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Lipschitz functions with prescribed blowups at many points

Andrea Marchese and Andrea Schioppa

Abstract. In this paper we prove generalizations of Lusin-type theorems for gradients due to Giovanni Alberti, where we replace the Lebesgue measure with any Radon measure μ. We apply this to go beyond the known result on the existence of Lipschitz functions which are non-differentiable at μ-almost every point x in any direction which is not contained in the decomposability bundle $V(\mu, x)$, recently introduced by Alberti and the first named author. More precisely, we prove that it is possible to construct a Lipschitz function which attains any prescribed *admissible blowup* at every point except for a closed set of points of arbitrarily small measure. Here a function is an admissible blowup at a point x if it is null at the origin and it is the sum of a linear function on $V(\mu, x)$ and a Lipschitz function on $V(\mu, x)^{\perp}$.

Keywords: Lipschitz function, Radon measure, blowup, Lusin type approximation.

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1. Introduction

In [Alb91], Alberti proved a "Lusin type theorem for gradients": roughly speaking, given any Borel vectorfield f on the Euclidean space $\mathbb{R}^N$ one can find a C^1 function g whose gradient coincides with f up to an exceptional closed set of arbitrarily small Lebesgue measure. In other words one can prescribe at many points the (unique) blowup of a C^1 function in an arbitrary (measurable) way. The Rademacher Theorem, which states that Lipschitz functions are differentiable almost everywhere, implies that, even if one weakens the assumptions on g, requiring it only to be locally Lipschitz, Alberti's result is still the best possible: no other blowups than the linear ones can be prescribed on a set of points of positive measure; moreover, one cannot get rid of the small exceptional set without any further assumption on f, i.e. in general one cannot find a Lipschitz function g such that the measure of the set $\{Dg \neq f\}$ is zero. However a continuous function g with such property can be found (see [MP08]).

In the present paper, we prove a generalization of Alberti's result, where the Lebesgue measure is replaced by any Radon measure. Since Rademacher's theorem does not hold in general with respect to a Radon measure (and in particular it fails with respect to any singular measure, as shown in Theorem 1.14 of [DR16]), then the following vague question is very natural in our setting. Given a measure μ on $\mathbb{R}^N$, which blowups is it possible to prescribe for a Lipschitz function, at many points with respect to μ, besides the linear ones?

Let us introduce some basic notations to make the question more precise. We denote by $B^N(0,1)$ the unit ball of $\mathbb{R}^N$, centred at the origin with radius 1.

Definition 1.1 (Blowups of a Lipschitz function)**.** Given a Lipschitz function g defined on an open subset $\Omega \subset \mathbb{R}^N$, and a point $x \in \Omega$, we denote by $\mathrm{Tan}(g,x)$ the set of all the possible limits, with respect to the uniform convergence,

$$\lim_{j\to\infty} \mathrm{T}_{x,r_j} f,$$

where $r_j \searrow 0$ and for every $r \leq \mathrm{dist}(x,\Omega^c)$, $\mathrm{T}_{x,r}f : B^N(0,1) \to \mathbb{R}$ is defined by

$$\mathrm{T}_{x,r}f(y) := r^{-1}(f(x+ry) - f(x)), \quad \text{for every } y \in B^N(0,1).$$

Definition 1.2 (Prescribing blowups)**.** Let μ be a positive Radon measure on an open set $\Omega \subset \mathbb{R}^N$. Denote by $\mathrm{Lip}(\mathrm{B}^\mathrm{N}(0,1),0)$ the space of Lipschitz functions on $B^N(0,1)$ which vanish at the origin, endowed with the supremum distance, and let $f : \Omega \subset \mathbb{R}^N \to \mathrm{Lip}(\mathrm{B}^\mathrm{N}(0,1),0)$ be a Borel function. We say that f *prescribes the blowups* of a Lipschitz function with respect to μ:

(i) *Weakly*, if there exists a Lipschitz function $g : \Omega \to \mathbb{R}$ such that $f(x) \subset \mathrm{Tan}(g,x)$ for μ-a.e. $x \in \Omega$;

(ii) *Weakly in the Lusin sense*, if for every $\varepsilon > 0$ there exists a Lipschitz function $g : \Omega \to \mathbb{R}$ such that

$$\mu(\{x \in \Omega : f(x) \not\subset \mathrm{Tan}(g,x)\}) < \varepsilon;$$

(iii) *Strongly in the Lusin sense*, if for every $\varepsilon > 0$ there exists a Lipschitz function $g : \Omega \to \mathbb{R}$ such that

$$\mu(\{x \in \Omega : f(x) \neq \mathrm{Tan}(g,x)\}) < \varepsilon.$$

In this paper we mainly address to the following question.

Question 1.1. *Given a positive Radon measure μ on an open set $\Omega \subset \mathbb{R}^N$ with $\mu(\Omega) < \infty$, for which choice of f is it possible to say that f prescribes the blowups of a Lipschitz function wrt. μ weakly/strongly/in the Lusin sense?*

As we already observed, when μ is the Lebesgue measure, Rademacher's Theorem is a constraint on the possible choices of a function f for which Question 1.1 may have a positive answer. Namely, in this case, for a.e. point x, the corresponding function $f(x)$ must be linear (more precisely, the restriction to $B^N(0,1)$ of a linear function). Using the notation that we have introduced above, the content of Alberti's result, or at least part of it, can be rephrased as follows: if $\Omega \subset \mathbb{R}^N$ is an open set with finite Lebesgue measure, then every Borel function $f : \Omega \to \mathrm{Lip}(\mathrm{B}^\mathrm{N}(0,1),0)$ whose values are (restrictions to $B^N(0,1)$ of) linear functions almost everywhere, prescribes the blowups of a Lipschitz function wrt. the Lebesgue measure strongly in the Lusin sense.

For a general measure μ, the Rademacher theorem does not hold. In particular De Philippis and Rindler proved in [DR16] that there are Lipschitz functions which are non-differentiable at μ_{sing}-a.e. point, where μ_{sing} is the singular part of μ wrt. Lebesgue. Nevertheless a suitable weaker version of Rademacher's theorem holds. Indeed, letting $\mathrm{Gr}(\mathbb{R}^N)$ denote the union of the Grasmannians of all vector subspaces of $\mathbb{R}^N$, in [AM16] it is proved that to every Radon measure μ on $\mathbb{R}^N$ it is possible to associate a Borel function $V(\mu,\cdot):\mathbb{R}^N\to\mathrm{Gr}(\mathbb{R}^N)$ called the *decomposability bundle* of μ, with the property that for every Lipschitz function g, the restriction of g to the affine subspace $x+V(\mu,x)$ is differentiable at μ-a.e. point x and moreover the bundle is maximal with respect to this property, meaning that there exists a Lipschitz function which is non-differentiable at μ-a.e. point x along any direction which is not in $V(\mu,x)$.

Clearly this is also a constraint on the possible choices of a function f for which one can expect a positive answer to Question 1.1, indeed one should at least require that $f(x)$ is linear on $V(\mu,x)$ for μ-a.e. point x. This observation partially motivates the introduction of the following subset of $\mathrm{Lip}(\mathrm{B^N}(0,1),0)$. Given a vector space V and a point $y\in\mathbb{R}^N$ we denote respectively y_V and $y_{V^\perp}$ the projections on V and on its orthogonal complement $V^\perp$. Finally we denote the class of *admissible blowups* by

$$C(\mu,x):=\{h\in\mathrm{Lip}(\mathrm{B^N}(0,1),0):h(y)=L(y_{V(\mu,x)})+m(y_{V(\mu,x)^\perp})\},\qquad(1.3)$$

where L is a linear function on $V(\mu,x)$ and m is a Lipschitz function on $V(\mu,x)^\perp$. A reason for the choice of such class is given in Section 4; roughly speaking, this class is, for a generic measure having a given decomposability bundle, the largest class of blow-ups that one can expect to be able to prescribe.

Now we are ready to state the main results of the paper.

1.2. Theorem. *Let μ be a Radon measure on $\mathbb{R}^N$, let $\Omega\subset\mathbb{R}^N$ be an open set with $\mu(\Omega)<\infty$, and let f be as in Definition 1.2. Then f prescribes the blowups of a Lipschitz function with respect to μ*

- *(I) strongly in the Lusin sense: if $f(x)=\{L(x)\}$ at μ-a.e x, where $L(x)$ is the restriction to $B^N(0,1)$ of a linear function;*
- *(II) weakly in the Lusin sense: if $f(x)\in C(\mu,x)$ for μ-a.e. x;*

For $N=1$ we can prove a stronger statement. In particular we don't need the restriction that Ω has finite measure. Firstly we can characterize those measures for which it is possible to prescribe strongly any reasonable blowup. Secondly we prove that any blowup can be prescribed weakly wrt. a singular measure μ. More precisely, for the typical 1-Lipschitz function g (in the sense of Baire categories) $\mathrm{Tan}(g,x)$ coincides with the set of (the restrictions to $B^1(0,1)$ of) all 1-Lipschitz functions, at μ-a.e. point x. Given a Borel set E, we denote by $\mu\llcorner E$ the measure defined by

$$\mu\llcorner E(A)=\mu(A\cap E),$$

for every Borel set E.

1.3. Theorem. *Let $\Omega\subset\mathbb{R}$ be an open set, and let μ and f be as in Definition 1.2. Then f prescribes the blowups of a Lipschitz function with respect to μ*

(I) *strongly in the Lusin sense, for for* $f(x) = \{H(x)\}$ *at* μ*-a.e.* x*, where* $H(x)$ *is the restriction to* $B(0,1)$ *of a positively homogeneous function: if and only if* $\mu \llcorner NL$ *is atomic, where*

$$NL = \{x \in \Omega : H(x) \; is \; not \; the \; restriction \; to \; B(0,1) \; of \; a \; linear \; function\}.$$

(II) *weakly: if* μ *is singular, for any Lipschitz function* f*;*

1.4. Remark. (i) Point (I) in Theorem 1.2 is the generalization of Theorem 1 of [Alb91]. In Section 2 we prove a more precise version of this statement, including the possibility to choose the Lipschitz function g in point (iii) of Definition 1.2 of class C^1, with arbitrarily small L^∞ norm and with L^p estimates on its gradient for every $p \in [1, \infty]$. A similar result was recently proved by David in [Dav15] in the setting of PI spaces: a class of metric measure spaces which admit a differentiable structure. We point out that Point (I) can also be extended to doubling metric measure spaces, where the differentiable structure is defined using operators called derivations: we will pursue this somewhere else.

(ii) The difference between point (II) of Theorem 1.2 and point (II) of Theorem 1.3 is twofold. Firstly, in the second case there is no restriction on the function f, due to the fact that the decomposability bundle of a singular measure in $\mathbb{R}$ is always trivial. Secondly, the fact that the blowups in the first case are prescribed weakly in the Lusin sense, while in the second case are prescribed weakly, can be explained as follows. In dimension $N = 1$, we are able to prove that, for every fixed ε a residual set of 1-Lipschitz functions attains every 1-Lipschitz function in the set blowups, outside a set of measure at most ε. Since the intersection of countably may residual sets remains residual, we can clearly reinforce the previous statement, saying that residually many 1-Lipschitz functions attain, in a set of full measure, every 1-Lipschitz function as a blowups. In higher dimension, for fixed ε it would be possible to prove the residuality of the set of Lipschitz functions attaining every "admissible" blowup outside a set of measure ε, with a technique similar to that used in the proof of Theorem 1.1 (ii) of [AM16]. The issue is that with such technique one could only prove that the above set is residual in a complete metric space of Lipschitz functions, which is suitably defined depending on ε itself. Hence, the intersection of a countable family such sets, might in principle be empty.

(iii) Point (I) of Theorem 1.3 is a simple observation, which is already contained in Proposition 4.2 of [Mar]. The restriction to the family of positively homogeneous functions is clearly necessary in order to prescribe blowups strongly, because if $f \in \mathrm{Tan}(g, x)$ and $h \in \mathrm{Tan}(f, 0)$, then it also holds $h \in \mathrm{Tan}(g, x)$. Presumably, also in dimension larger than 1 the possibility to prescribe strongly some non-linear blowups in the Lusin sense should depend on some property of the measure and intuitively it should fail when the measure is "very diffused". On the other side, it sounds reasonable that if the measure μ is supported on a k-rectifiable set E in $\mathbb{R}^N$ ($k < N$) then any Borel function f which is μ-a.e. positively homogeneous and linear along the tangent bundle to E prescribes the blowups of a Lipschitz function strongly in the Lusin sense. However we do not pursue this issue in this paper.

On the structure of the paper. The proof of Theorem 1.2 is split in Section 2 for the point (I) and Section 3, for point (II). In Section 4 we provide an example of a measure μ for which every blowup of a Lipschitz function is the sum of a linear function on $V(\mu, x)$ and a Lipschitz function on its orthogonal, at μ-a.e. point x, in order to justify the choice of the class $C(\mu, \cdot)$ appearing in point (II) of Theorem 1.2. Finally, in Section 5, we prove Theorem 1.3.

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2. Proof of Theorem 1.2(I)

In this section we prove the point (I) of Theorem 1.2. As anticipated in Remark 1.4 (i) we will actually prove a stronger statement including some gradient estimates. In particular L^∞ estimates on the function g and its gradient are necessary to prove part (II) of the Theorem. The proof is very similar to the one presented in [Alb91]. The new main technical ingredient is Corollary 2.3, where we guarantee that for any measure μ we can find many cubes which behave similarly to the Lebesgue measure, in terms of the proportion between the measure of the cube and that of its "frame".

2.1. Theorem. *Let μ be a Radon measure on $\mathbb{R}^N$. Let $\Omega \subset \mathbb{R}^N$ open with $\mu(\Omega) < \infty$. Then for every Borel map $f : \Omega \to \mathbb{R}^N$ and for every $\varepsilon, \zeta > 0$ there exist a compact set K and a function $g \in C^1_c(\Omega)$ with $\|g\|_\infty \le \zeta$ such that*

$$\mu(\Omega \setminus K) < \varepsilon \mu(\Omega). \tag{2.1}$$

$$Dg(x) = f(x), \quad \text{for every } x \in K. \tag{2.2}$$

Moreover, there exists $C = C(N)$ such that

$$\|Dg\|_p \le C\varepsilon^{1/p-1}\|f\|_p, \quad \text{for every } p \in [1, \infty], \tag{2.3}$$

where $\|\cdot\|_p$ denotes the usual norm in $L^p(\Omega, \mu)$.

Let $\mathrm{B}_\mathbb{Z}$ denote the cubulation of $\mathbb{R}^N$ by cubes of the form $\prod_{i=1}^N [2n_i - 1, 2n_i + 1]$ for $(n_i)_{i=1}^N \in \mathbb{Z}^N$. For $r > 0$ let $\mathrm{B}_\mathbb{Z}(r)$ denote the transform of $\mathrm{B}_\mathbb{Z}$ when the dilation $x \mapsto rx$ is applied to $\mathbb{R}^N$. For $x \in \mathbb{R}^N$ and $r > 0$ we denote the box or cube by

$$\mathrm{Bx}\,(x, r) = \left\{ y \in \mathbb{R}^N : \max_{1 \le i \le N} |x_i - y_i| \le r \right\}, \tag{2.4}$$

and for $\varepsilon > 0$ we define the "frame":

$$\mathrm{Fr}\,(x,r,\varepsilon) = \{y \in \mathrm{Bx}\,(x,r) : \ |x_i - y_i| \geq (1-\varepsilon)r \text{ for some } i \in \{1,\cdots,N\}\}\,. \tag{2.5}$$

Finally for $\omega \in \mathrm{Bx}\,(0,r)$ we let $\mathrm{B}_{\mathbb{Z}}(r,\omega) = \mathrm{B}_{\mathbb{Z}}(r)+\omega$ and let $P = \frac{1}{(2r)^N}\mathcal{L}^N \llcorner \mathrm{Bx}\,(0,r)$.

