A sphere theorem on locally conformally flat even-dimensional manifolds

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ABSTRACT. In this paper, we prove that a closed even-dimensional manifold which is locally conformally flat with positive scalar curvature, positive Euler characteristic and which satisfies some additional condition on its curvature is diffeomorphic to the sphere or projective space.

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1 Introduction

A compact surface (Σ, g) with positive scalar curvature must have Euler-Poincaré characteristic $\chi(\Sigma) > 0$ by the Gauss-Bonnet formula. Then, by the classification of surfaces, Σ is either (up to diffeomorphisms) $\mathbb{S}^2$ or $\mathbb{RP}^2$, and then one can conclude using the classical uniformization theorem that the metric g is conformal to the canonical metric of constant curvature. Since this result uses the Gauss-Bonnet formula in a fundamental manner (say as a bridge between topological and geometrical informations), the situation in higher dimension should be more complicated. Nevertheless, due to M. Gursky and independently by E. Hebey and M. Vaugon in the four dimensional case, and remembering that surfaces are locally conformally flat, one can prove the following

Theorem 1.1 (Gursky, [13] and Hebey–Vaugon, [15] and [16]). *Let (M, g) be a closed, locally conformally flat, n-dimensional Riemannian manifold, $n = 4$ or 6, with non-negative scalar curvature. Then $\chi(M) \leq 2$. Furthermore, $\chi(M) = 2$ if and only if (M, g) is conformally diffeomorphic to the standard sphere, and $\chi(M) = 1$ if and only if (M, g) is conformally diffeomorphic to the standard real projective space.*

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As pointed out by Gursky, this result is not true for higher dimensions. Take for example the product of $\mathbb{S}^4$ equipped with the canonical metric and a four dimensional hyperbolic space form. So, in view of generalizing the classification result of Gursky in higher dimensions, one have to add some additional condition on the geometry of the manifold.
In order to state such a result we need to introduce some notations. Consider (M, g), a compact, smooth, n-dimensional Riemannian manifold without boundary. Given a section A of the bundle of symmetric two tensors, we can use the metric to raise an index and view A as a tensor of type $(1,1)$, or equivalently as a section of $End(TM)$. This allows us to define $\sigma_k(g^{-1}A)$ the k-th elementary function of the eigenvalues of $g^{-1}A$, namely, if we denote by $\lambda_1, \ldots, \lambda_n$ these eigenvalues

$$\sigma_k(g^{-1}A) = \sum_{1\le i_1<\cdots<i_k\le n} \lambda_{i_1}\ldots\lambda_{i_k}.$$

In this paper we choose the tensor (here t is a real number)

$$A_g^t = \frac{1}{n-2}\left(Ric_g - \frac{t}{2(n-1)}R_g g\right),$$

where Ric_g and R_g denote the Ricci and the scalar curvature of g respectively. Note that for $t = 1$, A_g^1 is the classical Schouten tensor $A_g^1 = \frac{1}{n-2}\left(Ric_g - \frac{1}{2(n-1)}R_g g\right)$ (see [1]). Hence, with our notations, $\sigma_k(g^{-1}A_g^t)$ denotes the k-th elementary symmetric function of the eigenvalues of $g^{-1}A_g^t$. We call Γ_k^+ the cone defined by

$$\Gamma_k^+ := \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n / \sigma_1(\lambda_1, \ldots, \lambda_n) > 0, \ldots, \sigma_k(\lambda_1, \ldots, \lambda_n) > 0\}.$$

We say that for some t, A_g^t is in Γ_k^+, if the eigenvalues of A_g^t are in Γ_k^+.

A Gauss-Bonnet type formula was proved by Viaclovsky in [18] for locally conformally flat manifolds, which relates the classical Gauss-Bonnet-Chern to the integral of the $\frac{n}{2}$-th elementary function of the eigenvalues of the Schouten tensor. More precisely we have

$$\frac{1}{(2\pi)^{(n-1)}\left((n-2)!!\right)^2}\int_M \sigma_{\frac{n}{2}}(g^{-1}A_g)\, dV_g = \chi(M),$$

where $\chi(M)$ denotes the Euler characteristic of M.

