ON THE BOUNDARY BEHAVIOR FOR THE BLOW UP SOLUTIONS OF THE SINH-GORDON EQUATION AND RANK N TODA SYSTEMS IN BOUNDED DOMAINS

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ABSTRACT. In this paper we are concerned with the blow up analysis of two classes of problems in bounded domains arising in mathematical physics: sinh-Gordon equation and some general rank n Toda systems. The presence of a residual mass in the blowing up limit makes the analysis quite delicate: nevertheless, by exploiting suitable Pohozaev identities and a detailed blow up analysis we exclude blow up at the boundary. This is the first result in this direction in the presence of a residual mass. As a byproduct we obtain general existence results in bounded domains.

1. INTRODUCTION

In the present paper we are concerned with the blow up analysis of two classes of problems defined in bounded domains: sinh-Gordon equation (1.1) and a general rank n Toda systems (1.8). Let us start by considering

$$\begin{cases} \Delta u + \rho_1 \dfrac{h_1 e^u}{\int_\Omega h_1 e^u} - \rho_2 \dfrac{h_2 e^{-u}}{\int_\Omega h_2 e^{-u}} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^2$ is a bounded domain with smooth boundary $\partial\Omega$ and Δ is the Euclidean Laplace operator, ρ_1, ρ_2 are two positive parameters and h_1, h_2 are two smooth positive functions in Ω.

Equation (1.1) arises in mathematical physics as a mean-field equation of the equilibrium turbulence with arbitrarily signed vortices and it was first derived by Joyce and Montgomery [29] and by Pointin and Lundgren [50]. For more discussions concerning the physical background we refer for example to [11, 40, 44, 45, 47] and references therein. The case $h_1 = h_2$, $\rho_1 = \rho_2$ has a close relationship with geometry and is related to the study of constant mean curvature surfaces, see [59, 60].

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When $\rho_2 = 0$ equation (1.1) reduces to the following well-known Liouville problem:

$$\begin{cases} \Delta u + \rho \dfrac{he^u}{\int_\Omega he^u} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Equation (1.2) has been extensively studied in the literature since it is related to the prescribed Gaussian curvature problem and Euler flows, see [1, 55] and [9, 30], respectively. We refer the interested readers to the survey [56]. There are by now many results concerning (1.2), some of which we recall here to highlight the difference with the general case (1.1). Suppose (u_k, ρ_k) is a sequence of blow up solutions to (1.2) with ρ_k uniformly bounded, then it is known that there is no boundary blow up, namely that u_k is uniformly bounded near a neighborhood of $\partial\Omega$, see [42, 46]. Furthermore, we have a clear understanding of the blowing up solution: in particular, it holds $\rho_k \to 8m\pi$, for some $m \in \mathbb{N}$ and, after passing to a subsequence,

$$u_k(x) \to 8\pi \sum_{j=1}^m G(x, p_j) \qquad \text{in } C^2_{\mathrm{loc}}(\Omega \setminus \{p_1, \cdots, p_m\}), \tag{1.3}$$

$$\rho_k \frac{he^{u_k}}{\int_\Omega he^{u_k}} \to 8\pi \sum_{j=1}^m \delta_{p_j} \qquad \text{in the sense of measures,} \tag{1.4}$$

for some $p_j \in \Omega$, where $G(\cdot, p_j)$ is the Green function of $-\Delta$ with pole at p_j and Dirichlet boundary condition, see [8, 32, 33, 42, 46] (see also the recent results [2, 3] concerning uniqueness and non degeneracy of blowing up solutions, respectively). Roughly speaking, the latter properties denote a concentration property of the blowing up solution: in particular, all the mass is concentrated around the points p_j and there is no residual mass in the region $\Omega \setminus \{p_1, \cdots, p_m\}$. In this respect, the general case (1.1) (and the Toda system (1.8)) presents drastic differences since there might be a residual mass in the blow up limit (1.4), see the discussion later on (see in particular (3.11)-(3.12)). The latter fact makes the analysis quite delicate and our goal is to address this aspect of the problem.

Let us now return to equation (1.1). The latter problem has attracted a lot of attention in the last decades: we refer to [25, 26, 28, 47] for blow up analysis, to [17] for uniqueness aspects and to [5, 18, 19, 20] for what concerns existence results. In this paper we shall consider the blow up analysis of solutions to (1.1) and we shall address the existence of possible boundary bubbles. Such study is of independent interest: moreover, the exclusion of boundary blow up allows to exploit the analysis developed for the internal region and hence to extend some results for (1.1) (and (1.8)) from the compact surface case to the bounded domain setting. In particular, we may deduce the topological degree counting computed in [26, 31] or the existence results obtained in [5, 4], see for instance Theorems 1.3, 1.4.

In the case of equation (1.2) boundary blow up is excluded by the method of moving planes and the use of Kelvin's transform, see for example [42]. We point out such argument can be suitably used to treat cooperative systems, see [58], where we say

$$\begin{cases} \Delta u + f(x, u, v) = 0, \\ \Delta v + g(x, u, v) = 0, \end{cases}$$

is cooperative if $\frac{\partial f(x,u,v)}{\partial v} \geq 0$, $\frac{\partial g(x,u,v)}{\partial u} \geq 0$. Concerning the general equation (1.1) let us point out that one can uniquely decompose u in $u = u_1 - u_2$, where u_1 and u_2 satisfy

$$\begin{cases} \Delta u_1 + \rho_1 \dfrac{h_1 e^{u_1 - u_2}}{\int_\Omega h_1 e^{u_1 - u_2}} = 0, \quad u_1 = 0 \text{ on } \partial\Omega, \\ \Delta u_2 + \rho_2 \dfrac{h_2 e^{u_2 - u_1}}{\int_\Omega h_2 e^{u_2 - u_1}} = 0, \quad u_2 = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.5}$$

However, the system (1.5) is not cooperative and the method of moving planes does not apply. Instead, our strategy is to use suitably Pohozaev identities and a detailed analysis of the behavior of the blow up solutions. Similar arguments have been used by Robert-Wei [51] in the fourth order mean field equation and then by Lin-Wei-Zhao [35] in the $SU(3)$ Toda system, which corresponds to the case $\mathbf{K} = \mathbf{A}_2$ below.

Some remarks are needed here. In the works [35, 51] the concentration property (1.4) plays an important role. In the seminal work [8], Brezis and Merle studied the blow up behavior of the standard Liouville equation alike (1.2) and showed the "bubbling implies mass concentration" result, i.e., if the blow up phenomena happens, then the concentration property (1.4) holds. For the fourth order equation considered in [51] it is not difficult to deduce this property by a similar argument as in [8]. However, as for the sinh-Gordon equation (1.1) (and the general Toda system (1.8)) we can not expect to have the corresponding property. Indeed, in [13] the authors exhibit bubbling solution for the Toda system (1.8) with non vanishing residual mass, that is with no concentration property (1.4). The latter result seems to applied to the equation (1.1) as well: in particular, we can not exclude the presence of bubbling solutions u_k of (1.1) which blow up at some points $p_j \in \Omega$ and such that

$$\frac{h_1 e^{u_k}}{\int_\Omega h_1 e^{u_k}}, \frac{h_2 e^{-u_k}}{\int_\Omega h_2 e^{-u_k}} \not\rightharpoonup 0 \qquad \text{in } \Omega \setminus \{p_1, \cdots, p_m\}, \tag{1.6}$$

see for instance (3.11)-(3.12). For the question whether concentration property holds or not, we shall pursue this in a forthcoming paper. In particular, we can not conclude the bubbling solutions u_k of (1.1) converge to a sum of Green functions away from the blow up points as in (1.3). This makes the study of equation (1.1) more complicated. In order to overcome this difficulty, we have to refine the argument from [35, 51] and carry out a delicate analysis for the bubbling solutions around the blow up point. In particular, the first part of the argument follows the ideas introduced in [51, 35] but the final part substantially differs from the previous strategies. To the best of our knowledge, this is the first paper in treating this kind of problems without using the concentration property (1.6) and it applies to very general problems as the sinh-Gordon equation (1.1) (see also the asymmetric case (1.7)) and some general rank n Toda systems (1.8). This is the main contribution of this work. It is worth pointing out that an analogue procedure was already used to rule out the blow up at the boundary for Sobolev critical problems in higher dimension, where the weak limit plays the role of the residual mass, see for instance [14] and the references therein.

The blow up set of $|u_k|$ is defined by

$$\mathcal{S} = \Big\{ p \in \overline{\Omega} \mid \exists \{x_k\},\ x_k \to p,\ |u_k|(x_k) \to +\infty \Big\}.$$

Our first main result is the following.

Theorem 1.1. *Let u_k be a bubbling sequence of solutions to* (1.1) *with $\frac{1}{C} \le \rho_{1k}, \rho_{2k} \le C$, that is*

$$\max_{x \in \Omega} |u_k| \to +\infty,$$

for $k \to \infty$. Then, the blow up set of $|u_k|$ is in the interior of $\overline{\Omega}$, that is $\mathcal{S} \cap \partial\Omega = \emptyset$.

Remark 1. The method can be also applied to the following asymmetric sinh-Gordon equation:

$$\begin{cases} \Delta u + \rho_1 \dfrac{h_1 e^{u}}{\int_\Omega h_1 e^{u}} - \rho_2 \dfrac{h_2 e^{-au}}{\int_\Omega h_2 e^{-au}} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

with $a > 0$. The latter equation arises in the context of the statistical mechanics description of $2D$-turbulence under a deterministic assumption on the vortex intensities, see [49, 54]. Concerning equation (1.7) we refer the interested readers to the recent results [21, 23, 24, 52, 53].

The second class of problems we consider is the following rank n Toda system:

$$\begin{cases} \Delta u_i + \displaystyle\sum_{j=1}^{n} K_{ij} \rho_j \frac{h_j e^{u_j}}{\int_\Omega h_j e^{u_j}} = 0 & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega, \end{cases} \qquad i = 1, \cdots, n, \tag{1.8}$$

where ρ_i are positive parameters, h_i are positive smooth functions and $\mathbf{K} = (K_{ij})_{n \times n}$ is one of the following rank n Cartan matrices: $\mathbf{A}_n$, $\mathbf{B}_n$ or $\mathbf{C}_n$, where

$$\mathbf{A}_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}, \quad \mathbf{B}_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -2 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix},$$

$$\mathbf{C}_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -2 & 2 \end{pmatrix}.$$

For $n = 2$ we have an extra Cartan type matrix $\mathbf{G}_2$ given by

$$\mathbf{G}_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

Therefore, from now on $\mathbf{K}$ will denote on of the above matrices $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n$ or $\mathbf{G}_2$. The problem (1.8) has been extensively studied in the literature since it has several applications both in mathematical physics and in geometry. For instance, it arises in the non-abelian Chern-Simons theory [15, 57, 61], while in geometry it appears in the description of holomorphic curves in $\mathbb{CP}^n$ [7, 10, 37]. For more background of (1.8) with a general Cartan matrix $\mathbf{K}$, one can see [15, 37] and the references therein. For what concerns the analytical studies about Toda-type systems we refer to [27, 31, 36] for blow up analysis, to [37] for classification issues and to [5, 22, 43] for existence results.

