $\lambda(t)$, which is a function of time t only, is an eigenvalue of the Laplacian $\Delta_{g(t)}$ with an eigenfunction $f(t) = f(x, t)$, that is,

$$(7-3) \quad -\Delta_{g(t)} f(t) = \lambda(t) f(t).$$

Taking the derivative with respect to t , we get

$$-\left(\frac{\partial}{\partial t} \Delta_{g(t)}\right) f(t) - \Delta_{g(t)} \left(\frac{\partial}{\partial t} f(t)\right) = \left(\frac{d}{dt} \lambda(t)\right) f(t) + \lambda(t) \frac{\partial}{\partial t} f(t).$$

Integrating the above equation with f yields

$$\begin{aligned} - \int_M f(t) \left(\frac{\partial}{\partial t} \Delta_{g(t)}\right) f(t) dV_{g(t)} - \int_M f(t) \Delta_{g(t)} \left(\frac{\partial}{\partial t} f(t)\right) dV_{g(t)} \\ = \frac{d}{dt} \lambda(t) \cdot \int_M f(t)^2 dV_{g(t)} + \lambda(t) \int_M f(t) \frac{\partial}{\partial t} f(t) dV_{g(t)}. \end{aligned}$$

Since

$$\begin{aligned} - \int_M f(t) \Delta \left(\frac{\partial}{\partial t} f(t)\right) dV_{g(t)} &= - \int_M \Delta_{g(t)} f(t) \cdot \frac{\partial}{\partial t} f(t) dV_{g(t)} \\ &= \lambda(t) \int_M f(t) \frac{\partial}{\partial t} f(t) dV_{g(t)}, \end{aligned}$$

it follows that

$$(7-4) \quad \frac{d}{dt} \lambda(t) \cdot \int_M f(t)^2 dV_{g(t)} = - \int_M f(t) \left(\frac{\partial}{\partial t} \Delta_{g(t)}\right) f(t) dV_{g(t)}.$$

If we set $v_{ij} = -2R_{ij} + 4\partial_i u \partial_j u$,

$$\frac{\partial}{\partial t} \Gamma_{ij}^k = \frac{1}{2} g^{k\ell} (\partial_i v_{\ell j} + \partial_j v_{i\ell} - \partial_\ell v_{ij}).$$

We temporarily omit all subscripts t . Multiplying with g^{ij} on both sides, we obtain

$$\begin{aligned} g^{ij} \frac{\partial}{\partial t} \Gamma_{ij}^k &= \frac{1}{2} g^{kl} (2\nabla^i v_{li} - \nabla_l (g^{ij} v_{ij})) = g^{kl} \nabla^i v_{il} + \nabla^k S \\ &= g^{kl} \nabla^i (-2R_{il} + 4\nabla_i u \nabla_l u) + \nabla^k (R - 2|\nabla u|^2) \end{aligned}$$

$$\begin{aligned}
&= -\nabla^k R + 4\Delta u \cdot \nabla^k u + 4\nabla_i u \cdot \nabla^i \nabla^k u + \nabla^k R - 4\nabla^k \nabla^i u \cdot \nabla_i u \\
&= 4\Delta u \cdot \nabla^k u.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial}{\partial t}(\Delta f) &= \frac{\partial}{\partial t}(g^{ij} \nabla_i \nabla_j f) \\
&= \left(\frac{\partial}{\partial t} g^{ij} \right) \nabla_i \nabla_j f + g^{ij} \left[\partial_i \partial_j \frac{\partial f}{\partial t} - \left(\frac{\partial}{\partial t} \Gamma_{ij}^k \right) \partial_k f - \Gamma_{ij}^k \partial_k \frac{\partial f}{\partial t} \right] \\
&= \left(\frac{\partial}{\partial t} g^{ij} \right) \nabla_i \nabla_j f + \Delta_{g(t)} \left(\frac{\partial}{\partial t} f \right) - g^{ij} \left(\frac{\partial}{\partial t} \Gamma_{ij}^k \right) \nabla_k f \\
&= (2R_{ij} - 4\nabla_i u \nabla_j u) \nabla^i \nabla^j f - 4\Delta u \cdot \nabla^k u \nabla_k f + \Delta_{g(t)} \left(\frac{\partial}{\partial t} f \right).
\end{aligned}$$

Plugging this into (7-4), we derive

$$\begin{aligned}
&\frac{d}{dt} \lambda(t) \cdot \int_M f(t)^2 dV_{g(t)} \\
&= -2 \int_M R_{ij} \nabla^i \nabla^j f dV + 4 \int_M f \nabla^i u \nabla^j u \nabla_i \nabla_j f dV + 4 \int_M f \Delta u \cdot \nabla^k u \nabla_k f dV.
\end{aligned}$$

The first term can be rewritten as

$$\begin{aligned}
-2 \int_M f R_{ij} \nabla^i \nabla^j f dV &= \int_M \nabla^i (2f R_{ij}) \nabla^j f dV \\
&= 2 \int_M (\nabla^i f \cdot R_{ij} + f \cdot \nabla^i R_{ij}) \nabla^j f dV \\
&= 2 \int_M R_{ij} \nabla^i f \nabla^j f dV + \int_M f \nabla_j R \nabla^j f dV \\
&= 2 \int_M R_{ij} \nabla^i f \nabla^j f dV - \int_M R \nabla_j (f \nabla^j f) dV \\
&= \lambda \int_R f^2 dV - \int_M R |\nabla f|^2 dV + 2 \int_M R_{ij} \nabla^i f \nabla^j f dV.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left(\frac{d}{dt} \lambda(t) \right) \int_M f(t)^2 dV_{g(t)} \\
&= \lambda(t) \int_M R_{g(t)} f(t)^2 dV_{g(t)} + 2 \int_M R_{ij} \nabla^i f \nabla^j f dV - \int_M R_{g(t)} |\nabla_{g(t)} f(t)|_{g(t)}^2 dV_{g(t)} \\
&\quad + 4 \int_M f (\nabla^i u \nabla^j u \nabla_i \nabla_j f + \Delta u \nabla^k u \nabla_k f) dV.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \int_M f \nabla^i u \nabla^j u \nabla_i \nabla_j f \, dV \\
&= - \int_M \nabla_i (f \nabla^i u \nabla^j u) \nabla_j f \, dV \\
&= - \int_M (\nabla_i f \nabla^i u \nabla^j u + f \Delta u \nabla^j u + f \nabla^i u \nabla_i \nabla^j u) \nabla_j f \, dV \\
&= - \int_M f \Delta u \langle \nabla u, \nabla f \rangle \, dV - \int_M \nabla^i u \nabla^j u \nabla_i f \nabla_j f \, dV - \int_M f \nabla^i u \nabla^j f \nabla_i \nabla_j u \, dV
\end{aligned}$$

and therefore

$$\begin{aligned}
\frac{d}{dt} \lambda(t) \int_M f(t)^2 \, dV_{g(t)} &= \lambda(t) \int_M R_{g(t)} f(t)^2 \, dV_{g(t)} - 4 \int_M f \nabla^i u \nabla^j f \nabla_i \nabla_j u \, dV \\
&\quad + 2 \int_M S_{ij} \nabla^i f \nabla_j f \, dV - \int_M R_{g(t)} |\nabla_{g(t)} f(t)|_{g(t)}^2 \, dV_{g(t)}.
\end{aligned}$$

The last term here can be simplified as follows:

$$\begin{aligned}
& - \int_M f \nabla^i u \nabla^j f \nabla_i \nabla_j u \, dV \\
&= \int_M \nabla^j (f \nabla_i u \nabla_j f) \nabla^i u \, dV \\
&= \int_M (\nabla^j f \nabla_i u \nabla_j f + f \nabla^j \nabla_i u \nabla_j f + f \nabla_i u \Delta f) \nabla^i u \, dV \\
&= \int_M |\nabla u|^2 |\nabla f|^2 \, dV + \int_M f \Delta f |\nabla u|^2 \, dV + \int_M f \nabla^i u \nabla^j f \nabla_i \nabla_j u \, dV.
\end{aligned}$$

Consequently,

$$-2 \int_M f \nabla^i u \nabla^j f \nabla_i \nabla_j u \, dV = \int_M |\nabla u|^2 |\nabla f|^2 \, dV - \lambda \int_M f^2 |\nabla u|^2 \, dV.$$

Therefore we derive the following.

Theorem 7.1. *If $(g(t), u(t))$ is a solution of the harmonic-Ricci flow on a compact Riemannian manifold M and $\lambda(t)$ denotes the eigenvalue of the Laplacian $\Delta_{g(t)}$, then*

$$\begin{aligned}
(7-5) \quad & \frac{d}{dt} \lambda(t) \cdot \int_M f(t)^2 \, dV_{g(t)} \\
&= \lambda(t) \int_M S_{g(t), u(t)} f(t)^2 \, dV_{g(t)} - \int_M S_{g(t), u(t)} |\nabla_{g(t)} f(t)|_{g(t)}^2 \, dV_{g(t)} \\
&\quad + 2 \int_M \langle \mathcal{S}_{g(t), u(t)}, df(t) \otimes df(t) \rangle \, dV_{g(t)}.
\end{aligned}$$

We set

$$(7-6) \quad S_{\min}(0) := \min_{x \in M} S(x, 0).$$

Theorem 7.2. *Let $(g(t), u(t))_{t \in [0, T]}$ be a solution of the harmonic-Ricci flow on a compact Riemannian manifold M and let $\lambda(t)$ denote the eigenvalue of the Laplacian $\Delta_{g(t)}$. Suppose that $\mathcal{S}_{g(t), u(t)} - \alpha S_{g(t), u(t)} g(t) \geq 0$ along the harmonic-Ricci flow for some $\alpha \geq \frac{1}{2}$.*

- (1) *If $S_{\min}(0) \geq 0$, $\lambda(t)$ is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.*
- (2) *If $S_{\min}(0) > 0$, the quantity*

$$\left(1 - \frac{2}{n} S_{\min}(0) t\right)^{n\alpha} \lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for $T \leq n/(2S_{\min}(0))$.

- (3) *If $S_{\min}(0) < 0$, the quantity*

$$\left(1 - \frac{2}{n} S_{\min}(0) t\right)^{n\alpha} \lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.

