THE CLASSICAL OBSTACLE PROBLEM FOR NONLINEAR VARIATIONAL ENERGIES

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Dedicated to Nicola Fusco, a mentor and a friend.

Abstract. We develop the complete free boundary analysis for solutions to classical obstacle problems related to nondegenerate nonlinear variational energies. The key tools are optimal $C^{1,1}$ regularity, which we review more generally for solutions to variational inequalities driven by nonlinear coercive smooth vector fields, and the results in [16] concerning the obstacle problem for quadratic energies with Lipschitz coefficients.

Furthermore, we highlight similar conclusions for locally coercive vector fields having in mind applications to the area functional, or more generally to area-type functionals, as well.

1. Introduction

Variational inequalities are a classical topic in partial differential equations starting with the seminal works of Fichera and Stampacchia in the early 60's, motivated by a wide variety of applications in mechanics and other applied sciences. This subject has been developed over the last 50 years by the works of many authors; it is not realistic to give here a complete account: we rather refer to the books and surveys [4, 14, 15, 19, 32, 43, 44, 46, 47] for a fairly vast bibliography and its historical developments.

To introduce the problem, let ψ and g be given functions in $W^{1,p}(\Omega)$, $p \in (1, \infty)$, with $g \geq \psi$ $\mathcal{L}^n$ a.e. on Ω and set

$$\mathbb{K}_{\psi,g} := \{v \in g + W_0^{1,p}(\Omega) : v \geq \psi \quad \mathcal{L}^n \text{ a.e. on } \Omega\}. \tag{1.1}$$

Consider a smooth *coercive vector field* $(a_0, \mathbf{a}) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ according to [32, Definition 3.1 of Chapter IV] and [46, Chapter 4] (cf. Section 3 for the precise definitions and the necessary assumptions). The existence of a solution $u \in \mathbb{K}_{\psi,g}$ of the problem

$$\int_\Omega \mathbf{a}(x, u, \nabla u) \cdot \nabla(v - u)\, dx + \int_\Omega a_0(x, u, \nabla u)(v - u) dx \geq 0 \qquad \text{for all } v \in \mathbb{K}_{\psi,g}, \tag{1.2}$$

is well-known (cf. [32, Section 4 of Chapter III] if $p = 2$ and [46, Chapter 4] otherwise) and shortly recalled in Section 3 below. Under suitable hypotheses on the fields, classical results ensure optimal regularity for u, i.e. $u \in C^{1,1}_{loc}(\Omega)$, as long as $\psi \in C^{1,1}_{loc}(\Omega)$ (cf. for instance [46, Sections 4.5-4.6] in the quadratic case, and [47] in general).

The prototype example we have in mind is that of nonlinear variational problems

$$\min_{v \in \mathbb{K}_{\psi,g}} \int_\Omega F(x, v, \nabla v)\, dx \tag{1.3}$$

that leads to a variational inequality of the form (1.2) with $\mathbf{a} = \nabla_\xi F$ and $a_0 = \partial_z F$, under suitable assumptions on $F = F(x, z, \xi)$ such as global smoothness, convexity and p-growth in the last variable (cf. Theorem 3.8 below for the precise assumptions on F).

The aim of this short note is to perform an exhaustive analysis of the *free boundary*, i.e. the set $\partial\{u = \psi\}$, for the broad class of obstacle problems introduced in (1.3), and to establish a parallel with the known results in the quadratic case as developed by Caffarelli [11], Weiss [48] and Monneau [40] (cf. Theorem 3.8 for the statement).

The sharp analysis and stratification of the free boundary we provide is an outcome of a suitable linearization argument (cf. Lemma 3.12 below) and of the analogous results for the classical obstacle problem for quadratic energies with Lipschitz coefficients recently proved in [16] that we state in Section 2 (cf. Theorem 2.1). It corresponds to the case $F(x,\xi) = \mathbb{A}(x)\xi \cdot \xi$ in (1.3), with $\mathbb{A} \in \mathrm{Lip}(\Omega, \mathbb{R}^{n\times n})$ defining a continuous and coercive quadratic form. The lack of smoothness and homogeneity of the matrix of coefficients $\mathbb{A}$ in Theorem 2.1 does not permit to exploit elementary freezing arguments to locally reduce the regularity problem above to the analogous one for smooth operators, for which a complete theory has been developed by Caffarelli in a long term program [8, 9, 10, 11]. Building upon the variational approach to the classical obstacle problem developed by Weiss [48] and Monneau [40], the strategy to prove Theorem 2.1 is energy-based and relies on quasi-mononicity formulas extending those of Weiss [48] and Monneau [40], on Weiss's epiperimetric inequality as well as on Caffarelli's fundamental blow up analysis [9].

As a direct outcome of Theorem 2.1 we shall deduce the analogous result for solutions of (1.3) (cf. Theorem 3.8). Furthermore, adding suitable assumptions on the data of the problem, we can provide similar conclusions in case the vector field $\nabla_\xi F$ is more generally *locally coercive*, thus including in our analysis the important case of the area functional.

A short summary of the contents of the paper is resumed in what follows: Section 2 is devoted to fix the notation and state the conclusions of the free boundary analysis in the quadratic case following [16]. In Section 3 we introduce the necessary definitions to state the main result of the paper, Theorem 3.8, and show how the latter follows directly from Theorem 2.1. In doing this, we shall first review almost optimal and then optimal regularity in the broader setting of solutions to variational inequalities driven by coercive vector fields as in (1.2) (cf. Theorems 3.4 and 3.6), and then develop in details the analysis of the free boundary in the variational case in (1.3). Finally, in Section 4 we highlight the required changes to deduce similar conclusions for the case of locally coercive vector fields, and also analyze the case of the area functional in a Riemannian manifold.

Non-optimal regularity for solutions is a classical topic well-known in literature at least in the quadratic case $p = 2$ that has been established in several fashions: by penalization methods (cf. [35], [7], [5]), by Lewy-Stampacchia inequalities (cf. [42], [41], [31], [18], [46]), by local comparison methods (cf. [25]), by introducing a substitute variational inequality (cf. [30]), and by the linearization method (see [20, 21]). By following the streamline of ideas of the latter technique introduced in [20], we provide here an elementary variational proof valid in the general framework of nonlinear variational inequalities under investigation. In particular, we show that solutions of (1.2) satisfies a nonlinear elliptic PDE in divergence form, in turn from this suboptimal regularity can be established (for further comments cf. Section 2).

Finally, we are able to establish optimal regularity following Gerhardt [23] (see [17, 6, 12] for the classical results). In addition, we remark that solutions to (1.2) are actually *Q-minima* of a related functional according to Giaquinta and Giusti [26, 27].

Furthermore, in the case of the area functional one can prove that solutions to the obstacle problem are actually almost minimizers of the perimeter, thus leading by a well-known theory of minimal surfaces (cf. [45]) to estimates on the gradient of the solutions which bypass the global approach by Hartman and Stampacchia [33] exploiting the bounded slope condition and the construction of barriers.

To conclude this introduction M.F. would like to add some personal annotations. I had the luck of attending a PhD course on Calculus of Variations taught by Nicola Fusco when I was still a graduate student in Florence trying to find my way through Mathematics. I clearly remember

Nicola's mastery of the subject, his enthusiasm in transmitting the beauty of many ideas, and the pleasant atmosphere in the classroom despite several difficult proofs and lengthy calculations. That course pushed my interest forward Calculus of Variations. The influence of Nicola on my professional life is still active nowadays: on one hand in a direct way having the possibility to collaborate with him, and on the other hand indirectly in studying and exploiting many important results of his. All this written, it is a great pleasure for me to contribute with this note to celebrate Nicola's birthday.

2. Preliminaries

The scalar product in $\mathbb{R}^n$ is denoted by $\xi \cdot \eta$ for all ξ, $\eta \in \mathbb{R}^n$, while $\langle \cdot, \cdot \rangle$ is generically used to indicate a duality pairing of the relevant function spaces. We use standard notation for Lebesgue and Hausdorff measures, for Lebesgue and Sobolev spaces.

With c we denote a positive constant that may vary from line to line, we shall always highlight the parameters on which the constant depends.

We state explicitly only the ensuing result since it will be instrumental for our purposes.

Theorem 2.1 (Theorem 1.1 [16]). *Let $\Omega \subset \mathbb{R}^n$ be smooth, bounded and open; $\mathbb{A} \in \mathrm{Lip}(\Omega, \mathbb{R}^{n\times n})$ be symmetric and uniformly elliptic, i.e. $\lambda^{-1}|\xi|^2 \le \mathbb{A}(x)\xi \cdot \xi \le \lambda|\xi|^2$ for all $x \in \Omega$ and all $\xi \in \mathbb{R}^n$ for some $\lambda \ge 1$; $f \in C^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1]$; $g \in W^{1/2,2}(\partial\Omega)$; $\psi \in C^{1,1}_{loc}(\Omega)$ such that $\psi \le g$ $\mathcal{H}^{n-1}$-a.e on $\partial\Omega$ with $\mathrm{div}(\mathbb{A}\nabla\psi) \in C^{0,\alpha}(\Omega)$ in the sense of distributions and with $f - \mathrm{div}(\mathbb{A}\nabla\psi) \ge c_0 > 0$ for some constant c_0.*

Let u be the (unique) minimizer of

$$\mathscr{E}[v] := \int_\Omega \big(\mathbb{A}(x)\nabla v(x) \cdot \nabla v(x) + 2f(x)\, v(x)\big)\, dx,$$

on the set $\mathbb{K}_{\psi,g}$ introduced in (1.1) (with $p = 2$).

