A Lagrangian approach to scalar conservation laws

Stefano Bianchini and Elio Marconi

Abstract We provide an informal presentation of the work mainly contained in [3]. We consider the entropy solution u of a scalar conservation law in one-space dimension. In particular we prove that the entropy dissipation is a measure concentrated on countably many Lipschitz curves. This follows from a detailed analysis of the structure of the characteristics. We will introduce a few notions of Lagrangian representations and we prove that characteristics are segments outside a countably 1-rectifiable set.
MSC: 35L65.

Key words: Conservation laws, entropy solutions, shocks, concentration, lagrangian representation

1 Introduction

We are interested in the structure of the entropy solution u to the scalar conservation law in one space dimension

$$u_t + f(u)_x = 0, \qquad f : \mathbb{R} \to \mathbb{R} \text{ smooth}, \tag{1}$$

with initial datum $u_0(x) \in L^\infty(\mathbb{R})$. Being an entropy solution, by definition for all convex entropies η it holds in the sense of distributions

$$\eta(u)_t + q(u)_x \leq 0, \tag{2}$$

Stefano Bianchini
SISSA, via Bonomea, 265, 34136, Trieste (Italy), e-mail: bianchin@sissa.it

Elio Marconi
SISSA, via Bonomea, 265, 34136, Trieste (Italy), e-mail: emarconi@sissa.it

where $q'(u) = f'(u)\eta'(u)$ is the entropy flux. In particular the r.h.s. of (2) is a non-positive locally bounded measure μ. Moreover, being the divergence of an L^∞ vector field, $\mu \ll \mathscr{H}^1$.

For BV solutions, if we denote by J the jump set of u, by Volpert's formula it holds

$$\eta(u)_t + q(u)_x = \eta'(u)\big(D_t^{\text{cont}}u + f'(u)D_x^{\text{cont}}u\big) + \mu \llcorner J = \mu \llcorner J,$$

where $D^{\text{cont}}u = (D_t^{\text{cont}}u, D_x^{\text{cont}}u)$ is the continuous part of the measure Du.

This argument immediately applies when the initial datum has bounded variation because $u_0 \in \mathrm{BV}(\mathbb{R}) \Rightarrow u \in \mathrm{BV}_{\text{loc}}([0+,\infty)\times\mathbb{R})$ and in the case of uniformly convex flux f with general $u_0 \in L^\infty(\mathbb{R})$. In fact by Oleinik estimate [8]

$$f'' \geq c > 0 \Rightarrow u \in \mathrm{BV}_{\text{loc}}((0+,\infty)\times\mathbb{R}).$$

If the flux f has finitely many inflection points (together with an additional regularity assumption on f around each inflection point) it has been proved in [6] that $f' \circ u \in \mathrm{BV}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$ and in [7] that μ is concentrated on the jump set of $f' \circ u$.

The main result of this presentation is the following.

Theorem 1. *There exists a 1-rectifiable set J such that for every entropy η the dissipation measure μ is concentrated on J.*

The flux f is only supposed to be smooth. The result is a consequence of a description of the structure of the solution u, in particular on the behavior of its characteristics.

Here Lagrangian representation means an extension of the method of characteristics. In the case of solutions with bounded variation a first formulation appears in [5], then it has been extended to systems in [4]. In this paper we present a suitable version to deal with the case of L^∞ solutions introduced in [3]. In all the cited works the strategy is to exhibit a Lagrangian representation for a dense set of solutions, or approximate solutions and pass it to the limit.

It turns out that it is possible to decompose the half-plane $\mathbb{R}^+ \times \mathbb{R} = A \cup B \cup C$, where

1. A is countably 1-rectifiable,
2. B is open and $u \llcorner B \in \mathrm{BV}_{\text{loc}}$,
3. C is the union of segments starting from 0 on which the solution u is constant. Moreover the slope of the segments is given by the characteristic speed $f'(u)$.

In order to conclude the proof of Theorem 1 it is sufficient to analyze $\mu \llcorner C$. In Section 3 we will see how this structure allows to compute μ by exploiting the balance of u and $\eta(u)$ in the regions delimited by these segments.

The most important tools and ideas of the proof of Theorem 1 are presented but details are often omitted or presented in a simplified setting to reduce technicalities to the essential ones. When it is not indicated where to find the details, we implicitly refer to [3].

