# A quasi-static evolution generated by local energy minimizers for an elastic material with a cohesive interface

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**Abstract.** We consider a model for an elastic material with a cohesive crack along a prescribed fracture set. In the framework of in-plane elasticity we consider a cohesive law with incompenetrability constraint and general loading-unloading regimes. We provide first a time-discrete evolution by means of local minimizers of the energy with respect to the $L^2$-norm of the crack opening displacement. The choice of this norm is due to technical reasons (the $\lambda$-convexity of the energy) and is in analogy with the classical approach in quasi-static brittle fracture, where the evolution of the system is condensed into the evolution of the crack. In the "time-continuous" limit we obtain a $BV$-evolution, in parametrized form, characterized by Karush-Kuhn-Tuker conditions for the internal variable, equilibrium and energy identity.

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# 1 Introduction

Starting from pioneering works in mechanics (see [10] [3], [23] for basic models) the analysis of cohesive zone models has obtained much attention in the recent years, especially due to the numerous applications in engineering. Analysis has been carried on by several authors under different settings and considering several mechanical aspects of the problem. Recently a common assumption relies in considering models where the fracture is confined to a prescribed interface between two elastic or visco-elastic bodies. This hypothesis is considered in many contributions ([9], [5], [6], [14], [15], [24], [1], [7], [26], [19], and references therein) and is also assumed in the present work.

Let us spend some words on the model we consider: we have a reference configuration $\Omega = (\Omega^+ \cup \Omega^-) \subset \mathbb{R}^2$ given by the cross section of two elastic bodies, separated by an interface $K = \partial\Omega^+ \cap \partial\Omega^-$ representing the crack. The displacement $u : \Omega \to \mathbb{R}^2$ has linearized strain $\boldsymbol{\epsilon}(u)$ and signed opening displacement $[\![u]\!] = u^+ - u^-$ on $K$. A fundamental feature of our model is the presence of an internal variable $\xi : K \to [0, +\infty]$ taking into account the history of the separation between the two bodies (see e.g. [22, 20]); $\xi(x)$ represents the maximum separation that has taken place at the point $x \in K$ during the evolution. In particular $\xi(x) = 0$ means that no opening of the crack has still happened at $x$. As a consequence, the internal variable satisfies the irreversibility condition $\dot{\xi} \geq 0$ while the crack opening satisfies $|[\![u]\!]| \leq \xi$. In particular dissipation occurs only when $\dot{\xi} > 0$ and $|[\![u]\!]| = \xi$ (loading) while a different non-dissipative mechanical behaviour occurs when $\dot{\xi} = 0$ and $|[\![u]\!]| < \xi$.

We study a quasi static evolution for the energy

$$F(u, \xi) = \int_\Omega W(\boldsymbol{\epsilon}(u))\, dx + \int_K \varphi([\![u]\!], \xi, \nu)\, d\mathcal{H}^1,$$

where $\nu$ is the unit normal on $K$, $W$ is the bulk elastic energy, and $\varphi$ is the cohesive potential. For the precise technical and mechanical assumptions the reader is referred to §2; we point out that

$\varphi$ is $\lambda$-convex w.r.t. $[\![u]\!]$, non-decreasing w.r.t. $\xi$ and that it takes into account both infinitesimal incompenetrability and different loading-unloading regimes. We remark that in this representation the potential $\varphi$ can be written also as $\varphi([\![u]\!],\xi,\nu) = \varphi_s([\![u]\!],\xi,\nu)+\varphi_d(\xi)$ where $\varphi_s$ is the stored energy while $\varphi_d$ is the dissipated energy; in particular, the system is rate-independent since dissipation, i.e. rate of dissipated energy, takes the form $\mathcal{D}(\xi(t),\dot{\xi}(t)) = \psi'_d(\xi(t))\,\dot{\xi}(t)$. For simplicity, we consider the case where the quasi static evolution is governed only by a time-dependent Dirichlet boundary condition $u = g(t)$ for $t \in [0,T]$ on $\partial_D\Omega$, the Dirichlet part of the boundary, whereas the external forces acting on the bodies and external tractions on $\partial_N\Omega = \partial\Omega \setminus \partial_D\Omega$ are set equal to 0. Therefore, the equilibrium equations of the system will be "qualitatively" of the form

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma}(t) = 0 & \text{on } \Omega, \\ \boldsymbol{\sigma}_\nu(t) = 0 & \text{on } \partial_N\Omega, \\ \boldsymbol{\sigma}_\nu(t) \in \partial_{[\![u]\!]}\varphi([\![u(t)]\!],\xi(t),\nu) & \text{on } K, \\ u(t) = g(t) & \text{on } \partial_D\Omega, \end{cases}$$

where $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\epsilon}(u)$ is the linear stress tensor, $\boldsymbol{\sigma}_\nu = \boldsymbol{\sigma}\nu$ is the normal tension, $\partial_{[\![u]\!]}\varphi$ denotes the subdifferential with respect to $[\![u]\!]$ (in particular $[\![u(t)]\!] \cdot \nu \geq 0$ on $K$). These equations will be given a rigorous and detailed formulation in §5. As for the flow rule, we have to specify which kind of quasi-static evolution we are interested in. More precisely, we look for a parametrized $BV$-evolution in the sense of [17]. In general, $BV$-evolutions for rate-independent problems are defined by vanishing viscosity, taking the limit of rate-dependent parabolic evolutions. In this work we follow a different approach, we endow the space of admissible configurations with a norm $\|\cdot\|$ and we generate the evolutions by means of a time-discrete scheme based on local minimizers of $F$ (similar schemes can be found in [11, 18, 19]). Given the initial configuration $u_0$, $\xi_0$ at time $t_0 = 0$ and $\Delta > 0$ the discrete evolution is defined by induction as follows:

- if $t_k < T$ and $u_k$ is a local minimizer of $F(t_k, u, \xi_k)$ (in some neighborhood $\|u - u_k\| \leq r$ for $r > 0$) then
$$\begin{cases} t_{k+1} = \min\{t_k + c\Delta, T\} \\ u_{k+1} = u_k, \\ \xi_{k+1} = \xi_k, \end{cases}$$
(for a suitable choice of $c > 0$)
- if $t_k \leq T$ and $u_k$ is not a local minimizer then
$$\begin{cases} t_{k+1} = t_k, \\ u_{k+1} \in \operatorname{argmin}\{F(t_k,u,\xi_k) : \|u - u_k\| \leq \Delta\}, \\ \xi_{k+1} = \xi_k \vee |[\![u_{k+1}]\!]|. \end{cases}$$

Several comments are due. If $\|u_{k+1} - u_k\| < \Delta$ then $u_{k+1}$ is a local minimizer of $F(t_k,\cdot,\xi_k)$ in a ball $B(u_{k+1}, r)$, for $r$ sufficiently smaller than $\Delta$, on the contrary, if $\|u_{k+1} - u_k\| = \Delta$ then $u_{k+1}$ is not necessarily a local minimizer of $F(t_k,\cdot,\xi_k)$ in a ball $B(u_{k+1}, r)$; in particular, some indices $k$ may not correspond to a local minimizer of $F(t_k,\cdot,\xi_k)$. Selecting the indices $k_i$ such that $u_{k_i}$ is a local minimizer at time $t_{k_i} = ic\Delta$ would give a discrete-in-time evolution $t_{k_i} \mapsto u(t_{k_i})$ in which, by definition, $u(t_{k_i})$ is a local minimizer in some ball $B(u(t_{k_i}), r_i)$. On the contrary, for $k_i < k < k_{i+1}$ the role of the "intermediate" configurations $u_k$ is to provide a path between the local minimizers $u(t_{k_i})$ and $u(t_{k_{i+1}})$. This is an important point for rate-independent systems: in general, being $F(t,\cdot,\xi)$ non-convex, local minimizers are many and, even selecting, they may not depend continuously on time. As a consequence, passing to the limit as the "time step" $\Delta \to 0$ could eventually give a discontinuous evolution $t \mapsto u(t)$. The behaviour in discontinuity points is what characterizes $BV$-evolutions: re-parametrizing (in a suitable way) allows indeed to obtain in the limit as $\Delta \to 0$ a "transition path", of gradient flow type, between $u^-$ and $u^+$. Actually, in our approach we employ a parametrization $[0,S] \ni s \mapsto (t(s), u(s), \xi(s))$ to describe the entire limit evolution. Loosely speaking we could say that in this framework continuity points (in time) correspond to parametrization points where $t'(s) > 0$ while discontinuity points (in time) correspond to parametrization intervals where $t(s)$ is constant, and thus $t'(s) = 0$. For our problem, the limit evolution satisfies the following properties (for the precise technical statement we refer to Definition 2.7)

- $\|u'\| \le 1$, $t' \ge 0$ and $t(S) = T$,
- $\xi'(s) \ge 0$, $|[\![u]\!](s)| \le \xi(s)$ and $\xi'(s)(|[\![u]\!](s)| - \xi(s)) = 0$ for a.e. $s \in [0,S]$,
- $|\partial_u F|(t(s), u(s), \xi(s)) = 0$ for every $s \in [0,S]$ with $t'(s) > 0$,
- for every $s \in [0,S]$ the following energy balance holds

$$F(t(s), u(s), \xi(s)) = F(0, u_0, \xi_0) - \int_0^s |\partial_u F|(t(r), u(r), \xi(r))\, dr + \\ + \int_0^s \partial_t F(t(r), u(r), \xi(r))\, t'(r)\, dr.$$

Let us briefly explain the meaning of the above properties. The first line is basically an "arc-length" normalization. The second provides the evolution of the internal variable $\xi$ in terms of Karush-Kuhn-Tucker conditions. The third represents instead equilibrium, expressed in terms of the slope $|\partial_u F|$; in "practice" this condition implies that the directional derivatives $F'(t(s), u(s), \xi(s); z)$ are non-negative for every admissible variation $z$, and thus the system is in equilibrium. We remark that the slope, together with $\lambda$-convexity, turns out to be very useful in the analysis since the energy $F(t, \cdot, \xi)$ is not assumed to be convex and differentiable for every $u$. It is fair to say that, at the moment, it is not known whether $u(s)$, for $t'(s) > 0$, enjoys some kind of minimality property, even if it is the limit of local minimizers in the discrete setting. This is essentially due to the fact that the size of the neighbourhoods where local minimality holds, for the discrete evolution, may vanish for $\Delta \to 0$. Finally, to explain the energy balance we make a couple of examples. First, assume that $t' > 0$ a.e. in $(s_1, s_2)$. Since $|\partial_u F|(t(r), u(r), \xi(r)) = 0$ for every $r \in (s_1, s_2)$ the energy balance in $[s_1, s_2]$ reads

$$F(t(s_2), u(s_2), \xi(s_2)) = F(t(s_1), u(s_1), \xi(s_1)) + \int_{s_1}^{s_2} \partial_t F(t(r), u(r), \xi(r))\, t'(r)\, dr.$$

Since $t' > 0$ we can define its inverse $s(t)$. Denote, $t_i = t(s_i)$ for $i = 1, 2$ and, by abuse of notation, $u(t) = u \circ s(t)$ etc. Then, by a change of variable, we get

$$F(t_2, u(t_2), \xi(t_2)) = F(t_1, u(t_1), \xi(t_1)) + \int_{t_1}^{t_2} \partial_t F(t, u(t), \xi(t))\, dt.$$

We have a "classical" energy balance in the interval $[t_1, t_2]$ since the integrand turns out to be the power of external forces. This equilibrium property can be written also in PDE form, see §5. On the contrary, assume that $(s_1, s_2)$ is a maximal interval in which $t' = 0$ almost everywhere; hence $s \mapsto u(s)$ represents the "jump-path" connecting $u^- = u(s_1)$ and $u^+ = u(s_2)$ . Since $t' = 0$ the energy balance in any subinterval $(a, b) \subset (s_1, s_2)$ becomes (for $t$ constant)

$$F(t, u(b), \xi(b)) = F(t, u(a), \xi(a)) - \int_a^b |\partial_u F|(t, u(r), \xi(r))\, dr.$$

On the other hand (under suitable regularity assumptions)

$$F(t, u(b), \xi(b)) - F(t, u(a), \xi(a)) = \int_a^b F'(t, u(r), \xi(r); u'(r))\, dr.$$

It follows that $F'(t, u(r), \xi(r); u'(r)) = -|\partial_u F|(t, u(r), \xi(r))$ a.e. in $(s_1, s_2)$ from which we get

$$u'(r) \in \operatorname{argmin} \{F'(t, u(r), \xi(r); z) : \|z\| \le 1\} \quad \text{a.e. in } (s_1, s_2).$$

Hence, the "velocity" $u'(s)$ is a steepest descent direction for the energy $F(t, \cdot, \xi(s))$ with respect to the norm $\| \cdot \|$. Roughly speaking, the "jump-transition" path is a normalized gradient flow.

In the previous part of this introduction we intentionally did not specify the norm $\| \cdot \|$, to stress the fact that this approach is potentially suitable for different choices of the norm. Actually, independently of the approach, the notion itself of $BV$-evolution requires to choose a norm, or at least a metric [17]. In many cases different norms or metrics can be employed, resulting in different mechanical or mathematical properties; an original examples in which the "norm" is instead intrinsically tailored to the energy can be found in [16]. For instance, for cohesive energies in the anti-plane setting [5] and [19] used respectively the $L^2(\Omega)$-norm and the $H^1(\Omega)$-norm of the displacement. Here, we made an "intermediate" choice, i.e., the $L^2(K; \mathbb{R}^2)$-norm of $[\![u]\!]$. Mathematically, it is motivated by the $\lambda$-convexity of the surface energy. Mechanically, it seems to be a natural choice comparing with the standard approach in quasi-static brittle fracture, in which the evolution of the system is described just in terms of the crack (or interface) since the elastic displacement, in the bulk, changes accordingly. Indeed, in the paper we use a reduced energy $\mathcal{F}(t, w, \xi)$, instead of $F(t, u, \xi)$, where $w$ represent the opening $[\![u]\!]$; once $w$ is given on $K$ the elastic deformation with $[\![u]\!] = w$ is uniquely determined by equilibrium in the bulk $\Omega$.

To conclude, we highlight some links with related works on $BV$-evolutions and make some technical considerations. On the abstract side, a discrete scheme based on local minimizers and parametrization has been proposed initially by [11] for a class of finite dimensional rate-independent problems. Later, this approach has been developed in [18] with implicit and explicit Euler scheme for a class of $C^1$ functionals in infinite dimensional Hilbert spaces. Our incremental procedure is qualitatively similar, however neither [11] nor [18] is directly applicable to our model: the main differences and difficulties comes from the complex dependence on the internal variable $\xi$, from the incompenetrability constraint and, above all, from the fact that the energy is not differentiable with respect to the $L^2(K; \mathbb{R}^2)$-norm. This lack of regularity is balanced by $\lambda$-convexity, which plays an important role in several proofs. As a whole, the hardest technical part is contained in §4.2 where we prove that discrete evolutions have bounded length in $L^2(K; \mathbb{R}^2)$ and are bounded in $W^{1,2}(0, S; H^{1/2}(K; \mathbb{R}^2))$. After the above mentioned abstract results, [11] and [18], our starting point has been the recent [19]. Qualitatively the present work and [19] have some similarities, however there are also substantial mechanical and mathematical differences. Here we employ in-plane displacements (rather then anti-plane) with incompenetrability on the interface; further the cohesive density $\varphi$ used here is more general, in loading and above all in unloading (see §2.2). In perspective, such a general behavior points toward more realistic models of fatigue fracture, see e.g. [20, 7]. Moreover, on the mathematical level, in [19] the existence of a quasi-static $BV$-evolution is proven by approximation, i.e., first for a $C^1$-regularization $F_\varepsilon$ of the energy $F$ and then passing to the limit in the evolutions $u_\varepsilon$ as $\varepsilon \to 0$ (the approach is similar to the $\Gamma$-convergence of gradient flows [25, 18]). Moreover, as we already mentioned, in [19] it is employed the $H^1(\Omega)$-norm, which mechanically corresponds to a visco-elastic Kelvin-Voigt system; a weaker norm makes compactness and energy balance more delicate. Finally, in [19] the discrete scheme selects at each time $t_{k_i}$ an equilibrium configuration $u_{k_i}$ which a priori is not necessarily a local minimizers.

# Contents



# 2 Setting

Let $\Omega^+$ and $\Omega^-$ be disjoint polygonal domains in $\mathbb{R}^2$ (in the sense of [13]). Accordingly, let $\Gamma_i^+$, for $i \in I^+ = \{0, ..., N^+\}$, and $\Gamma_i^-$, for $i \in I^- = \{0, ..., N^-\}$, be the $C^1$ curves parametrizing respectively $\partial\Omega^+$ and $\partial\Omega^-$. We assume $I^+$ to be the disjoint union of the sets of indices $I_K^+$, $I_D^+$ and $I_N^+$ (and similarly for $I^-$). The interface between the domains is given by

$$K := \partial\Omega^+ \cap \partial\Omega^- = \bigcup_{i \in I_K^+} \Gamma_i^+ = \bigcup_{i \in I_K^-} \Gamma_i^-.$$

In the same way, $\partial_D\Omega^+$ and $\partial_N\Omega^+$ are the union of the curves $\Gamma_i^+$ respectively for $i \in I_D^+$ and $i \in I_N^+$ (and similarly for $\partial_D\Omega^-$ and $\partial_N\Omega^-$).

For convenience we will denote also $\Omega := \Omega^+ \cup \Omega^-$, $\partial_D\Omega := \partial_D\Omega^+ \cup \partial_D\Omega^-$, $\partial_N\Omega := \partial_N\Omega^+ \cup \partial_N\Omega^-$ and by $\nu_K$ the unit normal to the interface $K$ pointing inside the set $\Omega^+$. The sets $\partial_D\Omega$ and $\partial_N\Omega$ are respectively the Dirichlet and Neumann part of the "boundary" $\partial\Omega \setminus K$. To avoid degenerate cases we assume that the distance between $K$ and $\partial_D\Omega$ is strictly positive; in this way the trace operator from $\{u \in H^1(\Omega^+; \mathbb{R}^2) : u = 0 \text{ on } \partial_D\Omega^+\}$ to $H^{1/2}(K; \mathbb{R}^2)$ is continuous, surjective and admits a continuous "inverse" (the lifting operator). Note also that functions $u \in H^1(\Omega; \mathbb{R}^2)$ may have different traces and thus a jump $[\![u]\!] := u^+ - u^- \neq 0$ on $K$.

## 2.1 Elastic energy

**Elastic bulk energy.** For $u \in H^1(\Omega, \mathbb{R}^2)$ we introduce the linear elastic energy

$$E(u) := \int_\Omega W(\boldsymbol{\epsilon}(u))\, dx = \tfrac{1}{2} \int_\Omega \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(u)\, dx$$

where $\boldsymbol{\epsilon}(u) := \frac{1}{2}(\nabla u + \nabla u^T)$ is the strain tensor, $\boldsymbol{\sigma}(u) := \mathbf{C}\boldsymbol{\epsilon}(u)$ is the stress tensor and $\mathbf{C}$ is the stiffness tensor, which is assumed to be symmetric and positive definite in $\mathbb{R}^{2\times 2}_{sym}$. This implies the existence of two constants $0 < c_1 < c_2$ such that

$$c_1 \|\boldsymbol{\epsilon}(u)\|^2_{L^2(\Omega;\mathbb{R}^4)} \le E(u) \le c_2 \|\boldsymbol{\epsilon}(u)\|^2_{L^2(\Omega;\mathbb{R}^4)}. \tag{1}$$

**Boundary conditions.** For sake of simplicity we will consider only Dirichlet boundary conditions. The class of admissible displacements will be the space

$$\{u \in H^1(\Omega;\mathbb{R}^2) : u = g(t) \text{ on } \partial_D\Omega\}$$

where $t \mapsto g(t) \in H^1(\Omega;\mathbb{R}^2)$ is assumed to be of class $C^1$ in the time interval $[0,T]$. By [21, Theorem 2.5] there exists $C > 0$ such that for every $v \in H^1(\Omega;\mathbb{R}^2)$ with $v = 0$ on $\partial_D\Omega$ the Korn-Poincaré inequality holds true

$$\|v\|_{H^1(\Omega;\mathbb{R}^2)} \le C\sqrt{E(v)}, \tag{2}$$

where $E(v)$ is the elastic energy.

**Reduced energy.** Since we are interested in quasi-static evolutions it is not restrictive to consider that the displacement field is always in equilibrium. To this end, we introduce the map $\boldsymbol{u} : H^{1/2}(\partial_D\Omega;\mathbb{R}^2) \times H^{1/2}(K;\mathbb{R}^2) \to H^1(\Omega;\mathbb{R}^2)$ defined by

$$\boldsymbol{u}(b,w) = \operatorname{argmin}\{E(u) : u \in H^1(\Omega;\mathbb{R}^2),\ u = b \text{ on } \partial_D\Omega \text{ and } [\![u]\!] = w \text{ on } K\}. \tag{3}$$

Note that the constraint on $K$ is for the jump $[\![u]\!]$ and not on the single traces $u^+$ and $u^-$. Existence of a minimizer, given $b$ and $w$, follows from the direct method since the boundary conditions are weakly closed in $H^1(\Omega;\mathbb{R}^2)$ and since the energy $E$ is weakly lower semi-continuous and coercive. Uniqueness follows from the strict convexity of $E$. Hence $\boldsymbol{u}$ is well defined; moreover, it is linear and continuous. Linearity follows for instance from the fact that $\boldsymbol{u}(b,w)$ is characterized by the variational formulation

$$\int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}(b,w)) : \boldsymbol{\epsilon}(\psi)\,dx = 0 \qquad \text{for } \psi \in H^1(\Omega;\mathbb{R}^2) \text{ with } \psi = 0 \text{ on } \partial_D\Omega \text{ and } [\![\psi]\!] = 0 \text{ on } K.$$

Let us check continuity. Denote by $s(b,w) \in H^1(\Omega;\mathbb{R}^2)$ a lifting of $b \in H^{1/2}(\partial_D\Omega;\mathbb{R}^2)$ with traces $s^\pm(b,w) = \pm w/2$ on $K^\pm$. Hence $s(b,w)$ is an admissible competitor in the minimum problem (3) and $\|s(b,w)\|_{H^1(\Omega;\mathbb{R}^2)} \le C(\|w\|_{H^{1/2}(K;\mathbb{R}^2)} + \|b\|_{H^{1/2}(\partial_D\Omega;\mathbb{R}^2)})$. Let $v = \mathbf{u}(b,w) - s(b,w)$. Write

$$\|v\|_{H^1(\Omega;\mathbb{R}^2)} = \|\mathbf{u}(b,w) - s(b,w)\|_{H^1(\Omega;\mathbb{R}^2)} \ge \|\mathbf{u}(b,w)\|_{H^1(\Omega;\mathbb{R}^2)} - \|s(b,w)\|_{H^1(\Omega;\mathbb{R}^2)}.$$

Since $v = 0$ on $\partial_D\Omega$ the Korn-Poincaré inequality (2) holds; hence, using the fact that the energy is quadratic and $\mathbf{u}(b,w)$ is a minimizer, we can write

$$c\|v\|^2_{H^1(\Omega;\mathbb{R}^2)} \le E(v) \le 2E(\mathbf{u}(b,w)) + 2E(s(b,w)) \le 4E(s(b,w)) \le C\|s(b,w)\|^2_{H^1(\Omega;\mathbb{R}^2)}.$$

Combining the two inequalities yields the continuity of the map $\boldsymbol{u}$.

