FINE REGULARITY RESULTS FOR MUMFORD-SHAH MINIMIZERS: POROSITY, HIGHER INTEGRABILITY AND THE MUMFORD-SHAH CONJECTURE

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ABSTRACT. We review some classical results and more recent insights about the regularity theory for local minimizers of the Mumford and Shah energy and their connections with the Mumford and Shah conjecture. We discuss in details the links among the latter, the porosity of the jump set and the higher integrability of the approximate gradient. In particular, higher integrability turns out to be related with an explicit estimate on the Hausdorff dimension of the singular set and an energetic characterization of the conjecture itself.

2010 MSC. 49J45 ; 49Q20.
KEYWORDS. Mumford and Shah variational model, Mumford and Shah conjecture, local minimizer, regularity, density lower bound, approximate gradient, jump set, porosity, higher integrability.

1. INTRODUCTION

The Mumford and Shah model is a prominent example of variational problem in image segmentation (see [69]). It is an algorithm able to detect the boundaries of the contours of the objects in a black and white digitized image. Representing the latter by a greyscale function $g \in L^\infty(\Omega, [0,1])$, a smoothed version of the original image is then obtained by minimizing the functional

$$(v, K) \to \mathscr{F}(v, K, \Omega) + \gamma \int_{\Omega \setminus K} |v - g|^2 dx, \tag{1.1}$$

with

$$\mathscr{F}(v, K, \Omega) := \int_{\Omega \setminus K} |\nabla v|^2 \, dx + \beta \, \mathcal{H}^1(K), \tag{1.2}$$

where $\Omega \subseteq \mathbb{R}^2$ is an open set, K is a relatively closed subset of Ω with finite $\mathcal{H}^1$ measure, $v \in C^1(\Omega \setminus K)$, β and γ are nonnegative parameters to be tuned suitably according to the applications. In our discussion we can set $\beta = 1$ without loss of generality.

The role of the squared L^2 distance in (1.1) is that of a fidelity term in order that the output of the process is close in an average sense to the original input image g. The set K represents the set of contours of the objects in the image, the length of which is kept controlled by the penalization of its $\mathcal{H}^1$ measure to avoid over segmentation, while the Dirichlet energy of v favors sharp contours rather than zones where a thin layer of gray is used to pass smoothly from white to black or vice versa.

We stress the attention upon the fact that the set K is not assigned a priori and it is not a boundary in general. Therefore, this problem is not a free boundary problem, and new ideas and techniques had to be developed to solve it. Since its appearance in the late 80's to today the research on the Mumford and Shah problem, and on related fields, has been very active and different approaches have been developed. In this notes we shall focus mainly on that proposed by De Giorgi and Ambrosio. This is only due to a matter of taste of the Author and it is also dictated by understandable reasons of space. Even more, it is not possible to be exhaustive in our (short) presentation, therefore we refer to the books by Ambrosio, Fusco and Pallara [7] and David [26] for the proofs of many results we shall only quote, for a more detailed account of the several contributions in literature, for the many connections with other fields and for complete lists of references (see also the recent survey [53] that covers several parts of the regularity theory that are not presented here).

Going back to the Mumford and Shah minimization problem and trying to follow the path of the Direct Method of the Calculus of Variations, it is clear that a weak formulation calls for a function space allowing for discontinuities of co-dimension 1 in which an existence theory can be established. Therefore, by taking into account the structure of the energy, De Giorgi and Ambrosio were led to consider the space SBV of *Special functions of Bounded Variation*, i.e. the subspace of BV functions with singular part of the distributional derivative concentrated on a 1-dimensional set called in what follows the *jump set* (throughout the paper we will use standard notations and results concerning the spaces BV and SBV, following the book [7]).

The purpose of the present set of notes is basically to resume and collect several of the regularity properties known at present for Mumford and Shah minimizers. More precisely, Section 2 is devoted to recalling basic facts about the functional setting of the problem and its weak formulation. The celebrated De Giorgi, Carriero and Leaci [33] regularity result implying the equivalence between the strong and weak formulations, is discussed in details. In subsection 2.3 we provide a recent proof by De Lellis and Focardi valid in the 2d case that gives an explicit constant in the density lower bound, and in subsection 2.4 we discuss the almost monotonicity formula by Bucur and Luckhaus. Next, we state the Mumford and Shah conjecture. The understanding of such a claim is the goal at which researchers involved in this problem are striving for. In this perspective well-established and more recent fine regularity results on the jump set of minimizers are discussed in Section 3. Furthermore, we highlight two different paths that might lead to the solution in positive of the Mumford and Shah conjecture: the complete characterization of blow ups in subsection 2.6 and a sharp higher integrability of the (approximate) gradient in Theorem 3.11 together with the uniqueness of blow up limits. In particular, we discuss in details the latter by following the ideas introduced by Ambrosio, Fusco and Hutchinson [4] linking higher integrability of the gradient of a minimizer with the size of the *singular set* of the minimizer itself, i.e. the subset of points of the jump set having no neighborhood in which the jump set itself is a regular curve. An explicit estimate shows that the bigger the integrability exponent of the gradient is, the lower the Hausdorff dimension of the singular set is (cf. Theorem 3.10). Pushing forward this approach, an energetic characterization of a slightly weaker form of the Mumford and Shah conjecture can be found beyond the scale of L^p spaces (cf. Theorem 3.11). In particular, the quoted estimate on the Hausdorff dimension of the full singular set reduces to the higher integrability property of the gradient and a corresponding estimate on a special subset of singular points: those for which the scaled Dirichlet energy is infinitesimal. The latter topic is dealt with in full details in Section 4 in the setting of Caccioppoli partitions as done by De Lellis and Focardi in [35]. The analysis of Section 4 allowed the same Authors to prove the higher integrability property in 2-dimensions as explained in Section 5. A different path leading to higher integrability in any dimension is to exploit the porosity of the jump set. This approach, due to De Philippis and Figalli [37], is the object of Section 7. Some preliminaries on porous sets are discussed in Section 6.

To conclude this introduction it is worth mentioning that the Mumford and Shah energy and the theory developed in order to study it, have been employed in many other fields. The applications to Fracture Mechanics, both in a static setting and for quasi-static irreversible crack-growth for brittle materials according to Griffith are important instances of that (see in particular [12], [7, Section 4.6.6] and [15], [21], [60]). It is also valuable to recall that several contributions in literature are devoted to the asymptotic analysis or the variational approximation of free discontinuity energies by means of De Giorgi's Γ-*convergence* theory. We refer to the books by Braides [13, 14, 15] for the analysis of several interesting problems arising from models in different fields (for a quick introduction to Γ-convergence see [40], for a more detailed account consult the treatise [20]).

The occasion to write this set of notes stems from the course "*Fine regularity results for Mumford-Shah minimizers: higher integrability of the gradient and estimates on the Hausdorff dimension of the singular set*" taught by the Author in July 2014 at Centro De Giorgi in Pisa within the activities of the "School on Free Discontinuity problems",

ERC Research Period on Calculus of Variations and Analysis in Metric Spaces. The material collected here covers entirely the six lectures of the course, additional topics and some more recent insights are also included for the sake of completeness and clarity. It is a pleasure to acknowledge the hospitality of Centro De Giorgi and to gratefully thank N. Fusco and A. Pratelli, the organizers of the school, for their kind invitation. Let me also thank all the people in the audience for their attention, patience, comments and questions. In particular, the kind help of R. Cristoferi and E. Radici who read a preliminary version of these notes is acknowledged. Nevertheless, the Author is the solely responsible for all the inaccuracies contained in them.

2. Existence Theory and first Regularity results

In this section we shall overview the first basic issues of the problem. More generally we discuss the n-dimensional case, though we shall often make specific comments related to the 2-dimensional setting of the original problem (and sometimes to the 3d case as well). We shall freely use the notation for BV functions and Caccioppoli sets adopted in the book by Ambrosio, Fusco and Pallara [7]. We shall always refer to it also for the many results that we shall apply or even only quote without giving a precise citation.

2.1. **Functional setting of the problem.** A function $v \in L^1(\Omega)$ belongs to $BV(\Omega)$ if and only if Dv is a (vector-valued) Radon measure on the non empty open subset Ω of $\mathbb{R}^n$. The distributional derivative of v can be decomposed according to

$$Dv = \nabla v \, \mathcal{L}^n \llcorner \Omega + (v^+ - v^-)\nu_v \, \mathcal{H}^{n-1} \llcorner S_v + D^c v,$$

where

(i) ∇v is the density of the absolutely continuous part of Dv with respect to $\mathcal{L}^n \llcorner \Omega$ (and the *approximate gradient* of v in the sense of Geometric Measure Theory as well);

(ii) S_v is the set of *approximate discontinuities* of v, an $\mathcal{H}^{n-1}$-rectifiable set (so that $\mathcal{L}^n(S_v) = 0$) endowed with approximate normal ν_v for $\mathcal{H}^{n-1}$ a.e. on S_v;

(iii) $v^\pm$ are the *approximate one-sided traces* left by v $\mathcal{H}^{n-1}$ a.e. on S_v;

(iv) $D^c v$ is the rest in the Radon-Nikodym decomposition of the singular part of Dv after the absolutely continuous part with respect to $\mathcal{H}^{n-1} \llcorner S_v$ has been identified. Thus, it is a singular measure both with respect to $\mathcal{L}^n \llcorner \Omega$ and to $\mathcal{H}^{n-1} \llcorner S_v$ (for more details see [7, Proposition 3.92]).

By taking into account the structure of the energy in (1.1), only volume and surface contributions are penalized, so that it is natural to introduce the following subspace of BV.

Definition 2.1 ([32], Section 4.1 [7])**.** *$v \in BV(\Omega)$ is a* Special function of Bounded Variation, *in short $v \in SBV(\Omega)$, if $D^c v = 0$, i.e. $Dv = \nabla v\, \mathcal{L}^n \llcorner \Omega + (v^+ - v^-)\nu_v\, \mathcal{H}^{n-1} \llcorner S_v$.*

No Cantor staircase type behavior is allowed for these functions. Simple examples are collected in the ensuing list:

(i) if $n = 1$ and $\Omega = (\alpha, \beta)$, $SBV\big((\alpha,\beta)\big)$ is easily described in view of the well known decomposition of BV functions of one variable. Indeed, any function in $SBV\big((\alpha,\beta)\big)$ is the sum of a $W^{1,1}\big((\alpha,\beta)\big)$ function with one of pure jump, i.e. $\sum_{i\in\mathbb{N}} a_i \chi_{(\alpha_i,\alpha_{i+1})}$, with $\alpha = \alpha_0$, $\alpha_i < \alpha_{i+1} < \beta$, $(a_i)_{i\in I} \in \ell^\infty$;

(ii) $W^{1,1}(\Omega) \subset SBV(\Omega)$. Clearly, $Dv = \nabla v\, \mathcal{L}^n \llcorner \Omega$. In this case ∇v coincides with the usual distributional gradient;

(iii) let $(E_i)_{i\in I}$, $I \subseteq \mathbb{N}$, be a *Caccioppoli partition* of Ω, i.e. $\mathcal{L}^n\big(\Omega \setminus \cup_i E_i\big) = 0$ and $\mathcal{L}^n(E_i \cap E_j) = 0$ if $i \neq j$, with the E_i's sets of finite perimeter such that
$$\sum_{i\in I} \mathrm{Per}(E_i) < \infty.$$
Then, $v = \sum_{i\in I} a_i\, \chi_{E_i} \in SBV(\Omega)$ if $(a_i)_{i\in I} \in \ell^\infty$. In this case, if $J_{\mathscr{E}} := \cup_i \partial^* E_i$ denotes the *set of interfaces* of $\mathscr{E}$, with $\partial^* E_i$ the *essential boundary* of E_i, then $\mathcal{H}^{n-1}(S_v \setminus J_{\mathscr{E}}) = 0$ and
$$Dv = (v^+ - v^-)\nu_v\, \mathcal{H}^{n-1} \llcorner J_{\mathscr{E}}.$$
Functions of this type have zero approximate gradient, they are called *piecewise constant* and form a subspace denoted by $SBV_0(\Omega)$ (cf. [7, Theorem 4.23]);

(iv) the function $v(\rho,\theta) := \sqrt{\rho} \cdot \sin(\theta/2)$ for $\theta \in (-\pi,\pi)$ and $\rho > 0$ is in $SBV(B_r)$ for all $r > 0$. In particular, $v \in SBV(B_r) \setminus \big(W^{1,1}(B_r) \oplus SBV_0(B_r)\big)$.

