

THE MAZ'YA-SHAPOSHNIKOVA LIMIT IN THE MAGNETIC SETTING

ANDREA PINAMONTI, MARCO SQUASSINA, AND EUGENIO VECCHI

ABSTRACT. We prove a magnetic version of the Maz'ya-Shaposhnikova singular limit of nonlocal norms with vanishing fractional parameter. This complements a general convergence result recently obtained by authors when the parameter approaches one.

1. INTRODUCTION

About fifteen years ago, V. Maz'ya and T. Shaposhnikova proved that for any $n \geq 1$ and $p \in [1, \infty)$,

$$\lim_{s \searrow 0} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+ps}} dx dy = \frac{4\pi^{n/2}}{p\Gamma(n/2)} \|u\|_{L^p(\mathbb{R}^n)}^p,$$

whenever $u \in D_0^{s,p}(\mathbb{R}^n)$ for some $s \in (0, 1)$. Here Γ denotes the Gamma function and the space $D_0^{s,p}(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the Gagliardo norm

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+ps}} dx dy.$$

Their motivation was basically that of complementing a previous result by Bourgain-Brezis-Mironescu [4, 5] providing new characterizations for functions in the Sobolev space $W^{1,p}(\Omega)$. Precisely, if $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, then for any $W^{1,p}(\Omega)$ there holds

$$\lim_{s \nearrow 1} (1 - s) \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+ps}} dx dy = Q_{p,n} \int_{\Omega} |\nabla u|^p dx,$$

where $Q_{p,n}$ is defined by

$$(1.1) \quad Q_{p,n} = \frac{1}{p} \int_{\mathbb{S}^{n-1}} |\omega \cdot h|^p d\mathcal{H}^{n-1}(h),$$

being \mathbb{S}^{n-1} the unit sphere in \mathbb{R}^n and ω an arbitrary unit vector of \mathbb{R}^n . The above singular limits are natural and also admit a physical relevance in the framework of the theory of Levy processes. Also, there is a developed theory of fractional s -perimeters [6] and there have been several contributions concerning their asymptotic analysis in the limits $s \nearrow 1$ and $s \searrow 0$ [1, 7, 9].

One of the latest generalizations of this kind of convergence results appeared recently in [16] in the context of *magnetic Sobolev spaces* $W_A^{1,2}(\Omega)$, see [12]. In fact, a relevant role in the study of particles which interact with a magnetic field $B = \nabla \times A$, $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$, is assumed by the *magnetic Laplacian* $(\nabla - iA)^2$ [3, 12, 15], yielding to nonlinear Schrödinger equations of the type $-(\nabla - iA)^2 u + u = f(u)$, which have been extensively studied (see [2] and the references therein). The operator is defined weakly as the differential of the energy

$$W_A^{1,2}(\Omega) \ni u \mapsto \int_{\Omega} |\nabla u - iA(x)u|^2 dx.$$

If $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a smooth field and $s \in (0, 1)$, a nonlocal magnetic counterpart of the magnetic laplacian,

$$(-\Delta)_A^s u(x) = c(n, s) \lim_{\varepsilon \searrow 0} \int_{B_\varepsilon^c(x)} \frac{u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)}{|x - y|^{n+2s}} dy,$$

2010 *Mathematics Subject Classification.* 49A50, 26A33, 82D99.

Key words and phrases. Fractional magnetic spaces, Maz'ya-Shaposhnikova limit, characterization of Sobolev spaces.

The authors are members of *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni* (GNAMPA) of the *Istituto Nazionale di Alta Matematica* (INdAM). E.V. receives funding from the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme FP7/2007-2013/ under REA grant agreement No. 607643 (ERC Grant MaNET 'Metric Analysis for Emergent Technologies').

