ON THE SOLUTIONS OF A $2+1$ DIMENSIONAL MODEL FOR EPITAXIAL GROWTH WITH AXIAL SYMMETRY

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ABSTRACT. The evolution equation derived by Xu and Xiang in [16] to describe heteroepitaxial growth in $2+1$ dimensions with elastic forces on vicinal surfaces is, in the radial case and uniform mobility,

$$\ell_t = -\Delta\left[\nabla\cdot(-\hat{r}+|\ell_r|\ell_r) + L(\ell_r) - (\nabla\cdot\hat{r})\log|\ell_r| - \frac{\ell_{rr}}{\ell_r}\right], \qquad (1)$$

where $\hat{r}$ denotes the unit vector, ℓ denotes the surface height of the film, ℓ_r is assumed to be negative, and

$$L(\ell_r)(r) := \int_0^{+\infty}\left(\frac{K(m)}{\rho+r}+\frac{E(m)}{\rho-r}\right)\ell_r(\rho)\,\mathrm{d}\rho, \qquad m := \frac{4\rho r}{(\rho+r)^2},$$

with

$$K(m) := \int_0^{\pi/2}\frac{1}{\sqrt{1-m\sin^2\theta}}\,\mathrm{d}\theta, \qquad E(m) := \int_0^{\pi/2}\sqrt{1-m\sin^2\theta}\,\mathrm{d}\theta.$$

The strong nonlinearity, and the presence of elliptic integrals and Cauchy principal values are the main difficulties of our analysis. In this paper we will show that (1) is formally (i.e., when sufficient regularity is assumed) equivalent to the parabolic evolution equation (3) below, and the main aim is to prove existence, uniqueness and regularity of strong solutions to (3). We will extensively use techniques from the theory of evolution equations governed by maximal monotone operators in Banach spaces.

Keywords: epitaxial growth, vicinal surfaces, evolution equations, monotone operators

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1. Introduction

Within the context of heteroepitaxial growth of a film onto a substrate, terraces and steps self-organize to accommodate misfit elasticity forces. Discrete one-dimensional models have been proposed by Duport, Politi and Villain [5], and Tersoff, Phang, Zhang and Lagally [13]. The associated continuum model was derived by Xiang [14]. Also related is the work by Xiang and E [15]. For two-dimensional models, a continuum model was derived by Xu and Xiang [16].

For simplicity, we assume uniform mobility. Moreover, the original model from [16] involves three separate scales, and this makes the analysis significantly more challenging. Thus, similarly to what done by Gao, Liu and Lu in [7], all terms are assumed to be of order $O(1)$. Further neglecting non influential material constants, the non-dimensional evolution equation, in the radially symmetric case, reads (in polar coordinates (r,θ))

$$\ell_t = -\Delta\left[\nabla\cdot(-\hat{r}+|\ell_r|\ell_r) + L(\ell_r) - (\nabla\cdot\hat{r})\log|\ell_r| - \frac{\ell_{rr}}{\ell_r}\right]. \qquad (2)$$

Here ℓ denotes the surface height of the film, ℓ_r is assumed to be nonpositive, and

$$L(\ell_r)(r) := PV\int_0^{+\infty}\left(\frac{K(m)}{\rho+r}+\frac{E(m)}{\rho-r}\right)\ell_r(\rho)\,\mathrm{d}\rho, \quad m := \frac{4\rho r}{(\rho+r)^2},$$

where PV denotes the Cauchy principal value, and

$$K(m) := \int_0^{\pi/2} \frac{1}{\sqrt{1-m\sin^2\theta}}\,\mathrm{d}\theta, \quad E(m) := \int_0^{\pi/2} \sqrt{1-m\sin^2\theta}\,\mathrm{d}\theta,$$

denote the complete elliptic integrals of the first and second kind respectively. The time domain is the interval $[0,T]$, with $T>0$ being a given datum, and the space domain is $\mathbb{R}^2$. Although ℓ is a radial function, equation (2) is still quite different from the true one-dimensional equation, which has been studied by Dal Maso, Fonseca and Leoni in [4], and by Fonseca, Leoni and the author in [6]. In particular, due to the presence of the elliptic integrals $E(m)$ and $K(m)$, and the fact that L is defined via Cauchy principal value, the analysis of (2) is quite involved.

Working on an unbounded domain (such as $\mathbb{R}^2$) carries the drawbacks of not having Poincaré's inequality, and having significantly weaker versions of Sobolev embeddings. Therefore, to avoid such unnecessary technical difficulties, and similarly to the what done in [4] and [6], we will restrict our domain to the unit disk $D := \{(r,\theta) : r \le 1\}$. Then let H be the operator "representing" L in D. The operator H is the "analogue" of the Hilbert transform in the one-dimensional case. The explicit expression of H will be irrelevant, and the crucial fact is that, since L is a bounded linear operator with norm not exceeding $2/\pi$ (see Lemma 6 below), so is H.

Finally, for future reference, the notation $\langle,\rangle$ (without subscripts) will denote the Euclidean scalar product of $\mathbb{R}^2$.

Now fix a constant a, and we introduce the parabolic equation

$$\begin{aligned} u_t = -\nabla\Big[&\nabla\cdot(-\hat{r} + |\langle\nabla\nabla\cdot u,\hat{r}\rangle + a|(\langle\nabla\nabla\cdot u,\hat{r}\rangle + a)\hat{r}) + H(\langle\nabla\nabla\cdot u,\hat{r}\rangle + a) \\ &- (\nabla\cdot\hat{r})\log|\langle\nabla\nabla\cdot u,\hat{r}\rangle + a| - (\log|\langle\nabla\nabla\cdot u,\hat{r}\rangle + a|)_r\Big], \end{aligned} \tag{3}$$

with time domain $[0,T]$, and spatial domain D. As will be proven in Lemma 5 below, (3) has a variational structure.

The relation between (2) and (3) is the following: given a radial function h, let u be a (radial) function such that $h = \nabla\cdot u + ar$ (such a u can be found by direct computation, but is not unique, since one can always add divergence-free functions to u). Then, neglecting all regularity issues (e.g., when u, h are $W^{2,2}$-regular, and $\langle\nabla\nabla\cdot u,\hat{r}\rangle + a$, h_r are uniformly bounded away from zero), it follows

$$h_r = (\nabla\cdot u)_r + a = \langle\nabla\nabla\cdot u,\hat{r}\rangle + a.$$

Now, if u is a radial solution of (3), then

$$\begin{aligned} u_t &= -\nabla\Big[\nabla\cdot(-\hat{r} + |\langle\nabla\nabla\cdot u,\hat{r}\rangle + a|(\langle\nabla\nabla\cdot u,\hat{r}\rangle + a)\hat{r}) + H(\langle\nabla\nabla\cdot u,\hat{r}\rangle + a) \\ &\qquad - (\nabla\cdot\hat{r})\log|\langle\nabla\nabla\cdot u,\hat{r}\rangle + a| - (\log|\langle\nabla\nabla\cdot u,\hat{r}\rangle + a|)_r\Big] \\ &= -\nabla\Big[\nabla\cdot(-\hat{r} + |h_r|h_r\hat{r}) + H(h_r) - (\nabla\cdot\hat{r})\log|h_r| - (\log|h_r|)_r\Big] \\ &= -\nabla\Big[\nabla\cdot(-\hat{r} + |h_r|h_r\hat{r}) + H(h_r) - (\nabla\cdot\hat{r})\log|h_r| - \frac{h_{rr}}{h_r}\Big], \end{aligned}$$

hence

$$(\nabla\cdot u + ar)_t = h_t = -\Delta\Big[\nabla\cdot(-\hat{r} + |h_r|h_r) + H(h_r) - (\nabla\cdot\hat{r})\log|h_r| - \frac{h_{rr}}{h_r}\Big],$$

i.e., h is solution of (2). Thus, we have full "equivalence" between formulations (2) and (3) when sufficient regularity (and uniform boundedness away from zero) is assumed (but is unclear if such "equivalence" holds in general). Therefore, since $h_r < 0$ by hypothesis, we will also impose

$$\langle \nabla\nabla \cdot u, \hat{r}\rangle + a < 0 \ a.e. \tag{4}$$

The choice to introduce the term "$+ar$" in "$h = \nabla \cdot u + ar$" is due to the following reasons:

(1) to fully exploit Poincaré's inequality, we will impose $\int_D \nabla \cdot u \, \mathrm{d}x = 0$ (when defining the space V in (7) below).
(2) However, in the original equation (2), h_r was always nonpositive, hence the necessity of adding $+ar$ to have (4) (this forces also $a < 0$).

Equation (3) will be the main object of our analysis. Clearly, since we have axial symmetry from the model, for future reference, any considered function will be tacitly assumed to be radial. Thus (3) can be rewritten as

$$\begin{aligned} u_t = -\nabla\Big[&\nabla \cdot \Big(-\hat{r} - |\langle \nabla\nabla \cdot u, \hat{r}\rangle + a|^2 \hat{r}\Big) + H(\langle \nabla\nabla \cdot u, \hat{r}\rangle + a) \\ &- (\nabla \cdot \hat{r}) \log |\langle \nabla\nabla \cdot u, \hat{r}\rangle + a| - (\log |\langle \nabla\nabla \cdot u, \hat{r}\rangle + a|)_r\Big]. \end{aligned} \tag{5}$$

The main result is:

Theorem 1. *Given $T > 0$, $a < 0$, an initial datum $u^0 \in V$ (defined in (7) below) such that*

$$\begin{aligned} -\nabla\Big[&\nabla \cdot \Big(-\hat{r} - |\langle \nabla\nabla \cdot u^0, \hat{r}\rangle + a|^2 \hat{r}\Big) + H(\langle \nabla\nabla \cdot u^0, \hat{r}\rangle + a) \\ &- (\nabla \cdot \hat{r}) \log |\langle \nabla\nabla \cdot u^0, \hat{r}\rangle + a| - (\log |\langle \nabla\nabla \cdot u^0, \hat{r}\rangle + a|)_r\Big] \in L^2(D; \mathbb{R}^2), \end{aligned} \tag{6}$$

then there exists a unique strong solution

$$u \in L^\infty(0, T; V) \cap C^0([0, T]; U), \qquad u_t \in L^\infty(0, T; U)$$

of (3), with U defined in (8) below. That is,

$$\begin{aligned} u_t = -\nabla\Big[&\nabla \cdot \Big(-\hat{r} - |\langle \nabla\nabla \cdot u, \hat{r}\rangle + a|^2 \hat{r}\Big) + H(\langle \nabla\nabla \cdot u, \hat{r}\rangle + a) \\ &- (\nabla \cdot \hat{r}) \log |\langle \nabla\nabla \cdot u, \hat{r}\rangle + a| - (\log |\langle \nabla\nabla \cdot u, \hat{r}\rangle + a|)_r\Big] \end{aligned}$$

for a.e. $t \in [0, T]$, $r \in [0, 1]$ and $\theta \in [0, 2\pi]$.