A general receipt to construct interesting examples of SBV functions can be obtained as follows (see [7, Proposition 4.4]).

Proposition 2.2. *If $K \subset \Omega$ is a closed set such that $\mathcal{H}^{n-1}(K) < +\infty$ and $v \in W^{1,1} \cap L^\infty(\Omega \setminus K)$, then $v \in SBV(\Omega)$ and*
$$\mathcal{H}^{n-1}(S_v \setminus K) = 0. \tag{2.1}$$

Clearly, property (2.1) above is not valid for a generic member of SBV, but it does for a significant class of functions: local minimizers of the energy under consideration (see below for the definition), actually satisfying even a stronger property (cf. Proposition 2.9).

2.2. **Tonelli's Direct Method and Weak formulation.** The difficulty in applying the Direct Method is related to the surface term for which it is hard to find a topology ensuring at the same time lower semicontinuity and pre-compactness for minimizing sequences. Using the Hausdorff local topology requires a very delicate study of the latter ones to rule out typical counterexamples as shown by Maddalena and Solimini in [56]. Here, we shall follow instead the original approach by De Giorgi and Ambrosio [32].

Keeping in mind the example in Proposition 2.2, the weak formulation of the problem under study is obtained naively by taking $K = S_v$. Loosely speaking in this approach the set of contours K is identified by the (Borel) set S_v of (approximate) discontinuities of the function v that is not fixed a priori. This is the reason for the terminology *free discontinuity* problem coined by De Giorgi. The (weak counterpart of the) Mumford and Shah energy $\mathscr{F}$ in (1.2) of a function v in $SBV(\Omega)$ on an open subset $A \subseteq \Omega$ then reads as

$$\mathscr{F}(v, A) = \mathrm{MS}(v, A) + \gamma \int_A |v - g|^2 dx, \tag{2.2}$$

where

$$\mathrm{MS}(v, A) := \int_A |\nabla v|^2 dx + \mathcal{H}^{n-1}(S_v \cap A). \tag{2.3}$$

For the sake of simplicity in case $A = \Omega$ we drop the dependence on the set of integration.

In passing, we note that, the class $\{v \in BV(\Omega) : Dv = D^c v\}$ of Cantor type functions is dense in BV w.r.to the L^1 topology, thus it is easy to infer that

$$\inf_{BV(\Omega)} \mathscr{F} = 0,$$

so that the restriction to SBV is needed in order not to trivialize the problem.

Ambrosio's SBV closure and compactness theorem (see [7, Theorems 4.7 and 4.8]) ensures the existence of a minimizer of $\mathscr{F}$ on SBV.

Theorem 2.3 (Ambrosio [2]). *Let $(v_j)_j \subset SBV(\Omega)$ be such that*

$$\sup_j \big(\mathrm{MS}(v_j) + \|v_j\|_{L^\infty(\Omega)}\big) < \infty,$$

then there exists a subsequence $(v_{j_k})_k$ and a function $v \in SBV(\Omega)$ such that $v_{j_k} \to v$ $L^p(\Omega)$, for all $p \in [1, \infty)$.

Moreover, we have the separated lower semicontinuity estimates

$$\int_\Omega |\nabla v|^2 dx \le \liminf_k \int_\Omega |\nabla v_{j_k}|^2 dx \tag{2.4}$$

and

$$\mathcal{H}^{n-1}(S_v) \le \liminf_k \mathcal{H}^{n-1}(S_{v_{j_k}}). \tag{2.5}$$

Ambrosio's theorem is the natural counterpart of Rellich-Kondrakov theorem in Sobolev spaces. Indeed, for Sobolev functions, it reduces essentially to that statement provided that an L^p rather than an L^∞ bound is assumed. More generally, Ambrosio's theorem holds true in the bigger space $GSBV$. In particular, (2.4) and (2.5) display a separate lower semicontinuity property for the two terms of the energy in a way that the two terms cannot combine to create neither a contribution for the other nor a Cantor type one.

By means of the chain rule formula for BV functions one can prove that the functional under consideration is decreasing under truncation, i.e. for all $k \in \mathbb{N}$

$$\mathscr{F}(\tau_k(v)) \leq \mathscr{F}(v) \quad \forall v \in SBV(\Omega),$$

if $\tau_k(v) := (v \wedge k) \vee (-k)$.

Therefore, being $g \in L^\infty(\Omega)$, we can always restrict ourselves to minimize it over the ball in $L^\infty(\Omega)$ of radius $\|g\|_{L^\infty(\Omega)}$. In conclusion, Theorem 2.3 always provides the existence of a (global) minimizer for the weak formulation of the problem.

Once the existence has been checked, necessary conditions satisfied by minimizers are deduced. Supposing $g \in C^1(\Omega)$, by means of internal variations, i.e. constructing competitors to test the minimality of u by composition with diffeomorphisms of Ω arbitrarily close to the identity of the type $\mathrm{Id} + \varepsilon\,\phi$, the Euler-Lagrange equation takes the form

$$\int_{\Omega\setminus S_u} \Big(\big(|\nabla u|^2 + \gamma(u-g)^2\big)\mathrm{div}\phi - 2\langle \nabla u, \nabla u \cdot \nabla\phi\rangle - 2\gamma\,(u-g)\langle \nabla g, \phi\rangle\Big)dx + \int_{S_u} \mathrm{div}^{S_u}\phi\, d\mathcal{H}^{n-1} = 0 \quad (2.6)$$

for all $\phi \in C^1_c(\Omega, \mathbb{R}^n)$, $\mathrm{div}^{S_u}\phi$ denoting the tangential divergence of the field ϕ on S_u (cf. [7, Theorem 7.35]).

Instead, by using outer variations, i.e. range perturbations of the type $u + \varepsilon(v-u)$ for $v \in SBV(\Omega)$ such that $\mathrm{spt}(u-v) \subset\subset \Omega$ and $S_v \subseteq S_u$, we find

$$\int_\Omega \big(\langle \nabla u, \nabla(v-u)\rangle + \gamma\,(u-g)\,(v-u)\big)\,dx = 0. \quad (2.7)$$

2.3. **Back to the strong formulation: the density lower bound.** Existence of minimizers for the strong formulation of the problem is obtained via a regularity property enjoyed by (the jump set of) the minimizers of the weak counterpart. The results obtained in this framework will be instrumental also to establish way much finer regularity properties in the ensuing sections.

We start off analyzing the scaling of the energy in order to understand the local behavior of minimizers. This operation has to be done with some care since the volume and length terms in MS scale differently under affine change of variables of the domain. Let $v \in SBV(B_\rho(x))$, set

$$v_{x,\rho}(y) := \rho^{-1/2} v(x + \rho\, y), \tag{2.8}$$

then $v_{x,\rho} \in SBV(B_1)$, with

$$\mathrm{MS}(v_{x,\rho}, B_1) = \rho^{1-n}\mathrm{MS}(v, B_\rho(x))$$

and

$$\int_{B_1} |v_{x,\rho} - g_{x,\rho}|^2 dz = \rho^{-1-n} \int_{B_\rho(x)} |v - g|^2 dy.$$

Thus,

$$\rho^{1-n}\Big(\mathrm{MS}(v, B_\rho(x)) + \int_{B_\rho(x)} |v - g|^2 dz\Big) = \mathrm{MS}(v_{x,\rho}, B_1) + \rho^2 \int_{B_1} |v_{x,\rho} - g_{x,\rho}|^2 dy.$$

By taking into account that $g \in L^\infty$ and that along the minimization process we are actually interested only in functions satisfying the bound $\|v\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)}$, we get

$$\rho^2 \int_{B_1} |v_{x,\rho} - g_{x,\rho}|^2 dy \leq 2\rho\, \|g\|^2_{L^\infty(\Omega)} = O(\rho) \qquad \rho \downarrow 0.$$

This calculation shows that, at the first order, the leading term in the energy $\mathscr{F}$ computed on $B_\rho(x)$ is that related to the MS functional, the other being a contribution of higher order that can be neglected in a preliminary analysis.

Motivated by this, we introduce a notion of minimality involving only the leading part of the energy. This corresponds to setting $\gamma = 0$ in the definition of $\mathscr{F}$ (cf. (2.2)).

Definition 2.4. *A function $u \in SBV(\Omega)$ with $\mathrm{MS}(u) < \infty$[1] is a local minimizer of MS if*

$$\mathrm{MS}(u) \leq \mathrm{MS}(v) \qquad \textit{whenever } \{v \neq u\} \subset\subset \Omega.$$

In what follows, u will always denote a local minimizer of MS unless otherwise stated, and the class of all local minimizers shall be denoted by $\mathcal{M}(\Omega)$. Actually, we shall often refer to local minimizers simply as minimizers if no confusion can arise. In particular, regularity properties for minimizers of the whole energy can be obtained by perturbing the theory developed for local minimizers (see for instance Corollary 2.13 and Theorem 2.16 below).

Harmonic functions with small oscillation are minimizers as a simple consequence of (2.7).

[1]The finite energy condition is actually not needed due to the local character of the notion introduced, it is assumed only for the sake of simplicity.

Proposition 2.5 (Chambolle, see Proposition 6.8 [7]). *If u is harmonic in Ω', then $u \in \mathcal{M}(\Omega)$, for all $\Omega \subset\subset \Omega'$, provided*

$$\big(\sup_{\Omega} u - \inf_{\Omega} u\big)\|\nabla u\|_{L^\infty(\Omega)} \leq 1. \tag{2.9}$$

Proof. Let $A \subset\subset \Omega$. By Theorem 2.3 it is easy to show the existence of a minimizer $w \in SBV(\Omega)$ of the Dirichlet problem $\min\{\mathrm{MS}(v) : v \in SBV(\Omega),\, v = u \text{ on } \Omega \setminus A\}$. Moreover, by truncation $\inf_\Omega u \leq w \leq \sup_\Omega u$ $\mathcal{L}^n$ a.e. on Ω.