where $c(n, s)$ is a normalization constant which behaves as follows

$$(1.2) \quad \lim_{s \searrow 0} \frac{c(n, s)}{s} = \frac{\Gamma(n/2)}{\pi^{n/2}}, \quad \lim_{s \nearrow 1} \frac{c(n, s)}{1-s} = \frac{2n\Gamma(n/2)}{\pi^{n/2}},$$

was introduced in [8, 11] for complex-valued functions, with motivations falling into the framework of the general theory of Lévy processes. Recently, the authors in [14] (see [16] for $p = 2$) proved that if $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a C^2 vector field, then, for any $n \geq 1$, $p \in [1, \infty)$ and any Lipschitz bounded domain $\Omega \subset \mathbb{R}^n$

$$(1.3) \quad \lim_{s \nearrow 1} (1-s) \int_{\Omega} \int_{\Omega} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+ps}} dx dy = Q_{p,n} \int_{\Omega} |\nabla u - iA(x)u|_p^p dx,$$

for all $u \in W_A^{1,p}(\Omega)$, where $Q_{p,n}$ is as in (1.1) and $|z|_p := (|\Re z_1, \dots, \Re z_n|^p + |\Im z_1, \dots, \Im z_n|^p)^{1/p}$. This has provided a new nonlocal characterization of the magnetic Sobolev spaces $W_A^{1,p}(\Omega)$.

The main goal of this paper is to complete the picture of [14] by providing a *magnetic counterpart* of the convergence result by *Maz'ya-Shaposhnikova* for vanishing fractional orders s , namely for $s \searrow 0$.

We consider a locally bounded vector potential field $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and the space of complex valued functions $D_{A,0}^{s,p}(\mathbb{R}^n, \mathbb{C})$ defined as the completion of $C_c^\infty(\mathbb{R}^n, \mathbb{C})$ with respect to the norm

$$\|u\|_{D_{A,0}^{s,p}} = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+ps}} dx dy \right)^{1/p}.$$

By combining Lemma 2.1 and Lemma 2.3, we shall prove the following result.

Theorem 1.1 (Magnetic Maz'ya-Shaposhnikova). *Let $n \geq 1$ and $p \in [1, \infty)$. Then for every*

$$u \in \bigcup_{0 < s < 1} D_{A,0}^{s,p}(\mathbb{R}^n, \mathbb{C}),$$

there holds

$$\lim_{s \searrow 0} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+ps}} dx dy = \frac{4\pi^{n/2}}{p\Gamma(n/2)} \|u\|_{L^p(\mathbb{R}^n)}^p.$$

In particular, while the singular limit as $s \nearrow 1$ generates the magnetic gradient $\nabla - iA$, the limit for vanishing s tends to destroy the magnetic effects yielding the $L^p(\mathbb{R}^n)$ -norm of the function u . We point out that, while in (1.3) the norm of complex numbers is $|\cdot|_p$, in Theorem 1.1 we use the usual norm $|\cdot| = |\cdot|_2$. In any case when $A = 0$ and u is real-valued the formulas are all consistent with the classical statements. In the case $p = 2$, combining the asymptotic formulas in (1.2) with Theorem 1.1 implies that

$$\frac{c(n, s)}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^{n+2s}} dx dy \approx \|u\|_{L^2(\mathbb{R}^n)}^2, \quad \text{as } s \searrow 0,$$

for any $u \in D_0^{s,2}(\mathbb{R}^n)$ for some $s \in (0, 1)$. Although the magnetic setting is mainly meaningful in the framework of nonlocal Schrödinger equations, we remark that for $E \subset \mathbb{R}^n$, if $E^c := \mathbb{R}^n \setminus E$, the quantity

$$P_s(E; A) := \frac{1}{2} \int_E \int_E \frac{|1 - e^{i(x-y) \cdot A(\frac{x+y}{2})}|}{|x-y|^{n+s}} dx dy + \int_E \int_{E^c} \frac{1}{|x-y|^{n+s}} dx dy$$

plays the role of a nonlocal s -perimeter of E depending on A , which reduces for $A = 0$ to the usual notion of fractional s -perimeter of $E \subset \mathbb{R}^n$. Then, if $\mathcal{L}^n(E)$ denotes the n -dimensional Lebesgue measure of $E \subset \mathbb{R}^n$, Theorem 1.1, applied with $p = 1$ and $u(x) = \mathbf{1}_E(x)$, reads as

$$\lim_{s \searrow 0} s P_s(E, A) = \frac{4\pi^{n/2}}{\Gamma(n/2)} \mathcal{L}^n(E),$$

provided that $P_{s_0}(E, A) < +\infty$, for some $s_0 \in (0, 1)$.

2. PROOF OF THE MAIN RESULT

The proof of Theorem 1.1 follows by combining Lemma 2.1 and Lemma 2.3 below.