2 Lagrangian representation and structure of the solution

In this section we introduce the notion of Lagrangian representation for bounded entropy solutions. In order to motivate Proposition 2 we present two previous formulations in the particular cases of $u_0 \in \mathrm{BV}$ and u_0 continuous. Once the Lagrangian representation, in the form of family of admissible boundaries, is available we present how it is possible to deduce a result on the structure of the solution.

2.1 Lagrangian representation

Consider as a motivation the case of a smooth solution: applying the chain rule we have

$$\dot{\gamma}(t) = f'(u(t,\gamma(t))) \qquad \Rightarrow \qquad \frac{d}{dt}u(t,\gamma(t)) = u_t + f'(u)u_x = 0$$

i.e. u is constant along the characteristic γ which is therefore a straight line. So we introduce a flow $\mathtt{X} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ where $\mathtt{X}(t,y)$ denotes the position of the characteristic starting from y at time t. We say that $(\mathtt{X}, u_0)$ represents the solution in the sense that

$$u(t,x) = u_0(\mathtt{X}(t)^{-1}(x)).$$

Let $v = u_x$, differentiating formally (1) with respect to x we get

$$v_t + (f'(u)v)_x = 0.$$

Since $\mathtt{X}$ is the flow relative to the vector field $f'(u)$ it holds

$$v(t) = \mathtt{X}(t)_\sharp(v_0 \mathscr{L}^1), \qquad \text{where } v_0 = (u_0)_x.$$

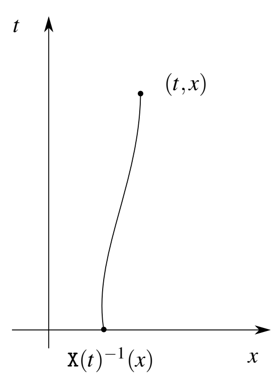

Fig. 1 The solution u at the point (t,x) is determined by the initial datum u_0 at the point $\mathtt{X}(t)^{-1}(x)$, i.e. the starting point of the characteristic passing through (t,x).

In particular for every $\varphi \in C_c^\infty(\mathbb{R})$

$$\int_{\mathbb{R}} u(t,x)\varphi'(x)dx = -\int_{\mathbb{R}} \varphi(\mathtt{X}(t,y))v_0(y)dy = \int_{\mathbb{R}} u_0(y)D_y(\varphi(\mathtt{X}(t,y)))dy. \tag{3}$$

The regularity of $\mathtt{X}$ that you have for free by compactness is monotonicity with respect to y and Lipschitz dependence on time. Moreover, in order to represent a rarefaction, it is convenient to renounce the usual assumption, in the linear case, $\mathtt{X}(0,\cdot) = \mathrm{Id}$. This is reflected in the fact that we need an auxiliary function $\mathtt{u}$ instead of u_0 in (3).

Definition 1. A Lagrangian representation is a pair $(\mathtt{X},\mathtt{u})$ such that

1. $\mathtt{X} : \mathbb{R}^+_t \times \mathbb{R}_y \to \mathbb{R}$ is Lipschitz with respect to t and non decreasing with respect to y;
2. $\mathtt{u} : \mathbb{R} \to \mathbb{R}$ is continuous;
3. for every $\varphi \in C_c^\infty(\mathbb{R})$

$$\int_{\mathbb{R}} u(t,x)\varphi'(x)dx = \int_{\mathbb{R}} \mathtt{u}(y)dD_y(\varphi \circ \mathtt{X}(t))(y). \tag{4}$$

Since $\mathtt{X}(t)$ is monotone for every t, the derivative in the sense of distributions $D_y(\varphi \circ \mathtt{X}(t))$ is a Radon measure and the integral on the r.h.s. of (4) is well defined.

We want to prove the existence of a Lagrangian representation in a dense class of solutions and obtain it for a general solution by approximation. We refer to [5] to see how it is possible to construct a Lagrangian representation for solutions with $u_0 \in \mathrm{BV}$ starting from wave-front tracking approximations and we discuss in which cases we can pass to the limit in the representation formula (4).