The key difficulty in the analysis is that the right-hand side term in (3) contains the elliptic integrals $K(m)$ and $E(m)$, a Cauchy principal value, and singularities due to the logarithm. To overcome this issue, we will prove that the linear term $\nabla H(\langle \nabla\nabla \cdot u, \hat{r}\rangle + a)$ is a bounded operator (Lemma 6), and the remaining (nonlinear) part is the sub-differential of some convex, proper, lower-semicontinuous functional (Lemmas 5 and 7). Then we will rely on the theory of parabolic evolution equations governed by maximal monotone operators.

2. Auxiliary results

The main aim of this section is to study the right-hand side operator in (5). Our main functional spaces through the paper will be

$$\begin{aligned} V := &\{u \in L^2(D;\mathbb{R}^2) : \nabla\cdot u \in L^2(D;\mathbb{R}),\ \nabla\nabla\cdot u \in L^2(D;\mathbb{R}^2),\\ &u \text{ is radial}, \langle u,\nu\rangle = \nabla\cdot u \equiv 0 \text{ on } \partial D\}, \end{aligned} \tag{7}$$

$$U := \{u \in L^2(D;\mathbb{R}^2) : u \text{ is radial}\}. \tag{8}$$

Here ν denotes the (exterior) unit normal vector to D. Here we identified U with its dual U', but in many instances, where it is important to consider U' as a dual space, we will continue to use the notation U' instead of U. It is straightforward to check that V is reflexive, and that the embeddings $V \hookrightarrow U$, $U' \hookrightarrow V'$ are continuous and dense. The duality pairing between V' and V will be denoted by $\langle,\rangle_{V',V}$, and given by

$$(\forall v' \in V',\ v \in V) \qquad \langle v', v\rangle_{V',V} := \int_D \langle v', v\rangle \,\mathrm{d}x.$$

Endow U with the L^2-norm of D, and V with the norm

$$\|v\|_V := \|\nabla\nabla\cdot v\|_U.$$

The absence of $\|\nabla\cdot v\|_U$ in the definition of $\|\cdot\|_V$ is due to the boundary condition $\langle v,\nu\rangle \equiv 0$ on ∂D, which gives $\int_D \nabla\cdot v\,\mathrm{d}x = 0$ for all $v \in V$. This, combined with Poincaré inequality, gives $\|\nabla\cdot v\|_U \le c\|\nabla\nabla\cdot v\|_V$ for some constant c. The absence of $\|v\|_U$ is due to Lemma 2 below.

Lemma 2. *There exists a constant $C > 0$ such that*

$$\|\nabla\cdot v\|_U \ge C\|v\|_U$$

for any $v \in V$.

Proof. The thesis follows from the fact that every radial function is a gradient, and from [1, Theorem 5.4] □

Remark 3. *As a consequence, the norms $\|v\|_V$ and $\|v\|_U + \|\nabla\cdot v\|_U + \|\nabla\nabla\cdot v\|_U$ are equivalent.*

Next let us recall the following definitions (see for instance [2, Chapter 2]):

Definition 4. *Given a Banach space X, denote by $\langle,\rangle_{X',X}$ the duality pairing between X' and X. A single-valued operator $A : X \longrightarrow X'$, whose domain we denote by*

$$\mathrm{dom}_X(A) := \{u \in X : Au \in X'\},$$

is:

(1) **monotone** *if for any $u, v \in \mathrm{dom}_X(A)$, it holds*

$$\langle Au - Av, u - v\rangle_{X',X} \ge 0.$$

Similarly, a set $G \subseteq X \times X'$ is "monotone" *if for any pair (u,u'), $(v,v') \in G$, it holds*

$$\langle u' - v', u - v\rangle_{X',X} \ge 0;$$

(2) **maximal monotone** *if the graph*

$$\Gamma_A(X) := \{(u, Au) : u \in X\} \subseteq X \times X'$$

is not a proper subset of any monotone set;

(3) **hemi-continuous** *if for any* $u, v, w \in X$ *the mapping* $t \longmapsto \langle Au + tJ_X v, w\rangle_{X',X}$ *is continuous, where* $J_X : X \longrightarrow X'$ *denotes the duality mapping.*

Moreover, the graph $\Gamma_A(X)$ *is* **demi-closed** *if for any sequence* $(x_n) \subseteq X$*, such that* $x_n \to x$ *strongly in* X*,* $Ax_n \rightharpoonup \xi \in X$*, it holds* $(x, \xi) \in A$*.*

Our next lemma proves a crucial monotonicity result.

Lemma 5. *Consider the operator*

$$B : V \longrightarrow V',$$

$$Bu := \nabla\Big[\nabla \cdot \Big(- \hat{r} - |\langle \nabla\nabla \cdot u, \hat{r}\rangle + a|^2 \hat{r}\Big) - (\nabla \cdot \hat{r}) \log |\langle \nabla\nabla \cdot u, \hat{r}\rangle + a| - (\log |\langle \nabla\nabla \cdot u, \hat{r}\rangle + a|)_r\Big],$$

that is (since $(\log |\langle \nabla\nabla \cdot u, \hat{r}\rangle + a|)_r = \langle \nabla \log |\langle \nabla\nabla \cdot u, \hat{r}\rangle + a|, \hat{r}\rangle$*),*

$$\langle Bu, v\rangle_{V',V} = -\int_D \langle \hat{r} + |\langle \nabla\nabla \cdot u, \hat{r}\rangle + a|^2 \hat{r}, \nabla\nabla \cdot v\rangle \,\mathrm{d}x + \int_D (\nabla \cdot \hat{r}) \log |\langle \nabla\nabla \cdot u, \hat{r}\rangle + a| \nabla \cdot v \,\mathrm{d}x - \int_D \log |\langle \nabla\nabla \cdot u, \hat{r}\rangle + a| \nabla \cdot (\hat{r}\nabla \cdot v) \,\mathrm{d}x.$$

for all $v \in V$*. Then* B *is the sub-differential of the proper, convex, lower-semicontinuous functional*

$$\phi : V \longrightarrow \mathbb{R}, \qquad \phi(u) := \int_D \Phi(-\langle \nabla\nabla \cdot u, \hat{r}\rangle - a) \,\mathrm{d}x, \tag{9}$$

with Φ *defined in* (10) *below. Consequently,* B *is maximal monotone and demi-closed.*

Proof. Note that

$$(\nabla \cdot \hat{r}) \log |\langle \nabla\nabla \cdot u, \hat{r}\rangle + a| + (\log |\langle \nabla\nabla \cdot u, \hat{r}\rangle + a|)_r = (\nabla \cdot \hat{r}) \log |\langle \nabla\nabla \cdot u, \hat{r}\rangle + a| + \langle \nabla \log |\langle \nabla\nabla \cdot u, \hat{r}\rangle + a|, \hat{r}\rangle = \nabla \cdot (\hat{r} \log |\langle \nabla\nabla \cdot u, \hat{r}\rangle + a|),$$

hence B is rewritten as

$$Bu = -\nabla\nabla \cdot (|\langle \nabla\nabla \cdot u, \hat{r}\rangle + a|^2 \hat{r}) - \nabla\nabla \cdot \hat{r} - \nabla\nabla \cdot (\hat{r} \log |\langle \nabla\nabla \cdot u, \hat{r}\rangle + a|).$$

The proof is essentially divided into four steps:

(1) first, we define the functions Φ and ϕ (see (10) below), and prove that

$$\partial\phi(u) \supseteq \{\nabla\nabla \cdot (-\Phi'(-\langle \nabla\nabla \cdot u, \hat{r}\rangle - a)\hat{r})\}$$

provided that $\partial\phi(u)$ is not empty.

(2) Second, we prove that the Gâteaux differential

$$\lim_{\varepsilon \to 0} \frac{\phi(u + v) - \phi(u)}{\varepsilon} = \int_D \langle \nabla\nabla \cdot (-\Phi'(-\langle \nabla\nabla \cdot u, \hat{r}\rangle - a))\hat{r}, v\rangle \,\mathrm{d}x$$

is well defined for all directions v belonging to $\bigcup_{C \geq 0} T_C$ (defined in (11) below).

(3) Third, we prove that $\bigcup_{C \geq 0} T_C$ is dense in V.

(4) Finally, we infer the uniqueness of the sub-differential $\partial\phi$, i.e.

$$\partial\phi(u) = \{\nabla\nabla \cdot (-\Phi'(-\langle \nabla\nabla \cdot u, \hat{r}\rangle - a)\hat{r})\}$$

provided that $\partial\phi(u)$ is not empty.

Step 1. Define the convex function

$$\Phi:\mathbb{R}\longrightarrow\mathbb{R}\cup\{+\infty\},\qquad \Phi(\xi):=\begin{cases}+\infty & \text{if } \xi<0,\\ 0 & \text{if } \xi=0,\\ \xi\log\xi+\xi^3/3 & \text{if } \xi>0,\end{cases}\tag{10}$$

and let

$$\phi:V\longrightarrow\mathbb{R}\cup\{+\infty\},\qquad \phi(u):=\int_D\Phi(-\langle\nabla\nabla\cdot u,\hat r\rangle-a)\,\mathrm{d}x.$$

It is straightforward to check that ϕ is lower-semicontinuous, proper (i.e., not identically equal to $+\infty$), and convex, as we imposed $\langle\nabla\nabla\cdot u,\hat r\rangle+a<0$ a.e.. Thus

$$\Phi(-\langle\nabla\nabla\cdot(u+\varepsilon v),\hat r\rangle-a)-\Phi(-\langle\nabla\nabla\cdot u,\hat r\rangle-a)\ge\varepsilon\Phi'(-\langle\nabla\nabla\cdot u,\hat r\rangle-a)\langle-\nabla\nabla\cdot v,\hat r\rangle\qquad\text{for a.e. } r,\theta$$

holds whenever $\Phi(-\langle\nabla\nabla\cdot u,\hat r\rangle-a)<+\infty$, and integrating on D gives

$$\begin{aligned}\phi(u+\varepsilon v)-\phi(u)&=\int_D[\Phi(-\langle\nabla\nabla\cdot(u+\varepsilon v),\hat r\rangle-a)-\Phi(-\langle\nabla\nabla\cdot u,\hat r\rangle-a)]\,\mathrm{d}x\\&\ge\varepsilon\int_D\Phi'(-\langle\nabla\nabla\cdot u,\hat r\rangle-a)\langle-\nabla\nabla\cdot v,\hat r\rangle\,\mathrm{d}x\\&=\varepsilon\int_D\langle-v,\nabla\nabla\cdot(\Phi'(-\langle\nabla\nabla\cdot u,\hat r\rangle-a)\hat r)\rangle\,\mathrm{d}x.\end{aligned}$$