For $\mathbf{K} = \mathbf{A}_2$ the authors in [35] proved there is no boundary blow up under some assumptions on the concentration property (1.4). We show here the boundary blow up is excluded in a general situation with no extra assumptions. Moreover, we can cover general rank n Toda systems. Our second main result is the following.

Theorem 1.2. *Let $\mathbf{K}$ be one of the Cartan matrices $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$ or $\mathbf{G}_2$, and let $(u_{ik})_i$, $i = 1, \cdots, n$ be a bubbling sequence of solutions to* (1.8) *with $\frac{1}{C} \leq \rho_{ik} \leq C$, $i = 1, \cdots, n$, that is*

$$\max_{x\in\Omega} \max\{u_{ik}, \cdots, u_{nk}\} \to +\infty,$$

for $k \to \infty$. Then, the blow up set of $u_{1k}, \cdots, u_{nk}$ is in the interior of $\overline{\Omega}$.

Finally, once the exclusion of boundary blow up is proven we may exploit the analysis developed for the internal region and extend some results for (1.1) and (1.8) from the compact surface case to the bounded domain setting. Let us focus on the existence results concerning these classes of problems. The strategy is mainly based on the variational structure of the problems (1.1) and (1.8). More precisely, by a Morse-theoretical approach it is possible to deduce existence of solutions by detecting a change of topology of the sublevels of the associated energy functional. On the other hand, to carry out the Morse theory one has to rule out the blow up solutions for which the problem presents a lack of compactness. To this end we have to avoid a critical set of the parameters ρ_i which is deeply connected with the quantized blow up local masses (see for example Lemma 3.5).

Concerning the sinh-Gordon equation (1.1) the latter program was carried out in [5], where the authors obtained existence of solutions on compact surfaces with non positive Euler characteristic for $\rho_1, \rho_2 \notin 8\pi\mathbb{N}$. Once the exclusion of boundary blow up is proven it is not difficult to adapt the latter argument for treating the Dirichlet problem (1.1). We have the following.

Theorem 1.3. *Let $\rho_1, \rho_2 \notin 8\pi\mathbb{N}$ and suppose $\chi(\Omega) \leq 0$, where $\chi(\Omega)$ denotes the Euler characteristic of Ω. Then, there exists a solution to* (1.1).

Concerning the Toda system (1.8) a first general existence result is presented in [5] for the case $\mathbf{K} = \mathbf{A}_2$ on compact surfaces with non positive Euler characteristic for $\rho_1, \rho_2 \notin 4\pi\mathbb{N}$. The argument is again based on the variational structure of the problem and it was next adapted in [4] for treating the case $\mathbf{B}_2$ ($= \mathbf{C}_2$) and $\mathbf{G}_2$. Moreover, by further assuming the non existence of blow up solutions to (1.8), the author in [4] generalizes the argument for treating general matrices $\mathbf{K}$, $n \geq 3$, provided they are symmetric, positive definite and with non-positive entries outside the diagonal. Very recently, C.S. Lin, X.X. Zhong and the third author of the present paper proved in [38] that the local masses of the bubbling solutions to (1.8) are multiple of 4π and thus that the solutions are uniformly bounded provided $\rho_i \notin 4\pi\mathbb{N}$ for all $i = 1, \ldots, n$, see also the discussion after Lemma 3.11. In conclusion, since we can rewrite the Toda system with $\mathbf{B}_n$ and $\mathbf{C}_n$ in a symmetric form through the following simple transformations, respectively,

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -2 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{n-1} \\ \rho_n \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{n-1} \\ 2\rho_n \end{pmatrix},$$

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{n-1} \\ \rho_n \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -2 \\ 0 & \cdots & 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{n-1} \\ \frac{1}{2}\rho_n \end{pmatrix},$$

we derive the following result.

Theorem 1.4. *Let $\mathbf{K}$ be one of the Cartan matrices $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$ or $\mathbf{G}_2$. Let $\rho_i \notin 4\pi\mathbb{N}$ for any $i = 1, \ldots, n$, and suppose $\chi(\Omega) \leq 0$, where $\chi(\Omega)$ denotes the Euler characteristic of Ω. Then, there exists a solution to* (1.8).

We point out Theorems 1.3, 1.4 are the first existence results for (1.1), (1.8) with Dirichlet boundary conditions and they hold for a general choice of the parameters ρ_i. In particular, Theorem 1.4 is new also for $n = 2$.

The present paper is organized as follows. In Section 2 we collect some useful results and derive the Pohozaev-type identities, in Section 3 we prove the main Theorems 1.1, 1.2 concerning the no boundary blow up, in Section 4 we provide the key steps for the proof of Theorems 1.3, 1.4 concerning the existence results.

Notation

Throughout this paper, without other explanations, the constant C will denote some generic constant which is independent of k and the value of C might change from one line to the other. The quantity $B = O(A)$ means that there exists $C > 0$ such that $|B| \leq CA$. All the convergence results are stated by passing to a subsequence. The symbol $B_r(p)$ will denote the open ball of radius r and center p.

2. Useful facts

In this section we list some useful results which will be used in the sequel. First, we collect some properties of the Green function in the following lemma, see for example [12, 16].

Lemma 2.1. *Let $G(x, y)$ be the Green function of $-\Delta$ with Dirichlet boundary condition. There exists $C > 0$ such that for all $x, y \in \Omega$, $x \neq y$, we have*

$$|G(x,y)| \leq C \log\left(2 + \frac{1}{|x-y|}\right), \quad |\nabla G(x,y)| \leq C|x-y|^{-1}.$$

Let now u_k be a sequence of solutions to (1.1) relative to (ρ_{1k}, ρ_{2k}). We set

$$\alpha_{1k} = \log\left(\frac{\int_\Omega h_1 e^{u_k}}{\rho_{1k}}\right), \quad \alpha_{2k} = \log\left(\frac{\int_\Omega h_2 e^{-u_k}}{\rho_{2k}}\right),$$

and

$$v_k = u_k - \alpha_{1k}, \quad w_k = -u_k - \alpha_{2k}. \tag{2.1}$$

Then, we write equation (1.1) into the following form

$$\begin{cases} \Delta v_k + h_1 e^{v_k} - h_2 e^{w_k} = 0, & v_k = -\alpha_{1k} \text{ on } \partial\Omega, \\ \Delta w_k - h_1 e^{v_k} + h_2 e^{w_k} = 0, & w_k = -\alpha_{2k} \text{ on } \partial\Omega. \end{cases} \tag{2.2}$$

The blow up set for problem (2.2) is defined by

$$\mathfrak{S}_1 = \left\{ p \in \overline{\Omega} \mid \exists \{x_k\}, x_k \to p,\ v_k(x_k) \to +\infty \right\},$$

$$\mathfrak{S}_2 = \left\{ p \in \overline{\Omega} \mid \exists \{x_k\}, x_k \to p,\ w_k(x_k) \to +\infty \right\}$$

and

$$\mathfrak{S} = \mathfrak{S}_1 \cup \mathfrak{S}_2. \tag{2.3}$$

Concerning the set $\mathfrak{S}$ we have the following result.

Lemma 2.2. *The set $\mathfrak{S}$ is finite.*

Proof. The finiteness of the number of blow up points is a very standard result for the standard Liouville equation (1.2). The argument was then carried out for similar problems by using the celebrated result of [8, Corollary 4], see [31, Lemma 2.1, Lemma 2.2]. We shall provide here a complete proof of this fact for the Sinh-Gordon type equation (1.1) to make this paper more self-contained.

For any $p \in \mathfrak{S}$ we define the local mass for u_k and $-u_k$ around p by

$$\sigma_{1,p} = \frac{1}{2\pi} \lim_{\delta\to 0} \lim_{k\to 0} \int_{B_\delta(p)} \frac{\rho_{1k} h_1 e^{u_k}}{\int_\Omega h_1 e^{u_k}} \quad \text{and} \quad \sigma_{2,p} = \frac{1}{2\pi} \lim_{\delta\to 0} \lim_{k\to 0} \int_{B_\delta(p)} \frac{\rho_{2k} h_2 e^{-u_k}}{\int_\Omega h_2 e^{-u_k}}.$$

Step 1. We claim that if $\sigma_{1,p}, \sigma_{2,p} < \frac{1}{3}$, then $p \notin \mathfrak{S}$.

