ENERGIES IN SBV AND VARIATIONAL MODELS IN FRACTURE MECHANICS

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Abstract: We describe some applications of special functions of bounded variation to problems in fracture mechanics.

1. Free Discontinuity Problems.

In the framework of Griffith's theory of fracture mechanics, the energy necessary to the production of a crack is proportional to the crack surface. If the medium under consideration is hyperelastic and brittle, *i.e.*, the elastic deformation outside the fracture surface can be modeled by the introduction of an elastic energy independent of the crack, then we can consider the problem of the existence of equilibria, under proper boundary conditions, in the framework of the calculus of variations. The simplest energy functional in the isotropic case, the study of whose minima leads to an existence result, takes the form

$$E(u,K) = \int_{\Omega\setminus K} W(\nabla u)\,dx + \lambda\,\mathcal{H}^2(K), \tag{1.1}$$

where u denotes the deformation, ∇u is the deformation gradient, $\Omega \subset \mathbb{R}^3$ is the reference configuration, K is the crack surface and $\mathcal{H}^2$ is the (Hausdorff) surface measure. The bulk energy density W accounts for elastic deformations outside the crack, while λ is a constant given by Griffith's criterion for fracture initiation (see [65], [62], [72], [20], [70], [54], [74]). The functional E makes sense in a classical way if K is a closed set, and $u \in C^1(\Omega \setminus K)$; a slightly more general approach could be to require that u belong to some Sobolev-Orlicz space in $\Omega \setminus K$ suitable for the bulk energy part. For non-isotropic materials the fracture initiation energy depends also on the orientation of the crack surface. In this case we must require that K be piecewise C^1, with a normal ν defined $\mathcal{H}^2$-a.e., so that we can study energies of the form

$$E'(u,K) = \int_{\Omega\setminus K} W(\nabla u)\,dx + \int_K \varphi(\nu)\,d\mathcal{H}^2\,, \tag{1.2}$$

with $0 < \alpha \le \varphi \le \beta$. Note that if $E(u,K) < +\infty$ then the Lebesgue measure of K is zero, u can be regarded as a measurable function defined on Ω, and the set K can be thought of as (a set containing) the set of discontinuity points for u. This explains why problems of this type are often called "free discontinuity problems". Note that in general K will not be the boundary of a set (in which case we talk of free *boundary* problems).

At a glance it is clear that the application of the direct methods of the calculus of variations to problems involving functionals of the form E or E' presents many difficulties, as no topology on closed sets is available that provides compactness for sequences of pairs (u_j, K_j) under the condition $\sup_j E(u_j, K_j) < +\infty$. It is therefore necessary to formulate a weak version of this kind of problems. The idea of De Giorgi and Ambrosio [50] has been to introduce a class of discontinuous functions and to replace the free surface K by the set of "discontinuity points" of these functions, which turns out to be "sufficiently regular" so that weak notions of surface area, orientation and traces can be given. We say that a function $u \in BV(\Omega;\mathbb{R}^m)$ belongs to $SBV(\Omega;\mathbb{R}^m)$ if its distributional derivative measure Du can be written as

$$Du = \nabla u\, \mathcal{L}^n \llcorner \Omega + (u^+ - u^-) \otimes \nu_u\, \mathcal{H}^{n-1} \llcorner S(u), \tag{1.3}$$

where ∇u is now the approximate gradient of u, $S(u)$ is the complement of the set of Lebesgue points of u, ν_u is the unit normal to $S(u)$, u^+, u^- are the approximate trace values of u on both sides of $S(u)$, the measures $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$ are the n-dimensional Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure, respectively. In this notation $\mu \llcorner A$ is the restriction of the measure μ to A; *i.e.*, $\mu \llcorner A(B) = \mu(A \cap B)$. If u is an arbitrary function in $BV(\Omega;\mathbb{R}^m)$ all the quantities above are well-defined; in general we have

$$Du = \nabla u\, \mathcal{L}^n \llcorner \Omega + (u^+ - u^-) \otimes \nu_u\, \mathcal{H}^{n-1} \llcorner S(u) + Cu, \tag{1.4}$$

where Cu is a measure which is singular with respect to the Lebesgue measure and does not charge sets of finite $\mathcal{H}^{n-1}$ measure. Hence, the requirement $u \in SBV(\Omega;\mathbb{R}^m)$ corresponds simply to forbid singular behaviour of the derivative of u outside of the set of discontinuity points. A precise definition of all the quantities above is given in Section 2 together with other (equivalent) definitions of SBV-spaces.

In the framework of the theory of SBV functions we have a weak formulation for energies of the type (1.1) or (1.2); for example E becomes

$$\mathcal{E}(u) = \int_\Omega W(\nabla u(x))\, dx + \lambda\, \mathcal{H}^2(S(u)), \qquad u \in SBV(\Omega;\mathbb{R}^3) \tag{1.5}$$

More in general we can deal with integral functionals of the form

$$\mathcal{F}(u) = \int_\Omega f(x, \nabla u(x))\, dx + \int_{S(u)} \varphi\big(x, u^+(x) - u^-(x), \nu_u(x)\big)\, d\mathcal{H}^{n-1}, \tag{1.6}$$

with $\Omega \subset \mathbb{R}^n$, f and φ suitable Borel functions, and $u \in SBV(\Omega;\mathbb{R}^m)$. Under natural growth conditions on f and φ the functional $\mathcal{F}$ is coercive, while suitable "convexity" conditions assure its lower semicontinuity, thanks to the compactness and semicontinuity results by Ambrosio, which are presented in Section 3. The application of the direct methods of the calculus of variations is now possible to obtain solutions to minimum problems in $SBV(\Omega;\mathbb{R}^m)$.

A general regularity theory for such "weak solutions" has yet to be developed, but important results have been obtained for some classes of integrals. De Giorgi, Carriero and Leaci ([51], [43]) have considered some minimum problems for the analog in $\mathbb{R}^n$ of the functional $\mathcal{E}$ in (1.5) with $f(\xi) = |\xi|^p$, showing that the jump set $S(u)$ of a (local) minimizer u differs from its closure by a set of $\mathcal{H}^{n-1}$-measure zero, and u is smooth outside $\overline{S(u)}$. It can be shown then that the pair $(u, \overline{S(u)})$ is a "classical" minimum point for the corresponding problem for the functional E in (1.1). Further regularity results have been obtained by Ambrosio, Fusco and Pallara who proved $\mathcal{H}^{n-1}$-a.e. smoothness of $S(u)$ ([16], [14], [15]).

2. Special functions of bounded variation.

In this section we introduce the class of SBV-functions, and present some equivalent definitions.

Notation. Let $m \geq 1$ and $n \geq 1$ be fixed integers. The set Ω is a bounded open subset of $\mathbb{R}^n$. If $x, y \in \mathbb{R}^n$ then $\langle x, y\rangle$ denotes their scalar product; $B_\rho(x)$ is the open ball with centre x and radius ρ; $M^{m\times n}$ is the space of the $m\times n$ real matrices. The usual product of a matrix $\xi \in M^{m\times n}$ and a vector $x \in \mathbb{R}^n$ is denoted by $\xi\cdot x$. The Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$ are denoted by $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$, respectively, but we also write $|E|$ in place of $\mathcal{L}^n(E)$. We use standard notation for the Lebesgue and Sobolev spaces $L^p(\Omega;\mathbb{R}^m)$ and $W^{1,p}(\Omega;\mathbb{R}^m)$, with norms $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$, rspectively.

Let $u : \Omega \to \mathbb{R}^m$ be a Borel function. We say that $z \in \mathbb{R}^m$ is the *approximate limit* of u in x and we write $z = \text{ap-}\lim_{y\to x} u(y)$ if for every $\varepsilon > 0$

$$\lim_{\rho\to 0} \rho^{-n}|\{y \in B_\rho(x)\cap\Omega : |u(y) - z| > \varepsilon\}| = 0\,.$$

We define the *jump set* of u $S(u)$ as the subset of Ω where the approximate limit of u does not exist. It turns out that $S(u)$ is a Borel set, $|S(u)| = 0$ and u is approximately continuous a.e. in Ω; more precisely, $u(x) = \text{ap-}\lim_{y\to x} u(y)$ for a.e. $x \in \Omega \setminus S(u)$.