Thus

$$\partial\phi(u)\neq\emptyset\Longrightarrow\partial\phi(u)\supseteq\{-\nabla\nabla\cdot(\Phi'(-\langle\nabla\nabla\cdot u,\hat r\rangle-a)\hat r)\}.$$

Note that

$$-\nabla\nabla\cdot(\Phi'(-\langle\nabla\nabla\cdot u,\hat r\rangle-a)\hat r)=-\nabla\nabla\cdot(|\langle\nabla\nabla\cdot u,\hat r\rangle+a|^2\hat r+\log|\langle\nabla\nabla\cdot u,\hat r\rangle+a|\hat r+\hat r)=Bu.$$

Step 2. Now we prove $\partial\phi(u)=\{-\nabla\nabla\cdot(\Phi'(-\langle\nabla\nabla\cdot u,\hat r\rangle-a)\hat r)\}$ (provided that $\partial\phi(u)\neq\emptyset$). We recall that $\partial\phi(u)$ is a singleton as soon as ϕ is Gâteaux differentiable at u. Let

$$\xi_1:=-\langle\nabla\nabla\cdot(u+\varepsilon v),\hat r\rangle-a,\qquad \xi_2:=-\langle\nabla\nabla\cdot u,\hat r\rangle-a,\qquad \xi_1-\xi_2=-\varepsilon\langle\nabla\nabla\cdot v,\hat r\rangle,$$

and we impose the extra condition

$$v\in T_C,\qquad T_C:=\left\{z\in U:\left|\frac{\langle\nabla\nabla\cdot z,\hat r\rangle}{\langle\nabla\nabla\cdot u,\hat r\rangle+a}\right|\le C<+\infty\right\}\tag{11}$$

for some C. The mean value theorem then gives

$$\phi(u+\varepsilon v)-\phi(u)=\int_D[\Phi(\xi_1)-\Phi(\xi_2)]\,\mathrm{d}x=\int_D\Phi'(\tilde\xi)(\xi_1-\xi_2)\,\mathrm{d}x,\tag{12}$$

where $\tilde\xi$ is a function satisfying

$$\min\{\xi_1,\xi_2\}\le\tilde\xi\le\max\{\xi_1,\xi_2\}\qquad\text{for a.e. } r,\theta.$$

Note that $\tilde\xi$ is Lebesgue measurable since Φ' is strictly increasing on $(0,+\infty)$, and $\tilde\xi=\xi_1=\xi_2$ on $\xi_1=\xi_2$, and

$$\tilde\xi=(\Phi')^{-1}\Big(\frac{\Phi(\xi_1)-\Phi(\xi_2)}{\xi_1-\xi_2}\Big)\qquad\text{on } \xi_1\neq\xi_2,$$

where the right-hand side term is clearly Lebesgue measurable. Since Φ' is strictly increasing, we have

$$\int_D\Phi'(\min\{\xi_1,\xi_2\})(\xi_1-\xi_2)\,\mathrm{d}x\le\int_D\Phi'(\tilde\xi)(\xi_1-\xi_2)\,\mathrm{d}x\le\int_D\Phi'(\max\{\xi_1,\xi_2\})(\xi_1-\xi_2)\,\mathrm{d}x.$$

We prove now

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_D \Phi'(\min\{\xi_1,\xi_2\})(\xi_1-\xi_2)\,\mathrm{d}x \\ = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_D \Phi'(\max\{\xi_1,\xi_2\})(\xi_1-\xi_2)\,\mathrm{d}x = \int_D \Phi'(\xi_2)\,\mathrm{d}x\langle\nabla\nabla\cdot v,\hat{r}\rangle\,\mathrm{d}x. \tag{13}$$

Note that both terms

$$\Phi'(\min\{\xi_1,\xi_2\}) = \Phi'(\min\{-\langle\nabla\nabla\cdot(u+\varepsilon v),\hat{r}\rangle - a, -\langle\nabla\nabla\cdot u,\hat{r}\rangle - a\}),$$
$$\Phi'(\max\{\xi_1,\xi_2\}) = \Phi'(\max\{-\langle\nabla\nabla\cdot(u+\varepsilon v),\hat{r}\rangle - a, -\langle\nabla\nabla\cdot u,\hat{r}\rangle - a\}),$$

converge to

$$\Phi'(-\langle\nabla\nabla\cdot u,\hat{r}\rangle - a) = \Phi'(\xi_2)$$

for a.e. r,θ. Since we have chosen $v\in T_C$ (defined in (11)), it follows

$$\varepsilon^{-1}|\xi_1-\xi_2| = |\langle\nabla\nabla\cdot v,\hat{r}\rangle| \le C|\langle\nabla\nabla\cdot u,\hat{r}\rangle + a| = C|\xi_2|.$$

As we will take the limit $\varepsilon\to 0$, without loss of generality we can consider only ε with $|\varepsilon| < 1/(2C)$. Thus

$$\xi_1 \ge \xi_2 - |\xi_1-\xi_2| \ge \xi_2 - C\varepsilon|\xi_2| > \xi_2/2, \qquad \xi_1 \le \xi_2 + |\xi_1-\xi_2| \le \xi_2 + C\varepsilon|\xi_2| < 3\xi_2/2,$$

and the monotonicity and positivity of Φ' gives

$$\varepsilon^{-1}|\xi_1-\xi_2|\Phi'(\max\{\xi_1,\xi_2\}),\ \varepsilon^{-1}|\xi_1-\xi_2|\Phi'(\max\{\xi_1,\xi_2\}) \le C|\xi_2|\Phi'(\xi_2).$$

Recalling that

$$\phi(u) = \int_D \Phi(-\langle\nabla\nabla\cdot u,\hat{r}\rangle - a)\,\mathrm{d}x = \int_D (\xi_2^3/3 + \xi_2\log\xi_2)\,\mathrm{d}x < +\infty, \tag{14}$$

$\xi_2 > 0$ a.e., and $\xi\mapsto\xi\log\xi$ is bounded from below, we infer $\int_D \xi_2^3\,\mathrm{d}x < +\infty$. Therefore,

$$\int_D |\xi_2|\Phi'(\xi_2)\,\mathrm{d}x = \int_D \xi_2\Phi'(\xi_2)\,\mathrm{d}x = \int_D [\xi_2^3 + \xi_2\log\xi_2 + \xi_2]\,\mathrm{d}x < +\infty,$$

as $\int_D [\xi_2^3 + \xi_2\log\xi_2]\,\mathrm{d}x < +\infty$ in view of (14), and $\int_D \xi_2\,\mathrm{d}x < +\infty$ in view of the continuity of the embedding $L^1(D;\mathbb{R}) \hookrightarrow L^3(D;\mathbb{R})$. Thus both functions

$$\Phi'(\min\{\xi_1,\xi_2\})(\xi_1-\xi_2), \qquad \Phi'(\max\{\xi_1,\xi_2\})(\xi_1-\xi_2)$$

are dominated by $C|\xi_2|\Phi'(\xi_2)\in L^1(D;\mathbb{R})$, and (13) follows from Lebesgue's dominated convergence theorem. Thus we have proven

$$\lim_{\varepsilon\to 0}\frac{\phi(u+\varepsilon v)-\phi(u)}{\varepsilon} = \int_D \Phi'(-\langle\nabla\nabla\cdot u,\hat{r}\rangle - a)\langle -\nabla\nabla\cdot v,\hat{r}\rangle\,\mathrm{d}x \\ = \int_D \langle\nabla\nabla\cdot(-\Phi'(-\langle\nabla\nabla\cdot u,\hat{r}\rangle - a))\hat{r}, v\rangle\,\mathrm{d}x$$

for any $v\in\bigcup_{C\ge 0} T_C$.

Step 3. Now we prove the set $\bigcup_{C\ge 0} T_C$ is dense in V. Consider an arbitrary $z\in V$, and construct the approximating sequence as follows: let

$$z_n := \begin{cases} \langle\nabla\nabla\cdot z,\hat{r}\rangle & \text{if } \left|\frac{\langle\nabla\nabla\cdot z,\hat{r}\rangle}{\langle\nabla\nabla\cdot u,\hat{r}\rangle}\right| \le n, \\ 0 & \text{otherwise.} \end{cases}$$

We look then for radial functions $z_n^*: D\longrightarrow\mathbb{R}$ (resp. $z_n^{**}: D\longrightarrow\mathbb{R}^2$) such that $\nabla z_n^* = z_n\hat{r}$ (resp. $\nabla\cdot z_n^{**} = z_n^*$) with the boundary conditions $z_n = 0$ on ∂D (resp. $z_n^{**} = (0,0)$ on ∂D). It suffices to construct z_n^* and z_n^{**} on the fiber $\{(r,\theta) : r \in [-1,0], \theta = 0\}$ of D, and the extend to D using the radial symmetry. For any $x \in [-1,0]$ let

$$z_n^*(x,0) := \int_{-1}^{x} z_n(r,0)\,\mathrm{d}r,$$

and clearly $z_n = \nabla z_n^*$, $z_n^* = 0$ on ∂D. To construct z_n^{**}, we recall that in radial coordinates the divergence of a radial function f takes the form $\nabla \cdot f = \frac{1}{r}\frac{\partial}{\partial r}(r\langle f, \hat{r}\rangle)$, and $\langle z_n^{**}(\cdot,0), \hat{r}\rangle$ can be found by integrating $z_n^*(\cdot,0)$. Then we extend the definition of z_n^* and z_n^{**} to D using their radial symmetry. As $\nabla\nabla \cdot z_n^{**} = z_n$, and $\|z_n - \langle \nabla\nabla \cdot z, \hat{r}\rangle\|_{L^2(D;\mathbb{R})} \to 0$, we infer $z_n^{**} \to z$ in V. Therefore,, for any n, $z_n^{**} \in T_n$, and $\bigcup_{C\geq 0} T_C^{**}$ is dense in V.

