# Asymptotically critical points and their multiplicity

Antonio Marino - Dimitri Mugnai*

**Abstract**

In this paper we study multiplicity results for the critical points of a functional via topological information which ensures multiplicity of critical points for a sequence of approximating functionals. The main statement is quite simple, and it seems it could be usefully compared with a large class of problems. In particular we mention some problems that can be studied in this framework.

## 1 Introduction: main concepts and aims

There are many variational problems whose solutions spontaneously come out as limit of solutions of approximating problems. This can happen in various situations.

A typical case is the one of studying a problem which is "irregular", for many reasons, in the sense that it lies (a little or a lot) beyond the classical framework of reference in which similar problems are studied (for degeneration of the coefficients, for lack of differentiability, ... ). Usually in this situation a sequence of "regular" functionals $(h_n)_n$ comes out in a very natural way. Such a sequence "tends" in some sense to an "irregular" functional $h$ whose "stationary" points, suitably defined, solve the initial problem.

An analogous situation arises while using approximation methods (of Galerkin type) for a problem (for example a differential equation) in an infinite dimensional space $H$, by a sequence of problems in finite dimensional

*Dipartimento di Matematica "L. Tonelli", Via F. Buonarroti 2, Pisa, Italy. e-mail: marino@dm.unipi.it & mugnai@mail.dm.unipi.it

subspaces $H_n$. In such a case, if $h : H \longrightarrow \mathbb{R}$ is the functional whose critical points are object of investigation, one can consider the functionals $h_n = h_{|H_n}$. If one wants to consider all $h_n$ defined in the same space $H$, one can set $h_n(u) = +\infty$ if $u \in H \setminus H_n$. The sequence $(h_n)_n$ defined in this way can be studied with the methods of [8], which extends the results of this paper.

But there are other interesting cases, in which not all the critical points of the "limit" functional $h$ have the properties of those which are limit of points $u_n$ such that $\nabla h_n(u_n) \to 0$.

An important problem of this type is the one of the bounce trajectories between two given points in a billiard with perfectly elastic walls, in presence, possibly, of a field of conservative forces (see [3] and [9]). If we describe the billiard with the closure of an open subset $\Omega$ of $\mathbb{R}^N$ and the potential of the field with a function $V : \mathbb{R}^N \longrightarrow \mathbb{R}$, given two points $A$ and $B$ in $\Omega$, it is natural to consider, for example, the functionals

$$f_n(\gamma) = \frac{1}{2}\int_0^1 |\dot\gamma(t)|^2\, dt - \int_0^1 V(\gamma(t))\, dt - n\int_0^1 U(\gamma(t))\, dt$$

defined on the space $X = \{\gamma \in H^1([0,1];\mathbb{R}^N) \,|\, \gamma(0) = A, \gamma(1) = B\}$, where $U : R^N \longrightarrow \mathbb{R}$ is a continuous function such that $U(x) = 0$ if $x \in \overline{\Omega}$ and $U(x) > 0$ if $x \in \mathbb{R}^N \setminus \overline{\Omega}$. Under suitable hypotheses, if $\forall n$ $\gamma_n$ in $X$ is a critical point of $f_n$ and if $(\gamma_n)_n$ converges in a suitable sense to a curve $\gamma^*$ of $X$, then $\gamma^*$ has image in $\overline{\Omega}$ and it is a bounce trajectory, in the sense that it verifies the "reversed" inequality

$$\int_0^1 \dot\gamma^* \cdot \dot\varphi - \int_0^1 \nabla V(\gamma^*) \cdot \varphi \le 0 \tag{1}$$