By the arbitrariness of A, the local minimality of u follows provided we show that $\mathrm{MS}(u,\Omega) \leq \mathrm{MS}(w,\Omega)$. To this aim, we use the Euler-Lagrange condition (2.7) with $\gamma = 0$, namely

$$\int_\Omega \langle \nabla w, \nabla(u-w)\rangle\, dx = 0 \Longleftrightarrow \int_\Omega |\nabla w|^2\, dx = \int_\Omega \langle \nabla w, \nabla u\rangle\, dx,$$

to get

$$\begin{aligned}\mathrm{MS}(u,\Omega) \leq \mathrm{MS}(w,\Omega) &\Longleftrightarrow \int_\Omega \langle \nabla u, \nabla(u-w)\rangle\, dx \leq \mathcal{H}^{n-1}(S_w)\\ &\Longleftrightarrow \int_\Omega \nabla u \cdot dD(u-w) - \int_{S_w} \langle \nabla u, \nu_w\rangle (w^+ - w^-) d\mathcal{H}^{n-1} \leq \mathcal{H}^{n-1}(S_w).\end{aligned}$$

An integration by parts, the harmonicity of u and the equality $w = u$ on $\Omega \setminus A$ give

$$\int_\Omega \nabla u \cdot dD(u-w) = -\int_\Omega (u-w)\triangle u\, dx = 0,$$

and therefore

$$\mathrm{MS}(u,\Omega) \leq \mathrm{MS}(w,\Omega) \Longleftrightarrow -\int_{S_w} \langle \nabla u, \nu_w\rangle (w^+ - w^-) d\mathcal{H}^{n-1} \leq \mathcal{H}^{n-1}(S_w).$$

The conclusion follows from condition (2.9) as $\inf_\Omega u \leq w \leq \sup_\Omega u$ $\mathcal{L}^n$ a.e. on Ω. □

By means of the slicing theory in SBV, i.e. the characterization of SBV via restrictions to lines, one can also prove that *pure jumps*, i.e. functions as

$$a\chi_{\{\langle x - x_o, \nu\rangle > 0\}} + b\chi_{\{\langle x - x_o, \nu\rangle < 0\}} \tag{2.10}$$

for a and $b \in \mathbb{R}$ and $\nu \in \mathbb{S}^{n-1}$, are local minimizers as well (cf. [7, Proposition 6.8]). Further examples shall be discussed in what follows (cf. subsection 2.5).

As established in [33] in all dimensions (and proved alternatively in [23] and [25] in dimension two), if $u \in \mathcal{M}(\Omega)$ then the pair $(u, \Omega \cap \overline{S_u})$ is a minimizer of $\mathscr{F}$ for $\gamma = 0$. The main point is the identity $\mathcal{H}^{n-1}(\Omega \cap \overline{S_u} \setminus S_u) = 0$, which holds for every $u \in \mathcal{M}(\Omega)$. The groundbreaking paper [33] proves this identity via the following *density lower bound* estimate (see [7, Theorem 7.21]).

Theorem 2.6 (De Giorgi, Carriero and Leaci [33])**.** *There exist dimensional constants* θ, $\varrho > 0$ *such that for every* $u \in \mathcal{M}(\Omega)$

$$\mathrm{MS}(u, B_r(z)) \geq \theta\, r^{n-1} \tag{2.11}$$

for all $z \in \Omega \cap \overline{S_u}$*, and all* $r \in (0, \varrho \wedge \mathrm{dist}(z, \partial\Omega))$*.*

Building upon the same ideas, in [17] it is proved a slightly more precise result (see again [7, Theorem 7.21]).

Theorem 2.7 (Carriero and Leaci [17])**.** *There exists a dimensional constant* θ_0, $\varrho_0 > 0$ *such that for every* $u \in \mathcal{M}(\Omega)$

$$\mathcal{H}^{n-1}(S_u \cap B_r(z)) \geq \theta_0\, r^{n-1} \tag{2.12}$$

for all $z \in \Omega \cap \overline{S_u}$*, and all* $r \in (0, \varrho_0 \wedge \mathrm{dist}(z, \partial\Omega))$*.*

In particular, from the latter we infer the so called *elimination property* for $\Omega \cap \overline{S_u}$, i.e. if $\mathcal{H}^{n-1}(S_u \cap B_r(z)) < \frac{\theta_0}{2^{n-1}}\, r^{n-1}$ then actually $\overline{S_u} \cap B_{r/2}(z) = \emptyset$.

Given Theorem 2.6 or 2.7 for granted we can easily prove the equivalence of the strong and weak formulation of the problem by means of the ensuing *density estimates*.

Lemma 2.8. *Let* μ *be a Radon measure on* $\mathbb{R}^n$*,* B *be a Borel set and* $s \in [0, n]$ *be such that*

$$\limsup_{r \downarrow 0} \frac{\mu(B_r(x))}{\omega_s r^s} \geq t \qquad \textit{for all } x \in B.$$

Then, $\mu(B) \geq t\, \mathcal{H}^s(B)$*.*

Proposition 2.9. *Let* $u \in \mathcal{M}(\Omega)$*, then* $\mathcal{H}^{n-1}(\Omega \cap \overline{S_u} \setminus S_u) = 0$*. In particular,* $\big(u, \Omega \cap \overline{S_u}\big)$ *is a local minimizer for* $\mathscr{F}$ *(with* $\gamma = 0$*).*

Proof of Proposition 2.9. In view of Theorem 2.7 we may apply the density estimates of Lemma 2.8 to $\mu = \mathcal{H}^{n-1} \llcorner S_u$ and to the Borel set $\overline{S_u} \setminus S_u$ with $t = \theta_0$. Therefore, we deduce that

$$\theta_0 \mathcal{H}^{n-1}(\Omega \cap \overline{S_u} \setminus S_u) \leq \mu(\Omega \cap \overline{S_u} \setminus S_u) = 0.$$

Clearly, $\mathrm{MS}(u) = \mathscr{F}(u, \Omega \cap \overline{S_u})$, and the conclusion follows at once. □

The argument for (2.11) used by De Giorgi, Carriero and Leaci in [33], and similarly in [17] for (2.12), is indirect: it relies on Ambrosio's SBV compactness theorem and Poincaré-Wirtinger type inequality in SBV established in [33] (see also [7, Theorem 4.14] and [8, Proposition 2] for a version in which boundary values are preserved) to analyze blow up limits of minimizers (see subsection 2.6 for the definition of blow ups) with vanishing jump energy and prove that they are harmonic functions (cf. [7, Theorem

7.21]). A contradiction argument shows that on small balls the energy of local minimizers inherits the decay properties as that of harmonic functions. Actually, the proof holds true for much more general energies (see [43], [7, Chapter 7]).

In the paper [34] an elementary proof valid only in 2-dimensions and tailored on the MS energy is given. No Poincaré-Wirtinger inequality, nor any compactness argument are required. Moreover, it has the merit to exhibit an explicit constant. Indeed, the proof in [34] is based on an observation of geometric nature and on a direct variational comparison argument. It also differs from those exploited in [23] and [25] to derive (2.12) in the 2-dimensional case.

Theorem 2.10 (De Lellis and Focardi [34]). *Let $u \in \mathcal{M}(\Omega)$. Then*

$$\mathrm{MS}(u, B_r(z)) \geq r \tag{2.13}$$

for all $z \in \Omega \cap \overline{S_u}$ and all $r \in (0, \mathrm{dist}(z, \partial\Omega))$.

More precisely, the set $\Omega_u := \{z \in \Omega : (2.13) \text{ fails}\}$ is open and $\Omega_u = \Omega \setminus \overline{S_u}$[2].

To the aim of establishing Theorem 2.10 we prove a consequence of (2.6), a monotonicity formula discovered independently by David and Léger in [27, Proposition 3.5] and by Maddalena and Solimini in [57]. The proof we present here is that given in [34, Lemma 2.1] (an analogous result holds true in any dimension with essentially the same proof).

Lemma 2.11. *Let $u \in \mathcal{M}(\Omega)$, $\Omega \subset \mathbb{R}^2$, then for every $z \in \Omega$ and for $\mathcal{L}^1$ a.e. $r \in (0, \mathrm{dist}(z, \partial\Omega))$*

$$r \int_{\partial B_r(z)} \left(\left(\frac{\partial u}{\partial \nu} \right)^2 - \left(\frac{\partial u}{\partial \tau} \right)^2 \right) d\mathcal{H}^1 + \mathcal{H}^1(S_u \cap B_z(r)) = \int_{S_u \cap \partial B_r(z)} |\langle \nu_u^\perp(x), x \rangle| d\mathcal{H}^0(x), \tag{2.14}$$

$\frac{\partial u}{\partial \nu}$ and $\frac{\partial u}{\partial \tau}$ being the projections of ∇u in the normal and tangential directions to $\partial B_r(z)$, respectively.[3]

Proof of Lemma 2.11. With fixed a point $z \in \Omega$, $r > 0$ with $B_r(z) \subseteq \Omega$, we consider special radial vector fields $\eta_{r,s} \in \mathrm{Lip} \cap C_c(B_r(z), \mathbb{R}^2)$, $s \in (0, r)$, in the first variation formula (2.6) (with $\gamma = 0$). Moreover, for the sake of simplicity we assume $z = 0$, and drop the subscript z in what follows. Let

$$\eta_{r,s}(x) := x\, \chi_{[0,s]}(|x|) + \frac{|x| - r}{s - r}\, x\, \chi_{(s,r]}(|x|),$$

[2] Actually, the very same proof shows also that $\Omega_u = \Omega \setminus \overline{J_u}$, where J_u is the subset of points of S_u for which one sided traces exist. Recall that $\mathcal{H}^{n-1}(S_v \setminus J_v) = 0$ for all $v \in BV(\Omega)$.

[3] For $\xi \in \mathbb{R}^2$, $\xi^\perp$ is the vector obtained by an anticlockwise rotation.

then a routine calculation leads to

$$\nabla\eta_{r,s}(x) := \mathrm{Id}\,\chi_{[0,s]}(|x|) + \left(\frac{|x|-r}{s-r}\mathrm{Id} + \frac{1}{s-r}\frac{x}{|x|}\otimes x\right)\chi_{(s,r]}(|x|)$$

$\mathcal{L}^2$ a.e. in Ω. In turn, from the latter formula we infer for $\mathcal{L}^2$ a.e. in Ω

$$\mathrm{div}\eta_{r,s}(x) = 2\,\chi_{[0,s]}(|x|) + \left(2\frac{|x|-r}{s-r} + \frac{|x|}{s-r}\right)\chi_{(s,r]}(|x|),$$

and, if $\nu_u(x)$ is a unit vector normal field in $x \in S_u$, for $\mathcal{H}^1$ a.e. $x \in S_u$

$$\mathrm{div}^{S_u}\eta_{r,s}(x) = \chi_{[0,s]}(|x|) + \left(\frac{|x|-r}{s-r} + \frac{1}{|x|(s-r)}|\langle x, \nu_u^\perp\rangle|^2\right)\chi_{(s,r]}(|x|).$$

Consider the set $I := \{\rho \in (0, \mathrm{dist}(0,\partial\Omega)) : \mathcal{H}^1(S_u \cap \partial B_\rho) = 0\}$, then $(0, \mathrm{dist}(0,\partial\Omega)) \setminus I$ is at most countable being $\mathcal{H}^1(S_u) < +\infty$. If ρ and $s \in I$, by inserting η_s in (2.6) we find

$$\frac{1}{s-r}\int_{B_r\setminus B_s}|x||\nabla u|^2 dx - \frac{2}{s-r}\int_{B_r\setminus B_s}|x|\langle\nabla u, \left(\mathrm{Id} - \frac{x}{|x|}\otimes\frac{x}{|x|}\right)\nabla u\rangle dx$$
$$= \mathcal{H}^1(S_u \cap B_s) + \int_{S_u\cap(B_r\setminus B_s)}\frac{|x|-r}{s-r}d\mathcal{H}^1 + \frac{1}{s-r}\int_{S_u\cap(B_r\setminus B_s)}|x||\langle\frac{x}{|x|}, \nu_u^\perp\rangle|^2 d\mathcal{H}^1.$$

