DYNAMIC PERFECT PLASTICITY AS CONVEX MINIMIZATION

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Abstract. We present a novel variational approach to dynamic perfect plasticity. This is based on minimizing over entire trajectories parameter-dependent convex functionals of Weighted-Inertia-Dissipation-Energy (WIDE) type. Solutions to the system of dynamic perfect plasticity are recovered as limit of minimizing trajectories as the parameter goes to zero. The crucial compactness is achieved by means of a time-discretization and a variational convergence argument.

1. Introduction

Plasticity is the macroscopic, inelastic behavior of a solid resulting from the accumulation of slip defects at its microscopic, crystalline level. As a result of these dislocations, the behavior of the material remains purely elastic (and hence reversible) as far as the magnitude of the stress remains *small*, and becomes irreversible as soon as a given stress-threshold is reached. When that happens, a plastic flow is developed such that, after unloading, the material remains permanently plastically deformed [27].

Referring to [22, 34] for an overview on plasticity models, we focus here on *dynamic perfect plasticity* in the form of the classical *Prandtl-Reuss* model [16]

$$\rho\ddot{u} - \nabla\cdot\sigma = 0, \tag{1.1}$$

$$\sigma = \mathbb{C}(Eu - p), \tag{1.2}$$

$$\partial H(\dot{p}) \ni \sigma_D \tag{1.3}$$

describing the basics of plastic behavior in metals [20]. Here $u(t) : \Omega \to \mathbb{R}^3$ denotes the (time-dependent) *displacement* of a body with reference configuration $\Omega \subset \mathbb{R}^3$ and density $\rho > 0$, and $\sigma(t) : \Omega \to \mathbb{M}^{3\times3}_{\rm sym}$ is its *stress*. In particular, relation (1.1) expresses the conservation of momenta. The constitutive relation (1.2) relates the stress $\sigma(t)$ to the *linearized strain* $Eu(t) = (\nabla u(t)+\nabla u(t)^\top)/2 : \Omega \to \mathbb{M}^{3\times3}_{\rm sym}$ and the *plastic strain* $p(t) : \Omega \to \mathbb{M}^{3\times3}_D$ (deviatoric tensors) via the fourth-order *elasticity tensor* $\mathbb{C}$. Finally, (1.3) expresses the plastic-flow rule: $H : \mathbb{M}^{3\times3}_D \to [0, +\infty)$ is a positively 1-homogeneous, convex *dissipation* function, σ_D stands for the deviatoric part of the stress, and the symbol ∂ is the subdifferential in the sense of Convex Analysis [9]. The system will be driven by imposing a nonhomogeneous boundary displacement. Details on notation and modeling are given in Section 2.

The focus of this paper is to recover weak solutions to the dynamic perfect plasticity system (1.1)-(1.3) by minimizing parameter-dependent convex functionals over entire trajectories, and by passing to the parameter limit. In particular, we consider the *Weighted-Inertia-Dissipation-Energy (WIDE)* functional of the form

$$I_\varepsilon(u,p) = \int_0^T\int_\Omega \exp\left(-\frac{t}{\varepsilon}\right)\left(\frac{\rho\varepsilon^2}{2}|\ddot{u}|^2 + \varepsilon H(\dot{p}) + \frac{1}{2}(Eu-p) : \mathbb{C}(Eu-p)\right)dx\,dt, \tag{1.4}$$

to be defined on suitable admissible classes of entire trajectories $t \in [0,T] \mapsto (u(t), p(t)) : \Omega \to \mathbb{R}^3 \times \mathbb{M}^{3\times3}_D$ fulfilling given boundary-displacement and initial conditions (on u and p, respectively). The functional bears its name from resulting from the sum of the inertial term $\rho|\ddot{u}|^2/2$, the dissipative term $H(\dot{p})$, and the energy term $(Eu-p) : \mathbb{C}(Eu-p)/2$, weighted by different powers of ε as well as the function $\exp(-t/\varepsilon)$.

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For all $\varepsilon > 0$ one can prove that (a suitable relaxation of) the convex functional I_ε admits minimizers $(u^\varepsilon, p^\varepsilon)$ which indeed approximate solutions to the dynamic perfect plasticity system (1.1)-(1.3). In particular, by computing the corresponding Euler-Lagrange equations one finds that the minimizers $(u^\varepsilon, p^\varepsilon)$ weakly solve the elliptic-in-time approximating relations

$$\varepsilon^2 \rho \dddot{u}^{\cdot\varepsilon} - 2\varepsilon^2 \rho \dddot{u}^\varepsilon + \rho \ddot{u}^\varepsilon - \nabla \cdot \sigma^\varepsilon = 0, \tag{1.5}$$

$$\sigma^\varepsilon = \mathbb{C}(Eu^\varepsilon - p^\varepsilon), \tag{1.6}$$

$$-\varepsilon(\partial H(\dot{p}^\varepsilon))^\cdot + \partial H(\dot{p}^\varepsilon) \ni \sigma_D^\varepsilon, \tag{1.7}$$

along with Neumann conditions at the final time T.

The dynamic perfect plasticity system (1.1)-(1.3) is formally recovered by taking $\varepsilon \to 0$ in system (1.5)-(1.7). The main result of this paper consists in making this intuition rigorous, resulting in a new approximation theory for dynamic perfect plasticity.

The interest in this variational-approximation approach is threefold. First, the differential problem (1.1)-(1.3) is reformulated on purely variational grounds. This opens the possibility of applying the powerful tools of the Calculus of Variations to the problem, from the Direct Method, to relaxation, and Γ-convergence [15].

Secondly, by addressing a time-discrete analogue of this approach we contribute a novel numerical strategy in order to approximate dynamic perfect plasticity by means of space-time optimization methods. We believe that this might be of potential interest in combination with global constraints or non-cylindrical domains.

Eventually, the variational formulation via WIDE functionals is easily open to be generalized by including more refined material effects, especially in terms of additional internal-variable descriptions. This indeed has been one of the main motivations for advancing the WIDE method in the first place, see in particular [10, 26] for applications in Materials Science. Having illustrated the details of the method in the case of dynamic perfect plasticity could then serve as basis for developing complete theories.

As a by-product of our analysis, we obtain a new proof of existence of weak solutions to dynamic perfect plasticity. Note that existence results for (1.1)-(1.3) are indeed quite classical. In the quasistatic case $\rho = 0$ they date back to Suquet [50] and have been subsequently reformulated by Dal Maso, DeSimone, and Mora [11] and Francfort and Giacomini [18] within the theory of rate-independent processes (see the recent monograph [39]). In the dynamic case $\rho > 0$ both the first existence results due to Anzellotti and Luckhaus [6, 35] and their recent revisiting by Babadjian and Mora [7] are based on viscosity techniques. Dimension reduction has been tackled both in the quasistatic and the dynamic case, in [13, 28, 29] and [36], respectively. Finally, in [12] convergence of solutions of the dynamic problem to solutions of the quasistatic problem as the density ρ tends to 0 has been shown. With respect to the available existence theories our approach is new, for it does not rely on viscous approximation but rather on a global variational method.

Before moving on, let us review here the available literature on WIDE variational methods. At the level of Euler-Lagrange equations, elliptic-regularization techniques are classical and have to be traced back to Lions [32, 33] and Oleinik [43]. Their variational version via global functionals is already mentioned in the classical textbook by Evans [17, Problem 3, p. 487] and has been used by Ilmanen [24], in the context of Brakke mean-curvature flow of varifolds, and by Hirano [23] in connection with periodic solutions to gradient flows.

The formalism has been then applied in the context of rate-independent systems ($\rho = 0$) by Mielke and Ortiz [38], see also the follow-up [40]. Viscous dynamics have been considered in many different settings, including gradient flows [41], curves of maximal slopes in metric spaces [44, 45], mean curvature flow [48], doubly-nonlinear equations [1, 2, 3, 4, 5], reaction-diffusion systems [37], and quasilinear parabolic equations [8].

The dynamic case $\rho > 0$ has been the object of a long-standing conjecture by De Giorgi on semilinear waves [14]. The conjecture was solved in the positive in [49] for finite-time intervals and then by Serra and Tilli in [46] for the whole time semiline, that is in its original formulation. De Giorgi himself pointed out in [14] the interest of extending the method to other dynamic problems. The task has been then taken up in [31] for mixed hyperbolic-parabolic equations, in [30] for Lagrangian Mechanics, and in [47] for other hyperbolic problems. The present paper delivers the first realization of De Giorgi's suggestion in the context of Continuum Mechanics.

The paper is organized as follows. We introduce notation and state our main result, namely Theorem 2.3 in Section 2. Then, we discuss in Section 3 the existence of minimizers of the WIDE functionals. In Section 4 a time discretization of the minimization problem is addressed. Its time-continuous limit is discussed in Section 5 by means of variational convergence arguments. A parameter-dependent energy inequality is derived in Section 6 and finally used in Section 7 in order to pass to the limit as $\varepsilon \to 0$ and prove Theorem 2.3.

2. Statement of the main result

We devote this section to the specification of the material model and its mathematical setting. Some notions from measure theory need to be recalled,and we introduce the notation and assumptions to be used throughout the article. The specific form of the WIDE functionals is eventually introduced in Subsection 2.9, and we conclude by stating our main result, namely Theorem 2.3.

2.1. **Tensors.** In what follows, for any map $f : [0,T]\times\mathbb{R}^3 \to \mathbb{R}$ we will denote by $\dot f$ its time derivative, and by ∇f its spatial gradient. The set of 3×3 real matrices will be denoted by $\mathbb{M}^{3\times 3}$. Given $M, N \in \mathbb{M}^{3\times 3}$, we will denote their scalar product by $M : N := \mathrm{tr}(M^\top N)$ where tr denotes the trace and the superscript stands for transposition, and we will adopt the notation M_D to identify the deviatoric part of M, namely $M_D := M - \mathrm{tr}(M)\mathrm{Id}/3$ where Id is the identity matrix. The symbol $\mathbb{M}^{3\times 3}_{\mathrm{sym}}$ will stand for the set of symmetric 3×3 matrices, whereas $\mathbb{M}^{3\times 3}_D$ will be the subset of $\mathbb{M}^{3\times 3}_{\mathrm{sym}}$ given by symmetric matrices having null trace.

2.2. **Measures.** Given a Borel set $B \subset \mathbb{R}^N$ the symbol $\mathcal{M}_b(B;\mathbb{R}^m)$ denotes the space of all bounded Borel measures on B with values in $\mathbb{R}^m$ ($m \in \mathbb{N}$). When $m = 1$ we will simply write $\mathcal{M}_b(B)$. We will endow $\mathcal{M}_b(B;\mathbb{R}^m)$ with the norm $\|\mu\|_{\mathcal{M}_b(B;\mathbb{R}^m)} := |\mu|(B)$, where $|\mu| \in \mathcal{M}_b(B)$ is the total variation of the measure μ.

If the relative topology of B is locally compact, by the Riesz representation Theorem the space $\mathcal{M}_b(B;\mathbb{R}^m)$ can be identified with the dual of $C_0(B;\mathbb{R}^m)$, which is the space of all continuous functions $\varphi : B \to \mathbb{R}^m$ such that the set $\{|\varphi| \geq \delta\}$ is compact for every $\delta > 0$. The weak* topology on $\mathcal{M}_b(B;\mathbb{R}^m)$ is defined using this duality.

2.3. **Functions with bounded deformation.** Let U be an open set of $\mathbb{R}^3$. The space $BD(U)$ of functions with *bounded deformation* is the space of all functions $u \in L^1(U;\mathbb{R}^3)$ whose symmetric gradient $Eu := \mathrm{sym}\, Du := (Du + Du^T)/2$ (in the sense of distributions) belongs to $\mathcal{M}_b(U;\mathbb{M}^{3\times 3}_{\mathrm{sym}})$. It is easy to see that $BD(U)$ is a Banach space endowed with the norm

$$\|u\|_{L^1(U;\mathbb{R}^3)} + \|Eu\|_{\mathcal{M}_b(U;\mathbb{M}^{3\times 3}_{\mathrm{sym}})}.$$

A sequence $\{u^k\}$ is said to converge to u weakly* in $BD(U)$ if $u^k \rightharpoonup u$ weakly in $L^1(U;\mathbb{R}^3)$ and $Eu^k \rightharpoonup Eu$ weakly* in $\mathcal{M}_b(U;\mathbb{M}^{3\times 3}_{\mathrm{sym}})$. Every bounded sequence in $BD(U)$ has a weakly* converging subsequence. If U is bounded and has a Lipschitz boundary, $BD(U)$ can be embedded into $L^{3/2}(U;\mathbb{R}^3)$ and every function $u \in BD(U)$ has a trace, still denoted by u, which belongs to $L^1(\partial U;\mathbb{R}^3)$. If Γ is a nonempty open subset of ∂U in the relative topology of ∂U, there exists a constant $C > 0$, depending on U and Γ, such that

$$\|u\|_{L^1(U;\mathbb{R}^3)} \leq C\|u\|_{L^1(\Gamma;\mathbb{R}^3)} + C\|Eu\|_{\mathcal{M}_b(U;\mathbb{M}^{3\times 3}_{\mathrm{sym}})}. \tag{2.1}$$

(see [51, Chapter II, Proposition 2.4 and Remark 2.5]). For the general properties of the space $BD(U)$ we refer to [51].

2.4. **The elasticity tensor.** Let $\mathbb{C}$ be the *elasticity tensor*, considered as a symmetric positive-definite linear operator $\mathbb{C} : \mathbb{M}^{3\times3}_{\rm sym} \to \mathbb{M}^{3\times3}_{\rm sym}$, and let $Q : \mathbb{M}^{3\times3}_{\rm sym} \to [0, +\infty)$ be the quadratic form associated with $\mathbb{C}$, given by

$$Q(\xi) := \tfrac{1}{2}\mathbb{C}\xi : \xi \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{\rm sym}.$$

Let the two constants $\alpha_{\mathbb{C}}$ and $\beta_{\mathbb{C}}$, with $0 < \alpha_{\mathbb{C}} \leq \beta_{\mathbb{C}}$, be such that

$$\alpha_{\mathbb{C}}|\xi|^2 \leq Q(\xi) \leq \beta_{\mathbb{C}}|\xi|^2 \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{\rm sym}, \tag{2.2}$$

and

$$|\mathbb{C}\xi| \leq 2\beta_{\mathbb{C}}|\xi| \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{\rm sym}. \tag{2.3}$$

2.5. **The reference configuration.** Let Ω be a bounded open set in $\mathbb{R}^3$ with C^2 boundary. Let Γ_0 be a connected open subset of $\partial\Omega$ (in the relative topology of $\partial\Omega$) such that $\partial_{\partial\Omega}\Gamma_0$ is a connected, one-dimensional, C^2 manifold. In the following we will assume Ω to be the reference configuration of our material, and Γ_0 to be the Dirichlet portion of $\partial\Omega$ where time-dependent boundary conditions are prescribed.

2.6. **The dissipation potential.** Let K be a closed convex set of $\mathbb{M}^{3\times3}_D$ such that there exist two constants r_K and R_K, with $0 < r_K \leq R_K$, satisfying

$$\{\xi \in \mathbb{M}^{3\times3}_D : \ |\xi| \leq r_K\} \subset K \subset \{\xi \in \mathbb{M}^{3\times3}_D : \ |\xi| \leq R_K\}.$$

The boundary of K is interpreted as the *yield surface*. The *plastic dissipation potential* is given by the support function $H : \mathbb{M}^{3\times3}_D \to [0, +\infty)$ of K, defined as

$$H(\xi) := \sup_{\sigma\in K} \sigma : \xi.$$

Note that $K = \partial H(0)$ is the subdifferential of H at 0 (see e.g. [9, Section 1.4]). The function H is convex and positively 1-homogeneous, with

$$r_K|\xi| \leq H(\xi) \leq R_K|\xi| \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_D. \tag{2.4}$$

In particular, H satisfies the triangle inequality

$$H(\xi + \zeta) \leq H(\xi) + H(\zeta) \quad \text{for every } \xi, \zeta \in \mathbb{M}^{3\times3}_D. \tag{2.5}$$

For every $\mu \in \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times3}_D)$ let $d\mu/d|\mu|$ be the Radon-Nikodým derivative of μ with respect to its variation $|\mu|$.

According to the theory of convex functions of measures [19], we introduce the nonnegative Radon measure $H(\mu) \in \mathcal{M}_b(\Omega \cup \Gamma_0)$ defined by

$$H(\mu)(A) := \int_A H\Big(\frac{d\mu}{d|\mu|}\Big)\, d|\mu|$$

for every Borel set $A \subset \Omega \cup \Gamma_0$. We also consider the functional $\mathcal{H} : \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times3}_D) \to [0, +\infty)$ defined by

$$\mathcal{H}(\mu) := H(\mu)(\Omega \cup \Gamma_0) = \int_{\Omega\cup\Gamma_0} H\Big(\frac{d\mu}{d|\mu|}\Big)\, d|\mu|$$

for every $\mu \in \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times3}_D)$. Notice that $\mathcal{H}$ is lower semicontinuous on $\mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times3}_D)$ with respect to weak* convergence. The following lemma is a consequence of [19, Theorem 4] and [51, Chapter II, Lemma 5.2] (see also [11, Subsection 2.2]).

Lemma 2.1. *Setting $\mathcal{K}_D(\Omega) := \{\tau \in C_0(\Omega \cup \Gamma_0; \mathbb{M}^{3\times3}_D) : \tau(x) \in K$ for every $x \in \Omega\}$, there holds*

$$\mathcal{H}(\mu) = \sup\{\langle \tau,\, \mu\rangle : \ \tau \in \mathcal{K}_D(\Omega)\}$$

for every $\mu \in \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times3}_D)$.

2.7. **The $\mathcal{H}$-dissipation.** Let $s_1, s_2 \in [0,T]$ with $s_1 \le s_2$. For every function $t \mapsto \mu(t)$ of bounded variation from $[0,T]$ into $\mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times3}_D)$, we define the *$\mathcal{H}$-dissipation* of $t \mapsto \mu(t)$ in $[s_1, s_2]$ as

$$D_{\mathcal{H}}(\mu; s_1, s_2) := \sup\left\{ \sum_{j=1}^n \mathcal{H}(\mu(t_j) - \mu(t_{j-1})) : \ s_1 = t_0 \le t_1 \le \dots \le t_n = s_2, \ n \in \mathbb{N} \right\}. \tag{2.6}$$

Denoting by $V_{\rm tot}$ the pointwise variation of $t \to \mu(t)$, that is,

$$V_{\rm tot}(\mu; s_1, s_2) := \sup\left\{ \sum_{j=1}^n \|\mu(t_j) - \mu(t_{j-1})\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)} : \ s_1 = t_0 \le \dots \le t_n = s_2, \, n \in \mathbb{N} \right\},$$

by (2.4) there holds

$$r_K V_{\rm tot}(\mu; s_1, s_2) \le D_{\mathcal{H}}(\mu; s_1, s_2) \le R_K V_{\rm tot}(\mu; s_1, s_2). \tag{2.7}$$

As in [38, Section 4.2],for every non-increasing and positive $a \in C([0,T])$ we define the *a-weighted $\mathcal{H}$-dissipation* of $t \mapsto \mu(t)$ in $[s_1, s_2]$ as

$$\begin{aligned} D_{\mathcal{H}}(a; \mu; s_1, s_2) := \sup\Bigg\{ \sum_{j=1}^n a(t_j)\mathcal{H}(\mu(t_j) - \mu(t_{j-1})) : \ t_0, t_n \in [s_1, s_2], \\ t_0 \le t_1 \le \dots \le t_n, \ n \in \mathbb{N} \Bigg\}, \end{aligned} \tag{2.8}$$

and for every $b \in C([0,T])$ we introduce the *b-weighted $\mathcal{H}$-dissipation* of $t \mapsto \mu(t)$ in $[s_1, s_2]$ as

$$\hat{D}_{\mathcal{H}}(b; \mu; s_1, s_2) := PMS \int_{s_1}^{s_2} b(t)\, dD_{\mathcal{H}}(\mu; 0, t),$$

namely, as the Pollard-Moore-Stieltjes integral (see [21, Sections 3 and 4]) of b with respect to the function of bounded variation

$$[0,T] \ni t \mapsto D_{\mathcal{H}}(\mu; 0, t) \in [0, D_{\mathcal{H}}(\mu; 0, T)].$$

Note that the integral above is well-defined owing to [21, Theorems 5.31 and 5.32], and that if b is non-increasing and positive then

$$\hat{D}_{\mathcal{H}}(b; \mu; s_1, s_2) = D_{\mathcal{H}}(b; \mu; s_1, s_2). \tag{2.9}$$

An adaptation of [11, Theorem 7.1] yields that if μ is absolutely continuous in time, then

$$V_{\rm tot}(\mu; s_1, s_2) = \int_{s_1}^{s_2} \mathcal{H}(\dot{\mu})\, dt,$$

and

$$D_{\mathcal{H}}(a; \mu; s_1, s_2) = \int_{s_1}^{s_2} a(t)\mathcal{H}(\dot{\mu})\, dt,$$

for every non-increasing and positive $a \in C([0,T])$.

2.8. **The equations of dynamic perfect plasticity.** On Γ_0 for every $t \in [0,T]$ we prescribe a boundary datum $w(t) \in W^{1/2,2}(\Gamma_0; \mathbb{R}^3)$. With a slight abuse of notation we also denote by $w(t)$ the $W^{1,2}(\Omega; \mathbb{R}^3)$-extension of the boundary condition to the set Ω.

The *set of admissible displacements and strains* for the boundary datum $w(t)$ is given by

$$\begin{aligned} \mathscr{A}(w(t)) := \Big\{ (u, e, p) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times3}_{\rm sym}) \times \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times3}_D) : \\ Eu = e + p \text{ in } \Omega, \quad p = (w(t) - u) \odot \nu \mathcal{H}^2 \text{ on } \Gamma_0 \Big\}, \end{aligned} \tag{2.10}$$

where $\odot$ stands for the symmetrized tensor product, namely

$$a \odot b := (a \otimes b + b \otimes a)/2 \quad \forall\, a, b \in \mathbb{R}^3,$$

ν is the outer unit normal to $\partial\Omega$, and $\mathcal{H}^2$ is the two-dimensional Hausdorff measure. The function u represents the *displacement* of the body, while e and p are called the *elastic* and *plastic strain*, respectively.

We point out that the constraint

$$p = (w(t) - u) \odot \nu \mathcal{H}^2 \text{ on } \Gamma_0 \tag{2.11}$$

is a relaxed formulation of the boundary condition $u = w(t)$ on Γ_0 (see also [42]). As remarked in [11], the mechanical meaning of (2.11) is that whenever the boundary datum is not attained a plastic slip develops, whose amount is directly proportional to the difference between the displacement u and the boundary condition $w(t)$.

Let $w \in W^{2,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3)) \cap C^3([0,T];L^2(\Omega;\mathbb{R}^3))$. A *solution to the equations of dynamic perfect plasticity* is a function $t \mapsto (u(t), e(t), p(t))$ from $[0,T]$ into $(L^2(\Omega;\mathbb{R}^3) \cap BD(\Omega)) \times L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym}) \times \mathcal{M}_b(\Omega \cup \Gamma_0;\mathbb{M}^{3\times 3}_D)$ with $(u,e,p) \in (W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^3)) \cap \mathrm{Lip}(0,T;BD(\Omega)) \times W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^3)) \times \mathrm{Lip}(0,T;\mathcal{M}_b(\Omega \cup \Gamma_0;\mathbb{M}^{3\times 3}_D))$ such that for every $t \in [0,T]$ there holds $(u(t), e(t), p(t)) \in \mathscr{A}(w(t))$, and the following conditions are satisfied:

- (c1) *equilibrium:* $\rho\ddot{u}(t) - \operatorname{div}\sigma(t) = 0$ in Ω and $\sigma(t)\nu = 0$ on $\partial\Omega \setminus \Gamma_0$, where $\sigma(t) := \mathbb{C}e(t)$ is the stress tensor, and $\rho > 0$ is the constant density;
- (c2) *stress constraint:* $\sigma_D(t) \in K$;
- (c3) *energy inequality:*

$$\int_\Omega Q(e(t))\,dx + \frac{\rho}{2}\int_\Omega |\dot{u}(t)|^2\,dx + \int_0^t \mathcal{H}(\dot{p}(t))\,dt \le \int_\Omega Q(e(0))\,dx$$
$$+ \frac{\rho}{2}\int_\Omega |\dot{u}(0)|^2\,dx + \int_0^t \int_\Omega \sigma(s) : E\dot{w}(s) + \rho\ddot{u}(s)\cdot\dot{w}(s)\,dx\,ds.$$

We remark that condition (c3) guarantees that the sum of the elastic and kinetic energies with the plastic dissipation at each time t is always smaller or equal to the sum of the initial energy with the work due to the time-dependent boundary condition.

Under suitable assumptions, when (c1) and (c2) are satisfied, condition (c3) is equivalent to the following *flow rule:*

(c3') $\dot{p}(t) = 0$ if $\sigma_D(t) \in \operatorname{int} K$, whilst $\dot{p}(t)$ belongs to the normal cone to K at $\sigma_D(t)$ if $\sigma_D(t) \in \partial K$.

A detailed analysis of the equivalence between (c1)–(c3), and (c1)–(c2), (c3') has been performed in [11, Section 6]. An adaptation of the argument yields the analogous statements in the dynamic setting.

The following existence and uniqueness result holds true (see [36, Theorem 3.1 and Remark 3.2]).

Theorem 2.2 (Existence of the evolution)**.** *Let Ω be a bounded open set in $\mathbb{R}^3$ with C^2 boundary. Let Γ_0 be a connected open subset of $\partial\Omega$ (in the relative topology of $\partial\Omega$) such that $\partial_{\partial\Omega}\Gamma_0$ is a connected, one-dimensional, C^2 manifold.*

Let $w \in W^{2,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3)) \cap C^3([0,T];L^2(\Omega;\mathbb{R}^3))$, and $(u^0, e^0, p^0) \in \mathscr{A}(w(0))$ be such that $\operatorname{div}\mathbb{C}e^0 = 0$ a.e. in Ω, $(\mathbb{C}e^0)\nu = 0$ $\mathcal{H}^2$-a.e. on $\partial\Omega \setminus \Gamma_0$, and $(\mathbb{C}e^0)_D \in K$ a.e. in Ω. Eventually, let $(u^1, e^1, 0) \in \mathscr{A}(\dot{w}(0))$.

Then there exist unique $u \in W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^3)) \cap \mathrm{Lip}(0,T;BD(\Omega))$, $e \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^3))$, and $p \in \mathrm{Lip}(0,T;\mathcal{M}_b(\Omega \cup \Gamma_0;\mathbb{M}^{3\times 3}_D))$ solving (c1), (c2) *and* (c3), *with $(u(0), e(0), p(0)) = (u^0, e^0, p^0)$, and $\dot{u}(0) = u^1$.*

2.9. **The WIDE functional.** Let the boundary datum $w \in W^{2,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3)) \cap C^3([0,T];L^2(\Omega;\mathbb{R}^3))$ be given. By reformulating the expression in (1.4) for the triple (u,e,p) one would be tempted to introduce the functional

$$(u,e,p) \mapsto \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\Big(\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot{u}|^2\,dx + \varepsilon\mathcal{H}(\dot{p}) + \int_\Omega Q(e)\,dx\Big)\,dt,$$

to be defined on the set $\mathcal{V}$, given by

$$\mathcal{V} := \{(u,e,p) \in W^{2,2}(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^1(0,T;BD(\Omega))$$

$$\times L^2((0,T)\times\Omega;\mathbb{M}^{3\times3}_{\rm sym})\times BV([0,T];\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)):$$
$$(u(t),e(t),p(t))\in\mathscr{A}(w(t))\text{ for a.e. } t\in[0,T],$$
$$Eu(t)=e(t)+p(t)\text{ in }\mathcal{D}'(\Omega;\mathbb{M}^{3\times3}_{\rm sym})\text{ for every } t\in[0,T],$$
$$u(0)=u^0,\ \dot u(0)=u^1,\ e(0)=e^0,\ p(0)=p^0\},\tag{2.12}$$

where $(u^0,e^0,p^0)\in\mathscr{A}(w(0))$, and $u^1\in BD(\Omega)$ is such that there exists a pair $(e^1,p^1)\in L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym})\times\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)$ satisfying $(u^1,e^1,p^1)\in\mathscr{A}(\dot w(0))$.

We observe that if $(u,e,p)\in\mathcal{V}$ then $Eu\in W^{2,2}(0,T;W^{-1,2}(\Omega;\mathbb{M}^{3\times3}_{\rm sym}))$. Thus $e(t)$ is defined for every $t\in[0,T]$ as a map in $W^{-1,2}(\Omega;\mathbb{M}^{3\times3}_{\rm sym})+\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)$, and the initial condition $e(0)=e^0$ is well justified. We recall that $BV([0,T];\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D))$ is the set of maps $\mu\in L^1(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D))$ such that $V_{\rm tot}(\mu;0,T)<+\infty$ (see also [11, Appendix]).