In the sequel, given $t \in [0,T]$ and the jump $w \in H^{1/2}(K;\mathbb{R}^2)$, we will often employ the displacement field $\boldsymbol{u}(g(t),w)$. Since $\boldsymbol{u}$ is linear (and not bilinear) we can write

$$\boldsymbol{u}(g(t),w) = \boldsymbol{u}(g(t),0) + \boldsymbol{u}(0,w).$$

Without loss of generality we will also assume that

$$g(t) = \operatorname{argmin}\left\{E(z) : z \in H^1(\Omega;\mathbb{R}^2),\ z = g(t) \text{ on } \partial_D\Omega \text{ and } [\![g(t)]\!] = 0 \text{ on } K\right\}$$

so that $\boldsymbol{u}(g(t),0) = g(t)$ and

$$\boldsymbol{u}(g(t),w) = g(t) + \boldsymbol{u}(0,w).$$

In conclusion, given the jump $w \in H^{1/2}(K;\mathbb{R}^2)$ the elastic energy for the equilibrium configuration at time $t$ is given by

$$\mathcal{E}(t,w) := E(\boldsymbol{u}(g(t),w)) = E(g(t) + \boldsymbol{u}(0,w)).$$

We will write also $\boldsymbol{\epsilon}(g(t),w) = \boldsymbol{\epsilon}(\boldsymbol{u}(g(t),w))$ and $\boldsymbol{\sigma}(g(t),w) = \boldsymbol{\sigma}(\boldsymbol{u}(g(t),w))$.

## 2.2 Cohesive energy

The cohesive potential (4) below, featuring incompenetration and separate loading-unloading regimes, will be defined starting from an auxiliary potential $\hat{\psi} : [0,+\infty) \to [0,+\infty)$ which is assumed to be bounded, concave, $\lambda$-convex (for some $\lambda < 0$), and such that $\hat{\psi}(0) = 0$ and $\hat{\psi}(\zeta) > 0$ if $\zeta > 0$. Under these assumptions $\hat{\psi}$ is continuous, non-decreasing and the function $\zeta \mapsto \hat{\psi}(\zeta) - \frac{\lambda}{2}|\zeta - \eta|^2$ is convex for every $\eta \in [0,+\infty)$ (see Appendix A). Since $\hat{\psi}$ is concave there exist its right and left derivative, denoted by $\hat{\psi}'_+$ and $\hat{\psi}'_-$, in every point of $(0,+\infty)$; moreover both $\hat{\psi}'_\pm$ are monotone non-increasing and it holds $\hat{\psi}'_+ \le \hat{\psi}'_-$. By $\lambda$-convexity there exist also the right and left derivative of $\zeta \mapsto \hat{\psi}(\zeta) - \frac{\lambda}{2}\zeta^2$ and it holds $\hat{\psi}'_+(\zeta) - \lambda\zeta \ge \hat{\psi}'_-(\zeta) - \lambda\zeta$ for every $\zeta \in (0,+\infty)$. Hence $\hat{\psi}'_+ = \hat{\psi}'_-$ and we can define $\hat{\psi}'$ in $(0,+\infty)$. Finally, we set $\hat{\psi}'(0) = \hat{\psi}'_+(0)$.

**Lemma 2.1.** $\hat{\psi}' : [0,+\infty) \to [0,+\infty)$ *is non-increasing, bounded and Lipschitz continuous.*

*Proof.* By concavity of $\hat{\psi}$ we know that $\hat{\psi}'$ is monotone non-increasing and hence there exists $\lim_{\zeta\to 0^+} \hat{\psi}'(\zeta) > -\infty$. By $\lambda$-convexity the function $\zeta \mapsto \hat{\psi}(\zeta) - \frac{\lambda}{2}\zeta^2$ is convex, hence $\zeta \mapsto \hat{\psi}'(\zeta) - \lambda\zeta$ is non-decreasing and $\lim_{\zeta\to 0^+} \hat{\psi}'(\zeta) - \lambda\zeta = \lim_{\zeta\to 0^+} \hat{\psi}'(\zeta) < +\infty$. By Lagrange Theorem $\lim_{\zeta\to 0^+} \hat{\psi}'(\zeta) = \hat{\psi}'_+(0)$.

Clearly $\hat{\psi}' \le \hat{\psi}'(0)$. If, by contradiction, $\hat{\psi}'(\zeta^*) < 0$ for some value $\zeta^*$ then $\hat{\psi}' \le \hat{\psi}'(\zeta^*) < 0$ in $[\zeta^*,+\infty)$. As a consequence $\hat{\psi}(\zeta) = \int_0^\zeta \hat{\psi}'(s)\, ds$ would be neither positive nor bounded. We conclude that $\hat{\psi}'$ is positive.

Since $\zeta \mapsto \hat{\psi}(\zeta) - \frac{\lambda}{2}\zeta^2$ is convex (for $\lambda < 0$) its derivative $\zeta \mapsto \hat{\psi}'(\zeta) - \lambda\zeta$ is non-decreasing; hence, for $0 < \zeta_1 < \zeta_2$ we have $\hat{\psi}'(\zeta_1) - \lambda\zeta_1 \le \hat{\psi}'(\zeta_2) - \lambda\zeta_2$ and by monotonicity of $\hat{\psi}'$

$$-\lambda(\zeta_1 - \zeta_2) \le \hat{\psi}'(\zeta_2) - \hat{\psi}'(\zeta_1) \le 0.$$

Taking the absolute values in this expression provides the Lipschitz continuity of $\hat{\psi}'$. □

**Remark 2.2.** *From the previous Lemma it follows that* $\hat{\psi}$ *is of class* $C^1$ *in* $[0,+\infty)$.

Now, in order to take into account the loading-unloading regimes, we define first an auxiliary surface energy $\psi(\zeta,\xi)$ (an example is given in the next section) and then the cohesive density $\varphi(w,\xi,\nu)$ which takes into account incompenetrability.

**Definition 2.3.** *Let* $\psi : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ *such that*

*(1)* $\psi$ *is continuous,*

*(2)* $\psi(\zeta,\cdot)$ *is monotone non-decreasing,*

*(3)* $\psi(\cdot,\xi) = \hat{\psi}(\cdot)$ *in* $[\xi,+\infty)$,

*(4)* $\psi(\cdot,\xi)$ *is monotone non-decreasing and convex in* $[0,\xi]$ *(for* $\xi > 0$*),*

*(5)* $\psi(\cdot,\xi)$ *is of class* $C^1$ *in* $[0,+\infty)$ *with* $\partial_\zeta\psi(0,\xi) = 0$ *for* $\xi > 0$.

The cohesive density $\varphi : \mathbb{R}^2 \times [0,+\infty) \times \mathbb{S}^1 \to \mathbb{R} \cup \{+\infty\}$ is given by

$$\varphi(w,\xi,\nu) := \begin{cases} \psi(|w|,\xi) & \text{if } w\cdot\nu \ge 0, \\ +\infty & \text{if } w\cdot\nu < 0. \end{cases} \tag{4}$$

Accordingly, the cohesive energy $\mathcal{K} : H^{1/2}(K;\mathbb{R}^2) \times L^2(K)$ is given by

$$\mathcal{K}(w,\xi) := \int_K \varphi(w,\xi,\nu_K)\, d\mathcal{H}^1,$$

where $\nu_K$ is the unit normal on $K$ pointing toward $\Omega^+$.

Let us list the main properties of $\psi$ and $\varphi$ which will be useful in the sequel.

**Lemma 2.4.** *The density $\psi$ enjoys the following properties:*

*(i)* $0 \leq \hat{\psi}(\zeta) \leq \psi(\zeta,\xi) \leq \lim_{\zeta\to+\infty} \hat{\psi}(\zeta)$,

*(ii)* $0 \leq \partial_\zeta \psi(\zeta,\xi) \leq \partial_\zeta \psi(\xi,\xi) = \hat{\psi}'(\xi) \leq \hat{\psi}'(0)$,

*(iii)* $\psi(\cdot,\xi)$ *is $\lambda$-convex in $[0,+\infty)$ with the same constant $\lambda < 0$ of $\hat{\psi}$.*

*Proof.* We prove (i). Let us write

$$\int_\zeta^\xi \partial_\zeta \psi(r,\xi)\, dr = \psi(\xi,\xi) - \psi(\zeta,\xi) = \hat{\psi}(\xi) - \psi(\zeta,\xi).$$

In the interval $[0,\xi]$ we have $\psi(\cdot,\xi)$ convex and $\hat{\psi}(\cdot)$ concave, hence for $r \in [0,\xi]$ we can write

$$\partial_\zeta \psi(r,\xi) \leq \partial_\zeta \psi(\xi,\xi) = \hat{\psi}'(\xi) \leq \hat{\psi}'(r). \tag{5}$$

Then

$$\hat{\psi}(\xi) - \psi(\zeta,\xi) = \int_\zeta^\xi \partial_\zeta \psi(r,\xi)\, dr \leq \int_\zeta^\xi \hat{\psi}'(r)\, dr = \hat{\psi}(\xi) - \hat{\psi}(r)$$

from which we deduce that $\psi(\zeta,\xi) \geq \hat{\psi}(\zeta)$ for $\zeta \in [0,\xi)$. For $\zeta \geq \xi$ there is nothing to prove since by definition $\psi(\zeta,\xi) = \hat{\psi}(\zeta)$. By monotonicity of $\psi(\cdot,\xi)$ and $\hat{\psi}(\cdot)$ it follows easily that $\lim_{\zeta\to+\infty} \hat{\psi}(\zeta)$ is an upper bound.

By (5) we have $\partial_\zeta \psi(\zeta,\xi) \leq \hat{\psi}'(\xi)$ for $\zeta \in [0,\xi]$, while for $\zeta \in (\xi,+\infty)$ we have $\partial_\zeta \psi(\zeta,\xi) = \hat{\psi}'(\zeta) \leq \hat{\psi}'(\xi)$ by concavity of $\hat{\psi}$, and also (ii) is proved.

To prove $\lambda$-convexity it is sufficient to see that (given $\xi > 0$) the function $\zeta \mapsto \psi(\zeta,\xi) - \frac{\lambda}{2}\zeta^2$ is convex in $[0,+\infty)$; this is easily checked because it is convex for $\zeta < \xi$ and $\zeta > \xi$ and the derivative is continuous at $\zeta = \xi$. □

In this setting we can write $\psi(\zeta,\xi) = \psi_d(\xi) + \psi_s(\zeta,\xi)$ where $\psi_d(\xi) = \psi(0,\xi)$ is the dissipated energy while $\psi_s(\zeta,\xi)$ is the stored energy. By Definition 2.3 the function $\psi_d$ is monotone non-decreasing in the internal variable $\xi$.

**Lemma 2.5.** *The density $\varphi$ enjoys the following properties:*

*(i)* $\varphi(\cdot,\xi,\nu)$ *is $\lambda$-convex in $\mathbb{R}^2$ with the same constant $\lambda < 0$ of $\hat{\psi}$,*

*(ii)* $\varphi(w,\cdot,\nu)$ *is monotone non-decreasing,*

*(iii)* $\varphi(\cdot,\cdot,\nu)$ *is bounded and continuous in the set $\{w \cdot \nu \geq 0\} \times [0,+\infty)$.*

*Proof.* Since $\zeta \mapsto \psi(\zeta,\xi) - \frac{\lambda}{2}\zeta^2$ is monotone non-decreasing and convex in $[0,+\infty)$ the composition with $|w|$ yields a convex function. Hence $\varphi(\cdot,\xi,\nu)$ is $\lambda$-convex in the set $\{w \cdot \nu \geq 0\}$ and thus in the whole $\mathbb{R}^2$.

The monotonicity of $\varphi(w,\cdot,\nu)$ follows from the monotonicity of $\psi(|w|,\cdot)$. Boundedness and continuity of $\varphi(\cdot,\cdot,\nu)$ in the closed set $\{w\cdot\nu \geq 0\}\times[0,+\infty)$ follows from the corresponding properties of $\psi$. □

Note that $\mathcal{K}(\cdot,\xi)$ turns out to be $\lambda$-convex in $L^2(K;\mathbb{R}^2)$, uniformly with respect to $\xi$, hence (see Appendix A) the functional

$$L^2(K;\mathbb{R}^2) \ni w \mapsto \int_K \varphi(w,\xi,\nu_K)\, d\mathcal{H}^1 - \tfrac{\lambda}{2}\|w - v\|^2_{L^2(K;\mathbb{R}^2)},$$

is convex for all fixed $v \in L^2(K;\mathbb{R}^2)$.

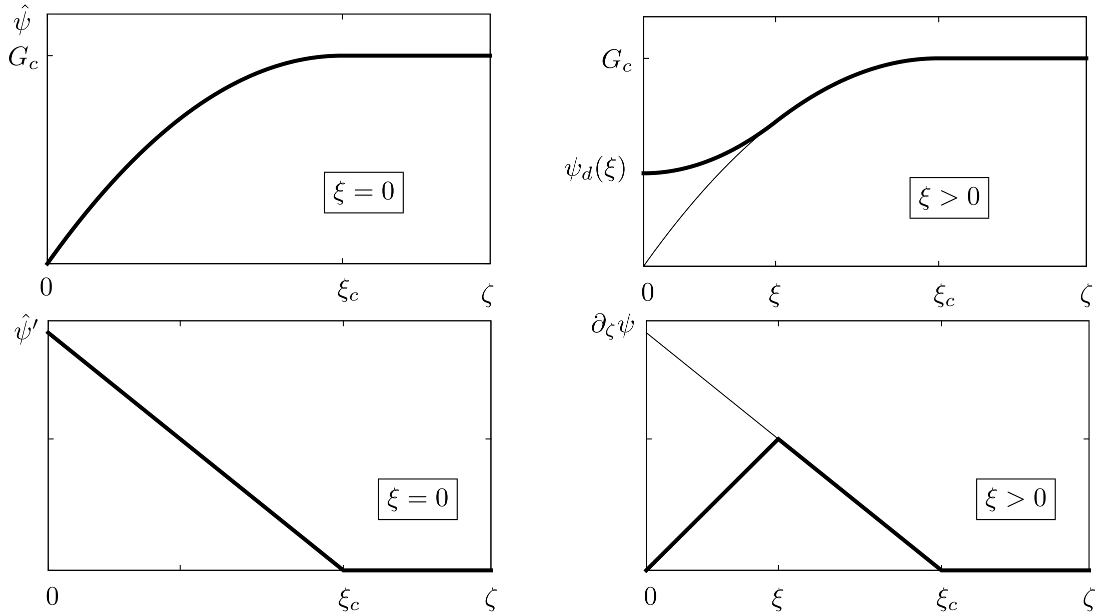


**Figure 1:** The energy density $\hat\psi$ (above) and the corresponding traction-separation law (below).

## 2.3 An example

In this section we provide a simple example of cohesive potential which satisfies Definition 2.3 (see Figure 1). For $G_c, \xi_c > 0$ define $\hat\psi : [0, +\infty) \to [0, +\infty)$ as

$$\hat\psi(\zeta) = \begin{cases} G_c(\zeta/\xi_c)[2 - (\zeta/\xi_c)] & 0 \le \zeta \le \xi_c \\ G_c & \zeta > \xi_c. \end{cases}$$

Note that $\hat\psi$ is of class $C^1$ and that

$$\hat\psi'(\zeta) = \begin{cases} 2(G_c/\xi_c)(1 - (\zeta/\xi_c)) & 0 \le \zeta \le \xi \\ 0 & \zeta \ge \xi_c. \end{cases}$$

Define, for $\xi > 0$,

$$\psi(\zeta, \xi) = \begin{cases} \frac{1}{2}(\hat\psi'(\xi)/\xi)\zeta^2 + \big(\hat\psi(\xi) - \frac{1}{2}\hat\psi'(\xi)\xi\big) & 0 \le \zeta \le \xi \\ \hat\psi(\zeta) & \zeta > \xi. \end{cases} \tag{6}$$

**Lemma 2.6.** *The density $\psi$ of this example satisfies Definition 2.3.*

*Proof.* It is immediate to check conditions (2), (3) and (5). The functions $\psi(\cdot, \xi)$ are uniformly Lipschitz continuous because their derivatives are bounded by $\hat\psi'(0)$, indeed, for $\xi > 0$

$$\partial_\zeta \psi(\zeta, \xi) = \begin{cases} (\hat\psi'(\xi)/\xi)\,\zeta & \text{if } \zeta \le \xi, \\ \hat\psi'(\xi) & \text{if } \zeta > \xi, \end{cases}$$

and $0 \le \hat\psi' \le \hat\psi'(0)$ by monotonicity (see Lemma 2.1). Next, let us check the continuity of $\psi(\zeta, \cdot)$. We distinguish two cases: $\zeta = 0$ and $\zeta > 0$. In the former, since $\zeta = 0 \le \xi$ by definition we have $\psi(0, \xi) = \hat\psi(\xi) - \frac{1}{2}\hat\psi'(\xi)\xi$, which is continuous. In the latter, from expression (6) is easy to see that $\psi(\zeta, \xi)$ is continuous with respect to $\xi$. Now, if $(\zeta_n, \xi_n) \to (\zeta, \xi)$ then by the uniform Lipschitz continuity of $\psi(\cdot, \xi)$

$$|\psi(\zeta_n, \xi_n) - \psi(\zeta, \xi)| \le |\psi(\zeta_n, \xi_n) - \psi(\zeta, \xi_n)| + |\psi(\zeta, \xi_n) - \psi(\zeta, \xi)| \le C|\zeta_n - \zeta| + o(1),$$

from which continuity follows.

Finally, let us prove (4), i.e. $\psi(\zeta,\xi_1) \le \psi(\zeta,\xi_2)$ for $\xi_1 \le \xi_2$. Being $\xi_1 \le \xi_2$, we can write

$$\psi(\zeta,\xi_1) = \psi(\xi_2,\xi_1) - \int_\zeta^{\xi_2} \partial_\zeta \psi(r,\xi_1)\,dr = \hat\psi(\xi_2) - \int_\zeta^{\xi_2} \partial_\zeta \psi(r,\xi_1)\,dr,$$
$$\psi(\zeta,\xi_2) = \psi(\xi_2,\xi_2) - \int_\zeta^{\xi_2} \partial_\zeta \psi(r,\xi_2)\,dr = \hat\psi(\xi_2) - \int_\zeta^{\xi_2} \partial_\zeta \psi(r,\xi_2)\,dr.$$

Thus, it is enough to check that $\partial_\zeta\psi(r,\xi_1) \ge \partial_\zeta\psi(r,\xi_2)$. Write

$$\partial_\zeta\psi(r,\xi_1) = \begin{cases} \hat\psi'(\xi_1)r/\xi_1 & \text{if } r<\xi_1,\\ \hat\psi'(r) & \text{if } \xi_1 \le r < \xi_2,\\ \hat\psi'(r) & \text{if } r \ge \xi_2,\end{cases} \qquad \partial_\zeta\psi(r,\xi_2) = \begin{cases} \hat\psi'(\xi_2)r/\xi_2 & \text{if } r<\xi_1,\\ \hat\psi'(\xi_2)r/\xi_2 & \text{if } \xi_1 \le r < \xi_2,\\ \hat\psi'(r) & \text{if } r \ge \xi_2.\end{cases}$$

If $r < \xi_1$ then $\hat\psi'(\xi_1)/\xi_1 > \hat\psi'(\xi_2)/\xi_2$ by monotonicity of $\hat\psi'$. If $\xi_1 \le r < \xi_2$ then (again by monotonicity) $\hat\psi'(r) > \hat\psi'(\xi_2) > \hat\psi'(\xi_2)r/\xi_2$, which concludes the proof. □

## 2.4 Definition of parametrized BV-evolution

We define the total energy $\mathcal{F} : [0,T] \times L^2(K;\mathbb{R}^2) \times L^2(K;[0,+\infty)) \to \mathbb{R}\cup\{+\infty\}$ by

$$\mathcal{F}(t,w,\xi) := \begin{cases} \mathcal{E}(t,w) + \mathcal{K}(w,\xi) & \text{if } w \in H^{1/2}(K;\mathbb{R}^2),\\ +\infty & \text{otherwise.}\end{cases}$$

We consider in $L^2(K;\mathbb{R}^2)$ and $L^2(K;[0,+\infty))$ the usual norm. We denote by $|\partial_w\mathcal{F}|(t,w,\xi)$ the slope of the energy, defined as

$$|\partial_w\mathcal{F}|(t,w,\xi) = \limsup_{v\to w} \frac{|\mathcal{F}(t,w,\xi) - \mathcal{F}(t,v,\xi)|^+}{\|w-v\|},$$

where $|\cdot|^+$ is the positive part. See Appendix A for the main properties of the slope.