Next we employ Co-Area formula and rewrite equality above as

$$\frac{1}{s-r}\int_s^r \rho\, d\rho\int_{\partial B_\rho}|\nabla u|^2 d\mathcal{H}^1 - \frac{2}{s-r}\int_s^r \rho\, d\rho\int_{\partial B_\rho}\left|\frac{\partial u}{\partial\tau}\right|^2 d\mathcal{H}^1$$
$$= \mathcal{H}^1(S_u \cap B_s) + \int_{S_u\cap(B_r\setminus B_s)}\frac{|x|-r}{s-r}d\mathcal{H}^1 + \frac{1}{s-r}\int_s^r d\rho\int_{S_u\cap\partial B_\rho}|\langle x, \nu_u^\perp\rangle| d\mathcal{H}^0$$

where $\nu := x/|x|$ denotes the radial unit vector and $\tau := \nu^\perp$ the tangential one. Lebesgue differentiation theorem then provides a subset I' of full measure in I such that if $r \in I'$ and we let $s \uparrow t^-$ it follows

$$-r\int_{\partial B_r}|\nabla u|^2 d\mathcal{H}^1 + 2r\int_{\partial B_r}\left|\frac{\partial u}{\partial\tau}\right|^2 d\mathcal{H}^1 = \mathcal{H}^1(S_u \cap B_r) - \int_{S_u\cap\partial B_r}|\langle x, \nu_u^\perp\rangle| d\mathcal{H}^0.$$

Formula (2.14) then follows straightforwardly. □

We are now ready to prove Theorem 2.10.

Proof of Theorem 2.10. Given $u \in \mathcal{M}(\Omega)$, $z \in \Omega$ and $r \in (0, \mathrm{dist}(z,\partial\Omega))$ let

$$e_z(r) := \int_{B_r(z)}|\nabla u|^2 dx, \quad \ell_z(r) := \mathcal{H}^1(S_u \cap B_r(z)),$$

and

$$m_z(r) := \mathrm{MS}(u, B_r(z)), \quad h_z(r) := e_z(r) + \frac{1}{2}\ell_z(r).$$

Clearly, $m_z(r) = e_z(r) + \ell_z(r) \leq 2h_z(r)$, with equality if and only if $e_z(r) = 0$.

Introduce the set $S_u^\star$ of points $x \in S_u$ for which

$$\lim_{r\downarrow 0} \frac{\mathcal{H}^1(S_u \cap B_r(x))}{2r} = 1\,. \tag{2.15}$$

Since S_u is rectifiable, $\mathcal{H}^1(S_u \setminus S_u^\star) = 0$. Next let $z \in \Omega$ be such that

$$m_z(R) < R \qquad \text{for some } R \in (0, \operatorname{dist}(z, \partial\Omega)). \tag{2.16}$$

We claim that $z \notin S_u^\star$.

W.l.o.g. we take $z = 0$ and drop the subscript z in e, ℓ, m and h.

In addition we can assume $e(R) > 0$. Otherwise, by the Co-Area formula and the trace theory of BV functions, we would find a radius $r < R$ such that $u|_{\partial B_r}$ is a constant (cf. the argument below). In turn, u would necessarily be constant in B_r because the energy decreases under truncations, thus implying $z \notin S_u^\star$. We can also assume $\ell(R) > 0$, since otherwise u would be harmonic in B_R and thus we would conclude $z \notin S_u^\star$.

We start next to compare the energy of u with that of an harmonic competitor on a suitable disk. The inequality $\ell(R) \le m(R) < R$ is crucial to select good radii.

Step 1: For any fixed $r \in (0, R - \ell(R))$, there exists a set I_r of positive length in (r, R) such that

$$\frac{h(\rho)}{\rho} \le \frac{1}{2} \cdot \frac{e(R) - e(r)}{R - r - (\ell(R) - \ell(r))} \qquad \text{for all } \rho \in I_r. \tag{2.17}$$

Define $J_r := \{t \in (r, R) : \mathcal{H}^0(S_u \cap \partial B_t) = 0\}$. We claim the existence of $J_r' \subseteq J_r$ with $\mathcal{L}^1(J_r') > 0$ and such that

$$\int_{\partial B_\rho} |\nabla u|^2 d\mathcal{H}^1 \le \frac{e(R) - e(r)}{R - r - (\ell(R) - \ell(r))} \qquad \text{for all } \rho \in J_r'. \tag{2.18}$$

Indeed, we use the Co-Area formula for rectifiable sets (see [7, Theorem 2.93]) to find

$$\mathcal{L}^1((r,R)\setminus J_r) \le \int_{(r,R)\setminus J_r} \mathcal{H}^0(S_u \cap \partial B_t)dt = \int_{S_u \cap (B_R \setminus \overline{B_r})} \left|\langle \nu_u^\perp(x), \frac{x}{|x|}\rangle\right| d\mathcal{H}^1(x) \le \ell(R) - \ell(r).$$

In turn, this inequality implies $\mathcal{L}^1(J_r) \ge R - r - (\ell(R) - \ell(r)) > 0$, thanks to the choice of r. Then, define J_r' to be the subset of radii $\rho \in J_r$ for which

$$\int_{\partial B_\rho} |\nabla u|^2 d\mathcal{H}^1 \le \fint_{J_r} \left(\int_{\partial B_t} |\nabla u|^2 d\mathcal{H}^1\right) dt\,.$$

Formula (2.18) follows by the Co-Area formula and the estimate $\mathcal{L}^1(J_r) \ge R - r - (\ell(R) - \ell(r))$.

We define I_r as the subset of radii $\rho \in J_r'$ satisfying both (2.14) and (2.18). Therefore,

$$\int_{\partial B_\rho} \left(\frac{\partial u}{\partial \tau}\right)^2 d\mathcal{H}^1 = \frac{1}{2} \int_{\partial B_\rho} |\nabla u|^2 d\mathcal{H}^1 + \frac{\ell(\rho)}{2\rho} \qquad \forall \rho \in I_r. \tag{2.19}$$

Clearly, I_r has full measure in J'_r, so that $\mathcal{L}^1(I_r) > 0$.

For any $\rho \in I_r$, we let w be the harmonic function in B_ρ with trace u on ∂B_ρ. Then, as $\frac{\partial w}{\partial \tau} = \frac{\partial u}{\partial \tau}$ $\mathcal{H}^1$ a.e. on ∂B_ρ, the local minimality of u entails

$$m(\rho) \leq \int_{B_\rho} |\nabla w|^2 dx \leq \rho \int_{\partial B_\rho} \left(\frac{\partial u}{\partial \tau}\right)^2 d\mathcal{H}^1 \stackrel{(2.19)}{=} \frac{\rho}{2} \int_{\partial B_\rho} |\nabla u|^2 d\mathcal{H}^1 + \frac{\ell(\rho)}{2}.$$

The inequality (2.17) follows from the latter inequality and from (2.18):

$$h(\rho) = e(\rho) + \frac{\ell(\rho)}{2} \leq \frac{\rho}{2} \int_{\partial B_\rho} |\nabla u|^2 d\mathcal{H}^1 \leq \frac{\rho}{2} \cdot \frac{e(R) - e(r)}{R - r - (\ell(R) - \ell(r))}.$$

Step 2: We now show that $0 \notin S_u^\star$.

Let $\varepsilon \in (0,1)$ be fixed such that $m(R) \leq (1-\varepsilon)R$, and fix any radius $r \in (0, R - \ell(R) - \frac{1}{1-\varepsilon} e(R))$. Step 1 and the choice of r then imply

$$\frac{h(\rho)}{\rho} \leq \frac{1}{2} \frac{e(R) - e(r)}{R - r - (\ell(R) - \ell(r))} \leq \frac{e(R)}{2(R - \ell(R) - r)} < \frac{1-\varepsilon}{2},$$

in turn giving $m(\rho) \leq 2h(\rho) < (1-\varepsilon)\rho$. Let $\rho_\infty := \inf\{t > 0 : m(t) \leq (1-\varepsilon)t\}$, then $\rho_\infty \in [0, \rho]$. Note that if ρ_∞ were strictly positive then actually ρ_∞ would be a minimum. In such a case, we could apply the argument above and find $\widetilde{\rho} \in (r_\infty, \rho_\infty)$, with $r_\infty \in (0, \rho_\infty - \ell(\rho_\infty) - \frac{1}{1-\varepsilon} e(\rho_\infty))$, such that $m(\widetilde{\rho}) < (1-\varepsilon)\widetilde{\rho}$ contradicting the minimality of ρ_∞. Hence, there is a sequence $\rho_k \downarrow 0^+$ with $m(\rho_k) \leq (1-\varepsilon)\rho_k$. Then, clearly condition (2.15) is violated, so that $0 \notin S_u^\star$.

Conclusion: We first prove that Ω_u is open. Let $z \in \Omega_u$ and let $R > 0$ and $\varepsilon > 0$ be such that $m_z(R) \leq (1-\varepsilon)R$ and $B_{\varepsilon R}(z) \subset \Omega$. Let now $x \in B_{\varepsilon R}(z)$, then

$$m_x(R - |x - z|) \leq m_z(R) \leq (1-\varepsilon)R < R - |x - z|,$$

therefore $x \in \Omega_u$.

As Ω_u is open and $S_u^\star \cap \Omega_u = \emptyset$ by Step 2, we infer $\Omega \cap \overline{S_u^\star} \subseteq \Omega \setminus \Omega_u$. Moreover, let $z \notin \overline{S_u^\star}$ and $r > 0$ be such that $B_r(z) \subseteq \Omega \setminus \overline{S_u^\star}$. Since $\mathcal{H}^1(S_u \setminus \overline{S_u^\star}) = 0$, $u \in W^{1,2}(B_r(z))$ and thus u is an harmonic function in $B_r(z)$ by minimality. Therefore $z \in \Omega_u$, and in conclusion $\Omega \setminus \Omega_u = \Omega \cap \overline{S_u^\star} = \Omega \cap \overline{S_u}$. □

A simple iteration of Theorem 2.10 gives a density lower bound as in (2.12) with an explicit constant θ_0 (see [34, Corollary 1.2]).

Corollary 2.12 (De Lellis and Focardi [34]). *If $u \in \mathcal{M}(\Omega)$, then $\mathcal{H}^1(\Omega \cap SS_u \setminus J_u) = 0$ and*

$$\mathcal{H}^1(S_u \cap B_r(z)) \geq \frac{\pi}{2^{23}} r \tag{2.20}$$

for all $z \in \Omega \cap SS_u$ *and all* $r \in (0, \mathrm{dist}(z, \partial\Omega))$.

A similar result can be established for quasi-minimizers of the Mumford and Shah energy, the most prominent examples being minimizers of the functional $\mathscr{F}$ in equation (1.1). More precisely, a *quasi-minimizer* is any function u in $SBV(\Omega)$ with $\mathrm{MS}(u) < \infty$ satisfying for some $\alpha > 0$ and $c_\alpha \geq 0$ for all balls $B_\rho(z) \subset \Omega$

$$\mathrm{MS}(u, B_\rho(z)) \leq \mathrm{MS}(w, B_\rho(z)) + c_\alpha\, \rho^{1+\alpha}$$

whenever $\{w \neq u\} \subset\subset B_\rho(z)$.