Proof. By [Theorem 7.1](#), we have

$$\frac{d}{dt} \lambda(t) \geq \frac{\int_M S_{g(t), u(t)} f(t)^2 dV_{g(t)}}{\int_M f(t)^2 dV_{g(t)}} \lambda(t) + (2\alpha - 1) \frac{\int_M S_{g(t), u(t)} |\nabla_{g(t)} f(t)|_{g(t)}^2}{\int_M f(t)^2 dV_{g(t)}}.$$

By definition we have $-f(t)\Delta_{g(t)} = \lambda(t)f(t)$. Integrating both sides yields that $\lambda(t) \geq 0$. Since

$$\frac{\partial}{\partial t} S_{g(t), u(t)} = \Delta_{g(t)} S_{g(t), u(t)} + 2|\mathcal{S}_{g(t), u(t)}|_{g(t)}^2 + 4|\Delta_{g(t)} u(t)|_{g(t)}^2$$

and $|\mathcal{S}_{g(t), u(t)}|^2 \geq (1/n)S_{g(t), u(t)}^2$, it follows that

$$\frac{\partial}{\partial t} S_{g(t), u(t)} \geq \Delta_{g(t)} S_{g(t), u(t)} + \frac{2}{n} S_{g(t), u(t)}^2.$$

The corresponding ODE

$$\frac{d}{dt} a(t) = \frac{2}{n} a(t)^2, \quad a(t) = S_{\min}(0)$$

has the solution

$$a(t) = \frac{S_{\min}(0)}{1 - (2/n)S_{\min}(0)t}.$$

Then the maximum principle implies $S_{g(t),u(t)} \geq a(t)$ and hence, using the assumption that $2\alpha - 1 \geq 0$,

$$\frac{d}{dt}\lambda(t) \geq a(t)\lambda(t) + (2\alpha - 1)a(t) \frac{\int_M |\nabla_{g(t)} f(t)|_{g(t)}^2 dV_{g(t)}}{\int_M f(t)^2 dV_{g(t)}}.$$

By integration by parts, we note that

$$\int_M |\nabla f|^2 dV = - \int_M f \cdot \Delta f dV = \lambda \int_M f^2 dV,$$

which shows that

$$\frac{d}{dt}\lambda(t) \geq a(t)\lambda(t) + (2\alpha - 1)a(t)\lambda = 2\alpha a(t)\lambda(t)$$

and

$$\frac{d}{dt} \left(\lambda(t) \cdot \exp \left(-2\alpha \int_0^t a(\tau) d\tau \right) \right) \geq 0.$$

This inequality clearly implies the desired result. If $S_{\min}(0) \geq 0$, by the nonnegativity of $\mathcal{S}_{g(t)}$ preserved along the harmonic-Ricci flow, we conclude that $d\lambda(t)/dt \geq 0$. \square

Corollary 7.3. *Let $(g(t), u(t))_{t \in [0, T]}$ be a solution of the harmonic-Ricci flow on a compact Riemannian surface Σ and let $\lambda(t)$ denote the eigenvalue of the Laplacian $\Delta_{g(t)}$.*

(1) *Suppose that $\text{Ric}_{g(t)} \leq \epsilon du(t) \otimes du(t)$ where*

$$(7-7) \quad \epsilon \leq 4 \frac{1 - \alpha}{1 - 2\alpha}, \quad \alpha > \frac{1}{2}.$$

(i) *If $S_{\min}(0) \geq 0$, $\lambda(t)$ is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.*

(ii) *If $S_{\min}(0) > 0$, the quantity*

$$(1 - S_{\min}(0)t)^{2\alpha} \lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for $T \leq 1/S_{\min}(0)$.

(iii) *If $S_{\min}(0) < 0$, the quantity*

$$(1 - S_{\min}(0)t)^{2\alpha} \lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.

(2) *Suppose that*

$$(7-8) \quad |\nabla_{g(t)} u(t)|_{g(t)}^2 g(t) \geq 2du(t) \otimes du(t).$$

(i) *If $S_{\min}(0) \geq 0$, $\lambda(t)$ is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.*

(ii) If $S_{\min}(0) > 0$, the quantity

$$(1 - S_{\min}(0)t)\lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for $T \leq 1/S_{\min}(0)$.

(iii) If $S_{\min}(0) < 0$, the quantity

$$(1 - S_{\min}(0)t)\lambda(t)$$

is nondecreasing along the harmonic-Ricci flow for any $t \in [0, T]$.

Proof. As above, we always omit subscripts t . In the surface case, we have $R_{ij} = \frac{1}{2}Rg_{ij}$. Then

$$\begin{aligned} T_{ij} &:= S_{ij} - \alpha Sg_{ij} = \frac{R}{2}g_{ij} - 2\nabla_i u \nabla_j u - \alpha(R - 2|\nabla u|^2)g_{ij} \\ &= \left(\frac{1}{2} - \alpha\right)Rg_{ij} - 2\nabla_i u \nabla_j u + 2\alpha|\nabla u|^2g_{ij}. \end{aligned}$$

For any vector $V = (V^i)$, we calculate

$$\begin{aligned} T_{ij}V^iV^j &= \left(\frac{1}{2} - \alpha\right)R|V|^2 - 2(\nabla_i u V^i)^2 + 2\alpha|\nabla u|^2|V|^2 \\ &\geq \left(\frac{1}{2} - \alpha\right)R|V|^2 - 2|\nabla u|^2|V|^2 + 2\alpha|\nabla u|^2|V|^2. \end{aligned}$$

If $R_{ij} \leq \epsilon \nabla_i u \nabla_j u$, then $T_{ij}V^iV^j = [(\frac{1}{2} - \alpha)\epsilon - 2 + 2\alpha]|\nabla u|^2|V|^2 \geq 0$.

For the second case, we note that

$$\begin{aligned} T_{ij}V^iV^j &= R_{ij}V^iV^j - 2\nabla_i u V^i \nabla_j u V^j - \frac{R}{2}|V|^2 + |\nabla u|^2|V|^2 \\ &\geq R_{ij}V^iV^j - |\nabla u|^2|V|^2 - \frac{R}{2}|V|^2 + |\nabla u|^2|V|^2 = 0. \end{aligned}$$

Hence the corresponding results follow by [Theorem 7.2](#). \square

When we consider the Ricci flow, we have the following two results derived from [Corollary 7.3](#).

Corollary 7.4. *Let $(g(t))_{t \in [0, T]}$ be a solution of the Ricci flow on a compact Riemannian surface Σ and let $\lambda(t)$ denote the eigenvalue of the Laplacian $\Delta_{g(t)}$.*

- (1) If $R_{\min}(0) \geq 0$, $\lambda(t)$ is nondecreasing along the Ricci flow for any $t \in [0, T]$.
- (2) If $R_{\min}(0) > 0$, the quantity $(1 - R_{\min}(0)t)\lambda(t)$ is nondecreasing along the Ricci flow for $T \leq 1/R_{\min}(0)$.
- (3) If $R_{\min}(0) < 0$, the quantity $(1 - R_{\min}(0)t)\lambda(t)$ is nondecreasing along the Ricci flow for any $t \in [0, T]$.

Remark 7.5. Let $(g(t))_{t \in [0, T]}$ be a solution of the Ricci flow on a compact Riemannian surface Σ with nonnegative scalar curvature and let $\lambda(t)$ denote the eigenvalue of the Laplacian $\Delta_{g(t)}$. Then $\lambda(t)$ is nondecreasing along the Ricci flow for $t \in [0, T]$.

8. Eigenvalues of the Laplacian-type under the harmonic-Ricci flow

Recall that

$$(8-1) \quad \mu(g, u) = \mu_1(g, u) = \inf \left\{ \mathcal{F}(g, u, f) \mid \int_M e^{-f} dV_g = 1 \right\}.$$

We showed that $\mu(g, u)$ is the smallest eigenvalue of the operator

$$-4\Delta_g + R_g - 2|\nabla_g u|_g^2.$$

Inspired by [Cao 2007; 2008], we define a Laplacian-type operators associated with quantities g, u, c :

$$(8-2) \quad \Delta_{g,u,c} := -\Delta_g + c(R_g - 2|\nabla_g u|_g^2),$$

$$(8-3) \quad \Delta_{g,u} := \Delta_{g,u,\frac{1}{2}} = -\Delta_g + \frac{1}{2}(R_g - 2|\nabla_g u|_g^2).$$

Then $\mu(g, u)$ is the smallest eigenvalue of the operator $4\Delta_{g,u,1/4}$.

To the operator $\Delta_{g,u}$ we associate the functional

$$(8-4) \quad C^\infty(M) \rightarrow \mathbb{R}, \quad f \mapsto \lambda_{g,u}(f) := \int_M f \Delta_{g,u} f dV_g.$$

When f is an eigenfunction of the operator $\Delta_{g,u}$ with the eigenvalue λ , that is, $\Delta_{g,u} f = \lambda f$ and is normalized by $\int_M f^2 dV_g = 1$, we obtain $\lambda_{g,u}(f) = \lambda$. The next lemma will deal with the evolution equation for $\lambda(f(t))$, where $f(t)$ is an eigenfunction of $\Delta_{g(t),u(t)}$ and the couple $(g(t), u(t))$ satisfies the harmonic-Ricci flow. Set

$$(8-5) \quad v_{ij} := -2S_{ij} = -2R_{ij} + 4\partial_i u \cdot \partial_j u, \quad v := g^{ij} v_{ij}.$$

The symmetric tensor field thus obtained is denoted by $\mathcal{V}_{g(t),u(t)} = (v_{ij})$.

Lemma 8.1. *Suppose that $(g(t), u(t))$ is a solution of the harmonic-Ricci flow on a compact Riemannian manifold M and $f(t)$ is an eigenfunction of $\Delta_{g(t),u(t)}$, that is, $\Delta_{g(t),u(t)} f(t) = \lambda(t) f(t)$ (where $\lambda(t)$ is only a function of time t only), with the normalized condition*

$$\int_M f(t)^2 dV_{g(t)} = 1.$$

Then we have

$$\begin{aligned}
(8-6) \quad & \frac{d}{dt} \lambda_{g(t), u(t)}(f(t)) \\
& = \int_M f(t) (\nabla^i v_{ik} - \frac{1}{2} \nabla_k v) \nabla^k f(t) dV_{g(t)} - \int_M f^2(t) \frac{\partial}{\partial t} |\nabla_{g(t)} u(t)|_{g(t)}^2 dV_{g(t)} \\
& \quad + \int_M \left(\langle \nabla_{g(t), u(t)}, \nabla_{g(t)}^2 f(t) \rangle_{g(t)} + \frac{1}{2} \left(\frac{\partial}{\partial t} R_{g(t)} \right) f(t) \right) f(t) dV_{g(t)}.
\end{aligned}$$

Before proving the lemma, we recall a formula that is an immediate consequence of the evolution equation:

$$\begin{aligned}
(8-7) \quad & \frac{\partial}{\partial t} (\Delta_{g(t)} f) \\
& = -g^{ip} g^{jq} v_{pq} \nabla_i \nabla_j f - g^{ij} g^{k\ell} \nabla_i v_{j\ell} \nabla_k f + \frac{1}{2} \langle \nabla_{g(t)} v_{g(t)}, \nabla_{g(t)} f(t) \rangle_{g(t)}
\end{aligned}$$

where the metric $g(t)$ evolves by $\partial g_{ij} / \partial t = v_{ij}$.