**Definition 2.7.** *Given $w_0 \in H^{1/2}(K;\mathbb{R}^2)$ and $\xi_0 \in L^2(K;[0,+\infty))$ with $|w_0| \le \xi_0$ such that $|\partial_w\mathcal{F}|(0,w_0,\xi_0) = 0$ a map $(t,w,\xi) : [0,S] \to [0,T] \times L^2(K;\mathbb{R}^2) \times L^2(K;[0,+\infty))$ is a parametrized BV-evolution for $\mathcal{F}$ with initial conditions $(0,w_0,\xi_0)$ if:*

*(i) $(t,w,\xi)$ is Lipschitz continuous with $t' \ge 0$ and $t(S) = T$,*

*(ii) $\xi'(s) \ge 0$, $|w(s)| \le \xi(s)$ and $\xi'(s)(|w(s)| - \xi(s)) = 0$ for a.e. $s \in [0,S]$,*

*(iii) $|\partial_w\mathcal{F}|(t(s),w(s),\xi(s)) = 0$ for every $s \in [0,S]$ with $t'(s) > 0$,*

*(iv) for every $s \in [0,S]$ the following energy balance holds*

$$\mathcal{F}(t(s),w(s),\xi(s)) = \mathcal{F}(0,w_0,\xi_0) - \int_0^s |\partial_w\mathcal{F}|(t(r),w(r),\xi(r))\,dr + \\ + \int_0^s \partial_t\mathcal{F}(t(r),w(r),\xi(r))\,t'(r)\,dr, \tag{7}$$

*where $|\partial_w\mathcal{F}|(t(r),w(r),\xi(r))$ denotes the slope in $L^2(K;\mathbb{R}^2)$, see Appendix A.*

In §5 we will provide a more explicit representation of the above conditions, in terms of PDEs.

# 3 Energies and slopes in $L^2$

## 3.1 Properties of the energy

**Lemma 3.1.** *If $t_n \to t$, $w_n \to w$ in $H^{1/2}(K;\mathbb{R}^2)$, $\xi_n \to \xi$ in $L^2(K;[0,+\infty))$ and if $\mathcal{F}(t_n, w_n, \xi_n) < +\infty$ for every $n \in \mathbb{N}$ then $\mathcal{F}(t, w, \xi) = \lim_{n\to+\infty} \mathcal{F}(t_n, w_n, \xi_n)$.*

*Proof.* The energy $\mathcal{E} = E \circ \boldsymbol{u}$ is the composition of continuous functionals and thus it is continuous. Since $\mathcal{F}(t_n, w_n, \xi_n) < +\infty$ then $\{w_n \cdot \nu \geq 0\}$ a.e. on $K$ for every $n \in \mathbb{N}$. Write

$$\mathcal{K}(w, \xi) = \int_K \varphi(w, \xi, \nu_K)\, d\mathcal{H}^1.$$

Consider a subsequence $n_k$. We can extract a further subsequence (not relabelled) such that $w_{n_k} \to w$ and $\xi_{n_k} \to \xi$ a.e. on $K$. Since $\varphi$ is bounded and continuous, by dominated convergence we get $\mathcal{K}(w_{n_k}, \xi_{n_k}) \to \mathcal{K}(w, \xi)$. By arbitrariness of the subsequence $w_{n_k}$ we get $\mathcal{K}(w_n, \xi_n) \to \mathcal{K}(w, \xi)$. □

**Lemma 3.2.** *If $t_n \to t$ and $w_n \rightharpoonup w$ in $L^2(K;\mathbb{R}^2)$ then $\mathcal{E}(t, w) \leq \liminf_{n\to+\infty} \mathcal{E}(t_n, w_n)$.*

*Proof.* If $\liminf_{n\to+\infty} \mathcal{E}(t_n, w_n) = +\infty$ there is nothing to prove. If the liminf is finite we can extract a subsequence (not relabelled) such that $\lim_{n\to+\infty} \mathcal{E}(t_n, w_n) = \liminf_{n\to+\infty} \mathcal{E}(t_n, w_n) < +\infty$. In particular there exists a constant $C > 0$ such that $E(\boldsymbol{u}(g(t_n), w_n)) \leq C$. Since $g(t_n) \to g(t)$ in $H^1(\Omega;\mathbb{R}^2)$ by Korn-Poincaré inequality it follows that $\boldsymbol{u}(g(t_n), w_n)$ is bounded in $H^1(\Omega;\mathbb{R}^2)$ and thus $w_n$ (as trace) is bounded in $H^{1/2}(K;\mathbb{R}^2)$. Up to subsequences (not relabelled) $w_n \rightharpoonup w$ weakly in $H^{1/2}(K;\mathbb{R}^2)$. By linearity and continuity $\boldsymbol{u}(g(t_n), w_n) \rightharpoonup \boldsymbol{u}(g(t), w)$ in $H^1(\Omega;\mathbb{R}^2)$. In particular, by definition of $\mathcal{E}$ and by weak lower semicontinuity of $E$ we deduce that

$$\mathcal{E}(t, w) = E \circ \boldsymbol{u}(g(t), w) \leq \liminf_{n\to+\infty} E \circ \boldsymbol{u}(g(t_n), w_n) = \liminf_{n\to+\infty} \mathcal{E}(t_n, w_n),$$

which concludes the proof. □

**Lemma 3.3.** *If $t_n \to t$, $w_n \rightharpoonup w$ weakly in $L^2(K;\mathbb{R}^2)$ and $\xi_n \to \xi$ strongly in $L^2(K;[0,+\infty))$ then*

$$\mathcal{F}(t, w, \xi) \leq \liminf_{n\to+\infty} \mathcal{F}(t_n, w_n, \xi_n). \tag{8}$$

*Proof.* By Lemma 3.2 it is sufficient to show that $\mathcal{K}(w, \xi) \leq \liminf_{n\to+\infty} \mathcal{K}(w_n, \xi_n)$. It is not restrictive to assume that the right-hand side of (8) is finite, then $\{w_n \cdot \nu \geq 0\}$ a.e. on $K$, moreover, arguing as in Lemma 3.2, $w_n$ is bounded in $H^{1/2}(K;\mathbb{R}^2)$ and thus (upon extracting a subsequence) we can assume that $w_n \to w$ strongly in $L^2(K;\mathbb{R}^2)$, $w_n \to w$ and $\xi_n \to \xi$ a.e. on $K$. The lower semicontinuity of $\mathcal{K}$ follows by Fatou's Lemma together with the continuity of $\varphi$. □

## 3.2 Properties of derivatives and slope

**Lemma 3.4.** *The elastic energy $\mathcal{E}(\cdot,\cdot)$ is Frèchet differentiable with*

$$\partial_t \mathcal{E}(t, w) = dE(\boldsymbol{u}(g(t), w))[\dot{g}(t)] = \int_\Omega \boldsymbol{\epsilon}(g(t), w) : \boldsymbol{\sigma}(\dot{g}(t))\, dx, \tag{9}$$

$$\partial_w \mathcal{E}(t, w)[z] = dE(\boldsymbol{u}(g(t), w))[\boldsymbol{u}(0, z)] = \int_\Omega \boldsymbol{\epsilon}(g(t), w) : \boldsymbol{\sigma}(\boldsymbol{u}(0, z))\, dx, \tag{10}$$

*for $z \in H^{1/2}(K;\mathbb{R}^2)$. Moreover, if $w_n \rightharpoonup w$ in $H^{1/2}(K;\mathbb{R}^2)$ and $t_n \to t$ then*

$$\lim_{n\to+\infty} \partial_t \mathcal{E}(t_n, w_n) = \partial_t \mathcal{E}(t, w). \tag{11}$$

*Finally, there exists a constant $C$ such that*

$$|\partial_t \mathcal{F}(t, w, \xi)| = |\partial_t \mathcal{E}(t, w)| \leq C(\mathcal{F}(t, w, \xi) + 1). \tag{12}$$

*Proof.* Write

$$\partial_t \mathcal{E}(t,w) = \lim_{h\to 0} \frac{\mathcal{E}(t+h,w) - \mathcal{E}(t,w)}{h} = \lim_{h\to 0} \frac{E \circ \boldsymbol{u}(g(t+h),w) - E \circ \boldsymbol{u}(g(t),w)}{h}.$$

By linearity of $\boldsymbol{u}$

$$\boldsymbol{u}(g(t+h),w) = \boldsymbol{u}(g(t),w) + \boldsymbol{u}(g(t+h)-g(t),0) = \boldsymbol{u}(g(t),w) + g(t+h) - g(t).$$

Remember that by assumption $g(t) = \boldsymbol{u}(g(t),0)$. Thus, by the Frèchet differentiability of the elastic energy $E$ we get

$$\begin{aligned}\partial_t \mathcal{E}(t,w) &= \lim_{h\to 0} \frac{E(\boldsymbol{u}(g(t),w) + g(t+h) - g(t)) - E(\boldsymbol{u}(g(t),w))}{h} \\ &= dE(\boldsymbol{u}(g(t),w))[\dot g(t)] = \int_\Omega \boldsymbol{\epsilon}(g(t),w) : \boldsymbol{\sigma}(\dot g(t))\, dx.\end{aligned}$$

In the same way,

$$\partial_w \mathcal{E}(t,w) = \lim_{h\to 0} \frac{\mathcal{E}(t,w+hz) - \mathcal{E}(t,w)}{h} = \lim_{h\to 0} \frac{E \circ \boldsymbol{u}(g(t),w+hz) - E \circ \boldsymbol{u}(g(t),w)}{h}$$

Writing $\boldsymbol{u}(g(t),w+hz) = \boldsymbol{u}(g(t),w) + h\boldsymbol{u}(0,z)$ we get (10).

If $w_n \rightharpoonup w$ in $H^{1/2}(K;\mathbb{R}^2)$ and if $t_n \to t$ then $\boldsymbol{u}(g(t_n),w_n) \rightharpoonup \boldsymbol{u}(g(t),w)$ in $H^1(\Omega;\mathbb{R}^2)$ by linearity and continuity. It follows that (11) holds true since $\dot g(t_n) \to \dot g(t)$ in $H^1(\Omega;\mathbb{R}^2)$ by the time regularity of $g$.

By the same argument one shows that the partial derivatives are continuous in $[0,T]\times H^{1/2}(K;\mathbb{R}^2)$ (endowed with the strong topology) and thus $\mathcal{E}$ is Frèchet differentiable.

Finally, estimate (12) easily follows from (9), the Taylor inequality, and (1) □

Now, let us prove the lower semicontinuity of the slope.

**Lemma 3.5.** *If $t_n \to t$, $w_n \rightharpoonup w$ in $H^{1/2}(K;\mathbb{R}^2)$ and $\xi_n \to \xi$ strongly in $L^2(K;[0,+\infty))$ then*

$$|\partial_w \mathcal{F}|(t,w,\xi) \le \liminf_{n\to+\infty} |\partial_w \mathcal{F}|(t_n,w_n,\xi_n). \tag{13}$$

*Proof.* Without loss of generality we can assume that the sequence $|\partial_w \mathcal{F}|(t_n,w_n,\xi_n)$ is bounded; in particular $\mathcal{F}(t_n,w_n,\xi_n)$ is finite. By Lemma A.5 write

$$|\partial_w \mathcal{F}|(t,w,\xi) = |\partial_w \mathcal{J}_w|(t,w,\xi) = \inf\{\|\eta\|_{L^2(K;\mathbb{R}^2)} : \eta \in \partial_w \mathcal{J}_w(t,w,\xi)\},$$

where

$$\mathcal{J}_w(t,z,\xi) = \mathcal{F}(t,z,\xi) - \tfrac{\lambda}{2}\|w-z\|^2_{L^2(K;\mathbb{R}^2)}.$$

Clearly the same identity holds true replacing $t$, $w$, and $\xi$ with $t_n$, $w_n$, and $\xi_n$. For every $n\in\mathbb{N}$ let $\eta_n \in \partial_w \mathcal{J}_{w_n}(t_n,w_n,\xi_n)$ be such that $\|\eta_n\|_{L^2(K;\mathbb{R}^2)} \le |\partial_w \mathcal{F}|(t_n,w_n,\xi_n) + 1/n$. It follows that $\eta_n$ is bounded in $L^2(K;\mathbb{R}^2)$, thus (up to subsequences) $\eta_n \rightharpoonup \eta$ and

$$\|\eta\|_{L^2(K;\mathbb{R}^2)} \le \liminf_{n\to+\infty} \|\eta_n\|_{L^2(K;\mathbb{R}^2)} \le \liminf_{n\to+\infty} |\partial_w \mathcal{F}|(t_n,w_n,\xi_n).$$

We will prove that $\eta \in \partial_w \mathcal{J}_w(t,w,\xi)$ from which (13) will follow. Remember that by convexity $\eta \in \partial_w \mathcal{J}_w(t,w,\xi)$ if and only if

$$\mathcal{J}_w(t,z,\xi) \ge \mathcal{J}_w(t,w,\xi) + \langle \eta, z-w\rangle_{L^2(K;\mathbb{R}^2)} \qquad \forall z \in L^2(K;\mathbb{R}^2). \tag{14}$$

In the same way $\eta_n \in \partial_w \mathcal{J}_{w_n}(t_n,w_n,\xi_n)$ if and only if

$$\mathcal{J}_{w_n}(t_n,z,\xi_n) \ge \mathcal{J}_{w_n}(t_n,w_n,\xi_n) + \langle \eta_n, z-w_n\rangle_{L^2(K;\mathbb{R}^2)} \qquad \forall z \in L^2(K;\mathbb{R}^2). \tag{15}$$

First, by Lemma 3.1 for every $z \in L^2(K;\mathbb{R}^2)$ we have

$$\lim_{n\to+\infty} \mathcal{J}_{w_n}(t_n, z, \xi_n) = \mathcal{J}_w(t, z, \xi)$$

(note that it is sufficient to consider $z \in H^{1/2}(K;\mathbb{R}^2)$). Next, since $\mathcal{J}_{w_n}(t_n, w_n, \xi_n) = \mathcal{F}(t_n, w_n, \xi_n)$ by Lemma 3.3 we get

$$\liminf_{n\to+\infty} \mathcal{J}_{w_n}(t_n, w_n, \xi_n) \geq \mathcal{J}_w(t, w, \xi).$$

Finally,

$$\lim_{n\to+\infty} \langle \eta_n, z - w_n\rangle_{L^2(K;\mathbb{R}^2)} = \langle \eta, z - w\rangle_{L^2(K;\mathbb{R}^2)}$$

because $\eta_n \rightharpoonup \eta$ while $w_n \to w$ in $L^2(K;\mathbb{R}^2)$. In conclusion, we can pass to the limit in (15) obtaining (14) and hence $\eta \in \partial_w \mathcal{J}_w(t, w, \xi)$. □

**Remark 3.6.** *If $|\partial_w \mathcal{F}|(t, w, \xi) = 0$ then $w$ is an equilibrium point for the energy $\mathcal{F}(t, \cdot, \xi)$.*

# 4 Parametrized BV evolution

## 4.1 Discrete evolution

In order to prove the existence of a quasi-static $BV$-evolution we employ an incremental scheme similar to the one adopted in [18, 19]. Let $c > 0$ (sufficiently small) to be fixed later. Let $\Delta s_n > 0$ be an infinitesimal sequence and let $s_{n,k} := k\Delta s_n$ for $k \in \mathbb{N}$. Let us fix $n \in \mathbb{N}$. We set $t_{n,0} = 0$, $w_{n,0} = w_0$ and $\xi_{n,0} = \xi_0$. We define by induction $w_{n,k+1}$, $\xi_{n,k+1}$ and $t_{n,k+1}$ as follows:

- if $t_{n,k} < T$ and $w_{n,k}$ is a local minimizer of $\mathcal{F}(t_{n,k}, \cdot, \xi_{n,k})$ (in some neighborhood $\|w - w_{n,k}\|_{L^2(K;\mathbb{R}^2)} \leq r$ for $r > 0$) then we update time, setting

$$\begin{cases} t_{n,k+1} = \min\{t_{n,k} + c\Delta s_n, T\} \\ w_{n,k+1} = w_{n,k}, \\ \xi_{n,k+1} = \xi_{n,k}, \end{cases} \tag{16}$$

- if $t_{n,k} \leq T$ and $w_{n,k}$ is not a local minimizer, then we look for a local minimizer, setting

$$\begin{cases} t_{n,k+1} = t_{n,k}, \\ w_{n,k+1} \in \operatorname{argmin}\{\mathcal{F}(t_{n,k+1}, w, \xi_{n,k}) : \|w - w_{n,k}\|_{L^2(K;\mathbb{R}^2)} \leq \Delta s_n\}, \\ \xi_{n,k+1} = \xi_{n,k} \vee |w_{n,k+1}|. \end{cases} \tag{17}$$

In the following section we will prove that for every $\Delta s_n > 0$ there exists a finite index $k_n$ such that $t_{n,k_n} = T$ and $w_{n,k_n}$ is a local minimizer. We do not define the evolution for $k > k_n$.

The existence of a minimizer in (17) (non necessarily unique) follows by the direct method: the ball $\|w - w_{n,k}\|_{L^2(K;\mathbb{R}^2)} \leq \Delta s_n$ is weakly closed in $L^2(K;\mathbb{R}^2)$ while the energy $\mathcal{F}(t_{n,k}, \cdot, \xi_{n,k})$ is weakly lower semicontinuous by Lemma 3.3 (notice that in the minimum problem $\xi_{n,k}$ is fixed).

**Remark 4.1.** Note that $t_{n,k}$, $\xi_{n,k}$ and $s_{n,k}$ are monotone non-decreasing with respect to $k$. Further, by definition $|w_{n,k}| \leq \xi_{n,k}$ and

$$\|w_{n,k+1} - w_{n,k}\|_{L^2(K;\mathbb{R}^2)} \leq \Delta s_n, \quad 0 \leq t_{n,k+1} - t_{n,k} \leq c\Delta s_n, \quad s_{n,k+1} - s_{n,k} = \Delta s_n.$$

Moreover,

$$0 \leq \xi_{n,k+1} - \xi_{n,k} \leq |w_{n,k+1} - w_{n,k}| \text{ a.e. in } K,$$

so that

$$\|\xi_{n,k+1} - \xi_{n,k}\|_{L^2(K;[0,+\infty))} \leq \|w_{n,k+1} - w_{n,k}\|_{L^2(K;\mathbb{R}^2)}.$$

Indeed, if $\xi_{n,k+1} = \xi_{n,k}$ there is nothing to prove, otherwise $|w_{n,k+1}| = \xi_{n,k+1} > \xi_{n,k} \geq |w_{n,k}|$ and hence $|w_{n,k+1} - w_{n,k}| \geq |w_{n,k+1}| - |w_{n,k}| \geq \xi_{n,k+1} - \xi_{n,k}$. Note also that $\|w_{n,k+1} - w_{n,k}\|_{L^2(K;\mathbb{R}^2)} + (t_{n,k+1} - t_{n,k}) > 0$ for every index $k$.

For all $n \in \mathbb{N}$ we define the length of the parametrization $S_n \in (0, +\infty]$ as $S_n := s_{n,k_n} = k_n \Delta s_n$, where $k_n$ is the first index such that $t_{n,k_n} = T$ and $w_{n,k_n}$ is a local minimizer, if it exists, otherwise we set $S_n = +\infty$. In the following section we will prove that $S_n$ is finite and uniformly bounded with respect to $n \in \mathbb{N}$. We define the discrete (parametrized) evolutions $(t_n, w_n, \xi_n) : [0, S_n) \to [0,T] \times L^2(K;\mathbb{R}^2) \times L^2(K;[0,+\infty))$ taking the affine interpolation of $t_{n,k}$, $w_{n,k}$, and $\xi_{n,k}$ in the points $s_{n,k}$. We will denote by $t'_n$, $w'_n$, and $\xi'_n$ the derivatives with respect to the parameter $s \in [0, S_n)$. In the sequel we will employ also the notation

$$w'_{n,k+1} = \frac{w_{n,k+1} - w_{n,k}}{\Delta s_n} = w'_n(s) \quad \text{for } s \in (s_{n,k}, s_{n,k+1})$$

and similarly for other variables. Note that, by definition, $|t'_n| \le c$, $\|w'_n\|_{L^2(K;\mathbb{R}^2)} \le 1$ and $\|\xi'_n\|_{L^2(K;\,[0,+\infty))} \le 1$.

## 4.2 Finite length and $W^{1,2}$-parametrization in $H^{1/2}$

In this section we will prove the following result.

**Theorem 4.2.** *For $c > 0$ sufficiently small there exist $S, C < +\infty$ such that for every $n \in \mathbb{N}$ we have $S_n \le S$ and*

$$\int_0^{S_n} \|w'_n\|_{H^{1/2}(K;\mathbb{R}^2)} + \|w'_n\|^2_{H^{1/2}(K;\mathbb{R}^2)}\, ds \le C.$$

*In particular there exists an index $k_n$ such that $t_{n,k_n} = t_n(S_n) = T$.*

In order to prove Theorem 4.2 we need several technical lemmas.

**Lemma 4.3.** *Let $k \in \mathbb{N}$ such that (17) occurs. If $\|w_{n,k+1} - w_{n,k}\|_{L^2(K;\mathbb{R}^2)} < \Delta s_n$ then*

$$\partial_w \mathcal{F}(t_{n,k+1}, w_{n,k+1}, \xi_{n,k}) \ni 0, \qquad |\partial_w \mathcal{F}|(t_{n,k+1}, w_{n,k+1}, \xi_{n,k}) = 0.$$

*Proof.* It is enough to note that by (17) $t_{n,k+1} = t_{n,k}$ and $w_{n,k+1}$ is a local minimizer of the energy $\mathcal{F}(t_{n,k+1}, \cdot, \xi_{n,k})$ in a ball centred in $w_{n,k+1}$ with radius $r > 0$ sufficiently small. □

Clearly the previous Lemma would no longer be true if $\|w_{n,k+1} - w_{n,k}\|_{L^2(K;\mathbb{R}^2)} = \Delta s_n$.