Our main result is the following:

Theorem 1.2. *Let (M, g) be a closed, locally conformally flat, n–dimensional Riemannian manifold, $n \ge 8$ even, with positive scalar curvature and with positive Euler–Poincaré characteristic.*
There exists a constant $t_0 = t_0(n, diam(M, g), \|\nabla^2 Rm\|) < 1$ such that, if

$$A_g^t \in \Gamma_{\frac{n}{2}}^+,$$

for some $t \in [t_0, 1]$, then there exists a metric $\tilde{g}$ conformal to g such that $A_{\tilde{g}}^1 \in \Gamma_{\frac{n}{2}}^+$. In particular $(M, \tilde{g})$ has non–negative Ricci curvature ($Ric_{\tilde{g}} \ge 0$).

Note that, since the manifold is assumed to be locally conformally flat, we can also write that $t_0 = t_0(n, diam(M, g), \|\nabla^2 Ric\|)$.

Using a classification result for compact, locally conformally flat manifolds with nonnegative Ricci curvature and a vanishing result, we can prove the following classification result

Theorem 1.3. *Let (M, g) be a closed, locally conformally flat, n–dimensional Riemannian manifold, $n \ge 8$ even, with positive scalar curvature and with positive Euler–Poincaré characteristic.*
There exists a constant $t_0 = t_0(n, diam(M, g), \|\nabla^2 Ric\|) < 1$ such that if

$$A_g^t \in \Gamma_{\frac{n}{2}}^+,$$

for some $t \in [t_0, 1]$, then M is diffeomorphic to either $\mathbb{S}^n$ or $\mathbb{RP}^n$.

Remark 1.4. *The assumption that there exists a constant* $t_0 = t_0(n, diam(M,g), \|\nabla^2 Ric\|) < 1$ *such that*

$$A_g^t \in \Gamma^+_{\frac{n}{2}},$$

for some $t \in [t_0, 1]$*, by the Guan–Viaclovsky–Wang inequality (see [12]), does not imply that the metric* g *has non–negative Ricci curvature. This would be true, if one could get* $t_0 = 1$*.*

We need to point out that there is another result in the same direction due to Guan-Lin-Wang [10], where they proved, as a corollary of a more general result, that if (M,g) is a locally conformally flat manifold of even dimension, with positive Euler-Poincaré characteristic and with $A_g^1 \in \Gamma_{\frac{n}{2}-1}$, then M is diffeomorphic to either $\mathbb{S}^n$ or $\mathbb{RP}^n$. Also in this case, there isn't a relation between their and our assumption.

2 Ellipticity

Following [14], we will discuss the ellipticity properties of equation (1).

Definition 2.1. *Let* $(\lambda_1, \cdot, \lambda_n) \in \mathbb{R}^n$*. We view the* k*-elementary symmetric function as a function on* $\mathbb{R}^n$*:*

$$\sigma_k(\lambda_1, \cdots, \lambda_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k},$$

and we define

$$\Gamma_k^+ = \bigcap_{1 \le j \le k} \{\sigma_j(\lambda_1, \cdots, \lambda_n) > 0\} \subset \mathbb{R}^n,$$

For a symmetric linear transformation $A : V \to V$, where V is an n-dimensional inner product space, the notation $A \in \Gamma_2^+$ will mean that the eigenvalues of A lie in the corresponding set. We note that this notation also makes sense for a symmetric 2-tensor on a Riemannian manifold. If $A \in \Gamma^+_{\frac{n}{2}}$, let $\sigma^{2/n}_{\frac{n}{2}}(A) = \{\sigma_{\frac{n}{2}}(A)\}^{2/n}$.