Proof. Using (8-7) and integration by parts, we get

$$\begin{aligned}
& \frac{d}{dt} \lambda_{g(t), u(t)}(f(t)) \\
& = \frac{\partial}{\partial t} \int_M \left(-\Delta_{g(t)} f(t) + \left(\frac{R_{g(t)}}{2} - |\nabla_{g(t)} u(t)|_{g(t)}^2 \right) f(t) \right) f(t) dV_{g(t)} \\
& = \int_M \left(g^{ip} g^{jq} v_{pq} \nabla_i \nabla_j f + g^{ij} g^{kl} \nabla_i v_{j\ell} \nabla_k f - \frac{1}{2} \langle \nabla_{g(t)} v_{g(t)}, \nabla_{g(t)} f(t) \rangle_{g(t)} \right) f(t) dV_{g(t)} \\
& \quad + \int_M \left(-\Delta_{g(t)} \left(\frac{\partial}{\partial t} f(t) \right) + \left(\frac{R_{g(t)}}{2} - |\nabla_{g(t)} u(t)|_{g(t)}^2 \right) \frac{\partial}{\partial t} f(t) \right) \\
& \quad + \left(\frac{\partial}{\partial t} \left(\frac{1}{2} R_{g(t)} \right) - \frac{\partial}{\partial t} (|\nabla_{g(t)} u(t)|_{g(t)}^2) \right) f(t) \right) f(t) dV_{g(t)} \\
& \quad + \int_M \left(-\Delta_{g(t)} f(t) + \left(\frac{R_{g(t)}}{2} - |\nabla_{g(t)} u(t)|_{g(t)}^2 \right) f(t) \right) \frac{\partial}{\partial t} (f(t) dV_{g(t)}) \\
& = \int_M \left(g^{ip} g^{jq} v_{pq} \nabla_i \nabla_j f + \frac{1}{2} \left(\frac{\partial}{\partial t} R_{g(t)} \right) f(t) \right) f(t) dV_{g(t)} \\
& \quad + \int_M \left(g^{ij} g^{kl} \nabla_i v_{j\ell} \nabla_k f - \frac{1}{2} g^{kl} \nabla_l v \nabla_k f \right) f(t) dV_{g(t)} \\
& \quad + \int_M \Delta_{g(t), u(t)} f(t) \left(\frac{\partial}{\partial t} f(t) dV_{g(t)} + \frac{\partial}{\partial t} (f(t) dV_{g(t)}) \right) \\
& \quad \quad \quad - \int_M \frac{\partial}{\partial t} (|\nabla_{g(t)} u(t)|_{g(t)}^2) f(t)^2 dV_{g(t)}.
\end{aligned}$$

Since $f(t)$ is an eigenfunction of $\Delta_{g(t), u(t)}$, it follows that

$$\int_M \Delta_{g(t), u(t)} f(t) \left(\frac{\partial}{\partial t} f(t) dV_{g(t)} + \frac{\partial}{\partial t} (f(t) dV_{g(t)}) \right) = \lambda(t) \frac{\partial}{\partial t} \int_M f(t)^2 dV_{g(t)} = 0$$

by the normalization condition. This completes the proof. \square

Using (3-6), we find that the first term in the right side of (8-6) can be written as

$$\begin{aligned}
& \int_M \left(v_{ij} \nabla^i \nabla^j f + \frac{1}{2} \left(\frac{\partial}{\partial t} R_{g(t)} \right) f(t) \right) f(t) dV_{g(t)} \\
&= \int_M \left(-2f(t) \langle \text{Ric}_{g(t)}, \nabla_{g(t)}^2 f(t) \rangle_{g(t)} + 4f(t) \langle du(t) \otimes du(t), \nabla_{g(t)}^2 f(t) \rangle_{g(t)} \right) dV_{g(t)} \\
&\quad + \int_M \left(\left(\frac{1}{2} \Delta_{g(t)} R_{g(t)} + |\text{Ric}_{g(t)}|_{g(t)}^2 \right) f(t)^2 + 2f(t)^2 |\Delta_{g(t)} u(t)|_{g(t)}^2 \right. \\
&\quad \left. - 2f(t)^2 |\nabla_{g(t)}^2 u(t)|_{g(t)}^2 - 4f(t)^2 \langle \text{Ric}_{g(t)}, du(t) \otimes du(t) \rangle_{g(t)} \right) dV_{g(t)} \\
&= \int_M \left(-2f(t) \langle \text{Ric}_{g(t)}, \nabla_{g(t)}^2 f(t) \rangle_{g(t)} + \left(\frac{1}{2} \Delta_{g(t)} R_{g(t)} + |\text{Ric}_{g(t)}|_{g(t)}^2 \right) f(t)^2 \right) dV_{g(t)} \\
&\quad + \int_M \left(4f(t) \langle du \otimes du, \nabla_{g(t)}^2 f(t) \rangle_{g(t)} - 4f^2 \langle du(t) \otimes du(t), \text{Ric}_{g(t)} \rangle_{g(t)} \right. \\
&\quad \left. + 2f(t)^2 |\Delta_{g(t)} u(t)|_{g(t)}^2 - 2f(t)^2 |\nabla_{g(t)}^2 u(t)|_{g(t)}^2 \right) dV_{g(t)}
\end{aligned}$$

For the second term in (8-6), using the contracted Bianchi identities, one has

$$\begin{aligned}
& \int_M \left(g^{ij} \nabla_i v_{jk} - \frac{1}{2} \nabla_k v \right) \nabla^k f \cdot f(t) dV_{g(t)} \\
&= \int_M \left(g^{ij} \nabla_i (-2R_{jk} + 4\partial_j u \partial_k u) \right. \\
&\quad \left. - \frac{1}{2} \nabla_k (-2R_{g(t)} + 4|\nabla_{g(t)} u(t)|_{g(t)}^2) \right) \nabla^k f \cdot f(t) dV_{g(t)} \\
&= \int_M 4f(t) \Delta_{g(t)} u(t) \langle \nabla_{g(t)} u(t), \nabla_{g(t)} f(t) \rangle_{g(t)} dV_{g(t)} \\
&\quad + \int_M \left(4g^{ij} \nabla_j u \cdot \nabla_i \nabla_k u - 2\nabla_k |\nabla_{g(t)} u(t)|_{g(t)}^2 \right) \nabla^k f \cdot f(t) dV_{g(t)} \\
&= \int_M 4f(t) \Delta_{g(t)} u(t) \langle \nabla_{g(t)} u(t), \nabla_{g(t)} f(t) \rangle_{g(t)} dV_{g(t)}
\end{aligned}$$

where in the last step we use the identity $\nabla_k |\nabla u|^2 = 2g^{pq} \nabla_k \nabla_p u \cdot \nabla_q u$. Therefore

$$\begin{aligned}
(8-8) \quad & \frac{d}{dt} \lambda_{g(t), u(t)}(f(t)) \\
&= \int_M \left(-2f(t) \langle \text{Ric}_{g(t)}, \nabla_{g(t)}^2 f(t) \rangle_{g(t)} + \left(\frac{1}{2} \Delta_{g(t)} R_{g(t)} + |\text{Ric}_{g(t)}|_{g(t)}^2 \right) f(t)^2 \right) dV_{g(t)} \\
&\quad + 4f(t) \int_M \left(\langle du(t) \otimes du(t), \nabla_{g(t)}^2 f(t) \rangle_{g(t)} - f(t) \langle du(t) \otimes du(t), \text{Ric}_{g(t)} \rangle_{g(t)} \right. \\
&\quad \left. + 2f(t)^2 |\Delta_{g(t)} u(t)|_{g(t)}^2 - 2f(t)^2 |\nabla_{g(t)}^2 u(t)|_{g(t)}^2 \right. \\
&\quad \left. + 4f(t) \Delta_{g(t)} u(t) \langle \nabla_{g(t)} u(t), \nabla_{g(t)} f(t) \rangle_{g(t)} \right) dV_{g(t)} \\
&\quad - \int_M \left(\Delta_{g(t)} |\nabla_{g(t)} u(t)|_{g(t)}^2 - 2|\nabla_{g(t)}^2 u(t)|_{g(t)}^2 - 4|\nabla_{g(t)} u(t)|_{g(t)}^4 \right) f(t)^2 dV_{g(t)}.
\end{aligned}$$

The above evolution equation can be simplified as follows.

