SINGULAR PARABOLIC EQUATIONS, MEASURES SATISFYING DENSITY CONDITIONS, AND GRADIENT INTEGRABILITY

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ABSTRACT. We consider solutions to singular parabolic equations with measurable dependence on the (x, t) variables and having on the right-hand side a measure satisfying a density condition. We prove that the less the measure is concentrated, the more the gradient is regular, in the Marcinkiewicz scale. We provide local estimates and recover some classic results.

To Nicola Fusco, on the occasion of his 60th birthday, with admiration and respect.

1. INTRODUCTION, ASSUMPTIONS, STATEMENTS

The aim of this paper is to give a natural integrability result for solutions to singular parabolic equations with measure data: we consider problems of the type

$$u_t - \operatorname{div} a(x, t, Du) = \mu \qquad \text{in } \Omega_T := \Omega \times (-T, 0), \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^n$, $n \geq 2$, and the vector field $a(\cdot)$ satisfies only minimal growth and monotonicity assumptions:

$$\begin{cases} \langle a(x,t,\xi_1) - a(x,t,\xi_2), \xi_1 - \xi_2 \rangle \geq \nu\big(|\xi_1| + |\xi_2|\big)^{p-2}|\xi_1 - \xi_2|^2, \\ |a(x,t,\xi)| \leq L\,|\xi|^{p-1}, \end{cases} \tag{1.2}$$

for almost every $(x, t) \in \Omega_T$, every $\xi_1, \xi_2, \xi \in \mathbb{R}^n$, with $0 < \nu \leq 1 \leq L$. The most prominent model we have in mind for (1.1) is the singular parabolic p-Laplacian equation with measurable coefficients. Being $N := n + 2$ the parabolic dimension, the exponent p is assumed to satisfy

$$2 - \frac{1}{N-1} < p < 2; \tag{1.3}$$

we are hence considering singular parabolic equations, following DiBenedetto [17, Chapters IV, VII, VIII]. In this note μ will be a signed Borel measure with finite total mass, in general not belonging to the dual of the energy space naturally associated to the operator on the left-hand side: in view of this fact the lower bound in (1.3) is natural in the whole theory (see [12, 13, 22, 25]) since it ensures the existence of a solution with $Du \in L^1(\Omega_T)$.

The phenomenon object of this investigation is *the improvement of integrability for the gradient* of solutions of (1.1) in the case the measure on the right-hand side satisfies certain density-type conditions. This is a general fact and it was first noted in [30] in the elliptic case, see the forthcoming lines for a detailed presentation of these results. The analog results in the parabolic direction can be found in [7, 6, 4]; the first two contributions include results in the non-degenerate case (that is, for vector fields satisfying (1.2) with $p = 2$) while the last contribution deals with the more difficult degenerate parabolic case

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($p > 2$), which needs very different techniques with respect to both the elliptic and the non-degenerate cases. To be more specific, for a signed Borel measure μ of finite total mass, one is lead to consider a Morrey-type condition on standard parabolic cylinders as follows:

$$\sup_{Q_R(z_0)\subset\Omega_T} \frac{|\mu|(Q_R(z_0))}{R^{N-\vartheta}} \le c_d < \infty, \tag{1.4}$$

where $Q_R(z_0) = B_R(x_0)\times(t_0 - R^2, t_0 + R^2)$: note that $|Q_R| = c(n)R^N$. This condition can be naturally extended to L^1 functions by setting $|\mu|(A) := \int_A |\mu|\,dx$ when A is a measurable set.

The improved integrability of the gradient is formulated in terms of Marcinkiewicz spaces, which are the optimal ones to be used when dealing with measure data problems (this is natural in view of the behavior of the fundamental solution in the elliptic case), see for instance [8, 15, 30, 31]: summarizing the results of [7, 4], we have that

$$\mu \text{ satisfies (1.4) for some } 2 \le \vartheta \le N \quad \Longrightarrow \quad Du \in \mathcal{M}^{p-1+\frac{1}{\vartheta-1}}_{\text{loc}}(\Omega_T, \mathbb{R}^n), \tag{1.5}$$

where u is a solution to (1.1)-(1.2) (in the sense described after Definition 1.2) with $p \ge 2$. We recall the reader that Marcinkiewicz spaces $\mathcal{M}^m(\Omega_T, \mathbb{R}^\ell)$ are defined via the following decay condition on level sets (for $f : \Omega_T \to \mathbb{R}^\ell$ a measurable map):

$$\sup_{\lambda>0} \lambda^m \big|\{z \in \Omega_T : |f(z)| > \lambda\}\big| =: \|f\|^m_{\mathcal{M}^m(\Omega_T,\mathbb{R}^\ell)} < \infty.$$

Their local variant is defined in the usual way.

The improvement of integrability in the case of Morrey data for the elliptic case has been presented in [30]:

$$\mu \in L^{1,\vartheta}(\Omega), \quad p \le \vartheta \le n \qquad \Longrightarrow \qquad Du \in \mathcal{M}^{\frac{\vartheta(p-1)}{\vartheta-1}}_{\text{loc}}(\Omega, \mathbb{R}^n).$$

$L^{1,\vartheta}(\Omega)$ is here the space of signed Borel measures satisfying

$$\sup_{B_R(x_0)\subset\Omega} \frac{|\mu|(B_R(x_0))}{R^{n-\vartheta}} \le c_d < \infty, \tag{1.6}$$

Note that for $\vartheta < p$ a classic result of Hedberg and Wolff states that $L^{1,\vartheta}$ embeds into the dual space of $W^{1,p}$. In the case $\vartheta = n$, the results in [30] give back the classic, sharp result

$$\mu \in L^{1,n}(\Omega) \equiv \mathcal{M}_b(\Omega) \qquad \Longrightarrow \qquad Du \in \mathcal{M}^{\frac{n(p-1)}{n-1}}(\Omega, \mathbb{R}^n),$$

for which we refer to [8, 15, 18]; notice that the class of measures satisfying (1.4) for $\vartheta = n$ is nothing else than the full space of signed Borel measures with finite total mass $\mathcal{M}_b(\Omega_T)$.

In this paper we prove the following singular counterpart of the result described in (1.5):

Theorem 1.1. *If $u \in V^{2,p}(\Omega_T)$ is a weak solution to (1.1) with $\mu \in L^1(\Omega_T)$ satisfying (1.4) for some $p \le \vartheta \le N$, then:*

- *if $p \le \vartheta \le n$, then $Du \in \mathcal{M}^{m_1}_{\text{loc}}(\Omega_T, \mathbb{R}^n)$ with*

$$m_1 := (p-1)\frac{\vartheta}{\vartheta-1};$$

- *if $n < \vartheta \le N$, then $Du \in \mathcal{M}^{m_2}_{\text{loc}}(\Omega_T, \mathbb{R}^n)$ with*

$$m_2 := \frac{1}{2}\Big(p - \frac{(2-p)n}{\vartheta}\Big)\frac{\vartheta}{\vartheta-1}.$$

Moreover, there exists a constant depending on n, p, ν, L and c_d such that the following local estimate holds for any parabolic cylinder $Q_{2R} \equiv Q_{2R}(z_0) \subset \Omega_T$:

$$R^{-\frac{N}{m}} \|Du\|_{\mathcal{M}^m(Q_R,\mathbb{R}^n)} \le c \left[\fint_{Q_{2R}} (|Du|+1)\,dz + \left[\frac{|\mu|(Q_{2R})}{|Q_{2R}|} \right]^{\frac{1}{m}} \right]^d; \tag{1.7}$$

the scaling deficit $d > 1$ is defined as

$$d := \frac{2}{2 - n(2-p)} \tag{1.8}$$

and

$$m := \left(p - 1 + \left[\frac{2-p}{2} \left(1 - \frac{n}{\vartheta} \right) \right]_+ \right) \frac{\vartheta}{\vartheta - 1}. \tag{1.9}$$

Note that $m = m_1$ if $\vartheta \le n$ and $m = m_2$ in the remaining case $\vartheta > n$.

Several remarks are now in order.

First of all, note that $V^{2,p}(\Omega_T)$ is the energy space for an operator satisfying (1.2) with parabolic p-growth, see (2.5) and (2.7). As in Theorem 1.1, we are going to state every result as *a priori* estimate for energy solution in view of (part of) the existence theory for problems as (1.1). Moreover we assume that $\mu \in L^1(\Omega_T)$ for technical reasons; we will show that this is not restrictive. In the general case of true measure data problems, one has the following definition:

Definition 1.2. *A very weak solution to* (1.1)*, under the structural assumptions* (1.2)-(1.3) *and being μ a signed Borel measure, is a function $u \in V^{1,1}(\Omega_T)$ (see the forthcoming* (2.5)*) such that the distributional formulation*

$$\int_{\Omega_T} \Big[-u\,\varphi_t + \langle a(x,t,Du), D\varphi \rangle \Big]\,dz = \int_{\Omega_T} \varphi\,d\mu, \tag{1.10}$$

holds true for every $\varphi \in C_c^\infty(\Omega_T)$.

Since the seminal works of Boccardo and Gallouet [12, 13], a solution to (1.1) (coupled with Cauchy-Dirichlet data) is usually found using an approximation procedure: one regularizes the datum μ (usually via mollification) and considers the energy solutions to the regularized problems (note that the Morrey condition in (1.4) remains essentially preserved under mollification, see [33]). Finally, one proves a.e. convergence of the gradients up to sub-sequences, and this allows to pass to the limit both in the weak formulation, obtaining a solution of (1.1) (see Definition 1.2), and in the estimates. This strategy leads to a *particular type of solution* usually called *Solution Obtained by Limits of Approximations*, *SOLA* in short. This is the reason why we can state our result in the form of a priori estimates; needless to say, *all our estimates will still hold (in a slightly different form) for SOLAs in the case μ is a signed Borel measure* satisfying (1.4), see [24, Paragraph 1.4] or [27, Section 1.2] for the necessary adaptations.

In the general case, where μ is a signed Borel measure with finite mass, the archetypal result is the existence of a SOLA, distributional solution (in the sense of Definition 1.2) to (1.1), for $p > 2 - 1/(N-1)$, such that

$$Du \in L^q(\Omega_T, \mathbb{R}^n) \qquad \text{for all } q \in [1, m_0), \qquad m_0 := p - 1 + \frac{1}{N-1}. \tag{1.11}$$

Note that in general the Cauchy-Dirichlet problem associated (1.1) does not have uniqueness; hence the SOLA is just *one solution* to the problem considered. Uniqueness holds only in special cases, as for instance $p = 2$ or $p = n$ in the elliptic case; for a comprehensive account of several results in this field, we refer to [32]. Moreover, also in the class of SOLA uniqueness is not guaranteed, in the sense that different approximation of μ could lead in general to different limit solutions. We will not go into details in those questions and we, again, refer to [20, 26, 32] for the elliptic case and to [4, 24, 25, 27] for the parabolic.

Note that if $d = 1$, then the estimate in (1.7) would be homogeneous and invariant under parabolic scaling. The fact that $d > 1$ if $p \neq 2$ is a well-known phenomenon that follows directly from the lack of natural homogeneous scalings for the equation. The explicit value in (1.8) is moreover characteristic when dealing with singular parabolic equations with measure data; for instance, observe that it also appears in [22, 25]. It is indeed naturally dictated by the lack of scaling of the equation together with the fact that we are forced to work below the energy level, and in particular with the L^1 norm of the gradient, since we are considering the singular case; see (5.6). This fact is in turn symptomatic of the fact that we need to produce estimates stable when passing to the limit, in view of what described above.

We remark here that $m_1 \geq m_2$ if $\vartheta \leq n$ while the inequality is reversed for $\vartheta > n$, since here $p < 2$. Moreover $\max\{m_1, m_2\} \leq m_0$ being m_0 defined in (1.11), since $p < 2$ and $\vartheta \leq N$; therefore the regularity of the gradient in the singular case is worse if compared with the one in the degenerate one. It is not clear if this fact is a consequence of the singular structure of the equation or it is a limit of the technique of our proof; this will be object of future investigations. One can on the other hand "force" the regularity $Du \in \mathcal{M}^{m_0}_{\mathrm{loc}}(\Omega_T, \mathbb{R}^n)$ in the singular case by artificially imposing an intrinsic condition of the type

$$\sup_{\substack{Q^\lambda_R(z_0) \subset \Omega_T \\ R>0, \lambda \geq 1}} \frac{|\mu|(Q^\lambda_R(z_0))}{[R^\lambda]^{N-\vartheta}} \leq c_d,$$

where the stretched cylinders $Q^\lambda_R(z_0)$ will be defined in (2.1); compare with (1.4). This condition is on the other hand difficult to verify in concrete cases and therefore we leave this remark just as a theoretical curiosity.

Note that once one takes $\vartheta = N$, (1.4) is satisfied for every signed Borel measure with finite mass. Thus this condition is essentially empty but we still get back (locally) the sharp improvement of the result in (1.11), which was already obtained in [3]; we also provide a natural local estimate.

Corollary 1.3. *If $u \in V^{2,p}(\Omega_T)$ is a weak solution to equation (1.1) with $\mu \in L^1(\Omega_T)$, then $Du \in \mathcal{M}^{m_0}_{\mathrm{loc}}(\Omega_T, \mathbb{R}^n)$. The local estimate coincides with (1.7) for $\vartheta = N$ and $m = m_0$.*

Remark 1.4. We stress now that we restrict to the case $\vartheta \geq p$ for simplicity. One could consider also $\vartheta < p$ but still "close" to p (see (5.29): m should only be smaller than $p(1 + \eta)$, with $\eta > 0$ depending on the data of the problem); we refer to the results and the discussion in [4]. Note that as soon as $\vartheta < p$, then $Du \in L^p_{\mathrm{loc}}(\Omega_T)$. The fact that the natural lower bound from below for ϑ differs from that appearing for degenerate equations is liked to the fact that the energy space in the singular case differs from that in the degenerate one, see [28, Page 166]. This will be object of future investigation.