Suppose $(\mathtt{X}^n,\mathtt{u}^n)$ are a family of Lagrangian representations for the solutions u^n with initial datum u_0^n. By Kružkov inequality

$$u_0^n \to u_0 \text{ s-}L^1 \qquad \Rightarrow \qquad \lim_{n\to\infty} \int_{\mathbb{R}} u^n(t,x)\varphi'(x)dx = \int_{\mathbb{R}} u(t,x)\varphi'(x)dx.$$

Since $\mathtt{X}^n$ can be constructed equi-bounded on compact sets and equi-Lipschitz with respect to y, up to subsequences,

$$D_y(\varphi \circ \mathtt{X}^n(t)) \rightharpoonup D_y(\varphi \circ \mathtt{X}(t))$$

as Radon measures. Therefore, in order to pass to the limit in the r.h.s. of (4), we need $\mathtt{u}^n \to \mathtt{u}$ uniformly. In particular it can be done when u_0 is continuous.

Proposition 1. *Every bounded entropy solution with continuous initial datum has a Lagrangian representation.*

This result is obtained in [2], where it is also shown how it is possible to deduce the rectifiability of the entropy dissipation measures μ_η. However it is hopeless to represent the entropy solution with $u_0 \in L^\infty$ with a continuous $\mathtt{u}$. Therefore we look for a more stable interpretation of Lagrangian representation. It can be proved in the BV setting that a characteristic γ with value w is an admissible boundary for the solution u in the following sense:

Definition 2. Let $\gamma : [0,+\infty) \to \mathbb{R}$ be a Lipschitz curve, $w \in \mathbb{R}$ and u be an entropy solution of (1). Denote by

$$\Omega^- = \{(t,x) \in (0,T) \times \mathbb{R} : x < \gamma(t)\}, \qquad \Omega^+ = \{(t,x) \in (0,T) \times \mathbb{R} : x > \gamma(t)\}.$$

Moreover let u^- be the solution of (1) in Ω^- with initial condition $u_0 \llcorner \{x < \gamma(0)\}$ and boundary datum constant equal to w on $\{(t,\gamma(t)) : t \in (0,T)\}$ and similarly for u^+. We say that (γ, w) is an *admissible boundary* in $(0,T)$ for u if

$$u^- = u \llcorner \Omega^- \qquad \text{and} \qquad u^+ = u \llcorner \Omega^+.$$

The notion of solution for the initial boundary value problem for scalar conservation laws has been introduced in [1] in the BV setting. For a more general treatment see [9].

Arguing by approximation, for example by wave-front tracking, we get the following result for solutions u with bounded variations.

Proposition 2. *There exists a family $\mathscr{K}$ of admissible boundaries (γ,w) for u and a function $T : \mathscr{K} \to \mathbb{R}^+ \cup \{+\infty\}$ such that the following hold.*

1. *For every $(\gamma,w), (\gamma',w') \in \mathscr{K}$*

$$\gamma(t) \le \gamma'(t)\ \forall t > 0 \qquad or \qquad \gamma'(t) \le \gamma(t)\ \forall t > 0.$$

   In particular the set $\mathscr{K}_\gamma = \{\gamma : \exists w((\gamma,w) \in \mathscr{K})\}$ is ordered.
2. *For every $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ and every $w \in \mathrm{conv}(u(t,x-),u(t,x+))$ there exists an admissible boundary $(\gamma,w) \in \mathscr{K}$ with $T(\gamma,w) \ge t$.*
3. *For every $(\gamma,w) \in \mathscr{K}$ and $t < T(\gamma,w)$,*

$$w \in \mathrm{conv}(u(t,\gamma(t)-),u(t,\gamma(t)+)).$$

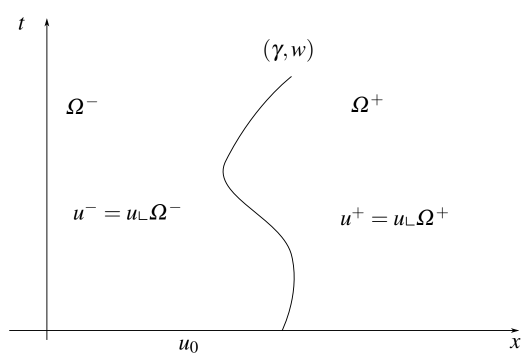

Fig. 2 Interpretation of a characteristic as an admissible boundary.