On the other hand, one readily sees that the term

$$\int_0^T\exp\Big(-\frac{t}{\varepsilon}\Big)\mathcal{H}(\dot p)\,dt$$

is not well defined in case p is not absolutely continuous with respect to time (see [11, Theorem 7.1]). We hence need to relax the form of the WIDE functional as

$$I_\varepsilon(u,e,p):=\int_0^T\exp\Big(-\frac{t}{\varepsilon}\Big)\Big(\frac{\varepsilon^2\rho}{2}\int_\Omega|\ddot u|^2\,dx+\int_\Omega Q(e)\,dx\Big)\,dt+\varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);p;0,T),\tag{2.13}$$

for every $(u,e,p)\in\mathcal{V}$. As pointed out in Subsection 2.7 an adaptation of [11, Theorem 7.1] yields

$$D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);p;0,T)=\int_0^T\exp\Big(-\frac{t}{\varepsilon}\Big)\mathcal{H}(\dot p)\,dt$$

whenever p is absolutely continuous with respect to time.

2.10. **Main result.** We are now ready to state the main result of the paper.

Theorem 2.3 (Dynamic perfect plasticity as convex minimization)**.** *Let Ω be a bounded open set in $\mathbb{R}^3$ with C^2 boundary. Let Γ_0 be a connected open subset of $\partial\Omega$ (in the relative topology of $\partial\Omega$) such that $\partial_{\partial\Omega}\Gamma_0$ is a connected, one-dimensional, C^2 manifold. Let $w\in W^{2,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3))\cap C^3([0,T];L^2(\Omega;\mathbb{R}^3))$, and $(u^0,e^0,p^0)\in\mathscr{A}(w(0))$ be such that $\operatorname{div}\mathbb{C}e^0=0$ a.e. in Ω, $(\mathbb{C}e^0)\nu=0$ $\mathcal{H}^2$-a.e. on $\partial\Omega\setminus\Gamma_0$, and $(\mathbb{C}e^0)_D\in K$ a.e. in Ω. Eventually, let $(u^1,e^1,0)\in\mathscr{A}(\dot w(0))$.*

For every $\varepsilon>0$ there exists $\{(u^\varepsilon,e^\varepsilon,p^\varepsilon)\}\subset\mathcal{V}$ solving

$$I_\varepsilon(u^\varepsilon,e^\varepsilon,p^\varepsilon)=\min_{(u,e,p)\in\mathcal{V}}I_\varepsilon(u,e,p),\tag{2.14}$$

such that for $\varepsilon\to0$ there holds

$$u^\varepsilon\rightharpoonup u\quad\text{weakly in } W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3)),$$
$$e^\varepsilon\rightharpoonup e\quad\text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym})).$$

Additionally, for every $t\in[0,T]$ we have

$$p^\varepsilon(t)\rightharpoonup^* p(t)\quad\text{weakly* in }\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D),$$

and for a.e. $t\in[0,T]$ there exists a t-dependent subsequence $\{\varepsilon_t\}$ such that

$$u^{\varepsilon_t}(t)\rightharpoonup^* u(t)\quad\text{weakly* in } BD(\Omega),$$
$$e^{\varepsilon_t}(t)\rightharpoonup e(t)\quad\text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym}),$$

where $u\in W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^3))\cap W^{1,\infty}(0,T;BD(\Omega))$, $e\in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^3))$, and $p\in W^{1,\infty}(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D))$ is the unique solution to the dynamic perfect plasticity problem (c1), (c2) *and* (c3)*, with $(u(0),e(0),p(0))=(u^0,e^0,p^0)$ and $\dot u(0)=u^1$.*

The rest of the paper is devoted to the proof of Theorem 2.3. Our argument runs as follows: we prove that minimizers $\{(u^\varepsilon, e^\varepsilon, p^\varepsilon)\}$ of Problem (2.14) exist in Section 3. Then, we devise an ε-independent a-priori estimate on $\{(u^\varepsilon, e^\varepsilon, p^\varepsilon)\}$ first in a discrete and then in a continuous setting (Section 4) by means of a Γ-convergence argument (Section 5). Then, we derive an energy inequality at level $\varepsilon > 0$ (Section 6) which allows discussing the limit $\varepsilon \to 0$ in Section 7.

We point out that the C^2 regularity of $\partial\Omega$ is needed in Theorem 2.3 in order to introduce a duality between stresses and plastic strains, along the footsteps of [25, Proposition 2.5]. Due to technical reasons it is not possible to use here the results in [18] and consider the case of a Lipschitz $\partial\Omega$. We refer to Remark 4.5 for some discussion of this point.

3. Minimizers of the WIDE functional

We start by focusing here on Problem (2.14) and show that the functional I_ε admits a minimizer in $\mathcal{V}$.

Proposition 3.1 (Existence of minimizers). *For every $\varepsilon > 0$ there exists a triple $(u^\varepsilon, e^\varepsilon, p^\varepsilon) \in \mathcal{V}$ such that*

$$I_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon) = \inf_{(u,e,p)\in\mathcal{V}} I_\varepsilon(u, e, p). \tag{3.1}$$

Proof. Fix $\varepsilon > 0$, and let $\{(u_n, e_n, p_n)\} \subset \mathcal{V}$ be a minimizing sequence for I_ε. We first observe that the triple

$$t \to (u^0 + tu^1 + w(t) - w(0) - t\dot{w}(0), e^0 + te^1 + Ew(t) - Ew(0) - tE\dot{w}(0), p^0 + tp^1)$$

belongs to $\mathcal{V}$. Hence,

$$\begin{aligned}\lim_{n\to+\infty} I_\varepsilon(u_n, e_n, p_n) \le \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\Big(\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot{w}|^2\,dx + \varepsilon\mathcal{H}(p^1)\\ + \int_\Omega Q(e^0 + te^1 + Ew(t) - Ew(0) - tE\dot{w}(0))\,dx\Big)\,dt \le C,\end{aligned}$$

thus yielding the uniform bound

$$\begin{aligned}\sup_{n\in\mathbb{N}}\Big\{\|\ddot{u}_n\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} + D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p_n; 0, T)\\ + \|e_n\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym}))}\Big\} \le C.\end{aligned} \tag{3.2}$$

Since $(u_n, e_n, p_n) \in \mathcal{V}$, there holds $p_n(0) = p^0$ for every $n \in \mathbb{N}$. In view of (2.7) and (2.8),

$$r_K \exp(-T/\varepsilon)V_{\rm tot}(p_n; 0, T) \le \exp(-T/\varepsilon)D_{\mathcal{H}}(p_n; 0, T) \le D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p_n; 0, T).$$

Therefore we are in a position of applying the variant of Helly's theorem in [11, Lemma 7.2] and to deduce the existence of a subsequence, still denoted by $\{p_n\}$ and a map $p^\varepsilon \in BV(0, T; \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times3}_D))$, such that

$$p_n(t) \rightharpoonup^* p^\varepsilon(t) \text{ weakly* in } \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times3}_D) \quad \text{for every } t \in [0, T], \tag{3.3}$$

and by the lower semicontinuity of the $\mathcal{H}$-dissipation,

$$D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p^\varepsilon; 0, T) \le \liminf_{n\to+\infty} D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p_n; 0, T). \tag{3.4}$$

By (3.2), there exist $e^\varepsilon \in L^2(0, T; L^2(\Omega; \mathbb{M}^{3\times3}_{\rm sym}))$ and $u^\varepsilon \in W^{2,2}(0, T; L^2(\Omega; \mathbb{R}^3))$ such that, up to the extraction of a (non-relabeled) subsequence,

$$e_n \rightharpoonup e^\varepsilon \quad \text{weakly in } L^2(0, T; L^2(\Omega; \mathbb{M}^{3\times3}_{\rm sym})), \tag{3.5}$$

and

$$u_n \rightharpoonup u^\varepsilon \quad \text{weakly in } W^{2,2}(0, T; L^2(\Omega; \mathbb{R}^3)). \tag{3.6}$$

This implies that $u^\varepsilon(0) = u^0$ and $\dot{u}^\varepsilon(0) = u^1$. By (3.3), (3.5), and (3.6) it follows that

$$e_n(t) \rightharpoonup e^\varepsilon(t) \quad \text{weakly in } \mathcal{D}'(\Omega; \mathbb{M}^{3\times3}_{\rm sym}) \tag{3.7}$$

for every $t \in [0,T]$, and hence $e^\varepsilon(0) = e^0$. In view of (3.5) and Fatou's lemma there holds

$$\int_0^T \liminf_{n\to+\infty} \int_\Omega |e_n|^2\,dx\,dt \le \liminf_{n\to+\infty} \int_0^T \int_\Omega |e_n|^2\,dx\,dt \le C.$$

Thus, by (3.7) for a.e. $t \in [0,T]$ there exists a t-dependent subsequence $\{n_t\}$ such that

$$e_{n_t}(t) \rightharpoonup e^\varepsilon(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym}). \tag{3.8}$$

Finally, by (2.1), (3.3), and (3.7), up to subsequences there holds

$$u_{n_t}(t) \rightharpoonup^* u^\varepsilon(t) \quad \text{weakly* in } BD(\Omega) \quad \text{for a.e. } t \in [0,T].$$

The fact that p^ε satisfies the boundary condition on Γ_0 for a.e. $t \in [0,T]$ follows arguing as in [11, Lemma 2.1]. The minimality of the limit triple $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ is a direct consequence of the lower semicontinuity of I_ε with respect to the convergences in (3.4), (3.5), and (3.6). □

We conclude this section with a conditional uniqueness result.

Proposition 3.2 (Uniqueness of minimizers given the plastic strain)**.** *Let (u_a, e_a, p_a) and (u_b, e_b, p_b) be two minimizers of I_ε in $\mathcal{V}$. Then there exists a constant C such that*

$$\begin{aligned} &\varepsilon^2 \|u_a - u_b\|^2_{W^{2,2}(0,T;L^2(\Omega;\mathbb{R}^3))} + \|e_a - e_b\|^2_{L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym}))} \\ &\quad \le C\varepsilon \exp\Big(\frac{T}{\varepsilon}\Big) V_{\rm tot}(p_a - p_b; 0, T). \end{aligned} \tag{3.9}$$

Proof. Arguing as in [11, Theorem 3.8], we set $v = u_a - u_b$, $f = e_a - e_b$, and $q = p_a - p_b$. Since $(v,f,q) \in \mathscr{A}(0)$, it follows that $(u_a, e_a, p_a) + \lambda(v,f,q) \in \mathcal{V}$ for every $\lambda \in \mathbb{R}$. Thus,

$$\begin{aligned} I_\varepsilon(u_a, e_a, p_a) &\le I_\varepsilon((u_a, e_a, p_a) + \lambda(v,f,q)) \\ &= \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\Big(\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot u_a + \lambda \ddot v|^2\,dx + \int_\Omega Q(e_a + \lambda f)\,dx\Big)\,dt \\ &\quad + \varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p_a + \lambda q; 0, T). \end{aligned}$$

By the arbitrariness of λ we deduce the inequality

$$\begin{aligned} &-\varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); q; 0,T) \le \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega \big(\varepsilon^2\rho \ddot u_a \ddot v + \mathbb{C}e_a : f\big)\,dx\,dt \\ &\quad \le \varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); -q; 0,T). \end{aligned} \tag{3.10}$$

Arguing analogously, the minimality of (u_b, e_b, p_b) yields

$$\begin{aligned} &-\varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); -q; 0,T) \le -\int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega \big(\varepsilon^2\rho \ddot u_b \ddot v + \mathbb{C}e_b : f\big)\,dx\,dt \\ &\quad \le \varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); q; 0,T). \end{aligned} \tag{3.11}$$

Summing (3.10) and (3.11) we obtain

$$\begin{aligned} &-\varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p_a - p_b; 0,T) - \varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p_b - p_a; 0,T) \\ &\quad \le \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega \big(\varepsilon^2\rho|\ddot u_a - \ddot u_b|^2 + 2Q(e_a - e_b)\big)\,dx\,dt \\ &\quad \le \varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p_a - p_b; 0,T) + \varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p_b - p_a; 0,T). \end{aligned}$$

The thesis follows now by (2.2), (2.7), and (2.8). □

Remark 3.3. Let us point out that the previous proposition can alternatively be read as a Lipschitz regularity result for the solution operator associated to the reduced problem $p \mapsto \operatorname{Argmin} I_\varepsilon(\cdot,\,\cdot,\,p)$.

4. Discrete energy estimate

With the aim of establishing an a-priori estimate on $\{(u^\varepsilon, e^\varepsilon, p^\varepsilon)\}$ independent of ε we start by analyzing a time-discrete version of the problem. Fix $n \in \mathbb{N}$, set $\tau := T/n$, and consider the time partition

$$0 = t_0 < t_1 < \cdots < t_n = T, \qquad t_i := i\tau.$$

We define $w_0 := w(0)$, $w_1 := w_0 + \tau \dot{w}(0)$, and, for $i = 2, \ldots, n$, we set $w_i := w(t_i)$. Our analysis will be set in the space

$$\begin{aligned}\mathscr{U}_\tau := \Big\{ &(u_0, e_0, p_0), \ldots, (u_n, e_n, p_n) \in \big(BD(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times 3}_{\rm sym}) \times \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D)\big)^{n+1} : \\ &(u_i, e_i, p_i) \in \mathscr{A}(w_i) \quad \text{for } i = 1, \ldots, n \Big\}.\end{aligned} \tag{4.1}$$

We define the *discrete energy functional* $I_{\varepsilon\tau} : \mathscr{U}_\tau \to [0, +\infty)$ as

$$\begin{aligned} I_{\varepsilon\tau}\big((u_0, e_0, p_0), \ldots, (u_n, e_n, p_n)\big) := &\frac{\varepsilon^2 \rho}{2} \sum_{i=2}^n \tau \eta_{\tau,i} \int_\Omega |\delta^2 u_i|^2 \, dx + \sum_{i=2}^{n-2} \tau \eta_{\tau,i+2} \int_\Omega Q(e_i) \, dx \\ &+ \varepsilon\tau \sum_{i=1}^{n-1} \eta_{\tau,i+1} \mathcal{H}(\delta p_i), \end{aligned} \tag{4.2}$$

where, given a vector $v = (v_1, \ldots, v_n)$, the operator δ denotes its discrete derivative,

$$\delta v_i := \frac{v_i - v_{i-1}}{\tau}, \qquad \delta^k v_i := \frac{\delta^{k-1} v_i - \delta^{k-1} v_{i-1}}{\tau},$$

for $k \in \mathbb{N}$, $k > 1$, and where the weights

$$\eta_{\tau,i} := \left(\frac{\varepsilon}{\varepsilon + \tau}\right)^i, \quad i = 0, \ldots, n,$$

are a discretization of the map $t \to \exp\big(-t/\varepsilon\big)$. Define the set

$$\begin{aligned}\mathscr{K}_\tau(u^0, e^0, p^0, u^1) := \{ &(u_0, e_0, p_0), \ldots, (u_n, e_n, p_n) \in \mathscr{U}_\tau : \, u_0 = u^0, \, e_0 = e^0, \, p_0 = p^0, \\ &\delta u_1 = u^1 \}.\end{aligned}$$

Arguing as in Proposition 3.1 we obtain the following result.

Lemma 4.1. *There exists a $(n+1)$-tuple of triples $(u^\varepsilon_k, e^\varepsilon_k, p^\varepsilon_k)$ such that $\big((u^\varepsilon_0, e^\varepsilon_0, p^\varepsilon_0), \ldots, (u^\varepsilon_n, e^\varepsilon_n, p^\varepsilon_n)\big) \in \mathscr{K}_\tau(u^0, e^0, p^0, u^1)$, and*

$$\begin{aligned} &I_{\varepsilon\tau}\big((u^\varepsilon_0, e^\varepsilon_0, p^\varepsilon_0), \ldots, (u^\varepsilon_n, e^\varepsilon_n, p^\varepsilon_n)\big) \\ &\quad = \min_{\big((u_0,e_0,p_0),\ldots,(u_n,e_n,p_n)\big) \in \mathscr{K}_\tau(u^0,e^0,p^0,u^1)} I_{\varepsilon\tau}\big((u_0, e_0, p_0), \ldots, (u_n, e_n, p_n)\big). \end{aligned} \tag{4.3}$$

4.1. **Discrete Euler-Lagrange equations.** We first compute the discrete Euler-Lagrange equations satisfied by a minimizing $(n+1)$-tuple $(u^\varepsilon_0, e^\varepsilon_0, p^\varepsilon_0), \ldots, (u^\varepsilon_n, e^\varepsilon_n, p^\varepsilon_n)$.

Proposition 4.2 (Discrete Euler-Lagrange equations)**.** *Let $(u^\varepsilon_0, e^\varepsilon_0, p^\varepsilon_0), \ldots, (u^\varepsilon_n, e^\varepsilon_n, p^\varepsilon_n)$ be a solution to (4.3). Then*

$$\sum_{i=2}^n \varepsilon^2 \rho \eta_{\tau,i} \int_\Omega \delta^2 u^\varepsilon_i \cdot \delta^2 \varphi_i \, dx + \sum_{i=2}^{n-2} \eta_{\tau,i+2} \int_\Omega \mathbb{C} e^\varepsilon_i : E\varphi_i \, dx = 0 \tag{4.4}$$

for every $\varphi_i \in W^{1,2}(\Omega; \mathbb{R}^3)$ such that $\varphi_i = 0$ $\mathcal{H}^2$-a.e. on Γ_0, $i = 2, \ldots, n$. In addition,

$$-\Big(\frac{\varepsilon}{\varepsilon + \tau}\Big)\mathcal{H}(\xi) - \mathcal{H}(-\xi) \le \Big(\frac{\tau}{\varepsilon + \tau}\Big) \int_\Omega \mathbb{C} e^\varepsilon_i : \xi \, dx \le \mathcal{H}(\xi) + \Big(\frac{\varepsilon}{\varepsilon + \tau}\Big)\mathcal{H}(-\xi), \tag{4.5}$$

for every $\xi \in L^2(\Omega; \mathbb{M}^{3\times 3}_D)$, $i = 2, \ldots, n-2$.

Proof. Let $(v_0, f_0, q_0), \dots, (v_n, f_n, q_n) \in \big(BD(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times3}_{\rm sym}) \times \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times3}_D)\big)^{n+1}$ be such that $(v_i, f_i, q_i) \in \mathscr{A}(0)$ for $i = 1, \dots, n$, with $v_0 = f_0 = q_0 = \delta v_1 = 0$. Consider the $(n+1)$-tuple

$$(u^\varepsilon_0 \pm \lambda v_0, e^\varepsilon_0 \pm \lambda f_0, p^\varepsilon_0 \pm \lambda q_0), \dots (u^\varepsilon_n \pm \lambda v_n, e^\varepsilon_n \pm \lambda f_n, p^\varepsilon_n \pm \lambda q_n),$$

with $\lambda > 0$. By the minimality of $(u^\varepsilon_0, e^\varepsilon_0, p^\varepsilon_0), \dots, (u^\varepsilon_n, e^\varepsilon_n, p^\varepsilon_n)$, there holds

$$\begin{aligned}\frac{1}{\lambda} I_{\varepsilon\tau}\big((u^\varepsilon_0 \pm \lambda v_0, e^\varepsilon_0 \pm \lambda f_0, p^\varepsilon_0 \pm \lambda q_0), \dots, (u^\varepsilon_n \pm \lambda v_n, e^\varepsilon_n \pm \lambda f_n, p^\varepsilon_n \pm \lambda q_n)\big)\\ - \frac{1}{\lambda} I_{\varepsilon\tau}\big((u^\varepsilon_0, e^\varepsilon_0, p^\varepsilon_0), \dots, (u^\varepsilon_n, e^\varepsilon_n, p^\varepsilon_n)\big) \geq 0.\end{aligned}$$

Therefore by (2.5) and (4.2) we deduce the inequality

$$\begin{aligned}-\varepsilon\tau \sum_{i=1}^{n-1} \eta_{\tau,i+1} \mathcal{H}(\delta q_i) &\leq \varepsilon^2 \rho \sum_{i=2}^{n} \tau \eta_{\tau,i} \int_\Omega \delta^2 u^\varepsilon_i \cdot \delta^2 v_i \, dx + \sum_{i=2}^{n-2} \tau \eta_{\tau,i+2} \int_\Omega \mathbb{C} e^\varepsilon_i : f_i \, dx\\ &\leq \varepsilon\tau \sum_{i=1}^{n-1} \eta_{\tau,i+1} \mathcal{H}(-\delta q_i). \end{aligned} \tag{4.6}$$

For $i = 0, \dots, n$, let $\varphi_i \in W^{1,2}(\Omega; \mathbb{R}^3)$ with $\varphi_i = 0$ $\mathcal{H}^2$-a.e. on Γ_0, and let $\xi_i \in L^2(\Omega; \mathbb{M}^{3\times3}_D)$. Choosing $v_i = \varphi_i$, $f_i = E\varphi_i$, and $q_i = 0$, for $i = 1, \dots, n$, by (4.6) we obtain

$$\varepsilon^2 \rho \sum_{i=2}^{n} \tau \eta_{\tau,i} \int_\Omega \delta^2 u^\varepsilon_i \cdot \delta^2 \varphi_i \, dx + \sum_{i=2}^{n-2} \tau \eta_{\tau,i+2} \int_\Omega \mathbb{C} e^\varepsilon_i : E\varphi_i \, dx = 0$$

for every $\varphi_1, \dots, \varphi_n \in W^{1,2}(\Omega; \mathbb{R}^3)$, $\varphi_i = 0$ $\mathcal{H}^2$-a.e. on Γ_0, $i = 0, \dots, n$, and hence (4.4). Choosing $v_i = 0$, $f_i = \xi_i$, and $q_i = -\xi_i$ for $i = 1, \dots, n$, estimate (4.6) yields

$$-\varepsilon\tau \sum_{i=1}^{n-1} \eta_{\tau,i+1} \mathcal{H}(-\delta\xi_i) \leq \sum_{i=2}^{n-2} \tau \eta_{\tau,i+2} \int_\Omega \mathbb{C} e^\varepsilon_i : \xi_i \, dx \leq \varepsilon\tau \sum_{i=1}^{n-1} \eta_{\tau,i+1} \mathcal{H}(\delta\xi_i),$$

for every $\xi_1, \dots, \xi_n \in L^2(\Omega; \mathbb{M}^{3\times3}_D)$, and thus (4.5). □

We observe that it follows from (4.5) that $(\mathbb{C} e^\varepsilon_i)_D \in L^\infty(\Omega; \mathbb{M}^{3\times3}_D)$ for every i and ε, although the bound is not uniform with respect to τ nor ε. Indeed, for every B Borel subset of Ω and for every $M \in \mathbb{M}^{3\times3}_D$ we can choose $\xi = M\chi_B$ in (4.5), where χ_B denotes the characteristic function of B. We have

$$-\Big(\frac{\varepsilon}{\varepsilon+\tau}\Big) H(M) - H(-M) \leq \Big(\frac{\tau}{\varepsilon+\tau}\Big) \mathbb{C} e^\varepsilon_i(x) : M \leq H(M) + \Big(\frac{\varepsilon}{\varepsilon+\tau}\Big) H(-M), \tag{4.7}$$

for $i = 2, \dots, n-2$, and a.e. $x \in \Omega$, which by (2.4) imply

$$-2R_K |M| \leq \Big(\frac{\tau}{\varepsilon+\tau}\Big) \mathbb{C} e^\varepsilon_i(x) : M \leq 2R_K |M|,$$

for $i = 2, \dots, n-2$, and every $M \in \mathbb{M}^{3\times3}_D$, for a.e. $x \in \Omega$. Thus we get the estimate

$$\|(\mathbb{C} e^\varepsilon_i)_D\|_{L^\infty(\Omega; \mathbb{M}^{3\times3}_D)} \leq 2\Big(\frac{\varepsilon+\tau}{\tau}\Big) R_K, \tag{4.8}$$

for $i = 2, \dots, n-2$.

As a consequence of inequality (4.7), the deviatoric parts of the discrete stresses $\sigma^\varepsilon_i := \mathbb{C} e^\varepsilon_i$, $i = 2, \dots, n-2$, belong to the subdifferential in 0 of suitable convex and positively 1-homogeneous functions. Indeed, by (4.7) we have

$$\Big(\frac{\tau}{\varepsilon+\tau}\Big) \sigma^\varepsilon_i(x) \in \partial F^\varepsilon_H(0), \quad \text{for a.e. } x \in \Omega,\ i = 2, \dots, n-2,$$

where $F^\varepsilon_H : \mathbb{M}^{3\times3}_D \to [0, +\infty)$ is defined as

$$F^\varepsilon_H(M) := H(M) + \Big(\frac{\varepsilon}{\varepsilon+\tau}\Big) H(-M),$$

for every $M \in \mathbb{M}^{3\times 3}_D$. The convexity and positive one-homogeneity of F^ε_H follow directly by the corresponding properties of H.

Equation (4.4) can be equivalently reformulated in the following useful form.

Proposition 4.3 (Discrete Euler-Lagrange equations 2)**.** *Let* $(u^\varepsilon_0, e^\varepsilon_0, p^\varepsilon_0), \dots, (u^\varepsilon_n, e^\varepsilon_n, p^\varepsilon_n)$ *be a solution to* (4.3)*. Then*

$$\delta^2 u^\varepsilon_n = \delta^3 u^\varepsilon_n = 0, \tag{4.9}$$

$$\int_\Omega \left[\rho(\varepsilon^2\delta^4 u^\varepsilon_{i+2} - 2\varepsilon\delta^3 u^\varepsilon_{i+1} + \delta^2 u^\varepsilon_i)\cdot\varphi + \mathbb{C}e^\varepsilon_i : E\varphi\right] dx = 0 \tag{4.10}$$

for $i = 2, \dots, n-2$*, and for every* $\varphi \in W^{1,2}(\Omega;\mathbb{R}^3)$ *with* $\varphi = 0$ $\mathcal{H}^2$*-a.e. on* Γ_0*.*

We omit the proof of this proposition as it follows arguing exactly as in [49, Subsection 2.3]. In view of (4.10) there holds

$$\begin{cases} \operatorname{div}\mathbb{C}e^\varepsilon_i = \rho(\varepsilon^2\delta^4 u^\varepsilon_{i+2} - 2\varepsilon\delta^3 u^\varepsilon_{i+1} + \delta^2 u^\varepsilon_i) & \text{a.e. in } \Omega,\\ \mathbb{C}e^\varepsilon_i\nu = 0 & \mathcal{H}^2\text{-a.e. on } \partial\Omega\setminus\Gamma_0, \end{cases} \tag{4.11}$$

and hence, $\operatorname{div}\mathbb{C}e^\varepsilon_i \in BD(\Omega)\cap L^2(\Omega;\mathbb{R}^3)$, $i = 2, \dots, n-2$.

4.2. **Stress-strain duality.** In order to establish a uniform discrete energy estimate we need to preliminary introduce a notion of duality for the discrete stresses σ^ε_i and the plastic strains p^ε_i.