Lemma 2.2 (Existence of cubes with negligible frames)**.** *Let K be compact with $\mu(K) > 0$ and $U \supset K$ open with $\mu(U) \leq \frac{3}{2}\mu(K)$. Assume that $r > 0$ is such that for each $\mathrm{Bx}\,(x,r)$ which intersects K one has $\mathrm{Bx}\,(x,r) \subset U$. For $\varepsilon > 0$ define*

$$\begin{aligned} B^{\mathrm{good}}_\omega = \Big\{ &\mathrm{Bx}\,(x,r) \in \mathrm{B}_{\mathbb{Z}}(r,\omega) : \mathrm{Bx}\,(x,r) \cap K \neq \emptyset \\ &\quad \text{and } \mu(\mathrm{Fr}\,(x,r,\varepsilon)) \leq 16N2^N\varepsilon\mu(\mathrm{Bx}\,(x,r)) \Big\}. \end{aligned} \tag{2.6}$$

Then for some $\omega \in \mathrm{Bx}\,(0,r)$ one has:

$$\mu(\bigcup B^{\mathrm{good}}_\omega \cap K) \geq \frac{3}{4}\mu(K). \tag{2.7}$$

Proof. We define the ε-boundaries of a family of cubes G

$$\partial_\varepsilon G = \{\mathrm{Fr}\,(x,r,\varepsilon) : \mathrm{Bx}\,(x,r) \in G\}\,, \tag{2.8}$$

the set of bad cubes

$$\begin{aligned} B^{\mathrm{bad}}_\omega = \Big\{ &\mathrm{Bx}\,(x,r) \in \mathrm{B}_{\mathbb{Z}}(r,\omega) : \mathrm{Bx}\,(x,r) \cap K \neq \emptyset \\ &\quad \text{and } \mu(\mathrm{Fr}\,(x,r,\varepsilon)) > 16N2^N\varepsilon\mu(\mathrm{Bx}\,(x,r)) \Big\}, \end{aligned} \tag{2.9}$$

and the set of bad ωs:

$$A_{\mathrm{bad}} = \Big\{ \omega \in \mathrm{Bx}\,(0,r) : \mu(\bigcup B^{\mathrm{bad}}_\omega) \geq \frac{1}{6}\mu(U) \Big\}. \tag{2.10}$$

The Lemma is proven by showing that $P(A_{\mathrm{bad}}) < 1$. Define:

$$I = \int \chi_{\bigcup \partial_\varepsilon B^{\mathrm{bad}}_\omega}(x)\chi_{A_{\mathrm{bad}}}(\omega)\, d\mu(x)\, dP(\omega), \tag{2.11}$$

and estimate it from below integrating first in $d\mu(x)$:

$$\begin{aligned} I &= \int \mu(\bigcup \partial_\varepsilon B^{\mathrm{bad}}_\omega)\chi_{A_{\mathrm{bad}}}(\omega) dP(\omega) \\ &\geq 16N2^N\varepsilon \int \mu(\bigcup B^{\mathrm{bad}}_\omega)\chi_{A_{\mathrm{bad}}}(\omega) dP(\omega) \\ &\geq \frac{16N2^N\varepsilon}{6}\mu(U)P(A_{\mathrm{bad}}). \end{aligned} \tag{2.12}$$

We estimate I from above integrating first in $dP(\omega)$:

$$I = \int P(\omega \in A_{\mathrm{bad}} : x \in \bigcup \partial_\varepsilon B^{\mathrm{bad}}_\omega)\, d\mu(x). \tag{2.13}$$

For fixed x the set $\{\omega : x \in \bigcup \partial_\varepsilon B_\omega^{\mathrm{bad}}\}$ has positive P-measure only if $x \in U$ and the one must also have $\omega \in \bigcup_{j=1}^{2^N} \mathrm{Fr}\,(\tilde{x}_j, r, \varepsilon)$ where the $\{\tilde{x}_j\}$ depend only on x. As the Lebesgue measure on $\mathrm{Fr}\,(\tilde{x}_j, r, \varepsilon)$ is at most $2N(2r)^N\varepsilon$ we get

$$I \le \varepsilon(2N)2^N\mu(U) \tag{2.14}$$

and so $P(A_{\mathrm{bad}}) \le \frac{12}{16} < 1$. Finally for $\omega \in A_{\mathrm{bad}}^c$ we observe:

$$\mu(K \cap \bigcup B_\omega^{\mathrm{good}}) \ge \mu(K) - \mu(\bigcup B_\omega^{\mathrm{bad}}) \ge \frac{3}{4}\mu(K). \tag{2.15}$$

□

By a standard covering argument, we deduce the following

Corollary 2.3 (Covering by good cubes). *Let $\varepsilon > 0$ and $r_0 > 0$; then for every open set $U \subset \mathbb{R}^N$ there is a sequence of disjoint cubes $\{\mathrm{Bx}\,(z_\lambda, r_\lambda)\}_\lambda$ contained in U such that:*

$$r_\lambda \le r_0 \tag{2.16}$$

$$\mu(\mathrm{Fr}\,(z_\lambda, r_\lambda, \varepsilon)) \le 16N2^N\varepsilon\mu(\mathrm{Bx}\,(z_\lambda, r_\lambda)) \tag{2.17}$$

$$\mu(U \setminus \bigcup_\lambda \mathrm{Bx}\,(z_\lambda, r_\lambda)) = 0. \tag{2.18}$$

To prove Theorem 2.1 it is sufficient to perform a straightforward iteration of Lemma 2.4 below. For the proof of the lemma, after we have established Corollary 2.3, we can easily adapt the proof given in [Alb91].

2.4. Lemma. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure μ. Let $f : \Omega \to \mathbb{R}^N$ be a bounded and continuous function. Then for every $\xi, \eta, \zeta > 0$ there exists a compact set $K \subset \Omega$ and a function $g \in C_c^1(\Omega)$ such that*

$$\mu(\Omega \setminus \mathrm{Int}(K)) < \xi\mu(\Omega), \tag{2.19}$$

$$\|g\|_\infty \le \zeta, \tag{2.20}$$

$$|f(x) - Dg(x)| \le \eta, \text{for every } x \in K. \tag{2.21}$$

$$\|Dg\|_p \le C\xi^{1/p-1}\|f\chi_{\mathrm{spt}(g)}\|_p, \text{ for every } p \in [1, \infty], \tag{2.22}$$

where χ_A denotes the characteristic function of the set A, with values 0 and 1 and $\mathrm{spt}(g)$ is the support of the function g.

Proof. Suppose $\xi < 1$. Let K' be a compact subset of Ω such that

$$\mu(\Omega \setminus K') < \mu(\Omega)\xi/3. \tag{2.23}$$

Let $d' := \mathrm{dist}(K', \mathbb{R}^N \setminus \Omega)$ and $d := \min\{1, d'/2\}$. Denote by K'' the compact set

$$K'' := \{x \in \Omega : \mathrm{dist}(x, K')\} \le d.$$

Since f is uniformly continuous in K'', there exists $0 < \delta < d$ such that for all $x \in K''$, $y \in \Omega$ it holds

$$|x - y| < \delta \implies |f(x) - f(y)| < \eta. \tag{2.24}$$

Consider the family of cubes $\{\mathrm{Bx}\,(x_i, r_i)\}_i$ obtained applying Corollary 2.3, with the parameters $r_0 := \min\{\delta/(2N); \zeta(N\|f\|_\infty)^{-1}\}$ and $\varepsilon := \xi/(48N2^N)$.

Let

$$\{\mathrm{Bx}\,(x_1, r_1)\,, \dots, \mathrm{Bx}\,(x_M, r_M)\}$$

be a finite subfamily such that $\mathrm{Bx}\,(x_i, r_i) \cap K' \neq \emptyset$ for $i = 1 \dots, M$ and

$$\mu\left(K' \setminus \bigcup_{i=1}^{M} \mathrm{Bx}\,(x_i, r_i)\right) < \mu(\Omega)\xi/3. \tag{2.25}$$

For $i = 1, \dots, M$, let $\phi_i \in C^1(\Omega)$ such that $0 \leq \phi_i \leq 1$, $\phi_i \equiv 1$ in the set $\mathrm{Bx}\,(x_i, r_i) \setminus \mathrm{Fr}\,(x_i, r_i, \varepsilon)$, $\phi_i \equiv 0$ outside $\mathrm{Bx}\,(x_i, r_i)$ and

$$\|D\phi_i\|_\infty \leq \frac{2}{r_i\varepsilon}. \tag{2.26}$$

Denoting

$$a_i = \frac{\int_{\mathrm{Bx}(x_i,r_i)} f\; d\mu}{\mu(\mathrm{Bx}\,(x_i, r_i))},$$

we set, for all $x \in \Omega$

$$g(x) = \sum_i \phi_i(x)\langle a_i, x - x_i\rangle.$$

We finally set

$$K := \bigcup_{i=1}^{M} \mathrm{cl}(\mathrm{Bx}\,(x_i, r_i) \setminus \mathrm{Fr}\,(x_i, r_i, \varepsilon)).$$

It is easy to see that $g \in C^1(\Omega)$ and $\|g\|_\infty \leq Nr_0\|f\|_\infty$, hence property (2.20) follows. Property (2.19) follows from the inequality

$$\mu(\Omega \setminus \mathrm{Int}(K)) \leq \mu(\Omega \setminus K') + \mu(K' \setminus (\bigcup_{i=1}^{M} \mathrm{Bx}\,(x_i, r_i))) + \mu((\bigcup_{i=1}^{M} \mathrm{Bx}\,(x_i, r_i)) \setminus \mathrm{Int}(K))$$

by applying (2.23), (2.25) and (2.17), paired with the choice of ε. Property (2.21) follows from (2.24) by the choice of r_0 and a_i. To prove (2.22), in the case $p \in [1, \infty)$, we compute, using (2.26) and the definition of g,

$$\|Dg\|_p^p \leq \sum_i \int_{\mathrm{Bx}(x_i,r_i)\setminus\mathrm{Fr}(x_i,r_i,\varepsilon)} |a_i|^p d\mu + \int_{\mathrm{Fr}(x_i,r_i,\varepsilon)} (2N|a_i|r_i)^p(2/(r_i\varepsilon))^p d\mu.$$

Combining with (2.17) we have

$$\|Dg\|_p^p \leq \sum_i \mu(\mathrm{Bx}\,(x_i, r_i))(|a_i|^p + 16N2^N\varepsilon^{1-p}(2N|a_i|)^p)$$

and by the definition of a_i, this implies

$$\|Dg\|_p^p \leq C\varepsilon^{1-p} \sum_i (\int_{\mathrm{Bx}(x_i,r_i)} |f|\; d\mu)^p.$$

Finally, by Jensen's inequality, we get (2.22). The case $p = \infty$ follows immediately from (2.26). □

Proof of Theorem 2.1. Suppose that $\varepsilon < 1$ and f is not μ-almost everywhere 0. Fix $p \in [1,\infty]$.

First case. f is continuous and bounded. For every $n \geq 1$, set

$$\eta_n := a\varepsilon^2 2^{-2(n+1)},$$

where

$$0 < a := \inf_{p\in[1,\infty]} \mu(\Omega)^{-1/p}\|f\|_p.$$

We define iteratively a sequence $(\Omega_n, g_n, K_n, f_n)_{n\in\mathbb{N}}$ as follows. Set $\Omega_0 := \Omega, g_0 := 0, K_0 := \emptyset, f_0 := f$. Let $n > 0$ and assume $\Omega_{n-1}, g_{n-1}, K_{n-1}, f_{n-1}$ are given. Apply Lemma 2.4, to obtain compact set $K_n \subset \Omega_{n-1}$ and a function $g_n \in C^1_c(\Omega_{n-1})$ such that

$$\mu(\Omega_{n-1} \setminus \mathrm{Int}(K_n)) < 2^{-n-1}\varepsilon\mu(\Omega_{n-1}), \tag{2.27}$$

$$\|g_i\|_\infty \leq 2^{-1}\zeta, \tag{2.28}$$

$$|f_{n-1}(x) - Dg_n(x)| \leq \eta_n, \text{ for every } x \in K_n. \tag{2.29}$$

$$\|Dg_n\|_p \leq C(2^{-n-1}\varepsilon)^{1/p-1}\|f_{n-1}\chi_{\mathrm{spt}(g_n)}\|_p, \text{ for every } p \in [1,\infty]. \tag{2.30}$$

Finally set $\Omega_n := \mathrm{int}(K_n)$. Define f_n on Ω_n as $f_n := f_{n-1} - Dg_n$. We set $K := \bigcap_{n>0} K_n$ and $g := \sum_{n>0} g_n$. The bound (2.20) is an immediate consequence of (2.28). We prove now that the set K and the function g satisfy (2.1), (2.2) and (2.3). To prove (2.1), notice that by (2.27) and (2.17) it holds

$$\mu(\Omega \setminus K) = \sum_{n\geq 0} \mu(\Omega_n \setminus \mathrm{Int}(K_{n+1})) \leq \varepsilon\mu(\Omega).$$

Since, for $n \geq 1$, $\mathrm{spt}(g_{n+1}) \subset K_n$, combining (2.29) and (2.30) with $p = \infty$, we get

$$\|Dg_{n+1}\|_\infty \leq C(2^{-(n+1)}\varepsilon)^{-1}\|f_n\chi_{\mathrm{spt}(g_{n+1})}\|_\infty \leq C(2^{-(n+1)}\varepsilon)^{-1}\eta_n = C(2^{-(n+1)}\varepsilon)a.$$

This implies that $(\sum_{i=1}^n Dg_i)_{n\in\mathbb{N}}$ (and hence also $(\sum_{i=1}^n g_i)_{n\in\mathbb{N}}$) converges uniformly, therefore $g \in C^1_c(\Omega)$. Since for $n \geq 1$ it holds $f = f_{n-1} + \sum_{i=0}^{n-1} Dg_i$ on K_n, then (2.2) follows immediately from (2.29). To prove (2.3), we compute, using (2.29) and (2.30)

$$\|Dg\|_p \leq \|Dg\chi_K\|_p + \|Dg\chi_{K^c}\|_p \leq \|f\chi_K\|_p + \sum_{n\geq 0}\|Dg\chi_{\Omega_n\setminus\Omega_{n+1}}\|_p$$

$$\leq \|f\|_p + \sum_{n\geq 0}\|Dg_{n+1}\chi_{\Omega_n\setminus\Omega_{n+1}}\|_p \leq \|f\|_p + C2^{n+1}\varepsilon^{1/p-1}\sum_{n\geq 0}\|f_n\chi_{\Omega_n}\|_p$$

$$\leq (1 + 2C\varepsilon^{1/p-1})\|f\|_p + C2^{n+1}\varepsilon^{1/p-1}\sum_{n\geq 1}\|(f_{n-1} - Dg_n)\chi_{\Omega_n}\|_p$$

$$\leq (1 + 2C\varepsilon^{1/p-1})\|f\|_p + C2^{n+1}\varepsilon^{1/p-1}\mu(\Omega)^{1/p}\sum_{n\geq 1}\eta_n \leq (1 + 3C\varepsilon^{1/p-1})\|f\|_p$$

□

Second case. f is Borel. Fix $\varepsilon > 0$. There exists $r > 0$ such that

$$\alpha := \mu(\{x \in \Omega : |f(x)| > r\}) \leq \varepsilon/4.$$

By Lusin's theorem there exists a continuous function $f_1 : \Omega \to \mathbb{R}^N$ which agrees with f outside a set of measure μ less than α. The function

$$f_2(x) = \begin{cases} f_1(x) & \text{if} |f_1(x)| \leq r \\ rf_1(x)/|f_1(x)| & \text{if} |f_1(x)| > r \end{cases}$$

is continuous and bounded and $\mu(\{x : f(x) \neq f_2(x)\}) \leq \varepsilon/2$. Moreover, for every $p \in [1, \infty]$, it holds $\|f_2\|_p \leq 2\|f\|_p$. The theorem follows easily applying the previous case to the function f_2.

3. Proof of Theorem 1.2(II)

The proof of part (II) of Theorem 1.2 is quite involved. The reader might find helpful to read Section 5 before proceeding: although the result in dimension 1 is stronger, the construction presented there requires a considerably smaller amount of technicalities.

3.1. Preliminary results.

Definition 3.1 (Local behaviour of a Lipschitz function)**.** Let S be a set and let $\alpha \geq 0$ and $r_0 > 0$. A real-valued Lipschitz function f whose domain contains S is said to be **α-Lipschitz on S below scale** r_0 if whenever $x, y \in X$ are such that $\operatorname{dist}(x, S), \operatorname{dist}(y, S) \leq r_0$ and $d(x, y) \leq r_0$ one has:

$$|f(x) - f(y)| \leq \alpha d(x, y). \tag{3.2}$$

The Lipschitz function f is said to be **asymptotically flat on** S if for each $\varepsilon > 0$ there is an $r_\varepsilon > 0$ such that f is ε-Lipschitz on S below scale r_ε.