An improvement of Theorem 1.1 can be obtained once considering more regular vector fields. In particular, if we replace the assumption of simple measurability and boundedness with respect to the (x, t) variables with a sort nonlinear version of VMO regularity (considered for instance in [11, 10, 23]), then we can consider values of ϑ close to one and at the same time obtain integrability of Du in any Lebesgue space. In particular, we consider now Carathéodory vector fields $a(x, t, \xi)$ differentiable with respect to the variable ξ and satisfying

$$\begin{cases} \langle \partial_\xi a(x,t,\xi)\tilde{\xi},\tilde{\xi}\rangle \geq \nu|\xi|^{p-2}|\tilde{\xi}|^2, \\ |a(x,t,\xi)| + |\partial_\xi a(x,t,\xi)|\,|\xi| \leq L\,|\xi|^{p-1}, \end{cases} \tag{1.12}$$

for all $x \in \Omega, t \in (-T,0), \xi, \tilde{\xi} \in \mathbb{R}^n$, with p as in (1.3) and structural constants $0 < \nu \leq L < \infty$. For what concerns the regularity with respect to the variable x, we assume that, denoting for $R > 0$ the nonlinear modulus of continuity

$$\omega(R) = \sup_{t\in(-T,0)} \sup_{\substack{B_r(x_0)\subset\Omega \\ 0<r\leq R}} \sup_{\substack{\xi\in\mathbb{R}^n \\ \xi\neq 0}} \frac{|a(x,t,\xi) - (a)_{B_r(x_0)}(t,\xi)|}{|\xi|^{p-1}},$$

we have

$$\lim_{R\searrow 0} \omega(R) = 0. \tag{1.13}$$

$(a)_{B_r(x_0)} : (-T,0)\times\mathbb{R}^n \to \mathbb{R}^n$ denotes the averaged vector field

$$(a)_{B_r(x_0)}(t,\xi) := \fint_{B_r(x_0)} a(x,t,\xi)\,dx$$

for $t \in (-T,0)$ and $\xi \in \mathbb{R}^n$. Note that a particular but fundamental instance of vector fields satisfying the assumptions in (1.12)-(1.13) are those VMO regular with respect to the space variables but only measurable and bounded with respect to time: this is to say, those of the form $a(x,t,\xi) = b(x)\tilde{a}(t,\xi)$ with $\tilde{a}(\cdot)$ a Carathéodory map, with $(t,\xi)\mapsto \partial_\xi\tilde{a}(t,\xi)$ Carathéodory regular too and satisfying

$$\begin{cases} \langle \partial_\xi \tilde{a}(t,\xi)\tilde{\xi},\tilde{\xi}\rangle \geq \sqrt{\nu}|\xi|^{p-2}|\tilde{\xi}|^2, \\ |a(t,\xi)| + |\partial_\xi a(t,\xi)|\,|\xi| \leq \sqrt{L}\,|\xi|^{p-1}, \end{cases} \tag{1.14}$$

for all $t \in (-T,0)$, $\xi,\tilde{\xi} \in \mathbb{R}^n$, p as in (1.3) and with ν, L as in (1.12) and $b : \Omega \to \mathbb{R}$ bounded from zero and infinity and VMO regular, i.e. $\sqrt{\nu} \leq b(\cdot) \leq \sqrt{L}$ and

$$\lim_{R\searrow 0} \omega(R) = 0, \quad \text{where} \quad \omega(R) := \sup_{\substack{B_r(x_0)\subset\Omega \\ 0<r\leq R}} \fint_{B_r(x_0)} \left|b - (b)_{B_r(x_0)}\right| dx.$$

In this case we have the following

Theorem 1.5. *If $u \in V_0^{2,p}(\Omega_T)$ is a weak solution to equation (1.1) under the assumptions in (1.12)-(1.13), with $\mu \in L^1(\Omega_T)$ satisfying the Morrey condition (1.4) for some $1 < \vartheta \leq n$, then*

$$Du \in \mathcal{M}^{m_1}_{\text{loc}}(\Omega_T, \mathbb{R}^n).$$

Splitting measures. We conclude this introduction with two results about splitting data. For the first one we assume that μ is of the following product-type:

$$\mu = \mu_1\cdot\mu_2, \qquad \mu_1 \in L^\infty(\Omega), \qquad |\mu_2|\big((\tau - R^2, \tau + R^2)\big) \leq c_d R^{2-\vartheta} \tag{1.15}$$

for every sub-interval $(\tau - R^2, \tau + R^2) \subset (-T,0)$ and for some $1 < \vartheta < 2$. In this case we can deduce a stronger result:

Theorem 1.6 (Elliptic/singular-parabolic regularity)**.** *If $u \in V^{2,p}(\Omega_T)$ is a weak solution to equation (1.1) under the assumptions (1.12)-(1.13) with p satisfying (1.3) and μ being as in (1.15), then*

$$Du \in \mathcal{M}^{m_3}_{\text{loc}}(\Omega_T, \mathbb{R}^n), \qquad \textit{where} \quad m_3 := \frac{p}{2}\frac{\vartheta}{\vartheta - 1}.$$

Again, for any $Q_{2R}(z_0) \subset \Omega_T$ a local estimate analogous to (1.7), with m_3 replacing m, holds true. The constant here does also depends on the Morrey constant c_d of μ_2 and on $\|\mu_1\|_{L^\infty}$.

Finally, for completeness, we present the analog of Theorem 1.6 for degenerate equations, which was not included in [4]. We assume in this case

$$\mu = \mu_3 \cdot \mu_4, \qquad \mu_3 \in L^1 \cap L^{1,\vartheta}(\Omega), \quad p < \vartheta < n; \qquad \mu_4 \in L^\infty(-T, 0), \tag{1.16}$$

with $L^{1,\vartheta}(\Omega)$ the elliptic Morrey space defined in (1.6).

Theorem 1.7 (Elliptic/degenerate-parabolic regularity). *If $u \in V^{2,p}(\Omega_T)$ is a weak solution to equation (1.1) with $2 \le p < n$ in (1.2) and μ being as in (1.16), then*

$$Du \in \mathcal{M}^{m_4}_{\rm loc}(\Omega_T, \mathbb{R}^n), \qquad \textit{where} \quad m_4 := (p-1)\frac{\vartheta}{\vartheta - 1} = m_1.$$

A local estimate similar to that in Theorem 1.6 holds true for any $Q_{2R}(z_0) \subset \Omega_T$, with the constant also depending on the Morrey constant of μ_3 and on $\|\mu_4\|_{L^\infty}$.

Note here in both cases μ satisfies (1.15) or (1.16) in particular we have that μ satisfies (1.4); however, $m_3 \ge m_1$ ($p < 2$) and $m_4 \ge m_0$ ($p \ge 2$); thus in both cases a more careful analysis of the geometry of the problem allows to obtain an improved integrability for the gradient of the solution. Moreover note that in order to be able to consider non-trivial cases, in Theorem 1.6 we chose to consider more regular vector fields as in (1.12), to allow for $\vartheta < 2$; however also in this case Remark 1.4 applies. Finally, we stress that assuming (1.15) in the degenerate case and (1.16) in the singular one does not lead to any better estimates using our approach, and this can be seen by our proofs.

2. Notation

This section is devoted to fix the notation we will use in the rest of the paper. $\mathbb{R}^{n+1}$ will always be thought as $\mathbb{R}^n \times \mathbb{R}$, so a point $z \in \mathbb{R}^{n+1}$ will be often also denoted as (x, t), z_0 as (x_0, t_0), and so on. The cylinders $Q^\lambda_R(z_0)$, for a scaling parameter $\lambda \ge 1$, are the natural cylinders associated to the equation in (1.1): they are defined as

$$\begin{aligned} Q^\lambda_R(z_0) &:= B_{R^\lambda}(x_0) \times (t_0 - R^2, t_0 + R^2) \\ &= B_{R^\lambda}(x_0) \times \big(t_0 - \lambda^{2-p}[R^\lambda]^2, t_0 + \lambda^{2-p}[R^\lambda]^2\big) \end{aligned} \tag{2.1}$$

with

$$R^\lambda := \lambda^{\frac{p-2}{2}} R.$$

R_λ is therefore the "spatial radius" of Q^λ_R; observe that $|Q^\lambda_R(z_0)| = c(n)\lambda^{2-p}[R^\lambda]^N$.

We recall the reader the different definition of the cylinders $Q^\lambda_R(z_0)$ for $\lambda \ge 1$ in the degenerate $p \ge 2$ case: we set

$$Q^\lambda_R(z_0) := B_R(x_0) \times \big(t_0 - \lambda^{2-p}R^2, t_0 + \lambda^{2-p}R^2\big) \tag{2.2}$$

and here we have $|Q^\lambda_R(z_0)| = c(n)\lambda^{2-p}R^N$.

Note that (in both cases), since we are always going to consider scaling parameters $\lambda \ge 1$, then

$$Q^\lambda_R(z_0) \subset Q_R(z_0). \tag{2.3}$$

Moreover note that for λ fixed, scaled cylinders are the balls of the metric given by the distance

$$d_\lambda(z_1, z_2) := \max\Big\{\lambda^{\frac{2-p}{2}}|x_1 - x_2|, \sqrt{|t_1 - t_2|}\Big\}. \tag{2.4}$$

When dealing with a several cylinders or Euclidean balls in the same context, if not otherwise stated, they will all have the same "vertex". For $\alpha > 0$ we shall write

$$\alpha Q_R^\lambda(x_0,t_0) := B_{\alpha R^\lambda}(x_0) \times (t_0 - (\alpha R)^2, t_0 + (\alpha R)^2).$$

By parabolic boundary of a cylindrical set $\mathcal{K} := A \times I$, with $A \subset \Omega$, $I \subset (-T,0)$, we will mean $\partial_{\mathcal{P}}\mathcal{K} := \overline{A} \times \{\inf I\} \cup \partial A \times I$. Being $C \in \mathbb{R}^m$ a measurable set with positive measure and $f : C \to \mathbb{R}^k$ an integrable map, with $m, k \geq 1$, we denote with $(f)_C$ the averaged integral

$$(f)_C := \fint_C f(\xi)\, d\xi := \frac{1}{|C|}\int_C f(\xi)\, d\xi.$$

Moreover for a measurable function $g : \mathcal{K} = A \times I \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^k$ over a cylinder we will use the notation

$$(g)_A(t) := \fint_A g(x,t)\, dx \qquad \text{for all } t \in I.$$

c will denote a generic constant greater than one, possibly varying from line to line. Constants we need to recall will be denoted with special symbols, such as $c_1, c_2, \tilde{c}, c_*$. Relevant dependencies will be highlighted between parentheses or after the equations; when non essential, the dependence on a parameter will be suppressed. If for a constant we will not mention its dependencies, then it will be a numerical constant.

For a cylinder $\mathcal{K} := A \times I \subset \mathbb{R}^n \times \mathbb{R}$, with $V^{\gamma,r}(\mathcal{K})$, $\gamma, r \geq 1$, we denote the spaces

$$V^{\gamma,r}(\mathcal{K}) := L^r(I; W^{1,r}(A)) \cap C(\overline{I}; L^\gamma(A)). \tag{2.5}$$

and

$$V_0^{\gamma,r}(\mathcal{K}) := L^r(I; W_0^{1,r}(A)) \cap C(\overline{I}; L^\gamma(A)).$$

Note that if $g \in L^p(I, W^{1,p}(A))$, $p \geq 1$, then $x \mapsto g(x,t) \in W^{1,p}(A)$ for a.e. $t \in I$. Hence we will be allowed to use Poincaré's inequality slice-wise. Finally, we introduce the auxiliary vector field $V : \mathbb{R}^n \to \mathbb{R}^n$ by

$$V(\xi) := |\xi|^{\frac{p-2}{2}}\xi$$

whenever $\xi \in \mathbb{R}^n$; it turns out to be a bijection of $\mathbb{R}^n$ and to encode the monotonicity properties of the vector field in (1.2): indeed it holds that

$$\frac{1}{c} \leq \frac{|V(\xi_1) - V(\xi_2)|^2}{(|\xi_1| + |\xi_2|)^{p-2}|\xi_1 - \xi_2|^2} \leq c$$

for $c \equiv c(p)$ and for all $\xi_1, \xi_2 \in \mathbb{R}^n$ not both zero if $s = 0$ (see [1, Lemma 2.3]); thus

$$\langle a(x,t,\xi_1) - a(x,t,\xi_2), \xi_1 - \xi_2 \rangle \geq \frac{1}{c(\nu,p)}\left|V(\xi_1) - V(\xi_2)\right|^2. \tag{2.6}$$

Moreover it holds

$$\langle a(x,t,\xi), \xi \rangle \geq c^{-1}\, |\xi|^p. \tag{2.7}$$

3. Estimates for homogeneous problems

Let us consider $v \in C^0(I; L^2(A)) \cap L^p(I; W^{1,p}(A))$ a solution to the problem

$$v_t - \operatorname{div} a(x,t,Dv) = 0 \qquad \text{in } A \times I \subset \Omega_T, \tag{3.1}$$

being A a domain, I an interval and with $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}$ the vector field appearing in (1.1), therefore satisfying (1.2)-(1.3) and (2.7).

In this section we collect some regularity results for weak solutions to (3.1).

The next Lemma is the standard energy estimate for solutions to (3.1); for its proof see [17, Chapter II, Proposition 3.1] or [21, Lemma 3.2].

Lemma 3.1 (Caccioppoli's inequality)**.** *Let $v \in V^{2,p}(A \times I)$ be a weak solution to (3.1) and let $Q_{\rho_2,\sigma_2} \equiv Q_{\rho_2,\sigma_2}(z_0) \subset A \times I$ be a cylinder. Then*

$$\sup_{t\in(t_0-\sigma_1,t_0+\sigma_1)} \int_{B_{\rho_1}(x_o)} |v(\cdot,t)-k|^2\,dx + \int_{Q_{\rho_1,\sigma_1}} |Dv|^p\,dz \leq c\int_{Q_{\rho_2,\sigma_2}} \left[\frac{|v-k|^p}{(\rho_2-\rho_1)^p} + \frac{|v-k|^2}{\sigma_2-\sigma_1}\right] dz \tag{3.2}$$

holds for all concentric cylinders $Q_{\rho_1,\sigma_1} \equiv Q_{\rho_1,\sigma_1}(z_0) \Subset Q_{\rho_2,\sigma_2}(z_0)$ and for all $k \in \mathbb{R}$; the constant depends only on n, p, ν and L.

The following is the $\sup$ bound for solutions to singular parabolic equations. It can be found in [34]; the local estimate we use here is in [17, Chapter V, Theorem 5.1]. Note in particular that we have by parabolic Sobolev's embedding and our assumption (1.3) that $w \in V^{2,p}(Q_1)$ implies $w \in L^1(Q_1)$; moreover (1.3) ensures that $n(p-2)+p > 0$. Recall that a sub-solution is a function such that the left-hand side of the weak formulation of (3.1) is non-positive, for every *positive* test function.

Proposition 3.2. *Any positive sub-solution $w \in V^{2,p}(A \times I)$ to (3.1) in $A \times I$ is locally bounded. Moreover, there exists a constant c depending only on n, p, ν, L such that the quantitative estimate*

$$\sup_{Q^\lambda_{3R/4}} w \leq c\Big(\big(R\,\lambda^{\frac{p}{2}}\big)^{n\frac{p-2}{p}} \fint_{Q^\lambda_R} |w|\,dz\Big)^{\frac{p}{n(p-2)+p}} + c\,R\,\lambda^{\frac{p}{2}}$$

holds for every cylinder $Q^\lambda_R \equiv Q^\lambda_R(z_0) \subset A \times I$.

Note that the lack of time regularity of solutions to (3.1) denies the possibility of using a standard Poincaré's inequality on cylinders. However, an amount of regularity can be retrieved by the equation itself; this allows to prove the following estimate. Similar estimates can be found in [4, 5, 35] and many other papers; for the proof, see the forthcoming Lemma 4.2 for $\mu \equiv 0$.

Lemma 3.3. *Let $v \in V^{2,p}(A \times I)$ be a solution to (3.1) in $A \times I$ and let $Q^\lambda_R \equiv Q^\lambda_R(z_0) \subset A \times I$ be a parabolic cylinder as in (2.1). Then*

$$\fint_{Q^\lambda_R} \left|\frac{v-(v)_{Q^\lambda_R}}{R^\lambda}\right| dz \leq c\fint_{Q^\lambda_R} |Dv|\,dz + c\,\lambda^{2-p}\Big(\fint_{Q^\lambda_R} |Dv|\,dz\Big)^{p-1}$$

for a constant c depending only on n, p and L.