We work along the footsteps of [25] and [11, Subsection 2.3]. Define the set

$$\Sigma(\Omega) := \{\sigma \in L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym}) : \sigma_D \in L^\infty(\Omega;\mathbb{M}^{3\times 3}_D) \text{ and } \operatorname{div}\sigma \in BD(\Omega)\cap L^2(\Omega;\mathbb{R}^3)\}. \tag{4.12}$$

By [25, Proposition 2.5] for every $\sigma \in \Sigma(\Omega)$ there holds

$$\sigma \in L^6(\Omega;\mathbb{M}^{3\times 3}_{\rm sym}),$$

and

$$\|\mathrm{tr}\sigma\|_{L^6(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})} \le C\big(\|\sigma\|_{L^1(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})} + \|\sigma_D\|_{L^\infty(\Omega;\mathbb{M}^{3\times 3}_D)} + \|\operatorname{div}\sigma\|_{L^2(\Omega;\mathbb{R}^3)}\big).$$

In addition, we can introduce the trace $[\sigma\nu] \in W^{-1/2,2}(\partial\Omega;\mathbb{R}^3)$ (see e.g. [51, Theorem 1.2, Chapter I]) as

$$\langle[\sigma\nu],\,\psi\rangle_{\partial\Omega} := \int_\Omega \operatorname{div}\sigma\cdot\psi\,dx + \int_\Omega \sigma : E\psi\,dx$$

for every $\psi \in W^{1,2}(\Omega;\mathbb{R}^3)$. Defining the normal and the tangential part of $[\sigma\nu]$ as

$$[\sigma\nu]_\nu := ([\sigma\nu]\cdot\nu)\nu \quad\text{and}\quad [\sigma\nu]^\perp_\nu := [\sigma\nu] - ([\sigma\nu]\cdot\nu)\nu,$$

by [25, Lemma 2.4] we have that $[\sigma\nu]^\perp_\nu \in L^\infty(\partial\Omega;\mathbb{R}^3)$, and

$$\|[\sigma\nu]^\perp_\nu\|_{L^\infty(\partial\Omega;\mathbb{R}^3)} \le \frac{1}{\sqrt{2}}\|\sigma_D\|_{L^\infty(\Omega;\mathbb{M}^{3\times 3}_D)}.$$

Let $\sigma \in \Sigma(\Omega)$ and let $u \in BD(\Omega)\cap L^2(\Omega;\mathbb{R}^3)$, with $\operatorname{div} u \in L^2(\Omega)$. We define the distribution $[\sigma_D : E_D u]$ on Ω as

$$\langle[\sigma_D : E_D u],\,\varphi\rangle := -\int_\Omega \varphi\operatorname{div}\sigma\cdot u\,dx - \frac{1}{3}\int_\Omega \varphi\,\mathrm{tr}\,\sigma\cdot\operatorname{div} u\,dx - \int_\Omega \sigma : (u\odot\nabla\varphi)\,dx \tag{4.13}$$

for every $\varphi \in C^\infty_c(\Omega)$. By [25, Theorem 3.2] it follows that $[\sigma_D : E_D u]$ is a bounded Radon measure on Ω, whose variation satisfies

$$|[\sigma_D : E_D u]| \le \|\sigma_D\|_{L^\infty(\Omega;\mathbb{M}^{3\times 3}_D)}|E_D u| \quad \text{in } \Omega.$$

Let $\Pi_{\Gamma_0}(\Omega)$ be the set of admissible plastic strains, namely the set of maps $p \in \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)$ such that there exist $u \in BD(\Omega)\cap L^2(\Omega;\mathbb{R}^3)$, $e \in L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})$, and $w \in W^{1,2}(\Omega;\mathbb{R}^3)$ with $(u,e,p) \in \mathscr{A}(w)$. Note that the additive decomposition $Eu = e + p$ implies that $\operatorname{div} u \in L^2(\Omega)$.

It is possible to define a duality between elements of $\Sigma(\Omega)$ and $\Pi_{\Gamma_0}(\Omega)$. To be precise, given $p \in \Pi_{\Gamma_0}(\Omega)$, and $\sigma \in \Sigma(\Omega)$, we fix (u, e, w) such that $(u, e, p) \in \mathscr{A}(w)$, with $u \in L^2(\Omega;\mathbb{R}^3)$, and we define the measure $[\sigma_D : p] \in \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D)$ as

$$[\sigma_D : p] := \begin{cases} [\sigma_D : E_D u] - \sigma_D : e_D & \text{in } \Omega \\ [\sigma\nu]^\perp_\nu \cdot (w-u)\mathcal{H}^2 & \text{on } \Gamma_0, \end{cases}$$

so that

$$\int_{\Omega\cup\Gamma_0} \varphi\, d[\sigma_D : p] = \int_\Omega \varphi\, d[\sigma_D : E_D u] - \int_\Omega \varphi\sigma_D : e_D\, dx + \int_{\Gamma_0} \varphi[\sigma\nu]^\perp_\nu \cdot (w-u)\, d\mathcal{H}^2$$

for every $\varphi \in C(\bar\Omega)$. Arguing as in [11, Section 2] one can prove that the definition of $[\sigma_D : p]$ is independent of the choice of (u, e, w), and that if $\sigma_D \in C(\bar\Omega; \mathbb{M}^{3\times 3}_D)$ and $\varphi \in C(\bar\Omega)$, then

$$\int_{\Omega\cup\Gamma_0} \varphi\, d[\sigma_D : p] = \int_{\Omega\cup\Gamma_0} \varphi\sigma_D : dp.$$

We finally rewrite [11, Proposition 2.2] in our framework.

Proposition 4.4. *Let $\sigma \in \Sigma(\Omega)$, $w \in W^{1,2}(\Omega;\mathbb{R}^3)$, and $(u, e, p) \in \mathscr{A}(w)$, with $u \in L^2(\Omega;\mathbb{R}^3)$. Then*

$$[\sigma_D : p](\Omega\cup\Gamma_0) + \int_\Omega \sigma : (e - Ew)\, dx = -\int_\Omega \operatorname{div}\sigma \cdot (u-w)\, dx + \int_{\partial\Omega\setminus\Gamma_0} [\sigma\nu]\cdot(u-w)\, dx.$$

Remark 4.5. We point out that the C^2 regularity of $\partial\Omega$ is needed here in order to apply [25, Proposition 2.5]. It is not possible to use here the results in [18] and extend the analysis to the case in which $\partial\Omega$ is Lipschitz, as (4.11) only implies that $\operatorname{div}\mathbb{C}e^\varepsilon_i \in L^2(\Omega;\mathbb{R}^3)$, whereas [18, Proposition 6.1] would require $\operatorname{div}\mathbb{C}e^\varepsilon_i \in L^3(\Omega;\mathbb{R}^3)$.

4.3. **Discrete energy estimate.** We preliminary establish a lower bound on the mass of the measures $[(\mathbb{C}e^\varepsilon_i)_D : q]$, $i = 2, \dots, n-2$, where $q \in \Pi_{\Gamma_0}(\Omega)$ is such that there exist $v \in BD(\Omega)\cap L^2(\Omega;\mathbb{R}^3)$ and $f \in L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})$ satisfying $(v, f, q) \in \mathscr{A}(0)$.

A caveat on notation: in the following we use the symbol C to indicate a generic constant, possibly depending on data and varying from line to line.

The following estimate holds true.

Proposition 4.6. *Let $q \in \Pi_{\Gamma_0}(\Omega)$, $v \in BD(\Omega)\cap L^2(\Omega;\mathbb{R}^3)$ and $f \in L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})$ be such that $(v, f, q) \in \mathscr{A}(0)$. Then*

$$\tau[(\mathbb{C}e^\varepsilon_i)_D : q](\Omega\cup\Gamma_0) + (\varepsilon+\tau)\mathcal{H}(\delta p^\varepsilon_i - q) + \varepsilon\mathcal{H}(q) \geq (\varepsilon+\tau)\mathcal{H}(\delta p^\varepsilon_i) \tag{4.14}$$

for every $i = 2, \dots, n-2$.

Proof. Let q be as in the statement of the proposition. By (4.8) and (4.11) it follows that $\mathbb{C}e^\varepsilon_i \in \Sigma(\Omega)$, $i = 2, \dots, n-2$. In view of the triangular inequality (2.5), since $(u^\varepsilon_0, e^\varepsilon_0, p^\varepsilon_0), \dots, (u^\varepsilon_n, e^\varepsilon_n, p^\varepsilon_n)$ is a solution to (4.3) it also solves the implicit minimum problem

$$I_{\varepsilon\tau}\big((u^\varepsilon_0, e^\varepsilon_0, p^\varepsilon_0), \dots, (u^\varepsilon_n, e^\varepsilon_n, p^\varepsilon_n)\big) = \min_{(u_0,e_0,p_0),\dots,(u_n,e_n,p_n)\in\mathscr{K}_\tau(u^0,e^0,p^0,u^1)} J_{\varepsilon\tau}\big((u_0, e_0, p_0), \dots, (u_n, e_n, p_n)\big)$$

where

$$J_{\varepsilon\tau}\big((u_0, e_0, p_0), \dots, (u_n, e_n, p_n)\big) := \frac{\varepsilon^2\rho}{2}\sum_{j=2}^n \tau\eta_{\tau,j}\int_\Omega |\delta^2 u_j|^2\, dx + \sum_{j=2}^{n-2}\tau\eta_{\tau,j+2}\int_\Omega Q(e_j)\, dx + \varepsilon\tau\sum_{j=1}^{n-1}\eta_{\tau,j+1}\bigg[\mathcal{H}\Big(\frac{p_j - p^\varepsilon_{j-1}}{\tau}\Big) + \mathcal{H}\Big(\frac{p^\varepsilon_{j-1} - p_{j-1}}{\tau}\Big)\bigg].$$

Arguing as in Proposition 4.2 we compute the Euler-Lagrange equations associated to the minimum problem above, and we perform variations $(u^\varepsilon_0 \pm \lambda v_0, e^\varepsilon_0 \pm \lambda f_0, p^\varepsilon_0 \pm \lambda q_0), \dots (u^\varepsilon_n \pm \lambda v_n, e^\varepsilon_n \pm \lambda f_n, p^\varepsilon_n \pm \lambda q_n)$, with $\lambda > 0$, and $(v_0, f_0, q_0), \dots, (v_n, f_n, q_n) \in \big(BD(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times 3}_{\rm sym}) \times \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D)\big)^{n+1}$ such that $(v_i, f_i, q_i) \in \mathscr{A}(0)$ for $i = 1, \dots, n$, with $v_0 = f_0 = q_0 = \delta v_1 = 0$. The convexity of $\mathcal{H}$ yields

$$\varepsilon^2 \rho \sum_{j=2}^{n} \tau \eta_{\tau,j} \int_\Omega \delta^2 u_j^\varepsilon \cdot \delta^2 v_j \, dx + \sum_{j=2}^{n-2} \tau \eta_{\tau,j+2} \int_\Omega \mathbb{C} e_j^\varepsilon : f_j \, dx$$
$$+ \varepsilon\tau \sum_{j=1}^{n-1} \eta_{\tau,j+1} \left[\mathcal{H}\Big(\delta p_j^\varepsilon + \frac{q_j}{\tau}\Big) - \mathcal{H}(\delta p_j^\varepsilon) + \mathcal{H}\Big(-\frac{q_{j-1}}{\tau}\Big) \right] \geq 0.$$

By combining Proposition 4.4 with the Euler-Lagrange equation (4.11), and performing the discrete integration by parts in [49, Subsection 2.3], we have

$$- \sum_{j=2}^{n-2} \tau \eta_{\tau,j+2} [(\mathbb{C} e_j^\varepsilon)_D : q_j](\Omega \cup \Gamma_0)$$
$$+ \varepsilon\tau \sum_{j=1}^{n-1} \eta_{\tau,j+1} \left[\mathcal{H}\Big(\delta p_j^\varepsilon + \frac{q_j}{\tau}\Big) - \mathcal{H}(\delta p_j^\varepsilon) + \mathcal{H}\Big(-\frac{q_{j-1}}{\tau}\Big) \right] \geq 0.$$

The thesis follows choosing $q_j = -\tau q$ for $j = i$, and $q_j = 0$ otherwise. $\square$

Given a vector $(w_0, \dots, w_n)$ we denote by $\bar{w}_\tau$ and w_τ its backward piecewise-constant and its piecewise-affine interpolants on the partition, that is

$$\bar{w}_\tau(0) = w_\tau(0) = w_0, \quad \bar{w}_\tau(t) = w_i, \quad w_\tau(t) := \alpha_\tau(t) w_i + (1 - \alpha_\tau(t)) w_{i-1} \tag{4.15}$$

for $t \in ((i-1)\tau, i\tau]$, $i = 1, \dots, n$, where

$$\alpha_\tau(t) := \frac{(t - (i-1)\tau)}{\tau} \quad \text{for } t \in ((i-1)\tau, i\tau], \quad i = 1, \dots, n.$$

In particular, $\dot{w}_\tau(t) = \overline{\delta w}_\tau(t)$ for almost every $t \in (0, T)$. Analogously, we define the piecewise constant maps

$$\bar{\eta}_\tau(t) := \eta_{\tau,i} \quad \text{for } t \in ((i-1)\tau, i\tau], \quad i = 1, \dots, n.$$

In addition, as in [49, Subsection 2.5.1] we denote by $\tilde{w}_\tau$ the piecewise quadratic interpolants, defined via

$$\tilde{w}_\tau(t) := w_\tau(t) \quad \text{in } [0, \tau]$$
$$\dot{\tilde{w}}_\tau(t) = \alpha_\tau(t) \dot{w}_\tau(t) + (1 - \alpha_\tau(t)) \dot{w}_\tau(t - \tau) \quad \text{in } (\tau, T]. \tag{4.16}$$

Notice that

$$\dot{\tilde{w}}_\tau(t) = \dot{w}_\tau(t - \tau) + \tau \alpha_\tau(t) \ddot{w}_\tau(t) \quad \text{for a.e. } t \in (\tau, T].$$

Theorem 4.7 (Discrete energy estimate)**.** *Let $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon), \dots, (u_n^\varepsilon, e_n^\varepsilon, p_n^\varepsilon)$, be a solution to (4.3). Assume in addition that $p^1 = 0$. Let $(\bar{u}_\tau^\varepsilon, \bar{e}_\tau^\varepsilon, \bar{p}_\tau^\varepsilon)$,$(u_\tau^\varepsilon, e_\tau^\varepsilon, p_\tau^\varepsilon)$ and $(\tilde{u}_\tau^\varepsilon, \tilde{e}_\tau^\varepsilon, \tilde{p}_\tau^\varepsilon)$ be the triples of associated piecewise-constant,piecewise-affine, and piecewise-quadratic interpolants, respectively. Then there exists a constant C (independent of ε and τ) such that*

$$\varepsilon\rho \int_{2\tau}^{T-2\tau} \int_\Omega |\ddot{\tilde{u}}_\tau^\varepsilon|^2 \, dx \, dt + \varepsilon\rho \int_{2\tau}^{T-2\tau} \int_{2\tau}^{t} \int_\Omega |\ddot{\tilde{u}}_\tau^\varepsilon|^2 \, dx \, ds \, dt + \rho \int_\tau^{T-2\tau} \int_\Omega |\dot{u}_\tau^\varepsilon|^2 \, dx \, dt$$
$$+ \int_\tau^{T-2\tau} \int_\Omega Q(\bar{e}_\tau^\varepsilon) \, dx \, dt + \int_\tau^{T-2\tau} \mathcal{H}(\dot{p}_\tau^\varepsilon) \, dt \leq C\Big(1 + \frac{T}{\varepsilon}\Big). \tag{4.17}$$

Proof. Take the map $\varphi = \tau(\delta u_i^\varepsilon - u^1 - \delta w_i + \dot{w}(0))$ as test function in (4.11). For $k = 2, \dots, n-2$ we obtain

$$\varepsilon^2 \rho \sum_{i=2}^{k} \tau \int_\Omega \delta^4 u_{i+2}^\varepsilon \cdot (\delta u_i^\varepsilon - u^1 - \delta w_i + \dot{w}(0)) \, dx$$

$$
\begin{aligned}
&- 2\varepsilon\rho \sum_{i=2}^{k} \tau \int_\Omega \delta^3 u^\varepsilon_{i+1} \cdot (\delta u^\varepsilon_i - u^1 - \delta w_i + \dot{w}(0))\, dx \\
&+ \rho \sum_{i=2}^{k} \tau \int_\Omega \delta^2 u^\varepsilon_i \cdot (\delta u^\varepsilon_i - u^1 - \delta w_i + \dot{w}(0))\, dx \\
&- \sum_{i=2}^{k} \tau \int_\Omega \operatorname{div} \mathbb{C} e^\varepsilon_i \cdot (\delta u^\varepsilon_i - u^1 - \delta w_i + \dot{w}(0))\, dx = 0. \qquad (4.18)
\end{aligned}
$$

Arguing as in [49, Subsection 2.4] we estimate the first three terms in the left-hand side of (4.18) from below as

$$
\begin{aligned}
&\varepsilon^2\rho \sum_{i=2}^{k} \tau \int_\Omega \delta^4 u^\varepsilon_{i+2} \cdot (\delta u^\varepsilon_i - u^1 - \delta w_i + \dot{w}(0))\, dx \geq \frac{\varepsilon^2\rho}{2} \int_\Omega |\delta^2 u^\varepsilon_2|^2\, dx \\
&+ \varepsilon^2\rho \int_\Omega \delta^3 u^\varepsilon_{k+2} \cdot (\delta u^\varepsilon_k - u^1 - \delta w_k + \dot{w}(0))\, dx - \frac{\varepsilon^2\rho}{2} \int_\Omega |\delta^2 u^\varepsilon_{k+1}|^2\, dx \\
&+ \frac{\varepsilon^2\rho}{2} \sum_{i=2}^{k} \int_\Omega |\delta^2 u^\varepsilon_{i+1} - \delta^2 u^\varepsilon_i|^2\, dx + \varepsilon^2\rho \int_\Omega \delta^2 u^\varepsilon_{k+1} \cdot \delta^2 w_k\, dx \\
&- \varepsilon^2\rho \int_\Omega \delta^2 u^\varepsilon_2 \cdot \delta^2 w_2\, dx - \frac{\varepsilon^2\rho}{2} \sum_{i=3}^{k} \tau \int_\Omega |\delta^2 u^\varepsilon_i|^2\, dx - \frac{\varepsilon^2\rho}{2} \sum_{i=4}^{k} \tau \int_\Omega |\delta^3 w_i|^2\, dx, \qquad (4.19)
\end{aligned}
$$

$$
\begin{aligned}
&- 2\varepsilon\rho \sum_{i=2}^{k} \tau \int_\Omega \delta^3 u^\varepsilon_{i+1} \cdot (\delta u^\varepsilon_i - u^1 - \delta w_i + \dot{w}(0))\, dx \geq -\varepsilon\rho \sum_{i=2}^{k} \tau \int_\Omega |\delta^2 w_i|^2\, dx \\
&- 2\varepsilon\rho \int_\Omega \delta^2 u^\varepsilon_{k+1} \cdot (\delta u^\varepsilon_k - u^1 - \delta w_k + \dot{w}(0))\, dx + \varepsilon\rho \sum_{i=2}^{k} \tau \int_\Omega |\delta^2 u^\varepsilon_i|^2\, dx, \qquad (4.20)
\end{aligned}
$$

and

$$
\begin{aligned}
&\rho \sum_{i=2}^{k} \tau \int_\Omega \delta^2 u^\varepsilon_i \cdot (\delta u^\varepsilon_i - u^1 - \delta w_i + \dot{w}(0))\, dx = \frac{\rho}{2} \int_\Omega |\delta u^\varepsilon_k - u^1|^2\, dx \\
&+ \frac{\rho}{2} \sum_{i=2}^{k} \int_\Omega |\delta u^\varepsilon_i - \delta u^\varepsilon_{i-1}|^2\, dx - \rho \sum_{i=2}^{k} \int_\Omega (\delta u^\varepsilon_i - \delta u^\varepsilon_{i-1}) \cdot (\delta w_i - \dot{w}(0))\, dx \\
&\geq \frac{\rho}{2} \int_\Omega |\delta u^\varepsilon_k - u^1|^2\, dx + \frac{\rho}{2} \sum_{i=2}^{k} \int_\Omega |\delta u^\varepsilon_i - \delta u^\varepsilon_{i-1}|^2\, dx - \frac{\rho}{16} \sum_{i=2}^{k-1} \tau \int_\Omega |\delta u^\varepsilon_i|^2\, dx \\
&- \frac{\rho}{2} \int_\Omega |\dot{w}(0)|^2\, dx - \rho \int_\Omega \delta u^\varepsilon_k \cdot \delta w_k\, dx + \rho \int_\Omega \delta u^\varepsilon_1 \cdot \delta w_2\, dx - 4\rho \sum_{i=3}^{k} \tau \int_\Omega |\delta^2 w_i|^2\, dx. \qquad (4.21)
\end{aligned}
$$

Regarding the fourth term in the right-hand side of (4.18), by (4.8) and (4.11) there holds $\mathbb{C}e^\varepsilon_i \in \Sigma(\Omega)$ for $i = 2, \ldots, n-2$ (see (4.12)). Therefore, in view of Proposition 4.4 and (4.11), we have

$$
\begin{aligned}
&- \sum_{i=2}^{k} \tau \int_\Omega \operatorname{div} \mathbb{C} e^\varepsilon_i : (\delta u^\varepsilon_i - u^1 - \delta w_i + \dot{w}(0))\, dx \\
&= \sum_{i=2}^{k} \tau \int_\Omega \mathbb{C} e^\varepsilon_i : (\delta e^\varepsilon_i - e^1 - E\delta w_i + E\dot{w}(0))\, dx + \sum_{i=2}^{k} \tau [(\mathbb{C} e^\varepsilon_i)_D : \delta p^\varepsilon_i](\Omega \cup \Gamma_0)
\end{aligned}
$$

for $k=2,\dots,n-2$. On the one hand

$$\sum_{i=2}^k \tau \int_\Omega \mathbb{C}e_i^\varepsilon : (-E\delta w_i + E\dot w(0))\,dx \geq -\frac{1}{4}\sum_{i=2}^k \tau\int_\Omega Q(e_i^\varepsilon)\,dx \\ -4\sum_{i=2}^k \tau\int_\Omega Q(E\delta w_i - E\dot w(0))\,dx,$$

and

$$\sum_{i=2}^k \tau\int_\Omega \mathbb{C}e_i^\varepsilon : (\delta e_i^\varepsilon - e^1)\,dx \geq \int_\Omega Q(e_k^\varepsilon)\,dx - \int_\Omega Q(e^1)\,dx - \sum_{i=2}^k \tau\int_\Omega \mathbb{C}e_i^\varepsilon : e^1\,dx.$$

By Proposition 4.6 we infer that

$$\sum_{i=2}^k \tau[(\mathbb{C}e_i^\varepsilon)_D : \delta p_i^\varepsilon](\Omega\cup\Gamma_0) \geq \sum_{i=2}^k \tau\mathcal{H}(\delta p_i^\varepsilon).$$

Therefore

$$\begin{aligned} &-\sum_{i=2}^k \tau\int_\Omega \operatorname{div}\mathbb{C}e_i^\varepsilon :\, (\delta u_i^\varepsilon - u^1 - \delta w_i + \dot w(0))\,dx \\ &\geq \int_\Omega Q(e_k^\varepsilon)\,dx - \int_\Omega Q(e^1)\,dx - \sum_{i=2}^k \tau\int_\Omega \mathbb{C}e_i^\varepsilon : e^1\,dx \\ &- \frac{1}{4}\sum_{i=2}^k \tau\int_\Omega Q(e_i^\varepsilon)\,dx - 4\sum_{i=2}^k \tau\int_\Omega Q(E\delta w_i - E\dot w(0))\,dx + \sum_{i=2}^k \tau\mathcal{H}(\delta p_i^\varepsilon). \end{aligned} \tag{4.22}$$

By combining (4.19)–(4.22), equality (4.18) yields

$$\begin{aligned} &\varepsilon^2\rho\int_\Omega \delta^3 u_{k+2}^\varepsilon\cdot(\delta u_k^\varepsilon - u^1 - \delta w_k + \dot w(0))\,dx - \frac{\varepsilon^2\rho}{2}\int_\Omega |\delta^2 u_{k+1}^\varepsilon|^2\,dx + \frac{\varepsilon^2\rho}{2}\int_\Omega |\delta^2 u_2^\varepsilon|^2\,dx \\ &+ \frac{\varepsilon^2\rho}{2}\sum_{i=2}^k\int_\Omega |\delta^2 u_{i+1}^\varepsilon - \delta^2 u_i^\varepsilon|^2\,dx + \varepsilon^2\rho\int_\Omega \delta^2 u_{k+1}^\varepsilon\cdot\delta^2 w_k\,dx + \frac{\rho}{2}\int_\Omega |\delta u_k^\varepsilon - u^1|^2\,dx \\ &- 2\varepsilon\rho\int_\Omega \delta^2 u_{k+1}^\varepsilon\cdot(\delta u_k^\varepsilon - u^1 - \delta w_k + \dot w(0))\,dx + \Big(\varepsilon - \frac{\varepsilon^2}{2}\Big)\rho\sum_{i=3}^k \tau\int_\Omega |\delta^2 u_i^\varepsilon|^2\,dx \\ &+ \frac{\rho}{2}\sum_{i=2}^k\int_\Omega |\delta u_i^\varepsilon - \delta u_{i-1}^\varepsilon|^2\,dx - \rho\int_\Omega \delta u_k^\varepsilon\cdot\delta w_k\,dx + \rho\int_\Omega \delta u_1^\varepsilon\cdot\delta w_2\,dx \\ &- \frac{\rho}{16}\sum_{i=2}^{k-1}\tau\int_\Omega |\delta u_i^\varepsilon|^2\,dx + \int_\Omega Q(e_k^\varepsilon)\,dx + \sum_{i=2}^k \tau\mathcal{H}(\delta p_i^\varepsilon) \\ &\leq \frac{\varepsilon^2\rho}{2}\int_\Omega |\delta^2 w_2|^2\,dx + \frac{\varepsilon^2\rho}{2}\sum_{i=4}^k \tau\int_\Omega |\delta^3 w_i|^2\,dx + \varepsilon\rho\sum_{i=2}^k \tau\int_\Omega |\delta^2 w_i|^2\,dx \\ &+ \frac{\rho}{2}\int_\Omega |\dot w(0)|^2\,dx + 4\rho\sum_{i=3}^k \tau\int_\Omega |\delta^2 w_i|^2\,dx + 4\sum_{i=2}^k \tau\int_\Omega Q(E\delta w_i - E\dot w(0))\,dx \\ &+ \int_\Omega Q(e^1)\,dx + \sum_{i=2}^k \tau\int_\Omega \mathbb{C}e_i^\varepsilon : e^1\,dx + \frac{1}{4}\sum_{i=2}^k \tau\int_\Omega Q(e_i^\varepsilon)\,dx. \end{aligned} \tag{4.23}$$

Since $w\in W^{2,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3))\cap C^3([0,T];L^2(\Omega;\mathbb{R}^3))$, by Hölder's inequality there holds

$$\begin{aligned}\frac{\varepsilon^2\rho}{2}\int_\Omega|\delta^2 w_2|^2\,dx &= \frac{\varepsilon^2\rho}{2}\int_\Omega\Big|\frac{w(t_2)-2\tau\dot w(0)-w(0)}{\tau^2}\Big|^2\,dx\\ &= \frac{\varepsilon^2\rho}{2}\int_\Omega\Big|\frac{1}{\tau^2}\int_0^{2\tau}\int_0^\xi \ddot w(\lambda)\,d\lambda\,d\xi\Big|^2\,dx \le C\varepsilon^2\rho,\end{aligned} \tag{4.24}$$

as well as the following

$$\begin{aligned}\sum_{i=2}^k\tau\int_\Omega|\delta^2 w_i|^2\,dx &= \sum_{i=3}^k\tau\int_\Omega\Big|\int_{(i-1)\tau}^{i\tau}\frac{\dot w(t)-\dot w(t-\tau)}{\tau^2}\,dt\Big|^2\,dx + C\tau\\ &\le \frac{1}{\tau}\sum_{i=3}^k\int_\Omega\int_{(i-1)\tau}^{i\tau}\int_{t-\tau}^t|\ddot w(\xi)|^2\,d\xi\,dt\,dx + C\tau \le C\int_0^T\int_\Omega|\ddot w|^2\,dx\,dt + C\tau.\end{aligned} \tag{4.25}$$

In addition, we have that

$$\begin{aligned}\sum_{i=4}^k\tau\int_\Omega|\delta^3 w_i|^2\,dx &= \frac{1}{\tau^5}\sum_{i=4}^k\int_\Omega\Big|\int_{(i-1)\tau}^{i\tau}\int_{\xi-\tau}^\xi(\ddot w(s)-\ddot w(s-\tau))\,ds\,d\xi\Big|^2\,dx\\ &\le C\int_\Omega\int_0^T|\dddot w|^2\,dt\,dx.\end{aligned} \tag{4.26}$$

Finally, in view of Jensen's inequality, we compute

$$\begin{aligned}&4\sum_{i=2}^k\tau\int_\Omega Q(E\delta w_i - E\dot w(0))\,dx \le 4\tau(k-2)\int_\Omega Q(E\dot w(0))\,dx\\ &+8\sum_{i=2}^k\tau\int_\Omega Q\Big(\frac{1}{\tau}\int_{(i-1)\tau}^{i\tau}Ew(\xi)\,d\xi\Big) + 8\tau\int_\Omega Q\Big(\frac{1}{\tau}\int_0^\tau(E\dot w(\xi)-E\dot w(0))\,d\xi\Big)\,dx\\ &\le 4\tau n\int_\Omega Q(E\dot w(0))\,dx + 8\int_\Omega\int_0^T Q(Ew)\,dt\,dx + 8\int_\Omega\int_0^\tau Q(E\dot w(t)-E\dot w(0))\,dt\,dx.\end{aligned} \tag{4.27}$$