Then, u is $C^{1,\tau}_{loc}(\Omega)$ for every $\tau \in (0,1)$, and the free boundary decomposes as $\partial\{u = \psi\} \cap \Omega = \mathrm{Reg}(u) \cup \mathrm{Sing}(u)$, where $\mathrm{Reg}(u)$ and $\mathrm{Sing}(u)$ are called its regular and singular part, respectively. Moreover, $\mathrm{Reg}(u) \cap \mathrm{Sing}(u) = \emptyset$ and

- *(i)* $\mathrm{Reg}(u)$ *is relatively open in $\partial\{u = \psi\}$ and, for every point $x_0 \in \mathrm{Reg}(u)$, there exist $r = r(x_0) > 0$ and $\beta = \beta(x_0) \in (0,1)$ such that* $\mathrm{Reg}(u) \cap B_r(x_0)$ *is a $C^{1,\beta}$ submanifold of dimension $n-1$;*
- *(ii)* $\mathrm{Sing}(u) = \cup_{k=0}^{n-1} S_k$, *with S_k contained in the union of at most countably many submanifolds of dimension k and class C^1.*

Remark 2.2. Following the generalization of the previous result by the second named author in [22], we can actually require f above to satisfy only a suitable Dini-type continuity condition to conclude an analogous free boundary analysis.

3. Coercive vector fields

Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded and open set. Consider $(a_0, \mathbf{a}) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ a smooth vector field satisfying (cf. [46, Section 4.3.2])

(H1) a_0 is Carathéodory, $\mathbf{a} \in C^{1,1}_{loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and there is $p \in (1,\infty)$, for which

- (i) $\big(\mathbf{a}(x,z,\xi)\cdot\xi\big) \wedge \big(a_0(x,z,\xi)z\big) \ge \lambda|\xi|^p + \lambda_1|z|^p - \phi_1(x)$ for $\mathcal{L}^n$ a.e. $x \in \Omega$, and for all $z \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, with $\phi_1 \in L^1(\Omega)$, $\lambda > 0$ and $\lambda_1 \ge 0$;
- (ii) $|a_0(x,z,\xi)| \vee |\mathbf{a}(x,z,\xi)| \le \Lambda(|z|^{p-1} + |\xi|^{p-1}) + \phi_2(x)$ for $\mathcal{L}^n$ a.e. $x \in \Omega$ and for all $(z,\xi) \in \mathbb{R} \times \mathbb{R}^n$, with $\Lambda > 0$ and $\phi_2 \in L^{\frac{p}{p-1}}(\Omega)$;

(iii) there is a constant $\Theta > 0$ such that for all $x \in \Omega$, z, $\zeta \in \mathbb{R}$, and $\xi \in \mathbb{R}^n$

$$|\mathbf{a}(x,z,\xi) - \mathbf{a}(x,\zeta,\xi)| \leq \Theta |z-\zeta|(1+|\xi|^{p-1});$$

(H2) for $\mathcal{L}^n$ a.e. $x \in \Omega$, and for all $z \in \mathbb{R}$, ξ, $\eta \in \mathbb{R}^n$

$$0 \leq \big(\mathbf{a}(x,z,\xi) - \mathbf{a}(x,z,\eta)\big) \cdot (\xi - \eta), \tag{3.1}$$

with strict inequality sign for $\xi \neq \eta$.

Note that strongly coercive vector fields as defined in [32, Definition 3.1 of Chapter IV] satisfy the assumptions above.

Under conditions (H1)-(H2) and supposing the obstacle ψ and the boundary datum g in $W^{1,p}(\Omega)$ and satisfying the compatibility condition $g \geq \psi$ $\mathcal{L}^n$ a.e. on Ω, the existence of solutions to (1.2) is a consequence of classical results. Indeed, consider the nonlinear operator $\mathscr{A} : W^{1,p}(\Omega) \mapsto W^{1,-p'}(\Omega)$ defined by

$$\langle \mathscr{A}(w), v\rangle := \int_\Omega \big(\widetilde{\mathbf{a}}(x,w,\nabla w)\cdot \nabla v + \widetilde{a}_0(x,w,\nabla w)\, v\big)\, dx \tag{3.2}$$

for $w \in W^{1,p}(\Omega)$ and $v \in W^{1,p}_0(\Omega)$, where for all $(x,z,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$

$$\widetilde{\mathbf{a}}(x,z,\xi) := \mathbf{a}(x, z+g(x), \xi + \nabla g(x)), \qquad \widetilde{a}_0(x,z,\xi) := a_0(x,z+g(x),\xi+\nabla g(x)).$$

Note that $\widetilde{\mathbf{a}}$ and $\widetilde{a}_0$ are Carathéodory functions on account of the regularity of $\mathbf{a}$ and a_0. Then, items (i) and (ii) in (H1) yield that $\mathscr{A}$ is coercive relative to the closed (in the norm topology of $W^{1,p}$) convex subset $\mathbb{K}_{\psi-g,0}$ of $W^{1,p}_0(\Omega)$ given by

$$\mathbb{K}_{\psi-g,0} := \{v \in W^{1,p}_0(\Omega) : \, v \geq \psi - g \quad \mathcal{L}^n \text{ a.e. on } \Omega\}.$$

More precisely, for some $w_0 \in \mathbb{K}_{\psi-g,0}$ (actually for any w_0 in this case)

$$\lim_{w\in W^{1,p}_0(\Omega),\, \|w\|_{W^{1,p}(\Omega)}\to\infty} \|w\|^{-1}_{W^{1,p}(\Omega)} \langle \mathscr{A}(w), w - w_0\rangle = +\infty.$$

Remark 3.1. Coercivity is clearly ensured under weaker conditions than those in item (i) of (H1) in view of Sobolev embedding theorems (cf. [28, Theorems 3.7 and 3.8])

In particular, [46, condition (4.26)] is fulfilled for any $w_0 \in \mathbb{K}_{\psi-g,0}$ and for any $R > 0$. Since the injection $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, assumption (H2) gives that $\mathscr{A}$ is a Leray-Lions operator (cf. [46, Theorem 4.21]). Existence of a solution $\widetilde{u} \in \mathbb{K}_{\psi-g,0}$ for

$$\int_\Omega \widetilde{\mathbf{a}}(x,\widetilde{u},\nabla\widetilde{u})\cdot\nabla(v-\widetilde{u})\,dx + \int_\Omega \widetilde{a}_0(x,\widetilde{u},\nabla\widetilde{u})(v-\widetilde{u})dx \geq 0 \qquad \text{for all } v \in \mathbb{K}_{\psi-g,0}$$

follows at once from [46, Lemma 4.13 and Theorem 4.17]. Therefore, $u := \widetilde{u} + g$ is a solution to (1.2).

Finally, uniqueness of solutions to (1.2) is guaranteed in case the ensuing more stringent monotonicity condition is satisfied

$$0 \leq \big(\mathbf{a}(x,z,\xi) - \mathbf{a}(x,\zeta,\eta)\big)\cdot(\xi-\eta) + \big(a_0(x,z,\xi) - a_0(x,\zeta,\eta)\big)(z-\zeta), \tag{3.3}$$

for $\mathcal{L}^n$ a.e. $x \in \Omega$, for all z, $\zeta \in \mathbb{R}$ and ξ, $\eta \in \mathbb{R}^n$, with strict inequality sign in (3.3) if $\xi \neq \eta$. Disregarding the characterization of the equality case in (3.3), the latter condition yields that the nonlinear operator $\mathscr{A}$ defined in (3.2) is monotone, actually T-monotone (cf. [46, p. 231]).

In the variational case in which $\mathbf{a} = \nabla_\xi F$ and $a_0 = \partial_z F$, (H2) follows from the convexity of the Lagrangian F in the gradient variable ξ, while (3.3) from the joint convexity of F in (z,ξ).

3.1. **Regularity of solutions.** In what follows we consider variational inequalities as in (1.2) for vector fields $(a_0, \mathbf{a})$ satisfying (H1)-(H2) and further assuming the following conditions on the obstacle function:

(H3) $\psi \in C^{1,1}_{loc}(\Omega)$.