Before moving on we need to recall something about the general differentiability theory for real-valued Lipschitz functions in the metric setting developed in [Sch16a] and the differentiability theory, wrt. singular Radon measures, for real-valued Lipschitz functions defined on Euclidean spaces studied in [AM16]. We do not want to dispirit the reader: both theories can essentially be treated as black-boxes to understand the results here, as they only intervene through the Localized Approximation Scheme, Theorem 3.2.

Definition 3.3 (Alberti representations)**.** Let μ be a Radon measure on a metric space X and let $\operatorname{Frag}(X)$ denote the set of 1-Lipschitz maps $\gamma : \operatorname{dom}\gamma \to X$ where $\operatorname{dom}\gamma$ is a compact subset of $\mathbb{R}$. We topologize $\operatorname{Frag}(X)$ with the Hausdorff distance between graphs. An Alberti representation of μ is a pair (Q, w) where Q is a Radon measure on $\operatorname{Frag}(X)$ and w is a locally bounded Borel map $w : X \to [0, \infty)$ such that:

$$\mu = \int_{\operatorname{Frag}(X)} w\gamma_\#(\mathcal{L}^1 \llcorner \operatorname{dom}\gamma)\, dQ(\gamma), \tag{3.4}$$

where $\gamma_\#(\mathcal{L}^1 \llcorner \operatorname{dom}\gamma)$ denotes the push-forward, using γ, of the 1-dimensional Lebesgue measure on $\operatorname{dom}\gamma$. More precisely, (3.4) should be understood as follows: for each $g : X \to \mathbb{R}$ continuous and in $L^1(\mu)$ one has:

$$\int_X g\, d\mu = \int_{\mathrm{Frag}(X)} dQ(\gamma) \int_{\operatorname{dom}\gamma} w \circ \gamma(t) g \circ \gamma(t)\, dt. \tag{3.5}$$

Definition 3.6 (The norm of the Weaver differential)**.** Let μ be a Radon measure on X and $f : X \to \mathbb{R}$ Lipschitz. We denote by $|df|_{\mathcal{E}(\mu)}$ the local norm of df [Sch16a, Defn. 2.101 & 2.123], which is an $L^\infty(\mu)$-function which is ≥ 0 μ-a.e. For this paper we do not need the explicit definition of $|df|_{\mathcal{E}(\mu)}$ but the following characterization [Sch16a, Sec. 3.3]: $|df|_{\mathcal{E}(\mu)} \geq \alpha$ on a Borel set $S \subset X$ if and only if for each $\varepsilon > 0$ the measure $\mu \llcorner S$ has an Alberti representation $(Q_\varepsilon, w_\varepsilon)$ such that for Q_ε-a.e. γ for $\mathcal{L}^1$-a.e. $t \in \operatorname{dom}\gamma$ one has $(f \circ \gamma)'(t) \geq \alpha - \varepsilon$. From the definition of the decomposability bundle in [AM16] in terms of the Alberti representations we see that if $f : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz for μ-a.e. x one has:

$$|df|_{\mathcal{E}(\mu)}(x) = \|d_{V(x,\mu)} f\|_2, \tag{3.7}$$

where $d_{V(x,\mu)} f$ is the derivative of f at x in the direction of $V(x, \mu)$.

Theorem 3.2 (Localized Approximation Scheme)**.** *Let (X, μ) be a locally compact metric measure space (μ being Radon). Let f be a real-valued Lipschitz function defined on X and $K \subset X$ a compact subset on which $|df|_{\mathcal{E}(\mu)} \leq \alpha$ for some $\alpha > 0$. Then for each $\varepsilon > 0$ there are an $r_\varepsilon > 0$, a compact $K_\varepsilon \subseteq K$ and a real-valued Lipschitz function f_ε defined on X such that:*

(Apx1): *For any open set $U \subseteq X$ containing K ($K \subset U$) the Lipschitz constant $\mathbf{L}(f_\varepsilon|U)$ of the restriction $f_\varepsilon|U$ is at most the Lipschitz constant $\mathbf{L}(f|U)$ of the restriction $f|U$. In particular, taking $U = X$, $\mathbf{L}(f_\varepsilon) \leq \mathbf{L}(f)$.*
(Apx2): *$\|f_\varepsilon - f\|_\infty \leq \varepsilon$ and f is α-Lipschitz on K_ε below scale r_ε.*
(Apx3): *$\mu(K \setminus K_\varepsilon) \leq \varepsilon$.*

Proof. The proof of this result is rather technical and corresponds to Theorem 3.66 of [Sch16a], proved in Section 5.1 of [Sch16a], in the special case where $q = 1$ (i.e. without discussing cones). However, here we need two slight modifications of that result: that X is locally compact and **(Apx1)**. We will refer to the notation and proof in Section 5.1. of [Sch16a].

That in Theorem 3.66 of [Sch16a] one can take X locally compact is not surprising because in the argument only the compactness of K is directly used.

On the other hand, to obtain **(Apx1)** we must inspect the construction more carefully. We have first constructed a convex metric space Z (i.e. any pair of points is joined by a geodesic) and obtained an isometric embedding $i : X \hookrightarrow Z$. Without loss of generality we have assumed $\mathbf{L}(f) = 1$, considered the cylinder $\mathrm{Cyl} = Z \times \mathbb{R}$ (here we use $\mathbb{R}$ instead of a finite interval because X is only known to be locally compact) with metric:

$$d_{\mathrm{Cyl}}((z_1, t_1), (z_2, t_2)) = \max(|t_1 - t_2|, d_Z(z_1, z_2)). \tag{3.8}$$

We now identify X with a subset of Cyl via $x \mapsto (i(x), f(x))$. Note that the projection

$$\begin{aligned} \tau : \mathrm{Cyl} &\to \mathbb{R} \\ (z,t) &\mapsto t, \end{aligned} \tag{3.9}$$

extends f as $\tau|X = f$. The goal has then become to approximate τ, and this has been accomplished by covering μ-a.e. point of K (thus in **(Apx3)** we pass to a subset K_ε) by strips whose union is $\mathcal{T}_\varepsilon$ (see the definition of $\mathcal{T}_n$ above equation (5.41) in [Sch16a]). As K is compact $\mathcal{T}_\varepsilon$ lies in $Z \times [a,b]$ for some a, b and the approximation τ_ε is obtained by setting $\tau_\varepsilon = \tau$ on $Z \times [-\infty, a)$ and

$$\tau_\varepsilon(z,t) = a + \int_a^t \chi_{\mathcal{T}_\varepsilon^c}(z,s)\, ds \quad \text{elsewhere.} \tag{3.10}$$

In (5.48) of [Sch16a] we have proved that τ_ε is 1-Lipschitz with respect to the distance:

$$D_\alpha((z_1,t_1),(z_2,t_2)) = \max(|t_1 - t_2|, \alpha d_Z(z_1,z_2)). \tag{3.11}$$

In particular, as $K \subset U$, $\alpha \le \mathbf{L}(f|U)$. Now pick $(z_i,t_i) \in U$ for $i \in \{1,2\}$ such that $z_i = i(x_i)$ and $t_i = f(x_i)$; then:

$$D_\alpha((z_1,t_1),(z_2,t_2)) \le \mathbf{L}(f|U) d_X(x_1,x_2), \tag{3.12}$$

which proves the theorem. □

Lemma 3.3 (Step 1 of Construction)**.** *Let K be a compact set and assume that the decomposability bundle of $\mu \llcorner K$ has constant dimension N_0 and let π_x denote its fiber at x and $\pi_x^\perp$ its orthogonal complement. Assume that for some N_0-dimensional hyperplane π one has $\|\pi_x - \pi\|_\infty \le \varepsilon_0$ and let $h : \pi^\perp \to \mathbb{R}$ be 1-Lipschitz. Then there is a constant $C = C(N, N_0)$ (indep. of h) such that for each choice of parameters $(\varepsilon_{\mathrm{s}}, \varepsilon_{\mathrm{m}}, \sigma, r_0) \in (0,1/2)^4$ there are a $\sqrt{3}$-Lipschitz function $g : X \to \mathbb{R}$, compact subsets $J^{\mathrm{good}} \subset J \subset K$ and a scale $r > 0$ such that:*

(a) *$\mu(K \setminus J) \le \varepsilon_{\mathrm{m}}\mu(K)$, $\|g\|_\infty \le \varepsilon_{\mathrm{s}}$ and g is $C\varepsilon_0$-Lipschitz on J below scale r.*
(b) *$\mu(J^{\mathrm{good}}) \ge C^{-1}\sigma^{N-N_0}\mu(J)$.*
(c) *One can decompose J^{good} as a finite disjoint union $\bigcup_{a=1}^M C_a$ such that for each $a \in \{1, \cdots, M\}$ there is an $0 < r_a \le r_0$ such that whenever $x \in C_a$ one has:*

$$\|\mathrm{T}_{x,r_a} g - h\|_{\infty, B(0,1)} \le C(\varepsilon_{\mathrm{s}} + \sigma), \tag{3.13}$$

where in (3.13) we have implicitly extended h as a map $h : \mathbb{R}^N = \pi \oplus \pi^\perp \to \mathbb{R}$ by letting $h(y, \tilde{y}) = h(\tilde{y})$.

Lemma 3.3 is proven using the following intermediate results.

Lemma 3.4 (A good rectangle)**.** *Let μ be a Radon measure on $\mathbb{R}^N$, $r_0 > 0$ and K a compact set with $\mu(K) > 0$. Fix parameters $(L, \sigma) \in [8, \infty) \times (0, 1/2)$ and* *define the following sets:*

$$\mathrm{E}\,(x,r) = x + \left[-\frac{L^2 r}{2}, \frac{L^2 r}{2}\right]^{N_0} \times [-2r, 2r]^{N-N_0}, \tag{3.14}$$

$$\mathrm{S}\,(x,r) = x + \left[-\frac{L^2 r}{2} + \frac{Lr}{2}, \frac{L^2 r}{2} - \frac{Lr}{2}\right]^{N_0} \times [-2\sigma r, 2\sigma r]^{N-N_0}. \tag{3.15}$$

Then there are $x \in K$ and $0 < r \le r_0$ such that:

$$\mu(\mathrm{S}\,(x,r)) \ge \sigma^{N-N_0}\frac{1}{3}2^{-2N-N_0-1}(1+1/6)^{-1}\mu(\mathrm{E}\,(x,r)) > 0. \tag{3.16}$$

Heuristically, $\mathrm{E}\,(x,r)$ is a rectangle at x at scale r which is L^2-times bigger in the direction of the first N_0 coordinates, while $\mathrm{S}\,(x,r)$ is a *core* of $\mathrm{E}\,(x,r)$ which is much smaller (generally $\sigma \ll 1$) in the transverse direction of the last $N - N_0$ coordinates. Even though μ is not the Lebesgue measure, Lemma 3.3 says that we can find a *good rectangle* $\mathrm{E}\,(x,r)$ such that $\mu(\mathrm{S}\,(x,r))/\mu(\mathrm{E}\,(x,r))$ behaves somewhat as good as for Lebesgue measure. Application of Besiscovitch's covering Lemma implies:

Corollary 3.5 (Covering by good rectangles)**.** *Let μ be a Radon measure on $\mathbb{R}^N$ and K compact with $\mu(K) > 0$ and $r_0 > 0$. Let L, σ, be as above. Then for any $\varepsilon > 0$ there are finitely many pairwise disjoint $\{\mathrm{E}\,(x_i, r_i)\}_i$ such that:*

$$0 < r_i \le r_0, \tag{3.17}$$

$$\mu(\mathrm{S}\,(x_i,r_i)\cap K) \ge \sigma^{N-N_0}\frac{1}{3}2^{-2N-N_0-1}(1+1/6)^{-1}\mu(\mathrm{E}\,(x_i,r_i)\cap K) > 0, \tag{3.18}$$

$$\mu(K \setminus \bigcup_i \mathrm{E}\,(x_i,r_i)) \le \varepsilon\mu(K). \tag{3.19}$$

Proof of Lemma 3.4. Let $c = \sigma^{N-N_0} \times 1/3 \times 2^{-2N-N_0-1}(1+1/6)^{-1}$ and choose an open set $U \supset K$ such that

$$\mu(U) \le \left(1 + \frac{1}{6}\right)\mu(K). \tag{3.20}$$

Then choose $r \le r_0$ such that $\mathrm{E}\,(x,4r)\cap K \ne \emptyset$ implies that $\mathrm{E}\,(x,4r) \subset U$. Define the bad set:

$$\mathrm{Bad}(K,r) = \left\{x \in K : 0 < \mu(\mathrm{S}\,(x,r)) \le c\mu(\mathrm{E}\,(x,r))\right\}. \tag{3.21}$$

Then let I denote the integral:

$$I = \int \chi_{\mathrm{Bad}(K,r)}(x_1)\chi_{\mathrm{S}(x_1,r/2)}(x_2)\,d\mu(x_1)d\mathcal{L}^N(x_2); \tag{3.22}$$

if we integrate first in x_2 we get:

$$I = \left(\frac{r}{2}\right)^N \sigma^{N-N_0}(L^2 - L)^{N_0}\mu(\mathrm{Bad}(K,r)). \tag{3.23}$$

If we integrate first in x_1 we get:

$$I = \int \mu\left(\mathrm{Bad}(K,r) \cap \mathrm{S}\left(x_2, r/2\right)\right) \, d\mathcal{L}^N(x_2). \tag{3.24}$$

Choose x_2 such that $\mathrm{Bad}(K,r) \cap \mathrm{S}\left(x_2, r/2\right) \neq \emptyset$ and let $x_1 \in K$ denote a point in this nonempty intersection. Then for $i \in \{1, \cdots, N_0\}$ we get:

$$|x_1^i - x_2^i| \leq \left(\frac{L^2}{2} - \frac{L}{2}\right) r; \tag{3.25}$$

for $i > N_0$ one has $|x_1^i - x_2^i| \leq \sigma r$. In particular, $\mathrm{Bad}(K,r) \cap \mathrm{S}\left(x_2, r/2\right) \subset \mathrm{S}\left(x_1, r\right)$; but as $x_1 \in \mathrm{Bad}(K,r)$ one has $\mu(\mathrm{S}\left(x_1, r\right)) \leq c\mu(\mathrm{E}\left(x_1, r\right))$. Using the triangle inequality we observe $\mathrm{E}\left(x_1, r\right) \subset \mathrm{E}\left(x_2, 2r\right)$ from which we conclude $\mathrm{E}\left(x_2, 2r\right) \subset \mathrm{E}\left(x_1, 4r\right) \subset U$. We thus obtain the upper bound:

$$\begin{aligned} I &\leq c \int \mu\left(U \cap \mathrm{E}\left(x_2, 2r\right)\right) \, d\mathcal{L}^N(x_2) \\ &= c \int \chi_U(x_1) \chi_{\mathrm{E}(x_2, 2r)}(x_1) \, d\mathcal{L}^N(x_2) d\mu(x_1) \\ &= c \int \chi_U(x_1) \chi_{\mathrm{E}(x_1, 2r)}(x_2) \, d\mathcal{L}^N(x_2) d\mu(x_1) \\ &= c(2rL^2)^N \mu(U) \leq c(2rL^2)^N \left(1 + \frac{1}{6}\right) \mu(K). \end{aligned} \tag{3.26}$$

The proof is completed combining (3.26) with (3.23) and the choice of c which gives $\mu(\mathrm{Bad}(K,r)) < \mu(K)$. □

Proof of Lemma 3.3. `Step1: Construction of helper functions.`
Without loss of generality we will assume that π_0 is the plane $\mathbb{R}^{N_0} \times \{0\}$.
Recall that the 1-Lipschitz retraction of $\mathbb{R}^{N_0}$ onto $B(0,1) \subset \mathbb{R}^{N_0}$ is given by:

$$J(x) = \begin{cases} x & \text{if } |x| \leq 1 \\ \frac{x}{|x|} & \text{otherwise.} \end{cases} \tag{3.27}$$

Fix the parameter $L \gg 1$; we define a $\frac{4}{L}$-Lipschitz cutoff function on $\mathbb{R}$:

$$\varphi(r) = \begin{cases} 1 & \text{if } |r| \in [0, L^2/2 - L/4], \\ 1 - \frac{4}{L}(|r| - L^2/2 + L/4) & \text{if } |r| \in (L^2/2 - L/4, L^2/2], \\ 0 & \text{otherwise.} \end{cases} \tag{3.28}$$

We also define the 1-Lipschitz cutoff function on $\mathbb{R}$:

$$\psi(r) = \begin{cases} 1 & \text{if } |r| \leq 1, \\ 2 - |r| & \text{if } |r| \in (1, 2], \\ 0 & \text{otherwise.} \end{cases} \tag{3.29}$$

We now replace h by $h \circ J$ so that we can assume $\|h\|_\infty \leq 1$ and $\partial_r h = 0$ on $\overline{B(0,1)}^c$.