One can then prove the ensuing infinitesimal version of (2.13) (cf. with [34, Corollary 1.3]).

Corollary 2.13 (De Lellis and Focardi [34]). *Let u be a quasi-minimizers of the Mumford and Shah energy, then*

$$\Omega \cap \overline{J_u} = \Omega \cap \overline{S_u} = \left\{ z \in \Omega : \liminf_{r \downarrow 0^+} \frac{m_z(r)}{r} \geq \frac{2}{3} \right\}. \tag{2.21}$$

The proof of this corollary, though, needs a blow up analysis and a new SBV Poincaré-Wirtinger type inequality of independent interest, obtained by improving upon some ideas contained in [41] (cf. with [34, Theorem B.6]); it is, therefore, much more technical.

Let us remark that it is possible to improve slightly Theorem 2.10 by combining the ideas of its proof hinted to above with the SBV Poincaré-Wirtinger type inequality in [34, Theorem B.6], and show that actually

$$\Omega \setminus \overline{S_u} = \{x \in \Omega : m_x(r) \leq r \quad \text{for some } r \in \mathrm{dist}(x, \partial\Omega)\}$$

(see [34, Remark 2.3]).

A natural question is the sharpness of the estimates (2.13) and (2.20). The analysis performed by Bonnet [10] suggests that $\pi/2^{23}$ in (2.20) should be replaced by 1, and 1 in (2.13) by 2. Note that the square root function $u(\rho, \theta) = \sqrt{\frac{2}{\pi}\rho} \cdot \sin(\theta/2)$ satisfies $\mathrm{MS}(u, B_\rho(0)) = \mathcal{H}^1(S_u \cap B_\rho(0)) = \rho$ for all $\rho > 0$ (its minimality will be discussed in subsection 2.5). Thus both the constants conjectured above would be sharp by [26, Section 62]. Unfortunately, none of them have been proven so far.

We point out that Bucur and Luckhaus [16], independently from [35], have been able to improve the ideas in Theorem 2.10 carrying on the proof without the 2-dimensional limitation via a delicate induction argument. Their approach leads to a remarkable monotonicity formula for (a truncated version of) the energy valid for a broad class of approximate minimizers that shall be the topic of the next subsection 2.4.

To conclude this paragraph we notice that the derivation of an energy upper bound for minimizers on balls is much easier as a result of a simple comparison argument.

Proposition 2.14. *For every $u \in \mathcal{M}(\Omega)$ and $B_r(z) \subseteq \Omega$*

$$\mathrm{MS}(u, B_r(z)) \leq n\,\omega_n\, r^{n-1}. \tag{2.22}$$

Proof. Let $\varepsilon > 0$, consider $u_\varepsilon := u\chi_{\Omega\setminus B_{r-\varepsilon}(z)}$ and test the minimality of u with u_ε. Conclude by letting $\varepsilon \downarrow 0$. □

2.4. **Bucur and Luckhaus' almost monotonicity formula.** Let us start off by introducing the notion of *almost-quasi minimizers* of the MS energy.

Definition 2.15. *Let $\Lambda \geq 1$, $\alpha > 0$ and $c_\alpha \geq 0$, a function $u \in SBV(\Omega)$ with $\mathrm{MS}(u) < \infty$ is a $(\Lambda, \alpha, c_\alpha)$-almost-quasi minimizer in Ω, we write $u \in \mathcal{M}_{\{\Lambda,\alpha,c_\alpha\}}(\Omega)$, if for all balls $B_\rho(z) \subset \Omega$*

$$\mathrm{MS}(u, B_\rho(z)) \leq \int_{B_\rho(z)} |\nabla v|^2\,dx + \Lambda\,\mathcal{H}^{n-1}\big(S_v \cap B_\rho(z)\big) + c_\alpha\,\rho^{n-1+\alpha},$$

whenever $\{u \neq v\} \subset\subset \Omega$.

Clearly, minimizers are $(1, \alpha, c_\alpha)$-almost minimizers for all α and c_α as in the definition above. Almost-quasi minimizers has turned out to be useful in studying the regularity properties of solutions to free boundary - free discontinuity problems with Robin conditions (cf. [16, Section 4]). We are now ready to state the Bucur and Luckhaus' almost monotonicity formula.

Theorem 2.16 (Bucur and Luckhaus [16])**.** *There is a (small) dimensional constant $C(n) > 0$ such that for all $u \in \mathcal{M}_{\{\Lambda,\alpha,c_\alpha\}}(\Omega)$ and $z \in \Omega$ the function*

$$\big(0, \mathrm{dist}(z, \partial\Omega)\big) \ni r \longmapsto E_z(r) := \Big(\frac{\mathrm{MS}(u, B_r(z))}{r^{n-1}} \wedge \frac{C(n)}{n-1}\Lambda^{2-n}\Big) + (n-1)\,\frac{c_\alpha}{\alpha}\,r^\alpha \tag{2.23}$$

is non decreasing.

To explain the strategy of proof of Theorem 2.16 we need to state two additional results on SBV functions defined on boundaries of balls. The first is related to the approximation with H^1 functions, the second to the problem of lifting. The tangential gradient on boundaries of balls shall be denoted by ∇_τ in what follows.

Lemma 2.17. *Let $n \geq 2$, then there is a (small) dimensional constant $C(n) > 0$ such that for all $v \in SBV(\partial B_r)$ with*

$$\mathcal{E}(v, \partial B_r) := \int_{\partial B_r} |\nabla_\tau v|^2 d\mathcal{H}^{n-1} + \mathcal{H}^{n-2}(S_v \cap \partial B_r) \leq C(n)\,\Lambda^{2-n}\,r^{n-2}, \tag{2.24}$$

there exists $w \in H^1(\partial B_r)$ such that

$$\mathcal{E}(w, \partial B_r) + (n-1)\,\frac{\Lambda}{r}\,\mathcal{H}^{n-1}\big(\{x \in \partial B_r : v \neq w\}\big) \leq \mathcal{E}(v, \partial B_r). \tag{2.25}$$

Lemma 2.18. *Let $n \geq 2$, suppose $v \in SBV(\partial B_r)$ satisfies (2.24) in Lemma 2.17. Then there exists $\widehat{w} \in H^1(B_r)$ harmonic in B_r such that*

$$\mathrm{MS}(\widehat{w}, B_r) + \Lambda\, \mathcal{H}^{n-1}\big(\{x \in \partial B_r : v \neq \widehat{w}\}\big) \leq \frac{r}{n-1}\, \mathcal{E}(v, \partial B_r). \tag{2.26}$$

The proof of Theorem 2.16 is based on a cyclic induction argument that starts with the validation of Lemma 2.17 in $\mathbb{R}^2$ and then runs as follows:

Lemma 2.17 in $\mathbb{R}^n \Rightarrow$Lemma 2.18 in $\mathbb{R}^n \Rightarrow$Theorem 2.16 in $\mathbb{R}^n \Rightarrow$Lemma 2.17 in $\mathbb{R}^{n+1}$.

The first inductive step can be established by using the geometric argument in the proof of Theorem 2.10 with the constant $C(2)$ being any number in $(0,1)$.

Before proving all the implications above we establish the essential closure of S_u for almost-quasi minimizers as a consequence of Theorem 2.16.

Theorem 2.19. *Let $u \in \mathcal{M}_{\{\Lambda,\alpha,c_\alpha\}}(\Omega)$, for some $\Lambda \geq 1$, $\alpha > 0$ and $c_\alpha \geq 0$. Then, $\mathcal{H}^{n-1}\big(\Omega \cap \overline{S_u} \setminus S_u\big) = 0$.*

Proof. Consider the subset $S_u^\star$ of points in S_u with density one, i.e. $x \in S_u$ such that

$$\lim_{\rho \downarrow 0} \frac{\mathcal{H}^{n-1}\big(S_u \cap B_\rho(x)\big)}{\omega_{n-1}\rho^{n-1}} = 1$$

(cf. (2.15) in Theorem 2.10), then clearly $\overline{S_u^\star} = \overline{S_u}$. Theorem 2.16 yields for $x \in S_u^\star$

$$E_x(\rho) \geq \lim_{\rho \downarrow 0} E_x(\rho) \geq \omega_{n-1} \wedge \frac{C(n)}{n-1}\Lambda^{2-n} \qquad \forall \rho \in \big(0, \mathrm{dist}(x, \partial\Omega)\big).$$

By approximation it is then easy to deduce the same estimate for all points $x \in \Omega \cap \overline{S_u}$. In turn, this implies that $\Omega \cap \overline{S_u} \setminus S_u \subset A_u := \{y \in \Omega \setminus S_u : \liminf_\rho E_x(\rho) > 0\}$.

In particular, the density estimates in Lemma 2.8 applied to the measure induced on Borel sets by $\mathrm{MS}(u, \cdot)$ gives that $\mathcal{H}^{n-1}(A_u) = 0$, and the conclusion $\mathcal{H}^{n-1}\big(\Omega \cap \overline{S_u} \setminus S_u\big) = 0$ follows at once (cf. the proof of Proposition 2.9). $\square$

Density lower bounds for $\mathrm{MS}(u, \cdot)$ as in Theorem 2.6 or for $\mathcal{H}^{n-1} \llcorner S_u$ as in Theorem 2.7 can also be recovered. Either by means of Theorem 2.16 and of a decay lemma due to De Giorgi, Carriero and Leaci or in an alternative direct way that provides explicit constants (cf. [7, Lemma 7.14] and [16, Section 3.3], respectively).

Let us now turn to the proofs of the implications among Theorem 2.16, Lemma 2.17 and Lemma 2.18. Recall that we have already commented on the validity of Lemma 2.17 in case $n = 2$.

Proof of Lemma 2.17 in $\mathbb{R}^n \Rightarrow$ Lemma 2.18 in $\mathbb{R}^n$. Denote by w the function provided by Lemma 2.17 and by $\widehat{w}$ its harmonic extension to B_ρ. Note that[4]

$$\frac{n-1}{\rho}\int_{B_\rho}|\nabla\widehat{w}|^2dx \le \int_{\partial B_\rho}|\nabla_\tau\widehat{w}|^2d\mathcal{H}^{n-1} \le \mathcal{E}(w,\partial B_\rho).$$

Hence, by (2.25) we conclude that

$$\begin{aligned}&\mathrm{MS}(\widehat{w},B_\rho)+\Lambda\,\mathcal{H}^{n-1}\big(\{x\in\partial B_\rho:\ v\neq\widehat{w}\}\big)\\&\quad\le\frac{\rho}{n-1}\Big(\mathcal{E}(w,\partial B_\rho)+(n-1)\frac{\Lambda}{\rho}\,\mathcal{H}^{n-1}\big(\{x\in\partial B_\rho:\ v\neq\widehat{w}\}\big)\Big)\le\frac{\rho}{n-1}\,\mathcal{E}(v,\partial B_\rho).\end{aligned}$$

□

Proof of Lemma 2.18 in $\mathbb{R}^n \Rightarrow$ Theorem 2.16 in $\mathbb{R}^n$. Set $I := \big(0,\mathrm{dist}(0,\partial\Omega)\big)$, then the energy function $m_z(\rho) := \mathrm{MS}(u,B_\rho(z)) : I \to [0,+\infty)$ is non-decreasing and thus it belongs to $BV_{loc}(I)$. Note that

$$E_z(\rho)=\Big(\frac{m_z(\rho)}{\rho^{n-1}}\wedge\frac{C(n)}{n-1}\,\Lambda^{2-n}\Big)+(n-1)\,\frac{c_\alpha}{\alpha}\rho^\alpha, \tag{2.27}$$

therefore, $E_z \in BV_{loc}(I)$. Denoting by m'_z and $D^s m_z$, respectively, the density of the absolutely continuous part and the singular part of Dm_z with respect to $\mathcal{L}^1 \llcorner I$ in the Radon-Nikodym decomposition, the Leibnitz rule for BV functions (cf. [7, Example 3.97]) yields

$$D\Big(\frac{m_z(\rho)}{\rho^{n-1}}\Big)=\Big(\frac{m'_z(\rho)}{\rho^{n-1}}-(n-1)\frac{m_z(\rho)}{\rho^n}\Big)+\frac{1}{\rho^{n-1}}D^s m_z(\rho).$$

Therefore, the latter equality, (2.27) and the Chain Rule formula for Lipschitz functions [7, Theorem 3.99] imply that E_z is non-decreasing if and only if the density E'_z of the absolutely continuous part of the distributional derivative DE_z is non-negative.