4. *The characteristic equation holds: for every $\gamma \in \mathscr{K}_\gamma$, for $\mathscr{L}^1$-a.e. $t > 0$*

$$\dot{\gamma}(t) = \begin{cases} f'(u(t,\gamma(t))) & \text{if } u(t) \text{ is continuous at } \gamma(t), \\ \dfrac{f(u(t,\gamma(t)+)) - f(u(t,\gamma(t)-))}{u(t,\gamma(t)+) - u(t,\gamma(t)-)} & \text{if } u(t) \text{ has a jump at } \gamma(t). \end{cases}$$

Cancellations occur in scalar conservation laws, the function T is introduced to take into account this phenomenon: $T(\gamma, w)$ denotes the time when the value w is canceled along γ.

In the next lemma we state the stability property that we need to pass to the limit in this formulation.

Lemma 1. *Let (γ^n, w^n) be admissible boundaries for entropy solutions u^n of (1) and assume that*

1. *$\gamma^n \to \gamma$ uniformly;*
2. *$w^n \to w$;*
3. *$u^n \to u$ strongly in $L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$.*

Then (γ, w) is an admissible boundary for u.

We can approximate $u_0 \in L^\infty(\mathbb{R})$ with a sequence $u_0^n \in \mathrm{BV}_{\text{loc}}$ with respect to the strong L^1_{loc} topology. As we already observed it implies the convergence of the relative solutions u^n to u in L^1_{loc}. Since the the curves $\gamma^n \in \mathscr{K}_\gamma^n$ satisfy the characteristic equation, they are equi-Lipschitz. So we have the compactness required to apply Lemma 1.

What we get in the limit is a priori much less than a representation as in Proposition 2. Monotonicity passes to the limit and we still have enough boundaries to cover the graph of u. Actually the set $\mathscr{K}$ of all the limit points of sequences of admissible boundaries is such that

$$\text{Graph } u \subset U \subset \{(t, \gamma(t), w) : (\gamma, w) \in \mathscr{K} \text{ and } T(\gamma, w) \geq t\},$$

where U is the Kuratowski limit of the sequence of the graphs of u_n. The first inclusion above can be strict and in general U does not identify a unique $u \in L^\infty$, but we will see in the next section that it does up to linearly degenerate components of the flux f, i.e. intervals where $f'' = 0$.

2.2 Structure of the solution

In this section we see that, being admissible boundaries of an entropy solution u, the elements of $\mathscr{K}$ enjoy some additional structure.

Let $(\bar{t}, \bar{x}) \in \mathbb{R}^+ \times \mathbb{R}$ and $\bar{\gamma} \in \mathscr{K}_\gamma$ be such that $\bar{\gamma}(\bar{t}) = \bar{x}$. We distinguish three situations, see Figure 3.

1. There exists $\gamma \in \mathscr{K}_\gamma$ and $t' < \bar{t}$ such that $\bar{\gamma}(\bar{t}) < \gamma(\bar{t})$ and $\bar{\gamma}(t') = \gamma(t')$.

2. Condition 1 does not holds and for every (x^n) convergent to $\bar{x}$ with $x^n > \bar{x}$ and $\gamma^n \in \mathscr{K}_\gamma$ with $\gamma^n(\bar{t}) = x^n$, γ^n converges uniformly to $\bar{\gamma}$ in $[0,\bar{t}]$.
3. Conditions 1 and 2 do not hold.

It is not difficult to prove that the set of points for which conditions 1 and 2 do not hold is contained in the graphs of countably many Lipschitz curves in $\mathscr{K}_\gamma$. In the next two lemmas we consider the first two cases.

Lemma 2. *Let $\bar{\gamma}$ and γ be as in Case 1 above. Then the solution u is monotone with respect to x in the region delimited by the two curves:*

$$\Omega = \{(t,x) \in (t',\bar{t}) \times \mathbb{R} : \bar{\gamma}(t) < x < \gamma(t)\}.$$

In the following lemma the linear degeneracy of the flux plays a role so we introduce the following notation: denote by $\mathscr{L}_f$ the set of maximal closed intervals (eventually singletons) on which f' is constant.

Lemma 3. *Let x^n and γ^n be as in Case 2 above and let w^n be the corresponding values. Then there exists $I \in \mathscr{L}_f$ such that*

$$\lim_{n\to\infty} \operatorname{dist}(w^n, I) = 0 \qquad \textit{and} \qquad \forall t \in (0,\bar{t})\big(\dot{\bar{\gamma}}(t) = f'(I)\big),$$

where $f'(I)$ denotes $f'(w)$ for one hence any $w \in I$. In particular $\bar{\gamma}\llcorner(0,\bar{t})$ is a segment.