Since $\sigma_{i,p} < \frac{1}{3}$, we can choose a small $r_0 > 0$ such that

$$\int_{B_{r_0}(p)} \rho_{ik} h_i e^{\widetilde{u}_{ik}} < \pi, \qquad i = 1, 2, \tag{2.4}$$

where

$$\widetilde{u}_{1k} = u_k - \log \int_\Omega h_1 e^{u_k} \quad \text{and} \quad \widetilde{u}_{2k} = -u_k - \log \int_\Omega h_2 e^{-u_k}. \tag{2.5}$$

From (2.4), by the assumptions on ρ_{ik} and h_i jointly with the Jensen inequality we can easily get $\int_{B_{r_0}(p)} \widetilde{u}_{ik}^+ \le C$, where we use the notation $u^+ = \max\{u, 0\}$. We next use the decomposition $\widetilde{u}_{1k} = \sum_{j=1}^2 \widetilde{u}_{1k,j}$, where $\widetilde{u}_{1k,j}$ satisfy the following equations:

$$\begin{cases} -\Delta \widetilde{u}_{1k,1} = \rho_{1k} h_1 e^{\widetilde{u}_{1k}} - \rho_{2k} h_2 e^{\widetilde{u}_{2k}} & \text{in } B_{r_0}(p), \quad \widetilde{u}_{1k,1} = 0 \quad \text{on } \partial B_{r_0}(p), \\ -\Delta \widetilde{u}_{1k,2} = 0 & \text{in } B_{r_0}(p), \quad \widetilde{u}_{1k,2} = \widetilde{u}_{1k} \quad \text{on } \partial B_{r_0}(p). \end{cases} \tag{2.6}$$

For the first equation in (2.6), since

$$\int_{B_{r_0}(p)} \left|\rho_{1k} h_1 e^{\widetilde{u}_{1k}} - \rho_{2k} h_2 e^{\widetilde{u}_{2k}}\right| < 3\pi,$$

by [8, Theorem 1], we have

$$\int_{B_{r_0}(p)} \exp\big((1+\delta)|\widetilde{u}_{1k,1}|\big)dx \le C, \tag{2.7}$$

for some $\delta \in (0, \frac{1}{3})$. Therefore, we have

$$\int_{B_{r_0}(p)} |\widetilde{u}_{1k,1}| \le C. \tag{2.8}$$

For the second equation in (2.6), by the mean value theorem for harmonic functions we have

$$\begin{aligned}\|\widetilde{u}^+_{1k,2}\|_{L^\infty(B_{r_0/2}(p))} &\le C\|\widetilde{u}^+_{1k,2}\|_{L^1(B_{r_0}(p))}\\ &\le C\Big[\|\widetilde{u}^+_{1k}\|_{L^1(B_{r_0}(p))} + \|\widetilde{u}_{1k,1}\|_{L^1(B_{r_0}(p))}\Big]\\ &\le C. \end{aligned}\tag{2.9}$$

From (2.9), we conclude

$$\rho_{1k} h_1 e^{\widetilde{u}_{1k,2}} \le C \qquad \text{in } B_{r_0/2}(p). \tag{2.10}$$

By (2.7), (2.10) and the Hölder inequality, we obtain

$$e^{\widetilde{u}_{1k}} \in L^{1+\delta_1}(B_{r_0}(p)),$$

for some $\delta_1 > 0$ independent of k. Similarly, we have

$$e^{\widetilde{u}_{2k}} \in L^{1+\delta_2}(B_{r_0}(p)),$$

for some $\delta_2 > 0$ independent of k. By using the standard elliptic estimates for the first equation in (2.6), we get $\|\widetilde{u}_{1k,1}\|_{L^\infty}(B_{r_0/2}(p))$ is uniformly bounded. Combined with and (2.9), we have $\widetilde{u}_{1k}$ is uniformly bounded above in $B_{\frac{r_0}{2}}(p)$. Following the same process we can also obtain $\widetilde{u}_{2k}$ is uniformly bounded above in $B_{\frac{r_0}{2}}(p)$. On the other hand, we note that

$$\widetilde{u}_{1k} = v_k - \log\rho_{1k}, \qquad \widetilde{u}_{2k} = w_k - \log\rho_{2k}.$$

As a consequence, we get $p \notin \mathfrak{S}$ as claimed.

Step 2. It follows that if $p \in \mathfrak{S}$, either $\sigma_{1,p} \ge \frac{1}{3}$ or $\sigma_{2,p} \ge \frac{1}{3}$: together with the fact that $\rho_{1k}, \rho_{2k} \le C$ by assumption, we deduce $|\mathfrak{S}| < \infty$. Hence, we finish the proof of the lemma. □

For the terms α_{1k}, α_{2k} the following holds.

Lemma 2.3. *There exists a constant* $C \in \mathbb{R}$ *independent of* k *such that* $\alpha_{ik} \ge C$, $i = 1, 2$.

Proof. Note that v_k, w_k satisfy

$$\begin{cases} \Delta v_k + h_1 e^{v_k} - h_2 e^{w_k} = 0, & v_k = -\alpha_{1k} \text{ on } \partial\Omega,\\ \Delta w_k - h_1 e^{v_k} + h_2 e^{w_k} = 0, & w_k = -\alpha_{2k} \text{ on } \partial\Omega. \end{cases}$$

Using Green's representation formula, we have

$$v_k = \int_\Omega G(x,z)\left(h_1 e^{v_k}(z) - h_2 e^{w_k}(z)\right)\mathrm{d}z - \alpha_{1k},$$
$$w_k = \int_\Omega G(x,z)\left(h_2 e^{w_k}(z) - h_1 e^{v_k}(z)\right)\mathrm{d}z - \alpha_{2k}.$$

Thus we get

$$\|v_k + \alpha_{1k}\|_{L^1(\Omega)} \le C, \quad \|w_k + \alpha_{2k}\|_{L^1(\Omega)} \le C. \tag{2.11}$$

We know by Lemma 2.2 that $\mathfrak{S} \subset \overline{\Omega}$ is finite, and both v_k and w_k are uniformly bounded from above in any compact subset of $\overline{\Omega}\setminus\mathfrak{S}$. Therefore, from (2.11) we see that α_{1k}, α_{2k} are bounded from below, which proves the lemma. □

Finally, we state the Pohozaev-type identities which are one of the key ingredients of the argument in the sequel.

Lemma 2.4. *Let u be a solution of equation* (1.1) *and let α_1, α_2 be defined analogously as before* (2.1). *Let $D \subseteq \Omega$ be a smooth domain. Then, for any $\xi \in \mathbb{R}^2$ it holds that*

$$\begin{aligned}
&\int_D 2\left(h_1 e^{u-\alpha_1} + h_2 e^{-u-\alpha_2}\right) + \int_D \langle x-\xi, \nabla h_1\rangle e^{u-\alpha_1} + \int_D \langle x-\xi, \nabla h_2\rangle e^{-u-\alpha_2}\\
&= \int_{\partial D}\left(h_1 e^{u-\alpha_1} + h_2 e^{-u-\alpha_2}\right)\langle x-\xi, \nu\rangle + \int_{\partial D}\frac{\partial u}{\partial \nu}\langle x-\xi, \nabla u\rangle\\
&\quad - \frac{1}{2}\int_{\partial D}|\nabla u|^2\langle x-\xi,\nu\rangle,
\end{aligned} \tag{2.12}$$

where ν stands for the outer normal unit vector on ∂D.

Moreover, let $(u_i)_i$ be a solution of equation (1.8) and let $\widetilde{u}_i = u_i - \log\left(\frac{\int_\Omega h_i e^{u_i}}{\rho_i}\right)$ for $i = 1, \cdots, n$. Then, for any $\xi \in \mathbb{R}^2$ we have

$$\begin{aligned}
&\int_D \sum_{i=1}^n \frac{K^{ni}}{K^{in}}\left(\langle x-\xi, \nabla h_i\rangle e^{\widetilde{u}_i} + 2h_i e^{\widetilde{u}_i}\right)\\
&= \int_{\partial D}\sum_{i=1}^n \frac{K^{ni}}{K^{in}}\sum_{j=1}^n K^{ij}\Big(\langle x-\xi, \nabla\widetilde{u}_i\rangle\langle\nabla\widetilde{u}_j, \nu\rangle - \frac{1}{2}\langle x-\xi,\nu\rangle\langle\nabla\widetilde{u}_i, \nabla\widetilde{u}_j\rangle\Big)\\
&\quad + \int_{\partial D}\sum_{i=1}^n \frac{K^{ni}}{K^{in}} h_i e^{\widetilde{u}_i}\langle x-\xi,\nu\rangle,
\end{aligned} \tag{2.13}$$

where $(K^{ij})_{n\times n}$ is the inverse matrix of $\mathbf{K}$ and $\mathbf{K}$ is either $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$ or $\mathbf{G}_2$.

Proof. Multiplying the equation (1.1) by $\langle x-\xi, \nabla u\rangle$ and integrating by parts we readily obtain (2.12).

On the other hand, we rewrite equation (1.8) as

$$\sum_{j=1}^n K^{ij}\Delta\widetilde{u}_j + h_i e^{\widetilde{u}_i} = 0, \qquad i = 1, \cdots, n. \tag{2.14}$$

Multiplying the $i-$th equation by $\langle x-\xi, \nabla\widetilde{u}_i\rangle$ and integrating by parts it is not difficult to derive (2.13) after some straightforward computations. We skip the details. □

3. Proof of the Theorems 1.1 and 1.2

In this section we shall prove the main Theorems 1.1 and 1.2 concerning the no boundary blow up. Since the proofs are very similar we shall give full details for deducing Theorem 1.1 in Subsection 3.1 and sketch the main steps to derive Theorem 1.2 in Subsection 3.2.

3.1. **Proof of Theorem 1.1.** Let u_k be a sequence of solutions for (1.1) relative to (ρ_{1k}, ρ_{2k}) and we recall the blow up set for $|u_k|$ is given by

$$\mathcal{S} = \left\{ p \in \overline{\Omega} \mid \exists \{x_k\},\ x_k \to p,\ |u_k|(x_k) \to +\infty \right\}. \tag{3.1}$$

Recall the definitions of v_k, w_k in (2.1) and the definition of $\mathfrak{S}$ in (2.3). We have the following.

Lemma 3.1. *It holds that* $\mathfrak{S} = \mathcal{S}$.

Proof. First we prove $\mathfrak{S} \subseteq \mathcal{S}$. It suffices to show that if $x \notin \mathcal{S}$, then $x \notin \mathfrak{S}$. Suppose $x \notin \mathcal{S}$, we have $|u_k(x)| \le C$ uniformly, by some constant $C > 0$. Using Lemma 2.3, we have

$$v_k(x) = u_k(x) - \alpha_{1k} \le C \quad \text{and} \quad w_k(x) = -u_k(x) - \alpha_{2k} \le C,$$

where $C > 0$ is a constant independent of k. Consequently $x \notin \mathfrak{S}$. Therefore we have $\mathfrak{S} \subseteq \mathcal{S}$.