By the locality of the distributional derivative (see [7, Remark 3.93]) it holds that $E'_z(\rho)=(n-1)\,c_\alpha\rho^{\alpha-1}>0$ at $\mathcal{L}^1$ a.e. $\rho\in I$ for which

$$m_z(\rho)\ge\frac{C(n)}{n-1}\,\Lambda^{2-n}\rho^{n-1}.$$

Instead, at $\mathcal{L}^1$ a.e. $\rho\in I$ at which

$$m_z(\rho)<\frac{C(n)}{n-1}\,\Lambda^{2-n}\rho^{n-1} \tag{2.28}$$

we have

$$E'_z(\rho)=\frac{m'_z(\rho)}{\rho^{n-1}}-(n-1)\frac{m_z(\rho)}{\rho^n}+(n-1)\,c_\alpha\rho^{\alpha-1}.$$

[4]The first inequality follows, for instance, from Almgren's monotonicity formula for the frequency function (cf. [39, Exercise 20 pg. 525]) and a direct comparison argument with the one-homogeneous extension of the boundary trace. Alternatively, one can expand $\widehat{w}$ in spherical harmonics.

Suppose by contradiction that $E_z' < 0$ on some subset J of I of positive measure, i.e.

$$m_z'(\rho) < (n-1)\frac{m_z(\rho)}{\rho} - (n-1)\, c_\alpha \rho^{n-2+\alpha}. \tag{2.29}$$

Since for $\mathcal{L}^1$ a.e. in I by the slicing theory $u|_{\partial B_\rho} \in SBV(\partial B_\rho)$ (cf. [7, Section 3.11]) and by the Co-Area formula $\mathcal{E}(u, \partial B_\rho) \le m_z'(\rho)$ (cf. [7, Theorem 2.93]), we conclude that for $\mathcal{L}^1$ a.e. $\rho \in J$

$$\mathcal{E}(u, \partial B_\rho) \le m_z'(\rho) \overset{(2.28),(2.29)}{<} C(n)\,\Lambda^{2-n}\,\rho^{n-2} - (n-1)\, c_\alpha\, \rho^{n-2+\alpha}.$$

Hence, with fixed $\rho \in J$ as above, Lemma 2.18 provides an harmonic function $\widehat{w} \in H^1(B_\rho)$ satisfying (2.26). In turn, the latter inequality and the assumption $E_z'(\rho) < 0$ give

$$\begin{aligned} &\mathrm{MS}(\widehat{w}, B_\rho) + \Lambda\, \mathcal{H}^{n-1}(\{x \in \partial B_\rho : \, u \ne \widehat{w}\}) \\ &\qquad \overset{(2.26)}{\le} \frac{\rho}{n-1}\mathcal{E}(u, \partial B_\rho) \le \frac{\rho}{n-1} m_z'(\rho) \overset{(2.29)}{<} m_z(\rho) - c_\alpha \rho^{n-1+\alpha}, \end{aligned}$$

leading to a contradiction to the almost-quasi minimality of u in Ω by taking the trial function $\widehat{w}\, \chi_{B_\rho(z)} + u\, \chi_{\Omega \setminus B_\rho(z)}$. $\square$

To conclude the cyclic induction argument we set some notation: in the following proof we denote by B_ρ the $(n+1)$-dimensional ball of radius ρ and by B_ρ^n its intersection with the hyperplane $\mathbb{R}^n \times \{0\}$. Moreover, we still denote by $\mathcal{E}$ the n dimensional version of the boundary energy in (2.24).

Proof of Theorem 2.16 in $\mathbb{R}^n \Rightarrow$ *Lemma 2.17 in* $\mathbb{R}^{n+1}$. Up to a scaling argument we may assume the radius r in the statement of Lemma 2.17 to be 1.

Then, consider $v \in SBV(\partial B_1)$ such that

$$\mathcal{E}(v, \partial B_1) \le C\, \Lambda^{1-n} \tag{2.30}$$

for some constant $C > 0$.

Claim: There exists $C(n+1) > 0$ such that if v satisfies (2.30) with $C \in (0, C(n+1)]$, every minimizer $w \in SBV(\partial B_1)$ of the problem

$$\inf_{\zeta \in SBV(\partial B_1)} F(\zeta, \partial B_1)$$

actually belongs to $H^1(\partial B_1)$, where for all open sets $A \subseteq \partial B_1$ and $\zeta \in SBV(\partial B_1)$ if $\mathcal{E}(\zeta, A)$ is defined as in (2.24) by integrating ζ on A, then

$$F(\zeta, A) := \mathcal{E}(\zeta, A) + n\, \Lambda\, \mathcal{H}^n\big(\{x \in A : \, \zeta \ne v\}\big). \tag{2.31}$$

Given the claim above for granted we conclude the thesis of Lemma 2.17 straightforwardly by comparing the values of the energy $F(\cdot, \partial B_1)$ in (2.31) on w and v respectively, namely

$$\int_{\partial B_1} |\nabla_\tau w|^2 d\mathcal{H}^n + n\,\Lambda\,\mathcal{H}^n\big(\{x \in \partial B_1 : w \neq v\}\big) \leq \int_{\partial B_1} |\nabla_\tau v|^2 d\mathcal{H}^n + \mathcal{H}^{n-1}(S_v).$$

We are then left with proving the claim above. Suppose by contradiction that for some constant $C > 0$ some minimizer w of (2.31) satisfies $\mathcal{H}^{n-1}(S_w) > 0$. Note that C can be taken arbitrarily small, it shall be chosen suitably in what follows. Even more, assume that the north pole $\mathscr{N} := (0, \ldots, 0, 1) \in \partial B_1$ is a point of density one for S_w. Set $\lambda := 2 - \sqrt{3}$ and denote by $\pi : \partial B_1 \cap B_\lambda(\mathscr{N}) \to B^n_{1/2}$ the orthogonal projection, then $\pi \in \mathrm{Lip}_1(\partial B_1 \cap B_\lambda(\mathscr{N}), B^n_{1/2})$ and $\pi^{-1} \in \mathrm{Lip}_\ell(B^n_{1/2}, \partial B_1 \cap B_\lambda(\mathscr{N}))$, for some $\ell > 0$. Actually, as $\pi \in C^1(\partial B_1 \cap B_\lambda(\mathscr{N}), B^n_{1/2})$ and $\pi^{-1} \in C^1(B^n_{1/2}, \partial B_1 \cap B_\lambda(\mathscr{N}))$

$$\mathrm{Lip}\big(\pi, \pi^{-1}(B^n_\rho)\big) \to 1, \quad \mathrm{Lip}\big(\pi^{-1}, B^n_\rho\big) \to 1 \qquad \text{as } \rho \downarrow 0. \tag{2.32}$$

For $\zeta \in SBV(\partial B_1 \cap B_\lambda(\mathscr{N}))$ let $\overline{\zeta} := \zeta \circ \pi^{-1}$, then $\overline{\zeta} \in SBV(B^n_{1/2})$ and $S_{\overline{\zeta}} = \pi(S_\zeta)$. Moreover, for all $\rho \in (0, 1/2)$, we have

$$\mathcal{H}^{n-1}\big(S_{\overline{\zeta}} \cap B^n_\rho\big) \leq \mathcal{H}^{n-1}\big(S_\zeta \cap \pi^{-1}(B^n_\rho)\big) \leq \ell^{n-1}\,\mathcal{H}^{n-1}(S_{\overline{\zeta}} \cap B^n_\rho), \tag{2.33}$$

and by the generalized Area Formula (see [7, Theorem 2.91])

$$\int_{B^n_\rho} |\nabla \overline{\zeta}|^2 dy \leq k_1(\rho) \int_{\pi^{-1}(B^n_\rho)} |\nabla_\tau \zeta|^2 d\mathcal{H}^n, \tag{2.34}$$

and

$$\int_{\pi^{-1}(B^n_\rho)} |\nabla_\tau \zeta|^2 d\mathcal{H}^n \leq k_2(\rho) \int_{B^n_\rho} |\nabla \overline{\zeta}|^2 dy, \tag{2.35}$$

for some $k_1(\rho)$ and $k_2(\rho) > 0$, with $k_1(\rho)$, $k_2(\rho) \downarrow 1$ as $\rho \downarrow 0$ by (2.32). In particular, for some $\tau_n > 0$, $0 \leq k_1(\rho) \vee k_2(\rho) \leq 1 + \tau_n \rho$ as $\rho \downarrow 0$.

We next prove that $\overline{w}$ is a $(\Lambda_1, 1, c_1)$-almost-quasi minimizer on $B^n_{1/2}$ of the n-dimensional Mumford and Shah energy for $\Lambda_1 := \ell^{n-1}$ and a suitable $c_1 = c_1(n, \ell, \Lambda) > 0$. Indeed, recalling that $\Lambda \geq 1$, if $\overline{\zeta}$ is a test function for $\overline{w}$, i.e. $\{y \in B^n_{1/2} : \overline{\zeta} \neq \overline{w}\} \subset\subset B^n_\rho$, $\rho \in (0, 1/2)$, we deduce from (2.33), (2.34), (2.35) and the minimality of w for the energy in (2.31) that

$$
\begin{aligned}
\mathrm{MS}(\overline{w}, B^n_\rho) &\le \int_{B^n_\rho} |\nabla \overline{w}|^2 dx + \mathcal{H}^{n-1}(S_{\overline{w}} \cap B^n_\rho) + \mathcal{H}^n(\{y \in B^n_\rho : \overline{w} \neq \overline{v}\}) \\
&\le k_1(\rho) \int_{\pi^{-1}(B^n_\rho)} |\nabla_\tau w|^2 d\mathcal{H}^n + \mathcal{H}^{n-1}\big(S_w \cap \pi^{-1}(B^n_\rho)\big) + \mathcal{H}^n\big(\{x \in \pi^{-1}(B^n_\rho) : w \neq v\}\big) \\
&\le k_1(\rho) F(w, \pi^{-1}(B^n_\rho)) \le k_1(\rho) F(\zeta, \pi^{-1}(B^n_\rho)) \qquad (2.36) \\
&\le k_1(\rho)\Big(k_2(\rho) \int_{B^n_\rho} |\nabla \overline{\zeta}|^2 dx + \Lambda_1 \mathcal{H}^{n-1}(S_{\overline{\zeta}} \cap B^n_\rho) + n\,\Lambda\,\Lambda_1^{\frac{n}{n-1}} \mathcal{H}^n(\{y \in B^n_\rho : \overline{\zeta} \neq \overline{v}\})\Big) \\
&\le (1+\tau_n\,\rho)^2 \Big(\int_{B^n_\rho} |\nabla \overline{\zeta}|^2 dx + \Lambda_1 \mathcal{H}^{n-1}(S_{\overline{\zeta}} \cap B^n_\rho)\Big) + (1+\tau_n) n\omega_n\,\Lambda\,\Lambda_1^{\frac{n}{n-1}} \rho^n, \qquad (2.37)
\end{aligned}
$$

being $\mathcal{H}^n(\{y \in B^n_\rho : \overline{\zeta} \neq \overline{v}\}) \le \omega_n \rho^n$.