Step 4. In Step 3 we proved that

$$\lim_{\varepsilon\to 0} \frac{\phi(u+\varepsilon v) - \phi(u)}{\varepsilon} = \int_D \langle \nabla\nabla \cdot (-\Phi'(-\langle \nabla\nabla \cdot u, \hat{r}\rangle - a))\hat{r}, v\rangle\,\mathrm{d}x$$

holds for all directions v belonging to $\bigcup_{C\geq 0} T_C$, which is dense in V. To prove that

$$\partial\phi(u) = \{\nabla\nabla \cdot (-\Phi'(-\langle \nabla\nabla \cdot u, \hat{r}\rangle - a)\hat{r})\}$$

whenever $\partial\phi(u)$ is not empty, assume that for some u, $\partial\phi(u)$ contains two elements $\eta_1, \eta_2 \in V'$. It is straightforward to check (using the same arguments from Step 2) that that for any $v \in T_C$ we have $\phi(u \pm \varepsilon v) < +\infty$ for sufficiently small ε (e.g., for all $\varepsilon < 1/(2T_C)$). By the definition of sub-gradient we then have

$$\lim_{\varepsilon\to 0} \frac{\phi(u+\varepsilon v) - \phi(u)}{\varepsilon} \geq \int_D \eta_1 v\,\mathrm{d}x, \qquad \lim_{\varepsilon\to 0} \frac{\phi(u-\varepsilon v) - \phi(u)}{\varepsilon} \geq -\int_D \eta_1 v\,\mathrm{d}x,$$
$$\lim_{\varepsilon\to 0} \frac{\phi(u+\varepsilon v) - \phi(u)}{\varepsilon} \geq \int_D \eta_2 v\,\mathrm{d}x, \qquad \lim_{\varepsilon\to 0} \frac{\phi(u-\varepsilon v) - \phi(u)}{\varepsilon} \geq -\int_D \eta_2 v\,\mathrm{d}x,$$

hence

$$\langle \eta_1, v\rangle_{V',V} = \int_D \eta_1 v\,\mathrm{d}x = \int_D \eta_2 v\,\mathrm{d}x = \langle \eta_2, v\rangle_{V',V}$$

for all $v \in \bigcup_{C\geq 0} T_C$. Thus $\eta_1 = \eta_2$, which gives

$$\partial\phi(u) = \{\nabla\nabla \cdot (-\Phi'(-\langle \nabla\nabla \cdot u, \hat{r}\rangle - a)\hat{r})\}$$

whenever $\partial\phi(u)$ is not empty.

Thus B is sub-differential of the proper, convex and lower-semicontinuous function ϕ. Then the maximal monotonicity of B follows from [2, Theorem 2.8], and the demi-closedness follows from [12, Theorem 1, Remarks 3-4]. □

Now we estimate the linear term $\nabla H(\langle \nabla\nabla \cdot u, \hat{r}\rangle + a)$. The main difficulties are due to the presence of the Cauchy principal value, and the elliptic integrals $K(m)$ and $E(m)$.

Lemma 6. *Consider the operator*

$$\mathcal{H} : V \longrightarrow V', \qquad \mathcal{H}(w) := -\nabla H(\nabla\nabla \cdot w),$$

i.e.,

$$(\forall w, v \in V) \qquad \langle \mathcal{H}(w), v\rangle_{V',V} := \int_D H(\nabla\nabla \cdot w)\nabla \cdot v\,\mathrm{d}x.$$

Then the operator norm of $\mathcal{H}$ is at most $\pi/2$, i.e.,

$$\sup_{v\in V}\frac{|\langle \mathcal{H}(w),v\rangle_{V',V}|}{\|\nabla\nabla\cdot v\|_U}\le \frac{\pi}{2}\|\nabla\nabla\cdot w\|_U.$$

Moreover, $\mathcal{H}(c)=0$ for any constant $c\in\mathbb{R}$.

Proof. Since we have no information on the structure of H, we must work with the original operator L. Note that

$$L(h_r)(r):=\int_0^{+\infty}\Big(\frac{K(m)}{\rho+r}+\frac{E(m)}{\rho-r}\Big)h_r(\rho)\,\mathrm{d}\rho$$

is a particular case (i.e., when h is radial) of the functional G given by

$$G(\nabla h)(x,y):=PV\int_{\mathbb{R}^2}\frac{(x-\xi)h_x(\xi,\eta)+(y-\eta)h_y(\xi,\eta)}{[(x-\xi)^2+(y-\eta)^2]^{3/2}}\,\mathrm{d}\xi\,\mathrm{d}\eta$$

We prove that the linear functional G_1 given by

$$G_1(k_x)(x,y):=PV\int_{\mathbb{R}^2}\frac{(x-\xi)k_x(\xi,\eta)}{[(x-\xi)^2+(y-\eta)^2]^{3/2}}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{15}$$

is bounded, and $G_1(k_x)\in L^2(\mathbb{R}^2;\mathbb{R})$, for any $k\in W^{1,2}(\mathbb{R}^2;\mathbb{R})$.

Consider an arbitrary $k\in W^{1,2}(\mathbb{R}^2;\mathbb{R})$. Setting

$$f:\mathbb{R}^2\longrightarrow\mathbb{R}^2,\qquad f(\xi,\eta):=\frac{\xi}{(\xi^2+\eta^2)^{3/2}}, \tag{16}$$

gives

$$G_1(k_x)(x,y)=PV\int_{\mathbb{R}^2}\frac{(x-\xi)k_x(\xi,\eta)}{[(x-\xi)^2+(y-\eta)^2]^{3/2}}\,\mathrm{d}\xi\,\mathrm{d}\eta=(f*k_x)(x,y),$$

and $\widehat{f*k_x}=\widehat{f}\cdot\widehat{k_x}$. Since $k_x\in L^2(\mathbb{R}^2;\mathbb{R})$, we have $\|k_x\|_{L^2(\mathbb{R}^2;\mathbb{R})}=\|\widehat{k_x}\|_{L^2(\mathbb{R}^2;\mathbb{R})}$.

It suffices to prove $\widehat{f}\in L^\infty(\mathbb{R}^2;\mathbb{R})$, which requires delicate manipulation of the principal values. Choose an arbitrary $(x,y)=(r_0,\theta_0)\in\mathbb{R}^2$. Using polar coordinates we get

$$\begin{aligned}\widehat{f}(r_0,\theta_0)&=PV\int_{\mathbb{R}^2}f(\xi,\eta)e^{-2i\pi\langle(x,y),(\xi,\eta)\rangle}\,\mathrm{d}\xi\,\mathrm{d}\eta\\&=PV\int_{\mathbb{R}^2}\frac{\xi}{(\xi^2+\eta^2)^{3/2}}e^{-2i\pi\langle(x,y),(\xi,\eta)\rangle}\,\mathrm{d}\xi\,\mathrm{d}\eta\\&=PV\int_0^{2\pi}\int_0^{+\infty}\frac{\cos\theta}{r}e^{-2i\pi r_0 r\cos(\theta_0-\theta)}\,\mathrm{d}r\,\mathrm{d}\theta\end{aligned}$$

Coupling the integrands at θ and $\theta+\pi$, we obtain

$$\begin{aligned}&PV\int_0^{\pi}\int_0^{+\infty}\Big(\frac{\cos\theta}{r}e^{-2i\pi r_0 r\cos(\theta_0-\theta)}+\frac{\cos(\theta+\pi)}{r}e^{-2i\pi r_0 r\cos(\theta_0-\theta-\pi)}\Big)\,\mathrm{d}r\,\mathrm{d}\theta\\&=-PV\int_{\theta_0}^{\theta_0-\pi}\int_0^{+\infty}\frac{\cos(\theta_0-\theta)}{r}\Big(e^{-2i\pi r_0 r\cos\theta}-e^{2i\pi r_0 r\cos\theta}\Big)\,\mathrm{d}r\,\mathrm{d}\theta\\&=PV\int_{\theta_0}^{\theta_0-\pi}\int_0^{+\infty}\frac{\cos(\theta_0-\theta)}{r}\frac{\sin(2\pi r_0 r\cos\theta)}{2i}\,\mathrm{d}r\,\mathrm{d}\theta\\&=\int_{\theta_0}^{\theta_0-\pi}\frac{\cos(\theta_0-\theta)}{2i}\Big(PV\int_0^{+\infty}\frac{\sin(2\pi r_0 r\cos\theta)}{r}\,\mathrm{d}r\Big)\,\mathrm{d}\theta\\&=\int_{\theta_0}^{\theta_0-\pi}\frac{\cos(\theta_0-\theta)}{2i}\,\mathrm{sgn}(\cos\theta)\Big(PV\int_0^{+\infty}\frac{\sin(\rho)}{\rho}\,\mathrm{d}\rho\Big)\,\mathrm{d}\theta.\end{aligned}$$

Recalling that

$$PV\int_0^{+\infty}\frac{\sin(\rho)}{\rho}\,\mathrm{d}\rho=\lim_{M\to+\infty}\int_0^M\frac{\sin(\rho)}{\rho}\,\mathrm{d}\rho=\frac{\pi}{2},$$

we infer

$$\Big|\int_{\theta_0}^{\theta_0-\pi}\frac{\cos(\theta_0-\theta)}{2i}\,\mathrm{sgn}(\cos\theta)\Big(PV\int_0^{+\infty}\frac{\sin(\rho)}{\rho}\,\mathrm{d}\rho\Big)\,\mathrm{d}\theta\Big|$$
$$\le\frac{\pi}{4}\int_{\theta_0}^{\theta_0-\pi}|\cos(\theta_0-\theta)|\,\mathrm{d}\theta=\frac{\pi}{4}\int_0^{\pi}|\cos\theta|\,\mathrm{d}\theta=\frac{\pi}{2},$$

hence $\|\hat{f}\|_{L^\infty(\mathbb{R}^2;\mathbb{R})}\le\pi/2$. Thus $G_1(k_x)\in L^2(\mathbb{R}^2;\mathbb{R})$. Moreover,

$$\|\widehat{f*k_x}\|_{L^2(\mathbb{R}^2;\mathbb{R})}=\|\widehat{f}\cdot\widehat{k_x}\|_{L^2(\mathbb{R}^2;\mathbb{R})}\le\frac{\pi}{2}\|\widehat{k_x}\|_{L^2(\mathbb{R}^2;\mathbb{R})}.$$

Therefore, the operator norm of G_1 is at most $\pi/2$.

Similarly, setting

$$G_2(k_y)(x,y):=PV\int_{\mathbb{R}^2}\frac{(y-\eta)k_y(\xi,\eta)}{[(x-\xi)^2+(y-\eta)^2]^{3/2}}\,\mathrm{d}\xi\,\mathrm{d}\eta$$

and

$$g:\mathbb{R}^2\longrightarrow\mathbb{R}^2,\qquad g(\xi,\eta):=\frac{\eta}{(\xi^2+\eta^2)^{3/2}},$$

we get $G_2(k_y)(x,y)=(g*k_y)(x,y)$, and the same arguments give that G_2 maps $L^2(\mathbb{R}^2;\mathbb{R})$ to $L^2(\mathbb{R}^2;\mathbb{R})$, and

$$\|\widehat{g*k_y}\|_{L^2(\mathbb{R}^2;\mathbb{R})}\le\frac{\pi}{2}\|k_y\|_{L^2(\mathbb{R}^2;\mathbb{R})}.$$

Now, we return to work again in the domain D. By Hölder and Poincaré inequalities, we get

$$|\langle\mathcal{H}w,v\rangle_{V',V}|\le\int_D|H(\nabla\nabla\cdot w)\nabla\cdot v|\,\mathrm{d}x\le\|H(\nabla\nabla\cdot w)\|_{L^2(D;\mathbb{R})}\|\nabla\cdot v\|_{L^2(D;\mathbb{R})}$$
$$\le\frac{\pi}{2}\|\nabla\nabla\cdot w\|_{L^2(D;\mathbb{R})}\|\nabla\nabla\cdot v\|_{L^2(D;\mathbb{R})},$$

for any $w,v\in V$.