To prove the other inclusion, we shall show that u_k is uniformly bounded in any compact subset of $\overline{\Omega} \setminus \mathfrak{S}$. More precisely, for any compact subset $K \Subset \overline{\Omega} \setminus \mathfrak{S}$, we shall prove that there is a constant $C_K > 0$ that depends on the compact set K such that

$$|u_k(x)| \le C_K, \qquad \forall x \in K.$$

By Green's representation formula we have

$$\begin{aligned} u_k(x) &= \int_\Omega G(x,z)\left(\rho_{1k}h_1 e^{\widetilde{u}_{1k}} - \rho_{2k}h_2 e^{\widetilde{u}_{2k}}\right) \\ &= \int_{\Omega_1} G(x,z)\left(\rho_{1k}h_1 e^{\widetilde{u}_{1k}} - \rho_{2k}h_2 e^{\widetilde{u}_{2k}}\right) \\ &\quad + \int_{\Omega\setminus\Omega_1} G(x,z)\left(\rho_{1k}h_1 e^{\widetilde{u}_{1k}} - \rho_{2k}h_2 e^{\widetilde{u}_{2k}}\right), \end{aligned} \tag{3.2}$$

where $\widetilde{u}_{ik}$, $i = 1,2$ are defined in (2.5), $\Omega_1 = \bigcup_{p \in \mathfrak{S}} B_{r_0}(p)$ and $r_0 > 0$ is small enough such that $K \Subset \Omega \setminus \Omega_1$. It is easy to see that

$$\int_{\Omega_1} G(x,z)\left(\rho_{1k}h_1 e^{\widetilde{u}_{1k}} - \rho_{2k}h_2 e^{\widetilde{u}_{2k}}\right) = O(1),$$

because $G(x,z)$ is bounded due to the distance $d(x,z) \ge \delta_0 > 0$ for $z \in \Omega_1$, and $x \in K$. In $\Omega \setminus \Omega_1$, we can see that $\widetilde{u}_{ik}$ are bounded above by some constant that depends on r_0. Then it is not difficult to obtain that

$$\int_{\Omega\setminus\Omega_1} G(x,z)\left(\rho_{1k}h_1 e^{\widetilde{u}_{1k}} - \rho_{2k}h_2 e^{\widetilde{u}_{2k}}\right) = O(1).$$

Therefore, by using (3.2) we can conclude that $|u_k(x)| \leq C$, where $C > 0$ depends on K only. Hence, $|u_k(x)|$ is bounded away from $\mathfrak{S}$ and $\mathcal{S} \subseteq \mathfrak{S}$. This completes the proof. □

Since $\mathfrak{S}$ is finite, see Lemma 2.2, we let

$$\mathfrak{S} = \{p_1, p_2, \cdots, p_m\}.$$

For any $p_i \in \overline{\Omega}$, let r be a positive number such that $B_r(p_i) \cap B_r(p_j) = \emptyset$ for $i \neq j$. Let $p_{i,k} \in \overline{\Omega}$ be such that

$$M_k(p_{i,k}) = \max_{\overline{\Omega} \cap B_r(p_i)} M_k(x),$$

where

$$M_k(x) = \max\{v_k(x), w_k(x)\}.$$

Define $\mu_{i,k}$ by

$$\mu_{i,k} = e^{-\frac{1}{2}M_k(p_{i,k})}.$$

Observe that $\mu_{i,k} \to 0$. On the other hand, $p_{i,k} \notin \partial\Omega$ because we know that $M_k(x) \mid_{\partial\Omega} \leq C$ from Lemma 2.3. Furthermore, we can estimate more precisely the distance of the point $p_{i,k}$ from the boundary.

Lemma 3.2. *It holds that, for $k \to +\infty$,*

$$\frac{\operatorname{dist}(p_{i,k}, \partial\Omega)}{\mu_{i,k}} \to \infty.$$

Proof. Suppose by contradiction the result is not true. Then, one can find a sequence $(p_{i,k}, \mu_{i,k})$, such that $\operatorname{dist}(p_{i,k}, \partial\Omega) = O(\mu_{i,k})$. Let us consider the following dilated domain:

$$\Omega_{i,k} = \frac{(\Omega - p_{i,k})}{\mu_{i,k}}.$$

We may assume without loss of generality that $\Omega_{i,k} \to (-\infty, t_0) \times \mathbb{R}$. Moreover, we may further assume $v_k(p_{i,k}) = -2\log \mu_{i,k}$ and define

$$\begin{aligned} \hat{v}_k(y) &= v_k(p_{i,k} + \mu_{i,k} y) + 2\log \mu_{i,k} + \log h_1(p_{i,k}), \\ \hat{w}_k(y) &= w_k(p_{i,k} + \mu_{i,k} y) + 2\log \mu_{i,k} + \log h_2(p_{i,k}). \end{aligned}$$

We note that

$$\begin{aligned} \hat{v}_k + \hat{w}_k &= v_k + w_k + 4\log \mu_{i,k} + \log h_1(p_{i,k}) + \log h_2(p_{i,k}) \\ &= 4\log \mu_{i,k} - \alpha_{1k} - \alpha_{2k} + C \leq 4\log \mu_{i,k} + C. \end{aligned}$$

Let $R > 0$ and $y \in B_R(0) \cap \Omega_{i,k}$. By the Green representation formula, with a little abuse of notation we have

$$\begin{aligned} |\nabla \hat{v}_k| &= |\mu_{i,k} \nabla v_k(p_{i,k} + \mu_{i,k} y)| \\ &= \mu_{i,k} \left| \int_\Omega \nabla G(p_{i,k} + \mu_{i,k} y, z) \, [h_1 e^{v_k}(z) - h_2 e^{w_k}(z)] \, \mathrm{d}z \right| \\ &\leq C\mu_{i,k} \left[\int_{B_{2R\mu_{i,k}}(p_{i,k})} + \int_{\Omega \setminus B_{2R\mu_{i,k}}(p_{i,k})} \right] \frac{\left| h_1 e^{v_k(z)} - h_2 e^{w_k(z)} \right|}{|p_{i,k} + \mu_{i,k} y - z|} \, \mathrm{d}z. \end{aligned} \tag{3.3}$$

In $B_{2R\mu_{i,k}}(p_{i,k})$ we have $e^{v_k}, e^{w_k} \le e^{v_k(p_{i,k})} = \mu_{i,k}^{-2}$. On the other hand, in $\Omega_{i,k} \setminus B_{2R\mu_{i,k}(p_{i,k})}$ we have

$$|p_{i,k} + \mu_{i,k} y - z| \ge |z - p_{i,k}| - \mu_{i,k}|y| \ge R\mu_{i,k}.$$

Hence,

$$|\nabla \hat{v}_k| \le C\mu_{i,k} \int_{B_{2R\mu_{i,k}}(p_{i,k})} \frac{|h_1 e^{v_k} - h_2 e^{w_k}|}{|p_{i,k} + \mu_{i,k} y - z|} + C(R) \int_\Omega |h_1 e^{v_k} - h_2 e^{w_k}| \le C(R).$$

Therefore, we get $|\nabla \hat{v}_k| = |\nabla \hat{w}_k| \le C(R)$ in $B_R(0) \cap \Omega_{i,k}$, which implies

$$|\hat{v}_k(y) - \hat{v}_k(0)| \le C|y| \le C \qquad \text{for any } y \in \overline{B_R(0) \cap \Omega_{i,k}}.$$

Choosing y_0 in $\partial\Omega_{i,k}$, we obtain

$$|v_k(p_{i,k}) + \alpha_{1k}| = |\hat{v}_k(y_0) - \hat{v}_k(0)| \le C.$$

Then, we have

$$-2\log \mu_{i,k} + \alpha_{1k} = O(1),$$

from which we get a contradiction to Lemma 2.3 and the fact $\mu_{i,k} \to 0$. Thus we proved the lemma. □

Next, we study the behavior of the blow up solutions around each blow up point p_i. The starting point is the following selection process for (2.2) which was carried out in [25, Proposition 2.1]. The idea is to select a bubbling area which consists of a finite number of disks and in each disk the blow up solution resembles (after dilation) a globally defined Liouville-type solution. The only difference here is that we have to take into account the presence of the boundary of Ω. However, we have from Lemma 3.2 an estimate about $\mathrm{dist}(p_{i,k}, \partial\Omega)$ for $p_{i,k} \to p_i$ (see also part (2) of the following proposition). As a consequence, after a suitable dilation we will not see the effect of the boundary of Ω and hence we still end up with globally defined solutions, see part (3) of the following proposition for more details. In particular, since the Liouville-type problem in $\mathbb{R}^2$ is completely classified we gain the information about its total mass which yields then (3.4).

Proposition 3.1. *Let u_k be a sequence of bubbling solutions of* (1.1) *and v_k, w_k be defined in* (2.1). *Let $\mathfrak{S}$ be defined in* (2.3). *Then around each point $p_i \in \mathfrak{S}$ there exists a finite sequence of points*

$$\Sigma_{k,i} := \left\{ p_{i,1}^k, p_{i,2}^k, \cdots, p_{i,m_i}^k \right\} \qquad \left(\lim_{k\to\infty} p_{i,j}^k \to p_i,\ j = 1, \cdots, m_i \right)$$

and positive numbers $l_{i,1}^k, \cdots, l_{i,m_i}^k \to 0$ such that

(1) *For $j = 1, \ldots, m_i$, $i = 1, \cdots, m$ we have*

$$\max\left\{ v_k(p_{i,j}^k), w_k(p_{i,j}^k) \right\} = \max_{x \in B_{l_{i,j}^k}(p_{i,j}^k)} \max\left\{ v_k(x), w_k(x) \right\}.$$

(2) *For $j = 1, \ldots, m_i$, $i = 1, \cdots, m$, we let $\varepsilon_{i,j,k} = e^{-\frac{1}{2}\max\left\{ v_k(p_{i,j}^k), w_k(p_{i,j}^k) \right\}}$. Then,*

$$\frac{l_{i,j}^k}{\varepsilon_{i,j,k}} \to \infty, \qquad \frac{\mathrm{dist}(p_{i,j}^k, \partial\Omega)}{\varepsilon_{i,j,k}} \to \infty.$$

(3) *In each* $B_{l^k_{i,j}}(p^k_{i,j})$ *we define the dilated functions*

$$\hat{v}_{k,i,j}(y) = v_k(p^k_{i,j} + \varepsilon_{i,j,k} y) + 2\log \varepsilon_{i,j,k},$$
$$\hat{w}_{k,i,j}(y) = w_k(p^k_{i,j} + \varepsilon_{i,j,k} y) + 2\log \varepsilon_{i,j,k}.$$

Then it holds that one of the $v_{k,i,j}, w_{k,i,j}$ *converges to a solution of Liouville equation* (1.2) *in the* $C^2_{\text{loc}}(\mathbb{R}^2)$ *norm, while the other one tends to minus infinity over all compact subsets of* $\mathbb{R}^2$*. In particular, it holds*

$$\lim_{r\to 0}\lim_{k\to\infty}\int_{B_r(p^k_{i,j})} (h_1 e^{v_k} + h_2 e^{w_k}) \geq 8\pi. \tag{3.4}$$

(4) *There exits a constant* $C > 0$ *independent of* k *such that*

$$\max\{v_k(x), w_k(x)\} + 2\log \operatorname{dist}(x, \Sigma_{k,i}) \leq C, \tag{3.5}$$

for all $x \in B_r(p_i)$*,* $i = 1, 2, \cdots, m$*.*

We observe that from (3.5) we deduce the following result.