Next we note that w satisfies the energy upper bound $F(w, \pi^{-1}(B^n_\rho)) \le k_n \rho^{n-1}$, for all $\rho \in (0, 1/2)$ and for some $k_n > 0$, by a direct comparison argument similar to that in Proposition 2.14. Therefore, (2.36) yields

$$
\mathrm{MS}(\overline{w}, B^n_\rho) \le (1+\tau_n) k_n \rho^{n-1}.
$$

Hence, we may assume $F(\zeta, \pi^{-1}(B^n_\rho)) \le 2(1+\tau_n) k_n \rho^{n-1}$ being otherwise the conclusion obvious. The latter condition and (2.37) then imply $\overline{w} \in \mathcal{M}_{\{\Lambda_1, 1, c_1\}}\big(B^n_{1/2}\big)$ with $c_1 := (1+\tau_n)\big(n\,\omega_n\,\Lambda\,\Lambda_1^{\frac{n}{n-1}} + 2(1+\tau_n)k_n\big)$.

By inductive assumption, Theorem 2.16 implies in particular that

$$
\begin{aligned}
&\Big(\frac{\mathrm{MS}(\overline{w}, B^n_\rho)}{\rho^{n-1}} \wedge \frac{C(n)}{n-1} \Lambda_1^{2-n}\Big) + (n-1) c_1 \rho \\
&\quad \ge \lim_{\rho \downarrow 0} \Big(\Big(\frac{\mathrm{MS}(\overline{w}, B^n_\rho)}{\rho^{n-1}} \wedge \frac{C(n)}{n-1} \Lambda_1^{2-n} \Big) + (n-1) c_1 \rho \Big) \ge \omega_{n-1} \wedge \frac{C(n)}{n-1} \Lambda_1^{2-n} =: \beta_n. \qquad (2.38)
\end{aligned}
$$

In the last inequality we have used that $\mathscr{N}$ is a point of density one for S_w and (2.32).

Given any $\rho \in (0, \frac{\beta_n}{2(n-1)c_1} \wedge 1/2)$, inequality (2.38) yields

$$
\frac{\mathrm{MS}(\overline{w}, B^n_\rho)}{\rho^{n-1}} \ge \frac{\beta_n}{2}.
$$

On the other hand, repeating the argument leading to the first and the second inequality in (2.36) imply for any $\rho \in (0, \frac{\beta_n}{2(n-1)c_1} \wedge 1/2)$

$$
\frac{\mathrm{MS}(\overline{w}, B^n_\rho)}{\rho^{n-1}} \le (1+\tau_n \rho) \frac{\mathcal{E}(v, \partial B_1)}{\rho^{n-1}} \overset{(2.30)}{\le} C\,(1+\tau_n\rho)\,(\rho\,\Lambda)^{1-n}.
$$

Thus, with fixed $\overline{\rho} \in (0, \frac{\beta_n}{2(n-1)c_1} \wedge 1/2)$, we infer a contradiction from the last two estimates by choosing the constant $C = C(\overline{\rho}) > 0$ in (2.30) so that $C\,(1+\tau_n)\,(\overline{\rho}\,\Lambda)^{1-n} < \beta_n/2$. $\square$

2.5. **The Mumford-Shah Conjecture.** Having established the existence of strong (local) minimizers, it is elementary to infer that u is harmonic on $\Omega \setminus \overline{S_u}$. Hence, we will focus in the rest of the note on the regularity properties of the set $\Omega \cap \overline{S_u}$ that will be instrumental also to gain further regularity on u itself (cf. Theorem 3.3).

The interest of the researchers in this problem is motivated by the Mumford and Shah conjecture (in 2-dimensions) that we recall below for the readers' convenience.

Conjecture 2.20 (Mumford and Shah [69])**.** *If $u \in \mathcal{M}(\Omega)$, $\Omega \subseteq \mathbb{R}^2$, then $\Omega \cap \overline{S_u}$ is the union of (at most) countably many injective C^1 arcs $\gamma_i : [a_i, b_i] \to \Omega$ with the following properties:*

(c1) *Any compact $K \subset \Omega$ intersects at most finitely many arcs;*
(c2) *Two arcs can have at most an endpoint p in common, and if this is the case, then p is in fact the endpoint of three arcs, forming equal angles of ${}^{2\pi}/_3$.*

So according to this conjecture only two possible singular configurations occur: either three arcs meet in an end forming angles equal to ${}^{2\pi}/_3$, or an arc has a free end in Ω. In what follows, we shall call *triple junction* the first configuration and *crack-tip* the second.

A suitable theory of calibrations for free discontinuity problems established by Alberti, Bouchitté and Dal Maso [1] shows that the model case of triple junction functions,

$$a\,\chi_{\{\vartheta\in(-\pi/6,\pi/2]\}} + b\,\chi_{\{\vartheta\in(\pi/2,7\pi/6]\}} + c\,\chi_{\{\vartheta\in(7\pi/6,11\pi/6]\}} \tag{2.39}$$

with $|a-b|\cdot|a-c|\cdot|b-c| > 0$, is indeed a local minimizer (for more results on calibrations in the setting of free discontinuity problems see [22, 65, 66, 67, 68]).

Instead, Bonnet and David [11] have shown that the model crack-tip functions, i.e. functions that up to rigid motions can be written as

$$C \pm \sqrt{\frac{2}{\pi}\rho}\cdot\sin(\theta/2) \tag{2.40}$$

for $\theta \in (-\pi, \pi)$, $\rho > 0$ and some constant $C \in \mathbb{R}$, are *global* minimizers of the Mumford and Shah energy[5].

More recently, second order sufficient conditions for minimality have been investigated. More precisely, Bonacini and Morini in [9] for suitable critical points, strictly stable and regular in their terminology, have proved that strict local minimality is implied by strict positivity of the second variation. This approach in the case of triple junctions is currently under investigation [19].

---

[5] The prefactor $\sqrt{\frac{2}{\pi}}$ results from a simple calculation to ensure stationarity for a crack-tip function by plugging it in the Euler-Lagrange equation (2.6).

Conjecture 2.20 has been first proven in some particular cases in the ensuing weaker form.

Conjecture 2.21. *If $u \in \mathcal{M}(\Omega)$, $\Omega \subseteq \mathbb{R}^2$, then $\Omega \cap \overline{S_u}$ is the union of (at most) countably many injective C^0 arcs $\gamma_i : [a_i, b_i] \to \Omega$ which are C^1 on $]a_i, b_i[$ and satisfy the two conditions of conjecture 2.20.*

The subtle difference between conjecture 2.20 and conjecture 2.21 above is in the following point: assuming conjecture 2.21 holds, if $y_0 = \gamma_i(a_i)$ is a "loose end" of the arc γ_i, i.e. it does not belong to any other arc, then the techniques in [10] show that any *blow up limit*, i.e. any limit of subsequences of the family $(u_{y_0,\rho})_\rho$ as in (2.8), is a crack-tip but do not ensure the uniqueness of the limit itself (for more details on the notion of blow up see the proof of Proposition 4.11).

2.6. **Blow up analysis and the Mumford and Shah conjecture.** The Mumford and Shah conjecture, in the form stated in conjecture 2.21, has been attacked first in the contribution by Bonnet [10]. Bonnet's approach is based on a weaker notion of minimality for the strong formulation of the problem, that includes a topological condition to be satisfied by the competitors, though still sufficient to develop a regularity theory.

Definition 2.22. *Let $\Omega \subseteq \mathbb{R}^2$ be an open set, a pair (u, K), $K \subset \Omega$ closed and $u \in W^{1,2}_{\mathrm{loc}}\big(\Omega \setminus K\big)$, is a Bonnet minimizer of $\mathscr{F}$ in Ω if $\mathscr{F}(u, K, \Omega) \leq \mathscr{F}(v, L, \Omega)$ among all couples (v, L), $L \subset \Omega$ closed and $v \in W^{1,2}_{\mathrm{loc}}\big(\Omega \setminus L\big)$, such that there is a ball $B_\rho(x) \subset \Omega$ for which*

- (i) *$u = v$ and $K = L$ on $\Omega \setminus \overline{B_\rho(x)}$,*
- (ii) *any pair of points in $\Omega \setminus \big(K \cup \overline{B_\rho(x)}\big)$ that lie in different connected components of $\Omega \setminus K$ are also in different connected components of $\Omega \setminus L$.*

Moreover, in case $\Omega = \mathbb{R}^2$, we say that (u, K) is a global minimizer of $\mathscr{F}$.

In particular, under the assumption that $\Omega \cap \overline{S_u}$ has a finite number of connected components Bonnet has classified all the blow up limits of minimizers as in Definition 2.22, and then also of local minimizers, as

- (i) constant functions,
- (ii) pure jumps (cf. (2.10)),
- (iii) triple junction functions (cf. (2.39)),
- (iv) crack-tip functions (cf. (2.40)),

establishing conjecture 2.21 in such a restricted framework. In particular, as already mentioned, Bonnet's result does not deal with the stronger conjecture 2.20 as it cannot exclude the possibility that γ_i "spirals" around y_0 infinitely many times (compare with the discussion at the end of [10, Section 1]). The method of Bonnet relies on a key monotonicity formula for the rescaled Dirichlet energy that so far has no counterpart in general. Recently, Lemenant [52] has exhibited another monotone quantity that plays the role of the rescaled Dirichlet energy in higher dimensions in case $\Omega \cap \overline{S_u}$ is contained in a sufficiently smooth cone. In view of this, a rigidity result for $\Omega \cap \overline{S_u}$ in the 3-dimensional case can be deduced. Let us also point out that the almost monotonicity formula established by Bucur and Luckhaus in Theorem 2.16 is of little use in the blow up analysis due to the truncation with the constant $C(n)$.

The contributions of Léger [50] and of David and Léger [27] improve upon Bonnet's results: the former addressing the case of $\Omega \cap \overline{S_u}$ satisfying a suitable flatness assumption, the latter identifying pure jumps and triple junction functions as the only minimizers in the sense of Bonnet for which $\mathbb{R}^2 \setminus \overline{S_u}$ is not connected. All these efforts are directed to push forward Bonnet's ideas. Indeed, [26, Proposition 71.2] shows in general, i.e. with no extra connectedness assumptions on $\Omega \cap \overline{S_u}$, that the complete classification of the blow up limits of local minimizers as in the list above turns out to be a viable strategy to establish conjecture 2.20. More precisely, coupling the latter piece of information with a detailed local description of the geometry of $\Omega \cap \overline{S_u}$, that is the topic of the ensuing subsection 3.1, would yield the conclusion. A further interesting consequence of the analysis in the paper [27] is that the Mumford and Shah conjecture turns out to be equivalent to the uniqueness (up to rotations and translations) of crack-tips as global minimizers of the MS energy as conjectured by De Giorgi in [30].