By (4.24)–(4.27), the first two rows of the right-hand side of (4.23) are uniformly bounded in terms of the boundary datum w, independently of τ and ε. Therefore we obtain the estimate

$$\begin{aligned}&\varepsilon^2\rho\int_\Omega\delta^2u^\varepsilon_{k+1}\cdot\delta^2w_k\,dx + \varepsilon^2\rho\int_\Omega\delta^3u^\varepsilon_{k+2}\cdot(\delta u^\varepsilon_k - u^1 - \delta w_k + \dot w(0))\,dx + \frac{\varepsilon^2\rho}{2}\int_\Omega|\delta^2u^\varepsilon_2|^2\,dx\\ &-\frac{\varepsilon^2\rho}{2}\int_\Omega|\delta^2u^\varepsilon_{k+1}|^2\,dx - 2\varepsilon\rho\int_\Omega\delta^2u^\varepsilon_{k+1}\cdot(\delta u^\varepsilon_k - u^1 - \delta w_k + \dot w(0))\,dx + \frac{\rho}{2}\int_\Omega|\delta u^\varepsilon_k - u^1|^2\,dx\\ &+\Big(\varepsilon - \frac{\varepsilon^2}{2}\Big)\rho\sum_{i=3}^k\tau\int_\Omega|\delta^2u^\varepsilon_i|^2\,dx + \int_\Omega Q(e^\varepsilon_k)\,dx + \tau\sum_{i=2}^k\mathcal{H}(\delta p^\varepsilon_i) - \rho\int_\Omega\delta u^\varepsilon_k\cdot\delta w_k\,dx\\ &+\rho\int_\Omega\delta u^\varepsilon_1\cdot\delta w_2\,dx - \frac{\rho}{16}\sum_{i=2}^{k-1}\tau\int_\Omega|\delta u^\varepsilon_i|^2\,dx \le C + \int_\Omega Q(e^1)\,dx\\ &+\sum_{i=2}^k\tau\int_\Omega\mathbb{C}e^\varepsilon_i : e^1\,dx + \frac{1}{4}\sum_{i=2}^k\tau\int_\Omega Q(e^\varepsilon_i)\,dx.\end{aligned} \tag{4.28}$$

Multiplying the previous inequality by τ and summing for $k = 2, \dots, n-2$, one obtains

$$\frac{\varepsilon^2\rho}{2}\tau(n-3)\int_\Omega|\delta^2u^\varepsilon_2|^2\,dx + \frac{\rho}{2}\sum_{k=2}^{n-2}\tau\int_\Omega|\delta u^\varepsilon_k - u^1|^2\,dx$$

$$+\varepsilon^2\rho\sum_{k=2}^{n-2}\tau\int_\Omega\delta^2u^\varepsilon_{k+1}\cdot\delta^2w_k\,dx+\varepsilon^2\rho\sum_{k=2}^{n-2}\tau\int_\Omega\delta^3u^\varepsilon_{k+2}\cdot(\delta u^\varepsilon_k-u^1-\delta w_k+\dot{w}(0))\,dx$$
$$-\frac{\varepsilon^2\rho}{2}\sum_{k=2}^{n-2}\tau\int_\Omega|\delta^2u^\varepsilon_{k+1}|^2\,dx-2\varepsilon\rho\sum_{k=2}^{n-2}\tau\int_\Omega\delta^2u^\varepsilon_{k+1}\cdot(\delta u^\varepsilon_k-u^1-\delta w_k+\dot{w}(0))\,dx$$
$$+\Big(\varepsilon-\frac{\varepsilon^2}{2}\Big)\rho\sum_{k=3}^{n-2}\sum_{i=3}^{k}\tau^2\int_\Omega|\delta^2u^\varepsilon_i|^2\,dx+\sum_{k=2}^{n-2}\tau\int_\Omega Q(e^\varepsilon_k)\,dx+\sum_{k=2}^{n-2}\sum_{i=2}^{k}\tau^2\mathcal{H}(\delta p^\varepsilon_i)$$
$$-\rho\sum_{k=2}^{n-2}\tau\int_\Omega\delta u^\varepsilon_k\cdot\delta w_k\,dx+\rho\tau(n-3)\int_\Omega\delta u^\varepsilon_1\cdot\delta w_2\,dx-\frac{\rho}{16}\sum_{k=2}^{n-2}\sum_{i=2}^{k-1}\tau^2\int_\Omega|\delta u^\varepsilon_i|^2\,dx$$
$$\le C+\tau(n-3)\int_\Omega Q(e^1)\,dx+\sum_{k=2}^{n-2}\sum_{i=2}^{k}\tau^2\int_\Omega\mathbb{C}e^\varepsilon_i:e^1\,dx+\frac{(n-3)}{4}\sum_{i=2}^{n-2}\tau^2\int_\Omega Q(e^\varepsilon_i)\,dx. \tag{4.29}$$

By choosing $k=n-2$ in (4.28) and by (4.9), we have

$$\frac{\varepsilon^2\rho}{2}\int_\Omega|\delta^2u^\varepsilon_2|^2\,dx+\frac{\rho}{2}\int_\Omega|\delta u^\varepsilon_{n-2}-u^1|^2\,dx+\Big(\varepsilon-\frac{\varepsilon^2}{2}\Big)\rho\sum_{i=3}^{n-2}\tau\int_\Omega|\delta^2u^\varepsilon_i|^2\,dx+\int_\Omega Q(e^\varepsilon_{n-2})\,dx$$
$$+\sum_{i=2}^{n-2}\tau\mathcal{H}(\delta p^\varepsilon_i)-\rho\int_\Omega\delta u^\varepsilon_{n-2}\cdot\delta w_{n-2}\,dx+\rho\int_\Omega\delta u^\varepsilon_1\cdot\delta w_2\,dx-\frac{\rho}{16}\sum_{i=2}^{n-3}\tau\int_\Omega|\delta u^\varepsilon_i|^2\,dx$$
$$\le C+\int_\Omega Q(e^1)\,dx+\sum_{i=2}^{n-2}\tau\int_\Omega\mathbb{C}e^\varepsilon_i:e^1\,dx+\frac{1}{4}\sum_{i=2}^{n-2}\tau\int_\Omega Q(e^\varepsilon_i)\,dx. \tag{4.30}$$

In view of (4.9) and (4.25) we deduce the lower bounds

$$\varepsilon^2\rho\sum_{k=2}^{n-2}\tau\int_\Omega\delta^3u^\varepsilon_{k+2}\cdot(\delta u^\varepsilon_k-u^1-\delta w_k+\dot{w}(0))\,dx$$
$$=-\varepsilon^2\rho\sum_{k=2}^{n-2}\tau\int_\Omega\delta^2u^\varepsilon_{k+1}\cdot(\delta^2u^\varepsilon_k-\delta^2w_k)\,dx\ge-\frac{3\varepsilon^2\rho}{2}\sum_{k=2}^{n-2}\tau\int_\Omega|\delta^2u^\varepsilon_k|^2\,dx$$
$$-\frac{\varepsilon^2\rho}{2}\sum_{k=2}^{n-2}\tau\int_\Omega|\delta^2w_k|^2\,dx\ge-\frac{3\varepsilon^2\rho}{2}\sum_{k=2}^{n-2}\tau\int_\Omega|\delta^2u^\varepsilon_k|^2\,dx-C, \tag{4.31}$$

and, analogously,

$$\varepsilon^2\rho\sum_{k=2}^{n-2}\tau\int_\Omega\delta^2u^\varepsilon_{k+1}\cdot\delta^2w_k\,dx\ge-\frac{\varepsilon^2\rho}{2}\sum_{k=3}^{n-2}\tau\int_\Omega|\delta^2u^\varepsilon_k|^2\,dx-\frac{\varepsilon^2\rho}{2}\sum_{k=2}^{n-2}\tau\int_\Omega|\delta^2w_k|^2\,dx$$
$$\ge-\frac{\varepsilon^2\rho}{2}\sum_{k=3}^{n-2}\tau\int_\Omega|\delta^2u^\varepsilon_k|^2\,dx-C. \tag{4.32}$$

In addition, arguing as in [49, Subsection 2.4],

$$-2\varepsilon\rho\sum_{k=2}^{n-2}\tau\int_\Omega\delta^2u^\varepsilon_{k+1}\cdot(\delta u^\varepsilon_k-u^1-\delta w_k+\dot{w}(0))\,dx=\varepsilon\rho\sum_{k=2}^{n-3}\int_\Omega|\delta u^\varepsilon_{k+1}-\delta u^\varepsilon_k|^2\,dx$$
$$-\varepsilon\rho\int_\Omega|\delta u^\varepsilon_{n-1}-u^1|^2\,dx+\varepsilon\rho\int_\Omega|\delta u^\varepsilon_2-u^1|^2\,dx-2\varepsilon\rho\int_\Omega(\delta u^\varepsilon_{n-1}-\delta u^\varepsilon_2)\cdot\dot{w}(0)\,dx$$
$$+2\varepsilon\rho\sum_{k=2}^{n-2}\int_\Omega(\delta u^\varepsilon_{k+1}-\delta u^\varepsilon_k)\cdot\delta w_k\,dx\ge-3\varepsilon\rho\int_\Omega|\delta u^\varepsilon_{n-1}-u^1|^2\,dx-2\varepsilon\rho\int_\Omega\delta u^\varepsilon_2\cdot\delta w_2\,dx$$

$$- \varepsilon\rho \sum_{k=2}^{n-2} \tau \int_\Omega |\delta u_k^\varepsilon|^2 \, dx - C, \tag{4.33}$$

where we used (4.9) and (4.25). Finally, using the elementary inequality

$$|\delta u_i^\varepsilon|^2 \leq 2|\delta u_i^\varepsilon - u^1|^2 + 2|u^1|^2 \quad \text{a.e. in } \Omega, \quad \text{for every } i,$$

we deduce that

$$\begin{aligned}
&- 3\varepsilon\rho \int_\Omega |\delta u_{n-1}^\varepsilon - u^1|^2 \, dx - \varepsilon\rho \sum_{k=2}^{n-2} \tau \int_\Omega |\delta u_k^\varepsilon|^2 \, dx - \rho \sum_{k=2}^{n-2} \tau \int_\Omega \delta u_k^\varepsilon \cdot \delta w_k \, dx \\
&\quad + \rho\tau(n-3) \int_\Omega \delta u_1^\varepsilon \cdot \delta w_2 \, dx - \frac{\rho}{16} \sum_{k=2}^{n-2} \sum_{i=2}^{k-1} \tau^2 \int_\Omega |\delta u_i^\varepsilon|^2 \, dx - \rho \int_\Omega \delta u_{n-2}^\varepsilon \cdot \delta w_{n-2} \, dx \\
&\quad + \rho \int_\Omega \delta u_1^\varepsilon \cdot \delta w_2 \, dx - \frac{\rho}{16} \sum_{i=2}^{n-3} \tau \int_\Omega |\delta u_i^\varepsilon|^2 \, dx - 2\varepsilon\rho \int_\Omega \delta u_2^\varepsilon \cdot \delta w_2 \, dx \\
&\quad \geq -\Big(\frac{1}{4} + 3\varepsilon\Big)\rho \int_\Omega |\delta u_{n-1}^\varepsilon - u^1|^2 \, dx - \frac{\varepsilon\rho}{2} \int_\Omega |\delta u_2^\varepsilon - u^1|^2 \, dx \\
&\quad - \Big(\varepsilon + \frac{1}{15}\Big)\rho \sum_{k=2}^{n-2} \tau \int_\Omega |\delta u_k^\varepsilon|^2 \, dx - \frac{\rho}{16} \sum_{k=2}^{n-2} \sum_{i=2}^{k} \tau^2 \int_\Omega |\delta u_i^\varepsilon|^2 \, dx - C \\
&\quad \geq -\Big(\frac{1}{4} + 3\varepsilon\Big)\rho \int_\Omega |\delta u_{n-1}^\varepsilon - u^1|^2 \, dx - \Big(2\varepsilon + \frac{2}{15}\Big)\rho \sum_{k=2}^{n-2} \tau \int_\Omega |\delta u_k^\varepsilon - u^1|^2 \, dx \\
&\quad - \frac{\rho}{8} \sum_{k=2}^{n-2} \sum_{i=2}^{k} \tau^2 \int_\Omega |\delta u_i^\varepsilon - u^1|^2 \, dx - \frac{\varepsilon\rho\tau^2}{2} \int_\Omega |\delta^2 u_2^\varepsilon|^2 \, dx - C.
\end{aligned} \tag{4.34}$$

Summing (4.29) with (4.30), in view of (4.9), estimates (4.31)–(4.34) yield the inequality

$$\begin{aligned}
&\Big(\frac{1}{8} - 2\varepsilon\Big)\rho \sum_{k=2}^{n-2} \tau \int_\Omega |\delta u_k^\varepsilon - u^1|^2 \, dx + \Big(\frac{1}{2} - 3\varepsilon\Big)\rho \int_\Omega |\delta u_{n-1}^\varepsilon - u^1|^2 \, dx + (\varepsilon - 3\varepsilon^2)\rho \sum_{k=3}^{n-2} \tau \int_\Omega |\delta^2 u_k^\varepsilon|^2 \, dx \\
&\quad + \frac{\varepsilon^2\rho(1 + \tau(n-3)) - \varepsilon\rho\tau^2}{2} \int_\Omega |\delta^2 u_2^\varepsilon|^2 \, dx + \Big(\varepsilon - \frac{\varepsilon^2}{2}\Big)\rho \sum_{k=3}^{n-2} \sum_{i=3}^{k} \tau^2 \int_\Omega |\delta^2 u_i^\varepsilon|^2 \, dx \\
&\quad + \sum_{k=2}^{n-2} \tau \int_\Omega Q(e_k^\varepsilon) \, dx + \sum_{k=2}^{n-2} \sum_{i=2}^{k} \tau^2 \mathcal{H}(\delta p_i^\varepsilon) + \int_\Omega Q(e_{n-2}^\varepsilon) \, dx + \tau \sum_{i=2}^{n-2} \mathcal{H}(\delta p_i^\varepsilon) \\
&\quad \leq (1 + \tau(n-3)) \int_\Omega Q(e^1) \, dx + \sum_{k=2}^{n-2} \sum_{i=2}^{k} \tau^2 \int_\Omega \mathbb{C} e_i^\varepsilon : e^1 \, dx + \frac{1}{4} \sum_{i=2}^{n-2} \tau \int_\Omega Q(e_i^\varepsilon) \, dx \\
&\quad + \frac{(n-3)}{4} \sum_{i=2}^{n-2} \tau^2 \int_\Omega Q(e_i^\varepsilon) \, dx + \sum_{i=2}^{n-2} \tau \int_\Omega \mathbb{C} e_i^\varepsilon : e^1 \, dx + C.
\end{aligned} \tag{4.35}$$

By the definition of τ, for τ and ε small enough we eventually obtain

$$\begin{aligned}
&\varepsilon\rho \sum_{k=3}^{n-2} \tau \int_\Omega |\delta^2 u_k^\varepsilon|^2 \, dx + \varepsilon\rho \sum_{k=3}^{n-2} \sum_{i=3}^{k} \tau^2 \int_\Omega |\delta^2 u_i^\varepsilon|^2 \, dx + \rho \sum_{k=2}^{n-2} \tau \int_\Omega |\delta u_k^\varepsilon - u^1|^2 \, dx \\
&\quad + \sum_{k=2}^{n-2} \tau \int_\Omega Q(e_k^\varepsilon) \, dx + \sum_{k=2}^{n-2} \tau \mathcal{H}(\delta p_k^\varepsilon) \leq C
\end{aligned} \tag{4.36}$$

and the assertion follows. □

5. Γ-Convergence from discrete to continuous

In this section we prove that for fixed $\varepsilon > 0$ the sequence of discrete energy functionals $\{I_{\varepsilon\tau}\}$ (see (4.2)) converges, as the time step τ tends to zero, to the functional I_ε. This will allow us to pass to the limit $\tau \to 0$ in the discrete energy estimate (4.17) in order to obtain its continuous analogue, see (5.40) below.

In order to state the convergence result we need to introduce a few auxiliary spaces and to extend the energy functionals I_ε and $I_{\varepsilon\tau}$. Let

$$\begin{aligned}\mathscr{U} :=&\{(u,e,p) \in W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^1(0,T;BD(\Omega))\\ &\times L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})) \times L^1(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D))\}\end{aligned}$$

and

$$\begin{aligned}\mathscr{U}^{\rm affine}_\tau :=&\{(u,e,p) : [0,T] \to (BD(\Omega)\cap L^2(\Omega;\mathbb{R}^3)) \times L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym}) \times \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)\\ &\text{piecewise affine on the time partition of step } \tau \text{ on } [0,T],\\ &\text{and such that } (u(0),e(0),p(0)),\ (u(\tau),e(\tau),p(\tau)),\dots,\\ &(u(T),e(T),p(T)) \in \mathscr{K}_\tau(u^0,e^0,p^0,u^1)\}.\end{aligned}$$

We set

$$G_\varepsilon(u,e,p) := \begin{cases} I_\varepsilon(u,e,p) & \text{if } (u,e,p)\in\mathcal{V},\\ +\infty & \text{otherwise in } \mathscr{U},\end{cases}$$

(where $\mathcal{V}$ is the space defined in (2.12)), and

$$G_{\varepsilon\tau}(u_\tau,e_\tau,p_\tau) := \begin{cases} I_{\varepsilon\tau}\big((u_\tau(0),e_\tau(0),p_\tau(0)),\ (u_\tau(\tau),e_\tau(\tau),p_\tau(\tau)),\dots,(u_\tau(T),e_\tau(T),p_\tau(T))\big)\\ \qquad \text{if } (u_\tau,e_\tau,p_\tau)\in\mathscr{U}^{\rm affine}_\tau,\\ +\infty \qquad \text{otherwise in } \mathscr{U}.\end{cases}$$

We now show that the sequence of energies $\{G_{\varepsilon\tau}\}$ converges to G_ε in the sense of Γ-convergence in $\mathscr{U}$ as $\tau\to 0$.

Theorem 5.1 (Liminf inequality)**.** *Let $\{(u_\tau,e_\tau,p_\tau)\}\subset\mathscr{U}^{\rm affine}_\tau$ and $(u,e,p)\in\mathscr{U}$ be such that*

$$u_\tau \rightharpoonup u \quad \text{weakly in } W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{5.1}$$

$$p_\tau(t) \rightharpoonup^* p(t) \quad \text{weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D) \text{ for every } t\in[0,T], \tag{5.2}$$

$$\bar{e}_\tau \rightharpoonup e \quad \text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})). \tag{5.3}$$

Then, we have that

$$G_\varepsilon(u,e,p) \le \liminf_{\tau\to 0} G_{\varepsilon\tau}(u_\tau,e_\tau,p_\tau).$$

Proof. Let $\{(u_\tau,e_\tau,p_\tau)\}$ and (u,e,p) be as in the statement of the theorem. If $\liminf_{\tau\to 0} G_{\varepsilon\tau}(u_\tau,e_\tau,p_\tau) = +\infty$ there is nothing to prove, therefore without loss of generality we can assume that

$$\begin{aligned}\liminf_{\tau\to 0} G_{\varepsilon\tau}(u_\tau,e_\tau,p_\tau) = \lim_{\tau\to 0}\Big[&\frac{\varepsilon^2\rho}{2}\sum_{i=2}^{n}\tau\eta_{\tau,i}\int_\Omega|\delta^2 u_\tau(i\tau)|^2\,dx\\ &+\sum_{i=2}^{n-2}\tau\eta_{\tau,i+2}\int_\Omega Q(e_\tau(i\tau))\,dx + \varepsilon\tau\sum_{i=1}^{n-1}\eta_{\tau,i+1}\mathcal{H}(\delta p_\tau(i\tau))\Big] < +\infty,\end{aligned} \tag{5.4}$$

In view of (5.1) and (5.2) it follows that $u(0)=u^0$ and $p(0)=p^0$. Denoting by $\bar{u}_\tau$ and $\tilde{u}_\tau$ the piecewise-constant and piecewise-quadratic interpolants associated to u_τ (see (4.15) and (4.16)), respectively, by (5.4), up to the extraction of a (not relabeled) subsequence, we have

$$\liminf_{\tau\to 0}\Big[\frac{\varepsilon^2\rho}{2}\int_\tau^T\bar{\eta}_\tau\int_\Omega|\ddot{\tilde{u}}_\tau|^2\,dx\,dt + \int_\tau^{T-2\tau}\bar{\eta}_\tau(\cdot+2\tau)\int_\Omega Q(\bar{e}_\tau)\,dx\,dt$$

$$+\varepsilon\int_0^T \bar{\eta}_\tau(\cdot+\tau)\mathcal{H}(\dot{p}_\tau)\,dt\Big]<+\infty. \tag{5.5}$$

In view of (5.5) for τ small there holds

$$\liminf_{\tau\to 0}\Big[\frac{\varepsilon^2\rho}{2}\int_\tau^T\int_\Omega(|\ddot{\tilde{u}}_\tau|^2+|\dot{u}_\tau|^2)\,dx\,dt+\int_\tau^{T-2\tau}\int_\Omega Q(\bar{e}_\tau)\,dx\,dt$$
$$+\varepsilon\int_0^T\mathcal{H}(\dot{p}_\tau)\,dt\Big]<+\infty. \tag{5.6}$$

Therefore, there exists a map $v\in W^{2,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$ such that

$$\tilde{u}_\tau\rightharpoonup v\quad\text{weakly in } W^{2,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3)). \tag{5.7}$$

Arguing as in [49, Subsection 2.5.1], we obtain that $u=v$, and $\dot{u}(0)=u^1$.

By (5.4) we deduce the upper bound

$$\lim_{\tau\to 0}D_{\mathcal{H}}(\bar{p}_\tau;0,T)\le C. \tag{5.8}$$

Since $\bar{p}_\tau(0)=p^0$ for every τ, by [11, Lemma 7.2] there exists a map $q\in BV([0,T];\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D))$ such that

$$\bar{p}_\tau(t)\rightharpoonup^* q(t)\quad\text{weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)\quad\text{for every } t\in[0,T], \tag{5.9}$$

and

$$D_{\mathcal{H}}(q;0,T)\le\liminf_\tau D_{\mathcal{H}}(\bar{p}_\tau;0,T).$$

By (5.5) and by Fatou's lemma, for a.e. $t\in[0,T]$ there exists $f^t\in L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})$, and a t-dependent subsequence τ_t such that

$$\bar{e}_{\tau_t}(t)\rightharpoonup f^t\quad\text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym}). \tag{5.10}$$

By (5.9) and (5.10), for a.e. $t\in[0,T]$, the sequence $\{E\bar{u}_{\tau_t}(t)\}$ is bounded in $\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)$ (see [11, Theorem 3.3]). This implies that for a.e. $t\in[0,T]$ there exists a map $v^t\in BD(\Omega)$ such that

$$\bar{u}_{\tau_t}(t)\rightharpoonup^* v^t\quad\text{weakly* in } BD(\Omega), \tag{5.11}$$
$$Ev^t=f^t+q(t), \tag{5.12}$$
$$q(t)=(w(t)-v^t)\odot\nu\mathcal{H}^2\quad\text{on } \Gamma_0. \tag{5.13}$$

In view of (5.1) there holds

$$u_\tau(t)\rightharpoonup u(t)\quad\text{weakly in } L^2(\Omega;\mathbb{R}^3)\quad\text{for every } t\in[0,T]. \tag{5.14}$$

In addition, for fixed $i\in\mathbb{N}$, and $t\in((i-1)\tau,i\tau]$, we have

$$\bar{u}_\tau(t)-u_\tau(t)=(i\tau-t)\dot{u}_\tau(t).$$

Thus by (5.6) we obtain the estimate

$$\|\bar{u}_\tau-u_\tau\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}=\frac{\tau}{\sqrt{3}}\|\dot{u}_\tau\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}\le C\tau,$$

which in turn by (5.14) implies that

$$\bar{u}_\tau(t)\rightharpoonup u(t)\quad\text{weakly in } L^2(\Omega;\mathbb{R}^3)\quad\text{for a.e. } t\in[0,T]. \tag{5.15}$$

By (5.9)–(5.11) we conclude that

$$v^t=u(t)\quad\text{for a.e. } t\in[0,T]. \tag{5.16}$$

Fix $i\in\mathbb{N}$ and $t\in((i-1)\tau,i\tau]$. Then,

$$\|\bar{p}_\tau(t)-p_\tau(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)}=\|(t-i\tau)\dot{p}_\tau(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)}. \tag{5.17}$$

Therefore, by (2.4) one has

$$\|\bar{p}_\tau-p_\tau\|_{L^1(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D))}=\frac{\tau}{2}\|\dot{p}_\tau\|_{L^1(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D))}$$

$$\leq \frac{\tau}{2r_K}\int_0^T \mathcal{H}(\dot{p}_\tau)\,dt \leq C\tau, \tag{5.18}$$

where the last inequality is due to (5.6). In view of (5.18),

$$\|\bar{p}_\tau(t) - p_\tau(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)} \to 0 \quad \text{for a.e. } t\in[0,T].$$

Thus, by (5.2) and (5.9) we deduce that

$$p(t) = q(t) \quad \text{for a.e. } t\in[0,T]. \tag{5.19}$$

By (5.7), (5.12), (5.16), and (5.19) we conclude that $f^t = e(t)$ for a.e. $t\in[0,T]$, and $(u,e,p)\in\mathcal{V}$. By (5.3), (5.7), and (5.16), since $\bar{\eta}_\tau$, $\bar{\eta}_\tau(\cdot+2\tau)\to \exp\Big(-\frac{\cdot}{\varepsilon}\Big)$ strongly in $L^\infty(0,T)$, we obtain that

$$\chi_{[\tau,T-2\tau]}\sqrt{\bar{\eta}_\tau(\cdot+2\tau)}\bar{e}_\tau \rightharpoonup \exp\Big(-\frac{\cdot}{\varepsilon}\Big)e \quad \text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym})), \tag{5.20}$$

and

$$\chi_{[\tau,T-2\tau]}\sqrt{\bar{\eta}_\tau}\ddot{\tilde{u}}_\tau \rightharpoonup \exp\Big(-\frac{\cdot}{\varepsilon}\Big)\ddot{u} \quad \text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{5.21}$$

where $\chi_{[\tau,T-\tau]}$ and $\chi_{[\tau,T-2\tau]}$ are the characteristic functions of the sets $[\tau,T-\tau]$ and $[\tau,T-2\tau]$, respectively. Therefore, by (5.20) and (5.21) one has that

$$\begin{aligned}
&\frac{\varepsilon^2\rho}{2}\int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega |\ddot{u}(t)|^2\,dx\,dt + \frac12\int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega Q(e(t))\,dx\,dt\\
&\leq \frac12\liminf_{\tau\to0}\int_0^T\Big[\varepsilon^2\rho\bar{\eta}_\tau(t)\chi_{[\tau,T-\tau]}(t)\int_\Omega|\ddot{\tilde{u}}_\tau(t)|^2\,dx\\
&\qquad+\bar{\eta}_\tau(t+2\tau)\chi_{[\tau,T-2\tau]}(t)\int_\Omega Q(\bar{e}_\tau(t))\,dx\Big]\,dt\\
&=\liminf_{\tau\to0}\Big[\frac{\varepsilon^2\rho}{2}\int_\tau^T\bar{\eta}_\tau\int_\Omega|\ddot{\tilde{u}}_\tau|^2\,dx\,dt + \frac12\int_\tau^{T-2\tau}\bar{\eta}_\tau(\cdot+2\tau)\int_\Omega Q(\bar{e}_\tau)\,dx\,dt\Big]\\
&=\liminf_{\tau\to0}\Big[\frac{\varepsilon^2\rho}{2}\sum_{i=2}^n\tau\eta_{\tau,i}\int_\Omega|\delta^2u_\tau(i\tau)|^2\,dx + \sum_{i=2}^{n-2}\tau\eta_{\tau,i+2}\int_\Omega Q(e_\tau(i\tau))\,dx\Big].
\end{aligned}\tag{5.22}$$

To conclude we need to prove a liminf inequality for the plastic dissipation. To this purpose, let $0\leq t_0<t_1<\dots<t_m\leq T$. In view of (5.8), (5.9) and (5.19), and since $\bar{p}_\tau$ only jumps in the points $i\tau$, $i=1,\dots,N$, we have

$$\begin{aligned}
&\sum_{i=1}^m\exp\Big(-\frac{t_i}{\varepsilon}\Big)\mathcal{H}(p(t_i)-p(t_{i-1}))\leq\liminf_{\tau\to0}\Big[\sum_{i=1}^m\exp\Big(-\frac{t_i}{\varepsilon}\Big)\mathcal{H}(\bar{p}_\tau(t_i)-\bar{p}_\tau(t_{i-1}))\Big]\\
&\leq\liminf_{\tau\to0}\Big[\sum_{i=1}^n\exp\Big(-\frac{i\tau}{\varepsilon}\Big)\mathcal{H}(\bar{p}_\tau(i\tau)-\bar{p}_\tau((i-1)\tau))+\frac{C\tau}{\varepsilon}D_{\mathcal{H}}(\bar{p}_\tau;0,T)\Big]\\
&=\liminf_{\tau\to0}\Big[\tau\sum_{i=1}^n\exp\Big(-\frac{i\tau}{\varepsilon}\Big)\mathcal{H}(\delta p_\tau(i\tau))\Big]\leq\liminf_{\tau\to0}\Big[\tau\sum_{i=1}^n\eta_{\tau,i+1}\mathcal{H}(\delta p_\tau(i\tau))\Big]\\
&+\lim_{\tau\to0}\tau\Big|\sum_{i=1}^n\Big(\exp\Big(-\frac{i\tau}{\varepsilon}\Big)-\eta_{\tau,i+1}\Big)\mathcal{H}(\delta p_\tau(i\tau))\Big|.
\end{aligned}$$

Since $\bar{\eta}_\tau(\cdot+\tau)\to\exp(-t/\varepsilon)$ strongly in $L^\infty(0,T)$ as $\tau\to0$, by (5.8), and using again the fact that $\bar{p}_\tau$ only jumps in the points $i\tau$, $i=1,\dots,N$ we deduce

$$\begin{aligned}
&\lim_{\tau\to0}\tau\Big|\sum_{i=1}^n\Big(\exp\Big(-\frac{i\tau}{\varepsilon}\Big)-\eta_{\tau,i+1}\Big)\mathcal{H}(\delta p_\tau(i\tau))\Big|\\
&\quad\leq\lim_{\tau\to0}\Big\|\exp\Big(-\frac{t}{\varepsilon}\Big)-\bar{\eta}_\tau(t+\tau)\Big\|_{L^\infty(0,T)}\tau\sum_{i=1}^n\mathcal{H}(\delta p_\tau(i\tau))
\end{aligned}$$

$$= \lim_{\tau\to 0}\Big\| \exp\Big(-\frac{t}{\varepsilon}\Big) - \bar\eta_\tau(t+\tau)\Big\|_{L^\infty(0,T)} D_{\mathcal{H}}(\bar p_\tau; 0,T)$$
$$\le \lim_{\tau\to 0} C\Big\| \exp\Big(-\frac{t}{\varepsilon}\Big) - \bar\eta_\tau(t+\tau)\Big\|_{L^\infty(0,T)} = 0.$$

Thus, we have checked that

$$\sum_{i=1}^m \exp\Big(-\frac{t_i}{\varepsilon}\Big)\mathcal{H}(p(t_i)-p(t_{i-1})) \le \liminf_{\tau\to 0}\Big[\sum_{i=1}^m \exp\Big(-\frac{t_i}{\varepsilon}\Big)\mathcal{H}(\bar p_\tau(t_i)-\bar p_\tau(t_{i-1}))\Big]$$
$$\le \liminf_{\tau\to 0}\Big[\tau\sum_{i=1}^n \eta_{\tau,i+1}\mathcal{H}(\delta p_\tau(i\tau))\Big].$$

The arbitrariness of the time partition $\{t_j\}_{j=0,\dots,m}$ yields that

$$D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);p;0,T) \le \liminf_{\tau}\Big[\tau\sum_{i=1}^{n-1} \eta_{\tau,i+1}\mathcal{H}(\delta p_\tau(i\tau))\Big]. \tag{5.23}$$

The thesis follows now by combining (5.22) and (5.23). □

We now prove that the lower bound identified in Theorem 5.1 is optimal.