Lemma 3.3. *Let* $\Sigma_k = \cup_{i=1}^m \Sigma_{k,i}$*. Then, there exists a constant* $C > 0$ *independent of* k *such that*

$$\max\{v_k(x), w_k(x)\} + 2\log \operatorname{dist}(x, \Sigma_k) \leq C, \quad \forall x \in \Omega. \tag{3.6}$$

Moreover, we derive the following estimates which will be used later on.

Lemma 3.4. *There exists a constant* $C > 0$ *independent of* k *such that*

$$\operatorname{dist}(x, \Sigma_k)|\nabla v_k(x)| \leq C, \qquad \operatorname{dist}(x, \Sigma_k)|\nabla w_k(x)| \leq C, \quad \forall x \in \Omega.$$

Proof. By Green's representation formula, we have

$$|\nabla v_k| \leq C\int_\Omega \frac{1}{|x-z|}\,|h_1 e^{v_k}(z) - h_2 e^{w_k}(z)|\,\mathrm{d}z.$$

To simplify the notation, we denote

$$\Sigma_k = \{q_{k,1}, \cdots, q_{k,n}\}.$$

Let

$$R_k(x) := \inf_{i=1,\cdots,n} |x - q_{k,i}|,$$

$$\Omega_{k,i} = \big\{x \in \Omega : |x - q_{k,i}| = R_k(x)\big\}, \qquad i = 1, \cdots, n.$$

It is easy to see that $\Omega = \cup_{i=1}^n \Omega_{k,i}$. By using (3.6), for any $z \in \Omega_{k,i} \setminus B_{\frac{|x-q_{k,i}|}{2}}(q_{k,i})$,

$$|x-z|^{-1} e^{v_k(z)} \leq \frac{C}{|x-z||z-q_{k,i}|^2} \leq \frac{C}{|x-z||x-q_{k,i}|^2}.$$

Then,

$$\int_{\Omega_{k,i}\setminus B_{\frac{|x-q_{k,i}|}{2}}(q_{k,i})} \frac{h_1 e^{v_k}(z)}{|x-z|}\,\mathrm{d}z \leq \frac{C}{|x-q_{k,i}|}. \tag{3.7}$$

On the other hand, for $z \in \Omega_{k,i} \cap B_{\frac{|x-q_{k,i}|}{2}}(q_{k,i})$, we have $|x-z| \geq \frac{1}{2}|x-q_{k,i}|$ and hence

$$\int_{\Omega_{k,i} \cap B_{\frac{|x-q_{k,i}|}{2}}(q_{k,i})} \frac{h_1 e^{v_k}(z)}{|x-z|}\,\mathrm{d}z \leq \frac{C}{|x-q_{k,i}|}\,. \tag{3.8}$$

By (3.7) and (3.8), we have

$$\int_{\Omega_{k,i}} \frac{h_1 e^{v_k}(z)}{|x-z|}\,\mathrm{d}z \leq \frac{C}{|x-q_{k,i}|}\,. \tag{3.9}$$

Similarly,

$$\int_{\Omega_{k,i}} \frac{h_2 e^{w_k}(z)}{|x-z|}\,\mathrm{d}z \leq \frac{C}{|x-q_{k,i}|}\,. \tag{3.10}$$

From (3.9) and (3.10), we can easily obtain that

$$\inf_{i=1,\cdots,n} |x-q_{k,i}||\nabla v_k(x)| \leq C.$$

Equivalently, we get

$$\mathrm{dist}(x,\Sigma_k)|\nabla v_k(x)| \leq C.$$

Finally, we note that $|\nabla v_k(x)| = |\nabla w_k(x)|$. Therefore, the proof is completed. □

Concerning the asymptotic behavior of the blowing up solutions we have the following.

Lemma 3.5. *Let u_k be a sequence of solutions to* (1.1)*, let $\mathcal{S}$ be defined in* (3.1) *and let v_k, w_k be defined in* (2.1)*. Then, in the sense of measures, we have*

$$h_1 e^{v_k} dx \to r_1(x)dx + \sum_{p \in \mathcal{S}\cap\Omega} m_1(p)\delta_p \quad \text{in} \quad \Omega, \tag{3.11}$$

$$h_2 e^{w_k} dx \to r_2(x)dx + \sum_{p \in \mathcal{S}\cap\Omega} m_2(p)\delta_p \quad \text{in} \quad \Omega, \tag{3.12}$$

where $r_i(x) \in L^1(\overline{\Omega}) \cap C^\infty_{\mathrm{loc}}(\overline{\Omega}\setminus\mathcal{S})$ and $m_i(p)$ are multiple of 8π for $i=1,2$. Moreover,

$$u_k \to \mathcal{G} + \mathcal{U}$$

in $C^\infty_{\mathrm{loc}}(\overline{\Omega}\setminus\mathcal{S})$ and in $W^{1,q}_0(\Omega)$ for any $q<2$, where $\mathcal{G},\mathcal{U}$ are defined by

$$\Delta\mathcal{G}(x) + \sum_{p \in \mathcal{S}\cap\Omega} \big(m_1(p) - m_2(p)\big)\delta_p = 0 \quad \text{in } \Omega, \qquad \mathcal{G}(x) = 0 \quad \text{on } \partial\Omega,$$

$$\Delta\mathcal{U}(x) + r_1(x) - r_2(x) = 0 \quad \text{in } \Omega, \qquad \mathcal{U}(x) = 0 \quad \text{on } \partial\Omega.$$

Proof. By [28, Lemma 3.4] we readily get (3.11)-(3.12). Using the quantization result of [28, Theorem 1.1] (see also [25] for a different proof) we have

$$\big(m_1(p), m_2(p)\big) = \big(4\pi(l+1)l, 4\pi(l-1)l\big) \text{ or } \big(4\pi(l-1)l, 4\pi(l+1)l\big),$$

for $l \in \mathbb{N}$ and $p \in \mathcal{S}$. The left conclusion of the Lemma 3.5 is a direct consequence of classical elliptic regularity theory and Lemma 3.3. □

We prove now the main result concerning the no boundary blow up.

Proof of Theorem 1.1. Let u_k be a sequence of solutions to (1.1). We have to prove that $\mathcal{S} \cap \partial\Omega = \emptyset$, where $\mathcal{S}$ is given in (3.1). We argue by contradiction. Let $x_0 \in \mathcal{S} \cap \partial\Omega$. Since $|\mathcal{S}|$ is finite, see Lemma 2.2, we may assume that $\mathcal{S} \cap B_r(x_0) = \{x_0\}$. Let $z_k = x_0 + \Theta_{k,r}\nu(x_0)$ with

$$\Theta_{k,r} = \frac{\int_{\partial\Omega\cap B_r(x_0)} \langle x - x_0, \nu\rangle \left|\frac{\partial u_k}{\partial \nu}\right|^2}{\int_{\partial\Omega\cap B_r(x_0)} \langle \nu(x_0), \nu\rangle \left|\frac{\partial u_k}{\partial \nu}\right|^2},$$

where $r > 0$ is small such that $\frac{1}{2} \le \langle \nu(x_0), \nu\rangle \le 1$ for $x \in \partial\Omega \cap B_r(x_0)$. Here $\nu(x)$ is the unit outer normal at $x \in \partial\Omega$. It is then easy to check that $|\Theta_{k,r}| \le 2r$ for $|\langle x - x_0, \nu\rangle| \le r$. Observing

$$x - z_k = x - x_0 - \Theta_{k,r}\nu(x_0),$$

we deduce that

$$\int_{\partial\Omega\cap B_r(x_0)} \langle x - z_k, \nu\rangle \left|\frac{\partial u_k}{\partial \nu}\right|^2 = 0. \tag{3.13}$$

Now, by applying the Pohozaev identity (2.12) of Lemma 2.4 in $\Omega \cap B_r(x_0)$ with $\xi = z_k$, we have that

$$\begin{aligned}
&\int_{\Omega\cap B_r(x_0)} 2h_1 e^{u_k-\alpha_{1k}} + \int_{\Omega\cap B_r(x_0)} 2h_2 e^{-u_k-\alpha_{2k}} + \int_{\Omega\cap B_r(x_0)} e^{u_k-\alpha_{1k}} \langle x - z_k, \nabla h_1\rangle \\
&\quad + \int_{\Omega\cap B_r(x_0)} e^{-u_k-\alpha_{2k}} \langle x - z_k, \nabla h_2\rangle \\
&= \int_{\partial(\Omega\cap B_r(x_0))} \left(h_1 e^{u_k-\alpha_{1k}} + h_2 e^{-u_k-\alpha_{2k}}\right) \langle x - z_k, \nu\rangle + \int_{\partial(\Omega\cap B_r(x_0))} \frac{\partial u_k}{\partial \nu} \langle x - z_k, \nabla u_k\rangle \\
&\quad - \frac{1}{2} \int_{\partial(\Omega\cap B_r(x_0))} |\nabla u_k|^2 \langle x - z_k, \nu\rangle,
\end{aligned} \tag{3.14}$$

where α_{1k}, α_{2k} are defined before (2.1). In view of the boundary conditions in (1.1), it is easy to see that

$$\lim_{k\to+\infty} \int_{\partial\Omega\cap B_r(x_0)} \left(h_1 e^{u_k-\alpha_{1k}} + h_2 e^{-u_k-\alpha_{2k}}\right) \langle x - z_k, \nu\rangle = O(r^2)$$

and, by (3.13),

$$\begin{aligned}
&\int_{\partial\Omega\cap B_r(x_0)} \frac{\partial u_k}{\partial \nu} \langle x - z_k, \nabla u_k\rangle - \frac{1}{2} \int_{\partial\Omega\cap B_r(x_0)} |\nabla u_k|^2 \langle x - z_k, \nu\rangle \\
=&\frac{1}{2} \int_{\partial\Omega\cap B_r(x_0)} \langle x - z_k, \nu\rangle |\nabla u_k|^2 = 0.
\end{aligned}$$