We define the building block of our construction:

$$F(y,\tilde{y}) = \varphi(|y|)\psi(|\tilde{y}|)h(\tilde{y}). \tag{3.30}$$

We now collect some properties of F:

(F1): F is $(4/L+\sqrt{2})$-Lipschitz (note that on $\overline{B(0,1)}^c$ $\nabla\psi$ and ∇h give orthogonal contributions to the derivative of F).

(F2): $F = 0$ outside of $B = \left[-\frac{L^2}{2}, \frac{L^2}{2}\right]^{N_0} \times [-2,2]^{N-N_0}$.

(F3): $F = h$ on the *core* $S = \left[-\frac{L^2}{2} + \frac{L}{4}, \frac{L^2}{2} - \frac{L}{4}\right]^{N_0} \times [-1,1]^{N-N_0}$.

(F4): $\|F\|_\infty \le \|h\|_\infty \le 1$.

(F5): We can assume $|df|_{\mathcal{E}(\mu)} \le C\varepsilon_0$, where C depends possibly only on N and N_0.

(F6): F decays linearly to 0 when approaching the boundary of B: $|F(y,\tilde{y})| \le d((y,\tilde{y}),\partial B)$.

Only **(F5)** and **(F6)** require justification. For **(F5)** observe that by the Leibniz rule $df = \psi h d\varphi + \varphi h d\psi + \varphi\psi dh$; observe also that $d\psi$ and dh give an $O(\varepsilon_0)$ contribution to the Weaver differential (we take μ as the reference measure) as $\|\pi_x - \pi_0\|_\infty \le \varepsilon_0$. Thus, $|df|_{\mathcal{E}(\mu)} \le \frac{4}{L} + C\varepsilon_0$, and, choosing L large enough and inflating C, we get **(F5)**. For **(F6)** we have three cases. The first: $|y| \ge L^2/2 - L/4$ so that:

$$\begin{aligned}
|F(y,\tilde{y})| &\le |\varphi(y)| \le 1 - \frac{4}{L}\left(|y| - \frac{L^2}{2} + \frac{L}{4}\right) \\
&= \frac{4}{L}\left(\frac{L^2}{2} - |y|\right) \\
&\le \frac{4}{L} d((y,\tilde{y}),\partial B),
\end{aligned} \tag{3.31}$$

and **(F6)** holds as long as $L \ge 4$. The second: $|y| \le L^2/2 - L/4$ and $|\tilde{y}| \ge 1$:

$$|F(y,\tilde{y})| \le \psi(\tilde{y}) = 2 - |\tilde{y}| \le d((y,\tilde{y}),\partial B). \tag{3.32}$$

The third: $|y| \le L^2/2 - L/4$ and $|\tilde{y}| \le 1$: $d((y,\tilde{y}),\partial B) \ge 1$ and so we conclude by **(F4)**.

`Step2: Covering` K `by good cubes.`

We apply Corollary 3.5 and find finitely many pairwise disjoint $\{\mathrm{E}(x_i,r_i)\}_{i=1}^M$ such that:

$$0 < r_i \le \min(r_0, \varepsilon_{\mathrm{s}}/2), \tag{3.33}$$

$$\mu(\mathrm{S}(x_i,r_i) \cap K) \ge C^{-1}\sigma^{N-N_0}\mu(\mathrm{E}(x_i,r_i) \cap K) > 0, \tag{3.34}$$

$$\mu(K \setminus \bigcup_i \mathrm{E}(x_i,r_i)) \le \frac{\varepsilon_{\mathrm{m}}}{2}\mu(K). \tag{3.35}$$

We let $\hat{J} = K \cap \bigcup_i \mathrm{E}(x_i,r_i)$ and $\hat{J}^{\mathrm{good}} = K \cap \bigcup_i \mathrm{E}(x_i,r_i)$ which give (b) and the first inequality in (a) if we replace J with $\hat{J}$: the set J will be chosen later to be a subset of $\hat{J}$ and J^{good} will be set to be $J \cap \hat{J}^{\mathrm{good}}$.

Write $x_i = (y_i, \tilde{y}_i)$ and define the $(4/L + \sqrt{2})$-Lipschitz function F_i supported on $\mathrm{E}\,(x_i, r_i)$:

$$F_i(y, \tilde{y}) = \sigma r_i F\left(\frac{y - y_i}{r_i}, \frac{\tilde{y} - \tilde{y}_i}{\sigma r_i}\right). \tag{3.36}$$

Because of **(F6)** the function F_i can be glued together to get a $(4/L + \sqrt{2})$-Lipschitz function f as in [Sch16b, Thm. 4.8]. Note that the choice of the r_i's implies $\|f\|_\infty \le \varepsilon_{\mathrm{s}}/2$.

If $x \in \mathrm{S}\,(x_i, r_i)$ lies on the core center, i.e. x is of the form $x = (y, \tilde{y}_i)$, then:

$$\|\mathrm{T}_{x,\sigma r_i} f - h\|_{\infty, B(0,1)} = 0. \tag{3.37}$$

Thus, as f is $(4/L + \sqrt{2})$-Lipschitz, for all $x \in \mathrm{S}\,(x_i, r_i)$ we have:

$$\|\mathrm{T}_{x,\sigma r_i} f - h\|_{\infty, B(0,1)} \le C\sigma. \tag{3.38}$$

`Step3: Applying the approximation scheme.`

Note that **(F5)** implies that $|df|_{\mathcal{E}(\mu)} \le C\varepsilon_0$ and applying Theorem 3.2 we can find a $(4/L + \sqrt{2})$-Lipschitz function g, a compact set $J \subset \hat{J}$ and $r > 0$ such that:

$$\|g - f\|_\infty \le \frac{\varepsilon_{\mathrm{s}}}{2} \min\left(\frac{1}{2}, \min_{1 \le i \le M} (\sigma r_i)\right), \tag{3.39}$$

$$\mu(\hat{J} \setminus J) \le \frac{\varepsilon_{\mathrm{m}}}{2} \mu(K), \tag{3.40}$$

$$\mu(J \cap \hat{J}^{\mathrm{good}}) \le \frac{C^{-1}\varepsilon_{\mathrm{m}}}{16} \sigma^{N - N_0} \mu(\hat{J}), \tag{3.41}$$

and g is $(C\varepsilon_0)$-Lipschitz on J below scale r. Thus, if we let $J^{\mathrm{good}} = \hat{J}^{\mathrm{good}} \cap J$, then (a) and (b) follow. For (c) we just combine (3.38) with:

$$\frac{\|f - g\|_\infty}{\sigma r_i} \le \frac{\varepsilon_{\mathrm{s}}}{2}. \tag{3.42}$$

We finally choose L large enough so that **(F6)** holds and $\frac{4}{L} \le \varepsilon_{\mathrm{s}}$. □

Lemma 3.6 (Step 2 of Construction). *Let $K \subset \mathbb{R}^N$ be a compact subset and $\alpha > 0$ be such that $|df|_{\mathcal{E}(\mu)} \le \alpha$ on K. Then there is a constant $C = C(N)$ (indep. of f) such that for any choice of parameters $(\varepsilon_{\mathrm{s}}, \varepsilon_{\mathrm{m}}) \in (0, 1/2)^2$ there are a Lipschitz function $\hat{f}$ and a compact set $\hat{K}$ such that:*

(a) $\left\|\hat{f} - f\right\|_\infty \le \varepsilon_{\mathrm{s}}$ *and* $\mu(K \setminus \hat{K}) \le \varepsilon_{\mathrm{m}} \mu(K)$.

(b) $\hat{f}$ *is asymptotically flat on* $\hat{K}$.

(c) $\hat{f}$ *is* $(\mathbf{L}(f) + C\frac{\alpha}{\varepsilon_{\mathrm{m}}})$*-Lipschitz.*

Proof. `Step1: Killing the gradient of` f.

We apply Theorem 2.1 with parameters $\varepsilon = \varepsilon_{\mathrm{m}}^{(1)} > 0$, $\zeta = \varepsilon_{\mathrm{s}}^{(1)} > 0$ to find a $\frac{C\alpha}{\varepsilon_{\mathrm{m}}^{(1)}}$-Lipschitz function f_1 and a compact $K_1 \subset K$ such that $\|f_1\|_\infty \le \varepsilon_{\mathrm{s}}^{(1)}$, $df_1 = df$ on K_1 and

$$\mu(K \setminus K_1) \le \varepsilon_{\mathrm{m}}^{(1)} \mu(K). \tag{3.43}$$

We let $g_1 = f - f_1$ and observe that f_1 is $(\mathbf{L}(f) + \frac{C\alpha}{\varepsilon_{\mathrm{m}}^{(1)}})$-Lipschitz with $dg_1 = 0$ on K_1. We fix the parameters $(\alpha_1, \eta_{\mathrm{s}}^{(1)}, \eta_{\mathrm{m}}^{(1)}) \in (0,1)^3$ and use Theorem 3.2 to find an $(\mathbf{L}(f) + \frac{C\alpha}{\varepsilon_{\mathrm{m}}^{(1)}})$-Lipschitz function $\hat{g}_1$ and a compact $H_1 \subset K_1$ and a scale $r_1 > 0$ such that:

$$\|\hat{g}_1 - g_1\|_\infty \leq \eta_{\mathrm{s}}^{(1)} \tag{3.44}$$

$$\mu(K_1 \setminus H_1) \leq \eta_{\mathrm{m}}^{(1)} \mu(K_1), \tag{3.45}$$

and $\hat{g}_1$ is α_1-Lipschitz on H_1 below scale r_1.

`Step2: The general iteration.`

We apply Theorem 2.1 with parameters $\varepsilon = \varepsilon_{\mathrm{m}}^{(j+1)} > 0$, $\zeta = \varepsilon_{\mathrm{s}}^{(j+1)} > 0$ to find a $\frac{C\alpha_j}{\varepsilon_{\mathrm{m}}^{(j+1)}}$-Lipschitz function f_{j+1} and a compact $K_{j+1} \subset H_j$ such that:

$$\|f_{j+1}\|_\infty \leq \varepsilon_{\mathrm{s}}^{(j+1)} \tag{3.46}$$

$$df_{j+1} = d\hat{g}_j \quad \text{on } K_{j+1} \tag{3.47}$$

$$\mu(H_j \setminus K_{j+1}) \leq \varepsilon_{\mathrm{m}}^{(j+1)} \mu(H_j). \tag{3.48}$$

We let $g_{j+1} = \hat{g}_j - f_{j+1}$ and observe that g_{j+1} satisfies:

$$\mathbf{L}(g_{j+1}) \leq \mathbf{L}(f) + \frac{C\alpha}{\varepsilon_{\mathrm{m}}^{(1)}} + \sum_{l \leq j} \frac{C\alpha_l}{\varepsilon_{\mathrm{m}}^{(l+1)}}, \tag{3.49}$$

and satisfies dg_{j+1} on K_{j+1}. Moreover, by the inductive step and because of **(Apx1)** in Theorem 3.2 we can assume that for $l \leq j$ the function g_{j+1} is $(\alpha_l + \sum_{k=l}^{j} \frac{C\alpha_k}{\varepsilon_{\mathrm{m}}^{(k+1)}})$-Lipschitz on H_l below scale r_l.

We fix the parameters $(\alpha_{j+1}, \eta_{\mathrm{s}}^{(j+1)}, \eta_{\mathrm{m}}^{(j+1)}) \in (0,1)^3$ and use Theorem 3.2 to find an $(\mathbf{L}(f) + \frac{C\alpha}{\varepsilon_{\mathrm{m}}^{(1)}} + \sum_{l \leq j} \frac{C\alpha_l}{\varepsilon_{\mathrm{m}}^{(l+1)}})$-Lipschitz function $\hat{g}_{j+1}$ and a compact $H_{j+1} \subset K_{j+1}$ and a scale $r_{j+1} \in (0, r_j)$ such that:

$$\|\hat{g}_{j+1} - g_{j+1}\|_\infty \leq \eta_{\mathrm{s}}^{(j+1)} \tag{3.50}$$

$$\mu(K_{j+1} \setminus H_{j+1}) \leq \eta_{\mathrm{m}}^{(j+1)} \mu(K_{j+1}), \tag{3.51}$$

and $\hat{g}_{j+1}$ is α_{j+1}-Lipschitz on H_{j+1} below scale r_{j+1}. Also by **(Apx1)** in Theorem 3.2 we see that $\hat{g}_{j+1}$ is $(\alpha_l + \sum_{k=l}^{j} \frac{C\alpha_k}{\varepsilon_{\mathrm{m}}^{(k+1)}})$-Lipschitz on H_l below scale r_l.

`Step3: Choice of parameters.`

The parameters α_j, $\varepsilon_{\mathrm{s}}^{(j)}$, $\eta_{\mathrm{s}}^{(j)}$ and $\eta_{\mathrm{m}}^{(j)}$ can be chosen arbitrarily small at each stage, while with the parameters $\varepsilon_{\mathrm{m}}^{(j)}$ we must be careful otherwise the Lipschitz constants of the functions involved at each stage will diverge to ∞. For $j \geq 1$ we thus let:

$$\varepsilon_{\mathrm{m}}^{(j+1)} = \sqrt{\alpha_j}. \tag{3.52}$$

We have the bound:

$$
\begin{aligned}
\mu(K \setminus H_j) &\leq \mu(K \setminus K_1) + \sum_{l \leq j} \mu(K_l \setminus H_l) + \sum_{k=1}^{j-1} \mu(H_l \setminus K_{l+1}) \\
&\leq \varepsilon_{\mathrm{m}}^{(1)} \mu(K) + \sum_{l \leq j} \eta_{\mathrm{m}}^{(l)} \mu(K_l) + \sum_{k=1}^{j-1} \varepsilon_{\mathrm{m}}^{(l+1)} \mu(H_l) \\
&\leq \left(\varepsilon_{\mathrm{m}}^{(1)} + \sum_{l \leq j} \eta_{\mathrm{m}}^{(l)} + \sum_{k=1}^{j-1} \sqrt{\alpha_l} \right) \mu(K).
\end{aligned}
\tag{3.53}
$$

We let $H_\infty = \bigcap_j H_j$ and want this set to contain a significant part of the measure of K. Thus, if we choose $\varepsilon_{\mathrm{m}}^{(1)} = \varepsilon_{\mathrm{m}}/3$ and the $\eta_{\mathrm{m}}^{(l)}$, α_l so that:

$$
\sum_l \eta_{\mathrm{m}}^{(l)} \leq \frac{\varepsilon_{\mathrm{m}}}{3} \tag{3.54}
$$

$$
\sum_l \sqrt{\alpha_l} \leq \frac{\varepsilon_{\mathrm{m}}}{3}, \tag{3.55}
$$

we get $\mu(K \setminus H_\infty) \leq \varepsilon_{\mathrm{m}} \mu(K)$ and can finally let $\hat{K} = H_\infty$.