BV-functions. We recall the main definition about functions of bounded variation. For the general theory we refer to [56], [61], [55] and [79]. We say that $u = (u^1,\ldots,u^m) \in L^1(\Omega;\mathbb{R}^m)$ is a *function of bounded variation* if its distributional first derivatives D_iu^j are (Radon) measures with finite total variation in Ω; that is, there exist finite measures μ_{ij} such that

$$\int_\Omega \frac{\partial\phi}{\partial x_j} u^i\,dx = -\int_\Omega \phi\,d\mu_{ij}\,.$$

This space will be denoted by $BV(\Omega;\mathbb{R}^m)$. We shall use Du to indicate the matrix-valued measure whose entries are D_iu^j. Functions of bounded variation can be introduced in various ways as subsets of $L^1(\Omega;\mathbb{R}^m)$. By Riesz's Theorem the requirement that Du be a vector measure is equivalent to requiring that

$$\Big|\int_\Omega \frac{\partial\phi}{\partial x_j} u^i\,dx\Big| \leq C\|\phi\|_\infty \qquad \phi \in C^1_0(\Omega)$$

for all i, j. Equivalently, it can be shown that $u \in L^1(\Omega; \mathbb{R}^m)$ belongs to $BV(\Omega; \mathbb{R}^m)$ if and only if there exists a sequence (u_j) of C^1 functions such that $u_j \to u$ in $L^1(\Omega; \mathbb{R}^m)$ and $|Du|(\Omega) = \lim_j \int_\Omega |\nabla u_j|\, dx < +\infty$. Both these characterization are very useful; for example the first one is stable under $L^1_{\rm loc}$-convergence of the u, while the second one often often enables the restriction of proofs to smooth functions.

If $u \in BV(\Omega; \mathbb{R}^m)$ then $S(u)$ is *countably $(n-1)$-rectifiable*, *i.e.*,

$$S(u) = N \cup \big(\bigcup_{i \in \mathbb{N}} K_i \big)\,, \tag{2.1}$$

where $\mathcal{H}^{n-1}(N) = 0$ and (K_i) is a sequence of compact sets, each contained in a C^1 hypersurface Γ_i. Moreover, there exist Borel functions $\nu_u : S(u) \to S^{n-1}$ and $u^+, u^- : S(u) \to \mathbb{R}^m$ such that for $\mathcal{H}^{n-1}$-a.e. $x \in S(u)$

$$\lim_{\rho \to 0} \rho^{-n} \int_{B^+_\rho(x) \cap \Omega} |u(y) - u^+(x)|\, dy = 0\,, \qquad \lim_{\rho \to 0} \rho^{-n} \int_{B^-_\rho(x) \cap \Omega} |u(y) - u^-(x)|\, dy = 0\,,$$

where $B^+_\rho(x) = \{y \in B_\rho(x) : \langle y - x, \nu_u(x)\rangle > 0\}$ and $B^-_\rho(x) = \{y \in B_\rho(x) : \langle y - x, \nu_u(x)\rangle < 0\}$. Hence, for $\mathcal{H}^{n-1}$-a.e. $x \in S(u)$

$$\lim_{\rho \to 0} \rho^{-n} |\{y \in B_\rho(x) \cap \Omega : \langle y - x, \pm\nu_u(x)\rangle > 0\,, |u(y) - u^\pm(x)| > \varepsilon\}| = 0$$

for every $\varepsilon > 0$. The triplet $(u^+(x), u^-(x), \nu_u(x))$ is uniquely determined up to a change of sign of $\nu_u(x)$ and a permutation of $u^+(x)$ and $u^-(x)$. The vector ν_u is normal to $S(u)$, in the sense that, if $S(u)$ is represented as in (2.1) then $\nu_u(x)$ is normal to Γ_i for $\mathcal{H}^{n-1}$-a.e. $x \in K_i$. In particular, it follows that $\nu_u(x) = \pm\nu_v(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in S(u) \cap S(v)$ and $u, v \in BV(\Omega; \mathbb{R}^m)$. If $x \notin S(u)$ we define $u^+(x) = u^-(x) = \text{ap-}\lim_{y \to x} u(y)$.

We denote by ∇u the density of the absolutely continuous part of Du with respect to the Lebesgue measure. $\nabla u(x)$ turns out to be the *approximate differential* of u at x for a.e. $x \in \Omega$, in the sense that

$$\lim_{\rho \to 0} \rho^{-n} \int_{B_\rho(x) \cap \Omega} \frac{|u(y) - u(x) - \nabla u(x) \cdot (y - x)|}{|y - x|}\, dy = 0\,.$$

We point out that the approximate differential is local on Borel sets, that is, if $u, v \in BV(\Omega; \mathbb{R}^m)$ then $\nabla u(x) = \nabla v(x)$ for a.e. $x \in \Omega$ such that $u(x) = v(x)$.

SBV-functions. We say that a function $u \in BV(\Omega; \mathbb{R}^m)$ is a *special function of bounded variation* if the singular part of Du is given by $(u^+ - u^-) \otimes \nu_u\, \mathcal{H}^{n-1} \llcorner S(u)$; *i.e.*.

$$Du = \nabla u\, \mathcal{L}^n + (u^+ - u^-) \otimes \nu_u\, \mathcal{H}^{n-1} \llcorner S(u)\,.$$

We denote the space of the special functions of bounded variation by $SBV(\Omega; \mathbb{R}^m)$. The introduction of this space is due to De Giorgi & Ambrosio [50]. In order to understand the meaning of SBV-functions it is instructive to consider the one-dimensional case $n = m = 1$, where belonging to SBV means being "piecewise $W^{1,1}$", while the Cantor-Vitali function, which is a continuous function and whose derivative is singular with respect to the Lebesgue measure, does not belong to SBV. For the properties of the functions $u \in SBV(\Omega;\mathbb{R}^m)$ we refer to [4] and [5]. The following characterization of $SBV(\Omega;\mathbb{R}^m)$ functions with $\mathcal{H}^{n-1}$-finite jump set has been obtained by Ambrosio [7].

Theorem 2.1 *Let $u \in BV(\Omega;\mathbb{R}^m)$. Then $u \in SBV(\Omega;\mathbb{R}^m)$ and $\mathcal{H}^{n-1}(S(u)) < +\infty$ if and only if there exists a constant C such that*

$$\Big|\int_\Omega \frac{\partial\varphi}{\partial x_i}(x,u) + \sum_{j=1}^m \frac{\partial\varphi}{\partial y_j}(x,u)\frac{\partial u^j}{\partial x_i}\,dx\Big| \le C\|\varphi\|_\infty\,, \tag{2.2}$$

for all $\varphi \in C^1_0(\Omega\times\mathbb{R}^m)$ and for all i.

Remark 2.2 The statement of Theorem 2.1 is actually improved requiring only the existence of a function $a \in L^1(\Omega;M^{m\times n})$ such that

$$\Big|\int_\Omega \frac{\partial\varphi}{\partial x_i}(x,u) + \sum_{j=1}^m \frac{\partial\varphi}{\partial y_j}(x,u)a_{ij}\,dx\Big| \le C\|\varphi\|_\infty \tag{2.3}$$

holds. The equality $a = \nabla u$ can be proved as a consequence of these formulas. Furthermore, $2\mathcal{H}^{n-1}(S(u))$ can be taken as the constant C.

Subspaces of $SBV(\Omega;\mathbb{R}^m)$ of particular importance are the domains of functionals of the form (1.5). If the bulk energy density W is of p-growth, then this space is given by

$$SBV^p(\Omega;\mathbb{R}^m) = \Big\{u \in SBV(\Omega;\mathbb{R}^m):\ \mathcal{H}^{n-1}(S(u)) < +\infty,\ \int_\Omega |\nabla u|^p\,dx < +\infty\Big\}\,,$$

which is roughly speaking the space of functions which are $W^{1,p}$ outside $S(u)$. For such functions it is possible to give a characterization which states that they can be approximated in a strong sense by piecewise smooth functions. The following result is by Braides and Chiadò Piat [35].