We now have the following simple corollary:

Corollary 3.4. *Let v be as in Lemma 3.3 and moreover suppose that the intrinsic relation*

$$\fint_{Q^\lambda_R} |Dv|\,dz \leq \kappa\lambda$$

holds for a constant $\kappa \geq 1$. Then we have

$$\fint_{Q^\lambda_R} \left|\frac{v-(v)_{Q^\lambda_R}}{R^\lambda}\right| dz \leq c(n,p,L,\kappa)\,\lambda.$$

At a certain point of the proof we will need to compare the intrinsic geometry for our solution u and the one for a certain comparison map, solution to a homogeneous equation as the one in (3.1). For the first one, the intrinsic relation will involve the L^1 norm of the gradient, since we need estimates stables for very weak solutions. For the second one, the natural geometry will involve the L^p norm of the gradient instead. The following Proposition will show that the *weak* geometry for Dv is equivalent to the standard one (note indeed that the equivalent implication for the bound from below is trivial in view of Hölder's inequality).

Proposition 3.5 (Intrinsic reverse Hölder's inequality)**.** *Let v be the weak solution to (3.1). If*

$$\fint_{Q_R^\lambda} |Dv|\,dz \le \kappa\lambda \tag{3.3}$$

holds for a constant $\kappa \ge 1$, then

$$\fint_{Q_{R/2}^\lambda} |Dv|^p\,dz \le c\,\lambda^p,$$

with c depending on n, p, ν, L and κ.

Proof. From Caccioppoli's inequality (3.2) we have

$$\begin{aligned}
\fint_{Q_{R/2}^\lambda} |Dv|^p\,dz &\le c\left[\fint_{Q_{3R/4}^\lambda}\left(\left|\frac{v-(v)_{Q_{3R/4}^\lambda}}{R^\lambda}\right|^p + \left|\frac{v-(v)_{Q_{3R/4}^\lambda}}{R}\right|^2\right)dz\right]\\
&= c\left[\fint_{Q_{3R/4}^\lambda}\left|\frac{v-(v)_{Q_{3R/4}^\lambda}}{R^\lambda}\right|^p dz\right.\\
&\qquad\qquad \left. + \lambda^{p-2}\fint_{Q_{3R/4}^\lambda}\left|\frac{v-(v)_{Q_{3R/4}^\lambda}}{R^\lambda}\right|^2 dz\right]\\
&\le c\left[\frac{1}{[R^\lambda]^p}\left[\operatorname*{osc}_{Q_{3R/4}^\lambda} v\right]^p + \frac{\lambda^{p-2}}{[R^\lambda]^2}\left[\operatorname*{osc}_{Q_{3R/4}^\lambda} v\right]^2\right],
\end{aligned}$$

where $c \equiv c(n, p, \nu, L)$. To estimate the oscillation we now use Proposition 3.2: indeed both $(v-(v)_{Q_{3R/4}^\lambda})_+$ and $(v-(v)_{Q_{3R/4}^\lambda})_-$ are positive sub-solutions to (3.1) and so

$$\begin{aligned}
\operatorname*{osc}_{Q_{3R/4}^\lambda} v &\le c\left[\left(\left(R\,\lambda^{\frac{p}{2}}\right)^{\frac{n}{p}(p-2)}\fint_{Q_R^\lambda}\left|v-(v)_{Q_R^\lambda}\right|dz\right)^{\frac{p}{n(p-2)+p}} + R\,\lambda^{\frac{p}{2}}\right]\\
&= c\left[\left(\left(R\,\lambda^{\frac{p}{2}}\right)^{\frac{n}{p}(p-2)+1}\lambda^{-1}\fint_{Q_R^\lambda}\left|\frac{v-(v)_{Q_R^\lambda}}{R^\lambda}\right|dz\right)^{\frac{p}{n(p-2)+p}} + R\,\lambda^{\frac{p}{2}}\right]\\
&\le c\left[\left(\left(R\,\lambda^{\frac{p}{2}}\right)^{\frac{n}{p}(p-2)+1}\right)^{\frac{p}{n(p-2)+p}} + R\,\lambda^{\frac{p}{2}}\right] = c\,R\,\lambda^{\frac{p}{2}}
\end{aligned}$$

with $c \equiv c(n, p, \nu, L)$; in the second-last line we used Poincaré's inequality in its intrinsic form of Corollary 3.4 (note indeed that we are assuming $(3.3)_2$). We thus conclude with

$$\fint_{Q_{R/2}^\lambda} |Dv|^p\,dz \le c\left[\frac{1}{[R^\lambda]^p}\left[R\,\lambda^{\frac{p}{2}}\right]^p + \frac{\lambda^{p-2}}{[R^\lambda]^2}\left[R\,\lambda^{\frac{p}{2}}\right]^2\right] \le c\,\lambda^p.$$

□

Next, a reverse-Hölder's inequality for solutions to (3.1). We stress that the important point in (3.5) is not the fact that the gradient is highly integrable - this fact has been proven, in different forms, in several papers, as [9, 21, 35] just to mention only the ones including the singular case - but the precise form of the estimate.

Proposition 3.6. *Let $v \in L^p(I; W^{1,p}(A))$ be a weak solution to (3.1) and let $Q_R^\lambda(z_0) \subset A \times I$ be a cylinder such that*

$$\left(\frac{\lambda}{\kappa}\right)^p \le \fint_{Q_{R/2}^\lambda} |Dv|^p\,dz \quad \textit{and} \quad \fint_{Q_R^\lambda} |Dv|^p\,dz \le (\kappa\lambda)^p \tag{3.4}$$

hold for a constant $\kappa \geq 1$. Then there exist an exponent $\eta \equiv \eta(n,p,\nu,L) > 0$ such that for any $\sigma > 0$ it holds

$$\left(\fint_{Q^\lambda_{R/2}} |Dv|^{p(1+\eta)}\,dz\right)^{\frac{1}{p(1+\eta)}} \leq c\left(\fint_{Q^\lambda_R} |Dv|^\sigma\,dz\right)^{\frac{1}{\sigma}}, \tag{3.5}$$

with the constant c depending only on n,p,ν,L,κ and σ.

Proof. We will be a bit sloppy in the following proof, in particular in the covering part, since the whole argument is quite standard: see its minor variants, for instance, in [2, 4, 5] and moreover the forthcoming Section 5.

We fix two intermediate radii $R_1 < R_2$, with $R_1, R_2 \in [R/2, R]$, and two further ones $R_1 \leq r_1 < r_2 \leq R_2$ and we fix

$$\mu_0^{\frac{p}{d}} := \lambda^{(1-d)\frac{p}{d}} \fint_{Q^\lambda_{R_2}} |Dv|^p\,dz, \qquad B^{\frac{p}{d}} := \left(\frac{10R_2}{r_2-r_1}\right)^N$$

with the scaling deficit at the energy scale defined as (see [2, 5, 9, 35])

$$d = \frac{2p}{p(n+2)-2n}.$$

We observe that we can build, for $\mu > B\mu_0$ fixed, a covering of

$$E(Q^\lambda_{r_1},\mu) = Q^\lambda_{r_1} \cap \big\{|Dv(z)| > \mu,\ z \text{ is a Lebesgue's point of } Dv\big\}$$

consisting in a family of cylinders $Q^\mu_{\rho_{\bar z}}(\bar z)$, $\bar z \in E(Q^\lambda_{r_1},\mu)$ so that $\rho_{\bar z} \leq (r_2-r_1)/10$. Indeed, defining

$$CZ\big(Q^\mu_\rho(z)\big) := \fint_{Q^\mu_\rho(z)} |Dv|^p\,dz,$$

if $\rho \in ((r_2-r_1)/10, (r_2-r_1)/2)$ and $\mu > B\mu_0$ we estimate enlarging the domain of integration

$$\begin{aligned}
CZ\big(Q^\mu_\rho(z)\big) &\leq \frac{|Q^\lambda_{R_2}|}{|Q^\mu_\rho(z)|}\lambda^{(d-1)\frac{p}{d}}\mu_0^{\frac{p}{d}}\\
&< \left(\frac{R_2}{\rho}\right)^N \left(\frac{\mu}{\lambda}\right)^{\frac{2-p}{2}n} \lambda^{(d-1)\frac{p}{d}}\mu^{\frac{p}{d}}B^{-\frac{p}{d}}\\
&\leq \left(\frac{10R_2}{r_2-r_1}\right)^N \left(\frac{\mu}{\lambda}\right)^{\frac{2-p}{2}n} \lambda^{\frac{2-p}{2}n}\mu^{p\frac{n+2}{n}-n}\left(\frac{10R_2}{r_2-r_1}\right)^{-N}\\
&\leq \mu^p.
\end{aligned}$$

On the other hand, if $\bar z \in E(Q^\lambda_{r_1},\mu)$, then $|Dv(\bar z)| > \mu$ and by Lebesgue's differentiation Theorem we have that $CZ(Q^\mu_\varrho(\bar z)) > \mu^p$ for small radii $0 < \varrho \ll 1$; thus, by absolute continuity we find a critical, maximal radius $\varrho_{\bar z} \leq (r_2-r_1)/10$ such that $CZ(Q^\mu_{\varrho_{\bar z}}(\bar z)) = \mu^p$; moreover, by maximality we have

$$\frac{\mu^p}{c(n)} \leq \fint_{Q^\mu_{\alpha\varrho_{\bar z}}(\bar z)} |Dv|^p\,dz \leq \mu^p$$

for $\alpha \in [1,10]$. Thus we are in position to use the reverse Hölder's inequality of [9, Lemma 13] to infer

$$\fint_{Q^\mu_{\varrho_{\bar z}}(\bar z)} |Dv|^p\,dz \leq c\left(\fint_{Q^\mu_{2\varrho_{\bar z}}(\bar z)} |Dv|^{\bar p}\,dz\right)^{p/\bar p}$$

with $\bar p = 2n/(n+2) < p$ and a constant depending only on n,p,ν,L and not on the energy of Du. Note that the cylinders in [9] differ from ours in the singular case $p < 2$ but with a simple change of variable ($\varrho = \varrho_{\bar z}\lambda^{(p-2)/2} = \varrho^{\lambda}_{\bar z}$, where ϱ is appearing in [9]) we can recover our situation. This implies, calling the constant appearing in the display above $\tilde c^p$,

$$
\begin{aligned}
\mu^{\bar p} &\le \left(\fint_{Q^{\mu}_{\varrho_{\bar z}}(\bar z)} |Dv|^p\,dz\right)^{\bar p/p}\\
&\le \frac{\tilde c^{\bar p}}{|Q^{\mu}_{2\varrho_{\bar z}}(\bar z)|}\Bigg[\int_{Q^{\mu}_{2\varrho_{\bar z}}(\bar z)\cap\{|Dv|\le\mu/[2\tilde c]\}} |Dv|^{\bar p}\,dz\\
&\qquad\qquad + \int_{Q^{\mu}_{2\varrho_{\bar z}}(\bar z)\cap\{|Dv|>\mu/[2\tilde c]\}} |Dv|^{\bar p}\,dz\Bigg]\\
&\le \left(\frac{\mu}{2}\right)^{\bar p} + \frac{c}{|Q^{\mu}_{2\varrho_{\bar z}}(\bar z)|}\int_{Q^{\mu}_{2\varrho_{\bar z}}(\bar z)\cap\{|Dv|>\mu/[2\tilde c]\}} |Dv|^{\bar p}\,dz;
\end{aligned}
$$

thus

$$
\mu^{\bar p} \le \frac{c}{|Q^{\mu}_{\varrho_{\bar z}}(\bar z)|}\int_{Q^{\mu}_{2\varrho_{\bar z}}(\bar z)\cap\{|Dv|>\mu/[2\tilde c]\}} |Dv|^{\bar p}\,dz
$$

and as consequence, calling $\varsigma \equiv \varsigma(n,p,\nu,L) := 1/[2\tilde c]$

$$
\fint_{5Q^{\mu}_{\varrho_{\bar z}}(\bar z)} |Dv|^p\,dz \le \mu^p \le \frac{c\,\mu^{p-\bar p}}{|Q^{\mu}_{\varrho_{\bar z}}(\bar z)|}\int_{E(Q^{\mu}_{2\varrho_{\bar z}}(\bar z),\varsigma\mu)} |Dv|^{\bar p}\,dz \tag{3.6}
$$

for a constant c depending on n,p,ν,L. Now we extract using Vitali's Lemma (recall that the cylinders $Q^{\mu}_{\varrho_{\bar z}}(\bar z)$ with λ fixed are the balls of the metric in (2.4)) a sub-collection $\{2Q_i\}_{i\in\mathcal I}$ of $\{Q^{\mu}_{2\varrho_{\bar z}}(\bar z)\}_{\bar z\in E(Q^{\lambda}_{r_1},\mu)}$, such that the 5-times enlarged cylinders $10Q_i$ cover almost all $E(Q^{\lambda}_{r_1},\mu)$ and the cylinders are pairwise disjoints. Note that $10Q_i\subset Q^{\lambda}_{r_2}$. We finally have, since the cylinders $2Q_i$ are disjoint, using (3.6)

$$
\begin{aligned}
\int_{E(Q^{\lambda}_{r_1},\mu)} |Dv|^p\,dz &\le \sum_{i\in\mathcal I}\int_{10Q_i} |Dv|^p\,dz\\
&\le c\,\mu^{p-\bar p}\sum_{i\in\mathcal I}\int_{E(2Q_i,\varsigma\mu)} |Dv|^{\bar p}\,dz\\
&\le c\,\mu^{p-\bar p}\int_{E(Q^{\lambda}_{r_2},\varsigma\mu)} |Dv|^{\bar p}\,dz
\end{aligned}
$$

Finally, as in the usual proof of higher integrability estimates via Fubini's theorem, we have for η small enough, using the estimate above and calling $\mu_1 := B\mu_0$

$$
\begin{aligned}
&\int_{Q^{\lambda}_{r_1}} |Dv|^{p(1+\eta)}\,dz = p\eta\int_0^\infty \mu^{p\eta}\int_{E(Q^{\lambda}_{r_1},\mu)} |Dv|^p\,dz\,\frac{d\mu}{\mu}\\
&\le \mu_1^{p\eta}\int_{Q^{\lambda}_{r_1}} |Dv|^p\,dz + c\,\eta\int_{\mu_1}^\infty \mu^{p\eta+p-\bar p}\int_{E(Q^{\lambda}_{r_2},\varsigma\mu)} |Dv|^{\bar p}\,dz\,\frac{d\mu}{\mu}\\
&\le \mu_1^{p\eta}\int_{Q^{\lambda}_{r_1}} |Dv|^p\,dz + \bar c\,\eta\int_{Q^{\lambda}_{r_2}} |Dv|^{p(1+\eta)}\,dz;
\end{aligned}
$$

note that we changed variable $\tilde\mu = \varsigma\mu$, recalling that $\varsigma\equiv\varsigma(n,p,\nu,L)$, to perform the estimate in the last line. Now we choose η small so that $\bar c\eta\le 1/2$ and this yields, recalling the definition of μ_1

$$
\begin{aligned}
\int_{Q^{\lambda}_{r_1}} |Dv|^{p(1+\eta)}\,dz &\le \frac12\int_{Q^{\lambda}_{r_2}} |Dv|^{p(1+\eta)}\,dz\\
&\qquad + c\left(\frac{R_2}{r_2-r_1}\right)^{Nd\eta}\mu_0^{p\eta}\int_{Q^{\lambda}_{r_2}} |Dv|^p\,dz.
\end{aligned}
$$

Now the standard iteration lemma [19, Lemma 6.1] allows to reabsorb the $L^{p(1+\eta)}$ norm on the right-hand side and to deduce

$$\int_{Q^\lambda_{R_1}} |Dv|^{p(1+\eta)}\,dz \le c\left(\frac{R_2}{R_2-R_1}\right)^{Nd\eta}\int_{Q^\lambda_{R_2}} |Dv|^p\,dz\,\times$$
$$\times\,\lambda^{p\eta}\lambda^{-pd\eta}\left[\fint_{Q^\lambda_{R_2}} |Dv|^p\,dz\right]^{d\eta}.$$

Note that the re-absorptions does not cause problems, since all the quantity in play are finite; the gradient higher integrability is a well-established fact and what we are interested in are precise local estimates. At this point notice that (3.4) implies

$$\frac{\lambda^p}{c(n,p,\kappa)} \le \fint_{Q^\lambda_{R_1}} |Dv|^p\,dz \quad\text{and}\quad \fint_{Q^\lambda_{R_2}} |Dv|^p\,dz \le c(n,p,\kappa)\lambda^p$$

since $R/2 \le R_1 < R_2 \le R$, and this implies in turn

$$\int_{Q^\lambda_{R_1}} |Dv|^{p(1+\eta)}\frac{dz}{|Q^\lambda_R|} \le c\left(\frac{R}{R_2-R_1}\right)^{Nd\eta}\left(\int_{Q^\lambda_{R_2}} |Dv|^p\frac{dz}{|Q^\lambda_R|}\right)^{1+\eta}.$$

Recall that the previous inequality holds for every $R_1, R_2 \in [R/2, R]$ with $R_1 < R_2$. We are now in position to precisely apply [22, Lemma 5.1] with $d\mu = dz/|Q^\lambda_R|$, which encodes the usual self-improving property of reverse-Hölder inequalities in a form fitting our context, to infer (3.5). □

4. Auxiliary results

In this section we approach the proof of Theorem 1.1, collecting some results: a comparison Lemma, a Poincaré-like inequality and a local “energy” estimate for measure data problems.