$\forall \varphi$ in $H_0^1([0,1];\mathbb{R}^N)$ such that $\varphi(t) \cdot \nu(\gamma^*(t)) \le 0$ $\forall t$ such that $\gamma^*(t) \in \partial\Omega$ (where $\nu(x)$ is the outward normal to $\Omega$ in $x$ of $\partial\Omega$), with the condition that it preserves its energy:

$$\frac{1}{2}|\gamma^*(t)|^2 + V(\gamma^*(t)) = \text{ constant.}$$

These two conditions characterize the bounce curves $\gamma^*$ in the billiard $\overline{\Omega}$ in a satisfactory way and we can assume them just to define such curves (we observe, *en passant*, that the usual elliptic variational inequalities have the sign "$\geq$" instead of the "$\leq$" of reference). Note that the fact that $\gamma^*$ solves (1), is equivalent to say that $\gamma^*$ is an "upper critical point" (see [4], [10]) for the functional $f : X \longrightarrow \mathbb{R} \cup \{-\infty\}$ defined by

$$f(\gamma) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 |\dot{\gamma}(t)|^2\, dt - \int_0^1 V(\gamma(t))\, dt & \text{if } \gamma([0,1]) \subset \overline{\Omega} \\ \\ -\infty & \text{elsewhere.} \end{cases}$$

Moreover it should be noted that the conservation of the energy is *not* a consequence of inequality (1), since it follows from the fact that $\gamma_n$ is a critical point for $f_n$.

Therefore, in this case it seems unnatural to introduce a functional whose "critical" points (in some sense) solve the bounce problem, while it is spontaneous to refer to all the sequence $(f_n)_n$.

Thus we are led to give the following definitions.

Let $(h_n)$ be a sequence of functionals defined, for example, on a Riemannian manifold $M$ and let us also consider a functional $h : M \longrightarrow \mathbb{R}$. Assume that $h_n$, $h$ and $M$ are regular, just for simplicity.

**Definition 1.1.** We say that $u$ in $M$ is asymptotically critical for the couple $\big((h_n)_n, h\big)$, if there exists a strictly increasing sequence of integers $(n_k)_k$ in $\mathbb{N}$ and there exists a sequence $(u_k)_k$ in $M$ such that

$$\nabla h_{n_k}(u_k) \to 0, \qquad u_k \to u \qquad \text{and} \qquad h_{n_k}(u_k) \to h(u).$$

We also say that $h(u)$ is an asymptotically critical value (level) for the couple $\big((h_n)_n, h\big)$.

Note that it is not necessary to impose that $u$ is a critical point for $h$. Indeed we could completely eliminate the function $h$, saying that the critical value is the limit (that in the definition we would assume to exist in $\mathbb{R}$) of $(h_{n_k}(u_k))_k$. But in this case we should add the hypothesis that for every $u$ in $M$ satisfying the modified definition, the limit of $(h_{n_k}(u_k))_k$ doesn't depend on $(n_k)_k$ and $(u_k)_k$.

In order to obtain multiplicity results related to asymptotically critical points for a couple $\big((h_n)_n, h\big)$, we introduce the following definition.

**Definition 1.2.** Let $c$ be a real number. We say that the couple $\big((h_n)_n, h\big)$ is $\nabla$–compact at level $c$, or that condition $\nabla(h_n, h; c)$ holds, if for every strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ and for every $(u_k)_k$ in $M$ such that

$$\nabla h_{n_k}(u_k) \to 0 \qquad \text{and} \qquad h_{n_k}(u_k) \to c,$$

there exists a strictly increasing sequence $(k_j)_j$ in $\mathbb{N}$ and there exists $u$ in $M$ such that $u_{k_j} \to u$ and $h(u) = c$.

If $a$ and $b$ are real numbers with $a \le b$ and $\nabla(h_n, h; c)$ holds $\forall\, c$ in $[a, b]$, we say that $\nabla(h_n, h; a, b)$ holds.

In §2 we will prove, in particular, the following Theorem.

**Theorem 1.3.** *Suppose $M$ is a complete Riemannian manifold of class $C^{1,1}$ and every $h_n$ is of class $C^1$; let $a$ and $b$ be in $\mathbb{R}$ with $a \le b$. If $\nabla(h_n, h; a, b)$ holds, then the number of asymptotically critical points for $\big((h_n)_n, h\big)$ with asymptotically critical value in $[a, b]$ is greater or equal to*

$$\limsup_{n\to\infty} \operatorname{cat}_M(h_n^b, h_n^a).$$

The notion of relative category we used to prove this theorem is quickly recalled in the appendix.