Theorem 2.3 *If $u \in SBV^p(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$ then there exists a sequence (u_j) in $SBV^p(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$ with $\|u_j\|_\infty \le \|u\|_\infty$, such that for each $j \in \mathbb{N}$ there exists a closed countably $(n-1)$-rectifiable set R_j such that $u_j \in C^1(\Omega\setminus R_j;\mathbb{R}^m)$, $\mathcal{H}^{n-1}(R_j) \to \mathcal{H}^{n-1}(S(u))$ and*

$$\begin{aligned} &u_j \to u \text{ in } L^1(\Omega;\mathbb{R}^m), && \nabla u_j \to \nabla u \text{ strongly in } L^p(\Omega;M^{m\times n}),\\ &\mathcal{H}^{n-1}(S(u_j)\triangle S(u)) \to 0, && \int_{S(u_j)\cap S(u)} (|u_j^+ - u^+| + |u_j^- - u^-|)d\mathcal{H}^{n-1} \to 0 \end{aligned}$$

(we choose the orientation $\nu_{u_j} = \nu_u$ $\mathcal{H}^{n-1}$-a.e. on $S(u_j)\cap S(u)$).

Such a characterization is useful when dealing with general functionals of the form (1.6) with discontinuous integrands, which may be very sensitive to variations of $S(u)$.

3. Compactness, lower semicontinuity and existence results.

The main property of SBV functions is the following compactness theorem due to L. Ambrosio (see [3], [7])

Theorem 3.1 (SBV Compactness Theorem) *Let (u_j) be a sequence in $SBV(\Omega;\mathbb{R}^m)$ such that*

$$\sup_j\Big(\int_\Omega |\nabla u_j|^p\,dx + \mathcal{H}^{n-1}(S(u_j)) + \|u_j\|_\infty\Big) < +\infty \tag{3.1}$$

for some $p > 1$; then there exists a subsequence $u_{j(k)}$ which converges in $L^1_{\rm loc}(\Omega;\mathbb{R}^m)$ to $u \in SBV(\Omega;\mathbb{R}^m)$. Moreover $\nabla u_{j(k)}$ weakly converges to ∇u in $L^p(\Omega;M^{m\times n})$.

Proof: the proof of this theorem is particularly simple after Theorem 2.1 and Remark 2.2. In fact, by (3.1) (u_j) is bounded in $BV(\Omega;\mathbb{R}^m)$ so that we may suppose without loss of generality that $u_j \to u$ weakly in BV, and in particular that $u_j \to u$ strongly in $L^1_{\rm loc}$ (actually, by the boundedness assumption of u_j and Ω, in any L^q), so that

$$\frac{\partial\varphi}{\partial x_i}(x,u_j) \to \frac{\partial\varphi}{\partial x_i}(x,u), \qquad \frac{\partial\varphi}{\partial y_j}(x,u_j) \to \frac{\partial\varphi}{\partial y_j}(x,u)$$

strongly in all L^q. We may suppose also that $\nabla u_j \to a$ weakly in $L^p(\Omega;M^{m\times n})$. By Remark 2.2, (2.2) holds for all (u_j) with $C = 2\sup_j \mathcal{H}^{n-1}(S(u_j))$, and hence, passing to the limit (2.3) holds with the same constant, so that $u \in SBV(\Omega;\mathbb{R}^m)$ and $a = \nabla u$. □

From the weak convergence of approximate gradients we immediately deduce the lower semicontinuity of convex integrals of the approximate gradients along sequences satisfying the thesis of Theorem 3.1 (and in particular along sequences satisfying (3.1) and converging to some function u in $L^1_{\rm loc}$). We also have a lower semicontinuity result valid for quasiconvex integrals as follows (see [6]). For the definition and properties of quasiconvex and polyconvex energies we refer to Ball [21], Acerbi and Fusco [1] and Dacorogna [45].

Theorem 3.2 (SBV Lower Semicontinuity Theorem - Bulk Integrals) *Let $(u_j) \subset SBV(\Omega;\mathbb{R}^m)$ be a sequence converging to $u \in SBV(\Omega;\mathbb{R}^m)$ in $L^1_{\rm loc}(\Omega;\mathbb{R}^m)$ such that $\sup_j \mathcal{H}^{n-1}(S(u_j)) < +\infty$, and let $f:\Omega\times M^{m\times n} \to [0,+\infty)$ be a Carathéodory function, quasiconvex in the second variable and satisfying the p-growth condition*

$$c_1(|A|^p - 1) \le f(x,A) \le c_2(|A|^p + 1) \tag{3.2}$$

for some strictly positive constants c_1 and c_2. Then

$$\int_\Omega f(x,\nabla u)\,dx \le \liminf_j \int_\Omega f(x,\nabla u_j)\,dx. \tag{3.3}$$

If f is polyconvex then the same lower semicontinuity result holds requiring a p-growth condition only from below with $p > n \wedge m$ (see [6] Corollary 4.9).

Necessary and sufficient conditions for the lower semicontinuity of the integrals on the free discontinuity set are studied in [10] and [5]. The general lower semicontinuity condition for the surface energy density is given by an integral inequality similar to quasiconvexity, at least for continuous integrands (for the case of discontinuous integrands some results are obtained in [35]). Here we state a sufficient condition.

Theorem 3.3 (SBV Lower Semicontinuity Theorem - Surface Integrals) *Let* $d : \mathbb{R}^m \to [0,+\infty)$ *be an even continuous function satisfying* $d(a+b) \le d(a) + d(b)$ (subadditivity), *and let* $\psi : \mathbb{R}^n \to [0,+\infty)$ *be an even convex function positively homogeneous of degree* 1. *Then we have*

$$\int_{S(u)} d(u^+ - u^-)\,\psi(\nu_u)\,d\mathcal{H}^{n-1} \le \liminf_j \int_{S(u_j)} d(u_j^+ - u_j^-)\,\psi(\nu_{u_j})\,d\mathcal{H}^{n-1} \tag{3.4}$$

if $u_j \to u$ *in* $L^1_{\rm loc}(\Omega;\mathbb{R}^m)$, $\nabla u_j \to \nabla u$ *weakly in* $L^1(\Omega; M^{m\times n})$ *and* $\sup_j \mathcal{H}^{n-1}(S(u_j)) < +\infty$. *In particular taking* $d = 1$ *and* $\psi(\nu) = |\nu|$ *we get* $\mathcal{H}^{n-1}(S(u)) \le \liminf_j \mathcal{H}^{n-1}(S(u_j))$.

From Theorems 3.1–3.3 it is easy now to obtain an existence result for confined problems in fracture mechanics.

Theorem 3.4 *Let* $W : M^{3\times 3} \to [0,+\infty)$ *be a polyconvex function satisfying* $W(A) \ge c|A|^p$ *with* $p > 3$ *and* $c > 0$ *(or a quasiconvex function satisfying a p-growth condition from below and from above for some* $p > 1$*), and let* $\varphi : \mathbb{R}^3 \to [0,+\infty)$ *be an even convex function positively homogeneous of degree* 1 *with* $\varphi(\nu) \ge c|\nu|$. *Let* K *be a non-empty compact set in* $\mathbb{R}^3$, *and let* $g \in L^1(\Omega;\mathbb{R}^3)$. *Then there exists a solution of the minimum problem*

$$\min\Big\{\int_\Omega W(\nabla u)\,dx + \int_{S(u)} \varphi(\nu_u)\,d\mathcal{H}^2 + \int_\Omega \langle g, u\rangle\,dx :\ u \in SBV(\Omega;\mathbb{R}^3),\ u \in K\ a.e.\Big\},$$

provided that this infimum is finite.