First of all, from now on we will choose as the set $A \times I$ a cylinder $Q^\lambda_R(z_0) \subset \Omega_T$ and we will introduce therein the comparison function solution to the Cauchy-Dirichlet problem

$$\begin{cases} v_t - \operatorname{div} a(x,t,Dv) = 0 & \text{in } Q^\lambda_R,\\ v = u & \text{on } \partial_p Q^\lambda_R, \end{cases} \tag{4.1}$$

where u is a solution to (1.1). Recall we are dealing with approximating, regular solutions $u \in V^{2,p}(\Omega_T)$; therefore existence and uniqueness of v are well known arguments (see [17]) and so it is the fact that $v \in u + V^{2,p}_0(Q^\lambda_R)$. The following comparison result is a generalization of [25, Lemma 4.1].

Lemma 4.1. *Let u be a weak solution to (1.1) and let v be the comparison function defined in (4.1). Then*

$$\left(\fint_{Q^\lambda_R} |Du-Dv|^q\,dz\right)^{\frac1q} \le c\left[\frac{|\mu|(Q^\lambda_R)}{|Q^\lambda_R|^{\frac{N-1}{N}}}\right]^{\frac{N}{(N-1)(p-1)+1}}$$
$$+\,c\,\frac{|\mu|(Q^\lambda_R)}{|Q^\lambda_R|^{\frac{N-1}{N}}}\left(\fint_{Q^\lambda_R} |Du|^q\,dz\right)^{\frac{2-p}{q}\frac{N-1}{N}} \tag{4.2}$$

for every

$$q \in \left[1, p-1+\frac{1}{N-1}\right) \tag{4.3}$$

and for a constant $c \equiv c(n,p,\nu,q)$.

Proof. For q as in the statement, we start from [24, Display (4.11)], that reads as

$$
\begin{aligned}
\alpha &:= \left(\fint_{Q_R^\lambda} |u-v|^{q\frac{N-1}{N-2}}\, dz \right)^{\frac{N-2}{q(N-1)}} \\
&\qquad \leq c(n,q)\, \left[|\mu|(Q_R^\lambda)\right]^{\frac{1}{N-1}} \left(\fint_{Q_R^\lambda} |Du-Dv|^q\, dz \right)^{\frac{N-2}{q(N-1)}}, \qquad (4.4)
\end{aligned}
$$

which actually holds in the full range $p > 2 - 1/(N-1)$: indeed, the only ingredients needed to prove the estimate are the equations solved by both u and v, together with the parabolic Sobolev's embedding (taking into account that $u-v$ has trace zero on the lateral boundary). Note that we can assume without loss of generality $\alpha > 0$; otherwise (4.2) would follow trivially. This implies, see again [24, Proof of Lemma 4.1, *Step 2*] or [25, Proof of Lemma 4.1]

$$
\begin{aligned}
&\fint_{Q_R^\lambda} \left|V(Du) - V(Dv)\right|^{\frac{2q}{p}}\, dz \\
&\qquad \leq c(n,p,q) \left[\frac{|\mu|(Q_R^\lambda)}{|Q_R^\lambda|} \alpha \right]^{\frac{q}{p}} \\
&\qquad \leq c \left[\frac{[|\mu|(Q_R^\lambda)]^{\frac{N}{N-1}}}{|Q_R^\lambda|} \left(\fint_{Q_R^\lambda} |Du-Dv|^q\, dz \right)^{\frac{N-2}{q(N-1)}} \right]^{\frac{q}{p}}; \qquad (4.5)
\end{aligned}
$$

again the only things needed to deduce such an estimate are the weak formulations of the equations for u and v and the usual monotonicity estimate in (2.6). To conclude the proof, we pointwise bound

$$
\begin{aligned}
|Du - Dv| &= \left[\big(|Du| + |Dv|\big)^{p-2} |Du-Dv|^2 \right]^{\frac{1}{2}} \big(|Du| + |Dv|\big)^{\frac{2-p}{2}} \\
&\leq c(n,p) \big|V(Du) - V(Dv)\big| \big(|Du| + |Dv|\big)^{\frac{2-p}{2}} \\
&\leq c \left|V(Du) - V(Dv)\right| \big(|Du-Dv| + |Du|\big)^{\frac{2-p}{2}} \\
&\leq c \left|V(Du) - V(Dv)\right|^{\frac{2}{p}} + \frac{1}{2}|Du-Dv| \\
&\qquad\qquad + \left|V(Du) - V(Dv)\right| |Du|^{\frac{2-p}{2}},
\end{aligned}
$$

using also Young's inequality with conjugate exponents $(2/p, 2/(2-p))$. Thus, reabsorbing, taking the q-power and averaging over Q_R^λ

$$
\begin{aligned}
\fint_{Q_R^\lambda} |Du-Dv|^q\, dz &\leq c(n,p,q) \Big[\fint_{Q_R^\lambda} \left|V(Du) - V(Dv)\right|^{\frac{2q}{p}}\, dz \\
&\qquad\qquad + \fint_{Q_R^\lambda} \left|V(Du) - V(Dv)\right|^q |Du|^{q\frac{2-p}{2}}\, dz \Big] \\
&\leq c(n,p,q) \Big[\fint_{Q_R^\lambda} \left|V(Du) - V(Dv)\right|^{\frac{2q}{p}}\, dz \\
&\qquad + \left(\fint_{Q_R^\lambda} \left|V(Du) - V(Dv)\right|^{\frac{2q}{p}}\, dx \right)^{\frac{p}{2}} \left(\fint_{Q_R^\lambda} |Du|^q\, dz \right)^{\frac{2-p}{2}} \Big] \\
&= c\big[I + II\big].
\end{aligned}
$$

If now $I \leq II$, then using (4.5)

$$
\fint_{Q_R^\lambda} |Du-Dv|^q\, dz \leq c \fint_{Q_R^\lambda} \left|V(Du) - V(Dv)\right|^{\frac{2q}{p}}\, dz
$$

$$\le c \left[\frac{[|\mu|(Q_R^\lambda)]^{\frac{N}{N-1}}}{|Q_R^\lambda|} \left(\fint_{Q_R^\lambda} |Du - Dv|^q \, dz \right)^{\frac{N-2}{q(N-1)}} \right]^{\frac{q}{p}};$$

and so

$$\fint_{Q_R^\lambda} |Du - Dv|^q \, dz \le c \left[\frac{[|\mu|(Q_R^\lambda)]^{\frac{N}{N-1}}}{|Q_R^\lambda|} \right]^{\frac{q(N-1)}{(N-1)(p-1)+1}};$$

on the other hand, if $II \le I$, then

$$\fint_{Q_R^\lambda} |Du - Dv|^q \, dz \le c \left[\frac{[|\mu|(Q_R^\lambda)]^{\frac{N}{N-1}}}{|Q_R^\lambda|} \left(\fint_{Q_R^\lambda} |Du - Dv|^q \, dz \right)^{\frac{N-2}{q(N-1)}} \right]^{\frac{q}{2}} \cdot$$
$$\cdot \left(\fint_{Q_R^\lambda} |Du|^q \, dz \right)^{\frac{2-p}{2}}$$

and thus

$$\fint_{Q_R^\lambda} |Du - Dv|^q \, dz \le c \left[\frac{[|\mu|(Q_R^\lambda)]^{\frac{N}{N-1}}}{|Q_R^\lambda|} \right]^{q\frac{N-1}{N}} \left(\fint_{Q_R^\lambda} |Du|^q \, dz \right)^{(2-p)\frac{N-1}{N}}.$$

□

Now it is the time for a Poincaré-type inequality valid for solutions to (1.1).

Lemma 4.2. *Let $u \in V^{2,p}(\Omega_T)$ be a solution to (1.1). Then, for every $q \in [1, p]$,*

$$\fint_{Q_R^\lambda} \left| \frac{u - (u)_{Q_R^\lambda}}{R^\lambda} \right|^q dz \le c \fint_{Q_R^\lambda} |Du|^q \, dz$$
$$+ c \left[\lambda^{2-p} \left(\fint_{Q_R^\lambda} |Du| \, dz \right)^{p-1} + c \, \frac{R^2}{R^\lambda} \frac{|\mu|(Q_R^\lambda)}{|Q_R^\lambda|} \right]^q, \quad (4.6)$$

where the constant c depends only on n, p and L.

Proof. We split the integral in (4.6) as follows:

$$\fint_{Q_R^\lambda} \left| \frac{u - (u)_{Q_R^\lambda}}{R^\lambda} \right| dz \le \fint_{Q_R^\lambda} \left| \frac{u - (u)^\eta_{B_R^\lambda}(t)}{R^\lambda} \right| dz$$
$$+ \frac{1}{R^\lambda} \fint_{t_0 - R^2}^{t_0 + R^2} \left| (u)^\eta_{B_R^\lambda}(t) - \fint_{t_0 - R^2}^{t_0 + R^2} (u)^\eta_{B_R^\lambda}(\tau) \, d\tau \right| dt$$
$$+ \frac{1}{R^\lambda} \left| \fint_{t_0 - R^2}^{t_0 + R^2} (u)^\eta_{B_R^\lambda}(\tau) \, d\tau - (u)_{Q_R^\lambda} \right| = I + II + III.$$

We used the notation

$$(u)^\eta_{B_R^\lambda}(t) := \fint_{B_R^\lambda} u(\cdot, t) \eta \, dx,$$

$\eta \in C_c^\infty(B_R^\lambda)$ being a positive weight function satisfying

$$\fint_{B_R^\lambda} \eta \, dx = 1, \qquad \eta(x) + R^\lambda |D\eta(x)| \le c(n).$$

Using slice-wise a variant of Poincaré's inequality (see [29, Corollary 1.64]) we infer

$$III \le I \le c(n) \fint_{Q_R^\lambda} |Du| \, dz.$$

To estimate II we use the equation solved by u: testing the weak formulation with the test function η independent of time, we get for a.e. $t_1, t_2 \in (t_0 - R^2, t_0 + R^2)$

$$
\begin{aligned}
&\big|(u)^{\eta}_{B^{\lambda}_R}(t_1) - (u)^{\eta}_{B^{\lambda}_R}(t_2)\big| \\
&\qquad = \left|\int_{t_1}^{t_2} \partial_t\big[(u)^{\eta}_{B^{\lambda}_R}(t)\big]\,dt\right| = \left|\int_{t_1}^{t_2} -\!\!\!\!\!\!\int_{B^{\lambda}_R} \partial_t u\,\eta\,dx\,dt\right| \\
&\qquad \le \frac{c(n,L)}{R^{\lambda}} \int_{t_0-R^2}^{t_0+R^2} -\!\!\!\!\!\!\int_{B^{\lambda}_R} |Du|^{p-1}\,dz + c(n)[R^{\lambda}]^{-n}|\mu|(Q^{\lambda}_R) \\
&\qquad = c\,\frac{R^2}{R^{\lambda}} -\!\!\!\!\!\!\int_{Q^{\lambda}_R} |Du|^{p-1}\,dz + c[R^{\lambda}]^{-n}|\mu|(Q^{\lambda}_R) \\
&\qquad \le c\,\frac{R^2}{R^{\lambda}}\left(-\!\!\!\!\!\!\int_{Q^{\lambda}_R} |Du|\,dz\right)^{p-1} + c\,R^2\,\frac{|\mu|(Q^{\lambda}_R)}{|Q^{\lambda}_R|};
\end{aligned}
$$

we used the growth conditions in $(1.2)_2$ and Hölder's inequality. Note that the previous estimate is just formal: a precise proof can be done using a regularizing in time procedure (recall that we assume $\mu \in L^1(\Omega_T)$); see for instance [9, Lemma 7]. $\square$

Now a reverse Hölder's inequality for measure data problems. The use of such inequalities is customary when dealing with regularity and, in particular, Calderón-Zygmund estimates for degenerate and non-uniform elliptic problems, see for instance the use in [2, 4, 5, 10, 11, 16, 30].