Definition 2.2. *Let* $A : V \to V$*, where* V *is an* n*-dimensional inner product space. The* $(\frac{n}{2}-1)$*-Newton transformation associated with* A *is*

$$T_{(\frac{n}{2}-1)}(A) := \sum_{j=0}^{\frac{n}{2}-1} (-1)^j \sigma_j(A) A^j.$$

Also, for $t \in \mathbb{R}$ *we define the linear transformation*

$$\mathcal{L}^t(A) := T_{(\frac{n}{2}-1)}(A) + \frac{1-t}{n-2}\sigma_1(T_{(\frac{n}{2}-1)}(A)) \cdot I.$$

We have the following:

Lemma 2.3. *(i)* $\Gamma^+_{\frac{n}{2}}$ *is an open convex cone with vertex at the origin.*
(ii) If $A \in \Gamma^+_{\frac{n}{2}}$*, then* $T_{\frac{n}{2}-1}(A)$ *is positive definite. Hence for all* $t \le 1$*,* $L^t(A)$ *is positive definite.*
(iii) If A *and* B *are symmetric linear transformations,* $A, B \in \Gamma^+_{\frac{n}{2}}$*, then* $\forall \rho \in [0,1]$*,* $\rho A + (1-\rho)B \in \Gamma^+_{\frac{n}{2}}$*, and*

$$\sigma^{\frac{2}{n}}_{\frac{n}{2}}(\rho A + (1-\rho)B) \ge \rho \sigma^{\frac{2}{n}}_{\frac{n}{2}}(A) + (1-\rho)\sigma^{\frac{2}{n}}_{\frac{n}{2}}(B).$$

Lemma 2.4. *If* $A : \mathbb{R} \to Hom(V,V)$*, then*

$$\frac{d}{ds}\sigma_{\frac{n}{2}}(A)(s) = \sum_{i,j} T_{(\frac{n}{2}-1)}(A)_{ij}(s)\frac{d}{ds}(A)_{ij}(s),$$

i.e, the $(\frac{n}{2}-1)$*-Newton transformation is what arises from differentiation of* $\sigma_{\frac{n}{2}}$*.*

Proof. The proof of this lemma is a consequence of an easy computation. See Gursky-Viaclovsky [14] □

Proposition 2.5 (Ellipticity property)**.** *Let $u \in C^2(M)$ be a solution of equation (1) for some $t \leq 1$ and let $\tilde{g} = e^{-2u}g$. Assume that $A^t_{\tilde{g}} \in \Gamma^+_{\frac{n}{2}}$. Then the linearized operator at u, $\mathcal{L}^t : C^{2,\alpha}(M) \to C^{\alpha}(M)$, is elliptic and invertible* $(0 < \alpha < 1)$.

Proof. Define the operator

$$F_t[u, \nabla_g u, \nabla^2_g u] = \sigma_{\frac{n}{2}}(g^{-1}A^t_{\tilde{g}}) - f(x)^{\frac{n}{2}} e^{nu},$$

so that solutions of the equation (1) are exactly the zeroes of F_t. Define the function $u_s = u + s\varphi$, then the linearization at u of the oerator F_t is defined by

$$\begin{aligned}\mathcal{L}^t(\varphi) &= \frac{d}{ds} F_t[u_s, \nabla_g u_s, \nabla_g 2u_s]\Big|_{s=0} \\ &= \frac{d}{ds}\left(\sigma_{\frac{n}{2}}(g^{-1}A^t_{\tilde{g}})\right)\Big|_{s=0} - \frac{d}{ds}\left(f(x)^{\frac{n}{2}} e^{nu_s}\right)\Big|_{s=0}.\end{aligned}$$

From Lemma 2.4 we have

$$\frac{d}{ds}\left(\sigma_{\frac{n}{2}}(g^{-1}A^t_{\tilde{g}}))\right)\Big|_{s=0} = T_{\frac{n}{2}-1}(g^{-1}A^t_{\tilde{g}})_{ij} \frac{d}{ds}\left((A^t_{\tilde{g}})_{ij})\right)\Big|_{s=0}.$$

We compute

$$\frac{d}{ds}\left((A^t_{\tilde{g}})_{ij})\right)\Big|_{s=0} = (\nabla^2_g\varphi)_{ij} + \frac{1-t}{n-2}(\Delta_g\varphi)g_{ij} - (2-t)\nabla_g u \cdot \nabla_g\varphi\, g_{ij} + 2du \otimes d\varphi.$$