Note then that

$$h := -\mathrm{div}\big(\mathbf{a}(x, \psi, \nabla\psi)\big) + a_0(x, \psi, \nabla\psi) \in L^\infty_{loc}(\Omega). \tag{3.4}$$

The key to establish optimal regularity is contained in Proposition 3.2 in which we switch from a variational inequality to a nonlinear elliptic PDE in divergence form. Indeed, on account of Proposition 3.2, in Theorem 3.4 we establish almost optimal regularity of solutions through classical elliptic regularity results and finally optimal regularity is achieved in Theorem 3.6 by means of Gerhardt's approach (cf. [23]).

Despite almost optimal regularity of solutions is a well-studied subject, we provide in Proposition 3.2 and Theorem 3.4 below a different proof that departs from the classical ones known in literature ([35, 7, 5, 42, 31, 25, 18, 30, 46]) by extending the linearization method to the general setting studied here (cf. [20, 21]). The idea is to reduce regularity for variational inequalities of the sort in (1.2) to the more standard setting of nonlinear elliptic PDEs. In the case of quadratic forms a similar argument has been established in [16] for the obstacle problem in Theorem 2.1, inspired by the case discussed in [48] for the Laplacian.

Proposition 3.2. *Let (H1)-(H3) hold true. Then, a solution $u \in \mathbb{K}_{\psi,g}$ to problem* (1.2) *solves*

$$-\,\mathrm{div}(\mathbf{a}(x, u, \nabla u)) + a_0(x, u, \nabla u) = \zeta(x) \tag{3.5}$$

$\mathcal{L}^n$ a.e. in Ω and in $\mathcal{D}'(\Omega)$, for some function $\zeta \in L^\infty_{loc}(\Omega)$ such that, for h defined in (3.4),

$$0 \le \zeta \le h^+ \chi_{\{u=\psi\}} \quad \mathcal{L}^n \text{ a.e. in } \Omega.$$

Proof. Let $\varphi \in C^\infty_c(\Omega)$ and for all $\varepsilon > 0$ take $v_\varepsilon := (u + \varepsilon\varphi) \vee \psi \in \mathbb{K}_{\psi,g}$ as test function in (1.2). Note that in case φ is a non-negative function we obtain

$$\int_\Omega \mathbf{a}(x, u, \nabla u) \cdot \nabla\varphi\, dx + \int_\Omega a_0(x, u, \nabla u)\varphi\, dx \ge 0. \tag{3.6}$$

Therefore, the distributional divergence $\mathrm{div}(\mathbf{a}(\cdot, u, \nabla u))$ of $\mathbf{a}(\cdot, u, \nabla u)$ satisfies

$$\langle -\mathrm{div}(\mathbf{a}(\cdot, u, \nabla u)) + a_0(\cdot, u, \nabla u)\mathcal{L}^n \llcorner \Omega, \varphi\rangle \ge 0 \qquad \text{for all } \varphi \in C^\infty_c(\Omega),\ \varphi \ge 0,$$

in turn implying that $\mu := -\mathrm{div}(\mathbf{a}(\cdot, u, \nabla u)) + a_0(\cdot, u, \nabla u)\mathcal{L}^n \llcorner \Omega$ is a non-negative Radon measure.

Next, consider v_ε as above with no sign assumptions on φ, set $\Omega_\varepsilon := \{u + \varepsilon\varphi < \psi\}$, and rewrite the two addends in (1.2) respectively as follows

$$\int_\Omega \mathbf{a}(x, u, \nabla u) \cdot \nabla(v_\varepsilon - u)dx = \varepsilon \int_\Omega \mathbf{a}(x, u, \nabla u) \cdot \nabla\varphi\, dx + \int_{\Omega_\varepsilon} \mathbf{a}(x, u, \nabla u) \cdot \nabla\big(\psi - (u + \varepsilon\varphi)\big)dx,$$

and

$$\int_\Omega a_0(x, u, \nabla u)(v_\varepsilon - u)dx = \varepsilon \int_\Omega a_0(x, u, \nabla u)\varphi\, dx + \int_{\Omega_\varepsilon} a_0(x, u, \nabla u)\big(\psi - (u + \varepsilon\varphi)\big)dx.$$

Thus, on account of the definition of the measure μ we conclude that

$$\varepsilon \int_\Omega \varphi\, d\mu \ge -\int_{\Omega_\varepsilon} \mathbf{a}(x, u, \nabla u) \cdot \nabla\big(\psi - (u + \varepsilon\varphi)\big)dx - \int_{\Omega_\varepsilon} a_0(x, u, \nabla u)\,\big(\psi - (u + \varepsilon\varphi)\big)dx.$$

By the monotonicity hypothesis on the field $\mathbf{a}$ in (H2) we have that

$$\varepsilon \int_\Omega \varphi\, d\mu \ge -\int_{\Omega_\varepsilon} \mathbf{a}(x, u, \nabla\psi) \cdot \nabla\big(\psi - u\big)dx$$

$$+\,\varepsilon\int_{\Omega_\varepsilon}\mathbf{a}(x,u,\nabla u)\cdot\nabla\varphi\,dx-\int_{\Omega_\varepsilon}a_0(x,u,\nabla u)\big(\psi-(u+\varepsilon\varphi)\big)dx$$

and therefore we infer that

$$\begin{aligned}\varepsilon\int_\Omega\varphi\,d\mu\geq&\underbrace{-\int_{\Omega_\varepsilon}\Big(\mathbf{a}(x,\psi,\nabla\psi)\cdot\nabla\big(\psi-(u+\varepsilon\varphi)\big)+a_0(x,\psi,\nabla\psi)\big(\psi-(u+\varepsilon\varphi)\big)\Big)dx}_{=:I_\varepsilon^{(1)}}\\&+\underbrace{\varepsilon\int_{\Omega_\varepsilon}\big(\mathbf{a}(x,u,\nabla u)-\mathbf{a}(x,\psi,\nabla\psi)\big)\cdot\nabla\varphi\,dx+\varepsilon\int_{\Omega_\varepsilon}\big(a_0(x,u,\nabla u)-a_0(x,\psi,\nabla\psi)\big)\varphi\,dx}_{=:I_\varepsilon^{(2)}}\\&+\underbrace{\int_{\Omega_\varepsilon}\big(\mathbf{a}(x,\psi,\nabla\psi)-\mathbf{a}(x,u,\nabla\psi)\big)\cdot\nabla(\psi-u)dx+\int_{\Omega_\varepsilon}\big(a_0(x,\psi,\nabla\psi)-a_0(x,u,\nabla u)\big)\big(\psi-u\big)dx}_{=:I_\varepsilon^{(3)}}.\end{aligned}\tag{3.7}$$

We deal with the three terms above separately. We start off with the first term that rewrites as

$$I_\varepsilon^{(1)}=-\int_\Omega\Big(\mathbf{a}(x,\psi,\nabla\psi)\cdot\nabla\big((\psi-(u+\varepsilon\varphi))\vee 0\big)+a_0(x,\psi,\nabla\psi)\big((\psi-(u+\varepsilon\varphi))\vee 0\big)\Big)dx.$$

Being $u\geq\psi$ $\mathcal{L}^n$ a.e. in Ω and $\varphi\in C_c^\infty(\Omega)$, we have $\Omega_\varepsilon\subset\subset\Omega$, so that $(\psi-(u+\varepsilon\varphi))\vee 0\in W_0^{1,p}(\Omega)$. By taking this into account, together with the condition $\psi\in C^{1,1}_{loc}(\Omega)$ (cf. (H3)), item (ii) in (H1) and an integration by parts yield, recalling that $h=-\mathrm{div}\big(\mathbf{a}(x,\psi,\nabla\psi)\big)+a_0(x,\psi,\nabla\psi)$,

$$\begin{aligned}I_\varepsilon^{(1)}&=\int_\Omega\big(\mathrm{div}(\mathbf{a}(x,\psi,\nabla\psi))-a_0(x,\psi,\nabla\psi)\big)\big((\psi-(u+\varepsilon\varphi))\vee 0\big)dx\\&=-\int_{\Omega_\varepsilon}h\,\big((\psi-(u+\varepsilon\varphi)\big)\,dx\geq-\int_{\Omega_\varepsilon}h^+\,\big(\psi-(u+\varepsilon\varphi)\big)\,dx\geq\varepsilon\int_{\Omega_\varepsilon}h^+\,\varphi\,dx\end{aligned}\tag{3.8}$$

where in the last but one equality we have used that $\psi-(u+\varepsilon\varphi)\geq 0$ $\mathcal{L}^n$ a.e. on Ω_ε and in the last one that $u\geq\psi$ $\mathcal{L}^n$ a.e. on Ω. In turn, the latter condition implies that

$$\mathcal{L}^n\big((\{u=\psi\}\cap\{\varphi<0\})\setminus\Omega_\varepsilon\big)=\mathcal{L}^n\big(\Omega_\varepsilon\setminus\big(\{0\leq u-\psi\leq\varepsilon\|\varphi\|_{L^\infty(\Omega)}\}\cap\{\varphi<0\}\big)\big)=0,$$

so that $\chi_{\Omega_\varepsilon}\to\chi_{\{u=\psi\}\cap\{\varphi<0\}}$ in $L^1(\Omega)$, for every $\varphi\in C_c^\infty(\Omega)$. Therefore, from (3.8) we infer

$$\liminf_{\varepsilon\to 0^+}\varepsilon^{-1}I_\varepsilon^{(1)}\geq\int_{\{u=\psi\}\cap\{\varphi<0\}}h^+\,\varphi\,dx.\tag{3.9}$$