*Therefore the condition $\nabla(h_n, h; a, b)$, which ensures the permanence of the (quasi)critical points on the "limit" function $h$, is also sufficient, as Theorem 1.3 shows, to evaluate the multiplicity of such points.*

Anyway it is interesting to note that the hypotheses of the Theorem don't imply that $(h_n)_n$ converges to $h$ according to any usual notion of convergence (pointwise, uniform, $\Gamma$, ... ).

**Remark 1.4.** We can give a weaker and simpler version of the condition of $\nabla$–compactness for a couple $\big((h_n)_n, h\big)$ at a level $c$:

$$\forall\ (u_n)_n \text{ in } M \text{ such that } \nabla h_n(u_n) \to 0 \text{ and } h_n(u_n) \to c,$$

there exists a subsequence $(u_{n_k})_k$ which converges to a point $u$ of $M$ with $h(u) = c$.

This definition is useful, too; in fact, if we replace it to the one given in Definition 1.2, Theorem 1.3 still holds, provided in the thesis we replace $\limsup_{n\to\infty} \operatorname{cat}_M(h_n^b, h_n^a)$ with $\liminf_{n\to\infty} \operatorname{cat}_M(h_n^b, h_n^a)$.

But in Proposition 2.5 and Theorem 2.6 we will use Definition 1.2 in an essential way.

In [8] we will also consider the case in which $h_n$ and $h$ are not differentiable in the classical sense and we will also give a nonsmooth version of this Theorem. In this nonsmooth version the theorem lets us study the following reversed variational inequality:

$$(2) \qquad \int_\Omega \Delta u \Delta(v-u) - c\int_\Omega Du \cdot D(v-u) - \alpha \int_\Omega u(v-u) \le 0$$

$\forall\, v$ in $H_0^1(\Omega) \cap H^2(\Omega)$ such that $v \ge \varphi$, where $\Omega$ is an open, bounded and regular subset of $\mathbb{R}^N$ with $N \le 3$ and $\varphi : \Omega \longrightarrow \mathbb{R}$ is a measurable function such that $\sup \varphi < 0$ (see [8]).

Finally we note that the nonsmooth version of Theorem 1.3 covers the approximation methods of Galerkin type described in [1] and [5]. This paper was also inspired by those techniques, further developed in [2].

We are grateful to Prof. C. Saccon for the conversations on the subject, and in particular for proposing to define the critical point, linking it to the sequence $(h_n)_n$: in this way one already includes in the definition richer information than the one expressed by the only fact that such a point is a critical point for the limit functional.

## 2 Multiplicity of asymptotically critical points

In this Section, in particular, we want to prove Theorem 1.3.

Hence let us consider a complete Riemannian $C^{1,1}$ manifold $M$ and a sequence $(h_n)_n$ of $C^1$ real functions defined on $M$.

We premise the following Lemmas.

**Lemma 2.1.** *Let $\alpha$ and $\beta$ be real numbers such that $\alpha \le \beta$ and let $A$ and $B$ be open subsets of $M$ such that*

- $A \subset B$,
- $\inf \big\{ d(x,y) \,\big|\, x \in A, y \in M \setminus B \big\} = \delta > 0$,
- $\inf\limits_{n \in \mathbb{N}} \inf \big\{ \|\nabla h_n(u)\| \,\big|\, u \in M \setminus A, \alpha \le h_n(u) \le \beta \big\} > 0.$

*Let $\gamma$ be a number in $[\alpha, \beta]$. Then there exists $\varepsilon > 0$ such that for every $n$ in $\mathbb{N}$*

- *$h_n^{\gamma-\varepsilon}$ is a strong deformation retract of $(h_n^{\gamma+\varepsilon} \setminus B) \cup h_n^{\gamma-\varepsilon}$ if $\alpha < \gamma < \beta$,*
- *$h_n^a$ is a strong deformation retract of $(h_n^{a+\varepsilon} \setminus B) \cup h_n^a$ if $\gamma = \alpha$,*
- *$h_n^{b-\varepsilon}$ is a strong deformation retract of $(h_n^b \setminus B) \cup h_n^{b-\varepsilon}$ if $\gamma = \beta$.*

The proof can be easily obtained from the classical one related to a single functional, by the uniformity with respect to $n$ of the hypothesis made here.