Theorem 8.2. *Suppose $(g(t), u(t))$ is a solution of the harmonic-Ricci flow on a compact Riemannian manifold M and $f(t)$ is an eigenfunction of $\Delta_{g(t), u(t)}$, that is, $\Delta_{g(t), u(t)} f(t) = \lambda(t) f(t)$ (where $\lambda(t)$ is only a function of time t only), with the normalized condition $\int_M f(t)^2 dV_{g(t)} = 1$. Then we have*

$$(8-9) \quad \frac{d}{dt} \lambda_{g(t), u(t)}(f(t)) = \int_M 2 \langle \mathcal{S}_{g(t)}, df(t) \otimes df(t) \rangle_{g(t)} dV_{g(t)} \\ + \int_M f(t)^2 (|\mathcal{S}_{g(t)}|_{g(t)}^2 + 2|\Delta_{g(t)} u(t)|_{g(t)}^2) dV_{g(t)}.$$

Proof. Calculate

$$\int_M 4f(t) \Delta_{g(t)} u(t) \langle \nabla_{g(t)} u(t), \nabla_{g(t)} f(t) \rangle_{g(t)} dV_{g(t)} \\ = -4 \int_M \nabla_i u [\nabla^i f \cdot \langle \nabla u, \nabla f \rangle + f(\nabla^i \langle \nabla u, \nabla f \rangle)] dV \\ = -4 \int_M |\langle \nabla u, \nabla f \rangle|^2 dV_g - 4 \int_M f \nabla_i u (\langle \nabla^i \nabla u, \nabla f \rangle + \langle \nabla u, \nabla^i \nabla f \rangle) dV.$$

By the same method, we have

$$\int_M -\Delta_{g(t)} |\nabla_{g(t)} u(t)|_{g(t)}^2 f(t)^2 dV_{g(t)} \\ = - \int_M |\nabla u|^2 (2f \Delta f + 2|\nabla f|^2) dV \\ = -2 \int_M |\nabla f|^2 |\nabla u|^2 dV - 2 \int_M f \Delta f |\nabla u|^2 dV.$$

However,

$$\int_M f \Delta f |\nabla u|^2 dV = \int_M -\nabla_i f \cdot \nabla^i (f |\nabla u|^2) dV \\ = - \int_M \nabla_i f (\nabla^i f |\nabla u|^2 + f \nabla^i |\nabla u|^2) dV \\ = - \int_M |\nabla u|^2 |\nabla f|^2 dV - \int_M f \nabla_i f \cdot \nabla^i |\nabla u|^2 dV.$$

Therefore we arrive at

$$\int_M -\Delta_{g(t)} |\nabla_{g(t)} u(t)|_{g(t)}^2 f(t)^2 dV_{g(t)} \\ = 2 \int_M f \nabla_i f \cdot \nabla^i |\nabla u|^2 dV \\ = 4 \int_M f(t) \langle du(t) \otimes df(t), \nabla_{g(t)}^2 u(t) \rangle_{g(t)} dV_{g(t)}.$$

Using the contracted Bianchi identities, we may simplify the term $\int_M \frac{1}{2} f^2 \Delta R dV$ as follows:

$$\begin{aligned}
& \int_N \frac{f(t)^2}{2} \Delta_{g(t)} R_{g(t)} dV_{g(t)} \\
&= -\frac{1}{2} \int_M \nabla_i R \cdot \nabla^i (f^2) dV \\
&= -\int_M \nabla_i R \cdot f \nabla^i f dV = -2 \int_M \nabla^k R_{ki} \cdot f \nabla^i f dV \\
&= 2 \int_M R_{ki} \nabla^k (f \nabla^j f) dV = 2 \int_M R_{ki} (\nabla^k f \cdot \nabla^j f + f \nabla^k \nabla^j f) dV \\
&= 2 \int_M \langle \text{Ric}_{g(t)}, df(t) \otimes df(t) \rangle_{g(t)} dV_{g(t)} + 2 \int_M f(t) \langle \text{Ric}_{g(t)}, \nabla_{g(t)}^2 f(t) \rangle_{g(t)} dV_{g(t)}.
\end{aligned}$$

Hence (8-8) becomes

$$\begin{aligned}
& \frac{d}{dt} \lambda_{g(t), u(t)}(f(t)) \\
&= \int_M (2 \langle \text{Ric}_{g(t)}, df(t) \otimes df(t) \rangle_{g(t)} + |\text{Ric}_{g(t)}|_{g(t)}^2 f(t)^2) dV_{g(t)} \\
&\quad + \int_M (2 |\Delta_{g(t)} u(t)|_{g(t)}^2 + 4 |\nabla_{g(t)} u(t)|_{g(t)}^4) f(t)^2 dV_{g(t)} \\
&\quad - \int_M 4 f(t)^2 \langle du(t) \otimes du(t), \text{Ric}_{g(t)} \rangle_{g(t)} dV_{g(t)} \\
&\quad - \int_M 4 |\langle \nabla_{g(t)} u(t), \nabla_{g(t)} f(t) \rangle_{g(t)}|^2 dV_{g(t)} \\
&= \int_M 2 \langle \mathcal{S}_{g(t)}, df(t) \otimes df(t) \rangle_{g(t)} dV_{g(t)} \\
&\quad + \int_M f(t)^2 (|\text{Ric}_{g(t)} - 2du(t) \otimes du(t)|_{g(t)}^2 + 2 |\Delta_{g(t)} u(t)|_{g(t)}^2) dV_{g(t)}
\end{aligned}$$

where, by definition, $S_{ij} = R_{ij} - 2\partial_i u \partial_j u$. □

List [2006] proved that the nonnegativity of the operator $\mathcal{S}_{g(t)}$ is preserved by the harmonic-Ricci flow. Hence we get the following.

Corollary 8.3. *If $\text{Ric}_{g(0)} - 2du(0) \otimes du(0) \geq 0$, the eigenvalues of the operator $\Delta_{g(t), u(t)}$ are nondecreasing under the harmonic-Ricci flow.*

Remark 8.4. If we choose $u(t) \equiv 0$, we obtain Cao's result [2007].

9. Another formula for $\frac{d}{dt}\lambda(f(t))$

In this section we give another formula for $\frac{d}{dt}\lambda(f(t))$ using a method similar to that in [Li 2007]. Recall the formula

$$\begin{aligned} \frac{d}{dt}\lambda_{g(t),u(t)}(f(t)) &= \int_M 2\langle \mathcal{S}_{g(t),u(t)}, df(t) \otimes df(t) \rangle_{g(t)} dV_{g(t)} \\ &\quad + \int_M f(t)^2 (|\mathcal{S}_{g(t),u(t)}|_{g(t)}^2 + 2|\Delta_{g(t)}u(t)|_{g(t)}^2) dV_{g(t)}. \end{aligned}$$

Consider the function φ determined by $f^2(t) = e^{-\varphi(t)}$. Then we have

$$df = \frac{-e^\varphi d\varphi}{2f}, \quad \frac{\nabla f}{f} = -\frac{\nabla \varphi}{2}, \quad \frac{\Delta f}{f} = -\frac{1}{2}\Delta\varphi + \frac{1}{4}|\nabla\varphi|^2.$$

Hence

$$\begin{aligned} 2\frac{d}{dt}\lambda_{g(t),u(t)}(f(t)) &= \int_M \langle \mathcal{S}_{g(t),u(t)}, d\varphi(t) \otimes d\varphi(t) \rangle_{g(t)} e^{-\varphi(t)} dV_{g(t)} \\ &\quad + 2 \int_M (|\mathcal{S}_{g(t),u(t)}|_{g(t)}^2 + 2|\Delta_{g(t)}u(t)|_{g(t)}^2) e^{-\varphi} dV_{g(t)}. \end{aligned}$$

Using integration by parts and contracted Bianchi identities yields

$$\begin{aligned} &\int_M \langle \mathcal{S}_{g(t),u(t)}, d\varphi(t) \otimes d\varphi(t) \rangle_{g(t)} e^{-\varphi(t)} dV_{g(t)} \\ &= \int_M S_{ij} \nabla^i \varphi \nabla^j \varphi e^{-\varphi} dV = - \int_M S_{ij} \nabla^j \varphi \nabla^i (e^{-\varphi}) dV \\ &= \int_M e^{-\varphi} \nabla^i (S_{ij} \nabla^j \varphi) dV \\ &= \int_M \nabla^i S_{ij} \cdot \nabla^j \varphi \cdot e^{-\varphi} dV + \int_M S_{ij} \nabla^i \nabla^j \varphi \cdot e^{-\varphi} dV \\ &= \int_M \nabla^i R_{ij} \cdot \nabla^j \varphi \cdot e^{-\varphi} dV_g + \int_M S_{ij} \nabla^i \nabla^j \varphi \cdot e^{-\varphi} dV \\ &\quad + \int_M \nabla^i (-2\nabla_i u \nabla_j u) \nabla^j \varphi \cdot e^{-\varphi} dV_g \\ &= \frac{1}{2} \int_M R \Delta(e^{-\varphi}) dV + \int_M S_{ij} \nabla^i \nabla^j \varphi \cdot e^{-\varphi} dV - 2 \int_M (\nabla^i u \nabla_j u) \nabla^i \nabla^j (e^{-\varphi}) dV. \end{aligned}$$

Thus

$$\begin{aligned} &\int_M S_{ij} \nabla^i \nabla^j \varphi \cdot e^{-\varphi} dV \\ &= \int_M S_{ij} \nabla^i \varphi \nabla^j \varphi e^{-\varphi} dV - \frac{1}{2} \int_M R \Delta(e^{-\varphi}) dV + 2 \int_M (\nabla^i u \nabla_j u) \nabla^i \nabla^j (e^{-\varphi}). \end{aligned}$$

On the other hand, one gets

$$\begin{aligned}
 \int_M |\nabla_{g(t)}^2 \varphi(t)|_{g(t)}^2 e^{-\varphi(t)} dV_{g(t)} &= \int_M \nabla_i \nabla_j \varphi \nabla^i \nabla_j \varphi \cdot e^{-\varphi} dV \\
 &= - \int_M \nabla_j \varphi \cdot \nabla_i \nabla^i \nabla^j \varphi \cdot e^{-\varphi} dV - \int_M \nabla_j \varphi \cdot \nabla^i \nabla^j \varphi \cdot \nabla_i (e^{-\varphi}) dV \\
 &= - \int_M \nabla_j \varphi \cdot \nabla_i \nabla^j \nabla^i \varphi \cdot e^{-\varphi} dV - \int_M \nabla_j \varphi \cdot \nabla^i \nabla^j \varphi \cdot \nabla_i (e^{-\varphi}) dV.
 \end{aligned}$$

Since

$$\begin{aligned}
 \int_M \nabla_j \varphi \cdot \nabla^i \nabla^j \varphi \cdot \nabla_i (e^{-\varphi}) dV &= - \int_M \nabla^i (\nabla_j \varphi \cdot \nabla_i (e^{-\varphi})) \nabla^j \varphi dV \\
 &= - \int_M \nabla^j \varphi \cdot \nabla^i \nabla_j \varphi \cdot \nabla_i (e^{-\varphi}) dV - \int_M |\nabla \varphi|^2 \Delta (e^{-\varphi}) dV,
 \end{aligned}$$

which implies

$$\int_M \nabla_j \varphi \cdot \nabla^i \nabla^j \varphi \cdot \nabla_i (e^{-\varphi}) dV = -\frac{1}{2} \int_M |\nabla \varphi|^2 \Delta (e^{-\varphi}) dV,$$

it follows that

$$\int_M |\nabla^2 \varphi|^2 e^{-\varphi} dV = - \int_M \nabla_j \varphi \cdot \nabla_i \nabla^j \nabla^i \varphi \cdot e^{-\varphi} dV + \frac{1}{2} \int_M |\nabla \varphi|^2 \Delta (e^{-\varphi}) dV.$$