In addition, by the Dominated convergence theorem and by the locality of the weak gradient, we conclude that for every $\varphi\in C_c^\infty(\Omega)$

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\varepsilon^{-1}I_\varepsilon^{(2)}=&\int_{\{u=\psi\}\cap\{\varphi<0\}}\big(\mathbf{a}(x,u,\nabla u)-\mathbf{a}(x,\psi,\nabla\psi)\big)\cdot\nabla\varphi\,dx\\&+\int_{\{u=\psi\}\cap\{\varphi<0\}}\big(a_0(x,u,\nabla u)-a_0(x,\psi,\nabla\psi)\big)\varphi\,dx=0.\end{aligned}\tag{3.10}$$

Finally, to deal with $I_\varepsilon^{(3)}$ we use item (iii) in (H1) to deduce that

$$I_\varepsilon^{(3)}\geq-\varepsilon\Theta\|\varphi\|_{L^\infty(\Omega)}\int_{\Omega_\varepsilon}(1+|\nabla\psi|^{p-1})|\nabla(\psi-u)|\,dx-\varepsilon\|\varphi\|_{L^\infty(\Omega)}\int_{\Omega_\varepsilon}|a_0(x,u,\nabla u)-a_0(x,\psi,\nabla\psi)|\,dx.$$

Therefore, by the quoted convergence of $\chi_{\Omega_\varepsilon}$ and by the locality of the weak gradient, as in (3.9) and (3.10), we conclude that

$$\liminf_{\varepsilon\to 0^+} \varepsilon^{-1} I_\varepsilon^{(3)} \geq 0. \tag{3.11}$$

Resuming, by (3.9), (3.10) and (3.11), passing to the limit as $\varepsilon \downarrow 0^+$ in (3.7) divided by $\varepsilon > 0$, we infer that

$$\int_\Omega \varphi\, d\mu \geq \int_{\{u=\psi\}\cap\{\varphi<0\}} h^+ \varphi\, dx.$$

By approximation (and by applying the argument above to $-\varphi$) we infer that for every $\varphi \in C_c^0(\Omega)$

$$\int_{\{u=\psi\}\cap\{\varphi<0\}} h^+ \varphi\, dx \leq \int_\Omega \varphi\, d\mu \leq \int_{\{u=\psi\}\cap\{\varphi>0\}} h^+ \varphi\, dx.$$

In turn, the latter inequalities imply that $\mu \ll \mathcal{L}^n \llcorner \Omega$. Thus, if $\mu = \zeta \mathcal{L}^n \llcorner \Omega$, with $\zeta \in L^1(\Omega)$, we infer that $0 \leq \zeta \leq h^+ \chi_{\{u=\psi\}}$ $\mathcal{L}^n$ a.e. Ω, so that $\zeta \in L^\infty_{loc}(\Omega)$ by (3.4).

In conclusion, as by definition $\mu = -\mathrm{div}(\mathbf{a}(\cdot, u, \nabla u)) + a_0(\cdot, u, \nabla u)\mathcal{L}^n \llcorner \Omega$, equation (3.5) follows at once. □

Remark 3.3. One can prove that a solution u of (1.2) is a Q-minimum of a lower order perturbation of the p-Dirichlet energy from the conclusions of Proposition 3.2 as argued in [27] (cf. also [28, Chapter 6]). More precisely, let $\mathscr{G} : \mathcal{B}(\Omega) \times W^{1,p}(\Omega) \to [0, \infty)$ be

$$\mathscr{G}(w, A) := \int_A G\big(x, w(x), \nabla w(x)\big)\, dx,$$

where $A \in \mathcal{B}(\Omega)$, the class of Borel subsets of Ω, and $G : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty)$ is the Carathéodory integrand defined by

$$G(x, z, \xi) := |\xi|^p + |z|^p + |\nabla\psi(x)|^p + |\phi_2(x)|^{\frac{p}{p-1}} + |\phi_1(x)| + |a_0(x, u(x), \nabla u(x))|^{\frac{p}{p-1}}.$$

Then, there is a constant $Q = Q(p, \lambda, \Lambda) > 1$ such that

$$\mathscr{G}(u, K) \leq Q\, \mathscr{G}(w, K) \tag{3.12}$$

for all $w \in g + W_0^{1,p}(\Omega)$ such that $K := \mathrm{spt}(w - u) \subset\subset \Omega$. Note that $|a_0(\cdot, u(\cdot), \nabla u(\cdot))|^{\frac{p}{p-1}} \in L^1(\Omega)$ by item (ii) in (H1). The direct methods for regularity introduced by Giaquinta and Giusti [26, 27] imply that $u \in C^{0,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0, 1]$ under suitable assumptions on ϕ_1, ϕ_2, a_0 and p (cf. [21] for instance).

Actually, we can establish (3.12) a priori, directly from (1.2) by taking the family of test functions $v = w \vee \psi$ with w as above by means of items (i) and (ii) in (H1).

Finally, we recall that under the standing assumptions on $(\mathbf{a}, a_0)$ upper semicontinuity and approximate continuity of ψ suffices to establish continuity of solutions (cf. [38]). In particular, this shows that the sets $\{u > \psi\}$ and Ω_ε, $\varepsilon > 0$ suitable, in the proof of Proposition 3.2 are actually open.

We are now ready to deduce almost optimal regularity for solutions to (1.2) from standard elliptic regularity provided item (iii) in (H1) and (H2) are substituted by the more restrictive

(iii)′ there is a constant $\Theta > 0$ such that for all x, $y \in \Omega$, z, $\zeta \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$

$$|\mathbf{a}(x, z, \xi) - \mathbf{a}(y, \zeta, \xi)| \leq \Theta(|x - y| + |z - \zeta|)(1 + |\xi|^{p-1})$$

(H2)′ there is $\nu > 0$ such that for $\mathcal{L}^n$ a.e. $x \in \Omega$, and for all $z \in \mathbb{R}$, ξ, $\eta \in \mathbb{R}^n$

$$\nu^{-1}(1 + |\xi| + |\eta|)^{p-2}\, |\xi - \eta|^2 \leq \big(\mathbf{a}(x, z, \xi) - \mathbf{a}(x, z, \eta)\big) \cdot (\xi - \eta) \leq \nu(1 + |\xi| + |\eta|)^{p-2}\, |\xi - \eta|^2; \tag{3.13}$$

On account of (3.5) in Proposition 3.2 suboptimal regularity follows.

Theorem 3.4 (Almost optimal regularity)**.** *Let (H1) (with (iii)′ in place of (iii)), (H2)′ and (H3) hold true. Let* $u \in \mathbb{K}_{\psi,g}$ *be a solution to problem* (1.2)*, then* $u \in W^{2,q}_{loc} \cap C^{1,\alpha}_{loc}(\Omega)$ *for all* $q \in [1, \infty)$ *and all* $\alpha \in (0, 1)$*.*

Proof. By taking into account that u solves (3.5) (cf. Proposition 3.2), classical elliptic regularity for nonlinear elliptic equations in divergence form yield that $u \in C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0, 1)$ (cf. [36, Section 3], [37, Chapter 5]).

It is also classical to prove that $u \in W^{2,2}_{loc}(\Omega)$ (cf. [34, Chapter 4, Theorem 5.2]) and by differentiation, on account of the $C^{1,\alpha}_{loc}$ regularity already established and (H1)-(H2)′, that the weak derivatives of u satisfy a linear uniformly elliptic PDE with Hölder coefficients and right hand side being the divergence of a field in $L^\infty_{loc}(\Omega, \mathbb{R}^n)$. Therefore, we may apply standard L^q-regularity estimates (cf. [28, Theorem 10.15]) to conclude that $u \in W^{2,q}_{loc} \cap C^{1,\alpha}_{loc}(\Omega)$ for all $q \in [1, \infty)$ and all $\alpha \in (0, 1)$. □

Corollary 3.5. *Under the assumptions of Theorem 3.4 the function* ζ *in* (3.5) *of Proposition 3.2 actually equals* $h^+\chi_{\{u=\psi\}}\ \mathcal{L}^n$ *a.e. on* Ω*.*

Proof. By the $W^{2,q}$ regularity of u and the $C^{1,1}_{loc}$ regularity of $\mathbf{a}$, one can compute the divergence in the definition of the measure μ and use the locality of weak derivatives to conclude. □

Optimal $C^{1,1}_{loc}$ regularity of solutions follows at once from Gerhardt's result [23] provided a_0 is locally Lipschitz continuous.