We now want to guarantee convergence of the sequence $\{\hat{g}_j\}$. First note:

$$
\hat{g}_j - f = \sum_{l \leq j} (\hat{g}_l - g_l) - \sum_{l \leq j} f_l, \tag{3.56}
$$

from which we deduce:

$$
\|\hat{g}_j - f\|_\infty \leq \sum_{l \leq j} (\eta_{\mathrm{s}}^{(l)} + \varepsilon_{\mathrm{s}}^{(l)}); \tag{3.57}
$$

we will choose the parameters $\eta_{\mathrm{s}}^{(l)}$, $\varepsilon_{\mathrm{s}}^{(l)}$ so that:

$$
\sum_l (\eta_{\mathrm{s}}^{(l)} + \varepsilon_{\mathrm{s}}^{(l)}) \leq \varepsilon_{\mathrm{s}}. \tag{3.58}
$$

Now $\hat{g}_{j+1} - \hat{g}_j = \hat{g}_{j+1} - g_{j+1} - f_j$ and thus:

$$
\|\hat{g}_{j+1} - \hat{g}_j\|_\infty \leq \eta_{\mathrm{s}}^{(j+1)} + \varepsilon_{\mathrm{s}}^{(j)}. \tag{3.59}
$$

Therefore for $j \to \infty$ we have $\hat{g}_j \to \hat{f}$ uniformly, $\hat{f}$ being a continuous function; but:

$$
\mathbf{L}(\hat{g}_j) \leq \mathbf{L}(f) + \frac{3C\alpha}{\varepsilon_{\mathrm{m}}} + C \sum_{l \leq j} \sqrt{\alpha_j}, \tag{3.60}
$$

and if we choose the α_j to satisfy:

$$
\sum_j \sqrt{\alpha_j} \leq \frac{\alpha}{\varepsilon_{\mathrm{m}}} \tag{3.61}
$$

and then inflate C we conclude that $\hat{f}$ is $(\mathbf{L}(f)+\frac{C\alpha}{\varepsilon_{\rm m}})$-Lipschitz. Finally for $l\le j$ the function $\hat{g}_j$ is $(\alpha_l+C\sum_{l\le k\le j}\sqrt{\alpha_k})$-Lipschitz on $H_l\supset H_\infty=\hat{K}$ below scale r_l, and thus $\hat{f}$ is asymptotically flat on $\hat{K}$. □

Lemma 3.7 (Step 3 of Construction)**.** *Let $K\subset\mathbb{R}^N$ be compact and assume that the decomposability bundle of $\mu\llcorner K$ has constant dimension N_0. Assume also that for some N_0-dimensional hyperplane π one has $\|\pi_x-\pi\|_\infty\le\varepsilon_0$ for each $x\in K$ where $\varepsilon_0>0$. Let $h:\pi^\perp\to\mathbb{R}^N$ be 1-Lipschitz with $h(0)=0$ and extend it to $\mathbb{R}^N$ as in Lemma 3.3. Then there are constants C_0,C_1, which depend only on N and N_0, such that the following holds: for each choice of parameters $(\varepsilon_{\rm s},\varepsilon_{\rm m},r_0)\in(0,1/2)^3$ there are a $(\sqrt{3}+C_0\frac{\varepsilon_0}{\varepsilon_{\rm m}^2})$-Lipschitz function $g:\mathbb{R}^N\to\mathbb{R}$ and a compact $J\subset K$ such that:*

(a) *g is asymptotically flat on J, $\|g\|_\infty\le C_1\varepsilon_{\rm s}$ and $\mu(K\setminus J)\le C_1\varepsilon_{\rm m}\mu(K)$.*
(b) *One can write J as a finite disjoint union $J=\bigcup_{a=1}^M C_a$ and for each $a\in 1,\cdots,M$ there is an $0<r_a\le r_0$ such that if $x\in C_a$ one has:*

$$\|{\rm T}_{x,r_a}g-h\|_{\infty,B(0,1)}\le C_1(\varepsilon_0+\varepsilon_{\rm m}^{1/(N-N_0)}).\tag{3.62}$$

Proof. `Step1: Applying Lemma 3.3.`

We let C denote the maximum of the constants C, C_0 and C_1 from Lemmas 3.3 and 3.6. We fix parameters $(\varepsilon_{\rm s}^{(1)},\varepsilon_{\rm m}^{(1)},\sigma,r^{(1)})\in(0,1)^4$ to be chosen later. For the moment we just remark we will need $r^{(1)}\le r_0$ and $\frac{C^{-1}\sigma^{N-N_0}}{2}>\varepsilon_{\rm m}^{(1)}$. We now apply Lemma 3.3 to obtain an $\sqrt{3}$-Lipschitz function g_1 and compact subsets $J_1^{\rm good}\subset J_1\subset K$ and a scale $0<\rho_1\le r^{(1)}$ such that:

(g_1:a): $\mu(K\setminus J_1)\le\varepsilon_{\rm m}^{(1)}\mu(K)$, $\|g_1\|_\infty\le\varepsilon_{\rm s}^{(1)}$ and g_1 is $C\varepsilon_0$-Lipschitz on J_1 below scale ρ_1.
(g_1:b): $\mu(J_1^{\rm good})\ge C^{-1}\sigma^{N-N_0}\mu(J_1)$.
(g_1:c): One can decompose $J_1^{\rm good}$ as a finite disjoint union $\bigcup_{a=1}^{M_1}C_a^{(1)}$ such that for each $a\in\{1,\cdots,M_1\}$ there is an $0<r_a^{(1)}\le r^{(1)}$ such that whenever $x\in C_a^{(1)}$ one has:

$$\left\|{\rm T}_{x,r_a^{(1)}}g_1-h\right\|_{\infty,B(0,1)}\le C(\varepsilon_{\rm s}^{(1)}+\sigma).\tag{3.63}$$

`Step2: Applying Lemma 3.6.`

We fix parameters $(\hat{\varepsilon}_{\rm s}^{(1)},\hat{\varepsilon}_{\rm m}^{(1)})\in(0,1)^2$ to be chosen later. We apply Lemma 3.6 (to the function g_1 and the compact set J_1) to find a Lipschitz function $\hat{g}_1$ and a compact set $\hat{J}_1\subset J_1$ such that:

($\hat{g}_1$:a): $\|\hat{g}_1-g_1\|_\infty\le\hat{\varepsilon}_{\rm s}^{(1)}$ and $\mu(J_1\setminus\hat{J}_1)\le\hat{\varepsilon}_{\rm m}^{(1)}\mu(J_1)$.
($\hat{g}_1$:b): $\hat{g}_1$ is asymptotically flat on $\hat{J}_1$.
($\hat{g}_1$:c): $\hat{g}_1$ is $(\sqrt{3}+C^2\frac{\varepsilon_0}{\hat{\varepsilon}_{\rm m}^{(1)}})$-Lipschitz.

As in **(g_1:c)** pick $x \in C_a^{(1)}$ and combine (3.63) with **($\hat{g}_1$:a)** to obtain:

$$\begin{aligned} \left\| \mathrm{T}_{x,r_a^{(1)}} \hat{g}_1 - h \right\|_{\infty,B(0,1)} &\le \left\| \mathrm{T}_{x,r_a^{(1)}} \hat{g}_1 - \mathrm{T}_{x,r_a^{(1)}} g_1 \right\|_{\infty,B(0,1)} + C(\varepsilon_{\mathrm{s}}^{(1)} + \sigma) \\ &\le \frac{2\hat{\varepsilon}_{\mathrm{s}}^{(1)}}{r_a^{(1)}} + C(\varepsilon_{\mathrm{s}}^{(1)} + \sigma), \end{aligned} \tag{3.64}$$

and observe that $\hat{\varepsilon}_{\mathrm{s}}^{(1)}$ will be chosen later to be insignificant next to $\min_{1\le a\le M_1} r_a^{(1)}$.

`Step3: The construction of` g_2`,` $\hat{g}_2$ `and` G_2`.`

We now want to follow the first two steps, but first need an intermediate construction. Fix parameters $(\alpha_1, \varepsilon_{\mathrm{p}}^{(1)}) \in (0,1)^2$ and choose $R_1 \le \alpha_1^2 \min_{1\le a\le M_1} r_a^{(1)}$ such that $\hat{g}_1$ is α_1-Lipschitz on $\hat{J}_1$ below scale R_1. Find a compact $J_{1\frac{1}{2}} \subset \hat{J}_1 \setminus J_1^{\mathrm{good}}$ such that:

$$\mu\left((\hat{J}_1 \setminus J_1^{\mathrm{good}}) \setminus J_{1\frac{1}{2}} \right) \le \varepsilon_{\mathrm{p}}^{(1)} \mu(\hat{J}_1 \setminus J_1^{\mathrm{good}}), \tag{3.65}$$

and find a $\tau_1 > 0$ such that the τ_1-neighbourhoods of $J_{1\frac{1}{2}}$ and J_1^{good} are disjoint.

We now fix parameters $(\varepsilon_{\mathrm{s}}^{(2)}, \varepsilon_{\mathrm{m}}^{(2)}, r^{(2)}) \in (0,1)^3$ (for the moment imposing the constraint that $2r^{(2)} < \tau_1$) and apply Lemma 3.3 using the parameters $(\varepsilon_{\mathrm{s}}^{(2)}, \varepsilon_{\mathrm{m}}^{(2)}, \sigma, r^{(2)})$ and the compact set $J_{1\frac{1}{2}}$ to obtain an $\sqrt{3}$-Lipschitz function g_2 and compact subsets $J_2^{\mathrm{good}} \subset J_2 \subset J_{1\frac{1}{2}}$ and a scale $0 < \rho_2 \le r^{(2)}$ such that:

(g_2:a): $\mu(J_{1\frac{1}{2}} \setminus J_2) \le \varepsilon_{\mathrm{m}}^{(2)} \mu(J_{1\frac{1}{2}})$, $\|g_2\|_\infty \le \varepsilon_{\mathrm{s}}^{(2)}$ and g_2 is $C\varepsilon_0$-Lipschitz on J_2 below scale ρ_2.
(g_2:b): $\mu(J_2^{\mathrm{good}}) \ge C^{-1}\sigma^{N-N_0}\mu(J_2)$.
(g_2:c): One can decompose J_2^{good} as a finite disjoint union $\bigcup_{a=1}^{M_2} C_a^{(2)}$ such that for each $a \in \{1, \cdots, M_2\}$ there is an $0 < r_a^{(2)} \le r^{(2)}$ such that whenever $x \in C_a^{(2)}$ one has:

$$\left\| \mathrm{T}_{x,r_a^{(2)}} g_2 - h \right\|_{\infty,B(0,1)} \le C(\varepsilon_{\mathrm{s}}^{(2)} + \sigma). \tag{3.66}$$

We fix parameters $(\hat{\varepsilon}_{\mathrm{s}}^{(2)}, \hat{\varepsilon}_{\mathrm{m}}^{(2)}) \in (0,1)^2$ to be chosen later. We apply Lemma 3.6 (to the function g_2 and the compact set J_2) to find a Lipschitz function $\hat{g}_2$ and a compact set $\hat{J}_2 \subset J_2$ such that:

($\hat{g}_2$:a): $\|\hat{g}_2 - g_2\|_\infty \le \hat{\varepsilon}_{\mathrm{s}}^{(2)}$ and $\mu(J_2 \setminus \hat{J}_2) \le \hat{\varepsilon}_{\mathrm{m}}^{(2)} \mu(J_2)$.
($\hat{g}_2$:b): $\hat{g}_2$ is asymptotically flat on $\hat{J}_2$.
($\hat{g}_2$:c): $\hat{g}_2$ is $(\sqrt{3} + C^2 \frac{\varepsilon_0}{\hat{\varepsilon}_{\mathrm{m}}^{(2)}})$-Lipschitz.

We now modify $\hat{g}_2$ so that it vanishes on the (τ_1)-neighbourhood of J_1^{good} and stays the same on the $(\tau_1/2)$-neighbourhood of $J_{1\frac{1}{2}}$. This is accomplished by replacing $\hat{g}_2$ with

$$\hat{g}_2 \max\left(0, 1 - \frac{2}{\tau_1} \mathrm{dist}(\cdot, (J_{1\frac{1}{2}})_{\tau_1/2}) \right), \tag{3.67}$$

where $(J_{1\frac{1}{2}})_{\tau_1/2}$ denotes the $(\tau_1/2)$-neighbourhood of $J_{1\frac{1}{2}}$. As $(J_1^{\mathrm{good}})_{\tau_1}$ and $(J_{1\frac{1}{2}})_{\tau_1}$ are disjoint, $\hat{g}_2$ now vanishes on $(J_1^{\mathrm{good}})_{\tau_1}$. One also obtains an upper bound on the Lipschitz constant of $\hat{g}_2$:

$$\mathbf{L}(\hat{g}_2) \leq \sqrt{3} + C^2 \frac{\varepsilon_0}{\hat{\varepsilon}_{\mathrm{m}}^{(2)}} + 2\frac{\varepsilon_{\mathrm{s}}^{(2)} + \hat{\varepsilon}_{\mathrm{s}}^{(2)}}{\tau_1}. \tag{3.68}$$

To get the last term in (3.68) to be $\leq \alpha_1$ we impose the restriction $2(\varepsilon_{\mathrm{s}}^{(2)} + \hat{\varepsilon}_{\mathrm{s}}^{(2)}) \leq \alpha_1 \tau_1$.

We now let $G_2 = \hat{g}_1 + \hat{g}_2$ and to get a good upper bound on $\mathbf{L}(G_2)$ impose the restriction $2(\varepsilon_{\mathrm{s}}^{(2)} + \hat{\varepsilon}_{\mathrm{s}}^{(2)}) \leq \alpha_1 R_1$. In fact, we now verify that

$$\mathbf{L}(G_2) \leq \sqrt{3} + C^2 \frac{\varepsilon_0}{\min(\hat{\varepsilon}_{\mathrm{m}}^{(1)}, \hat{\varepsilon}_{\mathrm{m}}^{(2)})} + 2\alpha_1; \tag{3.69}$$

the first case is when $d(x,y) \geq R_1$ in which we have:

$$\begin{aligned} |G_2(x) - G_2(y)| &\leq |\hat{g}_1(x) - \hat{g}_1(y)| + 2\,\|\hat{g}_2\|_\infty \\ &\leq \left(\sqrt{3} + C^2 \frac{\varepsilon_0}{\hat{\varepsilon}_{\mathrm{m}}^{(1)}} + \alpha_1\right) d(x,y), \end{aligned} \tag{3.70}$$

and the second case is when $d(x,y) \leq R_1$ in which case we have:

$$\begin{aligned} |G_2(x) - G_2(y)| &\leq |\hat{g}_2(x) - \hat{g}_2(y)| + |\hat{g}_1(x) - \hat{g}_1(y)| \\ &\leq \left(\sqrt{3} + C^2 \frac{\varepsilon_0}{\hat{\varepsilon}_{\mathrm{m}}^{(2)}} + 2\alpha_1\right) d(x,y). \end{aligned} \tag{3.71}$$

We now show that G_2 is asymptotically flat on $\hat{J}_2 \cup (J_1^{\mathrm{good}} \cap \hat{J}_1)$. In fact, $\hat{g}_1$ is asymptotically flat on $\hat{J}_1$ and hence on $\hat{J}_2 \cup (J_1^{\mathrm{good}} \cap \hat{J}_1)$, and $\hat{g}_2$ is asymptotically flat on $\hat{J}_2$ and vanishes on $(J_1^{\mathrm{good}})_{\tau_1}$. We finally establish analogs of (3.64) and (3.66). Pick $x \in C_a^{(1)}$; then:

$$\begin{aligned} \left\|\mathrm{T}_{x,r_a^{(1)}} G_2 - h\right\|_{\infty,B(0,1)} &\leq \|\mathrm{T}_{x,r_a} \hat{g}_1 - h\|_{\infty,B(0,1)} + \frac{2\,\|\hat{g}_2\|_\infty}{r_a^{(1)}} \\ &\leq \frac{2\hat{\varepsilon}_{\mathrm{s}}^{(1)}}{r_a^{(1)}} + C(\varepsilon_{\mathrm{s}}^{(1)} + \sigma) + \alpha_1. \end{aligned} \tag{3.72}$$

Pick $x \in C_a^{(2)}$; then:

$$\begin{aligned} \left\|\mathrm{T}_{x,r_a^{(2)}} G_2 - h\right\|_{\infty,B(0,1)} &\leq \frac{\|\hat{g}_1 - \hat{g}_1(x)\|_{\infty,B(x,r_a^{(2)})}}{r_a^{(2)}} + \left\|\mathrm{T}_{x,r_a^{(2)}} \hat{g}_2 - h\right\|_{\infty,B(0,1)} \\ &\leq \alpha_1 + C(\varepsilon_{\mathrm{s}}^{(2)} + \sigma) + 2\frac{\hat{\varepsilon}_{\mathrm{s}}^{(2)}}{r_a^{(2)}}. \end{aligned} \tag{3.73}$$

In connection with (3.73) we observe that $\hat{\varepsilon}_{\mathrm{s}}^{(2)}$ will be chosen later to be insignificant next to $\min_{1\le a\le M_2} r_a^{(2)}$.
`Step4: The construction of` G_{j+1}`, for` $j\ge 2$`.`

We fix parameters $(\alpha_j, \varepsilon_{\mathrm{p}}^{(j)}) \in (0,1)^2$ and choose

$$R_j \le \alpha_j^2 \min\{r_a^{(l)} : 1\le l\le j, 1\le a\le M_l\} \tag{3.74}$$

such that G_j is α_j-Lipschitz on $\hat{J}_j$ below scale R_j. We then find a compact set $J_{j\frac{1}{2}} \subset \hat{J}_j \setminus J_j^{\mathrm{good}}$ such that:

$$\mu\left((\hat{J}_j \setminus J_j^{\mathrm{good}}) \setminus J_{j\frac{1}{2}}\right) \le \varepsilon_{\mathrm{p}}^{(j)} \mu(\hat{J}_j \setminus J_j^{\mathrm{good}}), \tag{3.75}$$

and find a $\tau_j > 0$ such that $(J_{j\frac{1}{2}})_{\tau_j} \cap (J_j^{\mathrm{good}})_{\tau_j} = \emptyset$.