To prove the last part, i.e. $\mathcal{H}(c)=0$ for any constant $c\in\mathbb{R}$, it is again convenient to work with L instead of H. It suffices to prove that $G_1(c)=G_2(c)=0$ for any constant $c\in\mathbb{R}$. From (15), by change of variable $\tilde{\xi}:=x-\xi$, $\tilde{\eta}:=y-\eta$, we get

$$G_1(c)(x,y):=c\Big[PV\int_{\mathbb{R}^2}\frac{x-\xi}{[(x-\xi)^2+(y-\eta)^2]^{3/2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\Big]=c\Big[PV\int_{\mathbb{R}^2}\frac{\tilde{\xi}}{[\tilde{\xi}^2+\tilde{\eta}^2]^{3/2}}\,\mathrm{d}\tilde{\xi}\,\mathrm{d}\tilde{\eta}\Big]=0$$

as the function $\xi\mapsto\tilde{\xi}/[\tilde{\xi}^2+\tilde{\eta}^2]^{3/2}$ is odd for any $\tilde{\eta}$. The proof for $G_2(c)=0$ is completely analogous. □

Lemma 6 estimates the operator norm of $\mathcal{H}$, but does not give any information about monotonicity. The following results (Lemma 7 and Corollary (8)) will prove that both operators $\partial\phi\pm\mathcal{H}$ are maximal monotone.

Lemma 7. *Consider the operator*

$$\psi:V\longrightarrow\mathbb{R},\qquad\psi(u):=\int_D(\Phi(|\langle\nabla\nabla\cdot u,\hat{r}\rangle+a|)-|\langle\nabla\nabla\cdot u,\hat{r}\rangle|^2)\,\mathrm{d}x,\tag{17}$$

with Φ being the function from Lemma 5. Then its sub-differential $\partial\psi(u)$ is a maximal monotone operator.

Proof. It is straightforward to check that ψ is proper and lower-semicontinuous. To prove the convexity, set

$$\varphi : \mathbb{R} \longrightarrow \mathbb{R} \cup \{+\infty\}, \qquad \varphi(\xi) := \Phi(\xi + a) - \xi^2,$$

and it follows

$$\psi(u) = \int_D \varphi(|\langle \nabla\nabla \cdot u, \hat{r}\rangle|) \, \mathrm{d}x.$$

Note that φ is smooth on $(0, +\infty)$, and

$$\varphi''(\xi) = \Phi''(\xi + a) - 2 = 2(\xi + a) + (\xi + a)^{-1} - 2.$$

Since the minimum of $2(\xi + a) + (\xi + a)^{-1}$ is attained at $\xi + a = 1/\sqrt{2}$, it follows

$$\varphi''(\xi) = 2(\xi + a) + (\xi + a)^{-1} - 2 \geq 2\sqrt{2} - 2 > 0,$$

hence φ, and consequently ψ, is convex. Therefore, by [2, Theorem 2.8], its sub-differential $\partial\psi(u)$ is maximal monotone. □

Corollary 8. *The operators $\pm\mathcal{H} + \partial\phi$ are maximal monotone and coercive, i.e.*

$$(\forall u, v \in V) \qquad \langle \pm\mathcal{H}(u - v) + \partial\phi(u) - \partial\phi(v), u - v\rangle_{V',V} \geq \|u - v\|_V^2.$$

Although we will use only the operator $\partial\phi + \mathcal{H}$ in the following, we prove the maximal monotonicity for both operator $\partial\phi \pm \mathcal{H}$ (as the proof is identical), to show how the linear perturbation term $\mathcal{H}$ (independently of its sign) does not affect monotonicity.

Proof. Let ψ be the functional from Lemma 7. Note that

$$\partial\psi(u) = -\nabla\nabla \cdot [\hat{r}(\Phi'(|\langle \nabla\nabla \cdot u, \hat{r}\rangle + a)|)] - 2\nabla\nabla \cdot [\hat{r}|\langle \nabla\nabla \cdot u, \hat{r}\rangle + a|],$$

in view of the following facts:

(1) we already proved (in Lemma 5) that the sub-differential of $u \mapsto \int_D \Phi(|\langle \nabla\nabla \cdot u, \hat{r}\rangle + a|) \, \mathrm{d}x$ is $-\nabla\nabla \cdot [\hat{r}\Phi'(|\langle \nabla\nabla \cdot u, \hat{r}\rangle + a|)]$,

(2) direct computation then gives

$$\begin{aligned}
&\int_D [|\langle \nabla\nabla \cdot (u + v), \hat{r}\rangle|^2 - |\langle \nabla\nabla \cdot u, \hat{r}\rangle|^2] \, \mathrm{d}x \\
&\qquad = 2\int_D \langle \nabla\nabla \cdot (u + v), \hat{r}\rangle\langle \nabla\nabla \cdot v, \hat{r}\rangle \, \mathrm{d}x + \int_D |\langle \nabla\nabla \cdot v, \hat{r}\rangle|^2 \, \mathrm{d}x \\
&\qquad = \int_D \langle v, 2\nabla\nabla \cdot [\langle \nabla\nabla \cdot (u + v), \hat{r}\rangle\hat{r}]\rangle \, \mathrm{d}x + \|v\|_V^2,
\end{aligned}$$

hence the sub-differential of $\chi(u) := \int_D |\langle \nabla\nabla \cdot u, \hat{r}\rangle|^2 \, \mathrm{d}x$ is linear, and equal to $2\nabla\nabla \cdot [\hat{r}\langle \nabla\nabla \cdot u, \hat{r}\rangle]$.

Lemma 6 proved that the operator norm of $\mathcal{H}$ is at most $\pi/2$, and it has been proven in [10] that the Poincaré constant for L^2 norms on the unit disk is at most $2/\pi$. Thus it follows

$$\begin{aligned}
|\langle \mathcal{H}(u), u\rangle_{V',V}| &\leq \int_D |H(\langle \nabla\nabla \cdot u, \hat{r}\rangle)\nabla \cdot u| \, \mathrm{d}x \leq \frac{\pi}{2}\|\langle \nabla\nabla \cdot u, \hat{r}\rangle\|_U \|\nabla \cdot u\|_U \\
&\leq \|\langle \nabla\nabla \cdot u, \hat{r}\rangle\|_U \|\nabla\nabla \cdot u\|_U, \qquad (18)
\end{aligned}$$

and

$$
\begin{aligned}
&\langle \partial\phi(u) - \partial\phi(v), u - v\rangle_{V',V} - \langle \pm\mathcal{H}(u - v), u - v\rangle_{V',V} \\
&\qquad = \langle \partial\phi(u) - \partial\phi(v), u - v\rangle_{V',V} \mp \int_D H(\langle \nabla\nabla\cdot(u - v), \hat{r}\rangle)\nabla\cdot(u - v)\,\mathrm{d}x \\
&\qquad \geq \langle \partial\phi(u) - \partial\phi(v), u - v\rangle_{V',V} - \|\langle \nabla\nabla\cdot(u - v), \hat{r}\rangle\|_U \|\nabla\nabla\cdot(u - v)\|_U. \qquad (19)
\end{aligned}
$$

Since u is radial (and so is $\nabla\nabla\cdot u$), it follows

$$
\|\langle \nabla\nabla\cdot(u - v), \hat{r}\rangle\|_U = \|(\nabla\cdot(u - v))_r\|_U = \|\nabla\nabla\cdot(u - v)\|_U,
$$

thus (18) becomes

$$
\begin{aligned}
&\langle \partial\phi(u) - \partial\phi(v), u - v\rangle_{V',V} - \langle \pm\mathcal{H}(u - v), u - v\rangle_{V',V} \\
&\qquad \geq \langle \partial\phi(u) - \partial\phi(v), u - v\rangle_{V',V} - \|(\nabla\cdot(u - v))_r\|_U^2. \qquad (20)
\end{aligned}
$$

On the other hand, since we already proved that the the sub-differential of $\chi(u)$ is linear, and equal to $2\nabla\nabla\cdot[\hat{r}\langle \nabla\nabla\cdot u, \hat{r}\rangle]$, it follows

$$
\begin{aligned}
\langle \partial\chi(u) - \partial\chi(v), u - v\rangle_{V',V} &= 2\int_D \langle \nabla\nabla\cdot[\hat{r}\langle \nabla\nabla\cdot(u - v), \hat{r}\rangle], u - v\rangle\,\mathrm{d}x \\
&= 2\int_D \langle \hat{r}(\nabla\cdot(u - v))_r, \nabla\nabla\cdot(u - v)\rangle\,\mathrm{d}x = 2\int_D (\nabla\cdot(u - v))_r^2\,\mathrm{d}x = 2\|u - v\|_V^2.
\end{aligned}
$$

Combining with (20) then gives

$$
\begin{aligned}
&\langle \partial\phi(u) - \partial\phi(v), u - v\rangle_{V',V} - \langle \pm\mathcal{H}(u - v), u - v\rangle_{V',V} \\
&\qquad \geq \langle \partial\phi(u) - \partial\phi(v), u - v\rangle_{V',V} - \|\nabla\nabla\cdot(u - v)\|_U^2 \\
&\qquad = \langle \partial\psi(u) - \partial\psi(v), u - v\rangle_{V',V} + \langle \partial\chi(u) - \partial\chi(v), u - v\rangle_{V',V} - \|\nabla\nabla\cdot(u - v)\|_U^2 \\
&\qquad \geq \langle \partial\psi(u) - \partial\psi(v), u - v\rangle_{V',V} + \|u - v\|_V^2. \qquad (21)
\end{aligned}
$$

Since Lemma 7 proved that $\partial\psi$ is maximal monotone, and both operators $\partial\chi \pm \mathcal{H}$ are linear, bounded and monotone, it follows (by [11]) that both operators $\partial\phi \pm \mathcal{H} = \partial\psi + \partial\chi \pm \mathcal{H}$ are maximal monotone. □

3. Proof of Theorem 1

In this section we are going to prove Theorem 1.

3.1. **Variational Inequality.** The first step is to establish the variational inequality (24) below, that plays a key role in the proof of Theorem 1. This is a direct application of Kačur's result from [8, Section 5], but due to its relevance to our arguments, we report the proof.