Theorem 3.6 (Optimal regularity)**.** *Let (H1) (with (iii)′ in place of (iii)), (H2)′ and (H3) hold true, and assume* $g \in C^2(\overline{\Omega})$ *with* $\psi < g$ *on* $\partial\Omega$*, and* $a_0 \in C^{0,1}_{loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R})$*.*

If $u \in \mathbb{K}_{\psi,g}$ *is a solution to problem* (1.2)*, then* $u \in C^{1,1}_{loc}(\Omega)$*.*

Proof. The proof is essentially that of [23] despite the forcing term, i.e. $a_0(\cdot, u, \nabla u)$ in our case, is not in $C^{0,1}$ as required in the statement there. Nevertheless, a careful inspection of that proof shows that the slightly weaker assumption $a_0(\cdot, u, \nabla u) \in W^{1,q}_{loc}(\Omega)$ for all $q \in [1, \infty)$ actually suffices (cf. formula (16) there). In our setting this property is an immediate outcome of the regularity hypothesis on a_0 and Theorem 3.4 above. □

Remark 3.7. We point out that for $p \neq 2$ the study of degenerate fields $\mathbf{a}$ deserves additional efforts. Optimal regularity of solutions to (1.2) with $\mathbf{a}(\xi) = |\xi|^{p-2}\xi$ and $a_0(x, z) = f(x)z$, $f \in L^\infty(\Omega)$, has been established only recently in [1] (cf. the bibliography there for more detailed references, and also the results in [20]). That paper deals also with the case $\psi \in C^{1,\beta}(\Omega)$, $\beta \in (0, 1)$, that is not covered by our methods. More precisely, it is established there that solutions are $C^{1,\beta\wedge 1/(p-1)}_{loc}(\Omega)$, $\beta \in (0, 1]$, and actually $C^{1,\beta}_{loc}$ in the homogeneous setting $f \equiv 0$.

Building upon Proposition 3.2 and a careful analysis of the estimates in [37, Chapter 5] one can actually show that $u \in C^{1,\alpha}_{loc}(\Omega)$, for all $\alpha \in (0, \frac{1}{p-1}] \cap (0, 1)$ for fields satisfying (H1) and the degenerate analogue of (H2)′.

We end this subsection pointing out that the conclusions of Proposition 3.2 and Theorems 3.4 and 3.6 extend to the more general setting of fields a_0 satisfying the so called *unnatural* growth conditions following the terminology of Giusti [28] (cf. formula (6.15) there), of which item (ii) in (H1) is a simple instance.

This claim is also true in case a_0 satisfies the *natural* growth conditions (cf. [28, formula (6.18)]) provided bounded solutions are taken into account. Existence of such solutions is guaranteed for bounded obstacles and bounded boundary data, for instance.

3.2. **Free boundary regularity in the variational case.** We are now ready to state and prove the main result of the paper. From now on we restrict to the variational case, in which $\mathbf{a} = \nabla_\xi F$ and $a_0 = \partial_z F$ for suitable integrands F. We need to rephrase assumptions (H1), and (H2)$'$ terms of the energy density F itself. In passing we note that item (i) in (H1) is not needed provided F satisfies suitable convexity and growth conditions in view of the Direct Method of the Calculus of Variations. Indeed, item (i) in (H1) has been used only in the proof of existence of solutions to (1.2).

Theorem 3.8. *Let $\Omega \subset \mathbb{R}^n$ be smooth, bounded and open, and $p \in (1,\infty)$. Assume (H3) for ψ, and $g \in C^2(\overline{\Omega})$ with $\psi < g$ on $\partial\Omega$.*

Let $F \in C^{2,1}_{loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ be satisfying

$$c_1|\xi|^p - \phi(x) \leq F(x,z,\xi) \leq c_2|\xi|^p + c_3|z|^{p^*} + \phi(x) \tag{3.14}$$

for all $z \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, for $\mathcal{L}^n$ a.e. $x \in \Omega$, where $\phi \in L^1(\Omega)$, c_1, $c_2 > 0$ and $c_3 \geq 0$, and p^* is the Sobolev exponent of p (thus p^* is any exponent if $p \geq n$).

Suppose that items (ii), (iii)$'$ in (H1) are satisfied by $\mathbf{a} = \nabla_\xi F$ and $a_0 = \partial_z F$, and in addition assume $F(x,z,\cdot)$ to be uniformly convex uniformly in (x,z) w.r.to ξ, i.e. there exists $\nu > 1$ such that for all $x \in \Omega$, $z \in \mathbb{R}$ and ξ, $\eta \in \mathbb{R}^n$

$$\nu^{-1}(1+|\eta|)^{p-2}|\xi|^2 \leq \nabla^2_\xi F(x,z,\eta)\xi \cdot \xi \leq \nu(1+|\eta|)^{p-2}|\xi|^2. \tag{3.15}$$

Then, the minimum problem in (1.3) *has (at least) a solution $u \in \mathbb{K}_{\psi,g}$, and, moreover, every solution belongs to $C^{1,1}_{loc}(\Omega)$.*

Let $u \in \mathbb{K}_{\psi,g}$ be a solution. If, moreover, ψ satisfies

(H4) for some constant $c_0 > 0$ we have for $\mathcal{L}^n$ a.e. on Ω

$$h = -\mathrm{div}\big(\nabla_\xi F(x,\psi,\nabla\psi)\big) + \partial_z F(x,\psi,\nabla\psi) \geq c_0 > 0;$$

(H5) for some $\alpha \in (0,1)$

$$\mathrm{div}\big(\nabla_\xi F(\cdot,u,\nabla\psi)\big) \in C^{0,\alpha}_{loc}(\Omega),$$

then the free boundary decomposes as $\partial\{u = \psi\} \cap \Omega = \mathrm{Reg}(u) \cup \mathrm{Sing}(u)$, where $\mathrm{Reg}(u)$ and $\mathrm{Sing}(u)$ are called its regular and singular part, respectively. Moreover, $\mathrm{Reg}(u) \cap \mathrm{Sing}(u) = \emptyset$ and

- *(i) $\mathrm{Reg}(u)$ is relatively open in $\partial\{u = \psi\}$ and, for every point $x_0 \in \mathrm{Reg}(u)$, there exist $r = r(x_0) > 0$ and $\beta = \beta(x_0) \in (0,1)$ such that $\mathrm{Reg}(u) \cap B_r(x_0)$ is a $C^{1,\beta}$ submanifold of dimension $n-1$;*
- *(ii) $\mathrm{Sing}(u) = \cup_{k=0}^{n-1} S_k$, with S_k contained in the union of at most countably many submanifolds of dimension k and class C^1.*

Remark 3.9. In case $F = F(x,\xi)$ the structural conditions imposed on F, i.e. convexity and (3.14), imply item (ii) in (H1) (cf. [28, Lemma 5.2]). Therefore, besides uniform convexity, the only nontrivial assumption on F is (iii)$'$ in (H1). In turn, the latter is clearly satisfied in the autonomous case $F = F(\xi)$.

Remark 3.10. Assumption (H4) corresponds to the well-known concavity assumption on the obstacle function ψ in the case of the Laplacian, or better to the localized form of such a condition introduced in [13]. Simple examples show that (H4) is a necessary request to expect regular free boundaries.

Remark 3.11. In view of the regularity assumptions on F and the optimal regularity of u, assumption (H5) is basically an hypothesis on the obstacle ψ that can be enforced by assuming more regularity on ψ itself. For instance, it is implied by taking $\psi \in C^{2,\alpha}_{loc}(\Omega)$.

Finally, non trivial examples show that a qualified continuity hypothesis on the relevant operator calculated on the obstacle function, weaker than Hölder continuity imposed in (H5), is actually necessary to conclude free boundary regularity already in the classical case of the Laplacian (cf. [3, 40]).

To establish Theorem 3.8 we introduce the ensuing linearization; in this way we rewrite the PDE in (3.5) as a locally uniform elliptic equation with suitable locally Lipschitz continuous matrix coefficients in case the gradient of the solution itself shares such a regularity.