Proposition 4.3. *Let u be a weak solution to (1.1) under the assumptions (1.2) with (1.3) in force and with $\mu \in L^1(\Omega_T)$; suppose moreover that*

$$
\frac{\lambda}{\kappa} \le -\!\!\!\!\!\!\int_{Q^{\lambda}_R} |Du|\,dz \qquad \text{and} \qquad -\!\!\!\!\!\!\int_{Q^{\lambda}_{2R}} |Du|\,dz \le \kappa\,\lambda \tag{4.7}
$$

hold in some cylinder $Q^{\lambda}_R \equiv Q^{\lambda}_R(z_0)$ such that $Q^{\lambda}_R(z_0) \subset \Omega_T$, for some constant $\kappa \ge 1$. Then for every q as in (4.3), there exists a constant $c \equiv c(n,p,\nu,q,\kappa)$ such that

$$
\left(-\!\!\!\!\!\!\int_{Q^{\lambda}_R} |Du|^q\,dz\right)^{\frac{1}{q}} \le c -\!\!\!\!\!\!\int_{Q^{\lambda}_{2R}} |Du|\,dz + c\left[\frac{|\mu|(Q^{\lambda}_{2R})}{|Q^{\lambda}_{2R}|^{\frac{N-1}{N}}}\right]^{\frac{N}{(N-1)(p-1)+1}}.
$$

Proof. Many of the forthcoming estimates are true for any cylinder $Q = \mathcal{K} \times I \subset \Omega_T$; only at a certain point we will specify the estimates to the situation we are considering in the statement of the Proposition. We test the weak formulation in (1.10) with

$$
\varphi_1 := \pm\min\{1, u_{\pm}/\epsilon\}\zeta,
$$

for $\epsilon > 0$ and with $\zeta \in C^{\infty}_c(Q)$, $\|\zeta\|_{L^\infty} \le 1$. Note that this is possible only at a formal level, due to the lack of time regularity of u; this choice can be anyway made rigorous by using Steklov averages - we will proceed formally. Following [24, Lemma 4.1], we compute

$$
D\varphi_1 = \frac{1}{\epsilon} Du\,\chi_{\{0<u_{\pm}<\epsilon\}}\zeta \pm \min\{1, u_{\pm}/\epsilon\}D\zeta
$$

and thus we have (notice that $\|\varphi_1\|_{L^\infty} \le 1$ and recall (1.2))

$$
\begin{aligned}
&-\int_Q u\,\partial_t\varphi_1\,dz + \frac{1}{\epsilon}\int_Q \langle a(x,t,Du), Du\rangle \chi_{\{0<u_{\pm}<\epsilon\}}\zeta\,dz \\
&\qquad\qquad\qquad\qquad \le L\,\|D\zeta\|_{L^\infty}\int_Q |Du|^{p-1}\,dz + |\mu|(Q).
\end{aligned}
$$

We have, integrating twice by parts with respect to the time variable

$$
\begin{aligned}
-\int_Q u\,\partial_t\varphi_1\,dz &= \pm\int_Q \partial_t u\,\min\{1,u_\pm/\epsilon\}\zeta\,dz\\
&= \int_Q \partial_t \int_0^{u_\pm}\min\{1,\sigma/\epsilon\}\,d\sigma\,\zeta\,dz\\
&= -\int_Q\int_0^{u_\pm}\min\{1,\sigma/\epsilon\}\,d\sigma\,\partial_t\zeta\,dz
\end{aligned}
$$

(see [24, (4.8)]). This leads to

$$
\begin{aligned}
-\int_Q\int_0^{u_\pm}\min\{1,\sigma/\epsilon\}\,d\sigma\,\partial_t\zeta\,dz + \frac{1}{\epsilon}\int_Q\langle a(x,t,Du),Du\rangle\chi_{\{0<u_\pm<\epsilon\}}\zeta\,dz&\\
\le c\,\|D\zeta\|_{L^\infty}\int_Q |Du|^{p-1}\,dz + |\mu|(Q);&
\end{aligned}
$$

we note that by dominated convergence Theorem

$$
-\int_Q\int_0^{u_\pm}\min\{1,\sigma/\epsilon\}\,d\sigma\,\partial_t\zeta\,dz \to -\int_Q u_\pm\,\partial_t\zeta\,dz
$$

as $\epsilon\searrow 0$ and thus

$$
\begin{aligned}
-\int_Q |u|\,\partial_t\zeta\,dz + \sup_{\epsilon>0}\frac{1}{\epsilon}\int_Q\langle a(x,t,Du),Du\rangle\chi_{\{0<u_\pm<\epsilon\}}\zeta\,dz&\\
\le c\,\|D\zeta\|_{L^\infty}\int_Q |Du|^{p-1}\,dz + |\mu|(Q).&
\end{aligned}
$$

Now we choose

$$
\zeta(x,t) := \zeta_1(x,t)\zeta_{2,\varepsilon,\tau}(t) \tag{4.8}
$$

with $\zeta_1\in C^\infty_c(Q)$ such that $\zeta_1\le 1$ and, for any $\tau\in I$ being fixed and ε small enough so that also $\tau+\varepsilon\in I$, $\zeta_{2,\varepsilon,\tau}$ is continuous and piecewise linear, with $\zeta_{2,\varepsilon,\tau}\equiv 1$ on $(-\infty,\tau)$ and $\zeta_{2,\varepsilon,\tau}\equiv 0$ on $(\tau+\varepsilon,\infty)$. Note that $\int_{\mathbb{R}}|\partial_t\zeta_{2,\varepsilon,\tau}|\,dt = 1$ for any τ and ε and $\partial_t\zeta_{2,\varepsilon,\tau}\le 0$. With this choice we have

$$
\int_Q |u|\big(-\partial_t(\zeta_1\zeta_{2,\varepsilon,\tau})\big)\,dz \to \int_{\mathcal{K}} |u|(\cdot,\tau)\,dx + \int_Q |u|\,(-\partial_t\zeta_1)\chi_{\mathbb{R}^n\times(-\infty,\tau)}\,dz
$$

as $\varepsilon\to 0$ for $\tau\in I$ fixed, so

$$
\begin{aligned}
&\sup_{\tau\in I}\int_B |u|(\cdot,\tau)\,dx + \sup_{\epsilon>0}\frac{1}{\epsilon}\int_Q\langle a(x,t,Du),Du\rangle\chi_{\{0<u_\pm<\epsilon\}}\zeta\,dz\\
&\quad\le \|\partial_t\zeta_1\|_{L^\infty}\int_Q |u|\,dz + c\,\|D\zeta\|_{L^\infty}\int_Q|Du|^{p-1}\,dz + c\,|\mu|(Q) := c\,\mathcal{E}.
\end{aligned}
\tag{4.9}
$$

Now we fix two constants $\alpha>0$ and $\xi>1$ and we use in (1.10) the test function

$$
\varphi_2 := \frac{\varphi_1}{(\alpha+u_\pm)^{\xi-1}} = \frac{\pm\min\{1,u_\pm/\epsilon\}}{(\alpha+u_\pm)^{\xi-1}}\zeta,
$$

$\epsilon>0$, ζ as in (4.8); we use the weak formulation in the form

$$
\begin{aligned}
(\xi-1)&\int_Q\langle a(x,t,Du),Du_\pm\rangle\frac{\varphi_1}{(\alpha+u_\pm)^\xi}\,dz\\
&= \int_Q\langle a(x,t,Du),D\varphi_1\rangle\frac{dz}{(\alpha+u_\pm)^{\xi-1}} - \int_Q u\,\partial_t\varphi_2\,dz - \int_Q\varphi_2\,d\mu.
\end{aligned}
$$

We estimate the terms on the right-hand side: firstly, using (4.9)

$$
\begin{aligned}
\int_Q \langle a(x,t,Du), D\varphi_1\rangle \frac{dz}{(\alpha+u_\pm)^{\xi-1}}
&= \frac{1}{\epsilon}\int_Q \langle a(x,t,Du), Du\rangle \frac{\chi_{\{0<u_\pm<\epsilon\}}\zeta\,dz}{(\alpha+u_\pm)^{\xi-1}} \\
&\qquad \pm \int_Q \langle a(x,t,Du), D\zeta\rangle \frac{\min\{1,u_\pm/\epsilon\}\,dz}{(\alpha+u_\pm)^{\xi-1}} \\
&\le c\,\alpha^{1-\xi}\mathcal{E} + \alpha^{1-\xi}\|D\zeta\|_{L^\infty}\int_Q |Du|^{p-1}\,dz,
\end{aligned}
$$

so that

$$
\sup_{\epsilon>0}\int_Q \langle a(x,t,Du), D\varphi_1\rangle \frac{dz}{(\alpha+u_\pm)^{\xi-1}} \le c\,\alpha^{1-\xi}\mathcal{E}.
$$

Then we simply estimate the last term as follows:

$$
-\int_Q \varphi_2\,d\mu \le \int_Q |\varphi_2|\,d|\mu| \le \alpha^{1-\xi}|\mu|(Q) \le c\,\alpha^{1-\xi}\mathcal{E}.
$$

To estimate the parabolic term, we again use integration by parts and we deduce

$$
-\int_Q u\partial_t\varphi_2\,dz = \int_Q \partial_t u\,\frac{\varphi_1}{(\alpha+u_\pm)^{\xi-1}}\,dz = -\int_Q\int_0^{u_\pm}\frac{\min\{1,\sigma/\epsilon\}}{(\alpha+\sigma)^{\xi-1}}\,d\sigma\,\partial_t\zeta\,dz;
$$

again we choose ζ as above. We estimate the term containing $\zeta_{2,\varepsilon,\tau}$:

$$
\begin{aligned}
-\int_Q\int_0^{u_\pm}&\frac{\min\{1,\sigma/\epsilon\}}{(\alpha+\sigma)^{\xi-1}}\,d\sigma\,\zeta_1\,\partial_t\zeta_{2,\varepsilon,\tau}\,dz \\
&\le \alpha^{1-\xi}\sup_{\tau\in I}\int_{\mathcal{K}}\int_0^{u_\pm(\cdot,\tau)}\min\{1,\sigma/\epsilon\}\,d\sigma\,dx\int_I|\partial_t\zeta_{2,\varepsilon,\tau}|\,dt \\
&\le \alpha^{1-\xi}\sup_{\tau\in I}\int_{\mathcal{K}} u_\pm(\cdot,\tau)\,dx;
\end{aligned}
$$

thus, using (4.9),

$$
\sup_{\epsilon>0}\left(-\int_Q\int_0^{u_\pm}\frac{\min\{1,\sigma/\epsilon\}}{(\alpha+\sigma)^{\xi-1}}\,d\sigma\,\zeta_1\,\partial_t\zeta_{2,\varepsilon,\tau}\,dz\right) \le c\,\alpha^{1-\xi}\mathcal{E}
$$

and in turn

$$
\begin{aligned}
\sup_{\epsilon>0}\left(-\int_Q u\,\partial_t\varphi_2\,dz\right) &\le c\,\alpha^{1-\xi}\mathcal{E} \\
&\qquad + \sup_{\epsilon>0}\left(-\int_Q\int_0^{u_\pm}\frac{\min\{1,\sigma/\epsilon\}}{(\alpha+\sigma)^{\xi-1}}\,d\sigma\,\zeta_{2,\varepsilon,\tau}\partial_t\zeta_1\,dz\right) \\
&\le c\,\alpha^{1-\xi}\mathcal{E} \\
&\qquad + \sup_{\epsilon>0}\left(\int_Q\int_0^{u_\pm}\frac{\min\{1,\sigma/\epsilon\}}{(\alpha+\sigma)^{\xi-1}}\,d\sigma\,|\zeta_{2,\varepsilon,\tau}\partial_t\zeta_1|\,dz\right) \\
&\le c\,\alpha^{1-\xi}\mathcal{E} + \alpha^{1-\xi}\|\partial_t\zeta_1\|_{L^\infty}\int_Q |u|\,dz \\
&\le c\,\alpha^{1-\xi}\mathcal{E}.
\end{aligned}
$$

Merging all these estimates we obtain

$$
(\xi-1)\int_Q \langle a(x,t,Du), \pm Du_\pm\rangle \frac{\varphi_1}{(\alpha+u_\pm)^{\xi}}\,dz \le c\,\alpha^{1-\xi}\mathcal{E},
$$

that is

$$\int_Q \frac{\langle a(x,t,Du), \pm Du_\pm\rangle}{(\alpha+u_\pm)^\xi}\min\{1, u_\pm/\epsilon\}\zeta\,dz \le c\,\frac{\alpha^{1-\xi}}{1-\xi}\mathcal{E}.$$

Letting $\epsilon \searrow 0$ and performing some algebraic manipulations yields

$$\fint_Q \frac{\langle a(x,t,Du), Du\rangle}{(\alpha+|u|)^\xi}\zeta\,dz \le c\,\frac{\alpha^{1-\xi}}{1-\xi}\frac{\mathcal{E}}{|Q|}. \tag{4.10}$$

Now we specialize the previous estimate: we take $Q = Q^\lambda_{2R}$, intermediate radii $R \le R_1 < R_2 \le 2R$ and ζ_1 satisfying $\chi_{Q^\lambda_{R_1}} \le \zeta_1 \le \chi_{Q^\lambda_{R_2}}$ in such a way that

$$(R_2^\lambda - R_1^\lambda)\|D\zeta_1\|_{L^\infty} + (R_2-R_1)^2\|\partial_t\zeta_1\|_{L^\infty} \le c;$$

moreover we notice that if u solves (1.1), so does $u-(u)_{Q^\lambda_{2R}}$; thus we can apply (4.9) and (4.10) to $u-(u)_{Q^\lambda_{R_2}}$. Under our assumptions, we have

$$\begin{aligned}
\|D\zeta_1\|_{L^\infty}&\fint_{Q^\lambda_{2R}} |Du|^{p-1}\,dz + \frac{|\mu|(Q^\lambda_{2R})}{|Q^\lambda_{2R}|}\\
&\le \frac{c\,R^\lambda}{R_2^\lambda - R_1^\lambda}\frac{1}{R^\lambda}\left(\fint_{Q^\lambda_{2R}}|Du|\,dz\right)^{p-1} + \frac{|\mu|(Q^\lambda_{2R})}{|Q^\lambda_{2R}|}
\end{aligned}$$

and, using (4.6),

$$\begin{aligned}
\|\partial_t\zeta_1\|_{L^\infty}&\fint_{Q^\lambda_{2R}} \big|u-(u)_{Q^\lambda_{2R}}\big|\,dz\\
&\le c\Big(\frac{R^\lambda}{R_2^\lambda - R_1^\lambda}\Big)^2\frac{\lambda^{p-2}}{R^\lambda}\fint_{Q^\lambda_{2R}}\left|\frac{u-(u)_{Q^\lambda_{2R}}}{2R^\lambda}\right|dz\\
&\le c\Big(\frac{R}{R_2-R_1}\Big)^2\left[\frac{\lambda^{p-2}}{R^\lambda}\fint_{Q^\lambda_{2R}}|Du|\,dz\right.\\
&\qquad\left. + \frac{1}{R^\lambda}\left(\fint_{Q^\lambda_{R}}|Du|\,dz\right)^{p-1} + \frac{|\mu|(Q^\lambda_{2R})}{|Q^\lambda_{2R}|}\right].
\end{aligned}$$

We hence come up, denoting $\mathcal{R} := R/(R_2-R_1)$ with

$$\begin{aligned}
\fint_{Q^\lambda_{2R}}&\frac{\langle a(x,t,Du), Du\rangle}{(\alpha+|u-(u)_{Q^\lambda_{2R}}|)^\xi}\zeta_1\,dz\\
&\le c\,\frac{\alpha^{1-\xi}}{1-\xi}\Big(\frac{R}{R_2-R_1}\Big)^2\left[\frac{\lambda^{p-2}}{R^\lambda}\fint_{Q^\lambda_{2R}}|Du|\,dz\right.\\
&\qquad\left. + \frac{1}{R^\lambda}\left(\fint_{Q^\lambda_{2R}}|Du|\,dz\right)^{p-1} + \frac{|\mu|(Q^\lambda_{2R})}{|Q^\lambda_{2R}|}\right]\\
&=: c\,\frac{\alpha^{1-\xi}}{1-\xi}\mathcal{R}^2\tilde{\mathcal{E}}
\end{aligned} \tag{4.11}$$

and

$$\sup_{\tau\in(t_0-R^2,t_0+R^2)}\int_{B^\lambda_{2R}}\big|u-(u)_{Q^\lambda_{2R}}\big|(\cdot,\tau)\,dx \le c\,|Q^\lambda_{2R}|\,\mathcal{R}^2\tilde{\mathcal{E}}. \tag{4.12}$$