Easly we have also

$$\frac{d}{ds}\left(f(x)^{\frac{n}{2}} e^{nu_s})\right)\Big|_{s=0} = nf(x)^{\frac{n}{2}} e^{nu}\,\varphi.$$

Putting all togheter, we conclude

$$\mathcal{L}^t(\varphi) = T_{\frac{n}{2}-1}(g^{-1}A^t_{\tilde{g}})_{ij}\left((\nabla^2_g\varphi)_{ij} + \frac{1-t}{n-2}(\Delta_g\varphi)g_{ij}\right) - nf(x)^{\frac{n}{2}} e^{nu}\,\varphi + \cdots$$

where the last terms denote additional ones witch are linear in $\nabla_g\varphi$. The first term of the linearization is exactly the one defined in 2.1, i.e.

$$L^t(A^t_{\tilde{g}})_{ij} = T_{\frac{n}{2}-1}(A^t_{\tilde{g}})_{ij} + \frac{1-t}{n-2}T_{\frac{n}{2}-1}(A^t_{\tilde{g}})_{pp}\,\delta_{ij}.$$

So finally, we have

$$\mathcal{L}^t(\varphi) = L^t(A^t_{\tilde{g}})_{ij}(\nabla^2_g\varphi)_{ij} - nf(x)^{\frac{n}{2}} e^{nu}\,\varphi + \cdots$$

Since $A^t_{\tilde{g}} \in \Gamma^+_{\frac{n}{2}}$, by Lemma 2.3, we have that the tensor $L^t(A^t_{\tilde{g}})$ is positive definite. So the linearized opertator at any solution u must be elliptic. Note also that, by the previous formula, the operator is of the form

$$\mathcal{L}^t(\varphi) = E(\varphi) - c(x)\varphi,$$

where $E(\varphi)$ is a second order linear elliptic operator and $c(x)$ is a strictly positive function on M, since $c(x) = nf(x)^{\frac{n}{2}} e^{nu}$ and $f(x) > 0$. This allows us to invert this operator between the Hölder spaces $C^{2,\alpha}(M)$ and $C^{\alpha}(M)$. □

For the proof of Theorem 1.2 and Theorem 1.3, we will be concerned with the following equation for a conformal metric $\tilde{g} = e^{-2u}g$:

$$\left(\sigma_{\frac{n}{2}}(g^{-1}A^t_{\tilde{g}})\right)^{2/n} = fe^{2u}, \tag{1}$$

where f is a positive function on M. Let $\sigma_1(g^{-1}A^1_g)$ be the trace of A^1_g with respect to the metric g. We have the following formula for the transformation of A^t_g under this conformal change of metric:

$$A^t_{\tilde{g}} = A^t_g + \nabla^2_g u + \frac{1-t}{n-2}(\Delta_g u)g + du \otimes du - \frac{2-t}{2}|\nabla_g u|^2_g g. \tag{2}$$

Since

$$A^t_g = A^1_g + \frac{1-t}{n-2}\sigma_1(g^{-1}A^1_g)g,$$

this formula follows easily from the standard formula for the transformation of the Schouten tensor (see [18]):

$$A^1_{\tilde{g}} = A^1_g + \nabla^2_g u + du \otimes du - \frac{1}{2}|\nabla_g u|^2_g g. \tag{3}$$

Using this formula we may write (1) with respect to the background metric g

$$\sigma_{\frac{n}{2}}\left(g^{-1}\left(A^t_g + \nabla^2_g u + \frac{1-t}{n-2}(\Delta_g u)g + du \otimes du - \frac{2-t}{2}|\nabla_g u|^2_g g\right)\right)^{2/n} = f(x)e^{2u}.$$