Lemma 3.12. *Let (H1)-(H4) hold true, and let $u \in C^{1,1}_{loc}(\Omega)$ be a solution of* (1.3). *Then, there exists a symmetric matrix field $\mathbb{A} : \Omega \to \mathbb{R}^{n\times n}$ such that*

$$\operatorname{div}\big(\mathbb{A}(x)\nabla(u-\psi)\big) = \big(-\operatorname{div}(\nabla_\xi F(x,u,\nabla\psi)) + \partial_z F(x,u,\nabla u)\big)\,\chi_{\{u>\psi\}} \tag{3.16}$$

$\mathcal{L}^n$ a.e. in Ω and in $\mathcal{D}'(\Omega)$; with $\mathbb{A}$ satisfying

(i) $\mathbb{A} \in C^{0,1}_{loc}(\Omega, \mathbb{R}^{n\times n})$,

(ii) for all $K \subset\subset \Omega$ there is $\lambda_K \geq 1$ for which

$$\lambda_K^{-1}|\xi|^2 \leq \mathbb{A}(x)\xi\cdot\xi \leq \lambda_K|\xi|^2 \quad \text{for all } x\in K \text{ and for all } \xi\in\mathbb{R}^n. \tag{3.17}$$

Proof. We start off rewriting the Euler-Lagrange equation (3.5) as follows

$$\operatorname{div}\big(\nabla_\xi F(x,u,\nabla u) - \nabla_\xi F(x,u,\nabla\psi)\big) = \big(-\operatorname{div}(\nabla_\xi F(x,u,\nabla\psi)) + \partial_z F(x,u,\nabla u)\big)\,\chi_{\{u>\psi\}}. \tag{3.18}$$

In claiming the last equality we have used Corollary 3.5, assumption (H4) and the inclusion

$$\{u=\psi\} \subseteq \{\nabla u = \nabla\psi\},$$

consequence of the unilateral obstacle condition $u \geq \psi$ on Ω and the regularity of both u and ψ. Then set $w := u-\psi$, and note that for all x in Ω

$$\begin{aligned}\nabla_\xi F(x,u(x),\nabla u(x)) - \nabla_\xi F(x,u(x),\nabla\psi(x)) &= \nabla_\xi F(x,u(x),\nabla w(x)+\nabla\psi(x)) - \nabla_\xi F(x,u(x),\nabla\psi(x))\\ &= \Big(\int_0^1 \nabla^2_\xi F\big(x,u(x),\nabla\psi(x)+t\nabla w(x)\big)dt\Big)\nabla w(x) =: \mathbb{A}(x)\nabla w(x).\end{aligned} \tag{3.19}$$

From (3.18) and (3.19), we conclude that w satisfies (3.16). Moreover, being $u\,,\psi \in C^{1,1}_{loc}(\Omega)$ and $F \in C^{2,1}_{loc}(\Omega\times\mathbb{R}\times\mathbb{R}^n)$, we deduce that item (i) in the statement is satisfied, as well. Moreover, for all $x \in \Omega$ and for all $\xi \in K$, $K \subset \mathbb{R}^n$ a compact set, we have

$$\begin{aligned}\nu^{-1}(2^{p-2}\wedge 1)|\xi|^2 \int_0^1 \big(1+|\nabla\psi(x)+t\nabla w(x)|\big)^{p-2}\,dt &\leq \mathbb{A}(x)\xi\cdot\xi\\ &= \int_0^1 \nabla^2_\xi F\big(x,u(x),\nabla\psi(x)+t\nabla w(x)\big)\xi\cdot\xi\,dt \leq \|\nabla^2_\xi F\|_{L^\infty(K\times B_{r_K}\times B_{r_K},\mathbb{R}^{n\times n})}|\xi|^2,\end{aligned}$$

with $r_K := \sup_K(|u| + |\nabla\psi| + |\nabla w|)$. The inequality on the left hand side above is an easy consequence of the coercivity condition in (3.13). Ellipticity then easily follows if $p \geq 2$, for $p \in (1,2)$ instead we use that $u\,,\psi \in C^{1,1}_{loc}(\Omega)$. Finally, the upper bound in (3.17) follows easily in both cases. The conclusion then follows. □

We are ready to prove Theorem 3.8 as a direct consequence of Theorem 2.1 and Lemma 3.12.

Proof of Theorem 3.8. Existence of solutions to (1.3) follows from [28, Theorem 4.5] thanks to the convexity of $\xi \mapsto F(x,z,\xi)$ and the growth conditions (3.14). The former guarantees lower semicontinuity of the associated functional in the weak $W^{1,p}$ topology, the latter ensures its coercivity over $\mathbb{K}_{\psi,g}$. Therefore, the Direct Method of the Calculus of Variations applies.

Moreover, any minimizer u is $C^{1,1}_{loc}(\Omega)$. To this aim, it suffices to note that u satisfies the PDE in (3.5), since the derivation of the latter is independent from item (i) in (H1). Note that assumption (H2)$'$ corresponds to (3.15).

Hence, in view of Lemma 3.12, to conclude the free boundary analysis we only need to check that, locally in Ω, we may apply Theorem 2.1 with matrix field $\mathbb{A}$ as above, with

$$f := -\mathrm{div}(\nabla_\xi F(x,u,\nabla\psi)) + \partial_z F(x,u,\nabla u),$$

with 0 obstacle and with boundary datum $g-\psi$. Indeed, thanks to (3.16), $w = u-\psi$ is the minimizer of the quadratic energy

$$\mathscr{E}[v] = \int_\Omega \big(\mathbb{A}(x)\nabla v(x)\cdot\nabla v(x) + 2f(x)\,v(x)\big)\,dx$$

over $\mathbb{K}_{g-\psi,0}$. In addition, note that $\partial\{w=0\}\cap\Omega = \partial\{u=\psi\}\cap\Omega$.

With the aim of applying Theorem 2.1 we first recall that $\{u=\psi\}\subseteq\{\nabla u=\nabla\psi\}$, being $u\geq\psi$ on Ω. Thus, given $\Omega'\Subset\Omega$ and any $\varepsilon>0$, the set $\Omega'_\varepsilon := \{0\leq u-\psi<\varepsilon\}\cap\{|\nabla(u-\psi)|<\varepsilon\}\cap\Omega'$ is open and such that $\{u=\psi\}\cap\Omega'\subset\Omega'_\varepsilon$ in view of the remark above. Moreover, as $h = -\mathrm{div}(\nabla_\xi F(x,\psi,\nabla\psi)) + \partial_z F(x,\psi,\nabla\psi)\geq c_0>0$ (cf. (H4)), we have on Ω'_ε

$$\begin{aligned}
f &\geq h - \|h-f\|_{L^\infty(\Omega'_\varepsilon)} \\
&\geq c_0 - \|\partial_z F(\cdot,\psi,\nabla\psi) - \partial_z F(\cdot,u,\nabla u)\|_{L^\infty(\Omega'_\varepsilon)} - \|\mathrm{div}(\nabla_\xi F(\cdot,\psi,\nabla\psi)) - \mathrm{div}(\nabla_\xi F(\cdot,u,\nabla\psi))\|_{L^\infty(\Omega'_\varepsilon)} \\
&\geq c_0 - \omega_{\partial_z F}(2\varepsilon) - \omega_{\nabla^2_{x,\xi}F}(\varepsilon) - \|\nabla u\|_{L^\infty(\Omega'_\varepsilon,\mathbb{R}^n)}\omega_{\nabla^2_{z,\xi}F}(\varepsilon) \\
&\quad - \varepsilon\|\nabla^2_{z,\xi}F(\cdot,\psi,\nabla\psi)\|_{L^\infty(\Omega'_\varepsilon)} - \|\nabla^2\psi\|_{L^\infty(\Omega'_\varepsilon,\mathbb{R}^{n\times n})}\omega_{\nabla^2_\xi F}(\varepsilon),
\end{aligned}$$

denoting with ω_ϑ a modulus of continuity of the relevant function ϑ on Ω' (recall that $F\in C^{2,1}_{loc}$). Therefore, we can choose $\varepsilon>0$ sufficiently small in order to accomplish the condition $f\geq c_0/2>0$ on Ω'_ε. In addition, $f\in C^{0,\alpha}_{loc}(\Omega)$ by hypotheses (H3), (H5) and by Theorem 3.4. Hence, all the conditions in the statement of Theorem 2.1 are satisfied on the open set Ω'_ε, thus the conclusions follow straightforwardly. □

4. Locally coercive vector fields

The analysis in Section 3 does not cover many cases of interest, most relevantly that of the area functional where

$$F(\xi) = \sqrt{1+|\xi|^2}, \qquad \mathbf{a}(\xi) = \nabla F(\xi) = \frac{\xi}{\sqrt{1+|\xi|^2}}.$$

The latter vector field clearly does not fulfill (3.13) in (H2)$'$ being F strictly but not uniformly convex. Moreover, for such a vector field also the existence of solutions to the corresponding variational inequality is not guaranteed in general and requires additional conditions on the set Ω, on the obstacle ψ and on the boundary datum g (cf. [32, Section 4 of Chapter IV]), [28, Chapter 1] and the references therein). The same considerations hold more generally for *locally coercive* vector fields $\mathbf{a}$ (cf. [32, Section 4 of Chapter IV] in the autonomous case and Theorem 4.1 below).

Assuming a priori the existence of a solution and its global Lipschitz continuity, the next result due to Gerhardt implies its global $C^{1,1}$ regularity.