From Lemma 2 and 3 it follows the announced decomposition $\mathbb{R}^+ \times \mathbb{R} = A \cup B \cup C$ where

1. A is contained in the union of countably many graphs of curves in $\mathscr{K}_\gamma$.
2. B is open and $u\llcorner B \in \mathrm{BV}_{\mathrm{loc}}$.
3. C is the union of segments starting from 0 with characteristic speed.

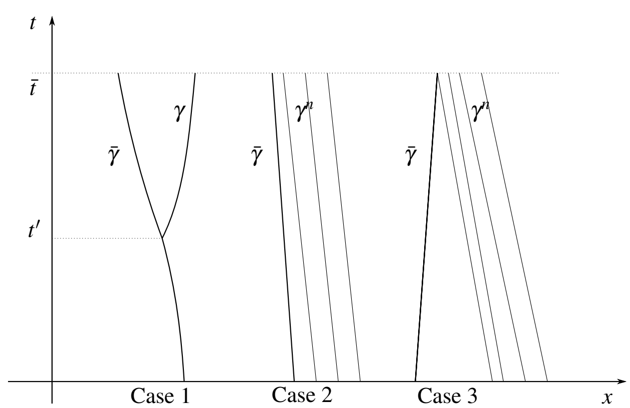

Fig. 3 The three possibilities for a point $(\bar{t},\bar{\gamma}(\bar{t}))$.

See figure 4. Moreover from the structure of the characteristics we can deduce a result on the structure of the solution u: it is continuous at every point except on countably many Lipschitz curves where it has jump type discontinuities. Everything holds up to linearly degenerate components of the flux.

To be more precise consider $\gamma \in \mathscr{K}_\gamma$, a differentiability point $\bar t$ of γ and $r, \delta > 0$ and let

$$B^{\delta+}_{\bar t,\gamma}(r) := \Big\{(t,x) \in B_{\bar t,\gamma(\bar t)}(r) : x > \gamma(\bar t) + \dot\gamma(\bar t)(t-\bar t) + \delta|t-\bar t|\Big\},$$

$$B^{\delta-}_{\bar t,\gamma}(r) := \Big\{(t,x) \in B_{\bar t,\gamma(\bar t)}(r) : x < \gamma(\bar t) + \dot\gamma(\bar t)(t-\bar t) - \delta|t-\bar t|\Big\}.$$

Accordingly we define

$$U^{\delta\pm}_{\bar t,\bar\gamma}(r) := \Big\{w \in \mathbb{R} : \exists t \in \mathbb{R}^+, (\gamma,w) \in \mathscr{K} \text{ such that } T(\gamma,w) > t, (t,\gamma(t)) \in B^{\delta\pm}_{\bar t,\bar\gamma}(r)\Big\}.$$

Proposition 3. *There exist J contained in countably many curves in $\mathscr{K}_\gamma$ and a representative of u such that*

1. *For every $(\bar t,\bar x) \in \mathbb{R}^+ \times \mathbb{R} \setminus J$ there exists $I \in \mathscr{L}_f$ such that for every $\varepsilon > 0$ there exists $r > 0$ for which*

$$\max_{(t,x)\in B_r(\bar t,\bar x)} \mathrm{dist}(u(t,x), I) < \varepsilon.$$

2. *For every $\gamma \in \mathscr{K}_\gamma$, for $\mathscr{L}^1$-a.e. $t > 0$, there exist $I^+, I^- \in \mathscr{L}_f$ such that*

$$\forall \delta > 0\, \forall \varepsilon > 0\, \exists r > 0 \left(U^{\delta\pm}_{t,\gamma}(r) \subset I^\pm + (-\varepsilon,\varepsilon)\right).$$

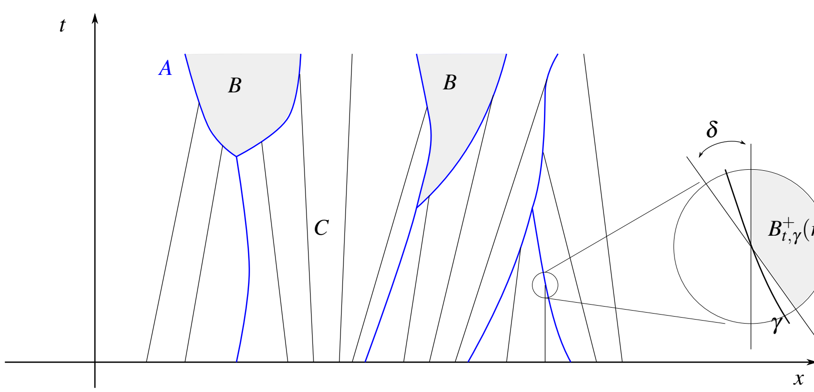

Fig. 4 The partition of the half-plane.