We then construct g_{j+1} and $\hat{g}_{j+1}$ as we did for g_2 and $\hat{g}_2$ in `Step3`: one needs only to adjust the indexes. We will refer to the variants of properties **(g_2:a)**–**(g_2:c)** and **($\hat{g}_2$:a)**–**($\hat{g}_2$:c)** by **(g_{j+1}:a)**–**(g_{j+1}:c)** and **($\hat{g}_{j+1}$:a)**–**($\hat{g}_{j+1}$:c)**.

We then have to modify $\hat{g}_{j+1}$ so that it vanishes on $(J_j^{\mathrm{good}})_{\tau_j}$ and stays the same on $(J_{j\frac{1}{2}})_{\tau_j/2}$. This is accomplished by replacing $\hat{g}_{j+1}$ with:

$$\hat{g}_{j+1} \max\left(0, 1 - \frac{2}{\tau_j}\mathrm{dist}(\cdot, (J_{j\frac{1}{2}})_{\tau_j/2})\right); \tag{3.76}$$

we also record the upper bound

$$\mathbf{L}(\hat{g}_{j+1}) \le \sqrt{3} + C^2 \frac{\varepsilon_0}{\hat{\varepsilon}_{\mathrm{m}}^{(j+1)}} + 2\frac{\varepsilon_{\mathrm{s}}^{(j+1)} + \hat{\varepsilon}_{\mathrm{s}}^{(j+1)}}{\tau_j}, \tag{3.77}$$

and impose the restriction $2(\varepsilon_{\mathrm{s}}^{(j+1)} + \hat{\varepsilon}_{\mathrm{s}}^{(j+1)}) \le \alpha_j \tau_j$ to get the last term in (3.77) to be $\le \alpha_j$.

We now let $G_{j+1} = G_j + \hat{g}_{j+1}$ and, akin to (3.69), we obtain the upper bound:

$$\mathbf{L}(G_{j+1}) \le \sqrt{3} + C^2 \frac{\varepsilon_0}{\min_{l\le j+1} \hat{\varepsilon}_{\mathrm{m}}^{(l)}} + 2\alpha_j, \tag{3.78}$$

after imposing the restriction $2(\varepsilon_{\mathrm{s}}^{(j+1)} + \hat{\varepsilon}_{\mathrm{s}}^{(j+1)}) \le \alpha_j R_j$.

Compared to `Step3`, the analogues of (3.72), (3.73) require some modifications because the errors cumulate additively; keep also in mind that for us an empty sum like $\sum_{j<1} \alpha_j$ defaults to 0. For $x \in C_a^{(l)}$ where $l \le j$ we obtain: then:

$$\begin{aligned} \left\| \mathrm{T}_{x, r_a^{(l)}} G_{j+1} - h \right\|_{\infty, B(0,1)} \le {} & \frac{\|G_{l-1} - G_{l-1}(x)\|_{\infty, B(x, r_a^{(l)})}}{r_a^{(l)}} + \left\| \mathrm{T}_{x, r_a^{(l)}} \hat{g}_l - h \right\|_{\infty, B(0,1)} \\ & + \frac{\sum_{l<k\le j+1} \|\hat{g}_k - \hat{g}_k(x)\|_{\infty, B(x, r_a^{(l)})}}{r_a^{(l)}}. \end{aligned} \tag{3.79}$$

The first term in (3.79) is bounded observing that G_{l-1} is α_{l-1} Lipschitz on $B(x, r_a^{(l)})$ for $k < l$. The second term is bounded using **(g_l:c)** and **($\hat{g}_l$:a)**. Finally the third term is bounded using $\|\hat{g}_k - \hat{g}_k(x)\|_{\infty,B(x,r_a^{(l)})} \leq 2\|\hat{g}_k\|_\infty$ and minding that we imposed the restriction $2(\varepsilon_{\mathrm{s}}^{(k+1)} + \hat{\varepsilon}_{\mathrm{s}}^{(k+1)}) \leq \alpha_k R_k$. We thus get from (3.79):

$$\begin{aligned}\left\|\mathrm{T}_{x,r_a^{(l)}} G_{j+1} - h\right\|_{\infty,B(0,1)} &\leq \alpha_{l-1} + C(\varepsilon_{\mathrm{s}}^{(l)} + \sigma) + 2\frac{\hat{\varepsilon}_{\mathrm{s}}^{(l)}}{r_a^{(l)}} + \sum_{l<k<j+1} \alpha_k \\ &= \sum_{\substack{l-1\leq k<j+1 \\ k\neq l}} \alpha_k + C(\varepsilon_{\mathrm{s}}^{(l)} + \sigma) + 2\frac{\hat{\varepsilon}_{\mathrm{s}}^{(l)}}{r_a^{(l)}}.\end{aligned} \tag{3.80}$$

Pick $x \in C_a^{(j+1)}$; then:

$$\left\|\mathrm{T}_{x,r_a^{(j+1)}} G_{j+1} - h\right\|_{\infty,B(0,1)} \leq \frac{\|G_j - G_j(x)\|_{\infty,B(x,r_a^{(j+1)})}}{r_a^{(j+1)}} + \left\|\mathrm{T}_{x,r_a^{(j+1)}} \hat{g}_{j+1} - h\right\|_{\infty,B(0,1)}. \tag{3.81}$$

Using that G_j is α_j-Lipschitz on $B(x, r_a^{(j+1)})$ and **(g_{j+1}:c)**, **($\hat{g}_{j+1}$:a)** we finally get:

$$\left\|\mathrm{T}_{x,r_a^{(j+1)}} G_{j+1} - h\right\|_{\infty,B(0,1)} \leq \alpha_j + C(\varepsilon_{\mathrm{s}}^{(j+1)} + \sigma) + 2\frac{\hat{\varepsilon}_{\mathrm{s}}^{(j+1)}}{r_a^{(j+1)}}. \tag{3.82}$$

`Step5: Choice of the parameters.`

There are two kinds of parameters:

- Parameters that can be chosen arbitrarily small: $\varepsilon_{\mathrm{s}}^{(l)}$, $\hat{\varepsilon}_{\mathrm{s}}^{(l)}$, $\varepsilon_{\mathrm{m}}^{(l)}$, $\varepsilon_{\mathrm{p}}^{(l)}$ and α_l.
- Parameters that can't be chosen arbitrarily small: $\hat{\varepsilon}_{\mathrm{m}}^{(l)}$ and σ. In particular, as $\hat{\varepsilon}_{\mathrm{m}}^{(l)} \to 0$ the Lipschitz constant of G_l blows-up.

We thus choose:

$$\hat{\varepsilon}_{\mathrm{m}}^{(l)} = \begin{cases} \min\left(\frac{\varepsilon_{\mathrm{m}}}{16}, \frac{1}{180C}\right) & \text{for } l \geq 2, \\ (\hat{\varepsilon}_{\mathrm{m}}^{(2)})^2 & \text{for } l = 1. \end{cases} \tag{3.83}$$

We now estimate:

$$\mu(J_1 \setminus J_1^{\mathrm{good}}) \leq \mu(J_1 \setminus \hat{J}_1) + \mu(\hat{J}_1 \setminus J_1^{\mathrm{good}}) \leq (\hat{\varepsilon}_{\mathrm{m}}^{(1)} + 1 - C^{-1}\sigma^{N-N_0})\mu(J_1); \tag{3.84}$$

thus, if we choose $\sigma = (1.5C\hat{\varepsilon}_{\mathrm{m}}^{(2)})^{1/(N-N_0)}$ we get:

$$\mu(J_1 \setminus J_1^{\mathrm{good}}) \leq \left(1 - \frac{\hat{\varepsilon}_{\mathrm{m}}^{(2)}}{2}\right) \mu(J_1). \tag{3.85}$$

For $l \geq 2$ the same argument for (3.85) yields:

$$\mu(J_l \setminus J_l^{\mathrm{good}}) \leq \left(1 - \frac{\hat{\varepsilon}_{\mathrm{m}}^{(2)}}{2}\right) \mu(J_l); \tag{3.86}$$

as $J_l \subset J_{l-1} \setminus J_{l-1}^{\mathrm{good}}$ induction grants:

$$\mu(J_l) \leq \left(1 - \frac{\hat{\varepsilon}_{\mathrm{m}}^{(2)}}{2}\right)^{l-1} \mu(K) \tag{3.87}$$

$$\mu(J_l \setminus J_l^{\mathrm{good}}) \leq \left(1 - \frac{\hat{\varepsilon}_{\mathrm{m}}^{(2)}}{2}\right)^{l} \mu(K). \tag{3.88}$$

The construction in `Step 3` will be iterated finitely many times and we just need an upper bound for the smallest number of iterations which will give the desired approximation in measure (third inequality in (a)). We get:

$$\begin{aligned}
\mu\left(K \setminus \bigcup_{k=1}^{l} (\hat{J}_k \cap J_k^{\mathrm{good}})\right) &\leq \mu(K \setminus J_1) + \sum_{k=1}^{l-1} \mu(J_{k\frac{1}{2}} \setminus J_{k+1}) + \sum_{k=1}^{l} \mu(J_k \setminus \hat{J}_k) \\
&+ \sum_{k=1}^{l-1} \mu\left((\hat{J}_k \setminus J_k^{\mathrm{good}}) \setminus J_{k\frac{1}{2}}\right) + \mu(J_l \setminus J_l^{\mathrm{good}}) \\
&\leq \Bigg(\varepsilon_{\mathrm{m}}^{(1)} + \sum_{k=1}^{l-1} \varepsilon_{\mathrm{m}}^{(k+1)} + \hat{\varepsilon}^{(1)} \sum_{k=1}^{l} \big(1 - \frac{\hat{\varepsilon}^{(2)}}{2}\big)^{k-1} \\
&+ \sum_{k=1}^{l-1} \varepsilon_{\mathrm{p}}^{(k)} + \big(1 - \frac{\hat{\varepsilon}^{(2)}}{2}\big)^{l}\Bigg)\mu(K).
\end{aligned} \tag{3.89}$$

Recall that at each stage we can choose $\varepsilon_{\mathrm{m}}^{(k)}$ and $\varepsilon_{\mathrm{p}}^{(k)}$ arbitrarily small and observe that:

$$\begin{aligned}
\hat{\varepsilon}_{\mathrm{m}}^{(1)} \sum_{k=1}^{l} \big(1 - \frac{\hat{\varepsilon}^{(2)}}{2}\big)^{k-1} &\leq 2\frac{\hat{\varepsilon}_{\mathrm{m}}^{(1)}}{\hat{\varepsilon}_{\mathrm{m}}^{(2)}} \\
&= 2\hat{\varepsilon}_{\mathrm{m}}^{(2)} \leq \varepsilon_{\mathrm{m}}.
\end{aligned} \tag{3.90}$$

Thus there is a universal constant C_1 such that if $l = \lceil |\log_{1-\hat{\varepsilon}_{\mathrm{m}}^{(2)}/2} \varepsilon_{\mathrm{m}}|\rceil$ one has:

$$\mu\left(K \setminus \bigcup_{k=1}^{l} (\hat{J}_k \setminus J_k^{\mathrm{good}})\right) \leq C_1 \varepsilon_{\mathrm{m}} \mu(K). \tag{3.91}$$

Thus we will let $J = \bigcup_{k=1}^{l} (\hat{J}_k \setminus J_k^{\mathrm{good}})$. We then have:

$$\|G_l\|_\infty \leq 2 \sum_{k=1}^{l} (\varepsilon_{\mathrm{s}}^{(k)} + \hat{\varepsilon}_{\mathrm{s}}^{(k)}) \tag{3.92}$$

which can be made $\leq C_1 \varepsilon_{\mathrm{s}}$ as the parameters $\varepsilon_{\mathrm{s}}^{(k)}$, $\hat{\varepsilon}_{\mathrm{s}}^{(k)}$ can be chosen arbitrarily small at each stage. Also the parameters α_k can be chosen arbitrarily small; thus, as $\hat{\varepsilon}^{(2)}_{\rm m} \simeq \varepsilon_{\rm m}^2$ we obtain a universal constant C_0 such that:

$$\mathbf{L}(G_l) \le \left(\sqrt{3} + C_0 \frac{\varepsilon_0}{\varepsilon_{\rm m}^2}\right). \tag{3.93}$$

We thus let $g = G_l$ and (3.62) now follows from (3.80), (3.82) if we choose at each step the parameters α_k, $\varepsilon_{\rm s}^{(k)}$ and $\hat{\varepsilon}_{\rm s}^{(k)}$ sufficiently small. □

3.8. **Proof of Theorem 1.2(II).** The proof will be achieved by an iteration of the Lemma 3.9, below which is a simple consequence of Lemma 3.7. More precisely, the function g required in Definition 1.2 is obtained as a sum of functions g_i given by Lemma 3.9. One of the subtle points in the process is that in principle it could be $\mathbf{L}(g_i) \ge \sqrt{3}$ for every i, hence the sum could fail to be Lipschitz in general. However, the fact that the g_i's can be chosen asymptotically flat and with arbitrarily small norm, allows one to control the Lipschitz constant of the sum.

Lemma 3.9. *Let $f : \Omega \to$ Lip be a Borel map such that, for μ-a.e. $x \in \Omega$, $f(x) \in C(\mu, x)$ and moreover the corresponding function L in (1.3) vanishes. Let $K \subset \Omega$ be a compact set such that $\mathbf{L}(f(x)) \le 1$ for every $x \in K$ and let $\varepsilon > 0$ be fixed. There are constants C_0 and C_1, depending only on N, a $(C_0+\sqrt{3})$-Lipschitz function $g : \Omega \to \mathbb{R}$ and a compact $J \subset K$ such that:*

(a) *g is asymptotically flat on J, $\|g\|_\infty \le \varepsilon$ and $\mu(K \setminus J) \le 3C_1\varepsilon\mu(K)$.*
(b) *There are $0 < r_1 \le r_0 \le \varepsilon$ and for every $x \in J$ there are $r_1 \le r(x) \le r_0$ such that:*

$$\left\|{\rm T}_{x,r(x)} g - f(x)\right\|_{\infty, B(0,1)} \le 3C_1\varepsilon. \tag{3.94}$$

(c) *g is supported on the tubular neighborhood of K with radius ε.*

Proof. Let C_0 and C_1 be the constants in Lemma 3.7. Since $f(x) \in C(\mu, x)$ for μ-a.e. x, by the Lusin's theorem we can find at most $N+1$ disjoint compact sets $K_j \subset K$ ($j = 0, \dots, N$) of positive measure, such that

$$\mu(K \setminus \bigcup_{j=0}^N K_j) < C_1\varepsilon\mu(K) \tag{3.95}$$

and $f(x) \in C(\mu, x)$ for every $x \in K_j$, for every j. Moreover, on each K_j both $V(\mu, x)$ and $f(x)$ vary continuously in x and $V(\mu, \cdot)$ has constant dimension j.