**Lemma 4.4.** *Let $k \in \mathbb{N}$ such that (17) occurs. Then*

$$w_{n,k+1} \in \operatorname{argmin} \{ \mathcal{F}(t_{n,k+1}, w, \xi_{n,k+1}) : \|w - w_{n,k}\|_{L^2(K;\mathbb{R}^2)} \le \Delta s_n \}$$

*Proof.* It is sufficient to adapt the proof of [19, Lemma 5.1]. □

**An auxiliary functional.** For technical reasons in the next few pages it is useful to employ the energy density $\tilde{\varphi}(w, \xi) = \psi(|w|, \xi)$ which does not take into account the incompenetrability constraint, and thus it does not depend on the normal vector $\nu_K$. Accordingly we define

$$\widetilde{\mathcal{K}}(w, \xi) = \int_K \tilde{\varphi}(w, \xi)\, d\mathcal{H}^1, \qquad \widetilde{\mathcal{F}}(t, w, \xi) = \mathcal{E}(t, w) + \widetilde{\mathcal{K}}(w, \xi). \tag{18}$$

To prove the next Lemma we need to introduce the directional derivatives of the energy $\widetilde{\mathcal{F}}(t, \cdot, \xi)$. Let $w, z \in H^{1/2}(K;\mathbb{R}^2)$. We set

$$\widetilde{\mathcal{K}}'(w, \xi; z) := \lim_{h \downarrow 0} \frac{\widetilde{\mathcal{K}}(w + hz, \xi) - \widetilde{\mathcal{K}}(w, \xi)}{h} = \int_K \tilde{\varphi}'(w, \xi; z)\, d\mathcal{H}^1,$$

where a.e. in $K$ we have

$$\tilde{\varphi}'(w, \xi; z) := \lim_{h \downarrow 0} \frac{\tilde{\varphi}(w + hz, \xi) - \tilde{\varphi}(w, \xi)}{h} = \begin{cases} \psi'(|w|, \xi) \langle \hat{w}, z \rangle & \text{if } |w| \neq 0, \\ \psi'_+(|w|, \xi) |z| & \text{if } |w| = 0, \end{cases} \tag{19}$$

with $\hat{w} = w/|w|$. In particular

$$\tilde{\varphi}'(w,\xi;z) = \hat{\psi}'(\xi)\langle \hat{w}, z\rangle \ \text{ if } |w| = \xi > 0, \qquad \tilde{\varphi}'(w,\xi;z) = \hat{\psi}'(0)|z| \ \text{ for } |w| = \xi = 0. \tag{20}$$

In general $|\tilde{\varphi}'(w,\xi;z)| \le \hat{\psi}'(\xi)|z| \le \hat{\psi}'(0)|z|$ . Finally, note that $\tilde{\varphi}(w,\xi;\cdot)$ is positively 1-homogeneous. Since the elastic energy is differentiable, by (10) we have

$$\widetilde{\mathcal{F}}'(t,w,\xi;z) = dE(\boldsymbol{u}(g(t),w))[\boldsymbol{u}(0,z)] + \widetilde{\mathcal{K}}'(w,\xi;z).$$

**Lemma 4.5.** *For every index $k \ge 0$ it holds, a.e. on $K$,*

$$\tilde{\varphi}'(w_{n,k},\xi_{n,k};w'_{n,k+1}) - \tilde{\varphi}'(w_{n,k+1},\xi_{n,k+1};w'_{n,k+1}) \le \big(\hat{\psi}'(\xi_{n,k}) - \hat{\psi}'(\xi_{n,k+1})\big)|w'_{n,k+1}|. \tag{21}$$

*Note that the right-hand side is non-negative, while the left-hand side can be negative.*

*Proof.* Since the expression in (21) is positively 1-homogeneous it suffices to prove that

$$\tilde{\varphi}'(w_{n,k},\xi_{n,k};\delta w_{n,k+1}) - \tilde{\varphi}'(w_{n,k+1},\xi_{n,k+1};\delta w_{n,k+1}) \le \big(\hat{\psi}'(\xi_{n,k}) - \hat{\psi}'(\xi_{n,k+1})\big)|\delta w_{n,k+1}|, \tag{22}$$

where $\delta w_{n,k+1} := w_{n,k+1} - w_{n,k}$. We will discuss several cases, taking into account the different behaviours of the cohesive law.

*Case 1:* $0 = \xi_{n,k} = \xi_{n,k+1}$. Since $|w_{n,k}| \le \xi_{n,k}$ and $|w_{n,k+1}| \le \xi_{n,k+1}$ it holds $w_{n,k} = w_{n,k+1} = 0$, hence $\delta w_{n,k+1} = 0$ and the thesis is trivial.

*Case 2:* $0 < \xi_{n,k} = \xi_{n,k+1}$. Now both $|w_{n,k}| \le \xi_{n,k}$ and $|w_{n,k+1}| \le \xi_{n,k+1} = \xi_{n,k}$. For $|w| \le \xi_{n,k}$ the energy density $\tilde{\varphi}(\cdot,\xi_{n,k}) = \psi(|\cdot|,\xi_{n,k})$ is convex, hence

$$\tilde{\varphi}'(w_{n,k},\xi_{n,k};\delta w_{n,k+1}) - \tilde{\varphi}'(w_{n,k+1},\xi_{n,k};\delta w_{n,k+1}) \le 0,$$

and formula (22) follows since its right-hand side is always non-negative.

*Case 3:* $0 \le \xi_{n,k} < \xi_{n,k+1}$ *and* $w_{n,k} = 0$. In this case it must be $|w_{n,k+1}| = \xi_{n,k+1} > 0$ and $\delta w_{n,k+1} = w_{n,k+1} \neq 0$. Note that by (20) if $\xi_{n,k} = 0$ then $\tilde{\varphi}'(w_{n,k},\xi_{n,k};\delta w_{n,k+1}) = \hat{\psi}'(0)|\delta w_{n,k+1}| = \hat{\psi}'(\xi_{n,k})|\delta w_{n,k+1}|$, while if $\xi_{n,k} \neq 0$ then $\tilde{\varphi}'(w_{n,k},\xi_{n,k};\delta w_{n,k+1}) \le \hat{\psi}'(\xi_{n,k})|\delta w_{n,k+1}|$. Thus, we have

$$\begin{aligned}
\tilde{\varphi}'(w_{n,k},&\xi_{n,k};\delta w_{n,k+1}) - \tilde{\varphi}'(w_{n,k+1},\xi_{n,k+1};\delta w_{n,k+1}) \le \\
&\le \hat{\psi}'(\xi_{n,k})|\delta w_{n,k+1}| - \psi'(|w_{n,k+1}|,\xi_{n,k+1})\langle \hat{w}_{n,k+1},\delta w_{n,k+1}\rangle = \\
&= \hat{\psi}'(\xi_{n,k})|\delta w_{n,k+1}| - \hat{\psi}'(\xi_{n,k+1})\langle \widehat{\delta w}_{n,k+1},\delta w_{n,k+1}\rangle = (\hat{\psi}'(\xi_{n,k}) - \hat{\psi}'(\xi_{n,k+1}))|\delta w_{n,k+1}|.
\end{aligned}$$

*Case 4:* $0 \le \xi_{n,k} < \xi_{n,k+1}$, $w_{n,k} \neq 0$ *and* $\langle \hat{w}_{n,k},\delta w_{n,k+1}\rangle < 0$. Note that $|w_{n,k+1}| = \xi_{n,k+1} > \xi_{n,k} \ge |w_{n,k}| > 0$. In this case

$$\begin{aligned}
\tilde{\varphi}'(w_{n,k},\xi_{n,k};\delta w_{n,k+1}) &- \tilde{\varphi}'(w_{n,k+1},\xi_{n,k+1};\delta w_{n,k+1}) = \\
&= \psi'(|w_{n,k}|,\xi_{n,k})\langle \hat{w}_{n,k},\delta w_{n,k+1}\rangle - \psi'(|w_{n,k+1}|,\xi_{n,k+1})\langle \hat{w}_{n,k+1},\delta w_{n,k+1}\rangle.
\end{aligned}$$

The first term is non-positive because $\psi'(|w_{n,k}|,\xi_{n,k}) \ge 0$. Also the second term is non-positive since $\psi'(|w_{n,k+1}|,\xi_{n,k+1}) \ge 0$ and

$$\langle w_{n,k+1},\delta w_{n,k+1}\rangle = \langle w_{n,k+1}, w_{n,k+1} - w_{n,k}\rangle \ge |w_{n,k+1}|^2 - |w_{n,k+1}||w_{n,k}| > 0.$$

As a consequence, the left-hand side of (22) is non-positive while the right-hand side is non-negative.

*Case 5:* $0 \le \xi_{n,k} < \xi_{n,k+1}$, $w_{n,k} \neq 0$ *and* $\langle \hat{w}_{n,k},\delta w_{n,k+1}\rangle \ge 0$. Note that $\xi_{n,k} \neq 0$. Let us define the two auxiliary vectors $\tilde{w}_{n,k}$ and $\tilde{w}_{n,k+1}$ as follows:

$$\tilde{w}_{n,k} := \hat{w}_{n,k}\,\xi_{n,k}, \qquad \tilde{w}_{n,k+1} := \hat{w}_{n,k+1}\,\xi_{n,k}.$$

We write

$$\begin{aligned}\tilde\varphi'(w_{n,k},\xi_{n,k};\delta w_{n,k+1})-\tilde\varphi'(w_{n,k+1},\xi_{n,k+1};\delta w_{n,k+1})=\\ =\tilde\varphi'(w_{n,k},\xi_{n,k};\delta w_{n,k+1})-\tilde\varphi'(\tilde w_{n,k},\xi_{n,k};\delta w_{n,k+1})+\\ +\tilde\varphi'(\tilde w_{n,k},\xi_{n,k};\delta w_{n,k+1})-\tilde\varphi'(\tilde w_{n,k+1},\xi_{n,k};\delta w_{n,k+1})+\\ +\tilde\varphi'(\tilde w_{n,k+1},\xi_{n,k};\delta w_{n,k+1})-\tilde\varphi'(w_{n,k+1},\xi_{n,k+1};\delta w_{n,k+1}).\end{aligned} \tag{23}$$

We treat separately the three lines in the right-hand side of (23). Note that $\tilde w_{n,k}/|\tilde w_{n,k}| = \hat w_{n,k}$, then the first line equals

$$\begin{aligned}\psi'(|w_{n,k}|,\xi_{n,k})\langle\hat w_{n,k},\delta w_{n,k+1}\rangle-\psi'(|\xi_{n,k}|,\xi_{n,k})\langle\hat w_{n,k},\delta w_{n,k+1}\rangle=\\ =(\psi'(|w_{n,k}|,\xi_{n,k})-\hat\psi'(\xi_{n,k}))\langle\hat w_{n,k},\delta w_{n,k+1}\rangle\le 0,\end{aligned}$$

because $\psi'(|w_{n,k}|,\xi_{n,k})\le\hat\psi'(\xi_{n,k})$ and in this case $\langle\hat w_{n,k},\delta w_{n,k+1}\rangle\ge 0$. As for the second line we have

$$\begin{aligned}\tilde\varphi'(\tilde w_{n,k},\xi_{n,k};\delta w_{n,k+1})-\tilde\varphi'(\tilde w_{n,k+1},\xi_{n,k};\delta w_{n,k+1}) &= \hat\psi'(\xi_{n,k})\langle\hat w_{n,k},\delta w_{n,k+1}\rangle-\hat\psi'(\xi_{n,k})\langle\hat w_{n,k+1},\delta w_{n,k+1}\rangle\\ &=-\hat\psi'(\xi_{n,k})\langle\hat w_{n,k+1}-\hat w_{n,k},\delta w_{n,k+1}\rangle\le 0,\end{aligned}$$

where the last inequality follows by

$$\begin{aligned}\langle\hat w_{n,k+1}-\hat w_{n,k},\delta w_{n,k+1}\rangle &=\langle\hat w_{n,k+1}-\hat w_{n,k},w_{n,k+1}-w_{n,k}\rangle\\ &=|w_{n,k+1}|+|w_{n,k}|-\langle\hat w_{n,k+1},w_{n,k}\rangle-\langle\hat w_{n,k},w_{n,k+1}\rangle\\ &\ge|w_{n,k+1}|+|w_{n,k}|-|w_{n,k+1}|-|w_{n,k}|=0.\end{aligned}$$

Since $\xi_{n,k+1}>\xi_{n,k}$ we have $|w_{n,k+1}|=\xi_{n,k+1}$ and then the last line of (23) gives

$$\begin{aligned}\tilde\varphi'(\tilde w_{n,k+1},\xi_{n,k};\delta w_{n,k+1})-\tilde\varphi'(w_{n,k+1},\xi_{n,k+1};\delta w_{n,k+1})=\\ =\hat\psi'(\xi_{n,k})\langle\hat w_{n,k+1},\delta w_{n,k+1}\rangle-\hat\psi'(\xi_{n,k+1})\langle\hat w_{n,k+1},\delta w_{n,k+1}\rangle\\ \le(\hat\psi'(\xi_{n,k})-\hat\psi'(\xi_{n,k+1}))|\delta w_{n,k+1}|,\end{aligned}$$

and the thesis is achieved. □

**Proposition 4.6.** *For every index $k\ge 1$ we have*

$$\widetilde{\mathcal F}'(t_{n,k},w_{n,k},\xi_{n,k};w'_{n,k})\le 0. \tag{24}$$

*Moreover there exists a constant $C_0>0$ (independent of $\Delta s_n$) such that for every index $k\ge 1$ we have*

$$\widetilde{\mathcal F}'(t_{n,k},w_{n,k},\xi_{n,k};w'_{n,k})\le\widetilde{\mathcal F}'(t_{n,k},w_{n,k},\xi_{n,k};w'_{n,k+1})+c\,C_0\|w_{n,k+1}-w_{n,k}\|_{H^{1/2}(K;\mathbb R^2)}, \tag{25}$$

*where $c$ is the constant appearing in* (16)*.*

*Proof.* We distinguish two cases: according to (17) and (16).

*Case 1:* (17) *occurs for $k-1$.* Denote $A_1=\{w\in H^{1/2}(K;\mathbb R^2):\|w-w_{n,k-1}\|_{L^2(K;\mathbb R^2)}\le\Delta s_n\}$ and $A_2=\{w\in H^{1/2}(K;\mathbb R^2): w\cdot\nu\ge 0$ a.e. on $K\}$. Let $I_i$ denote the indicator function of the set $A_i$ (the function equal to 0 in $A_i$ and $+\infty$ outside). Setting $I=I_1+I_2$ by Lemma 4.4 we have

$$w_{n,k}\in\operatorname{argmin}\{\widetilde{\mathcal F}(t_{n,k},w,\xi_{n,k})+I(w):w\in H^{1/2}(K;\mathbb R^2)\}.$$

This implies

$$0\in\partial(\widetilde{\mathcal F}+I)(t_{n,k},w_{n,k},\xi_{n,k}), \tag{26}$$

where the subdifferential is intended with respect to the scalar product of $H^{1/2}(K;\mathbb{R}^2)$. It is convenient to write $\widetilde{\mathcal{K}} = \widetilde{\mathcal{K}}_0 + \widetilde{\mathcal{K}}_+$, where

$$\widetilde{\mathcal{K}}_0(w,\xi_{n,k}) := \int_{\{\xi_{n,k}=0\}} \tilde{\varphi}(w,0)\, d\mathcal{H}^1 \qquad \text{and} \qquad \widetilde{\mathcal{K}}_+(w,\xi_{n,k}) := \int_{\{\xi_{n,k}>0\}} \tilde{\varphi}(w,\xi_{n,k})\, d\mathcal{H}^1. \tag{27}$$

Note that $\widetilde{\mathcal{K}}_+(\cdot,\xi)$ is differentiable because, for $\xi > 0$, the density $\tilde{\varphi}(\cdot,\xi)$ is differentiable. Since $\widetilde{\mathcal{F}}(t_{n,k},\cdot,\xi_{n,k})$ is continuous in $H^{1/2}(K;\mathbb{R}^2)$ as well as $I_1$ we can apply [12, Proposition 5.6], entailing that

$$\partial(\widetilde{\mathcal{F}}+I)(t_{n,k},w_{n,k},\xi_{n,k}) = d\mathcal{E}(t_{n,k},w_{n,k}) + d\widetilde{\mathcal{K}}_+(w_{n,k},\xi_{n,k}) + \partial\widetilde{\mathcal{K}}_0(w_{n,k},\xi_{n,k}) + \partial I_1(w_{n,k}) + \partial I_2(w_{n,k}).$$

Therefore, by (26), there exist $\eta_0 \in \partial\widetilde{\mathcal{K}}_0(w_{n,k},\xi_{n,k})$, $\eta_1 \in \partial I_1(w_{n,k})$ and $\eta_2 \in \partial I_2(w_{n,k})$ such that

$$0 = d\mathcal{E}(t_{n,k},w_{n,k}) + d\widetilde{\mathcal{K}}_+(w_{n,k},\xi_{n,k}) + \eta_0 + \eta_1 + \eta_2. \tag{28}$$

On the other hand, the directional derivative of $\widetilde{\mathcal{F}}$ along $\delta w_{n,k} = w_{n,k} - w_{n,k-1}$ reads

$$\begin{aligned}\widetilde{\mathcal{F}}'(t_{n,k},w_{n,k},\xi_{n,k};\delta w_{n,k}) &= d\mathcal{E}(t_{n,k},w_{n,k})[\delta w_{n,k}] + d\widetilde{\mathcal{K}}_+(w_{n,k},\xi_{n,k})[\delta w_{n,k}] + \widetilde{\mathcal{K}}_0'(w_{n,k},\xi_{n,k};\delta w_{n,k})\\ &= d\mathcal{E}(t_{n,k},w_{n,k})[\delta w_{n,k}] + d\widetilde{\mathcal{K}}_+(w_{n,k},\xi_{n,k})[\delta w_{n,k}],\end{aligned}$$

where we have used the fact that $\widetilde{\mathcal{K}}_0'(w_{n,k},\xi_{n,k};\delta w_{n,k}) = \int_{\{\xi_{n,k}=0\}} \hat{\psi}'(0)|\delta w_{n,k}|\, d\mathcal{H}^1 = 0$ since $\delta w_{n,k} = 0$ on the set $\{\xi_{n,k} = 0\}$ (indeed $|w_{n,k}| = |w_{n,k-1}| = \xi_{n,k} = 0$). Being $\eta_0 \in \partial\widetilde{\mathcal{K}}_0(w_{n,k},\xi_{n,k})$ we have

$$-\langle \eta_0, \delta w_{n,k}\rangle_{H^{1/2}} = \langle \eta_0, w_{n,k-1} - w_{n,k}\rangle_{H^{1/2}} \le \widetilde{\mathcal{K}}_0(w_{n,k-1},\xi_{n,k}) - \widetilde{\mathcal{K}}_0(w_{n,k},\xi_{n,k}) = 0$$

and hence, using (28), we get

$$\widetilde{\mathcal{F}}'(t_{n,k},w_{n,k},\xi_{n,k};\delta w_{n,k}) = -\langle \eta_0+\eta_1+\eta_2, \delta w_{n,k}\rangle_{H^{1/2}} \le -\langle \eta_1+\eta_2, \delta w_{n,k}\rangle_{H^{1/2}}. \tag{29}$$

Since $\eta_1 \in \partial I_1(w_{n,k})$ we have $\langle \eta_1, w - w_{n,k}\rangle_{H^{1/2}} \le 0$ for every $w \in A_1$. Choosing $w = w_{n,k-1}$ yields $\langle \eta_1, \delta w_{n,k}\rangle \ge 0$. Similarly for $\eta_2$. Hence from (29) we get (24).

In order to prove (25) we will show that

$$\widetilde{\mathcal{F}}'(t_{n,k},w_{n,k},\xi_{n,k};\delta w_{n,k}) \le \widetilde{\mathcal{F}}'(t_{n,k},w_{n,k},\xi_{n,k};\delta w_{n,k+1}),$$

where $\delta w_{n,k} = w_{n,k} - w_{n,k-1}$. Estimate (25) will follow by positive 1-homogeneity.

Arguing as before, a similar expression for the directional derivative $\widetilde{\mathcal{F}}'(t_{n,k},w_{n,k},\xi_{n,k};\delta w_{n,k+1})$ leads to

$$\widetilde{\mathcal{F}}'(t_{n,k},w_{n,k},\xi_{n,k};\delta w_{n,k+1}) = -\langle \eta_0+\eta_1+\eta_2, \delta w_{n,k+1}\rangle_{H^{1/2}} + \widetilde{\mathcal{K}}_0'(w_{n,k},\xi_{n,k};\delta w_{n,k+1}).$$

We first claim that

$$-\langle \eta_0, \delta w_{n,k+1}\rangle_{H^{1/2}} + \widetilde{\mathcal{K}}_0'(w_{n,k},\xi_{n,k};\delta w_{n,k+1}) \ge 0,$$

indeed, since $\eta_0 \in \partial\widetilde{\mathcal{K}}_0(w_{n,k})$, for all $h > 0$ we have

$$\widetilde{\mathcal{K}}_0(w_{n,k} + h\delta w_{n,k+1},\xi_{n,k}) \ge \widetilde{\mathcal{K}}_0(w_{n,k},\xi_{n,k}) + \langle \eta_0, h\delta w_{n,k+1}\rangle_{H^{1/2}},$$

and the claim follows taking the directional derivative. As a consequence

$$\widetilde{\mathcal{F}}'(t_{n,k},w_{n,k},\xi_{n,k};\delta w_{n,k+1}) \ge -\langle \eta_1+\eta_2, \delta w_{n,k+1}\rangle_{H^{1/2}}. \tag{30}$$

Now, thanks to (29) and (30), it is enough to show that

$$-\langle \eta_1+\eta_2, \delta w_{n,k+1}\rangle_{H^{1/2}} \ge -\langle \eta_1+\eta_2, \delta w_{n,k}\rangle_{H^{1/2}}.$$

The above inequality is a consequence of the three inequalities

$$\langle \eta_1, \delta w_{n,k+1} - \delta w_{n,k} \rangle_{H^{1/2}} \le 0, \qquad \langle \eta_2, \delta w_{n,k+1} \rangle_{H^{1/2}} \le 0, \qquad \langle \eta_2, -\delta w_{n,k} \rangle_{H^{1/2}} \le 0. \tag{31}$$

To see the first one, we write

$$\langle \eta_1, \delta w_{n,k+1} - \delta w_{n,k} \rangle_{H^{1/2}} = \langle \eta_1, (w_{n,k+1} - w_{n,k} + w_{n,k-1}) - w_{n,k} \rangle_{H^{1/2}}.$$

Since $\eta_1 \in \partial I_1(w_{n,k})$, we have

$$\langle \eta_1, z - w_{n,k} \rangle_{H^{1/2}} \le 0,$$

with $z = (w_{n,k+1} - w_{n,k} + w_{n,k-1}) \in A_1 = \{w : \|w - w_{n,k-1}\|_{L^2(K;\mathbb{R}^2)} \le \Delta s_n\}$, since $\|z - w_{n,k-1}\|_{L^2(K;\mathbb{R}^2)} = \|w_{n,k+1} - w_{n,k}\|_{L^2(K;\mathbb{R}^2)} = \Delta s_n$. To see the second inequality in (31) we simply observe that $w_{n,k+1} \in A_2 = \{w : w \cdot \nu \ge 0\}$, so that $\eta_2 \in \partial I_2(w_{n,k})$ implies

$$\langle \eta_2, w_{n,k+1} - w_{n,k} \rangle_{H^{1/2}} \le 0.$$

Similarly, the last inequality in (31) is proved since

$$\langle \eta_2, w_{n,k-1} - w_{n,k} \rangle_{H^{1/2}} \le 0.$$

The thesis is achieved in Case 1.