Lemma 2.1 (Liminf inequality). *Let $n \geq 1$, $p \in [1, \infty)$ and let*

$$u \in \bigcup_{0 < s < 1} D_{A,0}^{s,p}(\mathbb{R}^n, \mathbb{C}).$$

Then

$$\liminf_{s \searrow 0} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+ps}} dx dy \geq \frac{4\pi^{n/2}}{p\Gamma(n/2)} \|u\|_{L^p(\mathbb{R}^n)}^p.$$

Proof. If

$$\liminf_{s \searrow 0} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+ps}} dx dy = \infty,$$

the assertion follows. Otherwise, there exists a sequence $\{s_k\}_{k \in \mathbb{N}} \subset (0, 1)$ with $s_k \searrow 0$ and

$$\liminf_{s \searrow 0} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+ps}} dx dy = \lim_{k \rightarrow \infty} s_k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+ps_k}} dx dy,$$

the limit being finite. For a.e. $x, y \in \mathbb{R}^n$ we have the Diamagnetic inequality (cf. [8, Remark 3.2])

$$(2.1) \quad ||u(x)| - |u(y)|| \leq |u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|.$$

In particular, since $u \in D_{A,0}^{s_k,p}(\mathbb{R}^n, \mathbb{C})$, we have $|u| \in D_0^{s_k,p}(\mathbb{R}^n)$ and, for any $k \geq 1$,

$$s_k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{||u(x)| - |u(y)||^p}{|x-y|^{n+ps_k}} dx dy \leq s_k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+ps_k}} dx dy.$$

Taking the limit as $k \rightarrow \infty$ on both sides and invoking [13, Theorem 3] applied to $|u|$, yields

$$\frac{4\pi^{n/2}}{p\Gamma(\frac{n}{2})} \| |u| \|_{L^p(\mathbb{R}^n)}^p \leq \lim_{k \rightarrow \infty} s_k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+ps_k}} dx dy,$$

which concludes the proof. \square

Remark 2.2 (Magnetic Hardy inequality). By combining the pointwise Diamagnetic inequality (2.1) with the fractional Hardy inequality [10], for $n > ps$ the following *magnetic Hardy inequality* holds: there exists a positive constant $\mathcal{H}_{n,s,p}$ such that

$$(2.2) \quad \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{sp}} dx \leq \mathcal{H}_{n,s,p} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+ps}} dx dy,$$

for every $u \in D_{A,0}^{s,p}(\mathbb{R}^n, \mathbb{C})$. Similarly the following *magnetic Sobolev inequality* holds: there exists a positive constant $\mathcal{S}_{n,s,p}$ such that

$$\left(\int_{\mathbb{R}^n} |u(x)|^{\frac{np}{n-sp}} dx \right)^{\frac{n-sp}{n}} \leq \mathcal{S}_{n,s,p} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+ps}} dx dy,$$

for every $u \in D_{A,0}^{s,p}(\mathbb{R}^n, \mathbb{C})$.

Next we state a second lemma completing the proof of Theorem 1.1 when combined with Lemma 2.1.

Lemma 2.3 (Limsup inequality). *Let $n \geq 1$, $p \in [1, \infty)$ and let*

$$u \in \bigcup_{0 < s < 1} D_{A,0}^{s,p}(\mathbb{R}^n, \mathbb{C}).$$

Then

$$\limsup_{s \searrow 0} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+ps}} dx dy \leq \frac{4\pi^{n/2}}{p\Gamma(n/2)} \|u\|_{L^p(\mathbb{R}^n)}^p.$$

Proof. If $u \notin L^p(\mathbb{R}^n)$, there is nothing to prove. Hence, we may assume that $u \in L^p(\mathbb{R}^n)$. We observe that

$$\begin{aligned}
& s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+ps}} dx dy \\
&= s \int_{\mathbb{R}^n} \int_{\{|x| \leq |y| \leq 2|x|\}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dx dy \\
&+ s \int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dx dy \\
&+ s \int_{\mathbb{R}^n} \int_{\{|x| \geq |y|\}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dx dy \\
&= 2s \int_{\mathbb{R}^n} \int_{\{|x| \leq |y| \leq 2|x|\}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dx dy \\
&+ 2s \int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dx dy,
\end{aligned}$$

where the last equality follows noticing that since $|e^{i(x-y) \cdot A(\frac{x+y}{2})}| = 1$ then

$$\begin{aligned}
\int_{\mathbb{R}^n} \int_{\{|x| \geq |y|\}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dx dy &= \int_{\mathbb{R}^n} \int_{\{|y| \geq |x|\}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dx dy \\
&= \int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dx dy \\
&+ \int_{\mathbb{R}^n} \int_{\{|x| \leq |y| \leq 2|x|\}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dx dy.
\end{aligned}$$