Theorem 5.2 (Limsup inequality). *Let $(u,e,p)\in\mathcal{V}$. There exists a sequence of triples $(u_\tau,e_\tau,p_\tau)\in\mathscr{U}_\tau^{\rm affine}$ such that*

$$u_\tau \to u \quad \text{strongly in } W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{5.24}$$
$$p_\tau(t) \rightharpoonup^* p(t) \quad \text{weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D) \text{ for every } t\in[0,T], \tag{5.25}$$
$$\bar e_\tau \to e \quad \text{strongly in } L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym})), \tag{5.26}$$

and

$$\limsup_{\tau\to 0} G_{\varepsilon\tau}(u_\tau,e_\tau,p_\tau) \le G_\varepsilon(u,e,p). \tag{5.27}$$

Proof. Let u_τ be defined as the affine-in-time interpolant of the following values

$$\begin{cases} u_\tau(0) = u^0,\\ u_\tau(\tau) = u^0+\tau u^1,\\ u_\tau(i\tau) = M_\tau(u)(i\tau), \quad \text{for every } i=2,\dots,n,\end{cases}$$

where M_τ is the backward mean operator,

$$M_\tau(u)(t) := \frac{1}{\tau}\int_{t-\tau}^t u(s)\,ds \quad \text{for every } t>\tau.$$

Define e_τ accordingly, let $\bar e_\tau$ be its associated piecewise-constant interpolant, and let p_τ be the piecewise-affine in-time interpolant of the measure satisfying

$$\begin{cases} p_\tau(0) = p^0,\\ p_\tau(\tau) = p^0+\tau p^1,\\ p_\tau(i\tau) = M_\tau(p)(i\tau), \quad \text{for every } i=2,\dots,n,\end{cases}$$

where

$$\langle \varphi,\, M_\tau(p)(i\tau)\rangle := \frac{1}{\tau}\int_{t-\tau}^t\int_{\Omega\cup\Gamma_0}\varphi : \,dp(s)\,ds \quad \text{for every } \varphi\in C_0(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D).$$

The triple (u_τ,e_τ,p_τ) satisfies $(u_\tau,e_\tau,p_\tau)\in\mathscr{U}_\tau^{\rm affine}$, and (5.24) is obtained arguing as in [49, Subsection 2.5.2]. Property (5.26) follows by Lebesgue differentiation theorem once we observe that

$$\int_\Omega |e(t)-\bar e_\tau(t)|^2\,dx \le \frac{1}{\tau}\int_{(i-2)\tau}^{i\tau}\int_\Omega |e(t)-e(s)|^2\,dx\,ds \le \frac{1}{\tau}\int_{t-2\tau}^{t+2\tau}\int_\Omega |e(t)-e(s)|^2\,dx\,ds \quad \text{for every } t\in(2\tau,T].$$

Regarding the plastic strains, fix $t \in (0,T]$. For τ small enough, there exists $i>2$ such that $t \in ((i-1)\tau, i\tau]$. Thus, for every $\varphi \in C_0(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)$, there holds

$$\begin{aligned}
&\Big|\int_{\Omega\cup\Gamma_0}\varphi\,dp_\tau(t)-\int_{\Omega\cup\Gamma_0}\varphi\,dp(t)\Big|\\
&\quad=\frac{1}{\tau}\Big|\Big(\frac{t-(i-1)\tau}{\tau}\Big)\int_{(i-1)\tau}^{i\tau}\Big(\int_{\Omega\cup\Gamma_0}\varphi\,dp(s)-\int_{\Omega\cup\Gamma_0}\varphi\,dp(t)\Big)\,ds\\
&\qquad+\Big(1-\Big(\frac{t-(i-1)\tau}{\tau}\Big)\Big)\int_{(i-2)\tau}^{(i-1)\tau}\Big(\int_{\Omega\cup\Gamma_0}\varphi\,dp(s)-\int_{\Omega\cup\Gamma_0}\varphi\,dp(t)\Big)\,ds\Big|\\
&\quad\le\frac{\|\varphi\|_{L^\infty(\Omega\cup\Gamma_0)}}{\tau}\int_{t-2\tau}^{t+2\tau}\|p(s)-p(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)}\,ds. \qquad (5.28)
\end{aligned}$$

In particular, for τ small enough we have

$$\|p_\tau(t)-p(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)}\le\frac{1}{\tau}\int_{t-2\tau}^{t+2\tau}\|p(s)-p(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)}\,ds.$$

Since $t\mapsto\|p(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)}$ is $L^1(0,T)$, in view of Lebesgue differentiation theorem we obtain that

$$p_\tau(t)\to p(t)\quad\text{strongly in }\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)\quad\text{for a.e. }t\in[0,T]. \qquad (5.29)$$

In addition, by the definition of p_τ there holds

$$D_{\mathcal{H}}(p_\tau;0,T)\le D_{\mathcal{H}}(p;0,T)+\tau\|p^1\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)}+2\int_0^T\|p\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)}\,dt\le C. \qquad (5.30)$$

Since $p_\tau(0)=p^0$ for every τ, by (5.29), (5.30) and by [11, Lemma 7.2] we deduce (5.25). Arguing as in [49, Subsection 2.5.2] we obtain the inequality

$$\begin{aligned}
&\limsup_{\tau\to0}\Big[\frac{\varepsilon^2\rho}{2}\sum_{i=2}^n\tau\eta_{\tau,i}\int_\Omega|\delta^2u_\tau(i\tau)|^2\,dx+\sum_{i=2}^{n-2}\tau\eta_{\tau,i+2}\int_\Omega Q(e_\tau(i\tau))\,dx\Big]\\
&\quad\le\int_0^T\exp\Big(-\frac{t}{\varepsilon}\Big)\Big(\frac{\varepsilon^2\rho}{2}\int_\Omega|\ddot u|^2\,dx+\int_\Omega Q(e)\,dx\Big)\,dt.
\end{aligned}$$

To prove (5.27) it remains only to show that

$$\limsup_{\tau\to0}\Big[\tau\sum_{i=1}^n\eta_{\tau,i+1}\mathcal{H}(\delta p_\tau(i\tau))\Big]\le D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);p;0,T). \qquad (5.31)$$

We first observe that

$$\begin{aligned}
\tau\sum_{i=1}^n\eta_{\tau,i+1}\mathcal{H}(\delta p_\tau(i\tau))&=\sum_{i=1}^n\eta_{\tau,i+1}\mathcal{H}\big(p_\tau(i\tau)-p_\tau((i-1)\tau)\big)\\
&=\sum_{i=1}^n\Big(\eta_{\tau,i+1}-\exp\Big(-\frac{i\tau}{\varepsilon}\Big)\Big)\mathcal{H}\big(p_\tau(i\tau)-p_\tau((i-1)\tau)\big)\\
&\quad+\sum_{i=1}^n\exp\Big(-\frac{i\tau}{\varepsilon}\Big)\mathcal{H}\big(p_\tau(i\tau)-p_\tau((i-1)\tau)\big). \qquad (5.32)
\end{aligned}$$

By (5.30) the first term in the right-hand side of (5.32) can be bounded from above as follows

$$\begin{aligned}
&\Big|\sum_{i=1}^n\Big(\eta_{\tau,i+1}-\exp\Big(-\frac{i\tau}{\varepsilon}\Big)\Big)\mathcal{H}\big(p_\tau(i\tau)-p_\tau((i-1)\tau)\big)\Big|\\
&\quad\le\sum_{i=1}^n\mathcal{H}\big(p_\tau(i\tau)-p_\tau((i-1)\tau)\big)\|\bar\eta_\tau(\cdot+\tau)-\exp(-\,\cdot\,/\varepsilon)\|_{L^\infty(0,T)}\\
&\quad\le D_{\mathcal{H}}(p_\tau;0,T)\|\bar\eta_\tau(\cdot+\tau)-\exp(-\,\cdot\,/\varepsilon)\|_{L^\infty(0,T)}
\end{aligned}$$

$$\leq C\|\bar\eta_\tau(\cdot+\tau)-\exp(-\,\cdot\,/\varepsilon)\|_{L^\infty(0,T)} \tag{5.33}$$

and converges to zero as $\tau\to 0$.

To study the second term in the right-hand side of (5.32) we remark that for $i>2$

$$\mathcal{H}\big(p_\tau(i\tau)-p_\tau((i-1)\tau)\big)\leq \frac{1}{\tau^2}\int_{(i-1)\tau}^{i\tau}\int_{(i-2)\tau}^{(i-1)\tau}\mathcal{H}(p(t)-p(s))\,ds\,dt. \tag{5.34}$$

Indeed, for every $\varphi\in C_0(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)\cap\mathcal{K}_D(\Omega)$ by Lemma 2.1 there holds

$$\begin{aligned}\langle\varphi,\,p_\tau(i\tau)-p_\tau((i-1)\tau)\rangle&=\frac{1}{\tau}\int_{(i-1)\tau}^{i\tau}\int_{\Omega\cup\Gamma_0}\varphi\cdot\,dp(t)\,dt-\frac{1}{\tau}\int_{(i-2)\tau}^{(i-1)\tau}\int_{\Omega\cup\Gamma_0}\varphi\cdot\,dp(s)\,ds\\
&=\frac{1}{\tau^2}\int_{(i-1)\tau}^{i\tau}\int_{(i-2)\tau}^{(i-1)\tau}\int_{\Omega\cup\Gamma_0}\varphi\cdot\,d(p(t)-p(s))\,ds\,dt\\
&\leq\frac{1}{\tau^2}\int_{(i-1)\tau}^{i\tau}\int_{(i-2)\tau}^{(i-1)\tau}\mathcal{H}(p(t)-p(s))\,ds\,dt.\end{aligned}$$

A further application of Lemma 2.1 indeed yields (5.34). Analogously,

$$\mathcal{H}\big(p_\tau(2\tau)-p_\tau(\tau)\big)\leq\frac{1}{\tau}\int_{\tau}^{2\tau}\mathcal{H}(p(t)-p^0)\,dt+\tau\mathcal{H}(p^1)\leq D_{\mathcal{H}}(p;0,2\tau)+\tau\mathcal{H}(p^1). \tag{5.35}$$

In view of (5.34) and (5.35) we obtain

$$\begin{aligned}&\sum_{i=1}^{n}\exp\Big(-\frac{i\tau}{\varepsilon}\Big)\mathcal{H}\big(p_\tau(i\tau)-p_\tau((i-1)\tau)\big)\\
&\quad\leq\sum_{i=2}^{n}\exp\Big(-\frac{i\tau}{\varepsilon}\Big)D_{\mathcal{H}}(p;0,i\tau)+2\tau\mathcal{H}(p^1)\\
&\quad\leq\sum_{i=2}^{n}\exp\Big(-\frac{i\tau}{\varepsilon}\Big)\sup\Big\{\sum_{j=1}^{m}\mathcal{H}(p(s_j)-p(s_{j-1}))\,:\,0\leq s_1<\cdots<s_m\leq i\tau\Big\}+2\tau\mathcal{H}(p^1)\\
&\quad\leq D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);p;0,T)+2\tau\mathcal{H}(p^1).\end{aligned} \tag{5.36}$$

Estimate (5.31) follows now by combining (5.32)–(5.36). □

As a corollary of Theorems 5.1 and 5.2, we obtain a uniform energy estimate for minimizers of G_ε.

Corollary 5.3 (Uniform energy estimate)**.** *Let $p^1=0$. For every $\tau>0$, let $(u_\tau,e_\tau,p_\tau)\in\mathscr{U}_\tau^{\rm affine}$ be a minimizer of $G_{\varepsilon\tau}$. Then, there exists a a minimizer $(u^\varepsilon,e^\varepsilon,p^\varepsilon)$ of G_ε in $\mathcal{V}$, such that*

$$\tilde u_\tau\rightharpoonup u^\varepsilon\quad\text{weakly in } W^{2,2}(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{5.37}$$

$$p_\tau(t)\rightharpoonup^* p^\varepsilon(t)\quad\text{weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)\text{ for every } t\in[0,T], \tag{5.38}$$

$$\bar e_\tau\rightharpoonup e^\varepsilon\quad\text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym})), \tag{5.39}$$

where $\tilde u_\tau$ and $\bar e_\tau$ are the piecewise quadratic and piecewise constant interpolants of u_τ and e_τ, respectively (see (4.15) and (4.16)). In addition, there exists a constant C, independent of ε, and such that

$$\begin{aligned}&\varepsilon\rho\int_0^T\int_0^t\int_\Omega|\ddot u^\varepsilon|^2\,dx\,ds\,dt+\varepsilon\rho\int_0^T\int_\Omega|\ddot u^\varepsilon|^2\,dx\,dt\\
&\quad+\rho\int_0^T\int_\Omega|\dot u^\varepsilon|^2\,dx\,dt+\int_0^T\int_\Omega Q(e^\varepsilon)\,dx\,dt+D_{\mathcal{H}}(p^\varepsilon;0,T)\leq C.\end{aligned} \tag{5.40}$$

Proof. Let $\{(u_\tau, e_\tau, p_\tau)\}$ be as in the statement of the theorem. Let w_τ be the piecewise-affine in-time interpolant associated to the maps $\{w_0, \dots, w_n\}$ (see (4.1)). Since $(u^0 + tu^1 - w(0) - t\dot w(0) + w_\tau(t), e^0 + te^1 - Ew(0) - tE\dot w(0) + Ew_\tau(t), p^0) \in \mathscr{U}_\tau^{\rm affine}$ for every $\tau > 0$, there holds

$$\begin{aligned} G_{\varepsilon\tau}(u_\tau, e_\tau, p_\tau) &\le G_{\varepsilon\tau}(u^0 + tu^1 - w(0) - t\dot w(0) + w_\tau(t), e^0 + te^1 - Ew(0) - tE\dot w(0) + Ew_\tau(t), p^0) \\ &= \sum_{i=2}^{n-2} \tau\eta_{\tau,i+2} \int_\Omega Q(e^0 + i\tau e^1 - Ew(0) - i\tau E\dot w(0) + Ew_i)\, dx \le C \end{aligned} \tag{5.41}$$

for every $\tau > 0$. Arguing as in the proof of Theorem 5.1, in view of (5.41) there exists $(u^\varepsilon, e^\varepsilon, p^\varepsilon) \in \mathcal{V}$ such that (5.37)–(5.39) hold true, with

$$\chi_{[\tau, T-2\tau]} \sqrt{\bar\eta_\tau(\cdot + 2\tau)}\bar e_\tau \rightharpoonup \exp\Big(-\frac{\cdot}{\varepsilon}\Big) e^\varepsilon \quad \text{weakly in } L^2(0,T; L^2(\Omega; \mathbb{M}^{3\times 3}_{\rm sym})), \tag{5.42}$$

and

$$\frac{\varepsilon^2\rho}{2} \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big) \int_\Omega |\ddot u^\varepsilon(t)|^2\, dx\, dt \le \frac{\varepsilon^2\rho}{2} \liminf_{\tau\to 0} \int_0^T \bar\eta_\tau(t)\chi_{[\tau, T-\tau]}(t) \int_\Omega |\ddot{\tilde u}_\tau(t)|^2\, dx, \tag{5.43}$$

$$\frac{1}{2} \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big) \int_\Omega Q(e^\varepsilon(t))\, dx\, dt \le \frac{1}{2} \liminf_{\tau\to 0} \int_0^T \bar\eta_\tau(t + 2\tau)\chi_{[\tau, T-2\tau]}(t) \int_\Omega Q(\bar e_\tau(t))\, dx\, dt, \tag{5.44}$$

$$D_{\mathcal{H}}(\exp(-\cdot/\varepsilon); p^\varepsilon; 0, T) \le \liminf_\tau \Big[\tau \sum_{i=1}^{n-1} \eta_{\tau,i+1} \mathcal{H}(\delta p_\tau(i\tau))\Big]. \tag{5.45}$$

Hence,

$$G_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon) \le \liminf_{\tau\to 0} G_{\varepsilon\tau}(u_\tau, e_\tau, p_\tau). \tag{5.46}$$

Let now $(v, f, q) \in \mathcal{V}$. By Theorem 5.2 there exist maps $(v_\tau, f_\tau, q_\tau) \in \mathscr{U}_\tau^{\rm affine}$ such that

$$\limsup_{\tau\to 0} G_{\varepsilon\tau}(v_\tau, f_\tau, q_\tau) \le G_\varepsilon(v, f, q). \tag{5.47}$$

The minimality of $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ follows then by the minimality of (u_τ, e_τ, p_τ), and by combining (5.46) with (5.47). Using again Theorem 5.2 we get the existence of a sequence $\{(\hat u_\tau, \hat e_\tau, \hat p_\tau)\} \subset \mathscr{U}_\tau^{\rm affine}$ such that

$$\limsup_{\tau\to 0} G_{\varepsilon\tau}(u_\tau, e_\tau, p_\tau) \le \limsup_{\tau\to 0} G_{\varepsilon\tau}(\hat u_\tau, \hat e_\tau, \hat p_\tau) \le G_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon). \tag{5.48}$$

By combining (5.46) with (5.48) we conclude that

$$\lim_{\tau\to 0} G_{\varepsilon\tau}(u_\tau, e_\tau, p_\tau) = G_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon).$$

In view of Theorem 4.7, by (5.37) and (5.39) we have

$$\begin{aligned} &\varepsilon\rho \int_0^T \int_0^t \int_\Omega |\ddot u^\varepsilon|^2\, dx\, ds\, dt + \varepsilon\rho \int_0^T \int_\Omega |\ddot u^\varepsilon|^2\, dx\, dt + \rho \int_0^T \int_\Omega |\dot u^\varepsilon|^2\, dx\, dt \\ &\quad + \int_0^T \int_\Omega Q(e^\varepsilon)\, dx\, dt \le C. \end{aligned} \tag{5.49}$$

In addition, by (5.38), the lower semicontinuity of $\mathcal{H}$, and Theorem 4.7,

$$\sup_{a>0} D_{\mathcal{H}}(p^\varepsilon; a, T-a) \le \sup_{a>0} \liminf_{\tau\to 0} D_{\mathcal{H}}(p_\tau; a, T-a) \le C. \tag{5.50}$$

The thesis follows by combining (5.49) and (5.50). □

6. Energy inequality at level ε

The central results of this section are Propositions 6.4 and 6.6, delivering an ε-dependent energy inequality fulfilled by minimizers, and its integrated-in-time counterpart (see (6.12)). The proof strategy follows closely the one of [47, Theorem 2.5 (c)]. The additional difficulties in our setting are due to the fact that the dissipation potential satisfies linear growth conditions from above, and the triple $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ is required to fulfill the nonlinear constraint $(u^\varepsilon, e^\varepsilon, p^\varepsilon) \in \mathcal{V}$. Another crucial difference is that our analysis is performed on the finite interval $[0,T]$ instead of in the entire semiline $t \geq 0$.

As in [47, Section 4] we first introduce some auxiliary quantities, as well as the notion of *approximate energy*. Throughout this section we assume that $(u^1, e^1, 0) \in \mathscr{A}(\dot{w}(0))$, and we consider a minimizer $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ of G_ε. We set

$$\mathcal{K}_\varepsilon(t) := \frac{\rho\varepsilon^2}{2}\int_\Omega |\dot{u}^\varepsilon(t)|^2\,dx, \quad \text{and} \quad \mathcal{H}_\varepsilon(t) := \varepsilon D_{\mathcal{H}}(p^\varepsilon; 0, t)$$

for every $t \in [0,T]$, as well as

$$\mathcal{W}_\varepsilon(t) := \int_\Omega Q(e^\varepsilon(t))\,dx, \quad \text{and} \quad \mathcal{D}_\varepsilon(t) := \frac{\rho\varepsilon^2}{2}\int_\Omega |\ddot{u}^\varepsilon(t)|^2\,dx$$

for a.e. $t \in [0,T]$, and we define the *locally integrable lagrangian*

$$\mathcal{L}_\varepsilon(t) := \mathcal{D}_\varepsilon(t) + \mathcal{W}_\varepsilon(t) + \frac{\mathcal{H}_\varepsilon(t)}{\varepsilon} \quad \text{for a.e. } t \in [0,T].$$

Note that $\mathcal{K}_\varepsilon \in W^{1,1}(0,T)$, with

$$\dot{\mathcal{K}}_\varepsilon(t) = \rho\varepsilon^2\int_\Omega \ddot{u}^\varepsilon(t)\cdot\dot{u}^\varepsilon(t)\,dx \quad \text{for a.e. } t \in [0,T]. \tag{6.1}$$

For $f : [0,T] \to [0,+\infty]$ measurable we consider the operator

$$\mathcal{A}f(t) := \int_t^T \exp\Big(\frac{t-s}{\varepsilon}\Big)f(s)\,ds \quad \text{for every } t \in [0,T].$$

We point out that, if

$$\mathcal{A}f(0) = \int_0^T \exp\Big(-\frac{s}{\varepsilon}\Big)f(s)\,ds < +\infty,$$

then $\mathcal{A}f \in AC([0,T])$, and

$$\dot{\mathcal{A}}f(t) = \frac{\mathcal{A}f(t)}{\varepsilon} - f(t) \quad \text{for a.e. } t \in [0,T].$$

A direct computation yields

$$\mathcal{A}^2f(t) := \mathcal{A}(\mathcal{A}f)(t) = \int_t^T \exp\Big(\frac{t-s}{\varepsilon}\Big)(s-t)f(s)\,ds \quad \text{for every } t \in [0,T],$$

and

$$\mathcal{A}^2f(0) = \int_0^T \exp\Big(-\frac{s}{\varepsilon}\Big)sf(s)\,ds.$$

For every $\varepsilon > 0$ the *approximate energy* $\mathcal{E}_\varepsilon$ is defined as

$$\mathcal{E}_\varepsilon(t) := \mathcal{K}_\varepsilon(t) + \mathcal{A}^2\mathcal{W}_\varepsilon(t) + \frac{1}{\varepsilon}\mathcal{A}^2\mathcal{H}_\varepsilon(t) \quad \text{for every } t \in [0,T]. \tag{6.2}$$

The presence of the third term in the right-hand side of (6.2) is a key difference with respect to [47], and is needed due to the linear growth assumptions on the plastic dissipation potential in our setting.

We start by proving a preliminary inequality involving the quantities $\mathcal{D}_\varepsilon$, $\mathcal{K}_\varepsilon$, and $\mathcal{L}_\varepsilon$.

Lemma 6.1. *Let $(u^1, e^1, 0) \in \mathscr{A}(\dot w(0))$, and let $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ be a minimizer of G_ε. Then*

$$\begin{aligned}
&\int_0^T \exp\Big(-\frac{s}{\varepsilon}\Big)(\varepsilon\dot g(s) - g(s))\mathcal{L}_\varepsilon(s)\,ds - 4\varepsilon\int_0^T \exp\Big(-\frac{s}{\varepsilon}\Big)\dot g(s)\mathcal{D}_\varepsilon(s)\,ds\\
&\quad - \varepsilon\int_0^T \exp\Big(-\frac{s}{\varepsilon}\Big)\ddot g(s)\dot{\mathcal{K}}_\varepsilon(s)\,ds + \varepsilon^3\rho\int_0^T\int_\Omega \exp\Big(-\frac{s}{\varepsilon}\Big)\ddot u^\varepsilon(s)\cdot(\dot w(s)g(s))^{\cdot\cdot}\,dx\,ds\\
&\quad + \varepsilon\int_0^T\int_\Omega \exp\Big(-\frac{s}{\varepsilon}\Big)\mathbb{C}e^\varepsilon(s) : E\dot w(s)g(s)\,dx\,ds\\
&\quad + \varepsilon\int_0^T\int_\Omega \exp\Big(-\frac{s}{\varepsilon}\Big)\mathbb{C}e^\varepsilon(s) : (e^1 - E\dot w(0))s\dot g(0)\,dx\,ds \geq 0
\end{aligned}$$

for every $g \in C^2([0,T])$ such that $g(0) = 0$ and $g(t) \geq 0$ for every $t \in [0,T]$.