*Case 2: (16) occurs for $k-1$.* We have $t_{n,k} = t_{n,k-1} + c\Delta s_n$, $w_{n,k} = w_{n,k-1}$ and $\xi_{n,k} = \xi_{n,k-1}$; thus $w'_{n,k} = 0$ and (24) is true, whereas (25) is equivalent to

$$-\widetilde{\mathcal{F}}'(t_{n,k}, w_{n,k}, \xi_{n,k}; w'_{n,k+1}) \le c\, C_0 \|w_{n,k+1} - w_{n,k}\|_{H^{1/2}(K;\mathbb{R}^2)}.$$

We write

$$\begin{aligned} -\widetilde{\mathcal{F}}'(t_{n,k}, w_{n,k}, \xi_{n,k}; \delta w_{n,k+1}) = &- \widetilde{\mathcal{F}}'(t_{n,k}, w_{n,k}, \xi_{n,k}; \delta w_{n,k+1}) + \widetilde{\mathcal{F}}'(t_{n,k-1}, w_{n,k}, \xi_{n,k}; \delta w_{n,k+1}) \\ &- \widetilde{\mathcal{F}}'(t_{n,k-1}, w_{n,k}, \xi_{n,k}; \delta w_{n,k+1}). \end{aligned}$$

By (10) we have

$$\begin{aligned} -\widetilde{\mathcal{F}}'(t_{n,k}, &w_{n,k}, \xi_{n,k}; \delta w_{n,k+1}) + \widetilde{\mathcal{F}}'(t_{n,k-1}, w_{n,k}, \xi_{n,k}; \delta w_{n,k+1}) = \\ &= -d\mathcal{E}(t_{n,k}, w_{n,k})[\delta w_{n,k+1}] + d\mathcal{E}(t_{n,k-1}, w_{n,k})[\delta w_{n,k+1}] \\ &= -\int_\Omega \boldsymbol{\epsilon}(\mathbf{u}(g(t_{n,k}), w_{n,k}) - \mathbf{u}(g(t_{n,k-1}), w_{n,k})) : \boldsymbol{\sigma}(\mathbf{u}(0, w_{n,k+1} - w_{n,k}))dx \\ &= -\int_\Omega \boldsymbol{\epsilon}(\mathbf{u}(g(t_{n,k}) - g(t_{n,k-1}), 0)) : \boldsymbol{\sigma}(\mathbf{u}(0, w_{n,k+1} - w_{n,k}))dx \\ &\le C\|g(t_{n,k}) - g(t_{n,k-1})\|_{H^1} \, \|\mathbf{u}(0, w_{n,k+1} - w_{n,k})\|_{H^1(\Omega;\mathbb{R}^2)} \\ &\le c\, C_0 \Delta s_n \|w_{n,k+1} - w_{n,k}\|_{H^{1/2}(K;\mathbb{R}^2)}. \end{aligned} \tag{32}$$

Remembering that $w_{n,k} = w_{n,k-1}$ and $\xi_{n,k} = \xi_{n,k-1}$ we have

$$\widetilde{\mathcal{F}}'(t_{n,k-1}, w_{n,k}, \xi_{n,k}; \delta w_{n,k+1}) = \widetilde{\mathcal{F}}'(t_{n,k-1}, w_{n,k-1}, \xi_{n,k-1}; \delta w_{n,k+1}).$$

We show that $\mathcal{F}'(t_{n,k-1}, w_{n,k-1}, \xi_{n,k-1}; \delta w_{n,k+1}) \ge 0$, from which the required estimate follows dividing (32) by $\Delta s_n$. By convexity, $w_{n,k-1} + h(\delta w_{n,k+1}) = w_{n,k} + h(w_{n,k+1} - w_{n,k}) \in A_2$ for $h > 0$ sufficiently small (and thus the incompenetrability constraint is satisfied). By assumption $w_{n,k-1}$ is a local minimizer, hence

$$0 \le \mathcal{F}'(t_{n,k-1}, w_{n,k-1}, \xi_{n,k-1}; \delta w_{n,k+1}) = \widetilde{\mathcal{F}}'(t_{n,k-1}, w_{n,k-1}, \xi_{n,k-1}; \delta w_{n,k+1}).$$

The proof is concluded. □

**Proposition 4.7.** *There exist $C_1, C_2, C_3 > 0$ such that for every $n \in \mathbb{N}$ and every index $k \geq 0$*

$$\widetilde{\mathcal{F}}'(t_{n,k}, w_{n,k}, \xi_{n,k}; w'_{n,k+1}) - \widetilde{\mathcal{F}}'(t_{n,k+1}, w_{n,k+1}, \xi_{n,k+1}; w'_{n,k+1}) \leq C_1 \|\hat{\psi}'(\xi_{n,k}) - \hat{\psi}'(\xi_{n,k+1})\|_{L^1(K)} + \\ + c\, C_2 \|w_{n,k+1} - w_{n,k}\|_{H^{1/2}(K;\mathbb{R}^2)} - C_3 \Delta s_n^{-1} \|w_{n,k+1} - w_{n,k}\|^2_{H^{1/2}(K;\mathbb{R}^2)}, \tag{33}$$

*where $c$ is the constant appearing in* (16).

*Proof. First case:* (16) *occurs for $k$.* We have $t_{n,k+1} = t_{n,k} + c\Delta s_n$, $w_{n,k+1} = w_{n,k}$ and $\xi_{n,k+1} = \xi_{n,k}$. Thus $w'_{n,k+1} = 0$ and (33) is trivial for any choice of $C_1, C_2$ and $C_3$.

*Second case:* (17) *occurs for $k$.* We will prove by 1-homogeneity that

$$\widetilde{\mathcal{F}}'(t_{n,k}, w_{n,k}, \xi_{n,k}; \delta w_{n,k+1}) - \widetilde{\mathcal{F}}'(t_{n,k}, w_{n,k+1}, \xi_{n,k+1}; \delta w_{n,k+1}) \leq C_1 \Delta s_n \|\hat{\psi}'(\xi_{n,k}) - \hat{\psi}'(\xi_{n,k+1})\|_{L^1(K)} + \\ + c C_2 \Delta s_n \|w_{n,k+1} - w_{n,k}\|_{H^{1/2}(K;\mathbb{R}^2)} - C_3 \|w_{n,k+1} - w_{n,k}\|^2_{H^{1/2}(K;\mathbb{R}^2)} \tag{34}$$

Let us write explicitly

$$\begin{aligned} \widetilde{\mathcal{F}}'(t_{n,k}, w_{n,k}, \xi_{n,k}; \delta w_{n,k+1}) &- \widetilde{\mathcal{F}}'(t_{n,k}, w_{n,k+1}, \xi_{n,k+1}; \delta w_{n,k+1}) = \\ &= d\mathcal{E}(t_{n,k}, w_{n,k})[\delta w_{n,k+1}] - d\mathcal{E}(t_{n,k}, w_{n,k+1})[\delta w_{n,k+1}] + \\ &+ \widetilde{\mathcal{K}}'(w_{n,k}, \xi_{n,k}; \delta w_{n,k+1}) - \widetilde{\mathcal{K}}'(w_{n,k+1}, \xi_{n,k+1}; \delta w_{n,k+1}). \end{aligned}$$

The derivatives of the elastic energies yield

$$\begin{aligned} d\mathcal{E}(t_{n,k}, w_{n,k})[\delta w_{n,k+1}] &- d\mathcal{E}(t_{n,k}, w_{n,k+1})[\delta w_{n,k+1}] = \\ &= \int_\Omega \boldsymbol{\epsilon}\big(\mathbf{u}(g_{n,k}, w_{n,k}) - \mathbf{u}(g_{n,k}, w_{n,k+1})\big) : \boldsymbol{\sigma}\big(\mathbf{u}(0, w_{n,k+1} - w_{n,k})\big)\, dx \\ &= \int_\Omega \boldsymbol{\epsilon}(\mathbf{u}(0, w_{n,k} - w_{n,k+1})) : \boldsymbol{\sigma}(\mathbf{u}(0, w_{n,k+1} - w_{n,k}))\, dx \\ &\leq -C_3\, \|w_{n,k+1} - w_{n,k}\|^2_{H^{1/2}(K;\mathbb{R}^2)}. \end{aligned} \tag{35}$$

Now, let us consider the directional derivatives for the cohesive energy; by (21) we have

$$\begin{aligned} \widetilde{\mathcal{K}}'(w_{n,k}, \xi_{n,k}; \delta w_{n,k+1}) &- \widetilde{\mathcal{K}}'(w_{n,k+1}, \xi_{n,k+1}; \delta w_{n,k+1}) = \\ &= \int_K \tilde{\varphi}'(w_{n,k}, \xi_{n,k}; \delta w_{n,k+1}) - \tilde{\varphi}'(w_{n,k+1}, \xi_{n,k+1}; \delta w_{n,k+1}\,)\, d\mathcal{H}^1 \\ &\leq \int_K \big(\hat{\psi}'(\xi_{n,k}) - \hat{\psi}'(\xi_{n,k+1})\big)|w_{n,k+1} - w_{n,k}|\, d\mathcal{H}^1 \\ &\leq \|\hat{\psi}'(\xi_{n,k}) - \hat{\psi}'(\xi_{n,k+1})\|_{L^2(K)} \|w_{n,k+1} - w_{n,k}\|_{L^2(K;\mathbb{R}^2)} \\ &\leq \Delta s_n \|\hat{\psi}'(\xi_{n,k}) - \hat{\psi}'(\xi_{n,k+1})\|_{L^2(K)}. \end{aligned}$$

Since $\hat{\psi}'$ is bounded, by interpolation, with $1/2 = \alpha + (1-\alpha)/q$ we get

$$\|\hat{\psi}'(\xi_{n,k}) - \hat{\psi}'(\xi_{n,k+1})\|_{L^2(K)} \leq \|\hat{\psi}'(\xi_{n,k}) - \hat{\psi}'(\xi_{n,k+1})\|^\alpha_{L^1(K)} \|\hat{\psi}'(\xi_{n,k}) - \hat{\psi}'(\xi_{n,k+1})\|^{1-\alpha}_{L^q(K)}$$

By Young's inequality $ab \leq (a^p/p) + (b^{p'}/p')$, with $p = 1/\alpha$ and $p' = 1/(1-\alpha)$, we get

$$\|\hat{\psi}'(\xi_{n,k}) - \hat{\psi}'(\xi_{n,k+1})\|_{L^2(K)} \leq \alpha \|\hat{\psi}'(\xi_{n,k}) - \hat{\psi}'(\xi_{n,k+1})\|_{L^1(K)} + (1-\alpha) \|\hat{\psi}'(\xi_{n,k}) - \hat{\psi}'(\xi_{n,k+1})\|_{L^q(K)}$$

By Lemma 2.1 we known that $\hat{\psi}'$ is Lipschitz continuous and thus choosing $(1-\alpha) < c$ by (4.1) we conclude that

$$\begin{aligned} \|\hat{\psi}'(\xi_{n,k}) - \hat{\psi}'(\xi_{n,k+1})\|_{L^2(K)} &\leq \alpha \|\hat{\psi}'(\xi_{n,k}) - \hat{\psi}'(\xi_{n,k+1})\|_{L^1(K)} + C(1-\alpha) \|w_{n,k} - w_{n,k+1}\|_{L^q(K;\mathbb{R}^2)} \\ &\leq C_1 \|\hat{\psi}'(\xi_{n,k}) - \hat{\psi}'(\xi_{n,k+1})\|_{L^1(K)} + c\, C_2 \|w_{n,k} - w_{n,k+1}\|_{H^{1/2}(K;\mathbb{R}^2)}, \end{aligned}$$

for some positive constants $C_1, C_2$ independent of $n$ and $k$.

The last inequality, together with (35) yields (34) and the proof is concluded. □

Now, we have all the ingredients to prove Theorem 4.2.

**Proof of Theorem 4.2.** Combining Proposition 4.6 and Proposition 4.7 we get

$$\begin{aligned}\widetilde{\mathcal{F}}'(t_{n,k},w_{n,k},\xi_{n,k};w'_{n,k}) &\le \widetilde{\mathcal{F}}'(t_{n,k},w_{n,k},\xi_{n,k};w'_{n,k+1}) + c\,C_0\|w_{n,k+1}-w_{n,k}\|_{H^{1/2}(K;\mathbb{R}^2)}\\ &\le \widetilde{\mathcal{F}}'(t_{n,k+1},w_{n,k+1},\xi_{n,k+1};w'_{n,k+1}) + C_1\|\hat{\psi}'(\xi_{n,k})-\hat{\psi}'(\xi_{n,k+1})\|_{L^1(K)} +\\ &\quad + c\,(C_0+C_2)\|w_{n,k+1}-w_{n,k}\|_{H^{1/2}(K;\mathbb{R}^2)} - C_3\Delta s_n^{-1}\|w_{n,k+1}-w_{n,k}\|^2_{H^{1/2}(K;\mathbb{R}^2)}.\end{aligned}$$

Let us provide an algebraic estimate for the last line in the previous inequality. Assume that $c$ is sufficiently small, in such a way that $C'_2 = c(C_0+C_2) < C_3/2$. Let $C'_3 > 0$ such that $C_3 - C'_3 = 2C'_2$ then, there exists $C_4 > 0$ such that

$$c(C_0+C_2)z - C_3\Delta s_n^{-1}z^2 \le \begin{cases} -C'_2 z - C'_3\Delta s_n^{-1}z^2 + C'_4\Delta s_n & \text{if } 0\le z<\Delta s_n,\\ -C'_2 z - C'_3\Delta s_n^{-1}z^2 & \text{if } z\ge\Delta s_n.\end{cases}$$

If $0\le\|w_{n,k+1}-w_{n,k}\|_{H^{1/2}}<\Delta s_n$ then

$$\begin{aligned}\widetilde{\mathcal{F}}'(t_{n,k},w_{n,k},\xi_{n,k};w'_{n,k}) &\le \widetilde{\mathcal{F}}'(t_{n,k+1},w_{n,k+1},\xi_{n,k+1};w'_{n,k+1}) + C_1\|\hat{\psi}'(\xi_{n,k})-\hat{\psi}'(\xi_{n,k+1})\|_{L^1(K)} +\\ &\quad - C'_2\|w_{n,k+1}-w_{n,k}\|_{H^{1/2}(K;\mathbb{R}^2)} - C'_3\Delta s_n^{-1}\|w_{n,k+1}-w_{n,k}\|^2_{H^{1/2}(K;\mathbb{R}^2)} + C'_4\Delta s_n.\end{aligned}$$

If $\|w_{n,k+1}-w_{n,k}\|_{H^{1/2}}\ge\Delta s_n$ then

$$\begin{aligned}\widetilde{\mathcal{F}}'(t_{n,k},w_{n,k},\xi_{n,k};w'_{n,k}) &\le \widetilde{\mathcal{F}}'(t_{n,k+1},w_{n,k+1},\xi_{n,k+1};w'_{n,k+1}) + C_1\|\hat{\psi}'(\xi_{n,k})-\hat{\psi}'(\xi_{n,k+1})\|_{L^1(K)} +\\ &\quad - C'_2\|w_{n,k+1}-w_{n,k}\|_{H^{1/2}(K;\mathbb{R}^2)} - C'_3\Delta s_n^{-1}\|w_{n,k+1}-w_{n,k}\|^2_{H^{1/2}(K;\mathbb{R}^2)}.\end{aligned}$$

Let us introduce the sets of indices

$$\mathcal{K}_n^{(1)} = \{k : 0\le\|w_{n,k+1}-w_{n,k}\|_{H^{1/2}(K;\mathbb{R}^2)}<\Delta s_n\},\qquad \mathcal{K}_n^{(2)} = \{k : t_{n,k+1}>t_{n,k}\}.$$

By Remark 4.1 we know that $\|w_{n,k+1}-w_{n,k}\|_{L^2(K;\mathbb{R}^2)} + (t_{n,k+1}-t_{n,k}) > 0$. Hence, if $\|w_{n,k+1}-w_{n,k}\|_{H^{1/2}(K;\mathbb{R}^2)} = 0$ then $t_{n,k+1}>t_{n,k}$. If $0<\|w_{n,k+1}-w_{n,k}\|_{H^{1/2}(K;\mathbb{R}^2)}<\Delta s_n$ then $0<\|w_{n,k+1}-w_{n,k}\|_{L^2(K;\mathbb{R}^2)}<\Delta s_n$, thus $w_{n,k+1}$ is a local minimizer and $t_{n,k+2}>t_{n,k+1}$. In other terms, if $k\in\mathcal{K}_n^{(1)}$ then either $k$ or $k+1$ belong to $\mathcal{K}_n^{(2)}$. Hence $\#\mathcal{K}_n^{(1)}\le\#\mathcal{K}_n^{(2)}\le T/c\Delta s_n$.

For any $k^*>0$ fixed, we take the sum of the previous estimates over $k=0,...,k^*$. Since $\widetilde{\mathcal{F}}'(t_0,w_0,\xi_0;w'_{n,0}) = 0$, by assumption, and $\widetilde{\mathcal{F}}'(t_{n,k+1},w_{n,k+1},\xi_{n,k+1};w'_{n,k+1})\le 0$, by (24), we obtain the upper bound

$$\begin{aligned}C'_2\sum_{k=0}^{k^*}\|w_{n,k+1}-w_{n,k}\|_{H^{1/2}(K;\mathbb{R}^2)} + C'_3\Delta s_n^{-1}\sum_{k=0}^{k^*}\|w_{n,k+1}-w_{n,k}\|^2_{H^{1/2}(K;\mathbb{R}^2)} &\le\\ \le C_1\sum_{k=0}^{k^*}\|\hat{\psi}'(\xi_{n,k})-\hat{\psi}'(\xi_{n,k+1})\|_{L^1(K)} + C'_4\,\#\mathcal{K}_n^{(2)}\Delta s_n&\\ \le C_1\|\hat{\psi}'(\xi_{n,k^*+1})-\hat{\psi}'(\xi_0)\|_{L^1(K)} + C'_4 T/c \le C_1\hat{\psi}'(0)\mathcal{H}^1(K) + C'_4T/c,&\end{aligned}$$

where we have used the monotonicity of $\hat{\psi}'(\xi_{n,k})$ to write a single $L^1$-term. Since the right hand side is independent of $k^*$ the thesis is achieved. □

### 4.3 Discrete equilibrium and energy balance

In this section we prove some preliminary results in order to pass to the limit as $\Delta s_n \to 0$.

**Lemma 4.8.** *For all $n, k > 0$, $s \in (s_{n,k}, s_{n,k+1})$ it holds, a.e. in $K$,*

$$\xi_n'(s) \geq 0, \quad |w_n(s)| \leq \xi_n(s), \quad \xi_n'(s)(|w_{n,k+1}| - \xi_{n,k+1}) = 0. \tag{36}$$

*Proof.* This is an immediate consequence of the discrete scheme (16)-(17). □

**Proposition 4.9** (energy estimate)**.** *There exists $C > 0$ such that for every $k, n \in \mathbb{N}$ we have*

$$\begin{aligned} \mathcal{F}(t_{n,k+1}, w_{n,k+1}, \xi_{n,k+1}) \leq \mathcal{F}(t_{n,k}, w_{n,k}, \xi_{n,k}) + \int_{s_{n,k}}^{s_{n,k+1}} \partial_t \mathcal{F}(t_n(r), w_{n,k+1}, \xi_{n,k})\, t_n'(r)\, dr + \\ - \int_{s_{n,k}}^{s_{n,k+1}} |\partial_w \mathcal{F}|(t_{n,k}, w_{n,k+1}, \xi_{n,k})\, dr + C \Delta s_n \int_{s_{n,k}}^{s_{n,k+1}} \|w_n'(r)\|^2_{L^2(K;\mathbb{R}^2)}\, dr. \end{aligned} \tag{37}$$

*Proof.* If $(t_{n,k+1}, w_{n,k+1}, \xi_{n,k+1})$ is given by (16) the statement is easy to check. Assume that $w_{n,k+1}$ is given by (17). We will use the fact that $\mathcal{F}(t, \cdot, \xi)$ is $\lambda$-convex, for some $\lambda < 0$. Note that $w_{n,k+1}$ is the unique minimizer of the auxiliary functional

$$\mathcal{J}(w) = \mathcal{F}(t_{n,k}, w, \xi_{n,k}) - \tfrac{\lambda}{2} \|w - w_{n,k+1}\|^2_{L^2(K;\mathbb{R}^2)}$$

for $\|w - w_{n,k}\|_{L^2(K;\mathbb{R}^2)} \leq \Delta s_n$. We will now consider two possible cases: $0 < \|w_{n,k+1} - w_{n,k}\|_{L^2(K;\mathbb{R}^2)} < \Delta s_n$ and $\|w_{n,k+1} - w_{n,k}\|_{L^2(K;\mathbb{R}^2)} = \Delta s_n$. The case $0 = \|w_{n,k+1} - w_{n,k}\|_{L^2(K;\mathbb{R}^2)}$ is incompatible with (17).