Conclusion. Now we fix $q\in[1,m_0)$ with m_0 defined in (1.11) and we take

$$\xi=\frac{N-1}{N-2}(p-q)>1,\qquad \alpha^{q\frac{N-1}{N-2}}=\fint_{Q^\lambda_{2R}}\big|\big(u-(u)_{Q^\lambda_{2R}}\big)\zeta_1\big|^{q\frac{N-1}{N-2}}\,dz;$$

note that without loss of generality we can assume $\alpha\neq 0$, otherwise there would be nothing to prove. We estimate using parabolic Sobolev's embedding (see [17, Chapter I, Proposition 3.1] or [24, Lemma 4.1])

$$\begin{aligned}
\alpha^{q\frac{N-1}{N-2}}&\le c(n,q)\bigg(\sup_{\tau\in(t_0-R^2,t_0+R^2)}\int_{B_{R^\lambda}}\big|u-(u)_{Q^\lambda_{2R}}\big|(\cdot,\tau)\,dx\bigg)^{\frac{q}{N-2}}\cdot\\
&\qquad\cdot\bigg[\fint_{Q^\lambda_{R_2}}|Du|^q\,dz+\fint_{Q^\lambda_{2R}}\big|u-(u)_{Q^\lambda_{2R}}\big|^q|D\zeta_1|^q\,dz\bigg]\\
&\le c\,\big[|Q^\lambda_{2R}|\mathcal{R}^2\,\tilde{\mathcal{E}}\big]^{\frac{q}{N-2}}\bigg[\fint_{Q^\lambda_{R_2}}|Du|^q\,dz+\fint_{Q^\lambda_{2R}}\bigg|\frac{u-(u)_{Q^\lambda_{2R}}}{R^\lambda}\bigg|^q\,dz\bigg]\\
&\le c\,\big[|Q^\lambda_{2R}|\mathcal{R}^2\,\tilde{\mathcal{E}}\big]^{\frac{q}{N-2}}\bigg[\fint_{Q^\lambda_{R_2}}|Du|^q\,dz+\big[\lambda^{2-p}R^\lambda\tilde{\mathcal{E}}\big]^q\bigg],
\end{aligned}$$

after using (4.12). To conclude the proof, we compute using (2.7) and (4.11)

$$\begin{aligned}
\bigg(\fint_{Q^\lambda_{R_1}}|Du|^q\,dz\bigg)^{\frac1q}&\le c\bigg(\fint_{Q^\lambda_{R_1}}\big[\langle a(x,t,Du),Du\rangle\big]^{\frac qp}\,dz\bigg)^{\frac1q} \qquad (4.13)\\
&\le c\bigg(\fint_{Q^\lambda_{2R}}\frac{\langle a(x,t,Du),Du\rangle}{(\alpha+|u-(u)_{Q^\lambda_{2R}}|)^\xi}\zeta_1\,dz\bigg)^{\frac1p}\cdot\\
&\qquad\cdot\bigg(\fint_{Q^\lambda_{2R}}\big(\alpha+\big|\big(u-(u)_{Q^\lambda_{2R}}\big)\zeta_1\big|\big)^{\frac{q\xi}{p-q}}\,dz\bigg)^{\frac1q-\frac1p}\\
&\le c\bigg(\fint_{Q^\lambda_{2R}}\frac{\langle a(x,t,Du),Du\rangle}{(\alpha+|u-(u)_{Q^\lambda_{2R}}|)^\xi}\zeta_1\,dz\bigg)^{\frac1p}\alpha^{\frac{\xi}{p}}\\
&\le c\,\mathcal{R}^{\frac2p}\alpha^{\frac1p}\tilde{\mathcal{E}}^{\frac1p}\\
&\le c\,|Q^\lambda_{2R}|^{\frac1p\frac{1}{N-1}}\mathcal{R}^{\frac2p\frac{N}{N-1}}\tilde{\mathcal{E}}^{\frac1p\frac{N}{N-1}}\bigg(\fint_{Q^\lambda_{R_2}}|Du|^q\,dz\bigg)^{\frac{N-2}{N-1}\frac{1}{pq}}\\
&\qquad+c\,\mathcal{R}^{\frac2p\frac{N}{N-1}}|Q^\lambda_{2R}|^{\frac1p\frac{1}{N-1}}\big[\lambda^{2-p}R^\lambda\big]^{\frac1p\frac{N-2}{N-1}}\tilde{\mathcal{E}}^{\frac2p}\\
&\le\frac12\bigg(\fint_{Q^\lambda_{R_2}}|Du|^q\,dz\bigg)^{\frac1q}+c\,\mathcal{R}^{\bar\xi}|Q^\lambda_{2R}|^{\frac{1}{(N-1)(p-1)+1}}\tilde{\mathcal{E}}^{\frac{N}{(N-1)(p-1)+1}}\\
&\qquad+c\,\mathcal{R}^{\frac2p\frac{N}{N-1}}\big[\lambda^{2-p}R^\lambda\big]^{\frac1p\frac{N-2}{N-1}}|Q^\lambda_{2R}|^{-\frac1p\frac{N-2}{N(N-1)}}\cdot|Q^\lambda_{2R}|^{\frac{2}{pN}}\tilde{\mathcal{E}}^{\frac2p}\\
&\le\frac12\bigg(\fint_{Q^\lambda_{R_2}}|Du|^q\,dz\bigg)^{\frac1q}+c\,\mathcal{R}^{\bar\xi}|Q^\lambda_{2R}|^{\frac{1}{(N-1)(p-1)+1}}\tilde{\mathcal{E}}^{\frac{N}{(N-1)(p-1)+1}}+c\,\lambda,
\end{aligned}$$

for some $\bar\xi>0$ depending on n and p, after using Young's inequality twice with conjugate exponents, respectively,

$$\Big(\frac{p(N-1)}{(p-1)(N-1)+1},\frac{p(N-1)}{N-2}\Big)$$

and (if $p<2$)

$$\Big(\frac{Np}{(N-2)(2-p)},\frac p2\frac{N}{(p-1)(N-1)+1}\Big);$$

note that this choice is admissible since $1 < p < 2$. Indeed

$$
\begin{aligned}
\Big[\big[\lambda^{2-p}R^\lambda\big]^{\frac{1}{p}\frac{N-2}{N-1}}&|Q^\lambda_{2R}|^{-\frac{1}{p}\frac{N-2}{N(N-1)}}\Big]^{\frac{Np}{(N-2)(2-p)}}\\
&= \Big[\big[\lambda^{2-p}R^\lambda\big]^{\frac{1}{N-1}}|Q^\lambda_{2R}|^{-\frac{1}{N(N-1)}}\Big]^{\frac{N}{2-p}}\\
&= c(n,p)\Big[\big[\lambda^{2-p}R^\lambda\big]^{\frac{1}{N-1}}\big[\lambda^{\frac{2-p}{N}}R^\lambda\big]^{-\frac{1}{(N-1)}}\Big]^{\frac{N}{2-p}}\\
&= c(n,p)\Big[\lambda^{1-\frac{1}{N}}\Big]^{\frac{N}{N-1}} = c(n,p)\lambda.
\end{aligned}
$$

We conclude the proof first reabsorbing the L^q average on the right-hand side using a standard iteration Lemma (see [19, Lemma 6.1]) and then observing that by (4.7) we have

$$
\begin{aligned}
\mathcal{E} &= \frac{\lambda^{p-2}}{R^\lambda}\fint_{Q^\lambda_{2R}}|Du|\,dz + \frac{1}{R^\lambda}\bigg(\fint_{Q^\lambda_{2R}}|Du|\,dz\bigg)^{p-1} + \frac{|\mu|(Q^\lambda_{2R})}{|Q^\lambda_{2R}|}\\
&\le c(p,\kappa)\frac{\lambda^{p-1}}{R^\lambda} + \frac{|\mu|(Q^\lambda_{2R})}{|Q^\lambda_{2R}|}
\end{aligned}
$$

and thus

$$
\begin{aligned}
|Q^\lambda_{2R}|^{\frac{1}{(N-1)(p-1)+1}}\,&\mathcal{E}^{\frac{N}{(N-1)(p-1)+1}}\\
&\le c(n,p,\kappa)\bigg[|Q^\lambda_{2R}|^{\frac{1}{N}}\frac{\lambda^{p-1}}{R^\lambda} + |Q^\lambda_{2R}|^{\frac{1}{N}}\frac{|\mu|(Q^\lambda_{2R})}{|Q^\lambda_{2R}|}\bigg]^{\frac{N}{(N-1)(p-1)+1}}\\
&\le c\bigg[\lambda^{\frac{2-p}{N}}R^\lambda\frac{\lambda^{p-1}}{R^\lambda} + \frac{|\mu|(Q^\lambda_{2R})}{|Q^\lambda_{2R}|^{1-\frac{1}{N}}}\bigg]^{\frac{N}{(N-1)(p-1)+1}}\\
&\le c\,\lambda + c\bigg[\frac{|\mu|(Q^\lambda_{2R})}{|Q^\lambda_{2R}|^{\frac{N-1}{N}}}\bigg]^{\frac{N}{(N-1)(p-1)+1}}\\
&\le c\fint_{Q^\lambda_R}|Du|\,dz + c\bigg[\frac{|\mu|(Q^\lambda_{2R})}{|Q^\lambda_{2R}|^{\frac{N-1}{N}}}\bigg]^{\frac{N}{(N-1)(p-1)+1}}.
\end{aligned}
$$

□

With exactly the same proof (except for a different use of Young inequality in the last line of (4.13), namely with conjugate exponents $(p/2, p/(p-2))$) we have the degenerate counterpart of Proposition 4.3. Note however that we will not employ this estimate anywhere in this paper; we include it for completeness.

Proposition 4.4. *Let u be a weak solution to (1.1) under the assumptions (1.2), with now $p \ge 2$; assume that the inequalities*

$$
\frac{\lambda^{p-1}}{\kappa} \le \fint_{Q^\lambda_R}|Du|^{p-1}\,dz \quad \text{and} \quad \fint_{Q^\lambda_{2R}}|Du|^{p-1}\,dz \le \kappa\,\lambda^{p-1}
$$

hold in a cylinder $Q^\lambda_R \equiv Q^\lambda_R(z_0)$ as defined in (2.2), with $Q^\lambda_R(z_0) \subset \Omega_T$, for some $\kappa \ge 1$. For every q as in (4.3), there exists a constant $c \equiv c(n,p,\nu,q,\kappa)$ such that

$$
\begin{aligned}
\bigg(\fint_{Q^\lambda_R}|Du|^q\,dz\bigg)^{\frac{1}{q}} &\le c\fint_{Q^\lambda_{2R}}|Du|\,dz\\
&\qquad + c\bigg[\frac{|\mu|(Q^\lambda_{2R})}{|Q^\lambda_{2R}|^{\frac{N-1}{N}}}\bigg]^{\frac{N}{(N-1)(p-1)+1}} + c\,\frac{|\mu|(Q^\lambda_{2R})}{(2R)^{N-1}}.
\end{aligned}
$$

Note that the two terms taking into account the measure μ have the same dimensional character:

$$\left[\frac{|\mu|(Q^\lambda_{2R})}{|Q^\lambda_{2R}|^{\frac{N-1}{N}}}\right]^{\frac{N}{(N-1)(p-1)+1}} \lesssim \lambda \qquad \Longleftrightarrow \qquad \frac{|\mu|(Q^\lambda_{2R})}{R^{N-1}} \lesssim \lambda.$$

Now, using the result in Proposition 4.3 we can deduce an improved version of Lemma 4.1:

Corollary 4.5. *Let u be a weak solution to (1.1); suppose that $Q^\lambda_{2R} \equiv Q^\lambda_{2R}(z_0) \subset \Omega_T$ and let v be the solution to (4.1) in Q^λ_R. Assume that the inequalities*

$$\frac{\lambda}{\kappa} \le \fint_{Q^\lambda_R} |Du|\, dz, \qquad\qquad \fint_{Q^\lambda_{2R}} |Du|\, dz \le \kappa\, \lambda$$

hold for some $\kappa \ge 1$. Then

$$\left(\fint_{Q^\lambda_R} |Du - Dv|^q\, dz\right)^{\frac{1}{q}} \le c\left[\frac{|\mu|(Q^\lambda_{2R})}{|Q^\lambda_{2R}|^{\frac{N-1}{N}}}\right]^{\frac{N}{(N-1)(p-1)+1}} + c\,\frac{|\mu|(Q^\lambda_{2R})}{|Q^\lambda_{2R}|^{\frac{N-1}{N}}}\left(\fint_{Q^\lambda_{2R}} |Du|\, dz\right)^{(2-p)\frac{N-1}{N}} \tag{4.14}$$

for every q as in (4.3) and a constant c depending only on n, p, ν and κ.

Proof. (4.14) follows simply combining Lemma 4.1 and the reverse Hölder's inequality of Proposition 4.3, taking into account that

$$1 + \frac{N}{(N-1)(p-1)+1}\cdot(2-p)\frac{N-1}{N} = \frac{N}{(N-1)(p-1)+1}. \tag{4.15}$$

□

5. The proof of Theorem 1.1

We finally have in our hands all the tools needed to prove Theorem 1.1.

As already mentioned, the proof will be based on a covering argument more-or-less standard in the singular and degenerate parabolic setting, see [21, 2], which the reader could have already met in the proof of Proposition 3.6; for the implementation of the argument in the context of degenerate parabolic equations with measure data see our previous contribution in [4].

The "weighted" version we employ here (the weight is clearly represented by the large constant M, see few lines below, which we are going to choose later) has been developed in [2] to provide precise estimates of Calderón-Zygmund type in the setting of energy solutions and it is, in some sense, the PDE version of the "good-λ-inequality principle", see for instance [14].