On the other hand, since $\int_\Omega h_1 e^{u_k-\alpha_{1k}} \le C$ and $\int_\Omega h_2 e^{-u_k-\alpha_{2k}} \le C$, it holds that

$$\lim_{k\to+\infty} \int_{\Omega\cap B_r(x_0)} e^{u_k-\alpha_{1k}} \langle x - z_k, \nabla h_1\rangle = O(r)$$

and

$$\lim_{k\to+\infty} \int_{\Omega\cap B_r(x_0)} e^{-u_k-\alpha_{2k}} \langle x - z_k, \nabla h_2\rangle = O(r).$$

Finally, we claim

$$
\begin{aligned}
&\lim_{k\to+\infty}\int_{\Omega\cap\partial B_r(x_0)}\left(h_1e^{u_k-\alpha_{1k}}+h_2e^{-u_k-\alpha_{2k}}\right)\langle x-z_k,\nu\rangle=O(\varepsilon(r)),\\
&\int_{\Omega\cap\partial B_r(x_0)}\frac{\partial u_k}{\partial\nu}\langle x-z_k,\nabla u_k\rangle-\frac{1}{2}\int_{\Omega\cap\partial B_r(x_0)}|\nabla u_k|^2\langle x-z_k,\nu\rangle=O(\varepsilon(r)),
\end{aligned}
\tag{3.15}
$$

where $\varepsilon(r)\to 0$ as $r\to 0$. We postpone the proof of the latter claim in Lemma 3.6. Once we have (3.15) we conclude

$$
\lim_{r\to 0}\lim_{k\to\infty}\int_{\Omega\cap B_r(x_0)}\left(h_1e^{u_k-\alpha_{1k}}+h_2e^{-u_k-\alpha_{2k}}\right)=0,
$$

which is a contradiction to the lower bound on the local mass (3.4). The proof is completed, once we prove the claim (3.15). □

We are left with the proof of the claim (3.15) which we derive in the following lemma.

Lemma 3.6. *For any $\varepsilon>0$ there exists $r>0$ sufficiently small such that,*

$$
\lim_{k\to\infty}\int_{\Omega\cap\partial B_r(x_0)}\left(h_1e^{u_k-\alpha_{1k}}+h_2e^{-u_k-\alpha_{2k}}\right)|\langle x-z_k,\nu\rangle|=O(\varepsilon),\tag{3.16}
$$

and

$$
\int_{\Omega\cap\partial B_r(x_0)}\frac{\partial u_k}{\partial\nu}\langle x-z_k,\nabla u_k\rangle-\frac{1}{2}\int_{\Omega\cap\partial B_r(x_0)}|\nabla u_k|^2\langle x-z_k,\nu\rangle=O(\varepsilon).\tag{3.17}
$$

Proof. By Lemma 3.5 we know that $u_k\to\mathcal{G}+\mathcal{U}$ in $C^\infty_{\mathrm{loc}}(\Omega\setminus\mathcal{S})$. We let $r\in\left(0,\frac{1}{2}\mathrm{dist}(x_0,\mathcal{S}\setminus\{x_0\})\right)$. Then, we have $\|\mathcal{G}\|_{C^2(\Omega\cap\partial B_r(x_0))}\le C$ for some C independent of r. Note that

$$
|x-z_k|=|x-x_0-\Theta_{k,r}\nu(x_0)|\le|x-x_0|+|\Theta_{k,r}|=O(r)\text{ for }x\in\partial B_r(x_0)\cap\Omega.
$$

By the above facts and by Lemmas 2.3, 3.5, in order to get (3.16) and (3.17) it suffices to show that for any $\varepsilon>0$ there exists $r>0$ sufficiently small such that

$$
\int_{\Omega\cap\partial B_r(x_0)}re^{|\mathcal{U}|}=O(\varepsilon)\quad\text{and}\quad\int_{\Omega\cap\partial B_r(x_0)}r|\nabla\mathcal{U}|^2=O(\varepsilon).\tag{3.18}
$$

We recall that

$$
\Delta\mathcal{U}(x)+r_1(x)-r_2(x)=0\quad\text{in }\Omega,\qquad\mathcal{U}(x)=0\quad\text{on }\partial\Omega.
$$

By Green's representation formula and Lemma 2.1, for any $x\in\partial B_r(x_0)\cap\Omega$,

$$
\begin{aligned}
|\mathcal{U}(x)|&=\int_\Omega\left|G(x,y)\big(r_1(y)-r_2(y)\big)\right|\mathrm{d}y\\
&\le C\int_\Omega\log\left(2+\frac{1}{|x-y|}\right)\big(|r_1(y)|+|r_2(y)|\big)\mathrm{d}y\\
&\le C\int_{\Omega\cap B_{r'}(x_0)}\log\left(2+\frac{1}{|x-y|}\right)\big(|r_1(y)|+|r_2(y)|\big)\mathrm{d}y\\
&\quad+C\int_{\Omega\setminus B_{r'}(x_0)}\log\left(2+\frac{1}{|x-y|}\right)\big(|r_1(y)|+|r_2(y)|\big)\mathrm{d}y,
\end{aligned}
\tag{3.19}
$$

where $r' > 0$ is chosen so that $B_{3r'}(x_0) \cap (\mathcal{S} \setminus \{x_0\}) = \emptyset$ and

$$\int_{\Omega \cap B_{r'}(x_0)} (|r_1(y)| + |r_2(y)|) \mathrm{d}y \leq \delta,$$

with $\delta > 0$ to be determined later. Here we note that the choice of r', δ and the constants C in (3.19) are independent of r. For the last term on the right hand side of (3.19), we have

$$\int_{\Omega \setminus B_{r'}(x_0)} \log\left(2 + \frac{1}{|x-y|}\right) (|r_1(y)| + |r_2(y)|) \mathrm{d}y \leq C \log\left(1 + \frac{1}{r'}\right), \tag{3.20}$$

where C depends only on $\|r_i\|_{L^1(\Omega)}$, $i = 1, 2$ and the shape of the domain Ω, but independent of r. For the first term in (3.19), we rewrite it as

$$\begin{aligned}
&\int_{\Omega \cap B_{r'}(x_0)} \log\left(2 + \frac{1}{|x-y|}\right) (|r_1(y)| + |r_2(y)|) \mathrm{d}y \\
&= \int_{\Omega \cap B_{r'}(x_0) \cap B_{\frac{r}{N}}(x)} \log\left(2 + \frac{1}{|x-y|}\right) (|r_1(y)| + |r_2(y)|) \mathrm{d}y \\
&\quad + \int_{(\Omega \cap B_{r'}(x_0)) \setminus B_{\frac{r}{N}}(x)} \log\left(2 + \frac{1}{|x-y|}\right) (|r_1(y)| + |r_2(y)|) \mathrm{d}y \\
&= I_1 + I_2,
\end{aligned}$$

where $N > 0$ will be determined later on. From Lemma 3.3 and recalling that $B_{3r'}(x_0) \cap (\mathcal{S} \setminus \{x_0\}) = \emptyset$, we have

$$|x - x_0|^2 \max\{|r_1(x)|, |r_2(x)|\} \leq C \qquad \text{in } \Omega \cap B_{r'}(x_0). \tag{3.21}$$

Let us further impose $r < \min\left\{\frac{r'}{4}, \frac{1}{2}\mathrm{dist}(x_0, \mathcal{S} \setminus \{x_0\})\right\}$. For any $y \in B_{\frac{r}{N}}(x)$ we have

$$|y - x_0| \geq |x - x_0| - |x - y| = \frac{N-1}{N} r.$$

Then, by (3.21) and the latter fact we have

$$\max\{|r_1(y)|, |r_2(y)|\} \leq C\Big(\frac{N}{N-1}\Big)^2 \frac{1}{r^2}.$$

By the latter estimate we get

$$I_1 \leq C\Big(\frac{N}{N-1}\Big)^2 \Big(\frac{1}{N}\Big)^2 \log\left(2 + \frac{N}{r}\right) = C\Big(\frac{1}{N-1}\Big)^2 \log\left(2 + \frac{N}{r}\right). \tag{3.22}$$

On the other hand, for $y \in (\Omega \cap B_{r'}(x_0)) \setminus B_{\frac{r}{N}}(x)$ we have

$$\log\left(2 + \frac{1}{|x-y|}\right) \leq \log\left(2 + \frac{N}{r}\right).$$

Therefore,

$$I_2 \leq C \log\left(2 + \frac{N}{r}\right) \int_{\Omega \cap B_{r'}(x_0)} (|r_1(y)| + |r_2(y)|) \mathrm{d}y \leq C\delta \log\left(2 + \frac{N}{r}\right). \tag{3.23}$$

As a consequence of (3.19)-(3.23) we deduce

$$|\mathcal{U}|(x) \leq C \log\left(1 + \frac{1}{r'}\right) + C\left(\frac{1}{(N-1)^2} + \delta\right) \log\left(2 + \frac{N}{r}\right)$$

and

$$e^{|\mathcal{U}|}(x) \leq \left(1+\frac{1}{r'}\right)^{C}\left(2+\frac{N}{r}\right)^{C\left(\frac{1}{(N-1)^2}+\delta\right)}.$$

Finally, we first choose N sufficiently large and then r' sufficiently small such that $C\big(\frac{1}{(N-1)^2}+\delta\big)<2$. Here we note the choices of N and r' are independent of r. As a final step, for any given ε we choose r sufficiently small such that

$$r^2\left(1+\frac{1}{r'}\right)^{C}\left(2+\frac{N}{r}\right)^{C\left(\frac{1}{(N-1)^2}+\delta\right)} \leq \varepsilon.$$

This concludes the proof of the first estimate in (3.18). A similar argument yields the second estimate in (3.18). We skip the details to avoid repetitions. This finishes the proof of the lemma. □

3.2. **Proof of Theorem 1.2.** Let $u_{1k}, u_{2k}, \cdots, u_{nk}$ be a sequence of bubbling solutions to (1.8) relative to $\rho_{1k}, \cdots, \rho_{nk}$. We set

$$\widetilde{u}_{ik} = u_{ik} + \log\rho_{ik} - \log\int_{\Omega} h_i e^{u_{ik}}, \quad i=1,\cdots,n. \tag{3.24}$$

Then we can write (1.8) as

$$\begin{cases} \Delta\widetilde{u}_{ik} + \sum_{j=1}^{n} k_{ij}h_j e^{\widetilde{u}_{jk}} = 0 & \text{in } \Omega, \\ \widetilde{u}_{ik} = \log\rho_{ik} - \log\int_{\Omega} h_i e^{u_{ik}} & \text{on } \partial\Omega. \end{cases} \tag{3.25}$$

We set

$$\mathfrak{S}_{u_i} = \left\{p\in\overline{\Omega} \mid \exists\{x_k\}, x_k\to p, \widetilde{u}_{ik}\to+\infty\right\},\ i=1,\cdots,n, \tag{3.26}$$

and

$$\mathfrak{S}_u = \bigcup_{i=1}^{n}\mathfrak{S}_{u_i}. \tag{3.27}$$

Similarly, we define the blow up set for $u_{1k}, u_{2k}, \cdots, u_{nk}$:

$$\mathcal{S}_u = \left\{p\in\overline{\Omega} \mid \exists\{x_k\}, x_k\to p, \max_i u_{ik}(x_k)\to+\infty\right\}. \tag{3.28}$$

The same arguments of Lemmas 2.2 and 3.1 yield the following result.