Proof. We argue as in [47, Proposition 4.4], and for every $\delta > 0$ we consider the map

$$\varphi_\delta(t) := t - \delta\varepsilon g(t) \quad \text{for every } t \in [0,T].$$

For δ small, φ_δ is a C^2 diffeomorphism from $[0,T]$ to $[0,\varphi_\delta(T)]$, with inverse $\psi_\delta : [0,\varphi_\delta(T)] \to [0,T]$ satisfying

$$\psi_\delta(t) := t + \delta\varepsilon g(\psi_\delta(t)) \quad \text{for every } t \in [0,T].$$

We define the triple

$$\begin{aligned}
&\tilde u^\varepsilon(t) := u^\varepsilon(\varphi_\delta(t)) + t\delta\varepsilon\dot g(0)u^1 + w(t) - w(\varphi_\delta(t)) - t\delta\varepsilon\dot g(0)\dot w(0),\\
&\tilde p^\varepsilon(t) := p^\varepsilon(\varphi_\delta(t)),
\end{aligned}$$

for every $t \in [0,T]$, and

$$\tilde e^\varepsilon(t) := e^\varepsilon(\varphi_\delta(t)) + t\delta\varepsilon\dot g(0)e^1 + Ew(t) - Ew(\varphi_\delta(t)) - t\delta\varepsilon\dot g(0)E\dot w(0)$$

for a.e. $t \in [0,T]$. Since $(\tilde u^\varepsilon, \tilde e^\varepsilon, \tilde p^\varepsilon) \in \mathcal{V}$, by the minimality of $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ there holds

$$\limsup_{\delta\to 0} \frac{G_\varepsilon(\tilde u^\varepsilon, \tilde e^\varepsilon, \tilde p^\varepsilon) - G_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon)}{\delta} \geq 0. \tag{6.3}$$

We preliminary observe that

$$\dot{\tilde u}^\varepsilon(t) = \dot u^\varepsilon(\varphi_\delta(t))\dot\varphi_\delta(t) + \delta\varepsilon\dot g(0)u^1 + \dot w(t) - \dot w(\varphi_\delta(t))\dot\varphi_\delta(t) - \delta\varepsilon\dot g(0)\dot w(0)$$

for every $t \in [0,T]$, and

$$\ddot{\tilde u}^\varepsilon(t) = \ddot u^\varepsilon(\varphi_\delta(t))(\dot\varphi_\delta(t))^2 + \dot u^\varepsilon(\varphi_\delta(t))\ddot\varphi_\delta(t) + \ddot w(t) - \ddot w(\varphi_\delta(t))(\dot\varphi_\delta(t))^2 - \dot w(\varphi_\delta(t))\ddot\varphi_\delta(t)$$

for a.e. $t \in [0,T]$. Therefore a change of variable in inequality (6.3) yields

$$\begin{aligned}
&\limsup_{\delta\to 0}\frac{1}{\delta}\Big\{\int_0^{\varphi_\delta(T)} \dot\psi_\delta(t)\exp\Big(-\frac{\psi_\delta(t)}{\varepsilon}\Big)\Big[\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot u^\varepsilon(t)(\dot\varphi_\delta(\psi_\delta(t))^2 + \dot u^\varepsilon(t)\ddot\varphi_\delta(\psi_\delta(t)) + \ddot w(\psi_\delta(t))\\
&\quad - \ddot w(t)(\dot\varphi_\delta(\psi_\delta(t))^2 - \dot w(t)\ddot\varphi_\delta(\psi_\delta(t))|^2\,dx\\
&\quad + \int_\Omega Q\big(e^\varepsilon(t) + \psi_\delta(t)\delta\varepsilon\dot g(0)e^1 + Ew(\psi_\delta(t)) - Ew(t) - \psi_\delta(t)\delta\varepsilon\dot g(0)E\dot w(0)\big)\,dx\Big]\,dt\\
&\quad - \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\Big[\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot u^\varepsilon(t)|^2\,dx + \int_\Omega Q(e^\varepsilon(t))\,dx\Big]\,dt\\
&\quad + \varepsilon(D_{\mathcal{H}}(\exp(-\cdot/\varepsilon);\tilde p^\varepsilon;0,T) - D_{\mathcal{H}}(\exp(-\cdot/\varepsilon);p^\varepsilon;0,T))\Big\} \geq 0,
\end{aligned}$$

which in turn implies

$$\begin{aligned}
&\limsup_{\delta\to 0}\frac{1}{\delta}\Big\{\int_0^{\varphi_\delta(T)} \dot\psi_\delta(t)\exp\Big(-\frac{\psi_\delta(t)}{\varepsilon}\Big)\Big[\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot u^\varepsilon(t)(\dot\varphi_\delta(\psi_\delta(t))^2 + \dot u^\varepsilon(t)\ddot\varphi_\delta(\psi_\delta(t)) + \ddot w(\psi_\delta(t))\\
&\quad - \ddot w(t)(\dot\varphi_\delta(\psi_\delta(t))^2 - \dot w(t)\ddot\varphi_\delta(\psi_\delta(t))|^2\,dx
\end{aligned}$$

$$+\int_\Omega Q(e^\varepsilon(t)+\psi_\delta(t)\delta\varepsilon\dot g(0)e^1+Ew(\psi_\delta(t))-Ew(t)-\psi_\delta(t)\delta\varepsilon\dot g(0)E\dot w(0))\,dx\Big]\,dt$$
$$-\int_0^{\varphi_\delta(T)}\exp\Big(-\frac{t}{\varepsilon}\Big)\Big[\frac{\varepsilon^2\rho}{2}\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx+\int_\Omega Q(e^\varepsilon(t))\,dx\Big]\,dt$$
$$+\varepsilon(D_{\mathcal H}(\exp(-\cdot/\varepsilon);\tilde p^\varepsilon;0,T)-D_{\mathcal H}(\exp(-\cdot/\varepsilon);p^\varepsilon;0,\varphi_\delta(T)))\Big\}\geq 0. \tag{6.4}$$

The inertial terms satisfy

$$\limsup_{\delta\to 0}\frac{1}{\delta}\Big\{\int_0^{\varphi_\delta(T)}\dot\psi_\delta(t)\exp\Big(-\frac{\psi_\delta(t)}{\varepsilon}\Big)\Big[\frac{\varepsilon^2\rho}{2}\int_\Omega|\ddot u^\varepsilon(t)(\dot\varphi_\delta(\psi_\delta(t))^2+\dot u^\varepsilon(t)\ddot\varphi_\delta(\psi_\delta(t))+\ddot w(\psi_\delta(t))$$
$$-\ddot w(t)(\dot\varphi_\delta(\psi_\delta(t))^2-\dot w(t)\ddot\varphi_\delta(\psi_\delta(t))|^2\Big]\,dx\,dt-\frac{\varepsilon^2\rho}{2}\int_0^{\varphi_\delta(T)}\exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt\Big\}$$
$$=\frac{\varepsilon^2\rho}{2}\int_0^T(\varepsilon\dot g(t)-g(t))\exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt$$
$$+\varepsilon^3\rho\int_0^T\exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega\ddot u^\varepsilon(t)\cdot\Big(-2\ddot u^\varepsilon(t)\dot g(t)-\dot u^\varepsilon(t)\ddot g(t)+\dddot w(t)g(t)+2\ddot w(t)\dot g(t)+\dot w(t)\ddot g(t)\Big)\,dx\,dt, \tag{6.5}$$

where we used the Dominated Convergence Theorem, as well as the identities

$$\frac{\partial}{\partial\delta}\Big(\dot\psi_\delta(t)\exp\Big(-\frac{\psi_\delta(t)}{\varepsilon}\Big)\Big)\Big|_{\delta=0}=(\varepsilon\dot g(t)-g(t))\exp\Big(-\frac{t}{\varepsilon}\Big),$$
$$\frac{\partial}{\partial\delta}\psi_\delta(t)\Big|_{\delta=0}=\varepsilon g(t),$$
$$\frac{\partial}{\partial\delta}(\dot\varphi_\delta(\psi_\delta(t))^2\Big|_{\delta=0}=-2\varepsilon\dot g(t),$$
$$\frac{\partial}{\partial\delta}\ddot\varphi_\delta(\psi_\delta(t))\Big|_{\delta=0}=-\varepsilon\ddot g(t),$$

for every $t\in[0,T]$, and

$$\frac{\partial}{\partial\delta}\Big(\int_\Omega|\ddot u^\varepsilon(t)(\dot\varphi_\delta(\psi_\delta(t))^2+\dot u^\varepsilon(t)\ddot\varphi_\delta(\psi_\delta(t))+\ddot w(\psi_\delta(t))$$
$$-\ddot w(t)(\dot\varphi_\delta(\psi_\delta(t))^2-\dot w(t)\ddot\varphi_\delta(\psi_\delta(t))|^2\,dx\Big)\Big|_{\delta=0}$$
$$=2\varepsilon\int_\Omega\ddot u^\varepsilon(t)\cdot\Big(-2\ddot u^\varepsilon(t)\dot g(t)-\dot u^\varepsilon(t)\ddot g(t)+\dddot w(t)g(t)+2\ddot w(t)\dot g(t)+\dot w(t)\ddot g(t)\Big)\,dx,$$

for a.e. $t\in[0,T]$. Analogously,

$$\frac{\partial}{\partial\delta}\int_\Omega Q(e^\varepsilon(t)+\psi_\delta(t)\delta\varepsilon\dot g(0)e^1+Ew(\psi_\delta(t))-Ew(t)-\psi_\delta(t)\delta\varepsilon\dot g(0)E\dot w(0))\,dx\Big|_{\delta=0}$$
$$=\varepsilon\int_\Omega\mathbb{C}e^\varepsilon(t):\Big(E\dot w(t)g(t)+t\dot g(0)e^1-t\dot g(0)E\dot w(0)\Big)\,dx,$$

for a.e. $t\in[0,T]$, and hence

$$\limsup_{\delta\to 0}\frac{1}{\delta}\Big\{\int_0^{\varphi_\delta(T)}\dot\psi_\delta(t)\exp\Big(-\frac{\psi_\delta(t)}{\varepsilon}\Big)\int_\Omega Q\Big(e^\varepsilon(t)+\psi_\delta(t)\delta\varepsilon\dot g(0)e^1+Ew(\psi_\delta(t))$$
$$-Ew(t)-\psi_\delta(t)\delta\varepsilon\dot g(0)E\dot w(0)\Big)\,dx\,dt-\int_0^{\varphi_\delta(T)}\exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega Q(e^\varepsilon(t))\,dx\,dt\Big\}$$
$$=\int_0^T(\varepsilon\dot g(t)-g(t))\exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega Q(e^\varepsilon(t))\,dx\,dt$$
$$+\varepsilon\int_0^T\exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega\mathbb{C}e^\varepsilon(t):\Big(E\dot w(t)g(t)+t\dot g(0)e^1-t\dot g(0)E\dot w(0)\Big)\,dx\,dt. \tag{6.6}$$

To complete the proof of the lemma it remains to study the asymptotic behavior of the dissipation as $\delta \to 0$. Fix $\lambda > 0$ and let $0 \le t_0 < t_1 < \cdots < t_m \le T$ be such that

$$D_{\mathcal{H}}(\exp(-\cdot/\varepsilon);\tilde{p}^\varepsilon;0,T) \le \sum_{i=1}^m \exp\left(-\frac{t_i}{\varepsilon}\right)\mathcal{H}(\tilde{p}^\varepsilon(t_i)-\tilde{p}^\varepsilon(t_{i-1}))+\lambda.$$

For $i=1,\dots,m$, let $s_i \in [0,\varphi_\delta(T)]$ be such that $t_i = \psi_\delta(s_i)$. There holds

$$\begin{aligned}
&\sum_{i=1}^m \exp\left(-\frac{t_i}{\varepsilon}\right)\mathcal{H}(\tilde{p}^\varepsilon(t_i)-\tilde{p}^\varepsilon(t_{i-1})) = \sum_{i=1}^m \exp\left(-\frac{\psi_\delta(s_i)}{\varepsilon}\right)\mathcal{H}(p^\varepsilon(s_i)-p^\varepsilon(s_{i-1}))\\
&\quad= \sum_{i=1}^m \exp\left(-\frac{s_i}{\varepsilon}\right)\mathcal{H}(p^\varepsilon(s_i)-p^\varepsilon(s_{i-1})) + \sum_{i=1}^m \exp\left(-\frac{s_i}{\varepsilon}\right)\Big[\exp\left(-\delta g(\psi_\delta(s_i))\right)-1\Big]\mathcal{H}(p^\varepsilon(s_i)-p^\varepsilon(s_{i-1}))\\
&\quad\le D_{\mathcal{H}}(\exp(-\cdot/\varepsilon);p^\varepsilon;0,\varphi_\delta(T)) - \delta\sum_{i=1}^m \exp\left(-\frac{s_i}{\varepsilon}\right) g(\psi_\delta(s_i))\mathcal{H}(p^\varepsilon(s_i)-p^\varepsilon(s_{i-1})) + \mathrm{O}(\delta^2)\\
&\quad= D_{\mathcal{H}}(\exp(-\cdot/\varepsilon);p^\varepsilon;0,\varphi_\delta(T)) - \delta\sum_{i=1}^m \exp\left(-\frac{s_i}{\varepsilon}\right) g(s_i)\mathcal{H}(p^\varepsilon(s_i)-p^\varepsilon(s_{i-1})) + \mathrm{O}(\delta^2).
\end{aligned}$$

Thus,

$$\begin{aligned}
D_{\mathcal{H}}(\exp(-\cdot/\varepsilon);\tilde{p}^\varepsilon;0,T) &\le D_{\mathcal{H}}(\exp(-\cdot/\varepsilon);p^\varepsilon;0,\varphi_\delta(T)) + \lambda\\
&\quad - \delta\sum_{i=1}^m \exp\left(-\frac{s_i}{\varepsilon}\right) g(s_i)\mathcal{H}(p^\varepsilon(s_i)-p^\varepsilon(s_{i-1})) + \mathrm{O}(\delta^2).
\end{aligned}$$

By considering finer and finer refinements of $\{t_0,\dots,t_m\}$, in view of the definition of $\hat{D}_{\mathcal{H}}$ (see (2.9)), and by the arbitrariness of λ we conclude that

$$D_{\mathcal{H}}(\exp(-\cdot/\varepsilon);\tilde{p}^\varepsilon;0,T) \le D_{\mathcal{H}}(\exp(-\cdot/\varepsilon);p^\varepsilon;0,\varphi_\delta(T)) + \delta\hat{D}_{\mathcal{H}}(-\exp(-\cdot/\varepsilon)g(\cdot);0,\varphi_\delta(T)) + \mathrm{O}(\delta^2). \tag{6.7}$$

By (6.7) and by [21, Theorem 4.5] we obtain

$$\begin{aligned}
&\limsup_{\delta\to 0}\frac{\varepsilon}{\delta}\big(D_{\mathcal{H}}(\exp(-\cdot/\varepsilon);\tilde{p}^\varepsilon;0,T) - D_{\mathcal{H}}(\exp(-\cdot/\varepsilon);p^\varepsilon;0,\varphi_\delta(T))\big)\\
&\quad\le \limsup_{\delta\to 0}\varepsilon\hat{D}_{\mathcal{H}}(-\exp(-\cdot/\varepsilon)g(\cdot);0,\varphi_\delta(T)) = \limsup_{\delta\to 0} -\varepsilon PMS\int_0^{\varphi_\delta(T)}\exp\Big(-\frac{t}{\varepsilon}\Big)g(t)\,dD_{\mathcal{H}}(p^\varepsilon;0,t)\\
&\quad= -\liminf_{\delta\to 0}\Big\{\varepsilon g(\varphi_\delta(T))\exp\Big(-\frac{\varphi_\delta(T)}{\varepsilon}\Big)D_{\mathcal{H}}(p^\varepsilon;0,\varphi_\delta(T))\\
&\quad-\int_0^{\varphi_\delta(T)}(\varepsilon\dot{g}(t)-g(t))\exp\Big(-\frac{t}{\varepsilon}\Big)D_{\mathcal{H}}(p^\varepsilon;0,t)\,dt\Big\}\\
&\quad\le \int_0^T(\varepsilon\dot{g}(t)-g(t))\exp\Big(-\frac{t}{\varepsilon}\Big)D_{\mathcal{H}}(p^\varepsilon;0,t)\,dt.
\end{aligned}\tag{6.8}$$

The thesis follows by combining (6.1), (6.4)–(6.6), (6.8), and by the definition of $\mathcal{K}_\varepsilon$, $\mathcal{D}_\varepsilon$, and $\mathcal{L}_\varepsilon$. ☐

Setting

$$\begin{aligned}
R_\varepsilon(t) &:= -\varepsilon\int_\Omega \mathbb{C}e^\varepsilon(t):\big(e^1 - E\dot{w}(0)\big)\,dx,\\
\tilde{R}_\varepsilon(t) &:= -\varepsilon^3\rho\int_\Omega \ddot{u}^\varepsilon(t)\cdot\dddot{w}(t)\,dx - \varepsilon\int_\Omega \mathbb{C}e^\varepsilon(t):E\dot{w}(t)\,dx,\\
\hat{R}_\varepsilon(t) &:= -2\varepsilon^3\rho\int_\Omega \ddot{u}^\varepsilon(t)\cdot\ddot{w}(t)\,dx,\\
\mathring{R}_\varepsilon(t) &:= -\varepsilon^3\rho\int_\Omega \ddot{u}^\varepsilon(t)\cdot\dot{w}(t)\,dx,
\end{aligned}$$

for a.e. $t \in [0,T]$, and choosing $g(t) = t$ in Lemma 6.1, the same approximation argument as in [47, Corollary 4.5] yields

Corollary 6.2. *Let $(u^1, e^1, 0) \in \mathscr{A}(\dot{w}(0))$, and let $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ be a minimizer of G_ε. Then*

$$\varepsilon \mathcal{A} \mathcal{L}_\varepsilon(0) - \mathcal{A}^2 \mathcal{L}_\varepsilon(0) - 4\varepsilon \mathcal{A} \mathcal{D}_\varepsilon(0) \geq \mathcal{A}^2 R_\varepsilon(0) + \mathcal{A}^2 \tilde{R}_\varepsilon(0) + \mathcal{A} \hat{R}_\varepsilon(0).$$

Finally, by considering the sequence of maps $g_\delta : [0,T] \to [0,+\infty)$ defined as

$$g_\delta(s) := \begin{cases} 0 & \text{if } s \leq t, \\ \frac{(s-t)^2}{2\delta} & \text{if } t < s < t+\delta, \\ s - t - \frac{\delta}{2} & \text{if } s \geq t + \delta, \end{cases}$$

in Lemma 6.1, and by letting δ go to zero, we deduce the following inequality.

Corollary 6.3. *Let $(u^1, e^1, 0) \in \mathscr{A}(\dot{w}(0))$, and let $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ be a minimizer of G_ε. Then*

$$\varepsilon \mathcal{A} \mathcal{L}_\varepsilon(t) - \mathcal{A}^2 \mathcal{L}_\varepsilon(t) - 4\varepsilon \mathcal{A} \mathcal{D}_\varepsilon(t) - \varepsilon \dot{\mathcal{K}}_\varepsilon(t) \geq \mathcal{A}^2 \tilde{R}_\varepsilon(t) + \mathcal{A} \hat{R}_\varepsilon(t) + \mathring{R}_\varepsilon(t),$$

for a.e. $t \in [0,T]$.

We are now in a position to prove an ε-dependent energy inequality.

Proposition 6.4 (Energy inequality)**.** *Let $(u^1, e^1, 0) \in \mathscr{A}(\dot{w}(0))$, and let $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ be a minimizer of G_ε. Then*

$$\begin{aligned} &\frac{\mathcal{E}_\varepsilon(t)}{\varepsilon^2} - \rho \int_\Omega \dot{u}^\varepsilon(t) \cdot \dot{w}(t)\,dx + \frac{\mathcal{A}^2 \tilde{R}_\varepsilon(t)}{\varepsilon^2} + \frac{\mathcal{A} \hat{R}_\varepsilon(t)}{\varepsilon^2} + \frac{\mathcal{A} \tilde{R}_\varepsilon(t)}{\varepsilon} \\ &\quad \leq \frac{\mathcal{E}_\varepsilon(0)}{\varepsilon^2} - \rho \int_\Omega u^1 \cdot \dot{w}(0)\,dx + \frac{\mathcal{A}^2 \tilde{R}_\varepsilon(0)}{\varepsilon^2} + \frac{\mathcal{A} \hat{R}_\varepsilon(0)}{\varepsilon^2} + \frac{\mathcal{A} \tilde{R}_\varepsilon(0)}{\varepsilon} \\ &\quad\quad - \rho \int_0^t \int_\Omega \dot{u}^\varepsilon(s) \cdot \ddot{w}(s)\,dx\,ds + \int_0^t \frac{\tilde{R}_\varepsilon(s)}{\varepsilon}\,ds - \int_0^t \frac{\hat{R}_\varepsilon(s)}{\varepsilon^2}\,ds \end{aligned}$$

Proof. By the definition of the approximate energy (see (6.2)) there holds

$$\mathcal{E}_\varepsilon(t) := \mathcal{K}_\varepsilon(t) + \mathcal{A}^2(\mathcal{L}_\varepsilon(t) - \mathcal{D}_\varepsilon(t))$$

for every $t \in [0,T]$, which implies

$$\dot{\mathcal{E}}_\varepsilon(t) = \dot{\mathcal{K}}_\varepsilon(t) + \frac{\mathcal{A}^2 \mathcal{L}_\varepsilon(t)}{\varepsilon} - \frac{\mathcal{A}^2 \mathcal{D}_\varepsilon(t)}{\varepsilon} - \mathcal{A} \mathcal{L}_\varepsilon(t) + \mathcal{A} \mathcal{D}_\varepsilon(t)$$

for a.e. $t \in [0,T]$. In view of Corollary 6.3 we obtain the estimate

$$\begin{aligned} \dot{\mathcal{E}}_\varepsilon(t) &= -\frac{\mathcal{A}^2 \mathcal{D}_\varepsilon(t)}{\varepsilon} - 3\mathcal{A} \mathcal{D}_\varepsilon(t) - \frac{\mathcal{A}^2 \tilde{R}_\varepsilon(t)}{\varepsilon} - \frac{\mathcal{A} \hat{R}_\varepsilon(t)}{\varepsilon} - \frac{\mathring{R}_\varepsilon(t)}{\varepsilon} \\ &\leq -\frac{\mathcal{A}^2 \tilde{R}_\varepsilon(t)}{\varepsilon} - \frac{\mathcal{A} \hat{R}_\varepsilon(t)}{\varepsilon} - \frac{\mathring{R}_\varepsilon(t)}{\varepsilon} \end{aligned} \tag{6.9}$$

for a.e. $t \in [0,T]$. On the other hand,

$$-\frac{\mathring{R}_\varepsilon(t)}{\varepsilon^3} = \rho \Big(\int_\Omega \dot{u}^\varepsilon(t) \cdot \dot{w}(t)\,dx \Big)^{\cdot} - \rho \int_\Omega \dot{u}^\varepsilon(t) \cdot \ddot{w}(t)\,dx, \tag{6.10}$$

and

$$\begin{aligned} -\frac{\mathcal{A}^2 \tilde{R}_\varepsilon(t)}{\varepsilon^3} - \frac{\mathcal{A} \hat{R}^\varepsilon(t)}{\varepsilon^3} &= \Big(-\frac{\mathcal{A}^2 \tilde{R}_\varepsilon(t)}{\varepsilon^2} - \frac{\mathcal{A} \hat{R}_\varepsilon(t)}{\varepsilon^2} \Big)^{\cdot} - \frac{\mathcal{A} \tilde{R}_\varepsilon(t)}{\varepsilon^2} - \frac{\hat{R}_\varepsilon(t)}{\varepsilon^2} \\ &= \Big(-\frac{\mathcal{A}^2 \tilde{R}_\varepsilon(t)}{\varepsilon^2} - \frac{\mathcal{A} \hat{R}_\varepsilon(t)}{\varepsilon^2} - \frac{\mathcal{A} \tilde{R}_\varepsilon(t)}{\varepsilon} \Big)^{\cdot} + \frac{\tilde{R}_\varepsilon(t)}{\varepsilon} - \frac{\hat{R}_\varepsilon(t)}{\varepsilon^2} \end{aligned} \tag{6.11}$$

for a.e. $t \in [0,T]$. By combining (6.9)–(6.11) we deduce

$$\Big(\frac{\mathcal{E}_\varepsilon(t)}{\varepsilon^2} - \rho \int_\Omega \dot{u}^\varepsilon(t) \cdot \dot{w}(t)\,dx + \frac{\mathcal{A}^2 \tilde{R}_\varepsilon(t)}{\varepsilon^2} + \frac{\mathcal{A} \hat{R}_\varepsilon(t)}{\varepsilon^2} + \frac{\mathcal{A} \tilde{R}_\varepsilon(t)}{\varepsilon} \Big)^{\cdot}$$

$$\leq -\rho\int_\Omega \dot u^\varepsilon(t)\cdot\ddot w(t)\,dx+\frac{\tilde R_\varepsilon(t)}{\varepsilon}-\frac{\hat R_\varepsilon(t)}{\varepsilon^2}$$

for a.e. $t\in[0,T]$. An integration in time in $[0,T]$ yields the thesis. □

The same argument in [47, Lemma 6.1] provides the following technical result.

Lemma 6.5. *Let ℓ and m be two non-negative functions in $L^1(0,T)$ such that*

$$\mathcal{A}^2\ell(t)\leq m(t)\quad \textit{for a.e. } t\in[0,T].$$

Then, for every $a>0$, and $\delta\in(0,1)$, there holds

$$\Big(\int_0^{\delta a} s\exp\Big(-\frac{s}{\varepsilon}\Big)\,ds\Big)\int_{t+\delta a}^{t+a}\ell(s)\,ds\leq\int_t^{t+a}m(s)\,ds$$

for every $t\in[0,T-a]$.

In view of Proposition 6.4 and Lemma 6.5 we obtain an integrated-in-time version of the ε-energy inequality.

Proposition 6.6 (Integral energy inequality)**.** *Let $(u^1,e^1,0)\in\mathscr{A}(\dot w(0))$, and let $(u^\varepsilon,e^\varepsilon,p^\varepsilon)$ be a minimizer of G_ε. Then, for every $a>0$, and $\delta\in(0,1)$, there holds*

$$\begin{aligned}
&\Big(\frac{1}{\varepsilon^2}\int_0^{\delta a} s\exp\Big(-\frac{s}{\varepsilon}\Big)\,ds\Big)\int_{t+\delta a}^{t+a}\Big(\int_\Omega Q(e^\varepsilon(s))\,dx+D_{\mathcal H}(p^\varepsilon;0,s)\Big)\,ds+\frac{\rho}{2}\int_t^{t+a}\int_\Omega|\dot u^\varepsilon(s)|^2\,dx\,ds\\
&-\rho\int_t^{t+a}\int_\Omega\dot u^\varepsilon(s)\cdot\dot w(s)\,dx\,ds\leq-\int_t^{t+a}\Big(\frac{\mathcal{A}^2\tilde R_\varepsilon(s)}{\varepsilon^2}+\frac{\mathcal{A}\hat R_\varepsilon(s)}{\varepsilon^2}+\frac{\mathcal{A}\tilde R_\varepsilon(s)}{\varepsilon}\Big)\,ds\\
&+\frac{\mathcal{E}_\varepsilon(0)a}{\varepsilon^2}-\rho a\int_\Omega u^1\cdot\dot w(0)\,dx+a\frac{\mathcal{A}^2\tilde R_\varepsilon(0)}{\varepsilon^2}+a\frac{\mathcal{A}\hat R_\varepsilon(0)}{\varepsilon^2}+a\frac{\mathcal{A}\tilde R_\varepsilon(0)}{\varepsilon}\\
&-\rho\int_t^{t+a}\int_0^\xi\int_\Omega\dot u^\varepsilon(s)\cdot\ddot w(s)\,dx\,ds\,d\xi+\int_t^{t+a}\int_0^\xi\frac{\tilde R_\varepsilon(s)}{\varepsilon}\,ds\,d\xi-\int_t^{t+a}\int_0^\xi\frac{\hat R_\varepsilon(s)}{\varepsilon^2}\,ds\,d\xi
\end{aligned}\tag{6.12}$$

for every $t\in[0,T]$.

Proof. Owing to Proposition 6.4 we can apply Lemma 6.5 with

$$\ell(t):=\frac{\mathcal{W}_\varepsilon(t)}{\varepsilon^2}+\frac{\mathcal{H}_\varepsilon(t)}{\varepsilon^3},$$

and

$$\begin{aligned}
m(t):=&-\frac{\mathcal{K}_\varepsilon(t)}{\varepsilon^2}+\rho\int_\Omega\dot u^\varepsilon(t)\cdot\dot w(t)\,dx-\frac{\mathcal{A}^2\tilde R_\varepsilon(t)}{\varepsilon^2}-\frac{\mathcal{A}\hat R_\varepsilon(t)}{\varepsilon^2}-\frac{\mathcal{A}\tilde R_\varepsilon(t)}{\varepsilon}\\
&+\frac{\mathcal{E}_\varepsilon(0)}{\varepsilon^2}-\rho\int_\Omega u^1\cdot\dot w(0)\,dx+\frac{\mathcal{A}^2\tilde R_\varepsilon(0)}{\varepsilon^2}+\frac{\mathcal{A}\hat R_\varepsilon(0)}{\varepsilon^2}+\frac{\mathcal{A}\tilde R_\varepsilon(0)}{\varepsilon}\\
&-\rho\int_0^t\int_\Omega\dot u^\varepsilon(s)\cdot\ddot w(s)\,dx\,ds+\int_0^t\frac{\tilde R_\varepsilon(s)}{\varepsilon}\,ds-\int_0^t\frac{\hat R_\varepsilon(s)}{\varepsilon^2}\,ds
\end{aligned}$$

for a.e. $t\in[0,T]$. The thesis follows by the definitions of $\mathcal{W}_\varepsilon$, $\mathcal{H}_\varepsilon$, and $\mathcal{K}_\varepsilon$. □

We conclude this section by showing a further characterization of ε-minimizers.