*First case.* By minimality $|\partial_w \mathcal{F}|(t_{n,k}, w_{n,k+1}, \xi_{n,k}) = 0$ and thus we can write

$$\mathcal{J}(w_{n,k+1}) \leq \mathcal{J}(w_{n,k}) - |\partial_w \mathcal{F}|(t_{n,k}, w_{n,k+1}, \xi_{n,k})(s_{n,k+1} - s_{n,k}).$$

The last expression can be written also in the following form

$$\begin{aligned} \mathcal{F}(t_{n,k}, w_{n,k+1}, \xi_{n,k}) \leq \mathcal{F}(t_{n,k}, w_{n,k}, \xi_{n,k}) - |\partial_w \mathcal{F}|(t_{n,k}, w_{n,k+1}, \xi_{n,k})(s_{n,k+1} - s_{n,k}) + \\ - \tfrac{\lambda}{2} \|w_{n,k+1} - w_{n,k}\|^2_{L^2(K;\mathbb{R}^2)}. \end{aligned} \tag{38}$$

Now, using (38) we get (37) integrating in time as follows

$$\begin{aligned} \mathcal{F}(t_{n,k+1}, w_{n,k+1}, \xi_{n,k}) &= \mathcal{F}(t_{n,k}, w_{n,k+1}, \xi_{n,k}) + \int_{t_{n,k}}^{t_{n,k+1}} \partial_t \mathcal{F}(t, w_{n,k+1}, \xi_{n,k})\, dt \\ &= \mathcal{F}(t_{n,k}, w_{n,k+1}, \xi_{n,k}) + \int_{s_{n,k}}^{s_{n,k+1}} \partial_t \mathcal{F}(t_n(r), w_{n,k+1}, \xi_{n,k})\, t_n'(r)\, dr \end{aligned}$$

and then using the fact that $\mathcal{F}(t_{n,k+1}, w_{n,k+1}, \xi_{n,k}) = \mathcal{F}(t_{n,k+1}, w_{n,k+1}, \xi_{n,k+1})$, by Lemma 4.4.

*Second case.* Since the minimum $w_{n,k+1}$ is on the boundary of the ball $\|w - w_{n,k}\|_{L^2(K;\mathbb{R}^2)} \leq \Delta s_n$ we write $w_{n,k+1} \in \operatorname{argmin} \mathcal{J}(w) + I(w)$ where $I$ denotes the indicator function of the set $\|w - w_{n,k}\|_{L^2(K;\mathbb{R}^2)} \geq \Delta s_n$. As a consequence, there exists $\mu \geq 0$ such that $\mu(w_{n,k} - w_{n,k+1}) \in \partial \mathcal{J}(t_{n,k}, w_{n,k+1}, \xi_{n,k})$ where the subdifferential is in $L^2(K;\mathbb{R}^2)$. By Lemma A.5 we can write

$$\begin{aligned} \mathcal{J}(w_{n,k}) &\geq \mathcal{J}(w_{n,k+1}) + \langle \mu(w_{n,k} - w_{n,k+1}), w_{n,k} - w_{n,k+1} \rangle \\ &= \mathcal{J}(w_{n,k+1}) + \|\mu(w_{n,k} - w_{n,k+1})\|_{L^2(K;\mathbb{R}^2)} \|w_{n,k} - w_{n,k-1}\|_{L^2(K;\mathbb{R}^2)} \\ &\geq \mathcal{J}(w_{n,k+1}) + |\partial_w \mathcal{J}|(w_{n,k+1}) \|w_{n,k} - w_{n,k-1}\|_{L^2(K;\mathbb{R}^2)}. \end{aligned}$$

Therefore, since $\|w_{n,k+1} - w_{n,k}\|_{L^2(K;\mathbb{R}^2)} = s_{n,k+1} - s_{n,k}$, we get (38) and we conclude as in the first case. □

### 4.4 Limit evolution

Since by Theorem 4.2 the length of the parametrization interval is uniformly finite with respect to $\Delta s_n$, it is not restrictive, extending by a constant evolution, to consider all the discrete evolutions to be defined on a single parametrization interval, say $[0, S]$.

**Lemma 4.10** (compactness). *Let $(t_n, w_n, \xi_n) : [0, S] \to [0, T] \times L^2(K; \mathbb{R}^2) \times L^2(K)$ be a sequence of discrete evolutions. Up to subsequences (not relabelled)*

$$t_n \overset{*}{\rightharpoonup} t \text{ in } W^{1,\infty}(0, S), \qquad \xi_n \overset{*}{\rightharpoonup} \xi \text{ in } W^{1,\infty}(0, S; L^2(K)) \tag{39}$$

$$w_n \overset{*}{\rightharpoonup} w \text{ in } W^{1,\infty}(0, S; L^2(K, \mathbb{R}^2)), \qquad w_n \rightharpoonup w \text{ in } W^{1,2}(0, S; H^{1/2}(K; \mathbb{R}^2)). \tag{40}$$

*There exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ and $s \in [0, S]$ it holds*

$$\mathcal{F}(t_n(s), w_n(s), \xi_n(s)) \leq C. \tag{41}$$

*Finally, if $s_n \to s$, then*

$$t_n(s_n) \to t(s), \qquad w_n(s_n) \rightharpoonup w(s) \text{ in } H^{1/2}(K, \mathbb{R}^2), \qquad \xi_n(s_n) \rightharpoonup \xi(s) \text{ in } L^2(K). \tag{42}$$

*Proof.* By Remark 4.1 we know that $t_n$ is uniformly bounded in $W^{1,\infty}(0, S)$, $w_n$ is uniformly bounded in $W^{1,\infty}(0, S; L^2(K, \mathbb{R}^2))$, and $\xi_n$ is uniformly bounded in $W^{1,\infty}(0, S; L^2(K))$. By §4.2 $w_n$ is uniformly bounded in $W^{1,2}(0, S; H^{1/2}(K, \mathbb{R}^2))$. Weak and weak* convergence follows.

If (17) occurs, by Lemma 4.4 we have

$$\mathcal{F}(t_{n,k+1}, w_{n,k+1}, \xi_{n,k+1}) = \mathcal{F}(t_{n,k}, w_{n,k+1}, \xi_{n,k}) \leq \mathcal{F}(t_{n,k}, w_{n,k}, \xi_{n,k}). \tag{43}$$

If (16) occurs, we can write

$$\begin{aligned} &|\mathcal{F}^{1/2}(t_{n,k+1}, w_{n,k}, \xi_{n,k}) - \mathcal{F}^{1/2}(t_{n,k}, w_{n,k}, \xi_{n,k})| \\ &\qquad = |(\mathcal{E}(t_{n,k+1}, w_{n,k}) + \mathcal{K}(w_{n,k}, \xi_{n,k}))^{1/2} - (\mathcal{E}(t_{n,k}, w_{n,k}) + \mathcal{K}(w_{n,k}, \xi_{n,k}))^{1/2}| \\ &\qquad \leq |\mathcal{E}^{1/2}(t_{n,k+1}, w_{n,k}) - \mathcal{E}^{1/2}(t_{n,k}, w_{n,k})| \leq C(t_{n,k+1} - t_{n,k}) \end{aligned} \tag{44}$$

where, in the last estimate, we have used the fact that $t \mapsto \mathcal{E}^{1/2}(t, w)$ is Lipschitz continuous with constant $C > 0$ independent of $w$: indeed, $\mathcal{E}^{1/2}(t, w) = E^{1/2}(\boldsymbol{u}(g(t), w))$ where $E^{1/2}$ is Lipschitz continuous and $t \mapsto \boldsymbol{u}(t, w)$ is Lipschitz continuous as well since

$$\boldsymbol{u}(g(t + h), w) - \boldsymbol{u}(g(t), w) = \boldsymbol{u}(g(t + h) - g(t), 0) = g(t + h) - g(t).$$

From (43) and (44) we get for every (admissible) index $k \geq 0$

$$\mathcal{F}^{1/2}(t_{n,k+1}, w_{n,k+1}, \xi_{n,k+1}) \leq \mathcal{F}^{1/2}(t_{n,k}, w_{n,k}, \xi_{n,k}) + C(t_{n,k+1} - t_{n,k})$$

and thus

$$\mathcal{F}^{1/2}(t_{n,k+1}, w_{n,k+1}, \xi_{n,k+1}) \leq \mathcal{F}^{1/2}(t_0, w_0, \xi_0) + CT < +\infty.$$

By Korn's inequality it follows that $\boldsymbol{u}(g(t_{n,k}), w_{n,k})$ is bounded in $H^1(\Omega; \mathbb{R}^2)$ and thus $w_{n,k}$ is bounded in $H^{1/2}(K; \mathbb{R}^2)$. As a consequence the affine interpolant $w_n(s)$ is uniformly bounded as well. This implies (41).

If $s_n \to s$ we know that $w_n(s_n) \rightharpoonup w(s)$ in $L^2(K; \mathbb{R}^2)$. Since $w_n(s)$ is bounded in $H^{1/2}(K; \mathbb{R}^2)$ it follows that the convergence is weak in $H^{1/2}(K; \mathbb{R}^2)$. The convergence of $t_n(s_n)$ and $\xi_n(s_n)$ are straightforward. □

With the following Lemma we prove the Karush-Kuhn-Tucker conditions for the limit evolution; as a by-product we get also the strong convergence for the internal variable. The proof follows [19][Theorem 5.8].

**Lemma 4.11.** *Let $w$ and $\xi$ be the limit provided by Lemma 4.10. For a.e. $s \in (0, S)$ it holds*

$$\xi'(s) \geq 0, \qquad |w(s)| \leq \xi(s) \qquad \xi'(s)(|w|(s) - \xi(s)) = 0 \text{ a.e. on } K. \tag{45}$$

*Moreover, for every $s \in [0, S]$*

$$\xi_n(s) \to \xi(s) \text{ in } L^2(K). \tag{46}$$

*Proof.* Let us prove that conditions (36) pass to the limit. First we observe that, since $\xi_n(s) \rightharpoonup \xi(s)$ in $L^2(K)$ and $\xi_n(s_1) \leq \xi_n(s_2)$ if $s_1 < s_2$, we have $\xi(s_1) \leq \xi(s_2)$ whenever $s_1 < s_2$. Hence $\xi'(s) \geq 0$ for a.e. $s \in (0, S)$. Moreover, from $|w_n(s)| \to |w(s)|$ in $L^2(K)$ and $|w_n(s)| \leq \xi_n(s)$ we also infer $|w(s)| \leq \xi(s)$ for all $s \in [0, S]$. In particular we find $\xi'(s)(|w|(s) - \xi(s)) \leq 0$ for a.e. $s \in (0, S)$. To prove that equality holds it suffices to show that

$$\int_0^S \langle \xi'(s), |w(s)| - \xi(s) \rangle_{L^2(K)} ds \geq 0.$$

Now, let us introduce the piecewise constant functions $\bar{w}_n$ and $\bar{\xi}_n$ given by $\bar{w}_n(s) = w_{n,k+1}$ and $\bar{\xi}_n(s) = \xi_{n,k+1}$ for $s \in (s_{n,k}, s_{n,k+1}]$. Let us see that $\bar{w}_n \to w$ strongly in $L^1([0, S]; L^2(K; \mathbb{R}^2))$. By (42) we know that $w_n(s) \to w(s)$ strongly in $L^2(K; \mathbb{R}^2)$ while by (40) we know that $w_n(s)$ and $w(s)$ are bounded in $L^2(K; \mathbb{R}^2)$ uniformly with respect to $s \in [0, S]$ and $n \in \mathbb{N}$. Hence $w_n \to w$ strongly in $L^1([0, S]; L^2(K; \mathbb{R}^2))$. Further, for every $s \in (s_{n,k}, s_{n,k+1}]$

$$\|\bar{w}_n(s) - w_n(s)\|_{L^2(K;\mathbb{R}^2)} \leq \int_s^{s_{n,k+1}} \|w_n'(r)\|_{L^2(K;\mathbb{R}^2)} \, dr \leq \Delta s_n.$$

Hence $\bar{w}_n \to w$ strongly in $L^1([0, S]; L^2(K; \mathbb{R}^2))$.

Since $s \mapsto \|\xi_n(s)\|_{L^2(K)}$ is Lipschitz continuous (and thus bounded) it follows that $s \mapsto \|\xi(s)\|^2_{L^2(K)}$ is Lipschitz continuous and then for a.e. $s \in (0, S)$ we can write

$$(\|\xi(s)\|^2_{L^2(K)})' = 2\langle \xi(s), \xi'(s) \rangle_{L^2(K)}.$$

Being $\xi_n'(s)(|w_{n,k+1}| - \xi_{n,k+1}) = \xi_n'(s)(|\bar{w}_n(s)| - \bar{\xi}_n(s)) = 0$ and $\bar{\xi}_n(s) = \xi_{n,k+1} \geq \xi_n(s)$ for $s \in (s_{n,k}, s_{n,k+1}]$ we get

$$\begin{aligned} \int_0^S \langle \xi_n'(s), |\bar{w}_n(s)| \rangle_{L^2(K)} \, ds &= \int_0^S \langle \xi_n'(s), \bar{\xi}_n(s) \rangle_{L^2(K)} \, ds \geq \int_0^S \langle \xi_n'(s), \xi_n(s) \rangle_{L^2(K)} \, ds \\ &= \tfrac{1}{2}\|\xi_n(S)\|^2_{L^2(K)} - \tfrac{1}{2}\|\xi(0)\|^2_{L^2(K)}. \end{aligned}$$

Since, by Lemma 4.10, $\xi_n \overset{*}{\rightharpoonup} \xi$ in $L^\infty([0, S]; L^2(K))$ and $|\bar{w}_n| \to |w|$ strongly in $L^1([0, S]; L^2(K; \mathbb{R}^2))$ we can pass to the limit in the first term. As $\xi_n(S) \rightharpoonup \xi(S)$ in $L^2(K)$ we get

$$\int_0^S \langle \xi'(s), |w(s)| \rangle_{L^2(K)} \, ds \geq \tfrac{1}{2}\|\xi(S)\|^2_{L^2(K)} - \tfrac{1}{2}\|\xi(0)\|^2_{L^2(K)} = \int_0^S \langle \xi'(s), \xi(s) \rangle_{L^2(K)} \, ds,$$

and the claim is achieved. It remains to prove (46). Using (45) and $\xi_n'(s)(|\bar{w}_n(s)| - \bar{\xi}_n(s)) = 0$ we can write

$$\begin{aligned} \tfrac{1}{2}\|\xi(s)\|^2_{L^2(K)} &= \tfrac{1}{2}\|\xi(0)\|^2_{L^2(K)} + \int_0^s \langle \xi'(r), |w(r)| \rangle_{L^2(K)} \, dr \\ &= \tfrac{1}{2}\|\xi(0)\|^2_{L^2(K)} + \lim_{n\to\infty} \int_0^s \langle \xi_n'(r), |\bar{w}_n(r)| \rangle_{L^2(K)} \, dr \\ &= \tfrac{1}{2}\|\xi(0)\|^2 + \lim_{n\to\infty} \int_0^s \langle \xi_n'(r), \bar{\xi}_n(r) \rangle_{L^2(K)} \, dr \\ &\geq \tfrac{1}{2}\|\xi(0)\|^2 + \lim_{n\to\infty} \int_0^s \langle \xi_n'(r), \xi_n(r) \rangle_{L^2(K)} \, dr \geq \limsup_{n\to\infty} \tfrac{1}{2}\|\xi_n(s)\|^2_{L^2(K)}. \end{aligned}$$

As $\xi_n(s) \rightharpoonup \xi(s)$ in $L^2(K)$ we get $\|\xi(s)\|^2_{L^2(K)} \leq \liminf_{n\to+\infty} \|\xi_n(s)\|^2_{L^2(K)}$. Hence $\|\xi_n(s)\|_{L^2(K)} \to \|\xi(s)\|_{L^2(K)}$ and the strong convergence follows. □

**Theorem 4.12.** *Let $(t, w, \xi)$ be a limit of the discrete evolutions $(t_n, w_n, \xi_n)$ provided by Lemma 4.10. Then $(t, w, \xi)$ is a parametrized BV-evolution for $\mathcal{F}$ in the sense of Definition 2.3, i.e.*

*(i)* $(t, w, \xi)$ *is Lipschitz continuous with* $t' \geq 0$, $t(S) = T$ *and* $\|w'\|_{L^2(K;\mathbb{R}^2)} \leq 1$,

*(ii)* $\xi'(s) \geq 0$, $|w(s)| \leq \xi(s)$ *and* $\xi'(s)(|w(s)| - \xi(s)) = 0$ *for a.e.* $s \in [0, S]$,

*(iii)* $|\partial_w \mathcal{F}|(t(s), w(s), \xi(s)) = 0$ *for every* $s \in [0, S]$ *with* $t'(s) > 0$,

*(iv) for every* $s \in [0, S]$ *the following energy balance holds*

$$\mathcal{F}(t(s), w(s), \xi(s)) = \mathcal{F}(0, w_0, \xi_0) - \int_0^s |\partial_w \mathcal{F}|(t(r), w(r), \xi(r))\, dr + \\ + \int_0^s \partial_t \mathcal{F}(t(r), w(r), \xi(r))\, t'(r)\, dr.$$

*Proof.* By Lemma 4.10 and Lemma 4.11 it is clear that $(t, w, \xi)$ satisfies condition *(i)* and *(ii)*.

Let us check *(iii)*. If $t'(s) > 0$ then there exists a sequence $s_{n,k_n} \to s$ such that $t_{n,k_n+1} > t_{n,k_n}$ (cf. [18, Theorem 4.4]). By (16) we have $|\partial_w \mathcal{F}|(t_{n,k_n}, w_{n,k_n+1}, \xi_{n,k_n+1}) = 0$. By Lemma 4.10 and Lemma 4.11 we get $t_{n,k_n} = t_n(s_{n,k_n}) \to t(s)$, $w_{n,k_n+1} = w_n(s_{n,k_n+1}) \rightharpoonup w(s)$ in $H^{1/2}(K;\mathbb{R}^2)$ and $\xi_{n,k_n+1} = \xi_n(s_{n,k_n+1}) \to \xi(s)$ strongly in $L^2(K)$. We conclude by the lower semi-continuity of the slope proved in Lemma 3.5.

Let us prove *(iv)*. For convenience we will employ the auxiliary piecewise constant functions

$$\bar{w}_n(s) = w_{n,k+1} \quad \text{for } s \in (s_{n,k}, s_{n,k+1}], \\ \tilde{\xi}_n(s) = \xi_{n,k} \text{ and } \tilde{t}_n(s) = t_{n,k} \text{ for } s \in [s_{n,k}, s_{n,k+1}).$$

Arguing as in Lemma 4.11 we have $\tilde{t}_n(s) \to t(s)$, $\bar{w}_n(s) \rightharpoonup w(s)$ in $H^{1/2}(K;\mathbb{R}^2)$ and $\tilde{\xi}_n(s) \to \xi(s)$ strongly in $L^2(K)$ for a.e. $s \in [0, S]$.

Let $s \in [0, S)$ be arbitrary. For all $n > 0$ let $k_n \in \mathbb{N}$ such that $s_{n,k_n} \leq s < s_{n,k_n+1}$. Clearly $s_{n,k_n} \to s$. With the above notation, the sum of (37) for $k = 0, \ldots, k_n - 1$ reads

$$\mathcal{F}(t_{n,k_n}, w_{n,k_n}, \xi_{n,k_n}) \leq \mathcal{F}(0, w_0, \xi_0) + \int_0^{s_{n,k_n}} \partial_t \mathcal{F}(t_n(r), \bar{w}_n(r), \tilde{\xi}_n(r))\, t_n'(r)\, dr + \\ - \int_0^{s_{n,k_n}} |\partial_w \mathcal{F}|(\tilde{t}_n(r), \bar{w}_n(r), \tilde{\xi}_n(r))\, dr + C\Delta s_n \int_0^{s_{n,k_n}} \|w_n'(r)\|^2_{L^2(K;\mathbb{R}^2)}\, dr. \tag{47}$$

We aim to take the liminf as $n \to \infty$. Let us first observe that by Lemma 3.3

$$\mathcal{F}(t(s), w(s), \xi(s)) \leq \liminf_{n\to+\infty} \mathcal{F}(t_n(s_{n,k_n}), w_n(s_{n,k_n}), \xi_n(s_{n,k_n})) = \liminf_{n\to+\infty} \mathcal{F}(t_{n,k_n}, w_{n,k_n}, \xi_{n,k_n}).$$

Now we claim that

$$\lim_{n\to\infty} \int_0^{s_{n,k_n}} \partial_t \mathcal{F}(t_n(r), \bar{w}_n(r), \tilde{\xi}_n(r))\, t_n'(r)\, dr = \int_0^s \partial_t \mathcal{F}(t(r), w(r), \xi(r))\, t'(r)\, dr. \tag{48}$$

Indeed, remembering that $\partial_t \mathcal{F} = \partial_t \mathcal{E}$, for fixed $r \in (0, s)$ by (11) we have

$$\partial_t \mathcal{F}(t_n(r), \bar{w}_n(r), \tilde{\xi}_n(r)) \to \partial_t \mathcal{F}(t(r), w(r), \xi(r)).$$

By (12) and dominated convergence we get

$$\partial_t \mathcal{F}(t_n, \bar{w}_n, \tilde{\xi}_n) \to \partial_t \mathcal{F}(t, w, \xi) \text{ strongly in } L^1(0, s).$$

Then (48) follows by weak* convergence of $t'_n$. Next, the lower semi-continuity of the slope, see Lemma 3.5, together with Fatou's Lemma imply

$$
\begin{aligned}
-\liminf_{n\to\infty}\int_0^{s_{n,k_n}} |\partial_w\mathcal{F}|(\tilde t_n(r),\bar w_n(r),\tilde\xi_n(r))\,dr &\le -\int_0^s \liminf_{n\to\infty}|\partial_w\mathcal{F}|(\tilde t_n(r),\bar w_n(r),\tilde\xi_n(r))\,dr\\
&\le -\int_0^s |\partial_w\mathcal{F}|(t(r),w(r),\xi(r))\;dr.
\end{aligned}
$$

Finally, by the uniform Lipschitz continuity of $w_n$ we have

$$
\lim_{n\to\infty} C\Delta s_n\int_0^{s_{n,k_n}}\|w'_n(r)\|^2_{L^2(K;\mathbb{R}^2)}\,dr\le\lim_{n\to\infty}C'\Delta s_n=0.
$$

In conclusion, we can take the liminf in (47), obtaining

$$
\begin{aligned}
\mathcal{F}(t(s),w(s),\xi(s))\le\mathcal{F}(0,w_0,\xi_0)+&\int_0^s\partial_t\mathcal{F}(t(r),w(r),\xi(r))\,t'(r)\,dr+\\
&-\int_0^s|\partial_w\mathcal{F}|(t(r),w(r),\xi(r))\,dr.
\end{aligned}
$$