Proof: let (u_j) be a minimizing sequence for the problem above. We can suppose that

$$\sup_j\Big(\int_\Omega |\nabla u_j|^p\,dx + \mathcal{H}^{n-1}(S(u_j))\Big)$$

$$\le \frac{1}{c}\sup_j\Big(\int_\Omega W(\nabla u_j)\,dx + \int_{S(u_j)} \varphi(\nu_{u_j})\,d\mathcal{H}^2 + \int_\Omega \langle g, u_j\rangle\,dx\Big) + \|g\|_1 \sup\{|x| : x \in K\}$$

be finite. Hence, since $\|u_j\|_\infty \le \sup\{|x| : x \in K\}$, by Theorem 3.1 we can suppose that $u_j \to u \in SBV(\Omega;\mathbb{R}^3)$ in $L^1(\Omega;\mathbb{R}^3)$ and a.e., and $\nabla u_j \to \nabla u$ weakly in $L^1(\Omega; M^{3\times 3})$. Since $\int_\Omega \langle g, u_j\rangle\,dx \to \int_\Omega \langle g, u\rangle\,dx$ by Theorems 3.2 and 3.3 we get immediately that u is a minimum point as desired. □

A similar statement can be given for boundary value problems; in this case we have to reformulate the Dirichlet boundary conditions, penalizing the possible fracture at the boundary, considering minimum problems of the form

$$\min\Big\{\int_\Omega W(\nabla u)\,dx + \int_{S(u)} \varphi(\nu_u)\,d\mathcal{H}^2 + \int_{S(u\cup u_0)} \varphi(\nu)\,d\mathcal{H}^2 :\ u \in SBV(\Omega;\mathbb{R}^3)\Big\},$$

where Ω is a Lipschitz open set, ν is the inner normal to $\partial\Omega$, u_0 is the boundary datum, which we take suppose to be the outer trace on $\partial\Omega$ of a function in $u_0 \in SBV(\Omega', \mathbb{R}^3)$

($\Omega\subset\subset\Omega'$), and $S(u\cup u_0)=\{x\in\partial\Omega:\ u^+(x)\neq u_0(x)\}$ ($u^+(x)$ and $u_0(x)$ are the inner and outer traces at x, respectively, defined with respect to ν).

The confinement condition $u\in K$ does not seem to be natural for all problems in fracture mechanics, even though it may be a consequence of the geometry of the problem in the case of boundary values. To remove such a condition we have in general to state the problems in the larger space of the functions whose truncations are SBV (see [4]; some related techniques can be found in [42], [38] and [57]). For the sake of simplicity we will not treat this case.

4. Fracture mechanics in composite media.

We consider now the asymptotic behaviour of functionals of the type (1.6) modeling cellular elastic materials with fine microstructure. The study of this kind of nonlinear media, but without considering the possibility of fracture (*i.e.*, in the framework of Sobolev functions), has been carried on by S. Müller [68] and A. Braides [27] (see also [28], [29], [30], [33], [40], [59]). Here we consider functionals

$$\mathcal{F}_\varepsilon(u)=\int_\Omega f(\frac{x}{\varepsilon},\nabla u)\,dx+\int_{S(u)} g(\frac{x}{\varepsilon},(u^+-u^-)\otimes\nu_u)\,d\mathcal{H}^{n-1},\tag{4.1}$$

where f and g are Borel functions, periodic in the first variable, which model the response of the material to elastic deformation, and to fracture, respectively, at a microscopical scale (which is given by the small parameter ε). The behaviour of sequences of minima for problems involving $\mathcal{F}_\varepsilon$, and of the corresponding minimizers, can be deduced from the Γ-convergence of this sequence to a *homogenized* functional, which describes the overall response of the medium (see [52], [46] for an introduction to Γ-convergence, and [32] for Γ-convergence techniques for multiple integrals). The following result has been proven by Braides, Defranceschi and Vitali [38].

Theorem 4.1 (Homogenization Theorem) *Let f and $g:\mathbb{R}^n\times M^{m\times n}\to[0,+\infty)$ be Borel functions, periodic in the first variable; let $p>1$, C, α, $\beta>0$, and let f and g satisfy*

$$\begin{aligned}\alpha|\xi|^p\le f(x,\xi)\le\beta(1+|\xi|^p)\\ \alpha(1+C|\xi|)\le g(x,\xi)\le\beta(1+C|\xi|)\end{aligned}\qquad \textit{for all } x\in\mathbb{R}^n,\ \xi\in M^{m\times n}.\tag{4.2}$$

Let $f_{\rm hom}:M^{m\times n}\to[0,+\infty)$ and $g_{\rm hom}:M_1^{m\times n}\to[0,+\infty)$ ($M_1^{m\times n}$ denotes the set of matrices of rank one) be the homogenized bulk energy density *and the* homogenized surface energy density, *respectively, defined by*

$$f_{\rm hom}(\xi)=\lim_{T\to+\infty}\inf\Big\{\frac{1}{T^n}\int_{(0,T)^n} f(x,\nabla u+\xi)\,dx: u\in W_0^{1,p}((0,T)^n;\mathbb{R}^m)\Big\},\tag{4.3}$$

$$\begin{aligned}g_{\rm hom}(z\otimes\nu)=\lim_{T\to+\infty}\frac{1}{T^{n-1}}\inf\Big\{\int_{TQ_\nu\cap S(u)} g(x,(u^+-u^-)\otimes\nu_u)\,d\mathcal{H}^{n-1}:\\ u\in SBV(TQ_\nu;\mathbb{R}^m),\nabla u=0\ a.e.,\ u=u_{z,\nu}\ \textit{on}\ \partial(TQ_\nu)\Big\},\end{aligned}\tag{4.4}$$

where Q_ν is any unit cube in $\mathbb{R}^n$ with centre in the origin and one face orthogonal to ν, and

$$u_{z,\nu}(x) = \begin{cases} z & \text{if } \langle x,\nu\rangle \geq 0 \\ 0 & \text{if } \langle x,\nu\rangle < 0 . \end{cases} \tag{4.5}$$

Let $\mathcal{F}_{\rm hom}$ be the homogenized functional *defined by*

$$\mathcal{F}_{\rm hom}(u) = \int_\Omega f_{\rm hom}(\nabla u)\,dx + \int_{S(u)} g_{\rm hom}((u^+ - u^-)\otimes \nu_u)\,d\mathcal{H}^{n-1}. \tag{4.6}$$

Then the functionals $\mathcal{F}_\varepsilon$ Γ-converge in the $L^1(\Omega;\mathbb{R}^n)$-topology to $\mathcal{F}_{\rm hom}$ as $\varepsilon \to 0$.

As a consequence of this theorem we can deduce the convergence of minimum problems, also in the case $C = 0$ in (4.2) with a singular perturbation technique (see [38] Section 8).

Note that the homogenized bulk energy density is the same integrand computed in [27] in the case without fracture. From (4.3)–(4.6) we obtain that the overall behaviour of the medium described by (4.1) at the scale ε is the one of a homogeneous material whose bulk elastic response is given by the study of $\mathcal{F}_\varepsilon$ only on elastic deformations without cracks, and whose response to fracture can be derived by the examination of 'stiff deformations' (*i.e.*, where $\nabla u = 0$). In particular, note that the homogenized surface energy density is not influenced by f; this phenomenon is particular of the process of homogenization, since in general we do have an interaction (see [2] Theorem 4.1). We mention also that the homogenization under SBV-growth conditions (4.2) gives rise to a different type of phenomena than when a growth of order one is allowed; *i.e.*,

$$f(x,\xi) \leq \gamma|\xi| \qquad \text{or} \qquad g(x,\xi) \leq \gamma|\xi| \tag{4.7}$$

(*e.g.*, if $g(x,\cdot)$ is positively homogeneous of degree one), in which case the homogenized functional is defined and finite on the whole $BV(\Omega;\mathbb{R}^m)$, and the homogenized energy densities are determined by an interaction between f and g (see [30]).