Proposition 6.7 (Weak energy equality)**.** *Let $(u^\varepsilon,e^\varepsilon,p^\varepsilon)$ be a minimizer of G_ε. Then*

$$\begin{aligned}
&\int_0^T\dot\varphi(t)\Big[\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx+2\varepsilon\rho\int_0^t\int_\Omega|\ddot u^\varepsilon(s)|^2\,dx\,ds+\frac{\rho}{2}\int_\Omega|\dot u^\varepsilon(t)|^2\,dx\\
&+D_{\mathcal H}(p^\varepsilon;0,t)\Big]\,dt=\int_0^T\dot\varphi(t)\int_0^t\int_\Omega\mathbb{C}e^\varepsilon(s):E\dot w(s)\,dx\,ds\,dt
\end{aligned}$$

$$
\begin{aligned}
&-\frac{3\varepsilon^2\rho}{2}\int_0^T \dddot{\varphi}(t)\int_0^t\int_\Omega |\ddot{u}^\varepsilon(s)|^2\,dx\,ds\,dt-\varepsilon^2\rho\int_0^T\ddot{\varphi}(t)\int_\Omega \ddot{u}^\varepsilon(t)\cdot(\dot{w}(t)-\dot{u}^\varepsilon(t))\,dx\,dt\\
&+\rho\int_0^T\int_\Omega \dot{u}^\varepsilon(t)\cdot\partial_t[\dot{w}(t)(\varphi(t)+2\varepsilon\dot{\varphi}(t))]\,dx\,dt-\varepsilon^2\rho\int_0^T\int_\Omega \ddot{u}^\varepsilon(t)\cdot\ddot{w}(t)\varphi(t)\,dx\,dt\\
&-\varepsilon\rho\int_0^T\int_\Omega \ddot{\varphi}(t)|\dot{u}^\varepsilon(t)|^2\,dx\,dt+2\varepsilon\rho\int_0^T\int_\Omega \dot{u}^\varepsilon(t)\cdot\partial_t[\ddot{w}(t)(\varphi(t)+\varepsilon\dot{\varphi}(t))]\,dx\,dt \qquad (6.13)
\end{aligned}
$$

for every $\varphi\in C_c^\infty(0,T)$.

This last result of the section relies on the argument developed in [38, Proposition 4.1], which consists in comparing the energy associated to a minimizer $(u^\varepsilon,e^\varepsilon,p^\varepsilon)$ of G_ε with that of a suitably rescaled triple $(\tilde{u}^\varepsilon,\tilde{e}^\varepsilon,\tilde{p}^\varepsilon)$, obtained by composing $(u^\varepsilon,e^\varepsilon,p^\varepsilon)$ with a diffeomorphic reparametrization of $[0,T]$. We postpone the proof of Proposition 6.7 to Appendix A.

7. Proof of Theorem 2.3

Having established the uniform estimate (5.40) we are now ready to prove Theorem 2.3. For every $\varepsilon>0$, let $(u^\varepsilon,e^\varepsilon,p^\varepsilon)$ be a minimizer of G_ε satisfying (5.40). Since $p^\varepsilon(0)=p^0$ for every $\varepsilon>0$, by a generalization of Helly's Theorem [11, Theorem 7.2] there exists $p\in BV(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D))$ such that

$$
p^\varepsilon(t)\rightharpoonup p(t)\quad\text{weakly* in }\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)\quad\text{for every }t\in[0,T], \qquad (7.1)
$$

$$
D_{\mathcal{H}}(p;0,T)\le\liminf_{\varepsilon\to0}D_{\mathcal{H}}(p^\varepsilon;0,T). \qquad (7.2)
$$

In addition, (5.40) yields the existence of maps $u\in W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3))$ and $e\in L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym}))$ such that, up to subsequences,

$$
u^\varepsilon\rightharpoonup u\quad\text{weakly in }W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3)), \qquad (7.3)
$$

$$
e^\varepsilon\rightharpoonup e\quad\text{weakly in }L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym})). \qquad (7.4)
$$

In particular by (7.3), and by the embedding of $W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3))$ into $C_w([0,T];L^2(\Omega;\mathbb{R}^3))$ there holds

$$
u^\varepsilon(t)\rightharpoonup u(t)\quad\text{weakly in }L^2(\Omega;\mathbb{R}^3)\quad\text{for every }t\in[0,T], \qquad (7.5)
$$

and $u(0)=u^0$. In view of (7.1), (7.4), and (7.5) we obtain that

$$
e^\varepsilon(t)\rightharpoonup e(t)\quad\text{weakly in }\mathcal{D}'(\Omega;\mathbb{M}^{3\times3}_{\rm sym})\quad\text{for every }t\in[0,T]. \qquad (7.6)
$$

By (7.1), (7.5), and (7.6) for a.e. $t\in[0,T]$ there exists a t-dependent subsequence $\{\varepsilon_t\}$ such that

$$
e^{\varepsilon_t}(t)\rightharpoonup e(t)\quad\text{weakly in }L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym}), \qquad (7.7)
$$

$$
u^{\varepsilon_t}(t)\rightharpoonup^* u(t)\quad\text{weakly* in }BD(\Omega). \qquad (7.8)
$$

The fact that p satisfies the boundary condition on Γ_0 for a.e. $t\in[0,T]$ follows arguing as in [11, Lemma 2.1].

Let $v\in C_c^\infty((0,T)\times\Omega;\mathbb{R}^3)$. For $\lambda>0$, we have that

$$
\left(u^\varepsilon+\lambda\exp\left(\frac{t}{\varepsilon}\right)v,\,e^\varepsilon+\lambda\exp\left(\frac{t}{\varepsilon}\right)Ev,\,p^\varepsilon\right)\in\mathcal{V},
$$

thus by the minimality of $(u^\varepsilon,e^\varepsilon,p^\varepsilon)$,

$$
\frac{1}{\lambda}\left(G_\varepsilon\left(u^\varepsilon+\lambda\exp\left(\frac{t}{\varepsilon}\right)v,\,e^\varepsilon+\lambda\exp\left(\frac{t}{\varepsilon}\right)Ev,\,p^\varepsilon\right)-G_\varepsilon(u^\varepsilon,e^\varepsilon,p^\varepsilon)\right)\ge0,
$$

namely

$$
\rho\int_0^T\int_\Omega \ddot{u}^\varepsilon\cdot(v+\varepsilon\dot{v}+\varepsilon^2\ddot{v})\,dx\,dt+\int_0^T\int_\Omega \mathbb{C}e^\varepsilon:Ev\,dx\,dt=0 \qquad (7.9)
$$

for every $v \in C_c^\infty((0,T)\times\Omega;\mathbb{R}^3)$. Integrating by parts with respect to time, (7.3) and (7.4) yield

$$-\rho\int_0^T\int_\Omega \dot u\cdot\dot v\,dx\,dt+\int_0^T\int_\Omega \mathbb{C}e:\,Ev\,dx\,dt=0$$

for every $v \in C_c^\infty((0,T)\times\Omega;\mathbb{R}^3)$, that is

$$\rho\ddot u-\operatorname{div}\mathbb{C}e=0 \tag{7.10}$$

in the sense of distributions. Since the same procedure applies to every $v\in C_c^\infty(0,T;C^\infty(\bar\Omega;\mathbb{R}^3))$ with $v=0$ on Γ_0 for every $t\in[0,T]$, we also obtain

$$\mathbb{C}e\nu=0\quad\text{on }\partial\Omega\setminus\Gamma_0. \tag{7.11}$$

Let now $q\in C_c^\infty(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_D))$, $\lambda>0$, and consider the test triple

$$\Big(u^\varepsilon,\,e^\varepsilon-\lambda\exp\Big(\frac{t}{\varepsilon}\Big)q,p^\varepsilon+\lambda\exp\Big(\frac{t}{\varepsilon}\Big)q\Big).$$

By the minimality of $(u^\varepsilon,e^\varepsilon,p^\varepsilon)$,

$$\frac{1}{\lambda}\left(G_\varepsilon\left(u^\varepsilon,\,e^\varepsilon-\lambda\exp\left(\frac{t}{\varepsilon}\right)q,p^\varepsilon+\lambda\exp\Big(\frac{t}{\varepsilon}\Big)q\right)-G_\varepsilon(u^\varepsilon,e^\varepsilon,p^\varepsilon)\right)\geq 0. \tag{7.12}$$

On the other hand,

$$\begin{aligned}&\frac{1}{\lambda}\left(D_{\mathcal H}(\exp(-\,\cdot\,/\varepsilon);p^\varepsilon+\lambda\exp(\cdot/\varepsilon)q;0,T)-D_{\mathcal H}(\exp(-\,\cdot\,/\varepsilon);p^\varepsilon;0,T)\right)\\&\leq D_{\mathcal H}(\exp(-\,\cdot\,/\varepsilon);\exp(\cdot/\varepsilon)q;0,T),\end{aligned}$$

and by the in-time regularity of q,

$$\begin{aligned}&D_{\mathcal H}(\exp(-\,\cdot\,/\varepsilon);\exp(\cdot/\varepsilon)q;0,T)=\int_0^T\exp\Big(-\frac{t}{\varepsilon}\Big)\mathcal{H}\Big(\frac{1}{\varepsilon}\exp\Big(\frac{t}{\varepsilon}\Big)q(t)+\exp\Big(\frac{t}{\varepsilon}\Big)\dot q(t)\Big)\\&\leq\frac{1}{\varepsilon}\int_0^T\mathcal{H}(q(t))\,dt+\int_0^T\mathcal{H}(\dot q(t))\,dt.\end{aligned}$$

Thus (7.12) can be rewritten as

$$-\int_0^T\int_\Omega\mathbb{C}e^\varepsilon:\,q\,dx\,dt+\int_0^T\mathcal{H}(q(t))\,dt+\varepsilon\int_0^T\mathcal{H}(\dot q(t))\,dt\geq 0.$$

for every $q\in C_c^\infty(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_D))$, and by (7.4),

$$\int_0^T\int_\Omega\mathbb{C}e:\,q\,dx\,dt\leq\int_0^T\mathcal{H}(q(t))\,dt$$

for every $q\in C_c^\infty(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_D))$. By approximation, the previous inequality holds in particular by choosing $q=M\chi_I\chi_B$ with $M\in\mathbb{M}^{3\times3}_D$, I and B Borel subsets of $(0,T)$ and $\Omega\cup\Gamma_0$, respectively. Hence, we deduce that

$$[\mathbb{C}e(t)]_D\in\partial H(0) \tag{7.13}$$

for a.e. $t\in[0,T]$ and $x\in\Omega$.

It remains to show that the limit triple satisfies the energy inequality (c3). We first fix $a>0$ and $\delta\in(0,1)$, and we argue by passing to the limit as $\varepsilon\to0$ in (6.12). Since

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon^2}\int_0^{\delta a}s\exp\Big(-\frac{s}{\varepsilon}\Big)\,ds=1,$$

by (7.2), (7.3), and (7.4), we have

$$\begin{aligned}&\int_{t+\delta a}^{t+a}\Big(\int_\Omega Q(e(s))\,dx+D_{\mathcal H}(p;0,s)\Big)\,ds+\frac{\rho}{2}\int_t^{t+a}\int_\Omega|\dot u(s)|^2\,dx\,ds\\&-\rho\int_t^{t+a}\int_\Omega\dot u(s)\cdot\dot w(s)\,dx\,ds\leq-\rho a\int_\Omega u^1\cdot\dot w(0)\,dx-\rho\int_t^{t+a}\int_0^\xi\int_\Omega\dot u(s)\cdot\ddot w(s)\,dx\,ds\,d\xi\end{aligned}$$

$$+\limsup_{\varepsilon\to 0}\Big\{-\int_t^{t+a}\Big(\frac{\mathcal{A}^2\tilde R_\varepsilon(s)}{\varepsilon^2}+\frac{\mathcal{A}\hat R_\varepsilon(s)}{\varepsilon^2}+\frac{\mathcal{A}\tilde R_\varepsilon(s)}{\varepsilon}\Big)\,ds+\frac{\mathcal{E}_\varepsilon(0)a}{\varepsilon^2}+a\frac{\mathcal{A}^2\tilde R_\varepsilon(0)}{\varepsilon^2}$$
$$+a\frac{\mathcal{A}\hat R_\varepsilon(0)}{\varepsilon^2}+a\frac{\mathcal{A}\tilde R_\varepsilon(0)}{\varepsilon}+\int_t^{t+a}\int_0^\xi\frac{\tilde R_\varepsilon(s)}{\varepsilon}\,ds\,d\xi-\int_t^{t+a}\int_0^\xi\frac{\hat R_\varepsilon(s)}{\varepsilon^2}\,ds\,d\xi\Big\}. \tag{7.14}$$

By (5.40) there holds

$$\Big|\int_t^{t+a}\int_0^\xi\frac{\hat R_\varepsilon(s)}{\varepsilon^2}\,ds\,d\xi\Big|\le Ca\varepsilon\|\ddot u^\varepsilon\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}\|\ddot w\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))}\le Ca\sqrt{\varepsilon}, \tag{7.15}$$

and, analogously,

$$\Big|\int_t^{t+a}\int_0^\xi\frac{\tilde R_\varepsilon(s)}{\varepsilon}\,ds\,d\xi+\int_t^{t+a}\int_0^\xi\int_\Omega\mathbb{C}e^\varepsilon(s):E\dot w(s)\,dx\,ds\,d\xi\Big|\le Ca\varepsilon\sqrt{\varepsilon}. \tag{7.16}$$

for every $t\in[0,T]$. Thus, by (7.4)

$$\lim_{\varepsilon\to 0}\int_t^{t+a}\int_0^\xi\frac{\tilde R_\varepsilon(s)}{\varepsilon}\,ds\,d\xi-\int_t^{t+a}\int_0^\xi\frac{\hat R_\varepsilon(s)}{\varepsilon^2}\,ds\,d\xi=-\int_t^{t+a}\int_0^\xi\int_\Omega\mathbb{C}e(s):E\dot w(s)\,dx\,ds\,d\xi. \tag{7.17}$$

Arguing as in (7.16), and using again (5.40), we deduce

$$\begin{aligned}\Big|a\frac{\mathcal{A}^2\tilde R^\varepsilon(0)}{\varepsilon^2}\Big|&\le Ca\sqrt{\varepsilon}+a\Big|\frac{1}{\varepsilon}\int_0^T\exp\Big(-\frac{s}{\varepsilon}\Big)s\int_\Omega\mathbb{C}e^\varepsilon(s):E\dot w(s)\,dx\,ds\Big|\\&\le Ca\sqrt{\varepsilon}+C\frac{a}{\varepsilon}\Big\|\exp\Big(-\frac{s}{\varepsilon}\Big)s\Big\|_{L^2(0,T)}\le Ca\sqrt{\varepsilon},\end{aligned} \tag{7.18}$$

and

$$\begin{aligned}\Big|a\frac{\mathcal{A}\tilde R^\varepsilon(0)}{\varepsilon}\Big|&\le Ca\varepsilon\sqrt{\varepsilon}+a\Big|\int_0^T\exp\Big(-\frac{s}{\varepsilon}\Big)\int_\Omega\mathbb{C}e^\varepsilon(s):E\dot w(s)\,dx\,ds\Big|\\&\le Ca\varepsilon\sqrt{\varepsilon}+Ca\Big\|\exp\Big(-\frac{s}{\varepsilon}\Big)\Big\|_{L^2(0,T)}\le Ca\sqrt{\varepsilon}.\end{aligned} \tag{7.19}$$

The same argument yields

$$\Big|\frac{\mathcal{A}^2\tilde R^\varepsilon(t)}{\varepsilon^2}\Big|+\Big|\frac{\mathcal{A}\tilde R^\varepsilon(t)}{\varepsilon}\Big|\le C\sqrt{\varepsilon}\quad\text{for every } t\in[0,T]. \tag{7.20}$$

Finally, estimates analogous to (7.15) imply

$$\Big|\frac{\mathcal{A}\hat R^\varepsilon(t)}{\varepsilon^2}\Big|\le C\sqrt{\varepsilon}\quad\text{for every } t\in[0,T]. \tag{7.21}$$

By combining (7.14) with (7.17), (7.20) and (7.21) we conclude that

$$\begin{aligned}&\int_{t+\delta a}^{t+a}\Big(\int_\Omega Q(e(s))\,dx+D_{\mathcal{H}}(p;0,s)\Big)\,ds+\frac{\rho}{2}\int_t^{t+a}\int_\Omega|\dot u(s)|^2\,dx\,ds\\&-\rho\int_t^{t+a}\int_\Omega\dot u(s)\cdot\dot w(s)\,dx\,ds\le-\rho a\int_\Omega u^1\cdot\dot w(0)\,dx-\rho\int_t^{t+a}\int_0^\xi\int_\Omega\dot u(s)\cdot\ddot w(s)\,dx\,ds\,d\xi\\&-\int_t^{t+a}\int_0^\xi\int_\Omega\mathbb{C}e(s):E\dot w(s)\,dx\,ds\,d\xi+a\limsup_{\varepsilon\to 0}\frac{\mathcal{E}_\varepsilon(0)}{\varepsilon^2}\end{aligned} \tag{7.22}$$

for every $a>0$ and $\delta\in(0,1)$. In particular, letting $\delta\to 0$, dividing by a, and letting $a\to 0$, by Lebesgue differentiation theorem we deduce the inequality

$$\begin{aligned}&\int_\Omega Q(e(t))\,dx+D_{\mathcal{H}}(p;0,t)+\frac{\rho}{2}\int_\Omega|\dot u(t)|^2\,dx-\rho\int_\Omega\dot u(t)\cdot\dot w(t)\,dx\\&\le-\rho\int_\Omega u^1\cdot\dot w(0)\,dx-\rho\int_0^t\int_\Omega\dot u(s)\cdot\ddot w(s)\,dx\,ds\\&-\int_0^t\int_\Omega\mathbb{C}e(s):E\dot w(s)\,dx\,ds+\limsup_{\varepsilon\to 0}\frac{\mathcal{E}_\varepsilon(0)}{\varepsilon^2},\end{aligned} \tag{7.23}$$

for a.e. $t \in [0,T]$. In order to complete the proof of the energy inequality (c3) it remains to estimate from above the quantity $\limsup_{\varepsilon\to 0}\frac{\mathcal{E}_\varepsilon(0)}{\varepsilon^2}$. To this aim, we observe that, by the definition of the approximate energy, by Corollary 6.2, and by (7.20) and (7.21) there holds

$$\begin{aligned}\frac{\mathcal{E}_\varepsilon(0)}{\varepsilon^2} &= \frac{\rho}{2}\int_\Omega |u^1|^2\,dx + \frac{\mathcal{A}^2\mathcal{W}_\varepsilon(0)}{\varepsilon^2} \leq \frac{\rho}{2}\int_\Omega |u^1|^2\,dx + \frac{\mathcal{A}^2\mathcal{L}_\varepsilon(0)}{\varepsilon^2}\\ &\leq \frac{\rho}{2}\int_\Omega |u^1|^2\,dx + \frac{\mathcal{A}\mathcal{L}_\varepsilon(0)}{\varepsilon} - \frac{\mathcal{A}^2R_\varepsilon(0)}{\varepsilon^2} - \frac{\mathcal{A}^2\tilde{R}_\varepsilon(0)}{\varepsilon^2} - \frac{\mathcal{A}\hat{R}_\varepsilon(0)}{\varepsilon^2}\\ &\leq \frac{\rho}{2}\int_\Omega |u^1|^2\,dx + \frac{\mathcal{A}\mathcal{L}_\varepsilon(0)}{\varepsilon} - \frac{\mathcal{A}^2R_\varepsilon(0)}{\varepsilon^2} + C\sqrt{\varepsilon}\\ &= \frac{\rho}{2}\int_\Omega |u^1|^2\,dx + \frac{1}{\varepsilon}\int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\Big(\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot{u}^\varepsilon(t)|^2\,dx + \int_\Omega Q(e^\varepsilon(t))\,dx + D_{\mathcal{H}}(p^\varepsilon;0,t)\Big)\,dt\\ &\quad - \frac{\mathcal{A}^2R_\varepsilon(0)}{\varepsilon^2} + C\sqrt{\varepsilon}.\end{aligned}$$

By (5.40),

$$\begin{aligned}\Big|\frac{\mathcal{A}^2R_\varepsilon(0)}{\varepsilon^2}\Big| &= \frac{1}{\varepsilon}\Big|\int_0^T \exp\Big(-\frac{s}{\varepsilon}\Big)s\int_\Omega \mathbb{C}e^\varepsilon(s) : \big(e^1 - E\dot{w}(0)\big)\,dx\,ds\Big|\\ &\leq \frac{C}{\varepsilon}\Big\|\exp\Big(-\frac{s}{\varepsilon}\Big)s\Big\|_{L^2(0,T)} \leq C\sqrt{\varepsilon}.\end{aligned}$$

In view of [21, Theorem 4.5], we have

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)D_{\mathcal{H}}(p^\varepsilon;0,t)\,dt &= -\exp\Big(-\frac{T}{\varepsilon}\Big)D_{\mathcal{H}}(p^\varepsilon;0,T) + PMS\int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\,dD_{\mathcal{H}}(p^\varepsilon;0,t)\\ &\leq D_{\mathcal{H}}(\exp(-\cdot/\varepsilon);p^\varepsilon;0,T).\end{aligned}$$

Thus, we obtain

$$\frac{\mathcal{E}_\varepsilon(0)}{\varepsilon^2} \leq \frac{\rho}{2}\int_\Omega |u^1|^2\,dx + C\sqrt{\varepsilon} + \frac{1}{\varepsilon}G_\varepsilon(u^\varepsilon,e^\varepsilon,p^\varepsilon).$$

By the minimality of $(u^\varepsilon,e^\varepsilon,p^\varepsilon)$, and since the triple

$$t \to (u^0 + tu^1 + w(t) - w(0) - t\dot{w}(0), e^0 + te^1 + Ew(t) - Ew(0) - tE\dot{w}(0), p^0)$$

belongs to $\mathcal{V}$, we deduce the upper bound

$$\begin{aligned}\frac{\mathcal{E}_\varepsilon(0)}{\varepsilon^2} &\leq \frac{\rho}{2}\int_\Omega |u^1|^2\,dx + C\sqrt{\varepsilon} + \frac{1}{\varepsilon}G_\varepsilon(u^\varepsilon,e^\varepsilon,p^\varepsilon)\\ &\leq \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\Big(\frac{\varepsilon\rho}{2}\int_\Omega |\ddot{w}(t)|^2\,dx + \frac{1}{\varepsilon}\int_\Omega Q(e^0 + te^1 + Ew(t) - Ew(0) - tE\dot{w}(0))\,dx\Big)\,dt\\ &\quad + \frac{\rho}{2}\int_\Omega |u^1|^2\,dx + C\sqrt{\varepsilon}\\ &= \frac{\rho}{2}\int_\Omega |u^1|^2\,dx + C\sqrt{\varepsilon} + \int_0^{T/\varepsilon} \exp(-z)\int_\Omega Q(e^0 + \varepsilon ze^1 + Ew(\varepsilon z) - Ew(0) - \varepsilon zE\dot{w}(0))\,dx\Big)\,dz\\ &= \frac{\rho}{2}\int_\Omega |u^1|^2\,dx + C\sqrt{\varepsilon} + \Big(\int_0^{T/\varepsilon}\exp(-z)\,dz\Big)\Big(\int_\Omega Q(e^0)\,dx + \mathrm{O}(\varepsilon)\Big),\end{aligned}$$

which in turn implies

$$\limsup_{\varepsilon\to 0}\frac{\mathcal{E}_\varepsilon(0)}{\varepsilon^2} \leq \frac{\rho}{2}\int_\Omega |u^1|^2\,dx + \int_\Omega Q(e^0)\,dx. \tag{7.24}$$

By combining (7.23) with (7.24) we have

$$\int_\Omega Q(e(t))\,dx + D_{\mathcal{H}}(p;0,t) + \frac{\rho}{2}\int_\Omega |\dot{u}(t)|^2\,dx - \rho\int_\Omega \dot{u}(t)\cdot\dot{w}(t)\,dx$$

$$\leq \int_\Omega Q(e^0)\,dx + \frac{\rho}{2}\int_\Omega |u^1|^2\,dx - \rho\int_\Omega u^1\cdot\dot w(0)\,dx$$
$$- \int_0^t\int_\Omega \dot u(s)\cdot\ddot w(s)\,dx\,ds - \int_0^t\int_\Omega \mathbb{C}e(s) : E\dot w(s)\,dx\,ds$$

for a.e. $t\in[0,T]$. This completes the proof of condition (c3).

In order to show that u satisfies the first-order initial condition $\dot u(0)=u^1$ we argue as in [46, Theorem 4.2]. The minimality of the triple $(u^\varepsilon,e^\varepsilon,p^\varepsilon)$, yields the Euler-Lagrange equation

$$\varepsilon^2\rho\int_0^T\int_\Omega \exp\Big(-\frac{t}{\varepsilon}\Big)\ddot u^\varepsilon(t)\cdot\ddot\phi(t)\,dx\,dt + \int_0^T\int_\Omega \exp\Big(-\frac{t}{\varepsilon}\Big)\mathbb{C}e^\varepsilon(t) : E\phi(t)\,dx\,dt = 0 \tag{7.25}$$

for every $\phi\in W^{2,2}(0,T;W^{1,2}_0(\Omega;\mathbb{R}^3))$ satisfying $\phi(0)=\phi'(0)=0$. Let $\varepsilon_n\to0$, and let S be a countable dense subset of $W^{1,2}_0(\Omega;\mathbb{R}^3)$. Let $I\subset(0,T)$ be defined as the set of points $t_0\in(0,T)$ such that

$$\lim_{\delta\to0}\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\exp\Big(-\frac{t}{\varepsilon_n}\Big)\int_\Omega \ddot u^{\varepsilon_n}(t)\cdot h(x)\,dx\,dt = \exp\Big(-\frac{t_0}{\varepsilon_n}\Big)\int_\Omega \ddot u^{\varepsilon_n}(t_0)\cdot h(x)\,dx\,dt, \tag{7.26}$$

for every $n\in\mathbb{N}$, and for every $h\in S$. Note that by Lebesgue differentiation theorem the set $[0,T]\setminus I$ is negligible.