3 Concentration of entropy dissipation

Here we take advantage of the structure of the solution obtained in Proposition 3 to prove Theorem 1. We consider entropies η such that $\eta(0) = 0$ so that there exists a constant $L > 0$ for which $|\eta(u)| \leq L|u|$. This is not a restrictive assumption since

$$\mu_\eta = \mu_{\eta - \eta(0)}.$$

As we already observed in the introduction, it is sufficient to consider $\mu \llcorner C$.

Fix a positive time T. In order to avoid some technicalities we present the proof of Theorem 1 assuming that for every $x \in \mathbb{R}$ the point $(T,x) \in C$. Even if it is not trivial, it is just a technical issue to implement the following argument in the real setting proving that $\mu \llcorner C_T$ is concentrated on countably many characteristic segments, where

$$C_T := \{(t,\gamma(t)) \in [0,T] \times \mathbb{R} : \gamma \in \mathscr{K}_\gamma, (T,\gamma(T)) \in C\}.$$

By Lemma 3 it follows that each $\gamma \in \mathscr{K}_\gamma$ restricted to $[0,T]$ is a segment. Let $\varepsilon > 0$ be such that $2\varepsilon < T$. We parametrize the characteristic segments by their position y at time ε, i.e. $\gamma_y(\varepsilon) = y$. By Lemma 3 it also follows that for every $y \in \mathbb{R}$ there exist $I^-(y), I^+(y) \in \mathscr{L}_f$ such that the limits of admissible boundaries from the left and the right of γ_y are contained in $I^-(y)$ and $I^+(y)$ respectively. Moreover it is not difficult to prove that $I^-(y) = I^+(y) =: I(y)$ except at most countably many points. Finally for $\mathscr{L}^1$-a.e. y there exists $U(y) \in I(y)$ such that $u(t,\gamma_y(t)) = U(y)$ for $\mathscr{L}^1$-a.e. $t \in (0,T)$.

Let $P : [0,T] \times \mathbb{R} \to \mathbb{R}$ be such that $P(t,x) = y$ where $\gamma_y(t) = x$. The goal is to prove that $m := P_\sharp \mu$ is atomic. This immediately implies that μ is concentrated on at most countably many segments and concludes the proof of Theorem 1. The idea is to compute the balances

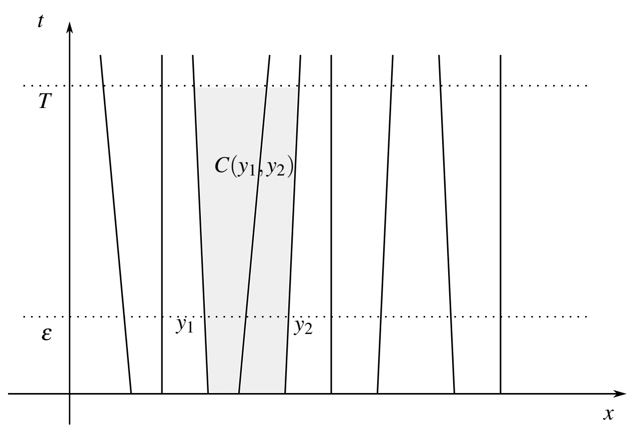

Fig. 5 A model set of segments parametrized by their position at time ε and a cylinder.

$$u_t + f(u)_x = 0, \qquad \eta(u)_t + q(u)_x = \mu_\eta$$

for the conserved quantity u and the entropy $\eta(u)$ on cylindrical regions of the form

$$C(y_1,y_2) = \{(t,x) \in (0,T) \times \mathbb{R} : \gamma_{y_1}(t) < x < \gamma_{y_2}(t)\}.$$