**Lemma 2.2.** *Let $a$ and $b$ be real numbers such that $a \le b$, let $F$ be a closed subset of $M$ which doesn't contain asymptotically critical points for $\big((h_n)_n, h\big)$ with asymptotically critical value in $[a, b]$. Assume $\nabla(h_n, h; a, b)$ holds. Then*

$$\liminf_{n\to\infty} \inf \left\{ \|\nabla h_n(u)\| \;\middle|\; u \in F, a \le h_n(u) \le b \right\} > 0.$$

*Proof.* If, by contradiction,

$$\liminf_{n\to\infty} \inf \left\{ \|\nabla h_n(u)| \;\middle|\; u \in F, a \le h_n(u) \le b \right\} = 0,$$

there would exist a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ and $(u_k)_k$ in $F$ such that $a \le h_{n_k}(u_k) \le b$ and $\nabla h_{n_k}(u_k) \to 0$. By $\nabla(h_n, h; a, b)$ there would exist a subsequence $(u_{k_j})_j$ which converges to a point $u$ in $F$. Evidently (passing to a subsequence), such a point $u$ would be an asymptotically critical point for $\big((h_n)_n, h\big)$ with asymptotically critical level in $[a, b]$. □

From now on, if $c \in \mathbb{R}$, we denote by $Z_c$ the set of asymptotically critical points for $\big((h_n)_n, h\big)$ at level $c$.

**Lemma 2.3.** *Let $a$ and $b$ be real numbers such that $a \le b$ and let $c_1, \dots, c_k$ be the unique asymptotically critical levels for $\big((h_n)_n, h\big)$ in $[a, b]$ and suppose that $a \le c_1 < \dots < c_k \le b$. Let $U_1, \dots, U_k$ be neighborhoods of radius $\delta$ of the sets $Z_{c_1}, \dots, Z_{c_k}$ respectively ($U_i = \big\{x \in M \;\big|\; d(x, Z_{c_i}) < \delta\big\}$). Suppose $\nabla(h_n, h; a, b)$ holds. Then*

$$\limsup_{n\to\infty} \operatorname{cat}_M(h_n^b, h_n^a) \le \sum_{i=1}^{k} \operatorname{cat}_M(U_i).$$

*Proof.*

**I.** For every $i = 1, \dots, k$ let $U_i'$ be the open neighborhood of radius $\delta/2$ of $Z_{c_i}$ and let $a_1, \dots, a_{k+1}$ be real numbers such that

$$a_1 = a \le c_1 < a_2 < \ldots < a_i < c_i < a_{i+1} < \ldots < c_k \le a_{k+1} = b.$$

By Lemma 2.2, for every $i = 1, \dots, k$

$$\liminf_{n\to\infty} \inf \left\{ \|\nabla h_n(u)\| \;\middle|\; u \in M \setminus U_i', a_i \le h_n(u) \le a_{i+1} \right\} > 0,$$

since in the closed set $M \setminus U_i'$ the couple $\big((h_n)_n, h\big)$ has no asymptotically critical points with asymptotically critical value contained in $[a_i, a_{i+1}]$ (it may have some at a level $c_j$ different from $c_i$).

**II.** For every $i+1, \dots, k$, by Lemma 2.1 (applied with $\gamma = c_i$, $\alpha = a_i$ and $\beta = a_{i+1}$) there exist $\overline{n}$ in $\mathbb{N}$ and $\varepsilon > 0$ (we can assume it is the same for every $i = 1, \dots, k$) such that $c_i + \varepsilon \le c_{i+1} - \varepsilon$ for every $i = 1, \dots, k-1$ and for every $n \ge \overline{n}$

- $h_n^{c_i-\varepsilon}$ is a strong deformation retract of $(h_n^{c_i+\varepsilon} \setminus U_i) \cup h_n^{c_i-\varepsilon}$ if $a < c_i < b$;
- $h_n^a$ is a strong deformation retract of $(h_n^{a+\varepsilon} \setminus U_1) \cup h_n^a$ if $c_1 = a$;
- $h_n^{b-\varepsilon}$ is a strong deformation retract of $(h_n^b \setminus U_k) \cup h_n^{b-\varepsilon}$ if $c_k = b$.