Proposition 9. *Let $\mathcal{H}$ and ϕ be the functionals from Lemma 6 and equation* (10) *respectively. Let $u^0 \in \mathrm{dom}_U(\mathcal{H} + \partial\phi)$ be a given initial datum, satisfying*

$$
(\mathcal{H} + \partial\phi)u^0 \in U. \qquad (22)
$$

Then there exists a function

$$
u \in L^\infty(0, T; V) \cap C^0([0, T]; U), \qquad u_t \in L^\infty(0, T; U) \qquad (23)
$$

such that $u(0) = u^0$ and

$$
\langle u_t(t), v - u(t)\rangle_{U',U} + \langle \mathcal{H}(u(t)), v - u(t)\rangle_{V',V} + \phi(v) - \phi(u(t)) \geq 0 \qquad (24)
$$

for a.e. time $t \in (0,T)$, and all $v \in V$. Moreover, it holds

$$\|u_t\|_{L^\infty(0,T;U)} \le \|(\mathcal{H} + \partial\phi)u^0\|_U. \tag{25}$$

Proof. The proof is essentially divided into three steps:

(1) first, using the classic method of time discretization (i.e., backward Euler's method), we construct an approximating sequence of piece-wise linear functions $u^\varepsilon : [0,T] \longrightarrow V$;
(2) then we prove that the sequence $(u^\varepsilon)_\varepsilon$ is uniformly bounded in $L^\infty(0,T;V) \cap W^{1,\infty}([0,T];U)$, and we obtain a limit function (in the weak-* topology of $L^\infty(0,T;V) \cap W^{1,\infty}([0,T];U)$)
$$u \in L^\infty(0,T;V) \cap C^0([0,T];U), \qquad u_t \in L^\infty(0,T;U);$$
(3) finally, we prove that such u is solution of (24).

Step 1. Let $\varepsilon > 0$ be given. Consider the partition

$$0 = t_0 < t_1 < \cdots < t_{n_\varepsilon - 1} < t_{n_\varepsilon} \le T \le t_{n_\varepsilon} + \varepsilon,$$
$$t_j - t_{j-1} = \varepsilon, \qquad j = 1, \cdots, n_\varepsilon := \left\lfloor \frac{T}{\varepsilon} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the integer part mapping.

We define the recursive sequence $(u_{\varepsilon,i})_i$ in the following way:

(1) $u_{\varepsilon,0} := u^0$,
(2) given $u_{\varepsilon,i-1} \in V$, let $u_{\varepsilon,i} \in V$ be a solution of
$$\left\langle \frac{u_{\varepsilon,i} - u_{\varepsilon,i-1}}{t_i - t_{i-1}} + (\mathcal{H} + \partial\phi)u_{\varepsilon,i}, v - u_{\varepsilon,i} \right\rangle_{V',V} \ge 0 \qquad \text{for all } v \in V.$$

Observe that this is equivalent to find $u_{\varepsilon,i} \in V$ such that

$$\langle (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi))u_{\varepsilon,i}, v - u_{\varepsilon,i} \rangle_{V',V} \ge \langle u_{\varepsilon,i-1}, v - u_{\varepsilon,i} \rangle_{V',V} \qquad \text{for all } v \in V, \tag{26}$$

which would surely hold if $u_{\varepsilon,i} \in V$ is (strong) solution of

$$u_{\varepsilon,i-1} = (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi))u_{\varepsilon,i}. \tag{27}$$

Since $\mathcal{H} + \partial\phi$ is maximal monotone, $\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi) : V \longrightarrow V'$ is surjective for all $\varepsilon > 0$. Hence there exists $u_{\varepsilon,i} \in V$ such that (27) holds. Moreover, we note that $\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi)$ is also injective since $\mathcal{H} + \partial\phi$ is strictly monotone: this because given $v_1, v_2 \in \mathrm{dom}_V(\mathcal{H} + \partial\phi)$, such that

$$[\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi)]v_1 = [\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi)]v_2,$$

it holds

$$\begin{aligned} 0 &= \langle [\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi)]v_1 - [\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi)]v_2, v_1 - v_2 \rangle_{V',V} \\ &= \|v_1 - v_2\|_U^2 + \varepsilon \langle (\mathcal{H} + \partial\phi)v_1 - (\mathcal{H} + \partial\phi)v_2, v_1 - v_2 \rangle_{V',V} \ge \|v_1 - v_2\|_U^2. \end{aligned}$$

Therefore, $u_{\varepsilon,i} = (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi))^{-1} u_{\varepsilon,i-1}$ is unique. Thus $u_{\varepsilon,i} \in V$ is a solution of (26).

Next we define the piece-wise linear functions u^ε such that

$$u^\varepsilon : [0,T] \longrightarrow V, \qquad u^\varepsilon(k\varepsilon) := u_{\varepsilon,k}, \quad k = 0, \cdots, \left\lfloor \frac{T}{\varepsilon} \right\rfloor.$$

Step 2. We first claim

$$\|u_{\varepsilon,i} - u_{\varepsilon,i-1}\|_U \le \|u_{\varepsilon,i-1} - u_{\varepsilon,i-2}\|_U, \qquad i = 2, 3, \cdots, \left\lfloor \frac{T}{\varepsilon} \right\rfloor. \tag{28}$$

By construction,

$$u_{\varepsilon,i} = (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi))^{-1} u_{\varepsilon,i-1} \Longrightarrow \langle (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi)) u_{\varepsilon,i} - u_{\varepsilon,i-1}, u_{\varepsilon,i-1} - u_{\varepsilon,i} \rangle_{V',V} \geq 0, \tag{29}$$

$$u_{\varepsilon,i-1} = (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi))^{-1} u_{\varepsilon,i-2} \Longrightarrow \langle (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi)) u_{\varepsilon,i-1} - u_{\varepsilon,i-2}, u_{\varepsilon,i} - u_{\varepsilon,i-1} \rangle_{V',V} \geq 0, \tag{30}$$

and summing (29) and (30) gives

$$\langle (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi)) u_{\varepsilon,i} - (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi)) u_{\varepsilon,i-1} - (u_{\varepsilon,i-1} - u_{\varepsilon,i-2}), u_{\varepsilon,i-1} - u_{\varepsilon,i} \rangle_{V',V} \geq 0,$$

hence

$$\begin{aligned} -\|u_{\varepsilon,i-1} - u_{\varepsilon,i}\|_U^2 &+ \varepsilon \langle (\mathcal{H} + \partial\phi) u_{\varepsilon,i} - (\mathcal{H} + \partial\phi) u_{\varepsilon,i-1}, u_{\varepsilon,i-1} - u_{\varepsilon,i} \rangle_{V',V} \\ &\geq \langle u_{\varepsilon,i-1} - u_{\varepsilon,i-2}, u_{\varepsilon,i-1} - u_{\varepsilon,i} \rangle_{V',V} \\ &\geq -\|u_{\varepsilon,i-1} - u_{\varepsilon,i-2}\|_U \|u_{\varepsilon,i-1} - u_{\varepsilon,i}\|_U. \end{aligned} \tag{31}$$

Since $\mathcal{H} + \partial\phi$ is monotone, it follows

$$\langle (\mathcal{H} + \partial\phi) u_{\varepsilon,i} - (\mathcal{H} + \partial\phi) u_{\varepsilon,i-1}, u_{\varepsilon,i-1} - u_{\varepsilon,i} \rangle_{V',V} \leq 0,$$

thus (31) yields

$$-\|u_{\varepsilon,i-1} - u_{\varepsilon,i}\|_U^2 \geq -\|u_{\varepsilon,i-1} - u_{\varepsilon,i-2}\|_U \|u_{\varepsilon,i-1} - u_{\varepsilon,i}\|_U \Longrightarrow \|u_{\varepsilon,i-1} - u_{\varepsilon,i}\|_U \leq \|u_{\varepsilon,i-1} - u_{\varepsilon,i-2}\|_U,$$

which proves (28).

Now observe that given $v_1, v_2 \in \mathrm{dom}_V(\mathcal{H} + \partial\phi)$, such that

$$(\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi)) v_1, \ (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi)) v_2 \in V,$$

then the monotonicity of $\mathcal{H} + \partial\phi$ gives

$$\begin{aligned} &\|(\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi)) v_1 - (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi)) v_2\|_U^2 \\ &\quad = \langle (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi)) v_1 - (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi)) v_2, (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi)) v_1 - (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi)) v_2 \rangle_{V',V} \\ &\quad = \|v_1 - v_2\|_U^2 + 2\varepsilon \langle (\mathcal{H} + \partial\phi) v_1 - (\mathcal{H} + \partial\phi) v_2, v_1 - v_2 \rangle_{V',V} \\ &\qquad + \varepsilon^2 \|(\mathcal{H} + \partial\phi) v_1 - (\mathcal{H} + \partial\phi) v_2\|_U^2 \geq \|v_1 - v_2\|_U^2. \end{aligned}$$

As $u_{\varepsilon,1} = (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi))^{-1} u^0$, we get

$$u_{\varepsilon,1} - u^0 = (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi))^{-1} u^0 - (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi))^{-1} (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi)) u^0,$$

hence

$$\begin{aligned} \|u_{\varepsilon,1} - u^0\|_U &= \|(\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi))^{-1} u^0 - (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi))^{-1} (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi)) u^0\|_U \\ &\leq \|u^0 - (\mathrm{id} + \varepsilon(\mathcal{H} + \partial\phi)) u^0\|_U = \varepsilon \|(\mathcal{H} + \partial\phi) u^0\|_U, \end{aligned}$$

which in turn gives

$$\frac{\|u_{\varepsilon,1} - u^0\|_U}{\varepsilon} \leq \|(\mathcal{H} + \partial\phi) u^0\|_U. \tag{32}$$

Combining (28) and (32) gives

$$\frac{\|u_{\varepsilon,i} - u_{\varepsilon,i-1}\|_U}{\varepsilon} \leq \|(\mathcal{H} + \partial\phi) u^0\|_U$$

for all $\varepsilon > 0$, and $i = 0, \cdots, \lfloor T/\varepsilon \rfloor$. Since

$$\frac{\|u_{\varepsilon,i} - u_{\varepsilon,i-1}\|_U}{\varepsilon} = \|u_t^\varepsilon(t)\|_U$$

for $t \in ((i-1)\varepsilon, i\varepsilon)$, it follows

$$\|u_t^\varepsilon\|_U \le \|(\mathcal{H}+\partial\phi)u^0\|_U \Longrightarrow \sup_\varepsilon \sup_{t\in[0,T]} \|u^\varepsilon(t)-u^0\|_U \le T\|(\mathcal{H}+\partial\phi)u^0\|_U. \tag{33}$$