3 Upper bound and higher order estimate

Throughout the sequel, (M, g) will be a closed n-dimensional Riemannian manifold (n even) with positive scalar curvature and locally conformally flat. Since $R_g > 0$, there exists $\delta > -\infty$ such that $A^\delta_g \in \Gamma^+_{\frac{n}{2}}$. (for example we can take δ such that A^δ_g is positive definite, i.e. $Ric_g - \frac{\delta}{2(n-1)}R_g g > 0$ on M). Note that δ only depends on $\|Ric\|$. For $t \in [\delta, 1]$, consider the path of equations (in the sequel we use the notation $A^t_{u_t} := A^t_{g_t}$ for g_t given by $g_t = e^{-2u_t}g$)

$$\sigma^{2/n}_{\frac{n}{2}}(g^{-1}A^t_{u_t}) = fe^{2u_t}, \tag{4}$$

where $f = \sigma^{2/n}_{\frac{n}{2}}(g^{-1}A^\delta_g) > 0$. Note that $u \equiv 0$ is a solution of (4) for $t = \delta$.

Proposition 3.1 (Upper bound)**.** *Let $u_t \in C^2(M)$ be a solution of (4) for some $t \in [\delta, 1]$, with $A^t_{u_t} \in \Gamma^+_{\frac{n}{2}}$. Then $u_t \leq \bar{\delta}$, where $\bar{\delta}$ depends only on $\|Ric\|$.*

Proof. From Newton's inequality we have

$$\sigma^{\frac{2}{n}}_{\frac{n}{2}} \leq C_n\sigma_1,$$

for some $C_n > 0$. So for all $x \in M$

$$fe^{2u_t} \leq C_n\sigma_1(g^{-1}A^t_{u_t}).$$

Let $p \in M$ be a maximum of u_t, then using (2), since the gradient terms vanish at p and $(\Delta u_t)(p) \leq 0$,

$$\begin{aligned} f(p)e^{2u_t(p)} &\leq C_n\sigma_1(g^{-1}A^t_{u_t})(p) \\ &= C_n\sigma_1(g^{-1}A^t_g)(p) + C_n\frac{2n-2-nt}{n-2}(\Delta u_t)(p) \\ &\leq C_n\sigma_1(g^{-1}A^t_g)(p) \\ &\leq C_n\sigma_1(g^{-1}A^\delta_g)(p). \end{aligned}$$

Since M is compact, then $u_t \leq \bar{\delta}$, for some $\bar{\delta}$ depending only on $\|Ric\|$. ∎

Once we have an upper bound for the solutions of equation (4), by the work of S. Chen [7], we immediately get C^1 and C^2 estimates:

Proposition 3.2 (C^1 and C^2 estimates)**.** *Let $u_t \in C^4(M)$ be a solution of (4) for some $t \in [\delta, 1]$, with $A^t_{u_t} \in \Gamma^+_2$. Then*

$$\sup_M \left(|\nabla_g u_t|^2_g + |\nabla^2_g u_t|_g \right) \le C_1,$$

where C_1 depens only on n, $diam(M,g)$ and $\|\nabla^2 Rm\|$.

Now, by the Yamabe equation for the conformal deformation of the scalar curvature, we have that

$$R_{g_t} e^{-2u_t} = \left(R_g + 2(n-1)\Delta_g u_t - (n-1)(n-2)|\nabla_g u_t|^2_g \right).$$

So we obtained an uniform estimates for the scalar curvature of g_t, i.e.

Proposition 3.3. *Let $u_t \in C^4(M)$ be a solution of (4) for some $t \in [\delta, 1]$, with $A^t_{u_t} \in \Gamma^+_2$. Then*

$$0 < R_{g_t} e^{-2u_t} \le \Lambda,$$

where Λ is a positive constant depending only on n $diam(M,g)$ and $\|\nabla^2 Rm\|$.