Fix a parabolic cylinders $Q_{2R} \equiv Q_{2R}(z_0) \subset \Omega_T$, $R > 0$ and let $M \ge 1$ be a free parameter to be chosen. Moreover we fix arbitrarily an exponent $\bar q$ such that

$$1 < \bar q < p - 1 + \frac{1}{N-1} = m_0 < p \tag{5.1}$$

so that it only depends on n and p: for instance, we can choose the midpoint of the interval $[1, m_0]$. We then define the functional

$$CZ(Q^\lambda_r) := \fint_{Q^\lambda_r} |Du|\, dz + \left[M\frac{|\mu|(Q^\lambda_r)}{|Q^\lambda_r|}\right]^{\frac{1}{m}} \tag{5.2}$$

for cylinders $Q_r^\lambda \equiv Q_r^\lambda(\bar z) \subset Q_{2R}$, where $m \geq m_0 > 1$ is as in (1.9). We fix two intermediate radii $R \leq r_1 < r_2 \leq 2R$ and we define

$$\lambda_0^{\frac{1}{d}} = \lambda_0^{1-\frac{n(2-p)}{2}} := \fint_{Q_{r_2}} |Du|\,dz + \left[M\frac{|\mu|(Q_{r_2})}{|Q_{r_2}|}\right]^{\frac{1}{m}} + 1; \tag{5.3}$$

recall the expression for d in (1.8). Moreover we take λ satisfying $\lambda > B\lambda_0$ with $B \geq 1$ defined as

$$B := \left(\frac{40r_2}{r_2 - r_1}\right)^{Nd} \tag{5.4}$$

and we consider radii satisfying

$$\frac{r_2 - r_1}{40} \leq r \leq \frac{r_2 - r_1}{2}. \tag{5.5}$$

Observe that $Q_r^\lambda(\bar z) \Subset Q_{r_2}$ for any $\bar z \in Q_{r_1}$ and for any radius r satisfying (5.5). Using Hölder's inequality, enlarging the domain of integration and using that $|\mu|$ is positive we have, also taking (5.3), (5.4) and $1/m < 1$ into account

$$\begin{aligned} CZ(Q_r^\lambda(\bar z)) &\leq \frac{|Q_{r_2}|}{|Q_r^\lambda(\bar z)|}\left[\fint_{Q_{r_2}} |Du|\,dz + \left[M\frac{|\mu|(Q_{r_2})}{|Q_{r_2}|}\right]^{\frac{1}{m}}\right] \\ &\leq \frac{|Q_{r_2}|}{|Q_r^\lambda(\bar z)|}\lambda_0^{\frac{1}{d}} \\ &< \lambda^{\frac{2-p}{2}n}\left(\frac{r_2}{r}\right)^N \lambda^{\frac{1-n(2-p)}{2}} B^{\frac{1}{d}} \\ &\leq \lambda\left(\frac{40r_2}{r_2 - r_1}\right)^N B^{-\frac{1}{d}} \leq \lambda. \end{aligned} \tag{5.6}$$

On the other hand, for $\lambda > 0$ and radii $\gamma \in [R, 2R]$, we define the level sets

$$E(\lambda, Q_\gamma) := \Big\{z \in Q_\gamma(z_0) : |Du(z)| > \lambda \text{ and } z \text{ is a Lebesgue's point of } |Du|\Big\};$$

then we take a point $\bar z \in E(\lambda, Q_{r_1})$ with $\lambda > B\lambda_0$. It holds

$$\lim_{r\searrow 0} CZ(Q_r^\lambda(\bar z)) \geq \lim_{r\searrow 0} \fint_{Q_r^\lambda(\bar z)} |Du|\,dx > \lambda.$$

Hence for small radii we have $CZ(Q_r^\lambda(\bar z)) > \lambda$. Hence, from this consideration and (5.6), continuity implies the existence of a maximal radius $r_{\bar z}$ such that

$$\fint_{Q_{r_{\bar z}}^\lambda(\bar z)} |Du|\,dx + \left[M\frac{|\mu|(Q_{r_{\bar z}}^\lambda(\bar z))}{|Q_{r_{\bar z}}^\lambda(\bar z)|}\right]^{\frac{1}{m}} = \lambda, \tag{5.7}$$

that is $CZ(Q_{r_{\bar z}}^\lambda(\bar z)) = \lambda$. The word "maximal" refers to the fact that for all radii $\tilde r \in (r_{\bar z}, (r_2 - r_1)/2]$ the inequality $CZ(Q_{\tilde r}^\lambda(\bar z)) \leq \lambda$ holds. In particular we have

$$\frac{\lambda}{3^{4N}} \leq \frac{\lambda}{40^N} \leq \fint_{Q_{\jmath r_{\bar z}}^\lambda(\bar z)} |Du|\,dx + \left[M\frac{|\mu|(Q_{\jmath r_{\bar z}}^\lambda(\bar z))}{|Q_{\jmath r_{\bar z}}^\lambda(\bar z)|}\right]^{\frac{1}{m}} \leq \lambda \tag{5.8}$$

for every $\jmath \in \{1, \dots, 40\}$; note that $r_{\bar z} < (r_2 - r_1)/40$ and hence $Q_{\jmath r_{\bar z}}^\lambda(\bar z) \subset Q_{r_2}$.

Now we single out one of the cylinders $Q_{2r_{\bar z}}^\lambda(\bar z)$ considered above and we note that one of the following two inequalities must hold true:

$$\frac{\lambda}{4^{4N}} \leq \fint_{Q_{2r_{\bar z}}^\lambda(\bar z)} |Du|\,dz \qquad \text{or} \qquad \left(\frac{\lambda}{4^{4N}}\right)^m \leq M\frac{|\mu|(Q_{2r_{\bar z}}^\lambda(\bar z))}{|Q_{2r_{\bar z}}^\lambda(\bar z)|}. \tag{5.9}$$

First case: A comparison map "λ-close" to u. Suppose that the first case in (5.9) holds true. Now, thinking $\bar{z} \in E(4\lambda, Q_{r_1})$ fixed and being $Q^{\lambda}_{20r_{\bar{z}}}(\bar{z})$ one of the cylinders considered above, we will denote in short

$$20Q := Q^{\lambda}_{20r_{\bar{z}}}(\bar{z}).$$

We introduce in $20Q$ the function solution to the Cauchy-Dirichlet problem

$$\begin{cases} v_t - \operatorname{div} a(x,t,Dv) = 0 & \text{in } 20Q, \\ v = u & \text{on } \partial_p(20Q); \end{cases} \tag{5.10}$$

we observe that (5.8)-(5.9) imply

$$\frac{\lambda}{4^{4N}} \le \fint_{20Q} |Du|\,dz, \qquad \fint_{40Q} |Du|\,dz \le \lambda. \tag{5.11}$$

Thus Corollary 4.5 implies

$$\begin{aligned} \left(\fint_{20Q} |Du - Dv|^q\,dz\right)^{\frac{1}{q}} &\le c\left[\frac{|\mu|(40Q)}{|40Q|^{\frac{N-1}{N}}}\right]^{\frac{N}{(N-1)(p-1)+1}} \\ &\quad + c\,\frac{|\mu|(40Q)}{|40Q|^{\frac{N-1}{N}}}\left(\fint_{40Q}|Du|\,dz\right)^{(2-p)\frac{N-1}{N}} \end{aligned} \tag{5.12}$$

for q as in (4.3); $c \equiv c(n,p,\nu)$.

We now distinguish the two cases considered in Theorem 1.1:

The good measure case. This is the case where $p \le \vartheta \le n$; using (5.8) and the value for $m = m_1$ in this case we have

$$\frac{|\mu|(40Q)}{|40Q|} \le \frac{\lambda^{\frac{\vartheta(p-1)}{\vartheta-1}}}{M}. \tag{5.13}$$

On the other hand, we notice that we can cover the cylinder $40Q$ with at most $2\lfloor\lambda^{2-p}\rfloor$ (possibly overlapping) standard parabolic cylinders with radius $40r^{\lambda}_{\bar{z}}$ (this exact number of cylinders is clearly not optimal):

$$40Q = B_{40r^{\lambda}_{\bar{z}}}(\bar{x}) \times (\bar{t} - r^2_{\bar{z}}, \bar{t} + r^2_{\bar{z}}) \subset \bigcup_{j=1}^{2\lfloor\lambda^{2-p}\rfloor} Q_{40r^{\lambda}_{\bar{z}}}(\bar{x}, t_j)$$

for points $t_j \in (\bar{t} - r^2_{\bar{z}}, \bar{t} + r^2_{\bar{z}})$. Thus using (1.4)

$$|\mu|(40Q) \le \sum_{j=1}^{2\lfloor\lambda^{2-p}\rfloor} |\mu|\big(Q_{40r^{\lambda}_{\bar{z}}}(\bar{x}, t_j)\big) \le 2\lfloor\lambda^{2-p}\rfloor\, c_d [40r^{\lambda}_{\bar{z}}]^{N-\vartheta}$$

and in turn

$$\frac{|\mu|(40Q)}{|40Q|} \le c(n, c_d)\frac{\lambda^{2-p}[40r^{\lambda}_{\bar{z}}]^{N-\vartheta}}{\lambda^{2-p}[40r^{\lambda}_{\bar{z}}]^{N}} = c(n,c_d)[r^{\lambda}_{\bar{z}}]^{-\vartheta}.$$

We can thus estimate, for every $\epsilon \in (0,1)$

$$\begin{aligned} \frac{|\mu|(40Q)}{|40Q|^{\frac{N-1}{N}}} &= \left[\frac{|\mu|(40Q)}{|40Q|}\right]^{\frac{\vartheta-1}{\vartheta}} \left[\frac{|\mu|(40Q)}{|40Q|}\right]^{\frac{1}{\vartheta}} |40Q|^{\frac{1}{N}} \\ &\le \frac{c}{M^{\frac{\vartheta-1}{\vartheta}}}\lambda^{p-1}[40r^{\lambda}_{\bar{z}}]^{-1}\lambda^{\frac{2-p}{N}}\,40r^{\lambda}_{\bar{z}} = \frac{c}{M^{\frac{\vartheta-1}{\vartheta}}}\,\lambda^{\frac{(N-1)(p-1)+1}{N}}. \end{aligned}$$

The bad measure case. This is the case where $n < \vartheta \le N$ (and thus the measure might be more concentrated). As in (5.13) and noting that now $m = m_2$ we infer

$$\frac{|\mu|(40Q)}{|40Q|} \le \frac{\lambda^{\frac{1}{2}(p-\frac{(2-p)n}{\vartheta})\frac{\vartheta}{\vartheta-1}}}{M};$$

on the other hand, simply enlarging the cylinder (remember (2.3)), again using (1.4) on the standard parabolic cylinder $Q_{40r_{\tilde z}}$ and taking in mind that $N = n+2$, we have

$$\begin{aligned}\frac{|\mu|(40Q)}{|40Q|} &\le \frac{|\mu|(Q_{40r_{\tilde z}})}{|40Q|} \le c(n,c_d)\frac{[40r_{\tilde z}]^{N-\vartheta}}{\lambda^{2-p}[40r^{\lambda}_{\tilde z}]^N}\\ &= c(n,c_d)\frac{\lambda^{\frac{2-p}{2}(N-\vartheta)}[40r^{\lambda}_{\tilde z}]^{N-\vartheta}}{\lambda^{2-p}[40r^{\lambda}_{\tilde z}]^N}\\ &= c(n,c_d)\,\lambda^{\frac{2-p}{2}(n-\vartheta)}[40r^{\lambda}_{\tilde z}]^{-\vartheta}.\end{aligned}$$

So, similarly to above, we here have

$$\begin{aligned}\frac{|\mu|(40Q)}{|40Q|^{\frac{N-1}{N}}} &= \left[\frac{|\mu|(40Q)}{|40Q|}\right]^{\frac{\vartheta-1}{\vartheta}}\left[\frac{|\mu|(40Q)}{|40Q|}\right]^{\frac{1}{\vartheta}}|40Q|^{\frac{1}{N}}\\ &\le \frac{c}{M^{\frac{\vartheta-1}{\vartheta}}}\lambda^{\frac{1}{2}(p-\frac{(2-p)n}{\vartheta})}\lambda^{\frac{2-p}{2}(\frac{n}{\vartheta}-1)}[40r^{\lambda}_{\tilde z}]^{-1}\lambda^{\frac{2-p}{N}}40r^{\lambda}_{\tilde z}\\ &= \frac{c}{M^{\frac{\vartheta-1}{\vartheta}}}\lambda^{p-1}\lambda^{\frac{2-p}{N}} = \frac{c}{M^{\frac{\vartheta-1}{\vartheta}}}\lambda^{\frac{(N-1)(p-1)+1}{N}}.\end{aligned}$$

Thus in both cases we have

$$\frac{|\mu|(40Q)}{|40Q|^{\frac{N-1}{N}}} \le \frac{c}{M^{\frac{\vartheta-1}{\vartheta}}}\lambda^{\frac{(N-1)(p-1)+1}{N}}. \tag{5.14}$$

If now we consider (5.12), (5.14) and $(5.11)_2$ imply that

$$\begin{aligned}\left(\fint_{20Q}|Du-Dv|^q\,dx\right)^{\frac{1}{q}} &\le \bar c\left[M^{\frac{N}{(N-1)(p-1)+1}}+M\right]^{-\frac{\vartheta}{\vartheta-1}}\lambda\\ &=: \bar c\,\zeta(M)\lambda\end{aligned} \tag{5.15}$$

for q as in (4.3), with the obvious definition of ζ; note that we used also the computation in (4.15). Now we fix as M_1 the large constant satisfying $\bar c\,\zeta(M_1) \le 4^{-8N}$; note that we can take $M_1 \equiv M_1(n,p,\nu)$. If we take $M \ge M_1$, we have, for $q = 1$ and $q = \bar q$ as defined in (5.1)

$$\fint_{20Q}|Du-Dv|\,dx \le \frac{\lambda}{4^{8N}}, \qquad \left(\fint_{20Q}|Du-Dv|^{\bar q}\,dx\right)^{\frac{1}{\bar q}} \le \frac{\lambda}{4^{8N}}. \tag{5.16}$$

First we have, considering the first of the previous inequalities and (5.8)

$$\fint_{20Q}|Dv|\,dz \le \fint_{20Q}|Du|\,dz + \fint_{20Q}|Du-Dv|\,dz \le 2\lambda; \tag{5.17}$$

using (5.11),

$$\begin{aligned}\fint_{5Q}|Dv|\,dz &\ge \left(\frac{2}{5}\right)^N\fint_{2Q}|Du|\,dz\\ &\qquad\qquad - 4^N\fint_{20Q}|Du-Dv|\,dz\\ &\ge \frac{1}{4^N}\cdot\frac{1}{4^{5N}}\lambda - \frac{1}{4^{7N}}\lambda \ge \frac{\lambda}{4^{7N}}\end{aligned} \tag{5.18}$$

too. Therefore, applying Proposition 3.5, we infer

$$\fint_{10Q}|Dv|^{\bar q}\,dz \le c\,\lambda^{\bar q}, \qquad \fint_{10Q}|Dv|^p\,dz \le c\,\lambda^p; \tag{5.19}$$

we recall that $\bar{q}$ is defined in (5.1) as a constant in $(1,p]$ depending only on n and p. The constant c thus depends on n,p,ν and L.

Finally, we use $(5.19)_2$ and (5.18) that allow to apply Proposition 3.6: (3.5) reads here as

$$\left(\fint_{5Q} |Dv|^{p(1+\eta)}\,dz\right)^{\frac{1}{p(1+\eta)}} \le c\fint_{10Q} |Dv|\,dz \le c\,\lambda \tag{5.20}$$

with $c \equiv c(n,p,\nu,L)$, using (5.17). Finally, again using quasi-sub-additivity exactly as in (5.17), using this time $(5.16)_2$ together with $(5.19)_1$, we finally conclude this paragraph with the last estimate we need:

$$\fint_{5Q} |Du|^{\bar{q}}\,dz \le c\,\lambda^{\bar{q}}. \tag{5.21}$$

The constant still depends on n,p,ν and L.

First case: On the measure of the super-level of Du. We briefly conclude here the study of the case where the first inequality in (5.9) holds.