Lemma 3.7. *$\mathfrak{S}_u = \mathcal{S}_u$ and $|\mathfrak{S}_u| = |\mathcal{S}_u|$ is finite.*

We then let

$$\mathfrak{S}_u = \{p_1,\cdots,p_l\}.$$

For any $p_i\in\overline{\Omega}$, and r be a positive number such that $B_r(p_i)\cap B_r(p_j)=\emptyset$ for $i\neq j$. Let $p_{j,k}\in\overline{\Omega}$ be such that

$$\lambda_{j,k} = \max_i \widetilde{u}_{i,k}(p_{j,k}) = \max_{\Omega\cap B_r(p_j)}\max_i \widetilde{u}_{i,k},\ i=1,\cdots,n,\ j=1,\cdots,l.$$

We have the following lemma.

Lemma 3.8. *It holds that, for $k \to +\infty$,*

$$\frac{\operatorname{dist}(p_{j,k}, \partial\Omega)}{\widetilde{\mu}_{j,k}} \to +\infty, \tag{3.29}$$

where $\widetilde{\mu}_{j,k} = e^{-\frac{\lambda_{j,k}}{2}}$, $j = 1, \cdots, l$.

Proof. We shall prove it by contradiction. Suppose the conclusion is not true and one can find a sequence $(p_{j,k}, \widetilde{\mu}_{j,k})$, such that $\operatorname{dist}(p_{j,k}, \partial\Omega) = O(\widetilde{\mu}_{j,k})$. Let us consider the following dilated domain:

$$\Omega_{j,k} = \frac{(\Omega - p_{j,k})}{\widetilde{\mu}_{j,k}}.$$

We may assume without loss of generality that $\Omega_{j,k} \to (-\infty, t_0) \times \mathbb{R}$. Moreover, we may further assume $\widetilde{u}_{1k}(p_{j,k}) = -2\log\widetilde{\mu}_{j,k}$ and define

$$\bar{u}_{ik}(y) = \widetilde{u}_{ik}(p_{j,k} + \widetilde{\mu}_{j,k}y) + 2\log\widetilde{u}_{j,k} + 2\log h_i(p_{j,k}),\ i = 1, \cdots, n.$$

We note that $\bar{u}_{ik}(y)$ is uniformly bounded above. Let $R > 0$ and $y \in B_R(0) \cap \Omega_{j,k}$. By the Green representation formula, with a little abuse of notation we have

$$\begin{aligned} |\nabla\bar{u}_{ik}| &= |\widetilde{\mu}_{j,k}\nabla\widetilde{u}_{ik}(p_{j,k})| \\ &= \widetilde{\mu}_{j,k}\left|\int_\Omega \nabla G(p_{j,k} + \widetilde{\mu}_{j,k}y, z)\sum_{\ell=1}^n k_{i\ell}h_\ell e^{\widetilde{u}_{\ell k}(z)}\mathrm{d}z\right| \\ &\le C\widetilde{\mu}_{j,k}\left[\int_{B_{2R\widetilde{\mu}_{j,k}}(p_{j,k})} + \int_{\Omega\setminus B_{2R\widetilde{\mu}_{j,k}}(p_{j,k})}\right]\frac{\left|\sum_{\ell=1}^n k_{i\ell}h_\ell e^{\widetilde{u}_{\ell k}(z)}\right|}{\left|p_{j,k} + \widetilde{\mu}_{j,k}y - z\right|}\mathrm{d}z. \end{aligned} \tag{3.30}$$

In $B_{2R\widetilde{\mu}_{j,k}}(p_{j,k})$ we have $e^{u_{ik}} \le e^{u_{1k}(p_{j,k})} = \widetilde{\mu}_{j,k}^{-2}$, $i = 1, \cdots, n$. On the other hand, in $\Omega_{j,k} \setminus B_{2R\widetilde{\mu}_{j,k}(p_{j,k})}$ we have

$$|p_{j,k} + \widetilde{\mu}_{j,k}y - z| \ge |z - p_{j,k}| - \widetilde{\mu}_{j,k}|y| \ge R\widetilde{\mu}_{j,k}.$$

Hence,

$$|\nabla\bar{u}_{ik}| \le C\widetilde{\mu}_{j,k}\int_{B_{2R\widetilde{\mu}_{j,k}}(p_{j,k})}\frac{|\sum_{\ell=1}^n k_{i\ell}h_\ell e^{\widetilde{u}_{\ell k}}|}{|p_{j,k} + \widetilde{\mu}_{j,k}y - z|} + C(R)\int_\Omega\left|\sum_{\ell=1}^n h_\ell e^{\widetilde{u}_{\ell k}}\right| \le C(R).$$

Therefore, we get $|\nabla\bar{u}_{ik}| \le C(R)$ in $B_R(0) \cap \Omega_{j,k}$, which implies

$$|\bar{u}_{ik}(y) - \bar{u}_{ik}(0)| \le C|y| \le C \quad \text{for any } y \in \overline{B_R(0) \cap \Omega_{j,k}}$$

and any $i \in \{1, \cdots, n\}$. Choosing y_0 in $\partial\Omega_{j,k}$, we obtain

$$|\widetilde{u}_{1k}(p_k) + \log\frac{\int_\Omega h_1 e^{u_{1k}}}{\rho_{1k}}| = |\bar{u}_{1k}(y_0) - \bar{u}_{1k}(0)| \le C.$$

Then, we have

$$-2\log\widetilde{\mu}_{j,k} + \log\frac{\int_\Omega h_1 e^{u_{1k}}}{\rho_{1k}} = O(1),$$

from which we get a contradiction to the fact that $\log\frac{\int_\Omega h_1 e^{u_{1k}}}{\rho_{1k}}$ is bounded from below and $\widetilde{\mu}_{j,k} \to 0$. Thus we proved the lemma. $\square$

As explained after Lemma 3.2, by means of Lemma 3.8 we will not see the effect of the boundary of Ω after suitable scaling. Therefore, A similar selection process as in Proposition 3.1 can be carried out also for the Toda-type system in (1.8).In particular, we can use the conclusions in [34, Proposition 2.1] and [36, Proposition 3A] to obtain the following result.

Lemma 3.9. *Let $u_{1k},\cdots,u_{nk}$ be a sequence of bubbling solutions of* (1.8) *and $\widetilde{u}_{ik}$ be defined in* (3.24)*. Let $\mathfrak{S}_u$ be defined in* (3.27)*. Then around each point $p_j\in\mathfrak{S}_u$ there exists a finite sequence of points*

$$\Sigma_{k,i}:=\{p_{j,1,k},\cdots,p_{j,m_j,k}\},\quad \lim_{k\to+\infty}p_{j,\ell,k}\to p_j,\ \ell=1,\cdots,m_j,$$

and positive numbers $r^k_{j,1},\cdots,r^k_{j,m_j}\to 0$ such that

(1) *For $\ell=1,\cdots,m_j$, $j=1,\cdots,l$ we have*

$$\max\left\{\widetilde{u}_{1k}(p_{j,\ell,k}),\cdots,\widetilde{u}_{nk}(p_{j,\ell,k})\right\}=\max_{x\in B_{r^k_{j,\ell}}(p_{j,\ell,k})}\max_i\{\widetilde{u}_{1k}(x),\cdots,\widetilde{u}_{nk}(x)\}.$$

(2) *For $\ell=1,\cdots,m_j$, $j=1,\cdots,l$, we let $\delta_{j,\ell,k}=e^{-\frac{1}{2}\max\left\{\widetilde{u}_{1k}(p_{j,\ell,k}),\cdots,\widetilde{u}_{1k}(p_{j,\ell,k})\right\}}$. Then,*

$$\frac{r^k_{j,\ell}}{\delta_{j,\ell,k}}\to\infty,\quad \frac{\mathrm{dist}(p^k_{j,\ell},\partial\Omega)}{\delta_{j,\ell,k}}\to\infty.$$

(3) *In each $B_{r^k_{j,\ell}}(p_{j,\ell,k})$ we define the dilated functions*

$$\hat{u}_{ik,j,\ell}(y)=\widetilde{u}_{ik}(p_{j,\ell,k}+\delta_{j,\ell,k}y)+2\log\delta_{j,\ell,k},\ i=1,\cdots,n.$$

Then $\hat{u}_{1k,j,\ell},\cdots,\hat{u}_{nk,j,\ell}$ either satisfies (a) or (b):
(a) the sequence is fully bubbling: along a subsequence, $\hat{u}_{1k,j,\ell},\cdots,\hat{u}_{nk,j,\ell}$ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ to $u_1,\cdots,u_n$ which satisfies

$$\Delta u_i+\sum_{\ell=1}^n k_{i\ell}h_\ell(p_j)e^{u_l}=0\text{ in }\mathbb{R}^2,$$

(b) $\{1,\cdots,n\}=J_1\cup\cdots\cup J_t\cup N$, where $J_1,\cdots,J_t$ and N are disjoint sets of indices, $N\neq\emptyset$ and each J_i, $1\le i\le t$ consists of consecutive indices. For each $i\in N$, $u_{ik,j,\ell}\to-\infty$ over any fixed compact subsets of $\mathbb{R}^2$. The components with index in J_i converge in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ to a low rank Toda system, where $|J_i|$ is the rank.

(4) *There exists a constant $C>0$ independent of k such that*

$$\max_i\widetilde{u}_{ik}(x)+2\log\mathrm{dist}(x,\Sigma_{k,j})\le C,$$

for all $x\in B_r(p_j)$, $j=1,\cdots,l$.