4 Lower bound

Proposition 4.1 (Lower Bound)**.** *There exists a constant $t_0 = t_0(n, diam(M,g), \|\nabla^2 Rm\|) < 1$ such that, if $u_t \in C^2(M)$ is a solution of (4) and if $A^t_{u_t} \in \Gamma^+_{\frac{n}{2}}$ for some $t \in [t_0, 1]$, then $u_t \ge \underline{\delta}$ for some uniform constant $\underline{\delta}$ depending only on $diam(M,g)$ and $\|\nabla^2 Rm\|$.*

Proof. It's easy to see that the following formula holds

$$\sigma_{\frac{n}{2}}(g^{-1}A^t_g) = \sigma_{\frac{n}{2}}(g^{-1}A_g) + C_1(1-t)^{\frac{n}{2}}\sigma_1(g^{-1}A_g)^{\frac{n}{2}} + \sum_{i=1}^{\frac{n}{2}-1} c_{n,i} \left(\frac{1-t}{n-2} \right)^{\frac{n}{2}-i} \sigma_i(g^{-1}A_g)(\sigma_1(g^{-1}A_g))^{\frac{n}{2}-i},$$

for some positive constants C_1 and $c_{n,i}$ depending only on n and i.

Since $A^t_{u_t} \in \Gamma^+_{\frac{n}{2}}$, we have $\sigma_i(g^{-1}_{u_t} A^t_{u_t}) > 0$ for all $1 \le i \le n/2$. So, iterating the previous formula, we can easily check that

$$\sigma_i(g^{-1}_{u_t} A^1_{u_t}) > -C_i(1-t)^i \left(\sigma_1(g^{-1}_{u_t} A^1_{u_t}) \right)^i,$$

for some positive constants C_i depending only on n. Hence, by the previous formula, we have

$$\sigma_{\frac{n}{2}}(g^{-1}_{u_t} A^t_{u_t}) \ge \sigma_{\frac{n}{2}}(g^{-1}_{u_t} A^1_{u_t}) + C_1(1-t)^{\frac{n}{2}}\sigma_1(g^{-1}_{u_t} A^1_{u_t})^{\frac{n}{2}} - C_2(1-t)^{\frac{n}{2}}\sigma_1(g^{-1}_{u_t} A^1_{u_t})^{\frac{n}{2}}.$$

On the other hand, since u_t is a solution of equation (4), we have

$$\sigma_{\frac{n}{2}}(g^{-1}_{u_t} A^t_{u_t}) = e^{nu_t}\sigma_{\frac{n}{2}}(g^{-1} A^t_{u_t}) = e^{2nu_t} f^{\frac{n}{2}},$$

or equivalently

$$e^{-nu_t}\sigma_{\frac{n}{2}}(g^{-1}_{u_t} A^t_{u_t}) = e^{nu_t} f^{\frac{n}{2}}.$$

Integrating on M this with respect to dV_g, we obtain

$$\begin{aligned} C\int_M e^{nu_t}\, dV_g &\ge \int_M e^{nu_t} f^{\frac{n}{2}}\, dV_g \\ &= \int_M e^{-nu_t}\sigma_{\frac{n}{2}}(g^{-1}_{u_t} A^t_{u_t})\, dV_g \\ &= \int_M \sigma_{\frac{n}{2}}(g^{-1}_{u_t} A^t_{u_t})\, dV_{g_{u_t}} \\ &\ge \int_M \sigma_{\frac{n}{2}}(g^{-1}_{u_t} A^1_{u_t})\, dV_{g_{u_t}} + C_1(1-t)^{\frac{n}{2}} \int_M R^{\frac{n}{2}}_{g_{u_t}}\, dV_{g_{u_t}} - C_2(1-t)^{\frac{n}{2}} \int_M R^{\frac{n}{2}}_{g_{u_t}}\, dV_{g_{u_t}}, \end{aligned}$$

where $C > 0$ is chosen so that $f^{\frac{n}{2}} \leq C$ (recall that, since $f = \sigma_{\frac{n}{2}}^{2/n}(g^{-1}A_g^\delta)$, C depends only on $\|Ric\|$). Using Hölder inequality and the definition of the Yamabe invariant (which is positive), we get

$$\int_M R_{g_{u_t}}^{\frac{n}{2}} \, dV_{g_{u_t}} \geq (Y(M,[g]))^{\frac{n}{2}}.$$

Moreover, by the result of Viaclovsky in [18], we have the conformal invariance

$$\int_M \sigma_{\frac{n}{2}}(g_{u_t}^{-1} A_{u_t}^1) \, dV_{g_{u_t}} = \int_M \sigma_{\frac{n}{2}}(g^{-1}A_g^1) \, dV_g.$$