5. Barenblatt-type materials.

Functionals of the form (1.5), which are related to Griffith materials, allow the modeling of quasi-static fracture propagation only if a pre-existing crack is present and require singular stresses at the crack tip. These drawbacks can be overcome if we consider energies of the form

$$I(u) = \int_\Omega W(\nabla u)\,dx + \int_{S(u)} \varphi(|u^+ - u^-|)\,d\mathcal{H}^{n-1}, \tag{5.1}$$

in whose framework can be included the models proposed by Barenblatt [22], which allow an interaction between the two sides of the fracture for small values of the crack opening. If $\varphi(t) \to 0$ as $t \to 0$ then the surface energy of the functional I does not satisfy the hypotheses of Theorem 3.3, and I may not be lower semicontinuous in natural topologies. As a consequence, minimizing sequences for problems related to I may have unbounded jump set $\mathcal{H}^{n-1}$-measure, and may converge to a function with a "plasticity zone", which means that the limit of the jump part of the derivatives may not be a measure concentrated on a set of Hausdorff dimension $n-1$. In order to give a choice criterion among minimizing sequences we can follow a singular perturbation approach, considering functionals

$$I_\varepsilon(u) = I(u) + \varepsilon\mathcal{H}^{n-1}(S(u)). \tag{5.2}$$

These functionals are of Griffith-type, with $\lambda = \varepsilon$ representing the energy necessary to create a fracture of unit length. If φ is subadditive then I_ε is lower semicontinuous, and we can consider the minimum points $u_\varepsilon \in SBV(\Omega;\mathbb{R}^m)$ related to I_ε (if an application of the existence theorems is possible). The limit points of converging subsequences in $SBV(\Omega;\mathbb{R}^m)$ of (u_ε) are precisely the limits of minimizing sequences for the corresponding problems for I which are in $SBV(\Omega;\mathbb{R}^m)$ and have minimal $\mathcal{H}^{n-1}$-measure. A precise statement of this fact can be found in a paper by Braides and Coscia [36], where more general functionals of the form

$$I_\varepsilon(u) = I(u) + \varepsilon\int_{S(u)} \psi(|u^+ - u^-|)\, d\mathcal{H}^{n-1} \tag{5.3}$$

are considered.

We can give a mechanical interpretation to the approximating approach outlined above: the length of the pre-existing fracture necessary to quasi-static crack growth in a medium related to I_ε is proportional to ε. The passage to the limit as $\varepsilon \to 0$ can be viewed then as the requirement of pre-existing infinitesimal fractures; *i.e.*, as a modeling of microfractures. The form of ψ is related to the properties of these microfractures.

6. Linear elasticity problems.

Linear elasticity is not compatible with the Griffith theory of fracture mechanics, since this one implies infinite stresses at crack tips. However it may be interesting as a first approximation to study the functional in (1.1) when W is a linear-elasticity energy density. In this case W is degenerate as a quadratic form with respect to ∇u, and the framework of SBV functions is not suitable. To deal with minimum problems related to bulk energy densities which are coercive quadratic form with respect to the linearized strain tensor $\mathcal{E}u = \frac{1}{2}(\nabla_j u^i + \nabla_i u^j)_{ij}$ it is convenient to work within the space of *functions of bounded deformation* (see [66], [75], [77], [78], [64])

$$BD(\Omega) = \{u \in L^1(\Omega;\mathbb{R}^n) :\ Eu \text{ bounded (matrix-valued) Radon measure}\},$$

(where $(Eu)_{ij} = \frac{1}{2}(D_i u^j + D_j u^i)$), and to introduce a new subspace of BD functions analogous to SBV. To this purpose Ambrosio, Coscia and Dal Maso [12] have proven the following structure theorem.

Theorem 6.1 (BD Structure Theorem) *Let* $u \in BD(\Omega)$ *and let*

$$J(u) = \Big\{x \in \Omega :\ \lim_{\rho\to 0+} \frac{|Eu|(B(x,\rho))}{\rho^{n-1}} = 0\Big\}.$$

Then $J(u)$ is rectifiable with normal ν_u and there exist the traces $u^{\pm}$ on both sides of $J(u)$. Moreover, we have

$$Eu = \mathcal{E}u\,\mathcal{L}^n + \frac{1}{2}\Big((u^+ - u^-)\otimes\nu_u + \nu_u\otimes(u^+ - u^-)\Big)\,\mathcal{H}^{n-1}\llcorner J(u) + Cu$$

with Cu singular with respect to the Lebesgue measure and vanishing on Borel sets of σ-finite $\mathcal{H}^{n-1}$-measure. Finally, $|Cu|$-a.e. point of Ω is a Lebesgue point for u.

Theorem 6.1 shows that a decomposition analogous to (1.4) holds in BD. The only difference is that for BD functions it is not known whether $\mathcal{H}^{n-1}(S(u)\setminus J(u)) = 0$ or not. Kohn proved in [64] that the set $L = S(u)\setminus J(u)$ has Hausdorff dimension $(n-1)$, has zero $(n-1)$-dimensional capacity and is purely $(n-1)$-unrectifiable. The last statement in Theorem 6.1 shows that $Eu\llcorner L = 0$, i.e., the set L does not charge any distributional measure.

We can define now the space of *special functions of bounded deformation* $SBD(\Omega)$ as the subset of $BD(\Omega)$ of functions such that $Cu = 0$. This space has been first introduced in a different form by Bellettini, Coscia and Dal Maso [24], who proved the following compactness result.

Theorem 6.2 (SBD Compactness Theorem) *Let (u_j) be a sequence of $SBD(\Omega)$ functions such that*

$$\sup_j\Big(\int_\Omega |\mathcal{E}u_j|^2\,dx + \mathcal{H}^{n-1}(J(u_j)) + \|u_j\|_\infty\Big) < +\infty\,.$$

Then there exists a subsequence $u_{j(k)}$ converging in $L^1_{\rm loc}(\Omega;\mathbb{R}^n)$ to a function $u\in SBD(\Omega)$. Moreover $\mathcal{E}u_j \to \mathcal{E}u$ weakly in $L^2(\Omega;\mathbb{R}^{n^2})$ and $\mathcal{H}^{n-1}(J(u)) \le \liminf_j \mathcal{H}^{n-1}(J(u_j))$.

In this framework it is therefore possible to repeat the proof of the analogue of the existence results of Section 3.

7. Hencky's Plasticity.

In this section we recover a classical functional of Hencky's plasticity with a *relaxation* procedure, starting from a somewhat simpler functional defined of $SBD(\Omega)$. In this case the growth conditions of bulk and surface energy densities required in Section 3 are not satisfied, and the functional $\mathcal{F}$ in (1.6) is not lower semicontinuous even if convexity and subadditivity conditions are satisfied. For an introduction to relaxation techniques we refer to the book by Buttazzo [41].

Let Ω be a bounded domain in $\mathbb{R}^3$ and let $u:\Omega\to\mathbb{R}^3$ represent the displacement field of an elastic perfectly plastic body occupying the domain Ω in unstrained position. Denote by $\sigma:\Omega\to {\rm M}^{\rm sym}$ the corresponding stress tensor field referred to the configuration Ω (${\rm M}^{\rm sym}$ stands for the space of 3×3 symmetric real matrices). Let $E^D u$ be the *deviator* of the linearized strain tensor : $E^D u = Eu - \frac{1}{3}({\rm div}u)I$, where I is the identity matrix. Then the variational formulation of the displacement and stress problems in the theory of Hencky's plasticity (see [76], [53], [71]) involves respectively the functionals

$$\int_\Omega \psi(Eu(x))dx\,,\qquad \int_\Omega \psi^*(\sigma(x))dx\,,$$

where ψ and ψ^* are non-negative convex functions conjugate each other (in the duality theory of Fenchel-Moreau), and

$$\psi^*(\sigma) = \begin{cases} \frac{1}{4\mu}|\sigma^D|^2 + \frac{1}{18\kappa}(\operatorname{tr}\sigma)^2\,, & \text{if } \sigma \in K \\ +\infty\,, & \text{otherwise}\,. \end{cases}$$