The fluxes F and Q of u and $\eta(u)$ respectively across γ_y per unit time are given by

$$F(y) = f(U(y)) - \lambda(y)U(y), \qquad Q(y) = q(U(y)) - \lambda(y)\eta(U(y)),$$

where $\lambda(y) = f'(U(y))$ is the slope of the segment γ_y. The crucial point is that they do not depend on time. Observe that since the segments do not intersect in $(0,T)$ thanks to monotonicity of the family of boundaries, the speed $\lambda(y)$ is $1/\varepsilon$-Lipschitz. Therefore the balance for $\eta(u)$ in $C(y_1,y_2)$ gives

$$\int_{\gamma_{y_1}(T)}^{\gamma_{y_2}(T)} \eta(u(T,x))dx - \int_{\gamma_{y_1}(0)}^{\gamma_{y_2}(0)} \eta(u_0(x))dx + T(Q(y_2) - Q(y_1)) = m((y_1,y_2)).$$

This implies that $Q \in \mathrm{BV}$ and

$$D_y Q = -\lambda'(y)\eta(U(y))\mathscr{L}^1 + \frac{m}{T}. \tag{5}$$

In particular F is Lipschitz and for $\mathscr{L}^1$-a.e. $y \in \mathbb{R}$

$$F'(y) = -\lambda'(y)U(y).$$

Notice that this is the chain rule corresponding to $(f(u) - f'(u)u)_y = -(f'(u))_y u$.

The following general lemma links the flux F to the flux Q. See Figure 6.

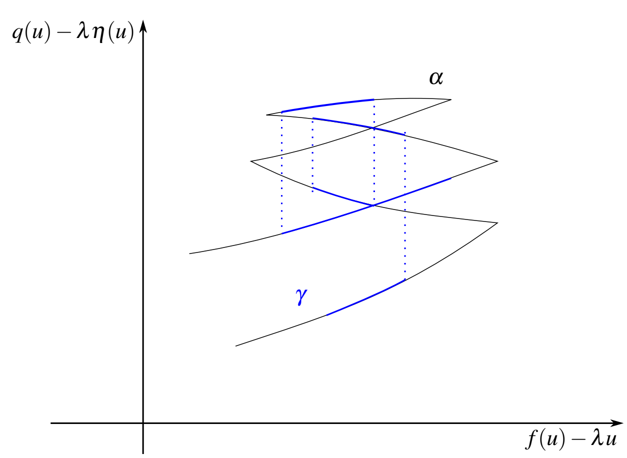

Fig. 6 The illustration of Lemma 4. Since the slope of α is bounded and the first projection of γ is Lipschitz, then $\gamma^2 \in \mathrm{SBV}$ and $\mathscr{L}^1$-a.e. the slope of γ coincides with the slope of α.

Lemma 4. *Let $\alpha = (\alpha^1, \alpha^2) : [-M,M] \to \mathbb{R}^2$ be a smooth curve such that there exists a constant $L > 0$ for which for every $w \in [-M,M]$ it holds $|\dot\alpha^2(w)| \le L|\dot\alpha^1(w)|$. Let $\gamma = (\gamma^1, \gamma^2) : \mathbb{R} \to \mathbb{R}^2$ be such that γ^1 is Lipschitz, γ^2 has bounded variation and* $\operatorname{Im}\gamma \subset \operatorname{Im}\alpha$. *Then $D\gamma^2$ has no Cantor part and for $\mathscr{L}^1$-a.e. $y \in \mathbb{R}$*

$$(\gamma^2)'(y)(\alpha^1)'(w(y)) = (\gamma^1)'(y)(\alpha^2)'(w(y)),$$

for some $w(y)$ such that $\gamma(y) = \alpha(w(y))$.

We can apply Lemma 4 to the curves

$$\alpha(w) = \begin{pmatrix} f(w) - f'(w)w \\ q(w) - f'(w)\eta(w) \end{pmatrix}, \qquad \gamma(y) = \begin{pmatrix} F(y) \\ Q(y) \end{pmatrix}$$

in fact

$$\alpha'(w) = \begin{pmatrix} -f''(w)w \\ -f''(w)\eta(w) \end{pmatrix}$$

and by assumption $|\eta(w)| \le L|w|$. So we get that D_yQ has no Cantor part and for $\mathscr{L}^1$-a.e. $y \in \mathbb{R}$

$$Q'(y) = -\lambda'(y)\eta(U(y)).$$

Therefore, comparing with (5), we get that m is atomic.

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