Theorem 4.1 (Theorem 0.1 [24])**.** *Let Ω be of class $C^{3,\alpha}$, for some $\alpha\in(0,1)$, $g\in C^{2,1}(\overline{\Omega})$ and $\psi\in C^{1,1}(\overline{\Omega})$. Let $a_0\in C^{1,1}(\Omega\times\mathbb{R}\times\mathbb{R}^n)$, and assume that $\mathbf{a}(\cdot,\cdot,\xi)$ is $C^{1,1}(\Omega\times\mathbb{R},\mathbb{R}^n)$ for all $\xi\in\mathbb{R}^n$, that $\mathbf{a}(x,z,\cdot)$ is $C^{2,1}(\mathbb{R}^n,\mathbb{R}^n)$ for all $(x,z)\in\Omega\times\mathbb{R}$, and that for all $(x,z,\eta)\in\Omega\times\mathbb{R}\times\mathbb{R}^n$*

$$\partial_\xi\mathbf{a}(x,z,\eta)\xi\cdot\xi>0 \qquad \textit{for all } \xi\neq 0.$$

If $u \in C^{0,1}(\Omega)$ *is a solution of the variational inequality in* (1.2) *over the set*

$$\{v \in C^{0,1}(\Omega) : v \geq \psi \text{ on } \Omega, \quad v = g \text{ on } \partial\Omega\}$$

then $u \in C^{1,1}(\Omega)$.

Therefore, with Theorem 4.1 at hand, if a locally coercive vector field corresponds to an integrand F satisfying hypothesis (H5) of Theorem 3.8 we can argue as in Lemma 3.12 and in the second part of the proof of Theorem 3.8 itself to conclude the same stratification result for the free boundary of a solution u. Note that, in particular, this claim holds for the area functional in the Euclidean space (cf. [32, Section 5 of Chapter V] for the two dimensional case, and [8]).

4.1. **The area functional in a Riemannian manifold.** Similarly, we would like to discuss here the case of the obstacle problem for the area functional in a Riemannian manifold, that naturally enters in several geometric applications (cf., *e.g.*, [39]). Indeed, to the best of our knowledge a comprehensive stratification result of the free boundary points in this case has not appeared elsewhere. Since we are aimed here for a local regularity result, we assume that

(M1) our manifold is parametrized by a single chart $\Sigma := B^n_{r_0} \times (-r_0, r_0) \subset \mathbb{R}^n \times \mathbb{R}$, for some $r_0 > 0$;

(M2) the metric tensor g satisfies $g(0) = I$ and $\nabla g(0) = 0$ (where ∇ denotes the Levi-Civita connection);

(M3) the obstacle $\psi \in C^{1,1}(B^n_{r_0}, (-r_0, r_0))$ with $\psi(0) = |\nabla\psi(0)| = 0$;

We consider the following obstacle problem:

$$\min_{v \in \mathbb{K}_{\psi,g}} \operatorname{vol}_g\big(\operatorname{graph}(v)\big), \tag{4.1}$$

where $\mathbb{K}_{\psi,g} := \big\{v \in C^{0,1}(B^n_{r_0}, (-r_0, r_0)) : v \geq \psi,\ v|_{\partial B^n_{r_0}} = g\big\}$ for some $g \in C^{0,1}(\partial B^n_{r_0})$ with $g \geq \psi|_{\partial B^n_{r_0}}$, $\operatorname{graph}(v) := \big\{(x, v(x)) : x \in B^n_{r_0}\big\} \subset \mathbb{R}^n \times \mathbb{R}$ and $\operatorname{vol}_g\big(\operatorname{graph}(v)\big)$ is the area (n-dimensional measure) of the Lipschitz submanifold associated to the graph of v. In local coordinates, one can express the area of graph(u) in the following way: let $G : B^n_{r_0} \to \Sigma$ be given by $G(x) = \big(x, u(x)\big)$ and

$$JG(x) := \sqrt{\det\big(DG(x)^T g(G(x)) DG(x)\big)};$$

then

$$\operatorname{vol}_g\big(\operatorname{graph}(u)\big) = \int_{B^n_{r_0}} JG(x)\, dx.$$

More explicitly, the matrix $M(x) := DG(x)^T g(G(x)) DG(x)$ has entries for $i, j = 1, \ldots, n$

$$M_{ij}(x) := g_{ij}\big(x, u(x)\big) + g_{j(n+1)}\big(x, u(x)\big)\, \partial_i u(x) + g_{i(n+1)}\big(x, u(x)\big)\, \partial_j u(x) + g_{(n+1)(n+1)} \partial_i u(x)\, \partial_j u(x).$$

As for the case of a flat metric, the existence of solutions to (4.1) is not always guaranteed and several conditions for it should be verified. However we do not investigate this problem in the present note, but we assume that we are given a solution $u \in C^{0,1}(B^n_{r_0}, (-r_0, r_0))$ and moreover we assume that

(M4) $u \in C^{1,\alpha}(B^n_{r_0}, (-r_0, r_0))$ for some $\alpha > 0$, and $u(0) = |\nabla u(0)| = 0$.

Remark 4.2. A comment regarding the assumption (M4) is in order. The natural setting for the study of obstacle problems in Riemannian manifolds is that of the so called “parametric minimal surfaces” theory, *i.e.* the theory of Caccioppoli sets minimizing the perimeter among all sets which contain (or are contained in) a given obstacle. In this setting the existence issue for the obstacle problem is a simple consequence of the compactness property of Caccioppoli sets, although in general the graphical property would not be ensured.

On the other hand, around points of the free boundary of the solutions it is simple to check that one can choose normal coordinates in such a way that hypotheses (M1)–(M4) are matched. In particular, the hypothesis (M4) is a consequence of the *almost minimizing* property of the solutions to the parametric obstacle problem and of a Bernstein theorem (cf. [39, Section 6.1.2] and [45]), and therefore it is not restrictive to assume it.

In order to better understand the structure of the area functional, we can follow the strategy in [39] and look at the first variations of the functional

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0^+}\mathrm{vol}_g\big(\mathrm{graph}(u+\varepsilon\phi)\big)\geq 0, \tag{4.2}$$

for every $\phi\in C_c^\infty(B^n_{r_0})$ such that $\phi|_{\Lambda_u}\geq 0$ where $\Lambda_u:=\{u=\psi\}$. By following the computations in [39] we infer that the inequality (4.2) reads as

$$\int_{B^n_{r_0}}\phi\, Lu\, dx\leq 0\quad \forall\,\phi\in C_c^\infty(B^n_{r_0}),\ \phi|_{\Lambda_u}\geq 0, \tag{4.3}$$

where

$$Lu(x):=\mathrm{div}\Big(A\big(x,u(x),\nabla u(x)\big)\nabla u(x)+b\big(x,u(x),\nabla u(x)\big)\Big)-f\big(x,u(x),\nabla u(x)\big),$$

and A, b and f are given by the following formulas (the Einstein convention of repeated indices is consistently employed in the sequel):

(1) $A=(a^{ij})_{i,j=1,\dots,n}:B^n_{r_0}\times(-r_0,r_0)\times\mathbb{R}^n\to\mathbb{R}^{n\times n}$ is given by

$$a^{ij}(x,z,\xi):=g_{(n+1)(n+1)}(x,z)h^{ij}(x,z,\xi),$$

and $(h^{ij})_{i,j=1,\dots,n}$ is the inverse of the matrix $(h_{ij})_{i,j=1,\dots,n}$ with

$$\begin{aligned}h_{ij}(x,z,\xi):=&\ g_{ij}(x,z)+\xi_i\, g_{j(n+1)}(x,z)+\xi_j\, g_{(n+1)i}(x,z)\\&+\xi_i\,\xi_j\, g_{(n+1)(n+1)}(x,z)\qquad\forall\, i,j=1,\dots,n,\end{aligned}$$

(note that $(h_{ij})_{i,j=1,\dots,n}$ is non-singular for small enough $|x|,|z|,|\xi|$);

(2) $b=(b^i)_{i=1,\dots,n}:B^n_{r_0}\times(-r_0,r_0)\times\mathbb{R}^n\to\mathbb{R}^n$ is given by

$$b^i(x,z,\xi):=g_{j(n+1)}(x,z)h^{ji}(x,z,\xi);$$

(3) $f:B^n_{r_0}\times(-r_0,r_0)\times\mathbb{R}^n\to\mathbb{R}$ is given by

$$\begin{aligned}f(x,z,\xi):=&\ h^{ij}\,\xi_i\,\Gamma^k_{(n+1)(n+1)}\,g_{jk}+h^{ij}\,\xi_j\,\xi_i\,\Gamma^k_{(n+1)(n+1)}\,g_{k(n+1)}\\&+h^{ij}\,\Gamma^k_{i(n+1)}\,g_{jk}+h^{ij}\,\xi_j\,\Gamma^k_{i(n+1)}\,g_{k(n+1)},\end{aligned}$$

where to simplify the notation we have written $h^{ij}=h^{ij}\big(x,z,\xi\big)$, $g_{ij}=g_{ij}\big(x,z\big)$ and $\Gamma^k_{ij}=\Gamma^k_{ij}\big(x,z\big)$ denote the Christoffel symbols.