By the Ricci identity the term $\nabla^i \nabla^j \nabla^i \varphi$ equals

$$\begin{aligned}
 \nabla_i \nabla^j \nabla^i \varphi &= g^{jk} g^{il} \nabla_i \nabla_k \nabla_l \varphi = g^{jk} g^{il} (\nabla_k \nabla_i \nabla_l \varphi - R_{ikl}^p \nabla_p \varphi) \\
 &= \nabla^j \nabla_i \nabla^i \varphi - g^{jk} g^{il} R_{iklp} \nabla^p \varphi \\
 &= \nabla^j \Delta \varphi + g^{jk} g^{il} R_{ikpl} \nabla^p \varphi = \nabla^j \Delta \varphi + g^{jk} R_{kp} \nabla^p \varphi.
 \end{aligned}$$

Hence

$$\begin{aligned}
 &- \int_M \nabla_j \varphi \cdot \nabla_i \nabla^j \nabla^i \varphi \cdot e^{-\varphi} dV \\
 &= - \int_M \nabla_i \varphi \cdot \nabla^j \Delta \varphi \cdot e^{-\varphi} dV - \int_M R_{kp} \nabla^k \varphi \cdot \nabla^p \varphi e^{-\varphi} dV \\
 &= \int_M \nabla^j \Delta \varphi \cdot \nabla_j (e^{-\varphi}) + \int_M R_{kp} \nabla^k \varphi \cdot \nabla^p (e^{-\varphi}) dV \\
 &= - \int_M \Delta \varphi \cdot \Delta (e^{-\varphi}) - \int_M e^{-\varphi} (\nabla^p R_{kp} \cdot \nabla^k \varphi + R_{kp} \nabla^p \nabla^k \varphi) \\
 &= - \int_M \Delta (e^{-\varphi}) \cdot \Delta \varphi dV + \frac{1}{2} \int_M \nabla_k R \cdot \nabla^k (e^{-\varphi}) dV - \int_M e^{-\varphi} R_{kp} \nabla^k \nabla^p \varphi dV \\
 &= - \int_M \Delta (e^{-\varphi}) (\Delta \varphi + \frac{1}{2} R) - \int_M R_{kp} \nabla^k \nabla^p \varphi \cdot e^{-\varphi} dV.
 \end{aligned}$$

Putting those formulas together, we obtain

$$\begin{aligned}
& \int_M 2S_{ij} \nabla^i \nabla^j \varphi \cdot e^{-\varphi} dV + \int_M |\nabla^2 \varphi|^2 e^{-\varphi} dV \\
&= \int_M S_{ij} \nabla^i \nabla_j \varphi \cdot e^{-\varphi} dV + \int_M (-2\nabla_i u \nabla_j u) \nabla^i \nabla^j \varphi \cdot e^{-\varphi} dV \\
&\quad - \int_M \Delta(e^{-\varphi}) \left(\Delta\varphi + \frac{R}{2} - \frac{1}{2} |\nabla\varphi|^2 \right) dV \\
&= \int_M S_{ij} \nabla^i \varphi \nabla^j \varphi \cdot e^{-\varphi} dV - \int_M \Delta(e^{-\varphi}) (\Delta\varphi + R - \frac{1}{2} |\nabla\varphi|^2) dV \\
&\quad + 2 \int_M (\nabla_i u \nabla_j u \cdot \nabla^i \nabla^j (e^{-\varphi}) - \nabla_i u \nabla_j u \cdot \nabla^i \nabla^j \varphi \cdot e^{-\varphi}) dV.
\end{aligned}$$

Since f is an eigenfunction of λ , it induces

$$\lambda = -\frac{\Delta f}{f} + \frac{R}{2} - |\nabla u|^2 = \frac{1}{2} \Delta\varphi - \frac{1}{4} |\nabla\varphi|^2 + \frac{R}{2} - |\nabla u|^2,$$

and therefore

$$\begin{aligned}
& \int_M 2S_{ij} \nabla^i \nabla^j \varphi \cdot e^{-\varphi} dV + \int_M |\nabla^2 \varphi|^2 e^{-\varphi} dV \\
&= \int_M S_{ij} \nabla^i \varphi \nabla^j \varphi \cdot e^{-\varphi} dV - 2 \int_M \Delta(|\nabla u|^2) \cdot e^{-\varphi} dV \\
&\quad + 2 \int_M \nabla_i u \nabla_j (\nabla^i \nabla^j (e^{-\varphi}) - \nabla^i \nabla^j \varphi \cdot e^{-\varphi}) dV.
\end{aligned}$$

Plugging this into the expression of $\frac{d}{dt} \lambda(f(t))$ yields

$$\begin{aligned}
& 2 \frac{d}{dt} \lambda_{g(t), u(t)}(f(t)) \\
&= \int_M S_{ij} \nabla^i \varphi \nabla^j \varphi \cdot e^{-\varphi} dV + \int_M |\mathcal{P}|^2 e^{-\varphi} dV + \int_M |\mathcal{P}'|^2 e^{-\varphi} dV + 4 \int_M |\Delta u|^2 e^{-\varphi} dV \\
&= \int_M |\mathcal{P}_{g(t), u(t)} + \nabla_{g(t)}^2 \varphi(t)|_{g(t)}^2 e^{-\varphi(t)} dV_{g(t)} + \int_M |\mathcal{P}'_{g(t), u(t)}|_{g(t)}^2 e^{-\varphi(t)} dV_{g(t)} \\
&\quad + 4 \int_M |\Delta_{g(t)} u(t)|_{g(t)}^2 e^{-\varphi(t)} dV_{g(t)} + 2 \int_M \Delta_{g(t)} |\nabla_{g(t)} u(t)|_{g(t)}^2 e^{-\varphi(t)} dV_{g(t)} \\
&\quad + 2 \int_M \nabla_i u \nabla_j u (-\nabla^i \nabla^j (e^{-\varphi}) + \nabla^i \nabla^j \varphi \cdot e^{-\varphi}) dV
\end{aligned}$$

Now define

$$\begin{aligned}
I &:= \int_M (\nabla_i u \nabla_j u \cdot \nabla^i \nabla^j \varphi) e^{-\varphi} dV = - \int_M \nabla^i (\nabla_i u \nabla_j u \cdot e^{-\varphi}) \nabla^j \varphi dV \\
&= - \int_M \nabla^j \varphi (\Delta u \cdot \nabla_j u \cdot e^{-\varphi} + \nabla_i u \nabla^i \nabla_j u \cdot e^{-\varphi} - \nabla_i u \nabla_j u \nabla^i \varphi \cdot e^{-\varphi}) dV \\
&= - \int_M \nabla_j u \nabla^j \varphi \Delta u \cdot e^{-\varphi} dV - \int_M \nabla_i u \nabla^j \varphi \nabla^i \nabla_j u \cdot e^{-\varphi} dV + \int_M |\langle du, d\varphi \rangle|^2 e^{-\varphi} dV,
\end{aligned}$$

$$\begin{aligned}
II &:= \int_M \nabla_i u \nabla_j u \nabla^i \nabla^j (e^{-\varphi}) dV = \int_M \nabla^i \nabla^j (\nabla_i u \nabla_j u) e^{-\varphi} dV \\
&= \int_M \nabla^i (\nabla^j \nabla_i u \cdot \nabla_j u + \nabla_i u \Delta u) e^{-\varphi} dV \\
&= \int_M (\Delta \nabla^i u \cdot \nabla_i u + \nabla^i \Delta u \cdot \nabla_i u + |\nabla^2 u|^2 + |\Delta u|^2) e^{-\varphi} dV, \\
III &:= \int_M \Delta (|\nabla u|^2) e^{-\varphi} dV = 2 \int_M \nabla^i (\nabla_i \nabla_j u \cdot \nabla^j u) e^{-\varphi} dV \\
&= 2 \int_M (\Delta \nabla_j u \cdot \nabla^j u + |\nabla^2 u|^2) e^{-\varphi} dV.
\end{aligned}$$

If we set

$$B := 2(III + I - II),$$

then

$$\begin{aligned}
\frac{B}{2} &= \int_M (\Delta \nabla_i u \cdot \nabla^i u - \nabla_i \Delta u \cdot \nabla^i u + |\nabla^2 u|^2 - |\Delta u|^2 + |\langle du, d\varphi \rangle|^2 \\
&\quad - \nabla_i u \cdot \nabla^i \varphi \cdot \Delta u - \nabla_i u \cdot \nabla^j \varphi \cdot \nabla^i \nabla_j u) e^{-\varphi} dV \\
&= \int_M (R_{ij} \nabla^i u \nabla^j u + |\nabla^2 u|^2 - |\Delta u|^2 + |\langle du, d\varphi \rangle|^2 \\
&\quad - \nabla_i u \cdot \nabla^i \varphi \cdot \Delta u - \nabla_i u \cdot \nabla^j \varphi \cdot \nabla^i \nabla_j u) e^{-\varphi} dV.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
- \int_M \nabla_i u \cdot \nabla^i \varphi \cdot \Delta u \cdot e^{-\varphi} dV &= \int_M (\nabla_i u \cdot \Delta u) \nabla^i (e^{-\varphi}) dV \\
&= - \int_M \nabla^i (\nabla_i u \cdot \Delta u) e^{-\varphi} dV \\
&= \int_M (-|\Delta u|^2 - \nabla_i u \cdot \nabla^i \Delta u) e^{-\varphi} dV
\end{aligned}$$

and

$$\begin{aligned}
- \int_M \nabla_i u \nabla^j \varphi \nabla^i \nabla_j u \cdot e^{-\varphi} dV &= \int_M \nabla_i u \nabla^i \nabla_j u \nabla^j (e^{-\varphi}) dV \\
&= - \int_M \nabla^j (\nabla_i u \nabla^i \nabla_j u) e^{-\varphi} dV \\
&= \int_M (-|\nabla^2 u|^2 - \nabla_i u \Delta \nabla^i u) e^{-\varphi} dV.
\end{aligned}$$

Therefore

$$(9-1) \quad \frac{B}{2} = \int_M (-2|\Delta u|^2 + |\langle du, d\varphi \rangle|^2 - 2\langle \nabla u, \nabla \Delta u \rangle) e^{-\varphi} dV.$$

By definition,

$$\Delta(|\nabla u|^2) = \Delta(\nabla^i u \cdot \nabla_i u) = 2\nabla^i u \cdot \Delta \nabla_i u + 2|\nabla^2 u|^2.$$