Now, we will show that for every $s\in[0,S)$ the opposite inequality holds, i.e.

$$
\begin{aligned}
\int_0^s\partial_t\mathcal{F}(t(r),w(r),\xi(r))\,t'(r)\,dr-&\int_0^s|\partial_w\mathcal{F}|(t(r),w(r),\xi(r))\,dr\le\\
&\le\mathcal{F}(t(s),w(s),\xi(s))-\mathcal{F}(0,w_0,\xi_0).
\end{aligned}
$$

We use an argument similar to the one adopted in [8] and [18]. Being $(t,w,\xi):[0,s]\to[0,T]\times H^{1/2}(K;\mathbb{R}^2)\times L^2(K)$ of class $W^{1,\infty}$, for every index $i>0$ we can choose a subdivision $0=s_{i,0}<\ldots<s_{i,j}<\cdots<s_{i,i}=s$ with $\Delta_i:=\max_j|s_{i,j+1}-s_{i,j}|\to0$ such that the Riemann sums

$$
Z_i=\sum_{j=0}^{i-1}(s_{i,j+1}-s_{i,j})|\partial_w\mathcal{F}|(t(s_{i,j}),w(s_{i,j}),\xi(s_{i,j}))\to\int_0^s|\partial_w\mathcal{F}|(t(s),w(s),\xi(s))\,ds. \tag{49}
$$

Note that the discrete points $s_{i,j}$ do not coincide, in general, with the points $s_{n,k}=k\Delta s_n$ defined in §4.1. It is convenient to introduce the notation $t_{i,j}:=t(s_{i,j})$, $w_{i,j}:=w(s_{i,j})$, and $\xi_{i,j}:=\xi(s_{i,j})$. Let us write, for all indices,

$$
\begin{aligned}
\mathcal{F}(t_{i,j},w_{i,j},\xi_{i,j})-\mathcal{F}(t_{i,j+1},w_{i,j+1},\xi_{i,j+1})=\;&\mathcal{F}(t_{i,j},w_{i,j},\xi_{i,j})-\mathcal{F}(t_{i,j},w_{i,j+1},\xi_{i,j})+\\
&+\mathcal{F}(t_{i,j},w_{i,j+1},\xi_{i,j})-\mathcal{F}(t_{i,j+1},w_{i,j+1},\xi_{i,j})+\\
&+\mathcal{F}(t_{i,j+1},w_{i,j+1},\xi_{i,j})-\mathcal{F}(t_{i,j+1},w_{i,j+1},\xi_{i,j+1}).
\end{aligned}\tag{50}
$$

We will analyse the three rows of the right-hand side separately. Recalling that $\mathcal{J}_{\bar w}(t,w,\xi)=\mathcal{F}(t,w,\xi)-\frac{\lambda}{2}\|w-\bar w\|^2_{L^2(K;\mathbb{R}^2)}$, the first row can be written as

$$
\begin{aligned}
&\mathcal{F}(t_{i,j},w_{i,j},\xi_{i,j})-\mathcal{F}(t_{i,j},w_{i,j+1},\xi_{i,j})=\\
&\qquad\mathcal{J}_{w_{i,j}}(t_{i,j},w_{i,j},\xi_{i,j})-\mathcal{J}_{w_{i,j}}(t_{i,j},w_{i,j+1},\xi_{i,j})-\tfrac{\lambda}{2}\|w_{i,j+1}-w_{i,j}\|^2_{L^2(K;\mathbb{R}^2)}.
\end{aligned}
$$

Now, by convexity of $\mathcal{J}_{w_{i,j}}$ for all $\eta\in\partial\mathcal{J}_{w_{i,j}}(t_{i,j},w_{i,j},\xi_{i,j})$ we have

$$
\begin{aligned}
\mathcal{J}_{w_{i,j}}(t_{i,j},w_{i,j},\xi_{i,j})-\mathcal{J}_{w_{i,j}}(t_{i,j},w_{i,j+1},\xi_{i,j})&\le-\langle\eta,w_{i,j+1}-w_{i,j}\rangle\\
&\le\|\eta\|_{L^2(K;\mathbb{R}^2)}\|w_{i,j+1}-w_{i,j}\|_{L^2(K;\mathbb{R}^2)}.
\end{aligned}
$$

Then, taking the infimum with respect to $\eta$, owing to Lemma A.5, and remembering that $\|w'\|_{L^2(K;\mathbb{R}^2)} \leq 1$, we find that

$$\begin{aligned}\mathcal{F}(t_{i,j}, w_{i,j}, \xi_{i,j}) - \mathcal{F}(t_{i,j}, w_{i,j+1}, \xi_{i,j}) \leq \\ &\leq |\partial_w \mathcal{J}_{w_{i,j}}|(t_{i,j}, w_{i,j}, \xi_{i,j}) \|w_{i,j+1} - w_{i,j}\|_{L^2(K;\mathbb{R}^2)} - \tfrac{\lambda}{2} \|w_{i,j+1} - w_{i,j}\|^2_{L^2(K;\mathbb{R}^2)} \\ &= |\partial_w \mathcal{F}|(t_{i,j}, w_{i,j}, \xi_{i,j}) \|w_{i,j+1} - w_{i,j}\|_{L^2(K;\mathbb{R}^2)} - \tfrac{\lambda}{2} \|w_{i,j+1} - w_{i,j}\|^2_{L^2(K;\mathbb{R}^2)} \\ &\leq |\partial_w \mathcal{F}|(t_{i,j}, w_{i,j}, \xi_{i,j})(s_{i,j+1} - s_{i,j}) - \tfrac{\lambda}{2} \Delta_i (s_{i,j+1} - s_{i,j}) \end{aligned} \tag{51}$$

(remember that $\Delta_i = \max_j |s_{i,j+1} - s_{i,j}|$ and that $\lambda < 0$). As for the term $\mathcal{F}(t_{i,j}, w_{i,j+1}, \xi_{i,j}) - \mathcal{F}(t_{i,j+1}, w_{i,j+1}, \xi_{i,j})$ we see that it is equal to

$$\begin{aligned}\mathcal{E}(t_{i,j}, w_{i,j+1}) - \mathcal{E}(t_{i,j+1}, w_{i,j+1}) &= \int_{s_{i,j}}^{s_{i,j+1}} \partial_t \mathcal{E}(t(r), w_{i,j+1})\, t'(r)\, dr \\ &= \int_{s_{i,j}}^{s_{i,j+1}} \partial_t \mathcal{E}(t(r), \bar{w}_i(r))\, t'(r)\, dr, \end{aligned} \tag{52}$$

where we have used the piecewise constant function $\bar{w}_i(r) := w(s_{i,j})$ for $s \in (s_{i,j}, s_{i,j+1}]$. Finally, the term $\mathcal{F}(t_{i,j+1}, w_{i,j+1}, \xi_{i,j}) - \mathcal{F}(t_{i,j+1}, w_{i,j+1}, \xi_{i,j+1})$ is equal to

$$\mathcal{K}(w_{i,j+1}, \xi_{i,j}) - \mathcal{K}(w_{i,j+1}, \xi_{i,j+1}) = \int_K \varphi(w_{i,j+1}, \xi_{i,j}, \nu_K) - \varphi(w_{i,j+1}, \xi_{i,j+1}, \nu_K)\, d\mathcal{H}^1,$$

which is non-positive since by Proposition 2.5 (*ii*) the energy density $\varphi(w_{i,j+1}, \cdot, \nu_K)$ is monotone non-decreasing and $\xi_{i,j} \leq \xi_{i,j+1}$. Thus, considering (51) and (52) and taking the sum in (50) for $j = 0, \ldots, i-1$, we get

$$\begin{aligned}\mathcal{F}(0, w_0, \xi_0) - \mathcal{F}(t(s), w(s), \xi(s)) \leq \\ &\leq Z_i - \tfrac{\lambda}{2} \sum_{j=0}^{i-1} (s_{i,j+1} - s_{i,j})^2 + \int_0^s \partial_t \mathcal{E}(t(r), w_i(r))\, t'(r)\, dr. \end{aligned} \tag{53}$$

where $Z_i$ is the Riemann sum in (49). We aim to pass to the limit as $i \to +\infty$. By dominated convergence, from Lemma 3.4, we get $\partial_t \mathcal{E}(t(\cdot), \bar{w}_i(\cdot)) \to \partial_t \mathcal{E}(t(\cdot), w(\cdot))$ strongly in $L^1(0, s)$ and thus

$$\lim_{i \to \infty} \int_0^s \partial_t \mathcal{E}(t(r), \bar{w}_i(r))\, t'(r)\, dr = \int_0^s \partial_t \mathcal{E}(t(r), w(r))\, t'(r)\, dr = \int_0^s \partial_t \mathcal{F}(t(r), w(r), \xi(r))\, t'(r)\, dr.$$

Recalling that $\lambda < 0$, we write

$$-\tfrac{\lambda}{2} \sum_{j=0}^{n-1} \Delta_i (s_{i,j+1} - s_{i,j}) \leq -\tfrac{\lambda}{2} s \Delta_i \to 0.$$

Passing to the limit in (53) we obtain

$$\mathcal{F}(0, w_0, \xi_0) - \mathcal{F}(t(s), w(s), \xi(s)) \leq \int_0^s |\partial_w \mathcal{F}|(t(r), w(r), \xi(r)\, dr + \int_0^s \partial_t \mathcal{F}(t(r), w(r))\, t'(r)\, dr,$$

which concludes the proof. □

# 5 Equilibrium conditions and evolution in PDE form

This section collects the Euler-Lagrange conditions, in the bulk and on the interface, and a characterization of the evolution in discontinuity points. The main results are summarized in Propositions 5.1, 5.2 and 5.3. The proofs, reported in short format, rely on [5, 19] and on duality arguments; the regularity of solutions is not considered. We will assume, for simplicity, that the crack $K$ is of class $W^{2,+\infty}$ and we will employ, for convenience, the notation $u(s) = \boldsymbol{u}(g \circ t(s), w(s))$, $\boldsymbol{\sigma}(s) = \boldsymbol{\sigma}(u(s))$ etc.

## 5.1 General Euler-Lagrange conditions

As $[\![u]\!](s) = w(s)$ by (3)

$$u(s) = \text{argmin}\,\{E(u) : u \in H^1(\Omega;\mathbb{R}^2),\ u = g \circ t(s) \text{ on } \partial_D\Omega \text{ and } [\![u]\!] = w(s) \text{ on } K\}. \tag{54}$$

Hence $dE(u(s))[z] = \int_\Omega \boldsymbol{\sigma}(s) : \boldsymbol{\epsilon}(z)\,dx = 0$ for every $z \in H^1_0(\Omega;\mathbb{R}^2)$; in other terms

$$\operatorname{div} \boldsymbol{\sigma}(s) = 0 \qquad \text{in } H^{-1}(\Omega;\mathbb{R}^2). \tag{55}$$

Introduce the space $\tilde H^{1/2}(\partial_N\Omega^+;\mathbb{R}^2)$ as the subspace of $H^{1/2}(\partial_N\Omega^+;\mathbb{R}^2)$ given by the functions $u$ such that $\tilde u \in H^{1/2}(\partial\Omega^+;\mathbb{R}^2)$, where $\tilde u$ is

$$\tilde u(x) := \begin{cases} u(x) & \text{if } x \in \partial_N\Omega, \\ 0 & \text{if } x \in \partial_D\Omega \cup K. \end{cases}$$

(A comprehensive reference for these spaces is provided in [13]). Denote by $\tilde H^{-1/2}(\partial_N\Omega^+;\mathbb{R}^2)$ the dual of $\tilde H^{1/2}(\partial_N\Omega^+;\mathbb{R}^2)$ endowed with the standard $H^{1/2}$-norm. Similarly are defined the spaces $\tilde H^{1/2}(\partial_N\Omega^-;\mathbb{R}^2)$, $\tilde H^{1/2}(K;\mathbb{R}^2)$ and their duals. Denoting by $\nu$ the exterior unit normal on $\partial_N\Omega^+$, we define, using (55), the Neumann operator $\boldsymbol{\sigma}_\nu(s) \in \tilde H^{-1/2}(\partial_N\Omega^+;\mathbb{R}^2)$ as

$$\langle \boldsymbol{\sigma}_\nu(s), z\rangle := \int_{\Omega^+} \boldsymbol{\sigma}(s) : \boldsymbol{\epsilon}(\bar z)\,dx,$$

where brackets denote the duality pairing between $\tilde H^{1/2}(\partial_N\Omega^+;\mathbb{R}^2)$ and $\tilde H^{-1/2}(\partial_N\Omega^+;\mathbb{R}^2)$ while $\bar z \in H^1(\Omega^+;\mathbb{R}^2)$ is a lifting of $\tilde z$, with $\tilde z = 0$ on $K$ and $\partial_D\Omega^+$. Setting $\bar z = 0$ in $\Omega^-$ we have $\bar z = 0$ on $\partial_D\Omega$ and on $K$, hence $[\![\bar z]\!] = 0$ and by (3) we have $dE(u(s))[\bar z] = \int_{\Omega^+} \boldsymbol{\sigma}(s) : \boldsymbol{\epsilon}(\bar z)\,dx = 0$; in other terms

$$\boldsymbol{\sigma}_\nu(s) = 0 \qquad \text{in } \tilde H^{-1/2}(\partial_N\Omega^+;\mathbb{R}^2).$$

Arguing in the same way for $\Omega^-$ we obtain $\boldsymbol{\sigma}_\nu(s) = 0$ in $\tilde H^{-1/2}(\partial_N\Omega^-;\mathbb{R}^2)$.

Similarly we can define the one-side Neumann operators $\boldsymbol{\sigma}^\pm_\nu(s) \in \tilde H^{-1/2}(K;\mathbb{R}^2)$ as

$$\langle \boldsymbol{\sigma}^-_\nu(s), z\rangle := \int_{\Omega^-} \boldsymbol{\sigma}(s) : \boldsymbol{\epsilon}(\bar z)\,dx, \qquad \langle \boldsymbol{\sigma}^+_\nu(s), z\rangle := -\int_{\Omega^+} \boldsymbol{\sigma}(s) : \boldsymbol{\epsilon}(\bar z)\,dx,$$

where $\bar z$ is an again a lifting of $\tilde z$. The minus sign appearing in the second is consistent with the fact that, in our setting, $\nu$ is the outer normal for $\Omega^-$. Since $[\![\bar z]\!] = 0$ the functions $u(s) + h\bar z$ for $h \in \mathbb{R}$ are admissible competitors in (54) and then, by minimality, $dE(u(s))[\bar z] = \int_\Omega \boldsymbol{\sigma}(s) : \boldsymbol{\epsilon}(\bar z)\,dx = 0$. Splitting the integrals in $\Omega^-$ and $\Omega^+$ it follows that

$$\boldsymbol{\sigma}^-_\nu(s) = \boldsymbol{\sigma}^+_\nu(s) \qquad \text{in } \tilde H^{-1/2}(K;\mathbb{R}^2). \tag{56}$$

This conditions provides, in general, equilibrium of tractions on the interface $K$; in the next section we will refine it by taking into account cohesive forces. Note that in terms of the Neumann-jump operator

$$\langle [\![\boldsymbol{\sigma}_\nu(s)]\!], z\rangle := \int_\Omega \boldsymbol{\sigma}(s) : \boldsymbol{\epsilon}(\bar z)\,dx = \langle [\![\boldsymbol{\sigma}^-_\nu(s)]\!], z\rangle - \langle [\![\boldsymbol{\sigma}^+_\nu(s)]\!], z\rangle,$$

(56) reads $[\![\boldsymbol{\sigma}_\nu(s)]\!] = 0$ in $\tilde H^{-1/2}(K;\mathbb{R}^2)$.

**Proposition 5.1.** *For every $s \in [0,S]$ we have*

$$\operatorname{div} \boldsymbol{\sigma}(s) = 0 \ \textit{in } H^{-1}(\Omega;\mathbb{R}^2), \quad \boldsymbol{\sigma}_\nu(s) = 0 \ \textit{in } \tilde H^{-1/2}(\partial_N\Omega;\mathbb{R}^2), \quad \boldsymbol{\sigma}^-_\nu(s) = \boldsymbol{\sigma}^+_\nu(s) \ \textit{in } \tilde H^{-1/2}(K;\mathbb{R}^2).$$

In the following we will write $\boldsymbol{\sigma}_\nu(s) = \boldsymbol{\sigma}^+_\nu(s) = \boldsymbol{\sigma}^-_\nu(s)$ for the Neumann operator on $K$.

### 5.2 Euler-Lagrange conditions for $t'(s) > 0$

Let $\tau = \nu^\perp$ be a tangent vector field on $K$. In order to give a better description of the forces on the interface $K$ we define the normal and tangent component of the traction $\boldsymbol{\sigma}_\nu$ on $K$ as follows: for $\zeta \in \tilde{H}^{1/2}(K)$ (note that it is a scalar function), $\boldsymbol{\sigma}_{\nu\nu}(s) \in \tilde{H}^{-1/2}(K)$ is the operator

$$\langle \boldsymbol{\sigma}_{\nu\nu}(s), \zeta \rangle = \langle \boldsymbol{\sigma}_\nu(s), \zeta\nu \rangle = -\int_{\Omega^+} \boldsymbol{\sigma}(s) : \boldsymbol{\epsilon}(\bar{z})\, dx,$$

where $\bar{z}$ is a lifting of $\widetilde{(\zeta\nu)}$ in $\Omega^+$, while $\boldsymbol{\sigma}_{\nu\tau}(s) \in \tilde{H}^{-1/2}(K)$ is the operator

$$\langle \boldsymbol{\sigma}_{\nu\tau}(s), \zeta \rangle = \langle \boldsymbol{\sigma}_\nu(s), \zeta\tau \rangle = -\int_{\Omega^+} \boldsymbol{\sigma}(s) : \boldsymbol{\epsilon}(\bar{z})\, dx,$$

where $\bar{z}$ is a lifting of $\widetilde{(\zeta\tau)}$ in $\Omega^+$.

Let $s \in [0, S]$ be such that $t'(s) > 0$. By Theorem 4.12 we have $|\partial\mathcal{F}|(t(s), w(s), \xi(s)) = 0$ and then for all $z \in L^2(K; \mathbb{R}^2)$ we have

$$\liminf_{h \to 0} \frac{\mathcal{F}(t(s), w(s) + hz, \xi(s)) - \mathcal{F}(t(s), w(s), \xi(s))}{h\|z\|_{L^2(K;\mathbb{R}^2)}} \geq 0.$$

In particular, let $z \in H^{1/2}(K; \mathbb{R}^2)$ such that $(w(s) + hz) \cdot \nu \geq 0$ for $h \geq 0$ sufficiently small. Let $\bar{z}$ be a lifting of $\tilde{z}$ in $\Omega$. Let $u = \boldsymbol{u}(g(t), w)$; by minimality

$$F(t(s), u(s) + h\bar{z}, \xi(s)) \geq \mathcal{F}(t(s), w(s) + hz, \xi(s)) = \widetilde{\mathcal{F}}(t(s), w(s) + hz, \xi(s)).$$

Note that we can employ the functional $\widetilde{\mathcal{F}}$, and its directional derivative, because $z$ is an admissible variation for the incompenetrability constraint. Hence, for the directional derivative of $F(t(s), u(s), \xi(s))$ we have (see §4.2)

$$\begin{aligned} 0 \leq F'(t(s), u(s), \xi(s); \bar{z}) &= dE(t(s), u(s))[\bar{z}] + \widetilde{\mathcal{K}}'(w(s), \xi(s); z) \\ &= \int_\Omega \boldsymbol{\sigma}(s) : \boldsymbol{\epsilon}(\bar{z})\, dx + \int_K \tilde{\varphi}'(w(s), \xi(s); z)\, d\mathcal{H}^1. \end{aligned} \tag{57}$$

**Tangential component.** Given $\zeta \in \tilde{H}^{1/2}(K)$ let $\bar{z}$ a lifting of $\widetilde{(\zeta\tau)}$ in $H^1(\Omega^+; \mathbb{R}^2)$ and let $\bar{z} = 0$ in $\Omega^-$. As $\bar{z}$ is an admissible variation (because $[\![\bar{z}]\!] \cdot \nu = 0$ on $K$) by (57) we have

$$F'(t(s), u(s), \xi(s); \bar{z}) = -\langle \boldsymbol{\sigma}_{\nu\tau}(s), \zeta \rangle + \int_K \tilde{\varphi}'(w(s), \xi(s); \zeta\tau)\, d\mathcal{H}^1 \geq 0.$$

Hence, as in (20),

$$\langle \boldsymbol{\sigma}_{\nu\tau}(s), \zeta \rangle \leq \int_K \tilde{\varphi}'(w(s), \xi(s); \zeta\tau)\, d\mathcal{H}^1 \leq \hat{\psi}'(0)\|\zeta\|_{L^1(K)}.$$

Since $-\bar{z}$ is as well an admissible variation, we get $|\langle \boldsymbol{\sigma}_{\nu\tau}(s), \zeta \rangle| \leq \hat{\psi}'(0)\|\zeta\|_{L^1(K)}$. Hence, by density of $\tilde{H}^{1/2}(K)$ in $L^1(K)$ there exists a unique function $h_{\nu\tau}(s) \in L^\infty(K)$ which represents $\boldsymbol{\sigma}_{\nu\tau}(s)$ as

$$\langle \boldsymbol{\sigma}_{\nu\tau}(s), \zeta \rangle = \int_K h_{\nu\tau}(s)\zeta\, d\mathcal{H}^1.$$

Following the arguments of [19, Theorem 7.8] we obtain that (a.e. in $K$) $|h_{\nu\tau}(s)| \leq \hat{\psi}'(0)$ if $\xi(s) = 0$ (and thus $w(s) = 0$) and that $h_{\nu\tau}(s) = \tilde{\varphi}'(w(s), \xi(s); \tau) = \varphi'(w(s), \xi(s), \nu; \tau)$ if $\xi(s) > 0$; in the second case there exists indeed the directional derivative of $\varphi$ and $\varphi'(w(s), \xi(s), \nu; \tau) = \nabla_\tau \varphi(w(s), \xi(s), \nu)$ (the gradient is in the first variable).