Since the Grassmannian of j-planes in $\mathbb{R}^N$ is totally bounded, and since $V(\mu, x)$ and $f(x)$ vary continuously in x on each K_j, we can find finitely many disjoint non-empty compact subsets $K_j^\ell \subset K_j$ ($\ell = 1, \dots, k_j$) such that

$$\|V(\mu, x) - V(\mu, y)\|_\infty \le \varepsilon^{2N} \tag{3.96}$$

and

$$\|f(x) - f(y)\|_\infty \le C_1\varepsilon \tag{3.97}$$

for every pair (x, y) of points in K_j^ℓ and moreover

$$\mu(K_j \setminus \bigcup_\ell K_j^\ell) \le \frac{C_1\varepsilon}{(N+1)}\mu(K). \tag{3.98}$$

Choose for each j, l a point x_j^ℓ and let $f_j^\ell := f(x_j^\ell) \in \mathrm{Lip}$. Notice that, by assumption, $\mathbf{L}(f_j^\ell) \le 1$. Let

$$d := \min\{1, \mathrm{dist}(K, (\mathbb{R}^N \setminus \Omega)), \min_{j,k}\{\mathrm{dist}(K_j^l, K_k^m) : l \ne m\}\}.$$

For every (j, ℓ), apply Lemma 3.7 with $K = K_j^\ell$, $\pi = V(\mu, x_j^\ell)$, $h = f_j^\ell \llcorner \pi^\perp$ and $\varepsilon_0 = \varepsilon^{2N}, r_0 = \frac{d}{4}\varepsilon, \varepsilon_{\mathrm{m}} = \varepsilon^N, \varepsilon_{\mathrm{s}} = C_1^{-1}\varepsilon\min\{1, \frac{d}{4}\}$. By the lemma for every (j, ℓ) there exist a $(\sqrt{3} + C_0)$-Lipschitz function $g_j^\ell : \mathbb{R}^N \to \mathbb{R}$ and a compact set $J_j^\ell \subset K_j^\ell$ such that:

(a) g_j^ℓ is asymptotically flat on J_j^ℓ, $\left\|g_j^\ell\right\|_\infty \le \varepsilon\min\{1, \frac{d}{4}\}$ and $\mu(K_j^a \setminus J_j^\ell) \le C_1\varepsilon^N(K_j^\ell)$.

(b) One can write every J_j^ℓ as a finite disjoint union $J_j^\ell = \bigcup_{a=1}^M J_{j,a}^\ell$ and for each $a \in 1, \cdots, M$ (M may depend on j and ℓ) there is an $0 < r_a := r_a(\ell, j) \le r_0$ such that if $x \in J_{j,a}^\ell$ one has:

$$\left\|\mathrm{T}_{x,r_a} g_j^\ell - f\right\|_{\infty, B(0,1)} \le 2C_1\varepsilon. \tag{3.99}$$

Via a simple cutoff, we can modify each g_j^ℓ to a $(\sqrt{3} + C_0)$-Lipschitz function $\bar{g}_j^\ell$ supported on Ω such that

$$\bar{g}_j^\ell = g_j^\ell \quad \text{on } \{x : \mathrm{dist}(x, K_j^\ell) < \frac{d}{4}\varepsilon\}$$

and

$$\bar{g}_j^\ell = 0 \quad \text{on } \{x : \mathrm{dist}(x, K_j^\ell) > \frac{d}{2}\varepsilon\}$$

Finally we define the function $g : \Omega \to \mathbb{R}$ by

$$g := \sum_{j,\ell} \bar{g}_j^\ell$$

and we observe that, by (a), $\|g\|_\infty \le \varepsilon$. Denoting $J := \bigcup_{j,\ell} J_j^\ell$, by (3.95), (3.98) and (a), we get

$$\mu(K \setminus J) \le 3C_1\varepsilon\mu(K).$$

Moreover, denoting $r_1 := \min_{j,\ell,a}\{r_a(\ell, j)\}$ and setting for every $x \in J_{j,a}^\ell$ $r(x) := r_a(\ell, j)$, we get by (3.97) and (b) that it holds $0 < r_1 \le r(x) \le r_0 \le \varepsilon$ and

$$\left\|\mathrm{T}_{x,r(x)} g - f(x)\right\|_{\infty, B(0,1)} \le 3C_1\varepsilon, \tag{3.100}$$

where we observe that by the choice of r_0, $\mathrm{T}_{x,r(x)} g = (g_j^\ell)_{x,r(x)}$, for every $x \in J_j^\ell$. □

proof of Theorem 1.2(II). `Step1: Prescribing the linear part.` Fix $\varepsilon > 0$. It is not restrictive to assume that $\mu(\Omega) = 1$. Let K be a compact set such that $\mu(\Omega \setminus K) < \varepsilon/4$, $f(x) \in C(\mu, x)$, and $\mathbf{L}(f(x)) \leq D$, for every $x \in K$ and for some $D > 0$. Without loss of generality we can assume that $D = 1$. Firstly, for every $x \in K$ we extend the linear function L which $f(x)$ defines on $V(\mu, x)$ (see (1.3)) to a linear function $\tilde{L}$ defined on $\mathbb{R}^N = (V(\mu, x), V(\mu, x)^\perp)$ as

$$\tilde{L}(x, y) = L(x).$$

Then we take any Borel measurable extension $\bar{L}$ of $\tilde{L}$ defined on the set Ω and preserving the bound $\mathbf{L}(\bar{L}) \leq 1$. By Theorem 2.1 applied to $f = \bar{L}$ and $\zeta = 1$, we can find a compact set $K^0 \subset K$ and a function $g_0 \in C^1_c(\Omega)$ with $\|g_0\|_\infty \leq C$ such that

$$\mu(\Omega \setminus K^0) < \varepsilon/2$$

$$Dg_0(x) = \bar{L}(x) = \tilde{L}, \quad \text{for every } x \in K^0$$

and $\mathbf{L}(g_0) \leq C$.

`Step2: Prescribing the non-linear part.` Consider the function $f_0 : \Omega \to$ Lip such that $f_0(x) \equiv 0$ for every $x \in \Omega \setminus K$ and $f_0(x) = f(x) - \tilde{L}(x)$ for every $x \in K$. We will apply Lemma 3.9 to a sequence of sets $(K_i)_{i\in\mathbb{N}}$ (with $K_0 := K$), the map f_0 and a sequence of parameters $(\varepsilon_i)_{i\in\mathbb{N}}$ with $\varepsilon_i \to 0$ and we will obtain respectively functions g^i, compact sets $J_i =: K_{i+1}$, and for every $x \in J_i$ a radius $r^i_1 \leq r^i(x) \leq r^i_0 \leq \varepsilon_i$.

Since we can choose the ε_i inductively, we can assume that for every i it holds

$$\varepsilon_i r^i_1 \geq \sum_{j>i} \varepsilon_j$$

so that, for every i it holds, for every $x \in J_i$

$$\begin{aligned}\left\| \mathrm{T}_{x,r^i(x)}(g^i + \sum_{j>i} g^j) - f_0(x) \right\|_{\infty,B(0,1)} &\leq \left\| \mathrm{T}_{x,r^i(x)} g^i - f_0(x) \right\|_{\infty,B(0,1)} \\ &\quad + \left\| \mathrm{T}_{x,r^i(x)} \sum_{j>i} g^j \right\|_{\infty,B(0,1)} \\ &\leq 3C_1 \varepsilon^i + \frac{\sum_{j>i} \varepsilon_j}{r^i(x)} \\ &\leq (3C_1 + 1)\varepsilon_i.\end{aligned} \tag{3.101}$$

Moreover, since the g^j's are asymptotically flat on the J_j's, we can add the further restriction on the inductive choice of the ε_i's that, for every j and for every $i > j$, g_j is $(\varepsilon_j)^i$-Lipschitz on a tubular neighbourhood of J_j of radius ε_i, below scale ε_i.

Hence, since $r^i(x) \le r_0^i \le \varepsilon_i$, we have, for every $x \in J_i$

$$\begin{aligned}\left\|\mathrm{T}_{x,r^i(x)}(g^i + \sum_{j<i} g^j) - f_0(x)\right\|_{\infty,B(0,1)} &\le \left\|\mathrm{T}_{x,r^i(x)}g^i - f_0(x)\right\|_{\infty,B(0,1)} \\ &\quad + \left\|\mathrm{T}_{x,r^i(x)} \sum_{j<i} g^j\right\|_{\infty,B(0,1)} \\ &\le \left\|\mathrm{T}_{x,r^i(x)}g^i - f_0(x)\right\|_{\infty,B(0,1)} + \sum_{j<i} \mathbf{L}(g^j \llcorner B_{r^i(x)}(x)) \\ &\le 3C_1\varepsilon_i + \sum_{j<i} (\varepsilon_j)^i.\end{aligned} \tag{3.102}$$

Denote $g_1 := \sum_{i\in\mathbb{N}} g^i$ and $J := \cap_i K_i$. Combining (3.101) and (3.102) we have that, provided (ε_i) respect the choices made above and provided $\sum_j (\varepsilon_j)^i \to 0$ as $i \to \infty$, and $\sum_j (\varepsilon_j) \le \varepsilon/2$ it holds that $\mu(\Omega \setminus J) \le \varepsilon/2$ and moreover, for every $x \in J$,

$$\left\|\mathrm{T}_{x,r^i(x)}g_1 - f_0(x)\right\|_{\infty,B(0,1)} \to 0 \quad \text{as } i \to \infty.$$

`Step3: Lipschitz estimates.` It remains to show that g_1 is Lipschitz. Let $x, y \in \Omega$ with $x \neq y$. Firstly observe that if $|x - y| \ge \frac{1}{2}\sum_i \varepsilon_i$, then the estimate is very simple, indeed

$$|g_1(y) - g_1(x)| \le 2\sum_{i=1}^{\infty} \|g^i\|_\infty \le 2\sum_i \varepsilon_i \le 4|x - y|. \tag{3.103}$$

Otherwise let us consider different cases. Here we make the following assumption on the sequence $(\varepsilon_j)_{j\in\mathbb{N}}$: for every j it holds $\varepsilon_j \ge 2\varepsilon_k$ for every $k > j$, hence in particular $\varepsilon_j \ge \sum_{k>j} \varepsilon_k$.

- if either $x \in J$ or $y \in J$. Let j_0 be the first index j such that $|y - x| \ge \varepsilon_j$. In particular x and y are in $B_{\varepsilon_{j_0-1}}(J^{j_0-1})$. Since we know that for every $j < j_0 - 1$, the function g^j is $(\varepsilon_j)^{j_0-1}$-Lipschitz on the tubular neighbourhood of $B_{\varepsilon_{j_0-1}}(J^j)$, below scale ε_{j_0-1}, this implies that $\sum_{j=1}^{j_0-2} g^j$ is $(\sum_{j=1}^{j_0-2}(\varepsilon_j)^{j_0-1})$-Lipschitz on $B_{\varepsilon_{j_0-1}}(J^{j_0-1})$ below scale ε_{j_0-1} and in particular $|(\sum_{j=1}^{j_0-2} g^j)(y) - (\sum_{j=1}^{j_0-2} g^j)(x)| \le |y - x|$. Moreover, since

$\varepsilon_{j_0} \geq \sum_{k>j_0} \varepsilon_k$, it holds $|y-x| \geq \frac{1}{2}\sum_{j=j_0}^{\infty} \varepsilon_j$. Hence we can write

$$\begin{aligned} |g_1(y)-g_1(x)| &\leq |(\sum_{j=1}^{j_0-2} g^j)(y) - (\sum_{j=1}^{j_0-2} g^j)(x)| \\ &\quad + |g^{j_0-1}(y) - g^{j_0-1}(x)| + |(\sum_{j=j_0}^{\infty} g^j)(y) - (\sum_{j=j_0}^{\infty} g^j)(x)| \\ &\leq (1+\sqrt{3}+C_0)|y-x| + 2\sum_{j=j_0}^{\infty} \varepsilon_j \leq (5+\sqrt{3}+C_0)|y-x|. \end{aligned} \tag{3.104}$$

- If $x \notin J$ and $y \notin J$, let i_0 (respectively j_0) be the first index i such that $x \notin B_{\varepsilon_i}(J^i)$ (respectively $y \notin B_{\varepsilon_i}(J^i)$). We can assume, without loss of generality, that $i_0 \leq j_0$.

  If $|x-y| < \varepsilon_{i_0+1}$, then necessarily $j_0 - i_0 \leq 1$. In this case (recalling (c) in Lemma 3.9) $g^j(x) = g^j(y) = 0$ for every $j \geq i_0+1$. Moreover for every $i < i_0 - 1$, the function g^i is $(\varepsilon_i)^{i_0-1}$-Lipschitz on the tubular neighbourhood of $B_{\varepsilon_{i_0-1}}(J^i)$, below scale ε_{i_0-1}, hence, as in the previous case, we can estimate

$$\begin{aligned} |g_1(y)-g_1(x)| &\leq |(\sum_{i=1}^{i_0-2} g^i)(y) - (\sum_{i=1}^{i_0-2} g^i)(x)| + |g^{i_0-1}(y) - g^{i_0-1}(x)| \\ &\quad + |g^{i_0}(y) - g^{i_0}(x)| \\ &\leq \sum_{i=1}^{i_0-1} (\varepsilon_i)^{i_0}|y-x| + 2(\sqrt{3}+C_0)|y-x| \leq (1+2\sqrt{3}+2C_0)|y-x|. \end{aligned} \tag{3.105}$$

  The last case to analyze is when $|x-y| \geq \varepsilon_{i_0+1}$. In this case let $k_0 \leq i_0+1$ be the first index k such that $|x-y| \geq \varepsilon_k$. Note that if $k_0 = 1$, then we fall in the first case considered, because we have required in particular that $\varepsilon_1 \geq \sum_{j>1} \varepsilon_j$, and hence (3.103) provides the Lipschitz estimate. Therefore we can consider only the case $k_0 \geq 2$.

  Since $|x-y| \leq \varepsilon_{k_0-1}$, then for every $i < k_0 - 1$, the function g^i is $(\varepsilon_i)^{k_0-1}$-Lipschitz on the tubular neighbourhood of $B_{\varepsilon_{k_0-1}}(J^i)$, below scale ε_{k_0-1}, hence

$$
\begin{aligned}
|g_1(y)-g_1(x)| &\le |(\sum_{j=1}^{k_0-2} g^j)(y) - (\sum_{j=1}^{k_0-2} g^j)(x)| + |g^{k_0-1}(y) - g^{k_0-1}(x)| \\
&\quad + |g^{k_0}(y) - g^{k_0}(x)| + 2\sum_{j=k_0+1}^{\infty} \varepsilon_j \\
&\le (\sum_{j=1}^{k_0-2} (\varepsilon_j)^{k_0-1} + 2\sqrt{3} + 2C_0)|y-x| + 2\varepsilon_{k_0} \\
&\le (4 + 2\sqrt{3} + 2C_0)|y-x|.
\end{aligned} \tag{3.106}
$$

`Step4: Conclusion of the proof.` Consider the Lipschitz function $g := g_0 + g_1$. It is easy to see that it holds

$$\mu(\{x \in \Omega : f(x) \not\subset \mathrm{Tan}(g,x)\}) \le \mu(\Omega \setminus (K^0 \cap J)) < \varepsilon.$$

Indeed $d_{V(\mu,x)} g_1 = 0$ on J and $d_{V^\perp(\mu,x)} g_0 = 0$ on K^0. Hence, since ε can be chosen arbitrarily small, f prescribes the blowups of a Lipschitz function weakly in the Lusin sense.

□

3.10. Remark. As we will show in the proof of Theorem 1.3 (II), once it is possible to prescribe weakly (in the Lusin sense) a blowup in a closed class of admissible functions, it is also possible to prescribe all the admissible blowups "simultaneously". By this we mean that it is possible to find a Lipschitz function attaining all admissible blowups on an arbitrarily large set of points. In the previous proof it is sufficient to select (in a measurable way) a countable dense set of admissible blowups $\{g_i(x)\}$ for every point x, and, selecting in a suitable way different blowups at different scales, one can build a function f attaining at many points all the g_i as blowups. To conclude, it is sufficient to observe that the set of all blowups at one point is closed.

4. Optmality of the class $C(\mu,\cdot)$

We now give an example of a measure for which one cannot prescribe more blowups than those contained in $C(\mu,\cdot)$. In general it seems a hard problem to characterize the largest set of blowups one can prescribe in terms of structural properties of the Radon measure. Given a Radon measure μ on $\mathbb{R}^N$, $r > 0$ and a point x we define the measure $\mathrm{T}_{x,r}\mu$ by

$$\mathrm{T}_{x,r}\mu(A) := \mu(x + rA), \text{ for every Borel set } A.$$

We denote by $\mathrm{Tan}(\mu,x)$ the set of the *blowups* of μ at x, i.e. all the possible limits of the form

$$\lim_{r_i \searrow 0} \kappa_i$$

where

$$\kappa_i := \frac{\mathrm{T}_{x,r_i}\mu \llcorner B^N(0,1)}{r_i^N}. \tag{4.1}$$

Fix $k \in \{1, \dots, N-1\}$ and let $\nu_{\mathbb{R}}$ be a doubling Radon measure on $\mathbb{R}$ such that $\nu_{\mathbb{R}}$ is singular with respect to the Lebesgue measure, its support is $\mathbb{R}$ and for $\nu_{\mathbb{R}}$-a.e $x \in \mathbb{R}$ the set $\mathrm{Tan}(\nu_{\mathbb{R}}, x)$ contains just positive multiples of the Lebesgue measure. Examples of such measures are discussed in [Pre87] or can be obtained modifying the example of [GKS10].