Fix $t_0\in I$, and let $\varphi_{\delta n}\in C^{1,1}(\mathbb{R})$ be defined as

$$\varphi_{\delta n}(t) := \begin{cases} 0 & t\leq t_0\\ \dfrac{(t-t_0)^2}{\delta\varepsilon_n^2} & t\in(t_0,t_0+\delta)\\ 2\dfrac{(t-t_0)}{\varepsilon_n^2} - \dfrac{\delta}{\varepsilon_n^2} & t\geq t_0+\delta.\end{cases}$$

We observe that

$$\varphi''_{\delta n}(t) = \frac{2}{\delta\varepsilon_n^2}\chi_{(t_0,t_0+\delta)}(t),$$

where $\chi_{(t_0,t_0+\delta)}$ is the characteristic function of $(t_0,t_0+\delta)$. In addition,

$$|\varphi_{\delta n}(t)| \leq \frac{2}{\varepsilon_n^2}(t-t_0)^+ \quad\text{and}\quad \varphi_{\delta n}(t)\to\frac{2}{\varepsilon_n^2}(t-t_0)^+$$

as $\delta\to0$ for all $t\in(0,T)$. Choosing $\phi(t,x)=\varphi_{\delta n}(t)h(x)$, with $h\in S$, by (7.25) we obtain

$$\frac{2\rho}{\delta}\int_{t_0}^{t_0+\delta}\int_\Omega \exp\Big(-\frac{t}{\varepsilon_n}\Big)\ddot u^{\varepsilon_n}(t)\cdot h(x)\,dx\,dt$$
$$+ \int_{t_0}^T\int_\Omega \exp\Big(-\frac{t}{\varepsilon_n}\Big)\varphi_{\delta n}(t)\mathbb{C}e^{\varepsilon_n}(t) : Eh(x)\,dx\,dt = 0.$$

Letting $\delta\to0$, (7.26) and the Dominated Convergence Theorem yield

$$\rho\int_\Omega \ddot u^{\varepsilon_n}(t_0)\cdot h(x)\,dx + \frac{1}{\varepsilon_n^2}\int_{t_0}^T\int_\Omega \exp\Big(\frac{t_0-t}{\varepsilon_n}\Big)(t-t_0)\mathbb{C}e^{\varepsilon_n}(t) : Eh(x)\,dx\,dt = 0.$$

By (5.40), there holds

$$\Big|\frac{1}{\varepsilon_n}\int_{t_0}^T\int_\Omega \exp\Big(\frac{t_0-t}{\varepsilon_n}\Big)(t-t_0)\mathbb{C}e^{\varepsilon_n}(t) : Eh(x)\,dx\,dt\Big|$$
$$\leq \frac{C}{\varepsilon_n}\|e^{\varepsilon_n}\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym}))}\|Eh\|_{L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym})}\Big(\int_{t_0}^T \exp\Big(\frac{2(t_0-t)}{\varepsilon_n}\Big)(t-t_0)^2\,dt\Big)^{\frac12}$$
$$\leq C\sqrt{\varepsilon_n}\|h\|_{W^{1,2}_0(\Omega;\mathbb{R}^3)}\Big(\int_0^{\frac{(T-t_0)}{\varepsilon_n}} t^2\exp(-2t)\,dt\Big)^{\frac12} \leq C\|h\|_{W^{1,2}_0(\Omega;\mathbb{R}^3)},$$

where in the last inequality we used the fact that for t big there holds $t^2\exp(-2t)\leq 1$. Thus

$$\rho\Big|\int_\Omega \ddot{u}^{\varepsilon_n}(t_0)h(x)\,dx\Big|\leq C\|h\|_{W^{1,2}_0(\Omega;\mathbb{R}^3)},$$

where the constant C is independent of ε_n and t_0. In particular, we obtain the uniform estimate

$$\rho\|\ddot{u}^{\varepsilon_n}\|_{L^\infty(0,T;W^{-1,2}(\Omega;\mathbb{R}^3))}\leq C. \tag{7.27}$$

By combining (5.40), (7.3), and (7.27), we deduce that

$$\|\dot{u}^{\varepsilon_n}\|_{W^{1,2}(0,T;W^{-1,2}(\Omega;\mathbb{R}^3))}\leq C.$$

In particular, by the embedding of $W^{1,2}(0,T;W^{-1,2}(\Omega;\mathbb{R}^3))$ into $C_w([0,T];W^{-1,2}(\Omega;\mathbb{R}^3))$, up to the extraction of a (non-relabeled) subsequence, there holds

$$\dot{u}^{\varepsilon_n}(t)\rightharpoonup\dot{u}(t)\quad\text{weakly in } W^{-1,2}(\Omega;\mathbb{R}^3),$$

for every $t\in[0,T]$, which in turn yields $\dot{u}(0)=u^1$.

The thesis follows now by the uniqueness of solutions for the dynamic plasticity problem (see Theorem 2.2). □

We point out that the assertion of Theorem 2.3 still holds if we generalize the minimum problem (2.14) by imposing ε-dependent initial data satisfying suitable compatibility assumptions. To be precise, for every ε, define the set

$$\begin{aligned}\mathcal{V}_\varepsilon:=\{&(u,e,p)\in W^{2,2}(0,T;L^2(\Omega;\mathbb{R}^3))\cap L^1(0,T;BD(\Omega))\\ &\times L^2((0,T)\times\Omega;\mathbb{M}^{3\times3}_{\rm sym})\times BV([0,T];\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)):\\ &(u(t),e(t),p(t))\in\mathscr{A}(w(t))\text{ for a.e. } t\in[0,T],\\ &Eu(t)=e(t)+p(t)\text{ in }\mathcal{D}'(\Omega;\mathbb{M}^{3\times3}_{\rm sym})\text{ for every } t\in[0,T],\\ &u(0)=u^0_\varepsilon,\ \dot{u}(0)=u^1_\varepsilon,\ e(0)=e^0_\varepsilon,\ p(0)=p^0_\varepsilon\},\end{aligned}$$

with $(u^0_\varepsilon,e^0_\varepsilon,p^0_\varepsilon)\in\mathscr{A}(w(0))$, and $u^1_\varepsilon\in BD(\Omega)$ such that there exists $e^1_\varepsilon\in L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym})$ satisfying $(u^1_\varepsilon,e^1_\varepsilon,0)\in\mathscr{A}(\dot{w}(0))$. Assuming that the initial data are *well-prepared*, namely

$$\begin{aligned}&u^0_\varepsilon\overset{*}{\rightharpoonup}u^0\quad\text{weakly* in } BD(\Omega),\\ &e^0_\varepsilon\rightharpoonup e^0\quad\text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym}),\\ &p^0_\varepsilon\overset{*}{\rightharpoonup}p^0\quad\text{weakly* in }\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D),\\ &u^1_\varepsilon\to u^1\quad\text{strongly in } W^{-1,2}(\Omega;\mathbb{R}^3),\end{aligned}$$

and

$$\begin{aligned}&\lim_{\varepsilon\to0}\Big[\int_\Omega Q(e^0_\varepsilon)\,dx+\frac{\rho}{2}\int_\Omega|u^1_\varepsilon|^2\,dx-\rho\int_\Omega u^1_\varepsilon\cdot\dot{w}(0)\,dx\Big]\\ &\quad=\int_\Omega Q(e^0)\,dx+\frac{\rho}{2}\int_\Omega|u^1|^2\,dx-\rho\int_\Omega u^1\cdot\dot{w}(0)\,dx,\end{aligned}$$

one can again prove that there exists a sequence of triples $\{(u^\varepsilon,e^\varepsilon,p^\varepsilon)\}$, with $(u^\varepsilon,e^\varepsilon,p^\varepsilon)\subset\mathcal{V}_\varepsilon$ for every ε, such that

$$I_\varepsilon(u^\varepsilon,e^\varepsilon,p^\varepsilon)=\min_{(v,f,q)\in\mathcal{V}_\varepsilon}I_\varepsilon(v,f,q),$$

such that $\{(u^\varepsilon,e^\varepsilon,p^\varepsilon)\}$ converges to the solution (u,e,p) of dynamic perfect plasticity, namely (c1), (c2) and (c3), in the sense of Theorem 2.3.

Appendix A.

This appendix is devoted to the proof of Proposition 6.7. We start with a somehow technical lemma.

Lemma A.1. *Let* $\mu \in BV(0,T;\mathcal{M}_b(\Omega \cup \Gamma_0;\mathbb{M}^{3\times 3}_D))$ *and let* $\varphi \in C^\infty_c(0,T)$*. Then*

$$\hat{D}_{\mathcal{H}}(\varphi;\mu;0,T) = -\int_0^T \dot{\varphi}(t) D_{\mathcal{H}}(\mu;0,t)\,dt.$$

Proof. In view of [21, Theorem 4.5] there holds (see also [21, Theorem 2.15])

$$\hat{D}_{\mathcal{H}}(\varphi;\mu;0,T) = -PMS\int_0^T D_{\mathcal{H}}(\mu;0,t)\,d\varphi = -RS\int_0^T D_{\mathcal{H}}(\mu;0,t)\,d\varphi = -\int_0^T D_{\mathcal{H}}(\mu;0,t)\dot{\varphi}(t)\,dt,$$

where $PMS\int$ and $RS\int$ denote the Pollard-Moore-Stieltjes and the Riemann-Stieltjes integrals, respectively (see [21, Section 4]), and where the last equality is due to the regularity of φ and to classical properties of the Riemann-Stieltjes integral. □

We are now in a position to prove Proposition 6.7.

Proof of Proposition 6.7. We argue as in [38, Proposition 4.1] by comparing the energy associated to $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ with that of a rescaled triple $(\tilde{u}^\varepsilon, \tilde{e}^\varepsilon, \tilde{p}^\varepsilon)$. Consider an increasing diffeomorphism

$$\beta : [0,T] \to [0,T]$$

such that $\beta \in C^2([0,T])$, $\beta(0) = 0$, $\beta(T) = T$, and $\dot{\beta}(0) = 1$. We set

$$\tilde{u}^\varepsilon(s) := u^\varepsilon(\beta^{-1}(s)) - w(\beta^{-1}(s)) + w(s), \quad \tilde{p}^\varepsilon(s) := p^\varepsilon(\beta^{-1}(s)),$$

for every $s \in [0,T]$, and

$$\tilde{e}^\varepsilon(s) := e^\varepsilon(\beta^{-1}(s)) - Ew(\beta^{-1}(s)) + Ew(s)$$

for a.e. $s \in [0,T]$. It is easy to check that $(\tilde{u}^\varepsilon, \tilde{e}^\varepsilon, \tilde{p}^\varepsilon) \in \mathcal{V}$. Hence, by the minimality of $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ there holds

$$G_\varepsilon(\tilde{u}^\varepsilon, \tilde{e}^\varepsilon, \tilde{p}^\varepsilon) - G_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon) \geq 0. \tag{A.1}$$

Using the definition of $(\tilde{u}^\varepsilon, \tilde{e}^\varepsilon, \tilde{p}^\varepsilon)$, we can rewrite its associated energy as

$$\begin{aligned} G_\varepsilon(\tilde{u}^\varepsilon, \tilde{e}^\varepsilon, \tilde{p}^\varepsilon) &= \int_0^T \exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot{\beta}(t)\int_\Omega Q\big(e^\varepsilon(t) - Ew(t) + Ew(\beta(t))\big)\,dx\,dt \\ &+ \frac{\varepsilon^2\rho}{2}\int_0^T \exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot{\beta}(t)\Big(\int_\Omega\Big|\frac{\ddot{u}^\varepsilon(t)}{(\dot{\beta}(t))^2} - \frac{\dot{u}^\varepsilon(t)\ddot{\beta}(t)}{(\dot{\beta}(t))^3} - \frac{\ddot{w}(t)}{(\dot{\beta}(t))^2} \\ &+ \frac{\dot{w}(t)\ddot{\beta}(t)}{(\dot{\beta}(t))^3} + \ddot{w}(\beta(t))\Big|^2\,dx\Big)\,dt + \varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);\tilde{p}^\varepsilon;0,T). \end{aligned}$$

Along the footsteps of [38, Proposition 4.1], we fix $\varphi \in C^\infty_c(0,T)$. Let $\delta \in (0,1)$ be such that $\varepsilon\delta\dot{\varphi}(t) < \exp(-t/\varepsilon)$ for every $t \in [0,T]$, and define β as the solution to

$$\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big) - \exp\Big(-\frac{t}{\varepsilon}\Big) = \delta\varphi(t). \tag{A.2}$$

It is immediate to see that $\beta(0) = 0$ and $\beta(T) = T$. In addition, deriving (A.2) with respect to time, we have

$$\dot{\beta}(t) = \exp\Big(\frac{\beta(t)}{\varepsilon}\Big)\Big(\exp\Big(-\frac{t}{\varepsilon}\Big) - \varepsilon\delta\dot{\varphi}(t)\Big) \tag{A.3}$$

for every $t \in [0,T]$, yielding $\dot{\beta}(t) > 0$ for every $t \in (0,T)$ and $\dot{\beta}(0) = 1$. As already observed in [38, Proposition 4.1],

$$\beta(t) = t - \varepsilon\delta\varphi(t)\exp\Big(\frac{t}{\varepsilon}\Big) + \mathrm{O}(\delta^2). \tag{A.4}$$

In addition, by (A.2) and (A.3),

$$\dot\beta(t)=1-\delta(\varphi(t)+\varepsilon\dot\varphi(t))\exp\left(\frac{t}{\varepsilon}\right)+\mathrm{O}(\delta^2), \tag{A.5}$$

and by performing a further derivation in time of (A.3),

$$\ddot\beta(t)=-\delta\Big(\frac{\varphi(t)}{\varepsilon}+2\dot\varphi(t)+\varepsilon\ddot\varphi(t)\Big)\exp\left(\frac{t}{\varepsilon}\right)+\mathrm{O}(\delta^2). \tag{A.6}$$

Let us firstly observe that

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\Big\{\int_0^T\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\int_\Omega Q\big(e^\varepsilon(t)-Ew(t)+Ew(\beta(t))\big)\,dx\,dt\\
&-\int_0^T\exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt\Big\}\\
&=\lim_{\delta\to 0}\frac{1}{\delta}\Big\{\int_0^T\Big(\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)-\exp\Big(-\frac{t}{\varepsilon}\Big)\Big)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt\\
&+\int_0^T\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\int_\Omega Q\big(Ew(t)-Ew(\beta(t))\big)\,dx\,dt\\
&-\int_0^T\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\int_\Omega \mathbb{C}e^\varepsilon(t):\Big(Ew(t)-Ew(\beta(t))\Big)\,dx\,dt\Big\}.
\end{aligned}\tag{A.7}$$

In view of (A.2) and (A.5), and by the Dominated Convergence Theorem, the first term in the right-hand side of (A.7) becomes

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\int_0^T\Big(\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)-\exp\Big(-\frac{t}{\varepsilon}\Big)\Big)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt\\
&=\lim_{\delta\to 0}\frac{1}{\delta}\int_0^T\Big(\Big(\delta\varphi(t)+\exp\Big(-\frac{t}{\varepsilon}\Big)\Big)\dot\beta(t)-\exp\Big(-\frac{t}{\varepsilon}\Big)\Big)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt\\
&=-\varepsilon\int_0^T\dot\varphi(t)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt.
\end{aligned}\tag{A.8}$$

By the regularity of w and by (A.4) there holds

$$|Ew(t)-Ew(\beta(t))|=\Big|\int_t^{\beta(t)}E\dot w(\xi)\,d\xi\Big|\leq\delta\|w\|_{W^{1,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3))}.$$

Hence, by (A.2) and (A.5) one obtains

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_0^T\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\int_\Omega Q\big(Ew(t)-Ew(\beta(t))\big)\,dx\,dt=0. \tag{A.9}$$

Finally, by (A.2), (A.5), and the mean value theorem we get

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\int_0^T\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\int_\Omega \mathbb{C}e^\varepsilon(t):\Big(Ew(t)-Ew(\beta(t))\Big)\,dx\,dt\\
&=\lim_{\delta\to 0}\frac{1}{\delta}\int_0^T\Big(\exp\Big(-\frac{t}{\varepsilon}\Big)+\delta\varepsilon\dot\varphi(t)\Big)\int_\Omega \mathbb{C}e^\varepsilon(t):\Big(\int_t^{\beta(t)}E\dot w(\xi)\,d\xi\Big)\,dx\,dt\\
&=\lim_{\delta\to 0}\frac{1}{\delta}\int_0^T\Big(\exp\Big(-\frac{t}{\varepsilon}\Big)+\delta\varepsilon\dot\varphi(t)\Big)\int_t^{\beta(t)}\int_\Omega \mathbb{C}e^\varepsilon(t):E\dot w(\xi)\,dx\,d\xi\,dt\\
&=-\varepsilon\lim_{\delta\to 0}\int_0^T\varphi(t)\int_\Omega\mathbb{C}e^\varepsilon(t):E\dot w(\xi^t)\,dx\,dt=-\varepsilon\int_0^T\varphi(t)\int_\Omega\mathbb{C}e^\varepsilon(t):E\dot w(t)\,dx\,dt,
\end{aligned}\tag{A.10}$$

where, in the second-last line, for every $t \in [0,T]$, ξ^t is an intermediate value between t and $\beta(t)$. By combining (A.7)–(A.10) we obtain

$$\begin{aligned}&\lim_{\delta\to 0}\frac{1}{\delta}\Big\{\int_0^T \exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\int_\Omega Q\big(e^\varepsilon(\beta(t)) - Ew(t) + Ew(\beta(t))\big)\,dx\,dt\\&\quad-\int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt\Big\}\\&\quad= -\varepsilon\int_0^T \dot\varphi(t)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt + \varepsilon\int_0^T\varphi(t)\int_\Omega \mathbb{C}e^\varepsilon(t) : E\dot w(t)\,dx\,dt.\end{aligned}\tag{A.11}$$

We proceed by performing the analogous computation for the inertial term. We seek to estimate

$$\begin{aligned}&\lim_{\delta\to 0}\frac{1}{\delta}\Big\{\frac{\varepsilon^2\rho}{2}\int_0^T \exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\Big(\int_\Omega\Big|\frac{\ddot u^\varepsilon(t)}{(\dot\beta(t))^2} - \frac{\dot u^\varepsilon(t)\ddot\beta(t)}{(\dot\beta(t))^3} - \frac{\ddot w(t)}{(\dot\beta(t))^2}\\&\quad+\frac{\dot w(t)\ddot\beta(t)}{(\dot\beta(t))^3} + \ddot w(\beta(t))\Big|^2\,dx\Big)\,dt - \frac{\varepsilon^2\rho}{2}\int_0^T\exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt\Big\}.\end{aligned}\tag{A.12}$$

By (A.2) and (A.5) we have

$$\begin{aligned}&\lim_{\delta\to 0}\frac{1}{\delta}\frac{\varepsilon^2\rho}{2}\int_0^T\Big(\frac{1}{(\dot\beta(t))^3}\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big) - \exp\Big(-\frac{t}{\varepsilon}\Big)\Big)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt\\&\quad=\frac{3\varepsilon^3\rho}{2}\int_0^T\dot\varphi(t)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt + 2\varepsilon^2\rho\int_0^T\varphi(t)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt.\end{aligned}\tag{A.13}$$

By (A.2), (A.5), and (A.6), there holds

$$\lim_{\delta\to 0}\frac{1}{\delta}\frac{\varepsilon^2\rho}{2}\int_0^T\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\int_\Omega\Big[\frac{(\ddot\beta(t))^2}{(\dot\beta(t))^5}(|\dot u^\varepsilon(t)|^2 + |\dot w(t)|^2 - 2\dot u^\varepsilon(t)\cdot\dot w(t))\Big]\,dx\,dt = 0,\tag{A.14}$$

as well as

$$\begin{aligned}&\lim_{\delta\to 0}\frac{1}{\delta}\varepsilon^2\rho\int_0^T\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\int_\Omega\frac{\ddot\beta(t)}{(\dot\beta(t))^4}\ddot u^\varepsilon(t)\cdot(\dot w(t) - \dot u^\varepsilon(t))\,dx\,dt\\&= -\varepsilon^3\rho\int_0^T\ddot\varphi(t)\ddot u^\varepsilon(t)\cdot(\dot w(t) - \dot u^\varepsilon(t))\,dx\,dt - \varepsilon\rho\int_0^T(\varphi(t) + 2\varepsilon\dot\varphi(t))\ddot u^\varepsilon(t)\cdot(\dot w(t) - \dot u^\varepsilon(t))\,dx\,dt.\end{aligned}\tag{A.15}$$

To estimate the remaining terms, we observe that by (A.5) and in view of the regularity of the boundary datum,

$$\begin{aligned}-\frac{\ddot w(t)}{(\dot\beta(t))^2} + \ddot w(\beta(t)) &= -\frac{\ddot w(t)}{(\dot\beta(t))^2}(1 - (\dot\beta(t))^2) + \int_t^{\beta(t)}\dddot w(\xi)\,d\xi\\&= -\frac{2\delta\ddot w(t)}{(\dot\beta(t))^2}(\varphi(t) + \varepsilon\dot\varphi(t))\exp\Big(\frac{t}{\varepsilon}\Big) + \int_t^{\beta(t)}\dddot w(\xi)\,d\xi + \mathrm{O}(\delta^2).\end{aligned}$$

By the regularity of w, by (A.4), and by Lebesgue's Theorem,

$$\begin{aligned}&\lim_{\delta\to 0}\Big\|\frac{1}{\delta}\int_t^{\beta(t)}\dddot w(\xi)\,d\xi + \varepsilon\dddot w(t)\varphi(t)\exp\Big(\frac{t}{\varepsilon}\Big)\Big\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}\\&\quad=\lim_{\delta\to 0}\Big\|\frac{1}{\delta}\int_t^{\beta(t)}(\dddot w(\xi) - \dddot w(t))\,d\xi\Big\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}\\&\quad\le\lim_{\delta\to 0}\frac{1}{\delta}\int_0^T\int_{t-\delta\varepsilon\|\varphi\|_{L^\infty(0,T)}\exp(T/\varepsilon)}^{t+\delta\varepsilon\|\varphi\|_{L^\infty(0,T)}\exp(T/\varepsilon)}\int_\Omega|\dddot w(\xi) - \dddot w(t)|\,dx\,d\xi\,dt = 0.\end{aligned}$$

Therefore, by (A.5) and (A.6),

$$\lim_{\delta\to 0}\frac{1}{\delta}\Big\{\frac{\varepsilon^2\rho}{2}\int_0^T\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\int_\Omega\Big[\Big| - \frac{\ddot w(t)}{(\dot\beta(t))^2} + \ddot w(\beta(t))\Big|^2$$

$$+2\Big(-\frac{\ddot w(t)}{(\dot\beta(t))^2}+\ddot w(\beta(t))\Big)\cdot\frac{(\dot w(t)-\dot u^\varepsilon(t))\ddot\beta(t)}{(\dot\beta(t))^3}\Big]\,dx\,dt\Big\}=0, \tag{A.16}$$

and

$$\begin{aligned}
&\lim_{\delta\to0}\frac{1}{\delta}\varepsilon^2\rho\int_0^T\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\int_\Omega\frac{\ddot u^\varepsilon(t)}{(\dot\beta(t))^2}\cdot\Big(-\frac{\ddot w(t)}{(\dot\beta(t))^2}+\ddot w(\beta(t))\Big)\,dx\,dt\\
&=-2\varepsilon^2\rho\int_0^T\int_\Omega\ddot u^\varepsilon(t)\cdot\ddot w(t)(\varphi(t)+\varepsilon\dot\varphi(t))\,dx\,dt\\
&\quad-\varepsilon^3\rho\int_0^T\int_\Omega\ddot u^\varepsilon(t)\cdot\dddot w(t)\varphi(t)\,dx\,dt.
\end{aligned}\tag{A.17}$$

By combining (A.12)–(A.17), we obtain

$$\begin{aligned}
&\lim_{\delta\to0}\frac{1}{\delta}\Big\{\frac{\varepsilon^2\rho}{2}\int_0^T\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\int_\Omega\Big|\frac{\ddot u^\varepsilon(t)}{(\dot\beta(t))^2}-\frac{\dot u^\varepsilon(t)\ddot\beta(t)}{(\dot\beta(t))^3}-\frac{\ddot w(t)}{(\dot\beta(t))^2}\\
&\quad+\frac{\dot w(t)\ddot\beta(t)}{(\dot\beta(t))^3}+\ddot w(\beta(t))\Big|^2\,dx\,dt-\frac{\varepsilon^2\rho}{2}\int_0^T\exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt\Big\}\\
&\quad=\frac{3\varepsilon^3\rho}{2}\int_0^T\dot\varphi(t)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt+2\varepsilon^2\rho\int_0^T\varphi(t)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt\\
&\quad-\varepsilon^3\rho\int_0^T\int_\Omega\ddot\varphi(t)\ddot u^\varepsilon(t)\cdot(\dot w(t)-\dot u^\varepsilon(t))\,dx\,dt\\
&\quad-\varepsilon\rho\int_0^T\int_\Omega(\varphi(t)+2\varepsilon\dot\varphi(t))\ddot u^\varepsilon(t)\cdot(\dot w(t)-\dot u^\varepsilon(t))\,dx\,dt\\
&\quad-2\varepsilon^2\rho\int_0^T\int_\Omega\ddot u^\varepsilon(t)\cdot\ddot w(t)(\varphi(t)+\varepsilon\dot\varphi(t))\,dx\,dt\\
&\quad-\varepsilon^3\rho\int_0^T\int_\Omega\ddot u^\varepsilon(t)\cdot\dddot w(t)\varphi(t)\,dx\,dt.
\end{aligned}\tag{A.18}$$

To complete the proof of the ε-energy inequality it remains to estimate from above the quantity

$$\limsup_{\delta\to0}\frac{1}{\delta}\big(D_{\mathcal H}(\exp(-\cdot/\varepsilon);\tilde p^\varepsilon;0,T)-D_{\mathcal H}(\exp(-\cdot/\varepsilon);p^\varepsilon;0,T)\big).\tag{A.19}$$

To this aim, fix $\lambda>0$ and let $0\le t_0<t_1<\cdots<t_m\le T$ be such that

$$D_{\mathcal H}(\exp(-\cdot/\varepsilon);\tilde p^\varepsilon;0,T)\le\sum_{i=1}^m\exp\Big(-\frac{t_i}{\varepsilon}\Big)\mathcal H(\tilde p^\varepsilon(t_i)-\tilde p^\varepsilon(t_{i-1}))+\lambda.$$

For $i=0,\dots,m$, let $s_i\in[0,T]$ be such that $\beta(s_i)=t_i$. By the properties of β, it follows that $0\le s_0<s_1<\cdots<s_n\le T$. In view of (A.2), we have

$$\begin{aligned}
&\sum_{i=1}^m\exp\Big(-\frac{t_i}{\varepsilon}\Big)\mathcal H(\tilde p^\varepsilon(t_i)-\tilde p^\varepsilon(t_{i-1}))=\sum_{i=1}^m\exp\Big(-\frac{\beta(s_i)}{\varepsilon}\Big)\mathcal H(p^\varepsilon(s_i)-p^\varepsilon(s_{i-1}))\\
&=\sum_{i=1}^m\exp\Big(-\frac{s_i}{\varepsilon}\Big)\mathcal H(p^\varepsilon(s_i)-p^\varepsilon(s_{i-1}))\\
&+\sum_{i=1}^m\Big(\exp\Big(-\frac{\beta(s_i)}{\varepsilon}\Big)-\exp\Big(-\frac{s_i}{\varepsilon}\Big)\Big)\mathcal H(p^\varepsilon(s_i)-p^\varepsilon(s_{i-1}))\\
&\le D_{\mathcal H}(\exp(-\cdot/\varepsilon);p^\varepsilon;0,T)+\delta\sum_{i=1}^m\varphi(s_i)\mathcal H(p^\varepsilon(s_i)-p^\varepsilon(s_{i-1})).
\end{aligned}$$

By considering finer and finer refinements of $\{t_0, \dots, t_m\}$, in view of the definition of $\hat{D}_{\mathcal{H}}$, and by the arbitrariness of λ we conclude that

$$D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); \tilde{p}^\varepsilon; 0, T) \leq D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p^\varepsilon; 0, T) + \delta \hat{D}_{\mathcal{H}}(\varphi; p^\varepsilon; 0, T).$$

Thus we can bound (A.19) from above as

$$\limsup_{\delta\to 0} \frac{1}{\delta}\big(D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); \tilde{p}^\varepsilon; 0, T) - D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p^\varepsilon; 0, T)\big) \leq \hat{D}_{\mathcal{H}}(\varphi; p^\varepsilon; 0, T), \tag{A.20}$$

where $\hat{D}_{\mathcal{H}}$ is the quantity defined in (2.8). Combining (A.1), (A.11), (A.18), (A.20) and Lemma A.1 we finally obtain the inequality

$$\begin{aligned}
&0 \leq \limsup_{\delta\to 0} \frac{1}{\varepsilon\delta}\left(G_\varepsilon(\tilde{u}^\varepsilon, \tilde{e}^\varepsilon, \tilde{p}^\varepsilon) - G(u^\varepsilon, e^\varepsilon, p^\varepsilon)\right) \\
&\leq -\int_0^T \dot{\varphi}(t) \int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt - \int_0^T \varphi(t)\int_\Omega \mathbb{C}e^\varepsilon(t) : E\dot{w}(t)\,dx\,dt \\
&\quad + \frac{3\varepsilon^2\rho}{2}\int_0^T \dot{\varphi}(t)\int_\Omega |\ddot{u}^\varepsilon(t)|^2\,dx\,dt + 2\varepsilon\rho\int_0^T \varphi(t)\int_\Omega |\ddot{u}^\varepsilon(t)|^2\,dx\,dt \\
&\quad - \varepsilon^2\rho\int_0^T\int_\Omega \ddot{\varphi}(t)\ddot{u}^\varepsilon(t)\cdot(\dot{w}(t) - \dot{u}^\varepsilon(t))\,dx\,dt - \rho\int_0^T\int_\Omega(\varphi(t) + 2\varepsilon\dot{\varphi}(t))\ddot{u}^\varepsilon(t)\cdot(\dot{w}(t) - \dot{u}^\varepsilon(t))\,dx\,dt \\
&\quad - 2\varepsilon\rho\int_0^T\int_\Omega \ddot{u}^\varepsilon(t)\cdot\ddot{w}(t)(\varphi(t) + \varepsilon\dot{\varphi}(t))\,dx\,dt - \varepsilon^2\rho\int_0^T\int_\Omega \ddot{u}^\varepsilon(t)\cdot\dddot{w}(t)\varphi(t)\,dx\,dt \\
&\quad - \int_0^T D_{\mathcal{H}}(p^\varepsilon; 0, t)\dot{\varphi}(t)\,dt
\end{aligned} \tag{A.21}$$

for every $\varphi \in C_c^\infty(0, T)$. The weak energy equality (6.13) follows now by performing an integration by parts. □

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