**III.** Finally, if $n \ge \overline{n}$, we get

$$\operatorname{cat}_M(h_n^b, h_n^a) \le \sum_{i=2}^{k-1} \operatorname{cat}_M(h_n^{c_i+\varepsilon}, h_n^{c_{i-1}+\varepsilon})$$

$$+\operatorname{cat}_M(h_n^{c_1+\varepsilon}, h_n^a) + \operatorname{cat}_M(h_n^b, h_n^{c_{k-1}+\varepsilon})$$

by **(a)** of **A.3**. The right hand side of this inequality is less or equal to

$$\sum_{i=2}^{k-1} \operatorname{cat}_M(h_n^{c_i+\varepsilon}, h_n^{c_i-\varepsilon}) + \operatorname{cat}_M(h_n^{c_1+\varepsilon}, h_n^{a'}) + \operatorname{cat}_M(h_n^{b'}, h_n^{c_k-\varepsilon})$$

(setting $a' = \max\{a, c_1 - \varepsilon\}$ and $b' = \min\{b, c_k + \varepsilon\}$), since, for example, for $i = 2, \dots, k-1$,

$$\operatorname{cat}_M(h_n^{c_i+\varepsilon}, h_n^{c_{i-1}+\varepsilon}) \le \operatorname{cat}_M(h_n^{c_i+\varepsilon}, h_n^{c_i-\varepsilon}) + \operatorname{cat}_M(h_n^{c_i-\varepsilon}, h_n^{c_{i-1}+\varepsilon}),$$

and $\mathrm{cat}_M(h_n^{c_i-\varepsilon}, h_n^{c_{i-1}+\varepsilon}) = 0$ by Lemma 2.1 and Lemma 2.2.

Then

$$\mathrm{cat}_M(h_n^b, h_n^a) \leq \sum_{i=2}^{k-1} \mathrm{cat}_M\big((h_n^{c_i+\varepsilon} \setminus U_i) \cup h_n^{c_i-\varepsilon}, h_n^{c_i-\varepsilon}\big) + \sum_{i=2}^{k-1} \mathrm{cat}_M(U_i)$$

$$+\mathrm{cat}_M\big((h_n^{c_1+\varepsilon} \setminus U_1) \cup h_n^{a'}, h_n^{a'}\big) + \mathrm{cat}_M(U_1)$$

$$+\mathrm{cat}_M\big((h_n^{b'} \setminus U_k) \cup h_n^{c_k-\varepsilon}, h_n^{c_k-\varepsilon}\big) + \mathrm{cat}_M(U_k) = \sum_{i=1}^{h} \mathrm{cat}_M(U_i)$$

by Lemma 2.1 and **(b)** and **(c)** of **A.3**. □

**Corollary 2.4.** *Suppose $a \leq b$ in $\mathbb{R}$ and assume:*

**(a)** *$M$ is connected and $\big((h_n)_n, h\big)$ has only a finite number of asymptotically critical points with asymptotically critical value in $[a, b]$,*

**(b)** *$\nabla(h_n, h; a, b)$ holds;*

*then the number of asymptotically critical values for $\big((h_n)_n, h\big)$ in $[a, b]$ is greater or equal to*

$$\limsup_{n\to\infty} \mathrm{cat}_M(h_n^b, h_n^a).$$

*In place of* **(a)** *we can, for example, assume that $M$ is contractible.*

*Proof.* We can assume that every neighborhood $U_i$ of the sets $Z_{c_i}$ of the previous Lemma is contractible. □

By Lemma 2.3 we deduce Theorem 1.3.

*Proof of Theorem 1.3.* If $\big((h_n)_n, h\big)$ has only a finite number of asymptotically critical points with asymptotically critical value in $[a, b]$, denoting by $c_1, \ldots, c_k$ the asymptotically critical values for $\big((h_n)_n, h\big)$ in $[a, b]$, there exists neighborhoods $U_i, \ldots, U_k$ of $Z_{c_1}, \ldots, Z_{c_k}$ respectively such that $\mathrm{cat}_M(U_i) =$ cardinality of $Z_{c_i}$. By Lemma 2.3 the thesis follows. □

We want to underline the fact that Definition 1.2 (instead of the modified one recalled in Remark 1.4), lets us extend another classical result to our case.