Using the triangle inequality for the L^p -norm on \mathbb{R}^{2n} and recalling that $|e^{i(x-y) \cdot A(\frac{x+y}{2})}| = 1$, yields

$$\begin{aligned}
& s \int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dx dy \\
&\leq \left\{ \left(s \int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} \frac{|u(x)|^p}{|x-y|^{n+sp}} dx dy \right)^{1/p} + \left(s \int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} \frac{|u(y)|^p}{|x-y|^{n+sp}} dx dy \right)^{1/p} \right\}^p.
\end{aligned}$$

We claim that

$$\lim_{s \searrow 0} s \int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} \frac{|u(y)|^p}{|x-y|^{n+sp}} dx dy = 0.$$

Observe that $2|x-y| \geq |y| + (|y| - 2|x|)$. Then, if $|y| \geq 2|x|$ we get $2|x-y| \geq |y|$. Now, if \mathcal{H}^{n-1} denotes the $(n-1)$ -dimensional Haudorff measure, it follows that

$$\begin{aligned}
s^{1/p} \left(\int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} \frac{|u(y)|^p}{|x-y|^{n+sp}} dx dy \right)^{1/p} &\leq s^{1/p} \left(2^{n+sp} \int_{\mathbb{R}^n} \frac{|u(y)|^p}{|y|^{n+sp}} \left(\int_{\{|x| \leq |y|/2\}} dx \right) dy \right)^{1/p} \\
&= 2^s \left(\frac{s}{n} \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) \right)^{1/p} \left(\int_{\mathbb{R}^n} \frac{|u(y)|^p}{|y|^{sp}} dy \right)^{1/p},
\end{aligned}$$

and the last term goes to zero as $s \searrow 0$. Notice that $y \mapsto |y|^{-s}u(y)$ remains bounded in $L^p(\mathbb{R}^n)$ as $s \searrow 0$ by the argument indicated here below. Observe now that, if $|y| \geq 2|x|$ we then get $|x - y| \geq |x|$ yielding

$$\begin{aligned} & \left(s \int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} \frac{|u(x)|^p}{|x - y|^{n+sp}} dx dy \right)^{1/p} \leq \left(s \int_{\mathbb{R}^n} \int_{\{|x-y| \geq |x|\}} \frac{|u(x)|^p}{|x - y|^{n+sp}} dx dy \right)^{1/p} \\ & = \left(s \int_{\mathbb{R}^n} |u(x)|^p \int_{B(0,|x|)^c} \frac{dz}{|z|^{n+sp}} dx \right)^{1/p} = \frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})^{1/p}}{p^{1/p}} \left(\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{sp}} dx \right)^{1/p}. \end{aligned}$$

Moreover $|x|^{-sp}|u(x)|^p = f_s(x) + g_s(x)$, where

$$f_s(x) := \frac{|u(x)|^p}{|x|^{sp}} \mathbf{1}_{B(0,1)}(x), \quad g_s(x) := \frac{|u(x)|^p}{|x|^{sp}} \mathbf{1}_{B(0,1)^c}(x) \leq |u(x)|^p \mathbf{1}_{B(0,1)^c}(x) \in L^1(\mathbb{R}^n),$$

and $s \mapsto f_s$ is decreasing and, moreover, by the Hardy inequality (2.2) and the assumption on u , it follows that $f_s \in L^1(\mathbb{R}^n)$ for some $\tilde{s} \in (0, 1)$. Hence, by monotone and dominated convergence, we conclude that

$$\limsup_{s \searrow 0} s \int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} \frac{|u(x)|^p}{|x - y|^{n+sp}} dx dy \leq \frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{p} \|u\|_{L^p(\mathbb{R}^n)}^p = \frac{2\pi^{n/2}}{p\Gamma(\frac{n}{2})} \|u\|_{L^p(\mathbb{R}^n)}^p.$$

Then, we conclude from the above inequalities that

$$(2.3) \quad \limsup_{s \searrow 0} 2s \int_{\mathbb{R}^n} \int_{\{|y| \geq 2|x|\}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x - y|^{n+sp}} dx dy \leq \frac{4\pi^{n/2}}{p\Gamma(\frac{n}{2})} \|u\|_{L^p(\mathbb{R}^n)}^p.$$