We split the integral of $|Du|$ over $2Q$ and then use Hölder's inequality as follows, for ς to be chosen:

$$\begin{aligned}\int_{2Q} |Du|\,dz &\le \varsigma\lambda|2Q \setminus E(\varsigma\lambda, Q_{r_2})| + \int_{2Q\cap E(\varsigma\lambda,Q_{r_2})} |Du|\,dz \\ &\le \varsigma\lambda|2Q| + |2Q\cap E(\varsigma\lambda, Q_{r_2})|^{1-\frac{1}{\bar{q}}}\left(\int_{2Q} |Du|^{\bar{q}}\,dz\right)^{\frac{1}{\bar{q}}} \\ &\le \varsigma\lambda|2Q| + \left(\frac{|2Q\cap E(\varsigma\lambda, Q_{r_2})|}{|2Q|}\right)^{1-\frac{1}{\bar{q}}}\left(\fint_{5Q} |Du|^{\bar{q}}\,dz\right)^{\frac{1}{\bar{q}}}.\end{aligned}$$

Dividing by $|2Q|$ and using $(5.9)_1$ and (5.21) yields

$$\frac{\lambda}{4^{4N}} \le \varsigma\lambda + \left(\frac{|2Q\cap E(\varsigma\lambda, Q_{r_2})|}{|2Q|}\right)^{1-\frac{1}{\bar{q}}}\lambda$$

Now we fix $\varsigma = 4^{-5N}$ and thus we have

$$|2Q| \le c\,|2Q\cap E(\varsigma\lambda, Q_{r_2})|$$

for a constant depending on n,p,ν,L and c_d.

Second case and conclusion. Clearly, if the second alternative $(5.9)_2$ holds, we have

$$|2Q| \le c(n)\frac{M}{\lambda^m}|\mu|(2Q)$$

Thus merging those two cases we finally get the estimate for $|2Q|$ we were looking for:

$$|20Q| \le c(n)\,|2Q| \le c\,|2Q\cap E(\lambda, r_2)| + c\,\frac{M}{\lambda^m}|\mu|(2Q). \tag{5.22}$$

We recall that $2Q \equiv Q^{\lambda}_{2r_{\bar{z}}}(\bar{z})$.

Covering and iteration. Summarizing what we have done up to now, we have that once we fix $\lambda > B\lambda_0$, with B and λ_0 defined in (5.3)-(5.4), then for every $\bar{z} \in E(\lambda, Q_{r_1})$ we can find a cylinder $Q^{\lambda}_{r_{\bar{z}}}(\bar{z})$ such that (5.7) holds.

Then we consider the collection of all such cylinders $\mathcal{E}_\lambda := \{Q^{\lambda}_{2r_{\bar{z}}}(\bar{z})\}_{\bar{z}\in E(\lambda,Q_{r_1})}$ and, by a Vitali-type argument, we extract a countable sub-collection $\mathcal{F}_\lambda \subset \mathcal{E}_\lambda$ such that the 5-times enlarged cylinders cover almost all $E(\lambda, Q_{r_1})$ and the cylinders are pairwise disjoints - note that we are working at λ fixed; thus those cylinders are metric balls (precisely, of the metric defined in (2.4)). This is to say, if we denote the cylinders of $\mathcal{F}_\lambda$ by $Q^0_i := Q^{\lambda}_{2r_{\bar{z}_i}}(\bar{z}_i)$, for $i \in \mathcal{I}_\lambda$, being possibly $\mathcal{I}_\lambda = \mathbb{N}$, and with $\bar{z}_i \in E(\lambda, Q_{r_1})$, we have

$$Q^0_i \cap Q^0_j = \emptyset \quad \text{whenever } i \neq j \quad \text{and} \quad E(\lambda, Q_{r_1}) \subset \bigcup_{i\in\mathcal{I}_\lambda} Q^1_i \cup \mathcal{N}, \tag{5.23}$$

with $|\mathcal{N}| = 0$ and with $Q^1_i := 5Q^0_i = Q^{\lambda}_{10r_{\bar{z}_i}}(\bar{z}_i)$; note that using (5.5) we see that $Q^1_i \subset Q_{r_2}$ for all $i \in \mathcal{I}_\lambda$. We now fix $H \geq 1$ to be chosen later and we estimate

$$\big|E(H\lambda, r_1)\big| \leq \sum_{i\in\mathcal{I}_\lambda} \big|Q^1_i \cap E(2H\lambda, r_2)\big|. \tag{5.24}$$

We split every term in the following way:

$$\begin{aligned}\big|Q^1_i \cap E(H\lambda, r_2)\big| &= \big|\{z \in Q^1_i : |Du(x)| > 2H\lambda\}\big| \\ &\leq \big|\{z \in Q^1_i : |Du(x) - Dv_i(x)| > H\lambda\}\big| \\ &\quad + \big|\{z \in Q^1_i : |Dv_i(x)| > H\lambda\}\big| =: I_i + II_i.\end{aligned} \tag{5.25}$$

Here v_i is the comparison function solution to (5.10) in $Q^2_i \equiv Q^{\lambda}_{20r_{\bar{z}_i}}(\bar{z}_i) = 2Q^1_i$. We estimate separately the two pieces: for the first one we take $\epsilon > 0$ arbitrary and we use (5.15) and subsequently (5.22) to infer

$$\begin{aligned}I_i \leq \frac{1}{H\lambda}\int_{Q^2_i} |Du - Dv_i|\,dz &\leq \frac{c\,\zeta(M)}{H\lambda}|Q^2_i|\lambda \\ &\leq \frac{c\,\zeta(M)}{H}\left[|Q^0_i \cap E(\lambda, Q_{r_2})| + M\frac{|\mu|(Q^0_i)}{\lambda^m}\right],\end{aligned} \tag{5.26}$$

where $\zeta(\cdot)$ is given in (5.15) and provided $M \geq M_1$. On the other hand we use the higher integrability (5.20) to get

$$\begin{aligned}II_i &\leq \left(\frac{1}{H\lambda}\right)^{p(1+\eta)}|Q^1_i| \fint_{Q^1_i} |Dv_i|^{p(1+\eta)}\,dz \\ &\leq \frac{c}{(H\lambda)^{p(1+\eta)}}|Q^2_i|\left(\fint_{Q^2_i} |Dv_i|\,dz\right)^{p(1+\eta)} \\ &\leq \frac{c}{H^{p(1+\eta)}}\left[|Q^0_i \cap E(\lambda, Q_{r_2})| + M\frac{|\mu|(Q^0_i)}{\lambda^m}\right].\end{aligned} \tag{5.27}$$

Connecting the two estimates (5.26) and (5.27) and plugging the result into (5.25), taking into account that $H \geq 1$, gives

$$\big|E(2H\lambda, Q_{r_2}) \cap Q^1_i\big| \leq \left[\frac{c\,\zeta(M)}{H} + \frac{c}{H^{p(1+\eta)}}\right]|Q^0_i \cap E(\lambda, r_2)| + c\,M\frac{|\mu|(Q^0_i)}{\lambda^m}.$$

At this point, since the $\{Q^0_i\}$ are disjoint, see (5.23), summing up and multiplying both sides of the previous inequality by $(2H\lambda)^m$, see also (5.24), gives

$$\begin{aligned}(2H\lambda)^m\big|E(2H\lambda, Q_{r_1})\big| \leq \left[\frac{c_*\,\zeta(M)}{H^{1-m}} + \frac{c_*}{H^{p(1+\eta)-m}}\right]\lambda^m|E(\lambda, Q_{r_2})| \\ + c\,|\mu|(Q_{2R})\end{aligned} \tag{5.28}$$

with c_* depending only on n, p, ν, L, c_d but nor on H neither on M; c instead depends also on H, M but this is not a problem. Finally, recall that $1 < m < p\chi$; first choose H so large that

$$\frac{c_*}{H^{p(1+\eta)-m}} = \frac{1}{4}. \tag{5.29}$$

At this point, being now $H \equiv H(n, p, \nu, L, c_d)$ fixed, we choose M_2 so large that

$$c_*\, H^{m-1}\zeta(M_2) \leq \frac{1}{4}$$

and this fixes the value of M in (5.2) as $\max\{M_1, M_2\}$. This choice makes also M depend only on n, p, ν, L and c_d. Having such choices at hand, after taking the supremum with respect to $\lambda > B\lambda_0$, (5.28) rewrites as

$$\begin{aligned} \sup_{\lambda > 2HB\lambda_0} \lambda^m |E(\lambda, Q_{r_1})| &\leq \frac{1}{2} \sup_{\lambda > B\lambda_0} \lambda^m |E(\lambda, Q_{r_2})| + c\,|\mu|(Q_{2R}) \\ &\leq \frac{1}{2} \|Du\|^m_{\mathcal{M}^m(Q_{r_2})} + c\,|\mu|(Q_{2R}) \end{aligned} \tag{5.30}$$

and therefore by the definition of Marcinkiewicz quasi-norm

$$\|Du\|^m_{\mathcal{M}^m(Q_{r_1})} \leq \frac{1}{2}\|Du\|^m_{\mathcal{M}^m(Q_{r_2})} + c\,\big[B\lambda_0\big]^m R^N + c\,|\mu|(Q_{2R})$$

for all $R \leq r_1 < r_2 \leq 2R$, since $B\lambda_0 \geq 1$. At this point (1.7) follows exactly as in [4], using for the last time [19, Lemma 6.1]. Note that reabsorption is possible since u is an approximate energy solutions, thus since $Du \in L^{p\chi}_{\mathrm{loc}}(\Omega_T)$, we have $\|Du\|_{\mathcal{M}^m(Q_{2R})} < \infty$ for $m < p\chi$.

6. Proof of Theorems 1.5, 1.6 and 1.7

The proof of Theorem 1.5 is very similar to the proof of Theorem 1.1: the changes concern the regularity of the reference problem, that is the integrability of the comparison map solution to (3.1) and hence of (4.1)-(5.10). If we consider vector fields a satisfying (1.14), then the solution is indeed such that $Du \in L^q$ for every $q \geq 1$; this is to say, (3.5) holds for every $\eta > 1$ (and clearly also the constant at this point depends on η), see [2, Theorem 1] and [10] taking into account that our right-hand side is zero, thus in L^q for all $q \geq 1$. The proof needed in order to obtain the explicit local estimate (3.5) from the results in [2, 10] is exactly the same described in the proof of Proposition 3.6 to obtain (3.5) starting from [9, Lemma 13]; the changes only concern the range of exponents η considered.

We can formally follow our proof, with the only difference that here η is not a constant depending on the data of the problem, but a free parameter; the constants will thus depend also on η (and they all will show a critical dependence on η, in the sense that they will blow-up as $\eta \to \infty$). To conclude, given $\vartheta > 1$, we will to follow the proof until (5.28) and there we will choose $\eta \equiv \eta(p, \vartheta)$ such that $p(1+\eta) = m+1$; this will be sufficient to prove that $|Du| \in \mathcal{M}^m_{\mathrm{loc}}(\Omega_T)$. Note that our choice justifies the critical dependence of the constant upon ϑ, as $\vartheta \to 1$. The aforementioned Theorems in [2, 10] also justify the reabsorption after (5.30): since the data of our approximating problems are regular, then the energy solutions u we consider are as integrable as needed.

For the proofs of Theorems 1.6-1.7, the only different point is the treatment of the second term in

$$\frac{|\mu|(40Q)}{|40Q|^{\frac{N-1}{N}}} = \left[\frac{|\mu|(Q^\lambda_{40r_{\bar z}}(\bar z))}{|Q^\lambda_{40r_{\bar z}}(\bar z)|}\right]^{\frac{\vartheta-1}{\vartheta}} \left[\frac{|\mu|(Q^\lambda_{40r_{\bar z}}(\bar z))}{|Q^\lambda_{40r_{\bar z}}(\bar z)|}\right]^{\frac{1}{\vartheta}} |Q^\lambda_{40r_{\bar z}}(\bar z)|^{\frac{1}{N}};$$

this will allow for a different value of the exponent m in the Calderón-Zygmund operator in (5.2), used to estimate the first term in the quantity above if we want to mimic the algebraic computations leading to (5.14) (this is the real core of the proof, since this is sufficient to give to the comparison estimates (5.15) their "homogeneous" form). The exponent m appearing in (5.2), in view of the rest of the proof, will become the Marcinkiewicz exponent appearing in the statements: respectively m_3 for Theorem 1.6 and m_4 for Theorem 1.7.

In particular for Theorem 1.6 we can estimate

$$\begin{aligned}|\mu|(40Q) &= |\mu_1|(B_{40r^\lambda_{\bar z}(\bar x)})|\mu_2|\big((\bar t-[40r_{\bar z}]^2,\bar t+[40r_{\bar z}]^2)\big)\\ &\le \|\mu_1\|_{L^\infty}\, c(n)[40r^\lambda_{\bar z}]^n\, c_d[40r_{\bar z}]^{2-\vartheta}\\ &= c(n,\|\mu_1\|_{L^\infty},c_d)\lambda^{\frac{2-p}{2}(2-\vartheta)}[40r_{\bar z}]^{N-\vartheta},\end{aligned}$$

while for Theorem 1.7 we have

$$\begin{aligned}|\mu|(40Q) &= |\mu_3|(B_{40r^\lambda_{\bar z}(\bar x)})|\mu_4|\big((\bar t-[40r_{\bar z}]^2,\bar t+[40r_{\bar z}]^2)\big)\\ &\le c_d[40r^\lambda_{\bar z}]^{n-\vartheta}\,\|\mu_4\|_{L^\infty}\, 2[40r_{\bar z}]^2\\ &= c(n,\|\mu_4\|_{L^\infty},c_d)\lambda^{2-p}[40r_{\bar z}]^{N-\vartheta}.\end{aligned}$$

Now for Theorem 1.6 we have

$$\begin{aligned}\frac{|\mu|(40Q)}{|40Q|^{\frac{N-1}{N}}} &\le c\left[\frac{\lambda^{m_3}}{M}\right]^{\frac{\vartheta-1}{\vartheta}}\left[\frac{\lambda^{\frac{2-p}{2}(2-\vartheta)}[40r_{\bar z}]^{N-\vartheta}}{\lambda^{2-p}[40r^\lambda_{\bar z}]^N}\right]^{\frac{1}{\vartheta}}\lambda^{\frac{2-p}{N}}\,40r^\lambda_{\bar z}\\ &= \frac{c}{M^{\frac{\vartheta-1}{\vartheta}}}\lambda^{\frac{p}{2}}\lambda^{\frac{p-2}{2}}[40r^\lambda_{\bar z}]^{-1}\,\lambda^{\frac{2-p}{N}}\,40r^\lambda_{\bar z} = \frac{c}{M^{\frac{\vartheta-1}{\vartheta}}}\lambda^{p-1}\lambda^{\frac{2-p}{N}}\end{aligned}$$

and (recall that here $p\ge 2$ and thus the intrinsic cylinders are different, see (2.2) and compare with [4])

$$\begin{aligned}\frac{|\mu|(40Q)}{|40Q|^{\frac{N-1}{N}}} &\le c\left[\frac{\lambda^{m_4}}{M}\right]^{\frac{\vartheta-1}{\vartheta}}\left[\frac{\lambda^{2-p}[40r_{\bar z}]^{N-\vartheta}}{\lambda^{2-p}[40r^\lambda_{\bar z}]^N}\right]^{\frac{1}{\vartheta}}\lambda^{\frac{2-p}{N}}\,40r^\lambda_{\bar z}\\ &= \frac{c}{M^{\frac{\vartheta-1}{\vartheta}}}\lambda^{p-1}[40r^\lambda_{\bar z}]^{-1}\,\lambda^{\frac{2-p}{N}}\,40r^\lambda_{\bar z} = \frac{c}{M^{\frac{\vartheta-1}{\vartheta}}}\lambda^{p-1}\lambda^{\frac{2-p}{N}}\end{aligned}$$

for Theorem 1.7; both these expressions lead exactly to (5.14). Now the proofs proceed exactly as after (5.14), taking into account the first lines of this Section when dealing, if $p<2$, with vector field satisfying the assumptions in (1.12)-(1.13).

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