First of all we can prove the following Proposition.

**Proposition 2.5.** *Let $c$ be in $\mathbb{R}$ and suppose that $\nabla(h_n, h; c)$ holds. Then $Z_c$ is compact.*

*Proof.* Let $(z_i)_i$ be a sequence in $Z_c$. Then $\forall\, i$ there exists a strictly increasing sequence $(n_k^i)_k$ in $\mathbb{N}$ and there exists $(u_k^i)_k$ in $M$ such that $u_k^i \to z_i$, $\nabla h_{n_k^i}(u_k^i) \to 0$ and $h_{n_k^i}(u_k^i) \to h(z_i) = c$.

Then $\forall\, i$ in $\mathbb{N}$ there exists $k_i$ in $\mathbb{N}$ such that

$$d(u_{k_i}^i, z_i) \leq \frac{1}{i}, \qquad \|\nabla h_{n_{k_i}^i}(u_{k_i}^i)\| \leq \frac{1}{i}, \qquad \left| h_{n_{k_i}^i}(u_{k_i}^i) - c \right| \leq \frac{1}{i},$$

where $d$ is the metric on $M$. Since $k_i$ can be taken as big as desired, we can suppose that $n_{k_i}^i < n_{k_{i+1}}^{i+1}$ $\forall i$ in $\mathbb{N}$.

Setting $n_{k_i}^i = m_i$ and $v_i = u_{k_i}^i$, we get:

$$d(v_i, z_i) \leq \frac{1}{i}, \qquad h_{m_i}(v_i) \to c \qquad \text{and} \qquad \nabla h_{m_i}(v_i) \to 0.$$

By $\nabla(h_n, h; c)$ there exists a strictly increasing sequence $(i_j)_j$ in $\mathbb{N}$ and there exists $z$ in $M$ such that $v_{i_j} \to z$ and $h(z) = c$. Then $z \in Z_c$ and $z_{i_j} \to z$. □

We recall that in every paracompact Banach manifold $M$ one can prove that for every closed subset $E$ of $M$ there exists a neighborhood $U$ of $E$ such that $\text{cat}_M(U)$=$\text{cat}_M(E)$.

If $k \in \mathbb{N}$ we set

$$c_k = \inf \left\{ b \in \mathbb{R} \,\middle|\, \limsup_{n \to \infty} \text{cat}_M(h_n^b) \geq k \right\}.$$

Evidently $c_k \leq c_{k+1}$ $\forall k$ in $\mathbb{N}$.

**Theorem 2.6.** *Let $M$ be a complete Riemannian paracompact manifold of class $C^{1,1}$ and suppose that $h_n$ is of class $C^1$ $\forall n$ in $\mathbb{N}$. If there exists $i$ in $\mathbb{N}$ such that $c = c_{k+1} = \ldots = c_{k+i} \in \mathbb{R}$ and if $\nabla(h_n, h; c)$ holds, then $cat_M(Z_c) \geq i$.*

*Proof.* $Z_c$ is compact, so there exists a neighborhood $U$ of $Z_c$ such that $\text{cat}_M(U)$ =$\text{cat}_M(Z_c)$. Moreover, $d(Z_c, M \setminus U) = \delta > 0$.

If $U'$ is the neighborhood of $Z_c$ of radius $\delta/2$, by $\nabla(h_n, h; c)$ we get that there exists $\varepsilon > 0$ and there exists $\overline{n}$ in $\mathbb{N}$ such that

$$\inf_{n \geq \overline{n}} \inf \left\{ \|\nabla h_n(u)\| \,\middle|\, u \in M \setminus U', c - \varepsilon \leq h_n(u) \leq c + \varepsilon \right\} > 0.$$

By Lemma 2.1 (possibly taking a smaller $\varepsilon$), we get that $\forall\, n \geq \overline{n}$

$$h_n^{c-\varepsilon} \text{ is a strong deformation retract of } (h_n^{c+\varepsilon} \setminus U) \cup h_n^{c-\varepsilon}.$$

Then, $\forall\, n \geq \overline{n}$

$$\mathrm{cat}_M(h_n^{c+\varepsilon} \leq \mathrm{cat}_M(h_n^{c-\varepsilon}) + \mathrm{cat}_M(U).$$