We claim that

$$(2.4) \quad \limsup_{s \searrow 0} 2s \int_{\mathbb{R}^n} \int_{\{|x| \leq |y| \leq 2|x|\}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x - y|^{n+sp}} dx dy = 0.$$

By assumption let $\tau \in (0, 1)$ such that $u \in D_{A,0}^{\tau,p}(\mathbb{R}^n)$. Now let $N \geq 1$ and $s < \tau$. Then

$$\begin{aligned} & 2s \int_{\mathbb{R}^n} \int_{\{|x| \leq |y| \leq 2|x|\}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x - y|^{n+sp}} dx dy \\ & = 2s \int_{\mathbb{R}^n} \int_{\substack{\{|x-y| \leq N\} \\ \{|x| \leq |y| \leq 2|x|\}}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x - y|^{n+sp}} dx dy \\ & + 2s \int_{\mathbb{R}^n} \int_{\substack{\{|x-y| > N\} \\ \{|x| \leq |y| \leq 2|x|\}}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x - y|^{n+sp}} dx dy =: \mathcal{I} + \mathcal{II}. \end{aligned}$$

Let us consider \mathcal{I} first. Since $|x - y| \leq N$, it holds that

$$\frac{1}{|x - y|^{n+sp}} = \frac{|x - y|^{p(\tau-s)}}{|x - y|^{n+\tau p}} \leq \frac{N^{p(\tau-s)}}{|x - y|^{n+\tau p}}.$$

Therefore \mathcal{I} goes to zero as $s \searrow 0$, since

$$\mathcal{I} \leq 2s N^{p(\tau-s)} \int_{\mathbb{R}^n} \int_{\substack{\{|x-y| \leq N\} \\ \{|x| \leq |y| \leq 2|x|\}}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x - y|^{n+\tau p}} dx dy.$$

Let us now move to \mathcal{II} . Since $|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p \leq 2^{p-1} (|u(x)|^p + |u(y)|^p)$, we get

$$\mathcal{II} \leq 2^p s \int_{\mathbb{R}^n} \int_{\substack{\{|x-y| \geq N\} \\ \{|x| \leq |y| \leq 2|x|\}}} \frac{|u(x)|^p}{|x - y|^{n+sp}} dx dy + 2^p s \int_{\mathbb{R}^n} \int_{\substack{\{|x-y| \geq N\} \\ \{|x| \leq |y| \leq 2|x|\}}} \frac{|u(y)|^p}{|x - y|^{n+sp}} dx dy =: \mathcal{II}' + \mathcal{II}''.$$

Regarding \mathcal{II}' , since $|x - y| \geq N$ and $|y| \leq 2|x|$, it holds

$$N \leq |x - y| \leq |x| + |y| \leq 3|x|,$$

which implies that $|x| \geq \frac{N}{3}$. In particular, this also implies that

$$\mathcal{I}\mathcal{I}' \leq 2^p s \int_{\{|x| \geq N/3\}} \left(\int_{\{|x-y| \geq N\}} \frac{|u(x)|^p}{|x-y|^{n+sp}} dy \right) dx \leq C(n, p) \int_{\{|x| \geq N/3\}} |u(x)|^p dx.$$

For $\mathcal{I}\mathcal{I}''$, since as before $|x-y| \geq N$ and $|x| \leq |y|$, we have

$$N \leq |x-y| \leq |x| + |y| \leq 2|y|,$$

which implies $|y| \geq \frac{N}{2} \geq \frac{N}{3}$. Therefore, we get

$$\mathcal{I}\mathcal{I}'' \leq 2^p s \int_{\{|y| \geq N/3\}} |u(y)|^p \left(\int_{\{|z| \geq N\}} \frac{1}{|z|^{n+sp}} dz \right) dy \leq C(n, p) \int_{\{|y| \geq N/3\}} |u(y)|^p dy.$$

Combining the estimates for $\mathcal{I}\mathcal{I}'$ and $\mathcal{I}\mathcal{I}''$, we get

$$\mathcal{I}\mathcal{I} \leq C(n, p) \int_{\{|x| \geq N/3\}} |u(x)|^p dx,$$

which is a bound independent of s . Now, going back to

$$\limsup_{s \searrow 0} 2s \int_{\mathbb{R}^n} \int_{\{|x| < |y| < 2|x|\}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dx dy \leq 2C(n, p) \|u\|_{L^p(B(0, N/3)^c)}^p,$$

and (2.4) follows letting $N \rightarrow \infty$, since $u \in L^p(\mathbb{R}^n)$. Collecting (2.3) and (2.4), the assertion follows. \square