Here $\operatorname{tr}\sigma = \sum_i \sigma_{ii}$, $\sigma^D = \sigma - \frac{1}{3}(\operatorname{tr}\sigma)\,I$, $\kappa = \lambda + \frac{2}{3}\mu$ is the bulk modulus of the material, λ and μ are the Lamè coefficients and K is a closed convex subset of $\mathrm{M}^{\mathrm{sym}}$ describing the elastic zone for the stress. Two classical examples of the set K are:

$$(7.1)\qquad \begin{aligned} K &= \{\sigma \in \mathrm{M}^{\mathrm{sym}} : |\sigma^D| \le \sqrt{2}k\} && \text{(von Mises' model)} \\ K &= \{\sigma \in \mathrm{M}^{\mathrm{sym}} : \lambda_M(\sigma) - \lambda_m(\sigma) \le C\} && \text{(Tresca's model)}, \end{aligned}$$

where k and C are fixed positive constants and $\lambda_M(\sigma)$ and $\lambda_m(\sigma)$ denote the maximum and the minimum eigenvalue of σ. Both these convex sets are of the form $K^D \oplus \mathbb{R}I$, where K^D is the orthogonal projection of K onto the space $\mathrm{M}^{\mathrm{sym}}_0$ of the matrices with null trace. This allows to write the functional of the displacement problem as

$$(7.2)\qquad \int_\Omega (\varphi(E^D u) + \frac{\kappa}{2}(\operatorname{div} u)^2)dx\,,$$

where $\varphi : \mathrm{M}^{\mathrm{sym}}_0 \to [0, +\infty)$ is the convex function in duality with

$$(7.3)\qquad \varphi^*(\sigma) = \begin{cases} \frac{1}{4\mu}|\sigma|^2\,, & \text{if } \sigma \in K \cap \mathrm{M}^{\mathrm{sym}}_0\,, \\ +\infty\,, & \text{otherwise} \end{cases} \qquad (\sigma \in \mathrm{M}^{\mathrm{sym}}_0)\,.$$

The application of the direct method of the Calculus of Variations to the minimum problem for the displacement, *i.e.*, involving the functional (7.2), requires a space where the functional is coercive and lower semicontinuous. Since φ grows only linearly as $|\xi^D| \to +\infty$ (note that $\varphi^* = +\infty$ outside $K \cap \mathrm{M}^{\mathrm{sym}}_0$) and no Korn's inequality holds on $W^{1,1}(\Omega; \mathbb{R}^3)$ (see [66], [64]), to gain coerciveness the definition of the functional must be extended to account for displacements whose strains are merely measures. This has led, in the general n-dimensional case, to the introduction of the space ([75], [77], [78], [63])

$$\mathcal{P} = \{u \in BD(\Omega) : \operatorname{div} u \in L^2(\Omega)\}\,.$$

The natural $L^1(\Omega; \mathbb{R}^3)$-lower semicontinuous extension to $\mathcal{P}$ of (7.2) turns out to be ([19])

$$(7.4)\qquad \int_\Omega (\varphi(\mathcal{E}^D u) + \frac{\kappa}{2}(\operatorname{div} u)^2)dx + \int_\Omega \varphi^\infty(\frac{E^D_s u}{|E^D_s u|})\,|E^D_s u|\,,$$

where $E^D u = \mathcal{E}^D u \mathcal{L}^n + E^D_s u$ is the Lebesgue decomposition of $E^D u$, $|E^D_s u|$ denotes the total variation of $E^D_s u$, $\frac{E^D_s u}{|E^D_s u|}$ stands for the Radon-Nikodym derivative, and $\varphi^\infty(\xi) = \lim_{t\to+\infty} \varphi(t\xi)/t$ is the recession function of φ.

We propose a different approach to problems involving the functional (7.4) introducing a somehow simpler model. Consider the space $SBD(\Omega) \cap \mathcal{P}$ whose elements are allowed to have "singularities" in the form of jump discontinuities along surfaces (the possible slippage surfaces of the material). It may be natural the attempt to obtain a functional of type (7.4) by relaxation of the following

$$(7.5)\quad F(u) = \int_\Omega (\mu|\mathcal{E}^D u|^2 + \frac{\kappa}{2}(\mathrm{div} u)^2)dx + c\int_{J(u)} |u^+ - u^-| d\mathcal{H}^2\,, \quad u \in SBD(\Omega)\cap\mathcal{P},$$

where $J(u)$ is the set defined in Theorem 6.1 and c is a positive constant. Indeed, the volume integral in the definition of F represents the (linearized) strain energy of an elastic material corresponding to a displacement u from the reference configuration Ω. The second term is a surface integral which takes into account the possible sliding of the material. While F models the microscopic form of the energy, from a macroscopic point of view, *i.e.*, as far as minimum problems are concerned, we can equivalently consider its lower semicontinuous envelope (or relaxation) $\overline{F}$ with respect to a suitable topology, which turns out to be the $L^1(\Omega;\mathbb{R}^3)$ topology. In [39] Braides, Defranceschi and Vitali have shown how in the relaxed functional $\overline{F}$ the effect of the volume and surface terms in (7.5) combine giving rise to a typical plastic behaviour: slippage is preferred to large strains $|\mathcal{E}^D u|$ and $\overline{F}$ can be written as

$$\overline{F}(u) = \int_\Omega (\phi(\mathcal{E}^D u) + \frac{\kappa}{2}(\mathrm{div} u)^2)dx + \int_\Omega \phi^\infty(\frac{E^D_s u}{|E^D_s u|})\,|E^D_s u|\,, \qquad u \in \mathcal{P}\,,$$

where ϕ is a convex function with linear growth at infinity. This relaxation procedure recovers *exactly* the functional (7.4) when adopting Tresca's yielding model. Indeed, the integrand function ϕ coincides with the function φ of (7.4) when the convex set K is the second one displayed in (7.1), with $C = 2c$.

Following Section 5, $\overline{F}$ can also be considered as the limit (in the sense of Γ-convergence) of the sequence of functionals

$$F_\varepsilon(u) = F(u) + \varepsilon\mathcal{H}^{n-1}(J(u))\,,$$

as $\varepsilon \to 0$, to whose minimum problems we can apply the existence results of Section 6. Such a singular perturbation approach may lead to a choice criterion among minima of boundary value problems involving the functional $\overline{F}$. For relaxation results in a BV setting we refer to [37], [26] and [23].

8. Problems involving tangential derivatives.

The viewpoint described above privileges the reference configuration, neglecting the effects of crack deformation. In a paper by Ambrosio, Braides and Garroni [11] it is discussed the possibility to define a sub-class of SBV functions which allow the statement (and solution) of problems taking into account also the deformation of $S(u)$, *i.e.*, the shape of the crack surface in the deformed configuration.

As an example we can think of an elastic body in two dimensions subject to fracture, so that a "hole" is formed bounded by two curves Γ^+ and Γ^- which are the images of $S(u)$

by u^+ and u^-, respectively. If the traces are sufficiently smooth then the length of (the boundary of the hole) $\Gamma^+ \cup \Gamma^-$ is given by

$$E_1(u) = \int_{S(u)} \Big(\Big| \frac{\partial u^+}{\partial \tau} \Big| + \Big| \frac{\partial u^-}{\partial \tau} \Big| \Big) \, d\mathcal{H}^1,$$

where τ is the tangent to $S(u)$. Similarly, if u is bounded and we have an "opening hole" (that is, $\Gamma^+ \cup \Gamma^-$ is compactly contained in $u(\Omega)$) we can also consider the "area of the hole", given by

$$E_2(u) = \int_{\rm hole} dy_1 dy_2 = - \int_{\Gamma^+ \cup \Gamma^-} y_1 dy_2 = - \int_{S(u)} \Big(u_1^+ \frac{\partial u_2^+}{\partial \tau} - u_1^- \frac{\partial u_2^-}{\partial \tau} \Big) \, d\mathcal{H}^1,$$

which again makes sense if the tangential derivatives of $u^\pm$ exist.