Let μ be the product measure

$$\mu = \mathcal{L}^k \otimes \nu_{\mathbb{R}}^{N-k}$$

and consider a Lipschitz function f on $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$. The decomposability bundle $V(\mu, \cdot)$ coincides with $\mathbb{R}^k$ as μ, by the choice of $\nu_{\mathbb{R}}$ is concentrated on a set which intersects each C^1-curve γ whose tangent vector does not lie in $\mathbb{R}^k$ in a set of zero 1-dimensional Hausdorff measure, and thus we have a well defined derivative $d_{\mathbb{R}^k} f$ in the direction of $\mathbb{R}^k$. Observe that μ is also doubling and fix a point P of approximate continuity for $d_{\mathbb{R}^k} f$. We can assume that at P all blowups of μ are positive multiples of $\mathcal{L}^N$.

Let g be a blowup of f at P and let $(r_i)_{i\in\mathbb{N}}$ be a sequence of radii for which $g = \lim_{i\to\infty} \mathrm{T}_{P,r_i} f$. Let $\tilde{\mu}$ be any subsequential limit of the sequence κ_i defined in (4.1) with $x = P$. Since the support of $\tilde{\mu}$ is the whole ball $B^N(0,1)$ and since $d_{\mathbb{R}^k} f$ is approximately continuous at P, for every $q \in \mathbb{R}^N$ and $v \in \mathbb{R}^k$ such that $q + v \in B^N(0,1)$ we get

$$g(q+v) - g(q) = \langle d_{R^k} f(P), v \rangle.$$

Let m be the (Lipschitz) restriction of g to $\mathbb{R}^{N-k} \cap B^N(0,1)$. Then for every $x \in \mathbb{R}^k, y \in \mathbb{R}^{N-k}$ such that $x + y \in B^N(0,1)$ it holds

$$g(x,y) = m(y) + \langle d_{R^k} f(P), x \rangle,$$

hence $g \in C(\mu, P)$.

5. Proof of theorem 1.3

As we already observed in Remark 1.4 (iii), point (I) of Theorem 1.3 is contained in Proposition 4.2 of [Mar]. Regarding point (II), we will prove a stronger (perhaps surprising) statement: namely we will prove that if μ is singular, then the generical 1- Lipschitz function (in the sense of Baire categories) attains every 1-Lipschitz function as blowup at μ-almost every point.

In this section we denote by X the complete metric space of 1-Lipschitz functions on $\mathbb{R}$ endowed with the supremum norm. By μ we denote a singular probability measure. We begin with the following lemma.

Lemma 5.1 (Covering by intervals with non-negligible centers). *Let $U \subset \mathbb{R}$ be an open set. For every $r_0 > 0$ and $n \in \mathbb{N}$ there is a sequence of closed intervals*

$\{[x_j - r_j, x_j + r_j]\}_j$ *contained in* U *with disjoint interiors such that:*

$$r_j \le r_0 \tag{5.1}$$

$$\mu(\bigcup_j [x_j - (8n)^{-1}r_j, x_j + (8n)^{-1}r_j]) \ge \frac{1}{16}n^{-1}\mu(\bigcup_j [x_j - r_j, x_j + r_j]) \tag{5.2}$$

$$\mu(U \setminus \bigcup_j [x_j - r_j, x_j + r_j]) = 0. \tag{5.3}$$

Proof. Through the proof the closed interval $[x - r, x + r]$ will be denoted by $I(x, r)$. Firstly we apply Corollary 2.3 with $\varepsilon = (2)^{-6}$ obtaining a sequence of disjoint intervals $\{I(z_\lambda, r_\lambda)\}_\lambda$. By the choice of ε, for every λ it holds that

$$\mu(I(z_\lambda, (1 - 2^{-6})r_\lambda) \ge \frac{1}{2}\mu(I(z_\lambda, r_\lambda)). \tag{5.4}$$

Now we "split" each interval $I(z_\lambda, (1 - 2^{-6})r_\lambda)$, into $2^7 - 2$ sub-intervals

$$\{I_\lambda^i := I(z_\lambda^i, 2^{-7}r_\lambda)\}_{i=1}^{2^7-2}.$$

with disjoint interiors and length $2^{-6}r_\lambda$. Denote by $\bar{I}_\lambda^i$ the "central part" of I_λ^i, i.e.

$$\bar{I}_\lambda^i := I(z_\lambda^i, 2^{-10}n^{-1}r_\lambda)$$

Observe that, for every λ, the family

$$\left\{\bigcup_{i=1}^{2^7-2} \bar{I}_\lambda^i + j2^{-9}n^{-1}r_\lambda\right\}_{j=0,\dots,8n-1}$$

covers the set $I(z_\lambda, (1 - 2^{-6})r_\lambda)$. Hence for at least one index j_0, the set

$$\bigcup_{i=1}^{2^7-2} I_\lambda^i + j_0 2^{-9}n^{-1}r_\lambda \tag{5.5}$$

satisfies

$$\mu(\bigcup_{i=1}^{2^7-2} \bar{I}_\lambda^i + j_0 2^{-9}n^{-1}r_\lambda) \ge \frac{1}{8}n^{-1}\mu(I(z_\lambda, (1 - 2^{-6})r_\lambda)) \overset{(5.4)}{\ge} \frac{1}{16}n^{-1}\mu(I(z_\lambda, r_\lambda)).$$

Moreover, for every $j = 0, \dots, 8n - 1$ and every $i = 1, \dots, 2^7 - 2$ the interval $I_\lambda^i + j2^{-9}n^{-1}r_\lambda$ is contained in the interior of $I(z_\lambda, r_\lambda)$ The result follows by adding to these intervals the two intervals

$$[z_\lambda - r_\lambda, a] \quad \text{and} \quad [b, z_\lambda + r_\lambda],$$

where a and b are respectively the minimum and the maximum of the set in (5.5). Of course the procedure above should be also repeated for every λ. □

5.2. Proposition. *Let* μ *be a singular probability measure on* $\mathbb{R}$. *Let* $f : [-1, 1] \to \mathbb{R}$ *be a 1-Lipschitz function with* $f(0) = 0$. *Then the set*

$$X_f := \{g \in X : f \in \mathrm{Tan}(g, x) \text{ for } \mu - \text{almost every } x\}$$

is residual in X (i.e. it contains the intersection of countably many open and dense sets).

Proof. For $n \in \mathbb{N}$ and $g \in X$, consider the set

$$E_g^n := \{x \in \mathbb{R} : \exists\, \rho < n^{-1} \text{ s.t. } |f - \mathrm{T}_{x,\rho} g| < n^{-1}\}$$

First of all we notice that E_g^n is Borel and in particular it is open. Indeed if $x \in E_g^n$, $\rho \le n^{-1}$ satisfies $|f - \mathrm{T}_{x,\rho} g| < n^{-1}$ and $y \in \mathbb{R}$ is so that

$$2|y - x| < \rho(n^{-1} - |f - \mathrm{T}_{x,\rho} g|),$$

then, using that g is 1-Lipschitz, we deduce

$$|f - \mathrm{T}_{y,\rho} g| \le |f - \mathrm{T}_{x,\rho} g| + |\mathrm{T}_{x,\rho} g - \mathrm{T}_{y,\rho} g| \le |f - \mathrm{T}_{x,\rho} g| + 2\rho^{-1}|y - x| < n^{-1},$$

hence $y \in E_f^n$. Now we define

$$A_n := \{g \in X : \mu(E_g^n) > 1 - n^{-1}\}.$$

`Step1:` A_n `is open.` Fix $g \in A_n$ and consider the multifunction $\varrho : E_g^n \to 2^{(0,n^{-1})}$ defined by

$$x \mapsto \{\rho \in (0, n^{-1}) \text{ s.t. } |f - \mathrm{T}_{x,\rho} g| < n^{-1}\}.$$

Notice that the values of ϱ are non-empty open sets because the function $(x, \rho) \mapsto |f - \mathrm{T}_{x,\rho} g|$ is continuous in the variable ρ. Moreover, since the function is continuous also in the variable x, for $\delta > 0$ the sets

$$U_\delta := \{x \in E_g^n : \rho_0(x) := \sup\{\varrho(x)\} > \delta \text{ and there exists } \rho(x) \text{ s.t. } \delta < \rho(x) \in \varrho(x) \text{ and } |f - \mathrm{T}_{x,\rho(x)} g| < n^{-1} - \delta\}$$

are open and $\bigcup_{\delta>0} U_\delta = E_g^n$. Moreover $\mu(E_g^n) > 1 - n^{-1}$, since $g \in A_n$. Then there exists $\delta > 0$, with $\mu(U_\delta) > 1 - n^{-1}$.

If we consider now $h \in X$ such that $2|g - h| < \delta^2$, we deduce that for every $x \in U_\delta$ it holds

$$\begin{aligned} |f - \mathrm{T}_{x,\rho(x)} h| &\le |f - \mathrm{T}_{x,\rho(x)} g| + |\mathrm{T}_{x,\rho(x)} g - \mathrm{T}_{x,\rho(x)} h| \\ &\le |f - \mathrm{T}_{x,\rho(x)} g| + 2|g - h|\delta^{-1} < (n^{-1} - \delta) + \delta. \end{aligned}$$

The former inequality guarantees that A_n is open.

`Step2:` A_n `is dense.` Let $g \in X$ and fix $\varepsilon > 0$. We want to show that there exists $h \in A_n$ such that $|h - g| \le \varepsilon$.

Consider inductively a sequence of functions h_i defined as follows. Let $h_0 := g$, $M_0 := \mathbb{R}$, $\alpha_0 := \varepsilon$ and for $i = 1, 2, \ldots$ let $U_i \subset M_{i-1}$ be an open set such that

$$\mathscr{L}^1(U_i) \le \frac{\alpha_{i-1}}{16n} \tag{5.6}$$

and

$$\mu(M_{i-1} \setminus U_i) < \frac{1}{n2^{i+2}}. \tag{5.7}$$

Moreover by Corollary 5.1 we can select $I^i_1, \dots, I^i_{m(i)} \subset U_i$ closed intervals with center x^i_j, length $4\ell^i_j$ and disjoint interiors such that firstly

$$\mu(U_i \setminus \bigcup_{j=1}^{m(i)} I^i_j) < \frac{1}{n2^{i+2}} \tag{5.8}$$

and secondly, denoting $\bar{I}^i_j$ the closed interval with center x^i_j and length $(2n)^{-1}\ell^i_j$,

$$\mu(\bigcup_{j=1}^{m(i)} \bar{I}^i_j) \geq (16n)^{-1}\mu(\bigcup_{j=1}^{m(i)} I^i_j), \quad \text{for every } i. \tag{5.9}$$

Or first aim is to perturb h_{i-1} obtaining a new function h_i such that all the points in $\bigcup_{j=1}^{m(i)} \bar{I}^i_j$ belong to $E^n_{h_i}$. Let f_i be the 1-Lipschitz function

$$f_i(x) := h_{i-1}(x - |(-\infty, x) \cap \bigcup_{j=1}^{m(i)} I^i_j|).$$

Observe that f_i is differentiable with $f'_i \equiv 0$ on the set $\bigcup_{j=1}^{m(i)} I^i_j$. Denote by k_i be the 1-Lipschitz function

$$k_i(x) := f_i(x) + \sum_{j=1}^{m(i)} f^j_i(x),$$

where $f^j_i : \mathbb{R} \to \mathbb{R}$ is any 1-Lipschitz function such that $f^j_i \equiv 0$ on $\mathbb{R} \setminus I^i_j$ and

$$\mathrm{T}_{x^i_j, \ell^i_j} f^j_i = f \tag{5.10}$$

(observe that such a function exists because f is 1-Lipschitz, $g(0) = 0$ and the length of the interval I^i_j is $4\ell^i_j$). Eventually we define the 1-Lipschitz function $h_i := f_i + k_i$.

Denote, for every i

$$\alpha_i := \min_{j=1,\dots,m(i)} \{\ell^i_j\}; \quad M_i := (\bigcup_{j=1}^{m(i)} I^i_j) \setminus (\bigcup_{j=1}^{m(i)} \bar{I}^i_j).$$

Note that the following properties hold, for every i, j

(i) $|h_i - h_{i-1}| \leq 2\mathcal{L}^1(U^i) \overset{(5.6)}{\leq} \frac{\alpha_{i-1}}{8n}$,
(ii) $|\mathrm{T}_{x,\ell^i_j} h_i - f| < (2n)^{-1}$, for every $x \in \bar{I}^i_j$.

Comparing (ii) with the definition of E^n_g, it is evident not only that every $x \in \bar{I}^i_j$ belongs to $E^n_{h_i}$, but also that there is still "room" for some additional perturbation. Namely for every function $\tilde h$ with $|h_i - \tilde h| < (4n)^{-1}\alpha_i$ it holds that every $x \in \bar I^i_j$ belongs to $E^n_{\tilde h}$, indeed

$$\begin{aligned} |\mathrm{T}_{x,\ell^i_j}\tilde h - f| &\overset{(i)}{\le} |\mathrm{T}_{x,\ell^i_j}\tilde h - \mathrm{T}_{x,\ell^i_j} h_i| + |\mathrm{T}_{x,\ell^i_j} h_i - f| \\ &\overset{(ii)}{\le} 2(\ell^i_j)^{-1}|\tilde h - h_i| + (2n)^{-1} < (2n)^{-1}(\ell^i_j)^{-1}\alpha_i + (2n)^{-1} \\ &\le n^{-1}. \end{aligned} \tag{5.11}$$

In particular, for every i, j, for every $x \in \bar I^i_j$ and for every $m > i$ it holds $x \in E^n_{h_m}$, since

$$|h_i - h_m| \le \sum_{j=i}^{m-1} \frac{\alpha_j}{8n} \overset{(5.6)}{\le} \frac{1}{8n}\alpha_i \sum_{j=0}^{m-j-1} (16)^{-i} < (4n)^{-1}\alpha_i.$$

Moreover, by (i) and the choice of α_0 it follows that $|g - h_m| < \varepsilon$, for every m. Combining (5.7), (5.8) and (5.9) we deduce that for i_0 large enough we have

$$\mu\Big(\bigcup_{i\le i_0}\Big(\bigcup_{j=1}^{m(i)} \bar I^i_j\Big)\Big) > 1 - n^{-1},$$

hence denoting $h := h_{i_0}$, we have that $h \in A_n$.

`Step3: Conclusion of the proof.` Clearly every function which belongs to the intersection of the A_n's is also in X_f, hence X_f is a residual set and in particular, by the Baire theorem, it is dense in X. □

Proof of theorem 1.3(II). Without loss of generality we can assume that the Lipschitz constant of $f(x)$ is bounded by 1 for μ-a.e. x and that $\Omega = \mathbb{R}$, because Proposition 5.2 also holds when $\mathbb{R}$ is replaced by an open subset Ω (clearly in this case the space X will be replaced by the space X^Ω of 1-Lipschitz functions on Ω).

First case. μ is a finite measure. Consider the metric space Z made by the 1-Lipschitz functions on $[-1, 1]$ with value 0 at the origin, endowed with the supremum distance. Let $(f_i)_{i\in\mathbb{N}}$ be dense in Z. Up to rescaling, we may assume that μ is a probability measure. By Proposition 5.2 each set X_{f_i} is residual in X, and so it is $Y := \bigcap_i X_{f_i}$. This means that for all $g \in Y$ and for μ-a.e. x, every f_i belongs to $\mathrm{Tan}(g, x)$. The theorem is then a consequence of the simple observation that $\mathrm{Tan}(g, x)$ is always a closed subset of Z.

Second case. μ is any Radon measure. Write $\mathbb{R}$ as a countable union of sets E_i, $i = 1, 2, \dots$ with finite measure. Consider for every i the space Y_i defined above relatively to the measure $\mu \llcorner E_i$. Since each Y_i is residual in X, so it is the set $Y_\infty := \cap_i Y_i$. □

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A.M.
Institut für Mathematik, Mathematisch-naturwissenschaftliche Fakultät, Universität Zürich
Winterthurerstrasse 190, CH-8057 Zürich, Switzerland
e-mail: `andrea.marchese@math.uzh.ch`

A.S.
Department of Mathematics, ETH Zürich
Rämistrasse 101 CH-8092 Zürich, Switzerland
e-mail: `andrea.schioppa@math.ethz.ch`