Note that (4.3) reads as a differential inequality of the form (1.2) where

$$\mathbf{a}(x,z,\xi)=A(x,z,\xi)\xi+b(x,z,\xi)\quad\text{and}\quad a_0(x,z,\xi)=f(x,z,\xi).$$

We now verify that there exists $s_0<r_0$ such that $\mathbf{a}$ and a_0 above satisfy the conditions of Theorem 3.6 as long as $|x|+|z|+|\xi|<s_0$, *i.e.* (H1) with (iii)′ replacing (iii) and $p=2$, (H2)′ for $p=2$.

For what concerns (H1), we note that $\mathbf{a}$ and a_0 are smooth functions in their domains and therefore (i), (ii) and (iii)′ clearly follows for $|x|+|z|+|\xi|<s_0$ after choosing ϕ_1 and ϕ_2 suitable constants.

Similarly, the upper bound of (H2)$'$ follows from the regularity of $\mathbf{a}$. For what concerns the coercivity condition we start estimating as follows (we write h^{-1} for the inverse of the matrix $h = (h_{ij})$):

$$\begin{aligned}\big(\mathbf{a}(z,x,\xi) - \mathbf{a}(z,x,\eta)\big)\cdot(\xi-\eta) &= \big(A(x,z,\xi)\xi - A(x,z,\eta)\eta\big)\cdot(\xi-\eta)\\ &\quad + \big(b(x,z,\xi) - b(x,z,\eta)\big)\cdot(\xi-\eta)\\ &= g_{(n+1)(n+1)}(x,z)\big(h^{-1}(x,z,\xi)\xi - h^{-1}(x,z,\eta)\eta\big)\cdot(\xi-\eta)\\ &\quad + g_{j(n+1)}(x,z)\big(h^{ji}(x,z,\xi) - h^{ji}(x,z,\eta)\big)\cdot(\xi_i-\eta_i). \end{aligned} \tag{4.4}$$

Next note that, since $g(0) = \mathrm{I}$, then for every $\kappa > 0$ one can find s_0 sufficiently small such that

$$\big|g_{j(n+1)}(x,z)\big(h^{ji}(x,z,\xi) - h^{ji}(x,z,\eta)\big)\cdot(\xi_i-\eta_i)\big| \le \kappa\,|\xi-\eta|^2. \tag{4.5}$$

On the other hand, we can estimate the first addendum in (4.4) in the following way:

$$\begin{aligned}\big(h^{-1}(x,z,\xi)\xi - h^{-1}(x,z,\eta)\eta\big)\cdot(\xi-\eta) &= h^{-1}(x,z,\xi)(\xi-\eta)\cdot(\xi-\eta)\\ &\quad + \big(h^{-1}(x,z,\xi) - h^{-1}(x,z,\eta)\big)\,\eta\cdot(\xi-\eta). \end{aligned} \tag{4.6}$$

We can use the fact that $h^{-1}(0,0,0) = \mathrm{I}$ and the regularity of h^{-1} to get that, if $|x|+|z|+|\xi| < s_0$ for some suitably small s_0, then

$$\begin{aligned}\big(h^{-1}(x,z,\xi)\xi - h^{-1}(x,z,\eta)\eta\big)\cdot(\xi-\eta) &\ge \frac{1}{2}\,|\xi-\eta|^2 - \big|h^{-1}(x,z,\xi) - h^{-1}(x,z,\eta)\big|\,|\eta|\,|\xi-\eta|\\ &\ge \Big(\frac{1}{2} - \mathrm{Lip}(h^{-1})\,s_0\Big)\,|\xi-\eta|^2. \end{aligned} \tag{4.7}$$

Using the fact that $g_{(n+1)(n+1)}(0,0) = 1$, we then conclude the lower bound in (H2)$'$ by choosing a suitable s_0 fulfilling all the requests above. Note also that (3.3) is also satisfied because a_0 does not depend on z.

Therefore, if we assume that (H3) is satisfied, in view of (M4) we can apply Theorem 3.6 to $u|_{B^n_{s_0}}$, and deduce that our solution $u|_{B^n_{s_0}}$ has the optimal regularity $C^{1,1}(B^n_{s_0})$.

Finally, we can consider the regularity of the free boundary of u in $B^n_{s_0}$, which can be now obtained by the use of classical arguments. Indeed, since now u has second derivatives almost everywhere, we can also rewrite the operator in the following form (the convention of summation over repeated indices is used):

$$Lu = c^{ij}\big(x,u(x),\nabla u(x)\big)\,\partial_{ij}u + d\big(x,u(x),\nabla u(x)\big), \tag{4.8}$$

where

$$c^{ij}(x,z,\xi) = \partial_{\xi_i}\mathbf{a}_j(x,z,\xi)$$

and

$$d(x,z,\xi) = \mathrm{div}_x\mathbf{a}(x,z,\xi) + \partial_z\mathbf{a}(x,z,\xi)\cdot\xi - a_0(x,z,\xi).$$

By a simple manipulation of the equation (3.5) it follows then that

$$\begin{aligned}-c^{ij}\big(x,\psi(x),\nabla\psi(x)\big)\partial_{ij}\big(u(x)-\psi(x)\big) &= \Big(L\psi(x) + d\big(x,u(x),\nabla u(x)\big) - d\big(x,\psi(x),\nabla\psi(x)\big)\Big)\chi_{\{u>\psi\}}\\ &\quad + \Big(c^{ij}\big(x,u(x),\nabla u(x)\big) - c^{ij}\big(x,\psi(x),\nabla\psi(x)\big)\Big)\partial_{ij}u(x)\,\chi_{\{u>\psi\}}. \end{aligned} \tag{4.9}$$

Moreover, we also deduce from the regularity of $\mathbf{a}$ and a_0 that, up to reducing eventually s_0, the function $w := u - \psi$ satisfies the following obstacle problem

$$\mathbb{A}^{ij}(x)\partial_{ij}w(x) = q(x)\chi_{\{w>0\}}, \tag{4.10}$$

where the matrix field $\mathbb{A}^{ij}(x) = c^{ij}\big(x, \psi(x), \nabla\psi(x)\big)$ is uniformly elliptic, and

$$q(x) = -L\psi(x) - \Big(d\big(x, u(x), \nabla u(x)\big) - d\big(x, \psi(x), \nabla\psi(x)\big)\Big)\chi_{\{u>\psi\}} - \Big(c^{ij}\big(x, u(x), \nabla u(x)\big) - c^{ij}\big(x, \psi(x), \nabla\psi(x)\big)\Big)\partial_{ij}u(x)\,\chi_{\{u>\psi\}}.$$

By additionally assuming (H4), we have that $-L\psi(x) \geq c_0 > 0$ and $q > c_0/2 > 0$. Furthermore, if the obstacle $\psi \in C^{2,\alpha}$ for some $\alpha > 0$ then $q \in C^{0,\alpha}$ (where, for the last claim, the Schauder estimates for the second derivatives of w in $\{w > 0\}$ are used (cf. [29, Theorem 6.2]), and the regularity of u which implies that $|\nabla u(x) - \nabla\psi(x)| \leq C\,\mathrm{dist}(x, \{u = \psi\})$).

Now, by using the regularity results for such obstacle problem in [8, 40] we can easily conclude the following final result.

Theorem 4.3. *Let (Σ, g) be a Riemannian manifold satisfying conditions (M1) and (M2), and let u be satisfying (M4) and be a solution to the obstacle problem for the area functional with respect to an obstacle $\psi \in C^{2,\alpha}(B^n_{r_0}, (-r_0, r_0))$ satisfying (M3) and such that $-L\psi(x) \geq c_0 > 0$.*

Then, there exists $s_0 > 0$ such that $u \in C^{1,1}(B^n_{s_0}, (-r_0, r_0))$ and the free boundary decomposes as $\partial\{u = \psi\} \cap B^n_{s_0} = \mathrm{Reg}(u) \cup \mathrm{Sing}(u)$, where $\mathrm{Reg}(u)$ and $\mathrm{Sing}(u)$ are called its regular and singular part, respectively. Moreover, $\mathrm{Reg}(u) \cap \mathrm{Sing}(u) = \emptyset$ and

- *(i)* $\mathrm{Reg}(u)$ *is relatively open in $\partial\{u = \psi\}$ and, for every point $x_0 \in \mathrm{Reg}(u)$, there exist $r = r(x_0) > 0$ and $\beta = \beta(x_0) \in (0, 1)$ such that $\mathrm{Reg}(u) \cap B_r(x_0)$ is a $C^{1,\beta}$ submanifold of dimension $n - 1$;*
- *(ii)* $\mathrm{Sing}(u) = \cup_{k=0}^{n-1} S_k$, *with S_k contained in the union of at most countably many submanifolds of dimension k and class C^1.*

Remark 4.4. Recalling that the operator L is the first variation of the area functional, the condition (H4) can be read as the geometric property of the obstacle ψ of having the mean curvature vector "pointing downward", i.e. on the opposite side with respect to the graph of u.

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