From the fourth conclusion of Lemma 3.9 we deduce the following result.

Lemma 3.10. *Let $\Sigma_k=\bigcup_{j=1}^l\Sigma_{k,i}$. Then, there exists a constant $C>0$ independent of k such that*

$$\max_i\widetilde{u}_{ik}(x)+2\log\mathrm{dist}(x,\Sigma_k)\le C,\quad\forall x\in\Omega. \tag{3.31}$$

With the above lemma we can derive the following crucial estimate.

Lemma 3.11. *There exists a constant $C > 0$ independent of k such that*

$$\operatorname{dist}(x, \Sigma_k)|\nabla u_{ik}| \leq C, \forall x \in \Omega, \quad i = 1, \cdots, n.$$

Proof. Using equation (3.25) and Green's representation formula, we have

$$|\nabla u_{ik}| \leq C \int_\Omega \frac{1}{|x-z|} |\sum_{\ell=1}^n e^{\widetilde{u}_{i\ell}(z)}| \mathrm{d}z.$$

For convenience, we set

$$\Sigma_k = \{q_{k1}, \cdots, q_{ks}\}.$$

Let

$$D_k(x) := \inf_j |x - q_{kj}|,$$

and

$$\Omega_{k,j} = \{x \in \Omega : |x - q_{kj}| = D_k(x)\}, \ j = 1, \cdots, s.$$

It is easy to see that $\Omega = \bigcup_{j=1}^s \Omega_{k,j}$. Using Lemma 3.10, for any $z \in \Omega_{k,j} \setminus B_{\frac{|x-q_{kj}|}{2}}(q_{kj})$,

$$|x-z|^{-1} e^{\widetilde{u}_{\ell k}} \leq \frac{C}{|x-z||z-q_{kj}|^2} \leq \frac{C}{|x-z||x-q_{kj}|^2}.$$

Then,

$$\int_{\Omega_{k,j} \setminus B_{\frac{|x-q_{kj}|}{2}}(q_{kj})} \frac{h_\ell e^{\widetilde{u}_{\ell k}}}{|x-z|} \mathrm{d}z \leq \frac{C}{|x-q_{kj}|}. \tag{3.32}$$

On the other hand, for $z \in \Omega_{k,j} \cap B_{\frac{|x-q_{kj}|}{2}}(q_{kj})$, we have $|x-z| \geq \frac{1}{2}|x-q_{kj}|$ and hence

$$\int_{\Omega_{k,j} \cap B_{\frac{|x-q_{kj}|}{2}}(q_{kj})} \frac{e^{\widetilde{u}_{\ell k}(z)}}{|x-z|} \mathrm{d}z \leq \frac{C}{|x-q_{kj}|}. \tag{3.33}$$

By (3.32) and (3.33), we have

$$\int_{\Omega_{k,j}} \frac{h_\ell e^{\widetilde{u}_{\ell k}(z)}}{|x-z|} \mathrm{d}z, \tag{3.34}$$

and the above inequality holds for any $\ell = 1, \cdots, n$. As a consequence, we obtain that there exists a constant independent of i such that

$$\inf_{j=1,\cdots,s} |x - q_{kj}||\nabla u_{ik}| \leq C, \quad i = 1, \cdots, n.$$

Therefore, we finish the proof. □

Finally, for what concerns the Toda system (1.8) we have a similar blow up picture as the one presented in Lemma 3.5 for the sinh-Gordon equation (1.1). Let $(u_{ik})_i$ be a sequence of solutions to (1.8) and let $(\widetilde{u}_{ik})_i$ be defined as in Lemma 2.4. Then, in the sense of measures, we have

$$h_i e^{\widetilde{u}_{ik}} dx \to f_i(x) + \sum_{p \in \mathcal{S} \cap \Omega} l_i(p) \delta_p \quad \text{in } \Omega, \quad i = 1, \cdots, n,$$

where $f_1, \cdots, f_n \in L^1(\Omega) \cap C^\infty_{\rm loc}(\overline{\Omega} \setminus \mathcal{S})$ and each $l_i(p)$ is a multiple of 4π. More precisely, for rank 2 Toda system the values of $l_i(p)$ are completely classified; for $\mathbf{K} = \mathbf{A}_2$ we have

$$\big(l_1(p), l_2(p)\big) \in \Big\{ (4\pi, 0), (0, 4\pi), (4\pi, 8\pi), (8\pi, 4\pi), (8\pi, 8\pi) \Big\},$$

see [34, Remark 1.2], for $\mathbf{K} = \mathbf{B}_2$ $(= \mathbf{C}_2)$ it holds

$$\big(l_1(p), l_2(p)\big) \in \Big\{ (4\pi, 0), (0, 4\pi), (8\pi, 4\pi), (4\pi, 12\pi), (8\pi, 16\pi), (12\pi, 12\pi), (12\pi, 16\pi) \Big\},$$

see [39, Theorem 1.1], while for $\mathbf{K} = \mathbf{G}_2$ we have

$$\begin{aligned} \big(l_1(p), l_2(p)\big) \in \Big\{ & (4\pi, 0), (0, 4\pi), (4\pi, 16\pi), (8\pi, 4\pi), (24\pi, 36\pi), (24\pi, 40\pi), \\ & (8\pi, 24\pi), (16\pi, 16\pi), (16\pi, 36\pi), (20\pi, 24\pi), (20\pi, 40\pi) \Big\}, \end{aligned}$$

see [36, Theorem 1.3]. For what concerns the A_n, B_n, C_n type Toda systems the classification of the values $l_i(p)$ is more involved. Nevertheless, recently the authors in [38] proved $l_i(p)$ is multiple of 4π.

Proof of Theorem 1.2. The proof is derived reasoning by contradiction and by using the Pohozaev identity (2.13) around a boundary blow up point. Since the arguments are local in nature, by exploiting Lemmas 3.7-3.11 we can reason exactly as in the proof of Theorem 1.1 to estimate each term in the Pohozaev identity and prove Theorem 1.2. We omit the details to avoid repetitions. □

4. Appendix: Proof of the Theorems 1.3 and 1.4

In this section we shall provide the proof of Theorems 1.3 and 1.4 concerning existence of solutions. Since their proof has become rather standard we will point out just the main steps and ideas.

Let us first consider Theorem 1.3. Its proof is based on the variational structure of the problem by considering the following functional associated to (1.1),

$$J_\rho(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \rho_1 \log \int_\Omega h_1(x)\, e^u - \rho_2 \log \int_\Omega h_2(x)\, e^{-u} \quad u \in H^1_0(\Omega), \tag{4.1}$$

where $\rho = (\rho_1, \rho_2)$. The goal is to exploit a Morse-type approach to detect critical points of the latter functional. However, standard deformation lemmas typically rely on some compactness assumption, for example the Palais-Smale condition. With respect to our problem, it is still unknown at the moment whether the functional (4.1) satisfies this condition. To bypass this obstruction one usually appeals to the compactness of the set of solutions to (1.1). To this end, we recall that for a sequence of blowing up solutions $(u_n)_n$ relative to ρ_n, we have by Lemma 3.5,

$$\rho_n \frac{h_1 e^{u_k}}{\int_\Omega h_1 e^{u_k}}\, dx \to r_1(x) dx + \sum_{p \in \mathcal{S} \cap \Omega} m_1(p) \delta_p \quad \text{in} \quad \Omega,$$

$$\rho_n \frac{h_2 e^{-u_k}}{\int_\Omega h_2 e^{-u_k}}\, dx \to r_2(x) dx + \sum_{p \in \mathcal{S} \cap \Omega} m_2(p) \delta_p \quad \text{in} \quad \Omega,$$

in the sense of measures, where $r_i(x) \in L^1(\overline{\Omega}) \cap C^\infty_{\text{loc}}(\overline{\Omega} \setminus \mathcal{S})$ and $m_i(p)$ are multiple of 8π for $i = 1, 2$. Concerning the residual mass, it is by now well known that either $r_1 \equiv 0$ or $r_2 \equiv 0$, see for instance [6] where the latter property is derived for a much more general problem. Therefore, the following result is granted.

Proposition 4.1. *Let $K \Subset (8\pi\mathbb{N} \times \mathbb{R}) \cup (\mathbb{R} \times 8\pi\mathbb{N}) \subset \mathbb{R}^2$. Then, the set of solutions $\{u_\rho\}_\rho \subset H^1_0(\Omega)$ with $\rho = (\rho_1, \rho_2) \in K$ is compact in $C^{2,\alpha}(\Omega)$ for some $\alpha > 0$.*

With the latter result at hand the Morse approach follows a common strategy so we will be sketchy. We will denote the sublevels of the functional J_ρ by

$$J^L_\rho = \big\{u \in H^1_0(\Omega) \,:\, J_\rho(u) \leq L\big\}, \quad L \in \mathbb{R}.$$

The arguments in [41] jointly with the compactnss property in Proposition 4.1 allow to derive the following deformation lemma.

Lemma 4.1. *Let a, $b \in \mathbb{R}$ be such that $a < b$ and let $\rho_1, \rho_2 \notin 8\pi\mathbb{N}$. Suppose that J_ρ has no critical points u_c with $a \leq J_\rho(u_c) \leq b$. Then, J^a_ρ is a deformation retract of J^b_ρ.*

We can now prove the main existence result in Theorem 1.3.

Proof of Theorem 1.3. The goal is to study the topology of the sublevels of the functional J_ρ and to apply Lemma 4.1. Roughly speaking, we aim to show a change of topology between high sublevels J^L_ρ and low sublevels J^{-L}_ρ for some $L \gg 0$.

Let us start with high sublevels. The compactness of solutions in Proposition 4.1 also implies boundedness from above of the energy on solutions, hence the following: if $\rho_1, \rho_2 \notin 8\pi\mathbb{N}$, then there exists $L \gg 0$ such that J^L_ρ is a deformation retract of $H^1_0(\Omega)$. In particular, it is contractible. On the other hand, since by assumption we have $\chi(\Omega) \leq 0$ and since the analysis in [5] concerning this part is local in nature, the same arguments as in [5] show that J^{-L}_ρ is not contractible. Therefore, by the latter change of topology between sublevels of J_ρ we conclude by Lemma 4.1 that there exists a critical point of J_ρ. This finishes the proof of Theorem 1.3.

□

The proof of Theorem 1.4 follows exactly the same steps introduced in the proof of Theorem 1.3 with obvious modifications. We omit the details to avoid repetitions.

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