So

$$\begin{aligned} \Delta|\nabla u|^2 &= 2|\nabla^2 u|^2 + 2(\nabla_i \Delta u + R_{ij} \nabla^j u) \nabla^i u \\ &= 2|\nabla^2 u|^2 + 2R_{ij} \nabla^i u \cdot \nabla^j u + 2\langle \nabla u, \nabla \Delta u \rangle. \end{aligned}$$

Plugging this into (9-1) yields

$$\frac{B}{2} = \int_M (-2|\Delta u|^2 + |\langle du, d\varphi \rangle|^2 + 2|\nabla^2 u|^2 - \Delta|\nabla u|^2 + 2R_{ij} \nabla^i u \nabla^j u) e^{-\varphi} dV.$$

Since

$$\begin{aligned} 2R_{ij} \nabla^i u \nabla^j u &= 2(S_{ij} + 2\nabla_i u \nabla_j u) \nabla^i u \nabla^j u \\ &= 2S_{ij} \nabla^i u \nabla^j u + 4|\nabla u|^4 \\ &= \frac{1}{4}|\mathcal{S} + 4 du \otimes du|^2 - \frac{1}{4}|\mathcal{S}|^2, \end{aligned}$$

it follows that

$$\begin{aligned} \frac{B}{2} &= III + I - II \\ &= \int_M (|\langle du, d\varphi \rangle|^2 - 2|\Delta u|^2 - \frac{1}{4}|\mathcal{S}|^2 + 2|\nabla^2 u|^2 + \frac{1}{4}|\mathcal{S} + 4 du \otimes du|^2) e^{-\varphi} dV - III. \end{aligned}$$

Hence

$$B = \int_M (-4|\Delta u|^2 + 2|\langle du, d\varphi \rangle|^2 - \frac{1}{2}|\mathcal{S}|^2 + 4|\nabla^2 u|^2 + \frac{1}{2}|\mathcal{S} + 4 du \otimes du|^2) e^{-\varphi} dV - 2III.$$

Theorem 9.1. *Suppose that $(g(t), u(t))$ is a solution of the harmonic-Ricci flow on a compact Riemannian manifold M and $f(t)$ is an eigenfunction of $\Delta_{g(t), u(t)}$, that is, $\Delta_{g(t), u(t)} f(t) = \lambda(t) f(t)$ (where $\lambda(t)$ is only a function of time t), with the normalized condition $\int_M f(t)^2 dV_{g(t)} = 1$. Then we have*

$$\begin{aligned} &\frac{d}{dt} \lambda(t) \\ &= \frac{d}{dt} \lambda_{g(t), u(t)}(f(t)) \\ &= \frac{1}{2} \int_M |\mathcal{S}_{g(t), u(t)} + \nabla_{g(t)}^2 \varphi(t)|_{g(t)}^2 e^{-\varphi(t)} dV_{g(t)} + \frac{1}{4} \int_M |\mathcal{S}_{g(t), u(t)}|_{g(t)}^2 e^{-\varphi(t)} dV_{g(t)} \\ &\quad + \int_M |\langle du(t), d\varphi(t) \rangle_{g(t)}|^2 e^{-\varphi(t)} dV_{g(t)} + 2 \int_M |\nabla_{g(t)}^2 u(t)|_{g(t)}^2 e^{-\varphi(t)} dV_{g(t)} \\ &\quad + \frac{1}{4} \int_M |\mathcal{S}_{g(t), u(t)} + 4 du(t) \otimes du(t)|_{g(t)}^2 e^{-\varphi(t)} dV_{g(t)} \\ &\quad - \int_M \Delta_{g(t)} (|\nabla_{g(t)} u(t)|_{g(t)}^2) e^{-\varphi(t)} dV_{g(t)}. \end{aligned}$$

Remark 9.2. When $u \equiv 0$, this equation reduces to Li's formula [2007].

10. The first variation of expander and shrinker entropies

Suppose that M is a closed manifold of dimension n . We define

$$\mathcal{W}_{\pm} : \mathcal{O}_{\pm}^2(M) \times C^{\infty}(M) \times C^{\infty}(M) \times \mathbb{R}^+ \rightarrow \mathbb{R}, \quad (g, u, f, \tau) \mapsto \mathcal{W}_{\pm}(g, u, f, \tau)$$

where

$$(10-1) \quad \mathcal{W}_{\pm}(g, u, f, \tau) := \int_M (\tau(S_{g,u} + |\nabla_g f|_g^2) \mp f \pm n) \frac{e^{-f}}{(4\pi\tau)^{n/2}} dV_g.$$

Set

$$\mu_{\pm}(g, u, \tau) := \inf \left\{ \mathcal{W}_{\pm}(g, u, f, \tau) \mid f \in C^{\infty}(M), \int_M \frac{e^{-f}}{(4\pi\tau)^{n/2}} dV_g = 1 \right\},$$

$$\nu_{\pm}(g, u) := \sup \{ \mu_{\pm}(g, u, \tau) \mid \tau > 0 \}.$$

Lemma 10.1. Suppose $\nu_{\pm}(g, u) = \mathcal{W}_{\pm}(g, u, f_{\pm}, \tau_{\pm})$ for some functions f_{\pm} and constants τ_{\pm} satisfying

$$\int_M \frac{e^{-f_{\pm}}}{(4\pi\tau_{\pm})^{n/2}} dV_g = 1, \quad \tau_{\pm} > 0.$$

Then we must have

$$\tau_{\pm}(-2\Delta_g f_{\pm} + |\nabla_g f_{\pm}|_g^2 - S_{g,u}) \pm f_{\pm} \mp n + \nu_{\pm}(g, u) = 0,$$

$$\int_M \frac{f_{\pm} e^{-f_{\pm}}}{(4\pi\tau)^{n/2}} dV_g = \frac{n}{2} \mp \nu_{\pm}(g, u).$$

Proof. Since g and u are fixed, we consider the corresponding Lagrangian multiplier function

$$\mathcal{L}_{\pm}(f, \tau; \lambda) := \mathcal{W}_{\pm}(g, u, f, \tau) - \lambda \left(\int_M \frac{e^{-f}}{(4\pi\tau)^{n/2}} dV_g - 1 \right).$$

Then the variation of \mathcal{L}_{\pm} in f direction is

$$\begin{aligned} \delta_f \mathcal{L}_{\pm}(f, \tau; \lambda) &= \int_M (2\tau \nabla^i f \nabla_i(\delta f) \mp \delta f + \lambda \delta f) \frac{e^{-f}}{(4\pi\tau)^{n/2}} dV_g \\ &\quad - \int_M (\tau(S_{g,u} + |\nabla_g f|_g^2) \mp f \pm n) \delta f \frac{e^{-f}}{(4\pi\tau)^{n/2}} dV_g. \end{aligned}$$

By the divergence theorem, we calculate

$$\begin{aligned} \int_M \nabla^i f \cdot \nabla_i(\delta f) \frac{e^{-f}}{(4\pi\tau)^{n/2}} dV_g &= - \int_M \nabla_i \left(\nabla^i f \frac{e^{-f}}{(4\pi\tau)^{n/2}} \right) \delta f dV_g \\ &= - \int_M (\Delta_g f - |\nabla_g f|_g^2) \delta f \frac{e^{-f}}{(4\pi\tau)^{n/2}} dV_g. \end{aligned}$$

Hence

$$\delta_f \mathfrak{L}_\pm(f, \tau; \lambda) = \int_M (\tau(-2\Delta_g f + |\nabla_g f|_g^2 - S_{g,u}) \pm f \mp n \mp 1 + \lambda) \delta f \frac{e^{-f}}{(4\pi\tau)^{n/2}} dV.$$

This implies that

$$\tau_\pm(-2\Delta_g f_\pm + |\nabla_g f_\pm|_g^2 - S_{g,u}) \pm f_\pm \mp n \mp 1 + \lambda_\pm = 0.$$

Since f_\pm satisfies the normalized condition, it follows that

$$0 = \lambda_\pm \mp 1 + \int_M (\tau_\pm(-2\Delta_g f_\pm + |\nabla_g f_\pm|_g^2 - S_{g,u}) \pm f_\pm \mp n) \frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}} dV_g.$$

From the identity

$$\int_M \Delta_g f \frac{e^{-f}}{(4\pi\tau)^{n/2}} dV_g = \int_M |\nabla_g f|_g^2 \frac{e^{-f}}{(4\pi\tau)^{n/2}} dV_g$$

and the definition (10-1), we obtain

$$v_\pm(g, u) = \mathfrak{W}_\pm(g, u, f_\pm, \tau_\pm) = \lambda_\pm \mp 1,$$

and, consequently,

$$\tau_\pm(-2\Delta_g f_\pm + |\nabla_g f_\pm|_g^2 - S_{g,u}) \pm f_\pm \mp n + v_\pm(g, u) = 0.$$

The variation of \mathfrak{L}_\pm with respect to τ indicates that

$$\begin{aligned} \delta_\tau \mathfrak{L}_\pm(f, \tau; \lambda) &= \int_M \delta\tau (S_{g,u} + |\nabla_g f|_g^2) \frac{e^{-f}}{(4\pi\tau)^{n/2}} dV_g - \lambda \int_M \left(-\frac{n}{2} \frac{\delta\tau}{\tau}\right) \frac{e^{-f}}{(4\pi\tau)^{n/2}} dV_g \\ &\quad + \int_M \left(-\frac{n}{2} \frac{\delta\tau}{\tau}\right) (\tau(S_{g,u} + |\nabla_g f|_g^2) \mp f \pm n) \frac{e^{-f}}{(4\pi\tau)^{n/2}} dV_g \\ &= \int_M \delta\tau \left(\left(1 - \frac{n}{2}\right) (S_{g,u} + |\nabla_g f|_g^2) + \frac{n}{2\tau} (\lambda \pm f \mp n) \right) \frac{e^{-f} dV_g}{(4\pi\tau)^{n/2}}. \end{aligned}$$

Using the first proved equation, we have

$$\begin{aligned} 0 &= \int_M ((v_\pm(g, u) \pm f_\pm \mp n) \left(1 - \frac{n}{2}\right) + \frac{n}{2} (v_\pm(g, u) \pm f_\pm \mp n \pm 1)) \frac{e^{-f_\pm} dV_g}{(4\pi\tau_\pm)^{n/2}} \\ &= \int_M \left(v_\pm \pm f_\pm \mp \frac{n}{2}\right) \frac{e^{-f_\pm}}{(4\pi\tau_\pm)^{n/2}} dV_g \end{aligned}$$

and therefore we obtain the second one. \square

For a symmetric 2-tensor $h = (h_{ij}) \in \odot^2(M)$, we set

$$g(s) := g + sh$$

Then the variation of $g(s)$ is

$$(10-2) \quad \frac{\partial}{\partial s} \Big|_{s=0} R_{g(s)} = -h^{ij} R_{ij} + \nabla^i \nabla^j h_{ij} - \Delta_g (\text{tr}_g h).$$