To estimate $\|u^\varepsilon(t)\|_V$, note that $(\mathrm{id}+\varepsilon(\mathcal{H}+\partial\phi))u_{\varepsilon,i} = u_{\varepsilon,i-1}$ gives, in view of the coercivity estimate (21),

$$\begin{aligned}\|u_{\varepsilon,i}\|_V^2 &\le \left|\langle(\mathcal{H}+\partial\phi)u_{\varepsilon,i}, u_{\varepsilon,i}\rangle_{V',V}\right| = \left|\left\langle \frac{u_{\varepsilon,i}-u_{\varepsilon,i-1}}{\varepsilon}, u_{\varepsilon,i}\right\rangle_{V',V}\right| \\ &\le \frac{\|u_{\varepsilon,i}-u_{\varepsilon,i-1}\|_U}{\varepsilon}\|u_{\varepsilon,i}\|_U \le \|(\mathcal{H}+\partial\phi)u^0\|_U(T\|(\mathcal{H}+\partial\phi)u^0\|_U + \|u^0\|_U),\end{aligned}$$

hence

$$\sup_\varepsilon \sup_{t\in[0,T]} \|u^\varepsilon(t)\|_V^2 \le \|(\mathcal{H}+\partial\phi)u^0\|_U(T\|(\mathcal{H}+\partial\phi)u^0\|_U + \|u^0\|_U). \tag{34}$$

Step 3. Note that the space $L^\infty(0,T;V)$ (resp. $L^\infty(0,T;U)$) is the dual of $L^1(0,T;V')$ (resp. $L^1(0,T;U)$). Thus by Banach-Alaoglu theorem, in view of the a priori estimates (33) and (34), we have

$$u^{\varepsilon_n} \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;V), \qquad u_t^{\varepsilon_n} \overset{*}{\rightharpoonup} u_t \text{ in } L^\infty(0,T;U),$$

Consequently, we have the lower-semicontinuity of the norms

$$\|u\|_{L^\infty(0,T;V)} \le \liminf_{n\to+\infty} \|u^{\varepsilon_n}\|_{L^\infty(0,T;V)}, \qquad \|u_t\|_{L^\infty(0,T;U)} \le \liminf_{n\to+\infty} \|u_t^{\varepsilon_n}\|_{L^\infty(0,T;U)},$$

and (23) follows. Moreover, we have also $u^{\varepsilon_n} \overset{*}{\rightharpoonup} u$ in $L^\infty(J;V)$ and $u_t^{\varepsilon_n} \overset{*}{\rightharpoonup} u_t$ in $L^\infty(J;U)$ for any time set $J \subseteq [0,T]$ of positive measure.

By construction, each u^{ε_n} satisfies

$$\langle u_t^{\varepsilon_n}(t) + (\mathcal{H}+\partial\phi)u^{\varepsilon_n}(t), v - u^{\varepsilon_n}(t)\rangle_{V',V} \ge 0 \tag{35}$$

for a.e. $t \in [0,T]$, and all $v \in V$. Due to the convexity of ϕ, (35) gives

$$\langle u_t^{\varepsilon_n}(t) + \mathcal{H}u^{\varepsilon_n}(t), v - u^{\varepsilon_n}(t)\rangle_{V',V} + \phi(v) - \phi(u^{\varepsilon_n}(t)) \ge 0$$

for a.e. $t \in [0,T]$, and all $v \in V$. Now consider an arbitrary time set $J \subseteq [0,T]$ of positive measure. Integrating on J gives

$$\int_J \Big[\langle u_t^{\varepsilon_n}(t) + \mathcal{H}u^{\varepsilon_n}(t), v - u^{\varepsilon_n}(t)\rangle_{V',V} + \phi(v) - \phi(u^{\varepsilon_n}(t))\Big]\,\mathrm{d}t \ge 0 \tag{36}$$

for all $v \in V$. We claim

$$\limsup_{n\to+\infty} -\int_J \phi(u^{\varepsilon_n}(t))\,\mathrm{d}t \le -\int_J \phi(u(t))\,\mathrm{d}t, \tag{37}$$

$$\lim_{n\to+\infty}\int_J \langle\mathcal{H}u^{\varepsilon_n}(t), v-u^{\varepsilon_n}(t)\rangle_{V',V}\,\mathrm{d}t = \int_J \langle\mathcal{H}u(t), v-u(t)\rangle_{V',V}\,\mathrm{d}t, \tag{38}$$

$$\lim_{n\to+\infty}\int_J \langle u_t^{\varepsilon_n}(t), v-u^{\varepsilon_n}(t)\rangle_{V',V}\,\mathrm{d}t = \int_J \langle u_t(t), v-u(t)\rangle_{V',V}\,\mathrm{d}t. \tag{39}$$

To prove (37), it suffices to note that $-\psi$ is concave, hence weakly-* upper-semicontinuous, and $u^{\varepsilon_n} \rightharpoonup u$ in $L^\infty(J;V)$.

To prove (38), note that

$$\int_J \langle\mathcal{H}u^{\varepsilon_n}(t), v-u^{\varepsilon_n}(t)\rangle_{V',V}\,\mathrm{d}t = \int_J \langle\mathcal{H}u^{\varepsilon_n}(t), v-u(t)\rangle_{V',V}\,\mathrm{d}t + \int_J \langle\mathcal{H}u^{\varepsilon_n}(t), u(t)-u^{\varepsilon_n}(t)\rangle_{V',V}\,\mathrm{d}t,$$

where

$$\lim_{n\to+\infty}\int_J \langle \mathcal{H}u^{\varepsilon_n}(t), v-u(t)\rangle_{V',V}\ \mathrm{d}t = \int_J \langle \mathcal{H}u(t), v-u(t)\rangle_{V',V}\ \mathrm{d}t, \tag{40}$$

since $\mathcal{H}$ is bounded and linear. To prove

$$\lim_{n\to+\infty}\int_J \langle \mathcal{H}u^{\varepsilon_n}(t), u(t)-u^{\varepsilon_n}(t)\rangle_{V',V}\ \mathrm{d}t = 0, \tag{41}$$

observe that, by Hölder's inequality and Lemma 6,

$$\begin{aligned}
&\left|\int_J \langle \mathcal{H}u^{\varepsilon_n}(t), u(t)-u^{\varepsilon_n}(t)\rangle_{V',V}\ \mathrm{d}t\right| \\
&\qquad \le \int_J \|H(\langle\nabla\nabla\cdot u^{\varepsilon_n}(t),\hat{r}\rangle)\|_U \|\nabla\cdot(u(t)-u^{\varepsilon_n}(t))\|_U\,\mathrm{d}t \\
&\qquad \le \frac{\pi}{2}\int_J \|\nabla\nabla\cdot u^{\varepsilon_n}(t)\|_U \|\nabla\cdot(u(t)-u^{\varepsilon_n}(t))\|_U\,\mathrm{d}t.
\end{aligned}$$

The sequence $u^{\varepsilon_n}(t)$ is uniformly bounded in V (by (34)), hence the sequence of norms $\|\nabla\nabla\cdot u^{\varepsilon_n}(t)\|_U$ is also uniformly bounded. Moreover, the embedding $V\hookrightarrow X$ is compact, where

$$X := \{u\in L^2(D;\mathbb{R}^2) : \nabla\cdot u\in L^2(D;\mathbb{R}), u \text{ is radial}, \langle u,\nu\rangle\equiv 0, \text{ on } \partial D\},$$

endowed with the norm $\|u\|_X := \|\nabla\cdot u\|_{L^2(D;\mathbb{R})}$. This because if a sequence $v_n\subseteq V$ is weakly converging in V, then both sequences $\nabla\nabla\cdot v_n$ and $\nabla\cdot v_n$ are weakly converging in $L^2(D;\mathbb{R}^2)$ and $L^2(D;\mathbb{R})$ respectively, hence $\nabla\cdot v_n$ is strongly converging in $L^2(D;\mathbb{R})$ (i.e., v_n is strongly converging in X). Thus

$$\|\nabla\nabla\cdot u^{\varepsilon_n}(t)\|_U\|\nabla\cdot(u(t)-u^{\varepsilon_n}(t))\|_U\to 0$$

for a.e. t. This proves (41). Combining (40) and (41) then gives (38).

To prove (39), note that

$$\int_J \langle u_t^{\varepsilon_n}(t), v-u^{\varepsilon_n}(t)\rangle_{V',V}\ \mathrm{d}t = \int_J \langle u_t^{\varepsilon_n}(t), v-u(t)\rangle_{V',V}\ \mathrm{d}t + \int_J \langle u_t^{\varepsilon_n}(t), u-u^{\varepsilon_n}(t)\rangle_{V',V}\ \mathrm{d}t,$$

and

$$\int_J \langle u_t^{\varepsilon_n}(t), v-u(t)\rangle_{V',V}\ \mathrm{d}t \to \int_J \langle u_t, v-u(t)\rangle_{V',V}\ \mathrm{d}t \tag{42}$$

since $u_t^{\varepsilon_n}\rightharpoonup u_t$ in $L^p(0,T;U)$. Note also that, by Hölder's inequality,

$$\int_J |\langle u_t^{\varepsilon_n}(t), u-u^{\varepsilon_n}(t)\rangle_{V',V}|\,\mathrm{d}t \le \int_J \|u_t^{\varepsilon_n}(t)\|_U\|u-u^{\varepsilon_n}(t)\|_U\,\mathrm{d}t,$$

where $\|u_t^{\varepsilon_n}(t)\|_U$ is uniformly bounded in view of (33), and $\|u-u^{\varepsilon_n}(t)\|_U\to 0$. Thus

$$\int_J \langle u_t^{\varepsilon_n}(t), u-u^{\varepsilon_n}(t)\rangle_{V',V}\ \mathrm{d}t \to 0,$$

and combining with (42) gives (39). Combining (37), (38), (39) and the arbitrariness of $J\subseteq[0,T]$ gives (24). □

3.2. **Proof of Theorem 1.** Before the proof of Theorem 1, a preliminary result is required.

Lemma 10. *For any* $h > 0$*, the operator* $\mathrm{id} + hB : \mathrm{dom}_U(B) \longrightarrow U'$ *is maximal monotone, and its graph is demi-closed.*

Note that although it is easy to check, using the maximal monotonicity of B, that $\mathrm{id} + hB$ is maximal monotone as functional from V to V', the same conclusion when considered as functional from U to U' is non trivial since $\mathrm{dom}_U(B) = \{u \in U : Bu \in U'\}$ is a proper subset of $\mathrm{dom}_V(B)$. As there is no inclusion relation between $U \times U'$ and $V \times V'$, proper extensions (which would contradict the maximality) of the graph of B in $U \times U'$ are not immediately excluded.