It is clear that the crucial point in order to apply the direct methods of the Calculus of Variations will be a weak definition of the tangential derivatives of u^+ and u^- on $S(u)$. To this purpose, the starting point is the characterization of the space SBV in Theorem 2.1. We can interpret formula (2.2) as a property of the graph of u, which is given for BV functions by

$$\Gamma = \{(x, u(x)) : \; x \in \Omega, \; \exists \nabla u(x)\},$$

and is oriented by the unit n-covector

$$\eta(x, u(x)) = \frac{1}{|\mathbf{M}(\nabla u)(x)|} (e_1 + \sum_j \frac{\partial u^j}{\partial x_1}(x) \varepsilon_j) \wedge \ldots \wedge (e_n + \sum_j \frac{\partial u^j}{\partial x_n}(x) \varepsilon_j) \, ,$$

where $\{e_1, \ldots, e_n\}$ and $\{\varepsilon_1, \ldots, \varepsilon_m\}$ are the standard orthonormal basis of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and $\mathbf{M}(\nabla u)$ denotes the vector of all minors of ∇u (see [60]). We can define the linear functional on n-forms (n-current) "integration on the graph", by

$$T_u(\omega) = \int_\Gamma \langle \omega, \eta \rangle d\mathcal{H}^n \, ,$$

and the *boundary of* T_u as the $(n-1)$-current given by

$$\partial T_u(\omega) = T_u(d\omega).$$

We can re-read formula (2.2) as a property of ∂T_u. In fact, using the area formula, we have

$$\int_\Omega \frac{\partial \varphi}{\partial x_i}(x, u) + \sum_{j=1}^m \frac{\partial \varphi}{\partial y_j}(x, u) \frac{\partial u^j}{\partial x_i} \, dx = \partial T_u(\varphi d\widehat{x}_i)$$

where $d\widehat{x}_i = (-1)^{i+1} dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n$, so that (2.5) states precisely that the boundary of T_u is a measure when computed on "horizontal forms" (*i.e.*, forms with no dy).

The class $\mathbf{SBV_0}$. Intuitively, tangential derivatives of $u^{\pm}$ give information on the "vertical part of the boundary of the graph of u". Following this reasoning Ambrosio, Braides and Garroni have defined in [11] a sub-class of $SBV(\Omega;\mathbb{R}^n)$ functions with $\mathcal{H}^{n-1}(S(u)) < +\infty$, called $SBV_0(\Omega;\mathbb{R}^n)$, simply requiring that ∂T_u be a measure also when computed on $(n-1)$-forms *with* a vertical part. This is equivalent to asking that in addition to the integration by parts formulas stated above, there exist measures $\mu_{\alpha\beta}$ (α and β multi-indices with $|\alpha|+|\beta| = n-1$) such that

$$\int_{\Omega\times\mathbb{R}} \phi(x)\psi(y)\,d\mu_{\alpha\beta} = \partial T_u\big(\phi(x)\psi(y)dx_\alpha\wedge dy_\beta\big) \tag{8.1}$$

for all $\phi\in C^1_0(\Omega)$, $\psi\in C^1_b(\mathbb{R})$.

It is worth noting that the property $u\in SBV_0(\Omega;\mathbb{R}^m)$ can be stated without using the language of currents, just as in the case of Theorem 2.1. The simplest case ($n=2, m=1$) gives, besides the conditions (2.2), also

$$\int_{\Omega\times\mathbb{R}} \varphi(x)\psi(y)\,d\mu = \int_{S(u)} \Big(\int_{u^-(x)}^{u^+(x)} \psi(y)\,dy\Big)\frac{\partial\phi}{\partial\tau}(x)\,d\mathcal{H}^1(x), \tag{8.2}$$

(τ the tangent to $S(u)$), for some finite measure μ (relative to $\alpha=0,\beta=1$). Roughly speaking, this is the requirement that the traces $u^{\pm}$ be functions of bounded variation on $S(u)$ (this is not precisely so since $S(u)$ may present a very complex structure, see the examples in [11]). In the physical case $n=m=3$ the integration by parts formulas obtained from (8.1) characterize the distributional and Jacobian determinants of ∇u and its (2-dimensional) adjoint matrices. An important property that can be deduced from (8.2) is that the *approximate tangential derivatives* $\nabla u^{\pm}$ exist $\mathcal{H}^{n-1}$-a.e. on $S(u)$, and $\int_{S(u)} |\nabla u^{\pm}|\,d\mathcal{H}^{n-1} < +\infty$.

We denote by $\partial_v T_u$ the vector of the measures $\mu_{\alpha\beta}$; *i.e.*, the components of ∂T_u corresponding to differential forms $\varphi dx_\alpha\wedge dy_\beta$, with $|\beta|>0$. The letter v refers to the fact that we have in mind "vertical components". The class $SBV_0(\Omega)$ has the following compactness property, which extends the results by Ball [21] to a class of functions "allowing for cavitation".

Theorem 8.1 *Let (u_j) be a sequence in SBV_0 such that*

$$\sup_{j\in\mathbb{N}}\Big(\|u_j\|_\infty + \mathcal{H}^1(S(u_j)) + \int_\Omega |\nabla u_j|^q\,dx + \|\partial_v T_{u_j}\|\Big) < +\infty\,,$$

where $q\geq \min\{n,m\}$, and assume in addition that $\Big(\det\dfrac{\partial(u_j)_\beta}{\partial x_\gamma}\Big)_j$ is a equi-integrable sequence for every pair of multi-indices β,γ of order $\min\{n,m\}$ if $q=\min\{n,m\}$.

Then, there exists a subsequence $(u_{j(k)})$ converging in $L^1_{\rm loc}(\Omega,\mathbb{R}^m)$ to $u\in SBV_0$, such that

$$\nabla u_{j(k)} \to \nabla u \text{ weakly in } L^q(\Omega,\mathbb{R}^{nm}),$$

$$\det\frac{\partial(u_{j(k)})_\beta}{\partial x_\gamma} \to \det\frac{\partial u_\beta}{\partial x_\gamma} \text{ weakly in } L^1(\Omega)$$

for every pair of multi-indices β, γ *of equal order not greater than* $\min\{n, m\}$, *and* $\partial T_{u_{j(k)}}$ *converges weakly to* ∂T_u. *In particular* $\partial_v T_{u_{j(k)}}$ *converges weakly to* $\partial_v T_u$ *in the sense of measures.*

$\mathbf{SBV_0}$-functions with Sobolev traces. As a subclass of $SBV_0(\Omega)$ (that is, "SBV-functions with BV-traces on $S(u)$") we can consider the family of "SBV-functions with Sobolev traces on $S(u)$", that is, those SBV_0 functions such that

$$\int_{S(u)} |\nabla u^{\pm}|^p \, dx < +\infty$$

for some $p \geq 1$, and such that the measure $\partial_v T_u$ is determined by $\nabla u^{\pm}$; *e.g.*, in the case $n = 2, m = 1$

$$\partial T_u(\phi\psi dy) = -\int_{S(u)} \big(\psi(u^+)\nabla u^+ - \psi(u^-)\nabla u^-\big)\phi(x)\, d\mathcal{H}^1.$$

Unfortunately, this subclass is not compact: it is possible to give an example such that all hypotheses of the compactness Theorem 8.1 are satisfied and in addition $\nabla u_j^{\pm}$ are equi-bounded, but the limit u does not possess Sobolev traces on $S(u)$. This phenomenon is due to the fact that $S(u_j)$ may converge only in a weak sense to $S(u)$; the phenomenon does not occur if we have strong convergence; *i.e.*, $\mathcal{H}^{n-1}(S(u_j)) \to \mathcal{H}^{n-1}(S(u))$.

Appendix.

It is worth spending a few words on another important application of SBV functions. A functional formally similar to the ones presented above has been proposed by Mumford and Shah as a variational model in computer vision (see [69], [67], [4], [47], [9], [44]). Minimum problems are considered, whose SBV formulation is

$$(A.1) \qquad \min\Big\{\int_\Omega |\nabla u|^2\, dx + c_1\mathcal{H}^{n-1}(S(u)) + c_2\int_\Omega |u-g|^2\, dx : \ u \in SBV(\Omega; \mathbb{R})\Big\},$$

where $g \in L^\infty(\Omega)$, called the "grey function", represents an "input picture" and $c_i > 0$ are constants. The solution u to (A.1) is the best "piecewise smooth" approximation of g, and its "jump set" is expected to detect the relevant contours of the objects in the picture. The existence of minimizers for (A.1) follows as in the proof of Theorem 3.4 once we remark that there is no restriction to confine ourselves to the case $\|u\|_\infty \leq \|g\|_\infty$. The functional

$$u \mapsto \int_\Omega |\nabla u|^2\, dx + c_1\mathcal{H}^{n-1}(S(u))$$

is in many cases a good simplified version of the energies (1.1), (1.2), and the techniques of computer vision developed for this functional, deriving from a different viewpoint, provide a good alternative to the methods of fracture mechanics. In this appendix we include some results whose study can be of help in the general case of fracture mechanics.