Theorem 10.2. *Suppose that (M, g) is a compact Riemannian manifold and u a smooth function on M . Let h be any symmetric covariant 2-tensor on M and set $g(s) := g + sh$. Let v be any smooth function on M and $u(s) := u + sv$. If*

$$v_{\pm}(g(s), u(s)) = {}^{\circ}W_{\pm}(g(s), u(s), f_{\pm}(s), \tau_{\pm}(s))$$

for some smooth functions $f_{\pm}(s)$ with

$$\int_M e^{-f_{\pm}(s)} dV / (4\pi \tau_{\pm}(s))^{n/2} = 1$$

and constants $\tau_{\pm}(s) > 0$,

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} v_{\pm}(g(s), u(s)) = & -\tau_{\pm} \int_M \left(\langle h, \mathcal{S}_{g,u} \rangle_g + \langle h, \nabla_g^2 f \rangle_{g \pm} \pm \frac{1}{2\tau_{\pm}} \text{tr}_g h \right) \frac{e^{-f_{\pm}}}{(4\pi \tau_{\pm})^{n/2}} dV_g \\ & + 4\tau_{\pm} \int_M v(\Delta_g u - \langle du, df_{\pm} \rangle_g) \frac{e^{-f_{\pm}}}{(4\pi \tau_{\pm})^{n/2}} dV_g, \end{aligned}$$

where $f_{\pm} := f_{\pm}(0)$ and $\tau_{\pm} := \tau_{\pm}(0)$. In particular, the critical points of $v_{\pm}(\cdot, \cdot)$ satisfy

$$\mathcal{S}_{g,u} + \nabla_g^2 f \pm \frac{1}{2\tau_{\pm}} g = 0, \quad \Delta_g u = \langle du, df_{\pm} \rangle_g.$$

Consequently, if ${}^{\circ}W_{\pm}(g, u, f, \tau)$ and $v_{\pm}(g, u)$ achieve their minimums, (M, g) is a gradient expanding and shrinker harmonic-Ricci soliton according to the sign.

Proof. By definition, one has

$$\begin{aligned} & \frac{d}{ds} v_{\pm}(g(s), u(s)) \\ &= \frac{d}{ds} {}^{\circ}W_{\pm}(g(s), u(s), f_{\pm}(s), \tau_{\pm}(s)) \\ &= \int_M \left(\frac{\partial}{\partial s} \tau_{\pm}(s) (\mathcal{S}_{g(s), u(s)} + |\nabla_{g(s)} f_{\pm}(s)|_{g(s)}^2) \right) \frac{e^{-f_{\pm}(s)}}{(4\pi \tau_{\pm}(s))^{n/2}} dV_{g(s)} \\ &+ \int_M \left(\tau_{\pm}(s) \frac{\partial}{\partial s} (\mathcal{S}_{g(s), u(s)} + |\nabla_{g(s)} f_{\pm}(s)|_{g(s)}^2) \mp \frac{\partial}{\partial s} f_{\pm}(s) \right) \frac{e^{-f_{\pm}(s)}}{(4\pi \tau_{\pm}(s))^{n/2}} dV_{g(s)} \\ &+ \int_M (\tau_{\pm}(s) (\mathcal{S}_{g(s), u(s)} + |\nabla_{g(s)} f_{\pm}(s)|_{g(s)}^2) \mp f_{\pm}(s) \pm n) \\ &\quad \cdot \frac{\partial}{\partial s} \left(\frac{e^{-f_{\pm}(s)}}{(4\pi \tau_{\pm}(s))^{n/2}} dV_{g(s)} \right). \end{aligned}$$

Since

$$\begin{aligned}
\frac{\partial}{\partial s} S_{g(s), u(s)} &= \frac{\partial}{\partial s} R_{g(s)} - 2 \frac{\partial}{\partial s} |\nabla_{g(s)} u(s)|_{g(s)}^2 \\
&= \frac{\partial}{\partial s} R_{g(s)} - 2 \left(\frac{\partial}{\partial s} g^{ij} \right) \nabla_i u \nabla_j u - 4 g^{ij} \frac{\partial}{\partial s} \nabla_i u \cdot \nabla_j u \\
&= \frac{\partial}{\partial s} R_{g(s)} - 2 (-g^{ip} g^{jq} h_{pq}) \nabla_i u \nabla_j u - 4 g^{ij} \nabla_i \left(\frac{\partial}{\partial s} u \right) \nabla_j u \\
&= \frac{\partial}{\partial s} R_{g(s)} + 2 h_{pq} \nabla^p u \nabla^q u - 4 \nabla_i \left(\frac{\partial}{\partial t} u \right) \nabla^i u
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial s} \left(\frac{e^{-f_{\pm}(s)}}{(4\pi \tau_{\pm}(s))^{n/2}} dV_{g(s)} \right) \\
&= \left(-\frac{\partial}{\partial s} f_{\pm}(s) - \frac{n}{2\tau_{\pm}(s)} \frac{\partial}{\partial s} \tau_{\pm}(s) \right) \frac{e^{-f_{\pm}(s)}}{(4\pi \tau_{\pm}(s))^{n/2}} dV_{g(s)} + \frac{e^{-f_{\pm}(s)}}{(4\pi \tau_{\pm}(s))^{n/2}} \frac{\partial}{\partial s} dV_{g(s)} \\
&= \left(-\frac{\partial}{\partial s} f_{\pm}(s) - \frac{n}{2\tau_{\pm}(s)} \frac{\partial}{\partial s} \tau_{\pm}(s) + \frac{1}{2} \operatorname{tr}_g h \right) \frac{e^{-f_{\pm}(s)}}{(4\pi \tau_{\pm}(s))^{n/2}} dV_{g(s)},
\end{aligned}$$

it follows that

$$\begin{aligned}
&\frac{d}{ds} v_{\pm}(g(s), u(s)) \\
&= \int_M \frac{\partial}{\partial s} \tau_{\pm}(s) (S_{g(s), u(s)} + |\nabla_{g(s)} f_{\pm}(s)|_{g(s)}^2) \frac{e^{-f_{\pm}(s)}}{(4\pi \tau_{\pm}(s))^{n/2}} dV_{g(s)} \\
&\quad + \int_M \left(\tau_{\pm}(s) \left(\frac{\partial}{\partial s} R_{g(s)} + 2 h_{pq} \nabla^p u \nabla^q u - 4 \nabla_i \left(\frac{\partial}{\partial s} u \right) \nabla^i u \right. \right. \\
&\quad \left. \left. - h_{pq} \nabla^p f \nabla^q f + 2 \nabla_i \left(\frac{\partial}{\partial s} f \right) \nabla^i f \right) \mp \frac{\partial}{\partial s} f_{\pm}(s) \right) \frac{e^{-f_{\pm}(s)}}{(4\pi \tau_{\pm}(s))^{n/2}} dV_{g(s)} \\
&\quad + \int_M \left(-\frac{\partial}{\partial s} f_{\pm}(s) - \frac{n}{2\tau_{\pm}(s)} \frac{\partial}{\partial s} \tau_{\pm}(s) + \frac{1}{2} \operatorname{tr}_g h \right) \\
&\quad \cdot \left(\tau_{\pm}(s) (S_{g(s), u(s)} + |\nabla_{g(s)} f_{\pm}(s)|_{g(s)}^2) \mp f_{\pm}(s) \pm n \right) \frac{e^{-f_{\pm}(s)}}{(4\pi \tau_{\pm}(s))^{n/2}} dV_{g(s)}.
\end{aligned}$$

From the equalities

$$\begin{aligned}
\int_M \Delta_g \operatorname{tr}_g h \cdot e^{-f} dV_g &= \int_M \operatorname{tr}_g h \cdot \Delta_g (e^{-f}) dV_g \\
&= \int_M \operatorname{tr}_g h (-\Delta_g f + |\nabla_g f|_g^2) e^{-f} dV_g,
\end{aligned}$$

$$\begin{aligned}
\int_M \nabla^i \nabla^j h_{ij} \cdot e^{-f} dV_g &= \int_M h_{ij} \nabla^i \nabla^j (e^{-f}) dV \\
&= \int_M h_{ij} (-\nabla^i \nabla^j f + \nabla^i f \nabla^j f) e^{-f} dV_g, \\
\int_M \nabla_i \left(\frac{\partial}{\partial s} f \right) \nabla^i f e^{-f} dV_g &= \int_M -\frac{\partial}{\partial s} f (\Delta_g f - |\nabla_g f|_g^2) e^{-f} dV_g, \\
\int_M \Delta_g (e^{-f}) dV_g &= \int_M (-\Delta_g f + |\nabla_g f|_g^2) e^{-f} dV_g,
\end{aligned}$$

and [Lemma 10.1](#), we obtain

$$\begin{aligned}
&\frac{d}{ds} \Big|_{s=0} v_{\pm}(g(s), u(s)) \\
&= \int_M \frac{\partial}{\partial s} \Big|_{s=0} \tau_{\pm}(s) (S_{g,u} + |\nabla_g f|_g^2) \frac{e^{-f_{\pm}}}{(4\pi \tau_{\pm})^{n/2}} dV_g \\
&\quad + \int_M \left(\tau_{\pm} \left(-h^{ij} R_{ij} + \nabla^i \nabla_j h_{ij} - \Delta_g(\operatorname{tr}_g h) + 2h_{pq} \nabla^p u \nabla^q u \right. \right. \\
&\quad \quad \left. \left. - 4\nabla_i v \nabla^i u - h_{pq} \nabla^p f \nabla^q f + 2\nabla_i \left(\frac{\partial}{\partial s} \Big|_{s=0} f(s) \right) \nabla^i f \right) \mp \frac{\partial}{\partial s} \Big|_{s=0} f(s) \Big) \\
&\quad \cdot \frac{e^{-f_{\pm}}}{(4\pi \tau_{\pm})^{n/2}} dV_g + \int_M \left(-\frac{\partial}{\partial s} \Big|_{s=0} f_{\pm}(s) - \frac{n}{2\tau_{\pm}}(s) \frac{\partial}{\partial s} \Big|_{s=0} \tau_{\pm}(s) + \frac{1}{2} \operatorname{tr}_g h \right) \\
&\quad \cdot (\tau_{\pm}(S_{g,u} + |\nabla_g f_{\pm}|_g^2) \mp f_{\pm} \pm n) \frac{e^{-f_{\pm}}}{(4\pi \tau_{\pm})^{n/2}} dV_g.
\end{aligned}$$