Proof. Lemma 5 gives that $B : V \longrightarrow V'$ is maximal monotone. The identity operator $\mathrm{id} : V \longrightarrow V'$ is hemi-continuous, bounded and monotone (thought not maximal), and by [11] the sum $\mathrm{id} + hB : V \longrightarrow V'$ is maximal monotone. Thus it is surjective, and

$$U' \subseteq V' = (\mathrm{id} + hB)(V) \subseteq (\mathrm{id} + hB)(U).$$

Thus for any $\eta' \in U'$ there exists $\eta \in V \subseteq U$ such that $\eta' = (\mathrm{id} + hB)(\eta)$, hence $\eta \in \mathrm{dom}_U(B)$, and the operator $\mathrm{id} + hB : \mathrm{dom}_U(B) \longrightarrow U'$ is surjective. By [3, 9] (see also [2, Theorem 2.2]) $\mathrm{id} + hB : \mathrm{dom}_U(B) \longrightarrow U'$ is also maximal monotone, and by [12, Remarks 3-4] its graph is demi-closed. □

Now we are ready to prove that the function u given by Proposition 9 is the desired solution given by the thesis of Theorem 1. The proof uses some ideas from [2]. However, crucial monotonicity estimates are achieved differently, since $\mathcal{H} + \partial\phi : V \longrightarrow V'$ is clearly not accretive.

Proof. (of Theorem 1) The proof is split in two parts: we first prove existence, and then uniqueness. For brevity, let $B := \mathcal{H} + \partial\phi$.

Existence. Let u be a solution of (24) given by Proposition 9. By construction, u_t is limit (in the weak-$*$ topology of $L^\infty(0,T;U)$) of a sequence $u_t^{\varepsilon_n}$ satisfying

$$\sup_n \|u_t^{\varepsilon_n}\|_{L^\infty(0,T;U)} \le \|Bu^0\|_U.$$

Thus

$$\|u_t\|_{L^\infty(0,T;U)} \le \liminf_{n\to+\infty} \|u_t^{\varepsilon}\|_{L^\infty(0,T;U)} \le \|Bu^0\|_U.$$

By construction, u also satisfies

$$u \in L^\infty(0,T;V) \cap C^0([0,T];U), \qquad u_t \in L^\infty(0,T;U), \qquad u(0) = u^0. \tag{43}$$

To conclude the proof of existence, it suffices to check that such u satisfies

$$u_t(t) = -Bu(t) \quad \text{for a.e. } t \in [0,T]. \tag{44}$$

Consider some $t > 0$ such that

$$u(t-h) = u(t) - hu_t(t) - hg(h), \qquad 0 < h \ll t, \tag{45}$$

for some function $g(h)$ satisfying

$$\lim_{h\to 0} \|g(h)\|_U = 0. \tag{46}$$

In view of (43), the set of such t that satisfies (45) has full measure.

Observe that the restriction $\mathrm{id}+hB_{|\,\mathrm{dom}_U(B)}:\mathrm{dom}_U(B)\longrightarrow U$ is maximal monotone by Lemma 10. For brevity, in the rest of the proof we will just write $\mathrm{id}+hB$ instead of $\mathrm{id}+hB_{|\,\mathrm{dom}_U(B)}$. Then, since $\mathrm{id}+hB:\mathrm{dom}_U(B)\longrightarrow U$ is bijective, we set

$$x^h := (\mathrm{id}+hB)^{-1}u(t-h)\in \mathrm{dom}_U(B),$$

hence $u(t-h)=(\mathrm{id}+hB)x^h$, and plugging into (45) gives

$$u(t)-x^h = h[Bx^h+u_t(t)+g(h)]. \tag{47}$$

Multiplying both sides of (47) by $u(t)-x^h$ then yields

$$\langle u(t)-x^h,u(t)-x^h\rangle_{V',V} = h\langle Bx^h+u_t(t),u(t)-x^h\rangle_{U',U}+h\langle g(h),u(t)-x^h\rangle_{V',V}. \tag{48}$$

We claim

$$\langle Bx^h+u_t(t),u(t)-x^h\rangle_{U',U}\le 0. \tag{49}$$

Since u is a solution of (24), taking $v=x^h$ gives

$$\langle u_t(t),x^h-u(t)\rangle_{U',U}+\langle\tilde{\mathcal{H}}u(t),x^h-u(t)\rangle_{V',V}+\psi(x^h)-\psi(u(t))\ge 0,$$

with ψ defined in (17) and $\tilde{\mathcal{H}}:=\mathcal{H}+\partial\phi-\partial\psi$. Recall that in the proof of Lemma 7 we proved that $\tilde{\mathcal{H}}$ is linear and monotone (in particular, using the same notations from the proof of Lemma 7 we have $\phi-\psi=\chi$, and $\partial\chi(u)=2\nabla\nabla\cdot[\hat r\langle\nabla\nabla\cdot u,\hat r\rangle]$). Thus, by the convexity of ψ and the monotonicity of $\tilde{\mathcal{H}}$, we get

$$\begin{aligned}
0&\le \langle u_t(t),x^h-u(t)\rangle_{U',U}+\langle\tilde{\mathcal{H}}u(t)+\partial\psi(x^h),x^h-u(t)\rangle_{V',V}\\
&=\langle u_t(t),x^h-u(t)\rangle_{U',U}+\langle\tilde{\mathcal{H}}x^h+\partial\psi(x^h),x^h-u(t)\rangle_{V',V}+\langle\tilde{\mathcal{H}}u(t)-\tilde{\mathcal{H}}x^h,x^h-u(t)\rangle_{V',V}\\
&\le \langle u_t(t),x^h-u(t)\rangle_{U',U}+\langle\tilde{\mathcal{H}}x^h+\partial\psi(x^h),x^h-u(t)\rangle_{V',V}\\
&=\langle u_t(t),x^h-u(t)\rangle_{U',U}+\langle Bx^h,x^h-u(t)\rangle_{V',V},
\end{aligned}$$

proving (49). Thus (48) gives

$$\langle u(t)-x^h,u(t)-x^h\rangle_{V',V}=h\langle Bx^h+u_t(t)+g(h),u(t)-x^h\rangle_{V',V}\le h\langle g(h),u(t)-x^h\rangle_{V',V},$$

hence $\dfrac{\|u(t)-x^h\|_U}{h}\to 0$ as $h\to 0$. Note that, by construction, we have

$$Bx^h=\frac{u(t-h)-x^h}{h}=\frac{u(t)-x^h}{h}+\frac{u(t-h)-u(t)}{h},$$

thus

$$Bx^h=\frac{u(t-h)-x^h}{h}=\frac{u(t)-x^h}{h}+\frac{u(t-h)-u(t)}{h}\to -u_t(t)\qquad\text{weakly in } U.$$

Since we proved that $x^h\to u(t)$, $Bx^h\to -u(t)$, and

$$\{(w,Bw): w\in\mathrm{dom}_U(B), Bw\in U\}$$

is demi-closed in $U\times U'$, we infer (by [12, Theorem 1] and [12, Remarks 3-4]) $Bu(t)=-u(t)$. Since this argument holds for a.e. $t\in[0,T]$, (44) is proven.

Uniqueness. Let u^1, u^2 be two solutions of $u_t=-Bu$, with $u^1(0)=\bar u^1$, $u^2(0)=\bar u^2$. Thus

$$\begin{aligned}
\frac12\frac{d}{dt}\|u^1(t)-u^2(t)\|_U^2&=\langle u^1_t(t)-u^2_t(t),u^1(t)-u^2(t)\rangle_{V',V}\\
&=-\langle Bu^1(t)-Bu^2(t),u^1(t)-u^2(t)\rangle_{V',V}\\
&\le -\|u^1(t)-u^2(t)\|_V^2\le -\|u^1(t)-u^2(t)\|_U^2,
\end{aligned}$$

with the last line following from Corollary 8. Gronwall's lemma thus gives

$$\|u^1(t) - u^2(t)\|_U^2 \leq \|\bar{u}^1 - \bar{u}^2\|_U^2 e^{-t}.$$

In particular, if $u^1(0) = u^2(0) = u^0$, then $\|u^1(t) - u^2(t)\|_U^2 = 0$ for a.e. t. □

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References

[1] Auchmuty, G.: *Divergence L^2-coercivity inequalities*, Numer. Funct. Anal. Opt., vol. 27(5-6), pp. 499–515, 2006

[2] Barbu V.: *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoof, Leydon, 1976

[3] Browder F., *Problèmes Nonlinéaires*, Les Presses de l'Université de Montréal, 1966

[4] Dal Maso G., Fonseca I. and Leoni G.: *Analytical validation of a continuum model for epitaxial growth with elasticity on vicinal surfaces*, Arch. Rational Mech. Anal., vol. 212, pp. 1037–1064, 2014

[5] Duport, C., Politi, P. and Villain, J.: *Growth instabilities induced by elasticity in a vicinal surface. Journal de Physique I* vol. 1(5), pp. 1317–1350, 1995

[6] Fonseca I., Leoni G. and Lu, X.Y.: *Regularity in time for weak solutions of a continuum model for epitaxial growth with elasticity on vicinal surfaces*, Commun. Part. Diff. Eq., vol. 40(10), pp. 1942–1957, 2015

[7] Gao Y., Liu J.-G. and Lu J.: *Continuum limit of a mesoscopic model of step motion on vicinal surfaces*, Preprint (arXiv:1606.08060), 2016

[8] Kačur, J.: *Method of Rothe in evolution equations*, Leipzig: Teubner Verlaggesellschaft, 1985

[9] Minty G., *Monotone (nonlinear) operators in Hilbert spaces*, Duke Math. J., vol. 29, pp. 341–346, 1962

[10] Payne, L.E. and Weinberger, H.F.: *An optimal Poincaré inequality for convex domains*, Arch. Rational Mech. Anal., vol. 5(1), pp. 286–292, 1960

[11] Rockafellar, R.T.: *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc., vol. 149, pp. 75–88, 1970

[12] Suryanarayana M.B.: *Monotonicity and upper semicontinuity*, Bull. Amer. Math. Soc., vol. 82(6), pp. 936–938, 1976

[13] Tersoff, J., Phang, Y.H., Zhang, Z. and Lagally, M.G.: *Step-bunching instability of vicinal surfaces under stress.* Physical Review Letters, vol. 75, pp. 2730–2733, 1995

[14] Xiang, Y.: *Derivation of a continuum model for epitaxial growth with elasticity on vicinal surface.* SIAM Journal on Applied Mathematics, vol. 63, pp. 241–258, 2002

[15] Xiang, Y. and E, W.: *Misfit elastic energy and a continuum model for epitaxial growth with elasticity on vicinal surfaces.* Physical Review B vol. 69, pp. 035409-1–035409-16, 2004

[16] Xu, H. and Xiang, Y.: *Derivation of a continuum model for the long-range elastic interaction on stepped epitaxial surfaces in 2+1 dimensions.* SIAM Journal on Applied Mathematics vol. 69(5), pp. 1393–1414, 2009

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