But $\mathrm{cat}_M(h_n^{c+\varepsilon}) \geq k+i$ for infinitely many $n$ ($c+\varepsilon > c_{k+1}$), $\mathrm{cat}_M(h_n^{c-\varepsilon}) \leq k$ definitely ($c - \varepsilon < c_{k+1}$) and $\mathrm{cat}_M(U)$ =$\mathrm{cat}_M(Z_c)$. The thesis follows. $\square$

# A Some recalls on relative category

As we said, for the readers' convenience here we recall the version of relative category that we used in the previous sections. For example see [1], [2], [5], [6], [7] and [11].

Let $X$ be a topological space and let $A$ and $B$ be two subsets of $X$. We first recall the following definition.

**Definition A.1.** We say that $B$ is a strong deformation retract of $A$ in $X$ if

- $B \subset A$;
- there exists $h : [0,1] \times A \longrightarrow X$ continuous and such that $h(0,u) = u$ $\forall\, u$ in $A$, $h(1,u) \in B$ $\forall\, u$ in $A$, $h(t,u) = u$ $\forall\, (t,u)$ in $[0,1] \times B$.

**Definition A.2.** If $B \subset A$, we say that the *relative category of $A$ with respect to $B$ in $X$* is $k$, and we write $\mathrm{cat}_X(A,B)$, if $k$ is the least integer such that there exist $k+1$ closed subsets $A_0, A_1, \dots, A_k$ of $X$ such that

- $A \subset \bigcup_{i=0}^{k} A_i$,
- $A_1, \dots, A_k$ are contractible in $X$,
- $B$ is a strong deformation retract of $A_0$.

If no such integer exists, we set $\mathrm{cat}_X(A,B) = +\infty$.

Note that, if $B = \emptyset$, then $\text{cat}_X(A, B) = \text{cat}_X(A)$, where $\text{cat}_X(A)$ denotes the classical category of Lusternik and Schnirelman.

For the proof of Theorem 1.3 we only need the following properties of the relative category.

**Properties A.3.**

**(a)** If $C \subset B \subset A \subset X$, then

$$\text{cat}_X(A, C) \leq \text{cat}_X(A, B) + \text{cat}_X(B, C).$$

**(b)** If $A \subset A' \subset X$, $A'$ is closed and $A$ is a strong deformation retract of $A'$, then $\text{cat}_X(A', A) = 0$.

**(c)** If $C \subset A \subset X$, $B \subset X$, then

$$\text{cat}_X(A \cup B, C) \leq \text{cat}_X(A, B) + \text{cat}_X(B).$$

It is interesting to note that from **(a)** of **A.3** we get (putting $C = \emptyset$) that, if $B \subset A \subset X$, then $\text{cat}_X(A, B) \geq \text{cat}_X(A) - \text{cat}_X(B)$.

**Remark A.4.** We also note the following interesting properties.

**(a)** If $C \subset B \subset A \subset X$, then $\text{cat}_X(B, C) \leq \text{cat}_X(A, C)$. If $B$ is a strong deformation retract of $A$, then equality holds.

**(b)** If $C \subset B \subset A \subset X$, $C$ is a strong deformation retract of $B$, then $\text{cat}_X(A, C) \leq \text{cat}_X(A, B)$.

Concerning the relations between the relative category and the category of Lusternik and Schnirelmann, we also note that if $B$ is a closed subset of $X$ contained in $A$, then

$$\text{cat}_X(A, B) \leq \text{cat}_X(A \setminus B)\ (\leq \text{cat}_X(A)).$$

It is also well known that by **A.3** and the usual variational techniques, one can prove the following Theorem, which is a particular case of Theorem 1.3.

**Theorem A.3.** *Let $M$ be a complete $C^{1,1}$ manifold and let $f : M \longrightarrow \mathbb{R}$ be a $C^1$ function. Suppose $a < b$ in $\mathbb{R}$ and $f$ satisfies $(PS)_c$ for every $c$ in $[a, b]$. Then $f$ has at least $\text{cat}_M(f^b, f^a)$ critical points in $f^{-1}([a, b])$.*

See [7].

## References


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