If we denote by B the last term and by A the remaining terms,

$$\begin{aligned}
A &= \int_M \left(\frac{\partial}{\partial s} \Big|_{s=0} \tau_{\pm}(s) (|\nabla_g f_{\pm}|_g^2 + S_{g,u}) \right. \\
&\quad \left. - \tau_{\pm}(h^{ij} \nabla_i \nabla_j f_{\pm} + h^{ij} S_{ij} + 4\nabla_i v \cdot \nabla^i u) \mp \frac{\partial}{\partial s} f_{\pm} \right) \frac{e^{-f_{\pm}}}{(4\pi \tau_{\pm})^{n/2}} dV_g \\
&\quad + \int_M \tau_{\pm} (\Delta_g f_{\pm} - |\nabla_g f_{\pm}|_g^2) \left(\operatorname{tr}_g h - 2 \frac{\partial}{\partial s} \Big|_{s=0} f(s) \right) \frac{e^{-f_{\pm}}}{(4\pi \tau_{\pm})^{n/2}} dV_g.
\end{aligned}$$

The normalized condition

$$1 = \int_M \frac{e^{-f_{\pm}(s)}}{(4\pi \tau_{\pm}(s))^{n/2}} dV_g$$

implies

$$0 = \int_M \left(-\frac{\partial}{\partial s} \Big|_{s=0} f_{\pm}(s) - \frac{n}{2\tau_{\pm}} \frac{\partial}{\partial s} \Big|_{s=0} \tau_{\pm}(s) + \frac{1}{2} \operatorname{tr}_g h \right) \frac{e^{-f_{\pm}(s)}}{(4\pi \tau_{\pm}(s))^{n/2}} dV_g.$$

From [Lemma 10.1](#), we conclude that

$$\tau_{\pm} S_{g,u} - \tau_{\pm} (|\nabla_g f_{\pm}|_g^2 - 2\Delta_g f_{\pm}) = \pm f_{\pm} \mp n + \nu_{\pm}(g, u).$$

Therefore

$$\tau_{\pm} (S_{g,u} + |\nabla_g f_{\pm}|_g^2) \mp f_{\pm} \pm n = 2\tau_{\pm} (|\nabla_g f_{\pm}|_g^2 - \Delta_g f_{\pm}) + \nu_{\pm}(g, u).$$

Plugging this into the definition of B yields

$$\begin{aligned} B &= \int_M \left(-\frac{\partial}{\partial s} \Big|_{s=0} f_{\pm}(s) - \frac{n}{2\tau_{\pm}} \frac{\partial}{\partial s} \Big|_{s=0} \tau_{\pm}(s) + \frac{1}{2} \operatorname{tr}_g h \right) \\ &\quad \cdot (2\tau_{\pm} (|\nabla_g f_{\pm}|_g^2 - \Delta_g f_{\pm}) + \nu_{\pm}(g, u)) \frac{e^{-f_{\pm}}}{(4\pi \tau_{\pm})^{n/2}} dV_g \\ &= \int_M \left(-\frac{\partial}{\partial s} \Big|_{s=0} f_{\pm}(s) - \frac{n}{2\tau_{\pm}} \frac{\partial}{\partial s} \Big|_{s=0} \tau_{\pm}(s) + \frac{1}{2} \operatorname{tr}_g h \right) \\ &\quad \cdot (2\tau_{\pm} (|\nabla_g f_{\pm}|_g^2 - \Delta_g f_{\pm})) \frac{e^{-f_{\pm}}}{(4\pi \tau_{\pm})^{n/2}} dV_g \\ &= \int_M \left(-\frac{\partial}{\partial s} \Big|_{s=0} f_{\pm}(s) + \frac{1}{2} \operatorname{tr}_g h \right) 2\tau_{\pm} (|\nabla_g f_{\pm}|_g^2 - \Delta_g f_{\pm}) \frac{e^{-f_{\pm}}}{(4\pi \tau_{\pm})^{n/2}} dV_g, \end{aligned}$$

where we use the fact that

$$\int_M \Delta_g (e^{-f}) dV_g = 0.$$

Hence B cancels with the last term in A . Therefore the above variation equals

$$\begin{aligned} &\frac{d}{ds} \Big|_{s=0} \nu_{\pm}(g(s), u(s)) \\ &= \int_M \left(\frac{\partial}{\partial s} \Big|_{s=0} \tau_{\pm}(s) \left(|\nabla_g f_{\pm}|_g^2 + S_{g,u} \pm \frac{n}{2\tau_{\pm}} \right) - \tau_{\pm} \left(h^{ij} \nabla_i \nabla_j f + h^{ij} S_{ij} \right. \right. \\ &\quad \left. \left. \pm \frac{1}{2\tau_{\pm}} \operatorname{tr}_g h + 4v(\langle du, df \rangle - \Delta_g u) \right) \right) \frac{e^{-f_{\pm}}}{(4\pi \tau_{\pm})^{n/2}} dV_g. \end{aligned}$$

To prove the theorem, it is sufficient to show that

$$\int_M \left(|\nabla_g f_{\pm}|_g^2 + S_{g,u} \pm \frac{n}{2\tau_{\pm}} \right) \frac{e^{-f_{\pm}}}{(4\pi \tau_{\pm})^{n/2}} dV = 0.$$

Since M is compact, we have

$$0 = \int_M \Delta_g (e^{-f_{\pm}}) = \int_M (-\Delta_g f_{\pm} + |\nabla_g f_{\pm}|_g^2) e^{-f_{\pm}} dV.$$

Hence

$$\begin{aligned} \int_M \left(|\nabla_g f_{\pm}|^2 + S_{g,u} \pm \frac{n}{2\tau_{\pm}} \right) \frac{e^{-f_{\pm}}}{(4\pi\tau_{\pm})^{n/2}} dV \\ = \int_M \left(2\Delta_g f_{\pm} - |\nabla_g f|_g^2 + S_{g,u} \pm \frac{n}{2\sigma_{\pm}} \right) \frac{e^{-f_{\pm}}}{(4\pi\tau_{\pm})^{n/2}} dV. \end{aligned}$$

Lemma 10.1 now indicates

$$\begin{aligned} \int_M \left(|\nabla_g f_{\pm}|^2 + S_{g,u} \pm \frac{n}{2\tau_{\pm}} \right) \frac{e^{-f_{\pm}}}{(4\pi\tau_{\pm})^{n/2}} dV \\ = \int_M \left(\frac{\pm f_{\pm} \mp n + v_{\pm}(g, u)}{\tau_{\pm}} \pm \frac{n}{2} \right) \frac{e^{-f_{\pm}}}{(4\pi\tau_{\pm})^{n/2}} dV \\ = \int_M \frac{1}{\tau_{\pm}} \left(\pm f_{\pm} \mp \frac{n}{2} + v_{\pm}(g, u) \right) \frac{e^{-f_{\pm}}}{(4\pi\tau_{\pm})^{n/2}} dV \\ = \frac{1}{\tau_{\pm}} \left(\pm \frac{n}{2} - v_{\pm}(g, u) \mp \frac{n}{2} + v_{\pm}(g, u) \right) = 0. \end{aligned}$$

The sign $+$ corresponds to the gradient expanding soliton and the sign $-$ to the gradient shrinker soliton. \square

Corollary 10.3. *Suppose that (M, g) is a compact Riemannian manifold and u is a smooth function on M . Let h be any symmetric covariant 2-tensor on M and set $g(s) := g + sh$. Let v be any smooth function on M and $u(s) := u + sv$. If $v_{\pm}(g(s), u(s)) = \mathcal{W}_{\pm}(g(s), u(s), f_{\pm}(s), \tau_{\pm}(s))$ for some smooth function $f_{\pm}(s)$ with $\int_M e^{-f_{\pm}(s)} dV / (4\pi\tau_{\pm}(s))^{n/2} = 1$ and a constant $\tau_{\pm}(s) > 0$, and (g, u) is a critical point of $v_{\pm}(\cdot, \cdot)$, then*

$$\mathcal{S}_{g,u} = \mp \frac{1}{2\tau_{\pm}} g, \quad f_{\pm} \equiv \text{constant}.$$

Thus, if $\mathcal{W}_{\pm}(g, u, \cdot, \cdot)$ achieve their minimum and (g, u) is a critical point of $v_{\pm}(\cdot, \cdot)$, (M, g, u) satisfies the static Einstein vacuum equation.

Proof. According to **Lemma 10.1** and **Theorem 10.2**, we have

$$\begin{aligned} \tau_{\pm}(-2\Delta_g f_{\pm} + |\nabla_g f_{\pm}|_g^2 - S_{g,u}) \pm f_{\pm} \mp n \\ = -v_{\pm} = - \int_M \left(\tau_{\pm}(S_{g,u} + |\nabla_g f|_g^2) \mp f_{\pm} \pm n \right) \frac{e^{-f_{\pm}}}{(4\pi\tau_{\pm})^{n/2}} dV_g, \end{aligned}$$

and hence

$$\begin{aligned} 2\Delta_g f_{\pm} - |\nabla_g f_{\pm}|_g^2 + S_{g,u} &= \int_M (S_{g,u} + |\nabla_g f_{\pm}|_g^2) \frac{e^{-f_{\pm}}}{(4\pi\tau_{\pm})^{n/2}} dV_g \\ &= \int_M (S_{g,u} + \Delta_g f_{\pm}) \frac{e^{-f_{\pm}}}{(4\pi\tau_{\pm})^{n/2}} dV_g \\ &= \mp \frac{n}{2\tau_{\pm}} = S_{g,u} + \Delta_g f_{\pm}. \end{aligned}$$

From this we get $\Delta_g f_{\pm} = |\nabla_g f_{\pm}|_g^2$. After integrating on both sides, the functions f_{\pm} must be constant, which implies $\mathcal{S}_g \pm (1/(2\tau_{\pm}))g = 0$. \square

Remark 10.4. In the situation of [Corollary 10.3](#), by normalization, we may choose $f_{\pm} = n/2$.

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