**Normal component.** In the case of normal tractions the reasoning is more delicate, due to the incompenetrability constraint. Given $\zeta \in \tilde{H}^{1/2}(K)$ with $\zeta \geq 0$ let $\bar{z}$ a lifting of $\widetilde{(\zeta\nu)}$ in $H^1(\Omega^+;\mathbb{R}^2)$ and let $\bar{z} = 0$ in $\Omega^-$. As $\bar{z}$ is an admissible variation (because $[\![\bar{z}]\!] \cdot \nu \geq 0$ on $K$) by (57) we have

$$\partial F(t(s), u(s), \xi(s); \bar{z}) = -\langle \boldsymbol{\sigma}_{\nu\nu}(s), \zeta \rangle + \int_K \tilde{\varphi}'(w(s), \xi(s); \zeta\nu)\, d\mathcal{H}^1 \geq 0.$$

Since $\zeta \geq 0$ by (20) we can write $\tilde{\varphi}'(w(s), \xi(s); \zeta\nu) = \tilde{\varphi}'(w(s), \xi(s); \nu)\zeta$. As a consequence the linear functional

$$-\langle \boldsymbol{\sigma}_{\nu\nu}(s), \zeta \rangle + \int_K \tilde{\varphi}'(w(s), \xi(s); \zeta\nu)\, d\mathcal{H}^1$$

is positive and thus, by a classical results on distributions, it is represented by a Radon measure. It follows that $\boldsymbol{\sigma}_{\nu\nu}(s)$ is a measure as well; we denote it simply by $\mu$. At this point, by Hahn decomposition we distinguish between positive and negative part $\mu^+$ and $\mu^-$. Since

$$\langle \mu, \zeta \rangle = \langle \boldsymbol{\sigma}_{\nu\nu}(s), \zeta \rangle \leq \int_K \tilde{\varphi}'(w(s), \xi(s); \zeta\nu)\, d\mathcal{H}^1 \leq \hat{\psi}'(0) \|\zeta\|_{L^1(K)}$$

it follows that the positive part $\mu_+$ is actually in $L^\infty(K)$, more precisely we will write $\mu_+ = h_{\nu\nu}(s)\mathcal{H}^1 \lfloor_K$ where $h_{\nu\nu}(s) \in L^\infty(K)$ and $h_{\nu\nu}(s) \geq 0$. Clearly $h_{\nu\nu}(s) = 0$ on the support of $\mu_-$.

Physically, $\boldsymbol{\sigma}_{\nu\nu}(s) = \mu$ gives the normal (elastic) force induced on $K$ by the elastic displacement on $\Omega^+$; since $\nu$ is the inward unit normal for $\Omega^+$ the positive part $\mu_+$ "opens the crack" and thus (as we will see) it is balanced by cohesive forces while the negative part $\mu_-$ is balanced by the reaction of the incompenetrability constraint.

Formally, we can argue in this way. If $\zeta \in \tilde{H}^{1/2}(K)$ is of the form $\zeta = v - w(s) \cdot \nu$ with $v \geq 0$ then, taking into account (57), we can write

$$-\int_K \zeta d\mu^- - \int_K h_{\nu\nu}(s)\zeta\, d\mathcal{H}^1 + \int_K \tilde{\varphi}'(w(s), \xi(s); \zeta\nu)\, d\mathcal{H}^1 \geq 0.$$

By density, if $\zeta \in L^1(K)$ is of the form $\zeta = z - w(s) \cdot \nu$ with $z \geq 0$ and $z = w(s) \cdot \nu$ (so that $\zeta = 0$) on the support of $\mu^-$ we get

$$-\int_K h_{\nu\nu}(s)\zeta\, d\mathcal{H}^1 + \int_K \tilde{\varphi}'(w(s), \xi(s); \zeta\nu) \geq 0.$$

Choosing properly the test function $\zeta$, it follows that (a.e. on $K$) we have $0 \leq h_{\nu\nu}(s) \leq \tilde{\varphi}'(w(s), \xi(s); \nu)$ if $w(s) \cdot \nu = 0$ and $h_{\nu\nu}(s) = \tilde{\varphi}'(w(s), \xi(s); \nu) = \varphi'(w(s), \xi(s), \nu; \nu)$ if $w(s) \cdot \nu > 0$. In particular, if $w(s) \cdot \nu = 0$ but $\xi(s) > 0$ then $\tilde{\varphi}'(w(s), \xi(s); \nu) = 0$ (by definition of $\varphi$) and thus $h_{\nu\nu}(s) = 0$.

Summarizing the results obtained so far, we can state the following Proposition.

**Proposition 5.2.** *Let $s \in [0, S]$ be such that $t'(s) > 0$. Denote by $\boldsymbol{\sigma}_{\nu\tau}(s)$ and $\boldsymbol{\sigma}_{\nu\nu}(s)$ respectively the tangential and normal components of the operator $\boldsymbol{\sigma}_\nu(s)$. Then there exists a function $h_{\nu\tau}(s) \in L^\infty(K)$ such that*

$$\langle \boldsymbol{\sigma}_{\nu\tau}(s), \zeta \rangle = \int_K h_{\nu\tau}(s)\zeta\, d\mathcal{H}^1 \qquad \text{with} \qquad \begin{cases} |h_{\nu\tau}(s)| \leq \hat{\psi}'(0) & \text{a.e. where } \xi(s) = 0, \\ h_{\nu\tau}(s) = \varphi'(w(s), \xi(s), \nu; \tau) & \text{a.e. where } \xi(s) > 0. \end{cases}$$

*Moreover, there exists a Radon measure $\mu$ on $K$ and a non-negative function $h_{\nu\nu} \in L^\infty(K)$ such that*

$$\langle \boldsymbol{\sigma}_{\nu\nu}(s), \zeta \rangle = \int_K \zeta\, d\mu = \int_K h_{\nu\nu}\zeta\, d\mathcal{H}^1 - \int_K \zeta\, d\mu^-,$$

*where $\mu^-$ is the negative part of the measure $\mu$ and*

$$\begin{cases} h_{\nu\nu}(s) \leq \hat{\psi}'(0) & \text{a.e. where } \xi(s) = 0, \\ h_{\nu\nu}(s) = \varphi'(w(s), \xi(s), \nu; \nu) & \text{a.e. where } \xi(s) > 0. \end{cases}$$

*If $w(s) \cdot \nu > 0$ then $\nabla\varphi(w(s), \xi(s), \nu) = \varphi'(w(s), \xi(s), \nu; \nu)\, \nu + \varphi'(w(s), \xi(s), \nu; \tau)\, \tau$.*

### 5.3 Normalized gradient flow for $t'(s) = 0$

In this section we show that in a point of discontinuity (in time) the parametrized velocity is the steepest descent direction for $\mathcal{F}(t(s), w(s), \xi(s))$ with respect to the $L^2(K;\mathbb{R}^2)$-norm.

**Proposition 5.3.** *Let $t(s) = t^*$ for $s \in (s_1, s_2)$. Then for a.e. $s \in (s_1, s_2)$ it holds*

$$w'(s) \in \operatorname{argmin} \{\mathcal{F}'(t^*, w(s), \xi(s); z) : z \in L^2(K;\mathbb{R}^2) \text{ with } \|z\|_{L^2(K;\mathbb{R}^2)} \leq 1\}.$$

*where $\mathcal{F}'(t^*, w(s), \xi(s); z)$ denotes the directional derivative.*

*Proof.* The proof shares some common points with [19]. First, we show that for a.e. $s \in (s_1, s_2)$ we have

$$\lim_{h\to 0^+} \frac{1}{h}\big(\mathcal{F}(t^*, w(s+h), \xi(s+h)) - \mathcal{F}(t^*, w(s), \xi(s)\big) \geq \mathcal{F}'(t^*, w(s), \xi(s); w'(s)). \tag{58}$$

By Lemma 4.10 and Theorem 4.12, $s \mapsto w(s)$ is absolutely continuous (taking values in $H^{1/2}(K;\mathbb{R}^2)$) and $s \mapsto \mathcal{F}(t^*, w(s), \xi(s))$ is absolutely continuous (taking values in $\mathbb{R}$). Let $s \in (s_1, s_2)$ be a differentiability point for both $w(\cdot)$ and $\mathcal{F}(t^*, w(\cdot), \xi(\cdot))$. By monotonicity of the density $\varphi(|w|, \cdot)$ (see Lemma 2.5 *(ii)*) we can write

$$\mathcal{F}(t^*, w(s+h), \xi(s+h)) = \mathcal{E}(t^*, w(s+h)) + \mathcal{K}(w(s+h), \xi(s+h)) \geq \mathcal{E}(t^*, w(s+h)) + \mathcal{K}(w(s+h), \xi(s))$$

Then, we will estimate $\mathcal{F}(t^*, w(s+h), \xi(s)) - \mathcal{F}(t^*, w(s), \xi(s))$. Consider the auxiliary convex functional

$$\widetilde{\mathcal{F}}_{w(s)}(t^*, w, \xi(s)) = \widetilde{\mathcal{F}}(t^*, w, \xi(s)) - \tfrac{1}{2}\lambda\|w - w(s)\|^2_{H^{1/2}(K;\mathbb{R}^2)},$$

where $\tilde{\mathcal{F}}$ is defined by (18). Denote by $\partial\widetilde{\mathcal{F}}_{w(s)}$ the subdifferential of $\widetilde{\mathcal{F}}_{w(s)}$ in $H^{1/2}(K;\mathbb{R}^2)$. For every $\eta \in \partial\widetilde{\mathcal{F}}_{w(s)}(t^*, w(s), \xi(s))$ we can write

$$\begin{aligned} \widetilde{\mathcal{F}}_{w(s)}(t^*, &w(s+h), \xi(s)) - \widetilde{\mathcal{F}}_{w(s)}(t^*, w(s), \xi(s)) = \\ &= \widetilde{\mathcal{F}}(t^*, w(s+h), \xi(s)) - \tfrac{1}{2}\lambda\|w(s+h) - w(s)\|^2_{H^{1/2}(K;\mathbb{R}^2)} - \widetilde{\mathcal{F}}(t^*, w(s), \xi(s)) \\ &\geq \langle \eta, w(s+h) - w(s)\rangle \end{aligned}$$

where brackets denote here the duality in $H^{1/2}(K;\mathbb{R}^2)$. Taking the limit we get

$$\lim_{h\to 0^+} \frac{1}{h}\big(\widetilde{\mathcal{F}}(t^*, w(s+h), \xi(s)) - \widetilde{\mathcal{F}}(t^*, w(s), \xi(s)\big) \geq \langle \eta, w'(s)\rangle$$

Note that the energy functional $\tilde{\mathcal{F}}(t^*, \cdot, \xi(s))$ is continuous in $H^{1/2}(K;\mathbb{R}^2)$. Hence we can apply [4][Theorem 17.19] obtaining

$$\begin{aligned} \lim_{h\to 0^+} \frac{1}{h}\big(\widetilde{\mathcal{F}}(t^*, w(s+h), \xi(s)) - \widetilde{\mathcal{F}}(t^*, w(s), \xi(s)\big) &\geq \widetilde{\mathcal{F}}'(t^*, w(s), \xi(s); w'(s)) \\ &= \mathcal{F}'(t^*, w(s), \xi(s); w'(s)), \end{aligned}$$

where the last equality follows from the fact that $\widetilde{\mathcal{F}}(t^*, w(\cdot), \xi(s)) = \mathcal{F}(t^*, w(\cdot), \xi(s))$ since the evolution $w$ satisfies the incompenetrability constraint $[\![w(s)]\!] \cdot \nu_K \geq 0$ on $K$ for every $s \in [0, S]$. We have proved (58).

By (58) we know that $\mathcal{F}'(t^*, w(s), \xi(s)) \geq \mathcal{F}'(t^*, w(s), \xi(s); w'(s))$ while the energy balance (7) (in the interval $(s_1, s_2)$ where $t(\cdot)$ is constant) provides

$$\mathcal{F}'(t^*, w(s), \xi(s)) = -|\partial_w \mathcal{F}|(t^*, w(s), \xi(s)).$$

By definition of the slope, it is clear that for every admissible variation $z$ with $\|z\|_{L^2(K;\mathbb{R}^2)} \leq 1$

$$-|\partial_w \mathcal{F}|(t^*, w(s), \xi(s)) \leq \mathcal{F}'(t^*, w(s), \xi(s); z) \leq |\partial_w \mathcal{F}|(t^*, w(s), \xi(s))\,.$$

Hence $w'(s) \in \operatorname{argmin} \{\mathcal{F}'(t^*, w(s), \xi(s); z) : z \in L^2(K;\mathbb{R}^2) \text{ with } \|z\|_{L^2(K;\mathbb{R}^2)} \leq 1\}$. □

# A Upper gradients, slopes and $\lambda$-convexity

For sake of completeness we collect here a couple of basic definitions [2]. We assume that $X$ is a Hilbert space, even if the definitions make sense in metric spaces. We will also identify its dual $X^*$ with $X$ (by means of Riesz Theorem).

**Definition A.1** (Slope). *If $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ and $\Phi(v) < +\infty$ the slope of $\Phi$ in $v$ is defined as*

$$|\partial\Phi|(v) := \limsup_{w\to v} \frac{|\Phi(v) - \Phi(w)|^+}{\|v - w\|}.$$

**Definition A.2.** *A functional $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ is $\lambda$-convex (for $\lambda < 0$) if for every $x, y \in X$ we have*

$$\Phi(tx + (1-t)y) \le t\Phi(x) + (1-t)\Phi(y) - \tfrac{\lambda}{2} t(1-t)|x - y|^2.$$

In Hilbert spaces $\lambda$-convexity is equivalent to the convexity of the functionals $\mathcal{I}_y(x) =\mapsto \Phi(x) - \frac{\lambda}{2}|x - y|^2$ for any $y \in X$. This property is stated in [2, Remark 2.4.4] for Euclidean spaces, but it can be easily checked also for Hilbert spaces.

**Definition A.3** (Subdifferential of nonconvex functions). *Let $\mathcal{G} : X \to \mathbb{R} \cup \{+\infty\}$. If $\mathcal{G}(w) = +\infty$ then $\partial\mathcal{G}(w) = \emptyset$. Otherwise, $\partial\mathcal{G}(w) \subseteq X$ consists of all $\xi \in X$ such that*

$$\liminf_{v\to w} \frac{\mathcal{G}(v) - \mathcal{G}(w) - \langle \xi, v - w\rangle}{\|v - w\|} \ge 0. \tag{59}$$

**Lemma A.4.** *If $\mathcal{G} : X \to \mathbb{R} \cup \{+\infty\}$ is $\lambda$-convex, for some $\lambda < 0$, then its subdifferential $\partial\mathcal{G}(w)$, in the sense of Definition A.3, coincides with the "classical" subdifferential $\partial\mathcal{J}_w(w)$ of the auxiliary convex function $\mathcal{J}_w(u) := \mathcal{G}(u) - \frac{\lambda}{2}\|u - w\|^2$.*

*Proof.* The inclusion $\partial\mathcal{J}_w(w) \subseteq \partial\mathcal{G}(w)$ is easy to see; indeed if $\xi \in \partial\mathcal{J}_w(w)$, for all $v \in X$ we write

$$\frac{\mathcal{G}(v) - \frac{\lambda}{2}\|v - w\|^2 - \mathcal{G}(w) - \langle \xi, v - w\rangle}{\|v - w\|} = \frac{\mathcal{J}_w(v) - \mathcal{J}_w(w) - \langle \xi, v - w\rangle}{\|v - w\|} \ge 0,$$

and thus

$$\liminf_{v\to w} \frac{\mathcal{G}(v) - \mathcal{G}(w) - \langle \xi, v - w\rangle}{\|v - w\|} \ge \liminf_{v\to w} \tfrac{\lambda}{2}\|v - w\| = 0.$$

Let us prove the opposite inclusion $\partial\mathcal{G}(w) \subseteq \partial\mathcal{J}_w(w)$. Let $\xi \in \partial\mathcal{G}(w)$, and let $v = w + \varepsilon\hat{v}$ for some $\hat{v} \in X$ with $\|\hat{v}\| = 1$. Then by (59)

$$\liminf_{\varepsilon\to 0} \frac{\mathcal{G}(w + \varepsilon\hat{v}) - \mathcal{G}(w) - \langle \xi, \varepsilon\hat{v}\rangle}{\varepsilon} \ge 0.$$

As a consequence being $\lambda < 0$ it holds

$$\lim_{\varepsilon\to 0^+} \frac{\mathcal{J}_w(w + \varepsilon\hat{v}) - \mathcal{J}_w(w)}{\varepsilon} - \langle \xi, \hat{v}\rangle \ge \liminf_{\varepsilon\to 0^+} \frac{\mathcal{G}(w + \varepsilon\hat{v}) - \mathcal{G}(w)}{\varepsilon} - \langle \xi, \hat{v}\rangle \ge 0,$$

that means that the directional derivative $\partial_{\hat{v}}\mathcal{J}_w(w) \ge \langle \xi, \hat{v}\rangle$. Hence, for $\hat{v} := \frac{v-w}{\|v-w\|}$ by (directional) convexity we have

$$\mathcal{J}_w(v) - \mathcal{J}_w(w) \ge \langle \xi, \hat{v}\rangle \|v - w\| = \langle \xi, v - w\rangle.$$

By arbitrariness of $v \in X$ we have concluded. □

**Lemma A.5.** *Let $\mathcal{G} : X \to \mathbb{R} \cup \{+\infty\}$ and $\mathcal{G}(w) < +\infty$. If $\mathcal{G}$ is convex then*

$$|\partial\mathcal{G}|(w) = \inf\{\|\eta\| : \eta \in \partial\mathcal{G}(w)\}$$

*where $|\partial\mathcal{G}|(w)$ denotes the slope (cf. Definition A.1). If $\mathcal{G}$ is $\lambda$-convex then*

$$|\partial\mathcal{G}|(w) = |\partial\mathcal{J}_w|(w) = \inf\{\|\eta\| : \eta \in \partial\mathcal{J}_w(w)\}, \tag{60}$$

*where $\mathcal{J}_w(z) := \mathcal{G}(z) - \frac{\lambda}{2}\|w - z\|^2$ is convex.*

*Proof.* It is well known that the representation $|\partial\mathcal{G}|(w) = \inf\{\|\eta\| : \eta \in \partial\mathcal{G}(w)\}$ holds for convex functionals.

Let us prove (60). Consider a sequence $w_n \to w$. Write

$$|\mathcal{J}_w(w) - \mathcal{J}_w(w_n)|^+ = |\mathcal{G}(w) - \mathcal{G}(w_n) + \tfrac{\lambda}{2}\|w - w_n\|^2|^+$$

and consider the three cases:

$$\begin{aligned} \mathcal{G}(w) - \mathcal{G}(w_n) &< 0, \\ 0 \le \mathcal{G}(w) - \mathcal{G}(w_n) &\le -\tfrac{\lambda}{2}\|w - w_n\|^2, \\ 0 \le -\tfrac{\lambda}{2}\|w - w_n\|^2 &< \mathcal{G}(w) - \mathcal{G}(w_n). \end{aligned}$$

In the first case $\mathcal{G}(w) - \mathcal{G}(w_n) + \frac{\lambda}{2}\|w - w_n\|^2 \le \mathcal{G}(w) - \mathcal{G}(w_n) < 0$ and then

$$\frac{|\mathcal{G}(w) - \mathcal{G}(w_n)|^+}{\|w - w_n\|} = \frac{|\mathcal{J}_w(w) - \mathcal{J}_w(w_n)|^+}{\|w - w_n\|} = 0. \tag{61}$$

In the second case

$$0 \le \frac{|\mathcal{G}(w) - \mathcal{G}(w_n)|^+}{\|w - w_n\|} \le -\tfrac{\lambda}{2}\|w - w_n\|, \qquad \frac{|\mathcal{J}_w(w) - \mathcal{J}_w(w_n)|^+}{\|w - w_n\|} = 0. \tag{62}$$

In the last case,

$$\begin{aligned} |\mathcal{G}(w) - \mathcal{G}(w_n)|^+ &= \big|\mathcal{G}(w) - \mathcal{G}(w_n) + \tfrac{\lambda}{2}\|w - w_n\|^2\big|^+ - \tfrac{\lambda}{2}\|w - w_n\|^2 \\ &= |\mathcal{J}_w(w) - \mathcal{J}_w(w_n)|^+ - \tfrac{\lambda}{2}\|w - w_n\|^2 \end{aligned}$$

and thus

$$\frac{|\mathcal{G}(w) - \mathcal{G}(w_n)|^+}{\|w - w_n\|} = \frac{|\mathcal{J}_w(w) - \mathcal{J}_w(w_n)|^+}{\|w - w_n\|} - \tfrac{\lambda}{2}\|w - w_n\|. \tag{63}$$

Consider a sequence $w_n \to w$ s.t.

$$|\partial\mathcal{G}|(w) = \lim_{n\to\infty} \frac{|\mathcal{G}(w) - \mathcal{G}(w_n)|^+}{\|w - w_n\|};$$

by (61)-(63)

$$|\partial\mathcal{G}|(w) = \lim_{n\to\infty} \frac{|\mathcal{G}(w) - \mathcal{G}(w_n)|^+}{\|w - w_n\|} = \lim_{n\to\infty} \frac{|\mathcal{J}_w(w) - \mathcal{J}_w(w_n)|^+}{\|w - w_n\|} \le |\partial\mathcal{J}_w|(w).$$

To prove the opposite inequality $|\partial\mathcal{J}_w|(w) \le |\partial\mathcal{G}|(w)$ it is sufficient to consider a sequence s.t.

$$|\partial\mathcal{J}_w|(w) = \lim_{n\to\infty} \frac{|\mathcal{J}_w(w) - \mathcal{J}_w(w_n)|^+}{\|w - w_n\|};$$

by (61)-(63) we have

$$|\partial\mathcal{G}|(w) \ge \lim_{n\to\infty} \frac{|\mathcal{G}(w) - \mathcal{G}(w_n)|^+}{\|w - w_n\|} = \lim_{n\to\infty} \frac{|\mathcal{J}_w(w) - \mathcal{J}_w(w_n)|^+}{\|w - w_n\|} = |\partial\mathcal{J}_w|(w),$$

and the proof is concluded. □

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