Thus we get

$$C \int_M e^{nu_t} \, dV_g \geq \int_M \sigma_{\frac{n}{2}}(g^{-1}A_g^1) \, dV_g + C_1(1-t)^{\frac{n}{2}}(Y(M,[g]))^{\frac{n}{2}} - C_2(1-t)^{\frac{n}{2}} \int_M R_{g_{u_t}}^{\frac{n}{2}} \, dV_{g_{u_t}}.$$

Now, by Proposition 3.3, we have an uniform estimate for the quantity

$$\int_M R_{g_t}^{\frac{n}{2}} \, dV_{g_t} = \int_M R_{g_t}^{\frac{n}{2}} e^{-nu_t} \, dV_g.$$

Hence for t_0 sufficientely close to 1, since the Euler-Poincaré characteristic of M is positive, we can always assume that

$$\int_M \sigma_{\frac{n}{2}}(g^{-1}A_g^1) \, dV_g + C_1(1-t)^{\frac{n}{2}}(Y(M,[g]))^{\frac{n}{2}} - C_2(1-t)^{\frac{n}{2}} \int_M R_{g_t}^{\frac{n}{2}} \, dV_{g_t} = \lambda_t > 0,$$

for every $t \in [t_0, 1]$. This gives

$$\max_M u_t \geq \frac{1}{n} \log \lambda_t - C(\mathrm{diam}(M,g), \|Ric\|) \, .$$

By Proposition 3.2, $\max_M |\nabla_g u_t|_g \leq C_1$. This implies the Harnack inequality

$$\max_M u_t \leq \min_M u_t + C(\mathrm{diam}(M,g), \|\nabla^2 Rm\|) \, ,$$

by simply integrating along a geodesic connecting points at which u_t attains its maximum and minimum. Combining these two inequalities, we obtain

$$u_t \geq \min_M u_t \geq \frac{1}{n} \log \lambda_t - C =: \underline{\delta} \, ,$$

where C only depends on $\mathrm{diam}(M,g)$ and $\|\nabla^2 Rm\|$. This ends the proof of the lemma. □

In the previous section we prove that for a solution u_t of the equation (4), with $A_u^t \in \Gamma_{\frac{n}{2}}^+$, we have a priori C^1 and C^2 estimates just depending on the upper bound of the function u_t. Now, if $t \in [t_0, 1]$, where t_0 is the one of Proposition 4.1, since $u_t \in C^2(M)$ has a lower bound too, from Proposition 3.2 we have

$$\|u_t\|_{L^\infty(M)} + \|\nabla_g u_t\|_{L^\infty(M)} + \|\nabla_g^2 u_t\|_{L^\infty(M)} \leq C,$$

where C depends only on n, $diam(M,g)$ and $\|\nabla^2 Rm\|$. By the works of Krylov [17] and Evans [8] we obtain $C^{2,\alpha}$ estimates, i.e

Proposition 4.2. *Let $u_t \in C^4(M)$ be a solution of (4) for some $t \in [t_0, 1]$, with $A_{u_t}^t \in \Gamma_{\frac{n}{2}}^+$. Then*

$$\|u\|_{C^{2,\alpha}(M)} \leq C,$$

where C is a positive constant depending only on n, $diam(M,g)$ and $\|\nabla^2 Rm\|$.

5 Proof of Theorem 1.2

Let $t_0 = t_0(n, \mathrm{diam}(M,g), \|\nabla^2 Rm\|) < 1$ be the one of Proposition 4.1. We assume by hypothesis that

$$A_g^{t_0} \in \Gamma^+_{\frac{n}{2}}.$$

The parameter t_0 will be the starting point in order to use the continuity method. Our 1-parameter family of equations, for $t \in [t_0, 1]$, is

$$\sigma^{2/n}_{\frac{n}{2}}(g^{-1}A^t_{u_t}) = f(x)e^{2u_t}, \tag{5}$$

with $f(x) = \sigma^{2/n}_{\frac{n}{2}}(g^{-1}A_g^{t_0}) > 0$. Define the set

$$\mathcal{S} = \left\{ t \in [t_0, 1] \mid \exists\, \text{a solution}\, u_t \in C^{2,\alpha}(M)\, \text{of}\, (5)\, \text{with}\, A^t_{u_t} \in \Gamma^+_{\frac{n}{2}} \right\}.$$