REFERENCES

- [1] L. Ambrosio, G. De Philippis, L. Martinazzi, Γ -convergence of nonlocal perimeter functionals, *Manuscripta Math.* **134** (2011), 377–403. [1](#)
- [2] G. Arioli, A. Szulkin, *A semilinear Schrödinger equation in the presence of a magnetic field*, *Arch. Ration. Mech. Anal.* **170** (2003), 277–295. [1](#)
- [3] J. Avron, I. Herbst, B. Simon, *Schrödinger operators with magnetic fields. I. General interactions*, *Duke Math. J.* **45** (1978), 847–883. [1](#)
- [4] J. Bourgain, H. Brezis, P. Mironescu, *Another look at Sobolev spaces*, in *Optimal Control and Partial Differential Equations. A Volume in Honor of Professor Alain Bensoussan's 60th Birthday* (eds. J. L. Menaldi, E. Rofman and A. Sulem), IOS Press, Amsterdam, 2001, 439–455. [1](#)
- [5] J. Bourgain, H. Brezis, P. Mironescu, *Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications*, *J. Anal. Math.* **87** (2002), 77–101. [1](#)
- [6] L. Caffarelli, J.-M. Roquejoffre, O. Savin, *Nonlocal minimal surfaces*, *Comm. Pure Appl. Math.* **63** (2010), 1111–1144. [1](#)
- [7] L. Caffarelli, E. Valdinoci, *Regularity properties of nonlocal minimal surfaces via limiting arguments*, *Adv. Math.* **248** (2013), 843–871. [1](#)
- [8] P. d'Avenia, M. Squassina, *Ground states for fractional magnetic operators*, preprint, <http://arxiv.org/abs/1601.04230> [2](#), [3](#)
- [9] S. Dipierro, A. Figalli, G. Palatucci, E. Valdinoci, *Asymptotics of the s -perimeter as $s \rightarrow 0$* , *Discrete Contin. Dyn. Syst.* **33** (2013), 2777–2790. [1](#)
- [10] R.L. Frank, R. Seiringer, *Non-linear ground state representation and sharp Hardy inequalities*, *J. Funct. Anal.* **255** (2008), 3407–3430. [3](#)
- [11] T. Ichinose, *Magnetic relativistic Schrödinger operators and imaginary-time path integrals*, *Mathematical physics, spectral theory and stochastic analysis*, 247–297, *Oper. Theory Adv. Appl.* **232**, Birkhäuser/Springer, Basel, 2013. [2](#)
- [12] E. Lieb and M. Loss, *Analysis*, *Graduate Studies in Mathematics* **14**, 2001. [1](#)
- [13] V. Maz'ya and T. Shaposhnikova, *On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces*, *J. Funct. Anal.* **195** (2002), 230–238. [3](#)
- [14] A. Pinamonti, M. Squassina, E. Vecchi, *Magnetic BV functions and the Bourgain-Brezis-Mironescu formula*, preprint, <https://arxiv.org/abs/1609.09714> [2](#)
- [15] M. Reed, B. Simon, *Methods of modern mathematical physics, I, Functional analysis*, Academic Press, Inc., New York, 1980 [1](#)
- [16] M. Squassina, B. Volzone, *Bourgain-Brezis-Mironescu formula for magnetic operators*, *C. R. Math. Acad. Sci. Paris* **354** (2016), 825–831. [1](#), [2](#)

(A. Pinamonti) DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DEGLI STUDI DI PADOVA, VIA TRIESTE 63, 35121 PADOVA, ITALY
E-mail address: `Pinamont@math.unipd.it`

(M. Squassina) DIPARTIMENTO DI MATEMATICA E FISICA
UNIVERSITÀ CATTOLICA DEL SACRO CUORE, VIA DEI MUSEI 41, I-25121 BRESCIA, ITALY
E-mail address: `marco.squassina@dmf.unicatt.it`

(E. Vecchi) DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI BOLOGNA, PIAZZA DI PORTA S. DONATO 5, 40126, BOLOGNA, ITALY
E-mail address: `eugenio.vecchi2@unibo.it`