Motion of Fracture. The functionals of the type (1.6) with $\varphi \geq c > 0$ are not differentiable on SBV (at least with respect to BV or L^p norms). In fact, if $u, v \in SBV(\Omega; \mathbb{R}^m)$ and $\mathcal{H}^{n-1}(S(v) \setminus S(u)) > 0$, then we have

$$\lim_{t \to 0+} \left| \frac{F(u + tv) - F(u)}{t} \right| = +\infty.$$

It is not possible therefore to define the motion of fracture by the flow of this functional. We can nevertheless study quasi-static motion by a minimization process at fixed time-steps, and then let the time-step tend to zero. An axiomatization of this procedure in an abstract setting has been proposed by De Giorgi [49] under the name of *minimizing movements.*

We illustrate this procedure considering the Mumford and Shah functional with the notation

$$(A.2) \qquad F(u, K) = \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^{n-1}(K) \qquad u \in SBV(\Omega)$$

as a model.

A function $u_0 \in SBV(\Omega)$ will be fixed, which we regard as the initial datum at $t = 0$. for the sake of simplicity we suppose $u_0 \in L^\infty(\Omega) \cap H^1(\Omega)$. Fixed $\lambda > 0$ we define by induction a sequence (u_j^λ) in $SBV(\Omega)$ and an increasing sequence of closed sets (K_j^λ) as follows: $u_0^\lambda = u_0$, $K_0^\lambda = \emptyset$, and $u_j^\lambda = w$, $K_j^\lambda = \overline{S(w)} \cup K_{j-1}^\lambda$, where w is a solution to

$$\min\Big\{ \int_\Omega |\nabla v|^2 \, dx + \mathcal{H}^{n-1}(S(v) \setminus K_{j-1}^\lambda) + \lambda \int_\Omega |v - u_{j-1}^\lambda|^2 \, dx : v \in SBV(\Omega), \ \|v\|_\infty \leq \|u_0\|_\infty \Big\}.$$

This solution exists since such a minimum problem is equivalent to the analogous one in $SBV(\Omega \setminus K_{j-1}^\lambda)$. Note that by the regularity result of [51] we have

$$\mathcal{H}^{n-1}\big(\overline{S(w)} \setminus [S(w) \cup K_{j-1}^\lambda]\big) = 0,$$

and this implies, taking $v = u_{j-1}^\lambda$ in (A.2), the energy estimate

$$F(u_j^\lambda, K_j^\lambda) + \lambda \int_\Omega |u_j^\lambda - u_{j-1}^\lambda|^2 \, dx \leq F(u_{j-1}^\lambda, K_{j-1}^\lambda);$$

in particular $F(u_j^\lambda, K_j^\lambda)$ is decreasing with j, and

$$\|u_j^\lambda - u_{j-1}^\lambda\|_2 \leq \lambda^{-1/2} \sqrt{F(u_{j-1}^\lambda, K_{j-1}^\lambda) - F(u_j^\lambda, K_j^\lambda)}.$$

If $k > j$ we obtain the estimate

$$\|u_k^\lambda - u_j^\lambda\|_2 \leq \sum_{i=j}^{k-1} \|u_{i+1}^\lambda - u_i^\lambda\|_2$$

$$
\begin{aligned}
&\le \sum_{i=j}^{k-1} \lambda^{-1/2} \sqrt{F(u_i^\lambda, K_i^\lambda) - F(u_{i+1}^\lambda, K_{i+1}^\lambda)} \\
&\le \sqrt{\frac{k-j}{\lambda}} \sqrt{\sum_{i=j}^{k-1} (F(u_i^\lambda, K_i^\lambda) - F(u_{i+1}^\lambda, K_{i+1}^\lambda))} \\
&= \sqrt{\frac{k-j}{\lambda}} \sqrt{F(u_j^\lambda, K_j^\lambda) - F(u_k^\lambda, K_k^\lambda)} \\
&\le \sqrt{\frac{k-j}{\lambda}} \sqrt{F(u_0, K_0)} = \sqrt{\frac{k-j}{\lambda}} \|\nabla u_0\|_2.
\end{aligned}
$$

We define then the piecewise constant functions $v_\lambda(t) : [0, +\infty) \to SBV(\Omega)$ by

$$v_\lambda(t) = u_\lambda([\lambda t]),$$

which satisfy the estimate

$$\|v_\lambda(t) - v_\lambda(s)\|_2 \le M \sqrt{t - s + \frac{1}{\lambda}} \qquad \text{if } t \ge s,$$

with $M = \|\nabla u_0\|_2$. Using the uniform estimate above it is not difficult to show that there exists a subsequence (λ_j) such that

$$v_{\lambda_j} \to u \qquad \text{uniformly in } L^\infty([0, T]; L^2(\Omega)) \ \text{ for all } T > 0$$

and the limit u belongs to $C^{0,1/2}([0, +\infty); L^2(\Omega))$. Moreover by Theorem 3.1 we have $u(t) \in SBV(\Omega)$ for all t. Functions u obtained in such a way for a particular choice of (λ_j) are called an *evolution by minimizing movements* of the Mumford-Shah functional with initial datum u_0.

Further information about minimizing movements can be found in [49] and [8].

Approximation with differentiable functionals. Another important issue in fracture mechanics is numerical approximation. Functionals of type (1.5) can be approximated by elliptic functionals via a procedure due to Ambrosio and Tortorelli [17], [18] which can be more easily handled. In the case of the Mumford and Shah functional we have the following result.

Theorem A.1 *Let Ω be a bounded Lipschitz set, let $g \in L^\infty(\Omega)$, and let $c_1, c_2 > 0$. For every $\varepsilon > 0$ and $\kappa_\varepsilon > 0$ consider the problem*

$$
\begin{aligned}
(A.3) \qquad \min\Big\{ &\int_\Omega \Big((\kappa_\varepsilon + v^2)|\nabla u|^2 + c_2(u-g)^2 \Big)\, dx + c_1 \int_\Omega \Big(\varepsilon |\nabla v|^2 + \frac{(1-v)^2}{\varepsilon} \Big)\, dx : \\
& u, v \in L^\infty(\Omega),\ 0 \le v \le 1 \ a.e.,\ \|u\|_\infty \le \|g\|_\infty,\ u, v \in H^{1,2}(\Omega) \Big\}.
\end{aligned}
$$

Then:

(i) *for every $\varepsilon > 0$ there exists at least a C^1 solution $(u_\varepsilon, v_\varepsilon)$ to $(A.3)$, and if (ε_j) is any sequence of positive numbers converging to 0, then the sequence u_{ε_j} is relatively compact in $L^2(\Omega)$;*

(ii) *if $\kappa_{\varepsilon_j} = o(\varepsilon_j)$, any limit point u of a subsequence $u_{\varepsilon_{j(k)}}$ belongs to $SBV(\Omega)$ and is a solution of $(A.1)$; we have also that $\nabla u_{\varepsilon_{j(k)}}$ strongly converges to ∇u in $L^2(\Omega)$ and the absolutely continuous measures with densities*

$$\varepsilon_{j(k)}|\nabla v_{\varepsilon_{j(k)}}|^2 + \frac{(1 - v_{\varepsilon_{j(k)}})^2}{\varepsilon_{j(k)}}$$

converge to $\mathcal{H}^{n-1} \llcorner S(u)$.

A discrete approximation of the Mumford and Shah functional using the Ambrosio and Tortorelli approach has been studied by Bellettini and Coscia [25].

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