Clearly, with our choice of f, $u \equiv 0$ is a solution for $t = t_0$. By assumption, $t_0 \in \mathcal{S}$, and $\mathcal{S} \neq \emptyset$. Let $t \in \mathcal{S}$, and u_t be a solution. By Proposition 2.5, the linearized operator at u_t, $\mathcal{L}^t : C^{2,\alpha}(M) \to C^\alpha(M)$, is invertible. The implicit function theorem tells us that $\mathcal{S}$ is open. From classical elliptic theory, since $u_t \in C^{2,\alpha}(M)$, by a bootstrap argument using classical Schauder estimates, it follows that $u_t \in C^\infty(M)$, since $f \in C^\infty(M)$. By Proposition 4.2 and the classical Ascoli-Arzela's Theorem, we will get that $\mathcal{S}$ must be closed, therefore $\mathcal{S} = [t_0, 1]$. The metric $\tilde{g} = e^{-2u_1}g$ then satisfies $\sigma_k(A^1_{\tilde{g}}) > 0$ for all $1 \leq k \leq \frac{n}{2}$. The inequality on the Ricci curvature in Theorem 1.2 follows from the following proposition for $k = \frac{n}{2}$ (see [12] for the proof)

Proposition 5.1. *If for some metric g_1 on M we have $A_{g_1} \in \Gamma^+_k$, then*

$$Ric(g_1) \geq \frac{2k-n}{2k(n-1)} R_{g_1} g_1.$$

6 Proof of Theorem 1.3

In Theorem 1.2 we have proved that if (M,g) is an even-dimensional, closed, locally conformally flat manifolds with positive scalar curvature, positive Euler-Poincaré characteristic, and "close to" be $\frac{n}{2}$-admissible, then M admits a metric $\tilde{g}$ which is $\frac{n}{2}$-admissible, i.e. with $A_{\tilde{g}} \in \Gamma^+_{\frac{n}{2}}$. In particular it turns out that the Ricci curvature ($Ric_{\tilde{g}}$) must be nonnegative. By the classification theorem for locally conformally flat manifolds with nonnegative Ricci curvature (for instance, see [2]), we have that M must be conformally equivalent to either a space form or a finite quotient of a Riemannian $\mathbb{S}^{n-1}(c) \times \mathbb{S}^1$, for some $c > 0$. Moreover, for locally conformally flat manifolds which admit k-admissible metrics (i.e. metrics g such that $A_g \in \Gamma^+_k$) we have topological restrictions (see, for example, [6], [9], [11]). In [9] they proved the following vanishing theorem

Proposition 6.1 (Gonzáles, [9])**.** *Let (M,g) be a closed, locally conformally flat manifold, with $A_g \in \Gamma^+_k$, $k < n/2$. Then the q-th Betti number $b_q = 0$, for*

$$\frac{n-2k}{2} + 1 \leq q \leq \frac{n+2k}{2} - 1.$$

Since $A_{\tilde{g}} \in \Gamma^+_{\frac{n}{2}} \subset \Gamma^+_{\frac{n}{2}-1}$, we can apply this proposition to the case $k = \frac{n}{2} - 1$ to get that the q-th Betti number

$$b_q = 0, \text{ for } 2 \leq q \leq n-2.$$

Since the Euler characteristic can be defined in terms of the Betti numbers, by Poincaré duality, we get that

$$\chi(M) = 2 - 2b_1.$$

Then $0 < \chi(M) \leq 2$, which forces the manifold M to be diffeomorphic to either $\mathbb{R}P^n$ (if $\chi(M) = 1$) or $\mathbb{S}^n$ (if $\chi(M) = 2$).

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