However, we shall not enter here into this streamline of results but rather refer for more details on them to the monograph [26] and to the recent review paper [53].

Instead, a different (but related) perspective to the goal of understanding conjecture 2.20 is taken in what follows. We shall link the latter to a sharp higher integrability property of the approximate gradient following Ambrosio, Fusco and Hutchinson [4]. In order to do this, in the next section we review the state of the art about the regularity properties of $\Omega \cap \overline{S_u}$.

3. Regularity of the jump set

The aim of this section is to survey on the regularity of $\Omega \cap \overline{S_u}$. In subsection 3.1 we shall first recall classical and more recent ε-regularity results, and then state an estimate on the size of the subset of singular points in $\Omega \cap \overline{S_u}$ that will be dealt with in details in Section 4. In particular, in subsection 3.2 the links of the higher integrability of the gradient with the Mumford and Shah conjecture 2.20 will be highlighted following the approach of

Ambrosio, Fusco and Hutchinson [4]. A slight improvement of the latter ideas leads to an energetic characterization of the conjecture 2.21 as exposed in subsection 3.3.

3.1. **ε-regularity theorems.** The starting point to address the regularity of $\Omega\cap\overline{S_u}$ for local minimizers is the ensuing ε-regularity result.

Theorem 3.1 (Ambrosio, Fusco and Pallara [5])**.** *Let $u\in\mathcal{M}(\Omega)$, then there exists $\Sigma_u\subset\Omega\cap\overline{S_u}$ relatively closed in Ω with $\mathcal{H}^{n-1}(\Sigma_u)=0$, and such that $\Omega\cap\overline{S_u}\setminus\Sigma_u$ is locally a $C^{1,\gamma}$ hypersurface for all $\gamma\in(0,1)$ and $C^{1,1}$ if $n=2$.*

More precisely, there exist $\varepsilon_0=\varepsilon_0(n)$, $\rho_0=\rho_0(n)>0$ such that

$$\Sigma_u=\{z\in\Omega\cap\overline{S_u}:\ \mathscr{D}(z,\rho)+\mathscr{A}(z,\rho)\geq\varepsilon_0\quad\forall\rho\in(0,\rho_0\wedge\operatorname{dist}(z,\partial\Omega))\}\tag{3.1}$$

where

$$\mathscr{D}_u(z,\rho):=\rho^{1-n}\int_{B_\rho(z)}|\nabla u|^2dy,\quad\textit{(scaled Dirichlet energy)}$$

$$\mathscr{A}_u(z,\rho):=\rho^{-n-1}\min_{T\in\Pi}\int_{S_u\cap B_\rho(z)}\operatorname{dist}^2(y,T)d\mathcal{H}^{n-1}(y),\quad\textit{(scaled mean flatness)},$$

with Π the class of $(n-1)$-affine planes.

For more details on Theorem 3.1 we refer to [7, Chapter 8], that is entirely devoted to the proof of it, and to [44] for a hint of the strategy of proof. Here we shall only comment on the quantities involved in (3.1).

First note that the affine change of variables mapping $B_\rho(x)$ into B_1 shows that $\mathscr{D}_u(x,\cdot)$ and $\mathscr{A}_u(x,\cdot)$ are equal to the Dirichlet energy and the mean flatness on B_1 of the rescaled maps $u_{x,\rho}$ in (2.8), respectively.

Further, the scaled mean flatness measures in an average sense the deviation of $\Omega\cap\overline{S_u}$ from being flat in z. For instance, if $\Omega\cap\overline{S_u}$ is a C^1 hypersurface in a neighborhood of z it is easy to check that $\mathscr{A}_u(z,\rho)=o(1)$ as $\rho\downarrow 0$. Actually , the density lower bound in Theorem 2.7 and the density upper bound in Proposition 2.14 allow us to show that the rescaled mean flatness $\mathscr{A}_u$ is equivalent to its L^∞ version, namely

$$\mathscr{A}_{u,\infty}(z,\rho):=\rho^{-1}\min_{T\in\Pi}\sup_{y\in S_u\cap B_\rho(z)}\operatorname{dist}(y,T).$$

Proposition 3.2. *Let $u\in\mathcal{M}(\Omega)$, then for all $z\in\overline{S_u}$ and $B_\rho(z)\subset\Omega$ we have*

$$\frac{\theta_0}{2^{n+1}}\mathscr{A}^{n+1}_{u,\infty}(z,{}^{\rho}/_{2})\leq\mathscr{A}_u(z,\rho)\leq n\,\omega_n\,\mathscr{A}^2_{u,\infty}(z,\rho).$$

where θ_0 is the constant in Theorem 2.7.

Proof. The estimate from above easily follows by the very definitions of $\mathscr{A}_u$ and $\mathscr{A}_{u,\infty}$ by taking into account (2.22).

Let then $\bar{T}$ be the affine hyperplane through z giving the minimum in the definition of $\mathscr{A}_u(z,\rho)$. If $\bar{y} \in S_u \cap B_{\rho/2}(z)$ is a point of almost maximum distance from $\bar{T}$, i.e. for a fixed $\delta \in (0,1)$

$$d := \mathrm{dist}(\bar{y},\bar{T}) \geq (1-\delta) \sup_{y \in S_u \cap B_{\rho/2}(z)} \mathrm{dist}(y,\bar{T}),$$

then we can estimate as follows thanks to (2.12)

$$\mathscr{A}_u(z,\rho) \geq \rho^{-n-1} \int_{S_u \cap B_{d/2}(\bar{y})} \mathrm{dist}^2(y,\bar{T}) d\mathcal{H}^{n-1}(y) \geq \rho^{-n-1} \frac{d^2}{4} \mathcal{H}^{n-1}\big(S_u \cap B_{d/2}(\bar{y})\big)$$
$$\geq \rho^{-n-1}\, \theta_0 \left(\frac{d}{2}\right)^{n+1} \geq (1-\delta)^{n+1} \frac{\theta_0}{2^{n+1}} \mathscr{A}_{u,\infty}^{n+1}(z,\rho/2).$$

The conclusion follows at once as $\delta \downarrow 0$. $\square$

The local graph property of $\Omega \cap \overline{S_u}$ established in Theorem 3.1, the Euler-Lagrange condition and the regularity theory for elliptic PDEs with Neumann boundary conditions determine the regularity of u close to $\Omega \cap \overline{S_u}$ (see [7, Theorem 7.49] or [26, Proposition 17.15] if $n = 2$).

Theorem 3.3 (Ambrosio, Fusco and Pallara [6])**.** *Let $u \in \mathcal{M}(\Omega)$ and $A \cap \overline{S_u}$, $A \subset \Omega$ open, be the graph of a $C^{1,\gamma}$ function ϕ, $\gamma \in (0,1)$. Then, $\phi \in C^\infty$ and u has C^∞ extension on each side of $A \cap \overline{S_u}$.*

Actually, Koch, Leoni and Morini [46] proved that if $A \cap \overline{S_u}$ is $C^{1,\gamma}$ then it is actually analytic, as conjectured by De Giorgi (cf. [29, 31]).

Going back to the regularity issue for $\overline{S_u}$ we resume below the outcomes of a different approach developed by David in the 2-dimensional case. In this setting it is also possible to address the situation in which $\Omega \cap \overline{S_u}$ is close in the Hausdorff distance to a triple-junction (cf. Section 6 for more comments in this respect).

Theorem 3.4 (David [25], Corollary 51.17 and Theorem 53.4 [26])**.** *There exists $\varepsilon > 0$ and an absolute constant $c \in (0,1)$ with the following properties. If $u \in \mathcal{M}(\Omega)$, $z \in \overline{S_u}$, $B_r(z) \subset \Omega$ and $\mathscr{C}$ is either a line or a triple junction such that*

$$\int_{B_r(z)} |\nabla u|^2\, dx + \mathrm{dist}_{\mathcal{H}}(\overline{S_u} \cap B_r(z), \mathscr{C} \cap B_r(z)) \leq \varepsilon\, r, \tag{3.2}$$

then there exists a C^1-diffeomorphism ϕ of $B_r(z)$ onto its image with

$$\overline{S_u} \cap B_{cr}(z) = \phi\,(\mathscr{C}) \cap B_{cr}(z).$$

In addition, for any given $\delta \in (0, 1/2)$, there is $\varepsilon > 0$ such that, if (3.2) holds, then $\overline{S_u} \cap (B_{(1-\delta)r}(z) \setminus B_{\delta r}(z))$ is δ-close, in the C^1 norm, to $\mathscr{C} \cap (B_{(1-\delta)r}(z) \setminus B_{\delta r}(z))$.

Remark 3.5. *The last sentence of Theorem 3.4 is not contained in* [26, Corollary 51.17, Theorem 53.4]. *However it is a simple consequence of the theory developed in there. By scaling, we can assume $r = 1$ and $x = 0$. Fix a cone $\mathscr{C}$, a $\delta > 0$ and a sequence $\{u_k\} \subset \mathcal{M}(B_1)$ for which the left hand side of (3.2) goes to 0. If $\mathscr{C}$ is a segment, then it follows from [26] (or [7]) that there are uniform $C^{1,\alpha}$ bounds on $\overline{S_{u_k}} \cap B_{1-\delta}$. We can then use the Ascoli-Arzelà Theorem to conclude that $\overline{S_{u_k}}$ is converging in C^1 to $\mathscr{C}$.*

In case the minimal cone $\mathscr{C}$ is a triple junction, then observe that $\mathscr{C} \cap (B_1 \setminus B_{\delta/2})$ consists of a three distinct segments at distance $\delta/2$ from each other. Covering each of these segments with balls of radius comparable to δ and centered in a point belonging to the segment itself, we can argue as above and conclude that, for k large enough, $\overline{S_{u_k}} \cap (B_{1-\delta} \setminus B_\delta)$ consist of three arcs, with uniform $C^{1,\alpha}$ estimates. Once again the Ascoli-Arzelà Theorem shows that $\overline{S_{u_k}} \cap (B_{1-\delta} \setminus B_\delta)$ is converging in C^1 to $\mathscr{C} \cap (B_{1-\delta} \setminus B_\delta)$.

Remark 3.6. *Actually assumption (3.2) can be relaxed to*

$$\int_{B_r(x)} |\nabla u|^2 \, dx \le \varepsilon \, r$$

(see [26, Proposition 60.1] *or Theorem 4.3 below).*

Remarkably, Lemenant [51] extended such a result to the 3-dimensional case with suitable changes in the statement (see also [53] for a sketch of the proof).

The techniques developed by David are also capable to describe in details the structure of $\overline{S_u}$ around points that corresponds to the model case of crack-tips despite uniqueness of blow ups is not ensured.

Theorem 3.7 (David, Theorem 69.29 [26])**.** *For all $\varepsilon_0 > 0$ there exists $\varepsilon > 0$ such that if $u \in \mathcal{M}(\Omega)$ and*

$$\mathrm{dist}_{\mathcal{H}}(\overline{S_u} \cap B_r(z), \sigma) < \varepsilon \, r$$

for some radius σ of $B_r(z) \subset \Omega$, then $\overline{S_u} \cap B_{r/2}(z)$ consists of a single connected arc which joins some point $y_0 \in B_{r/4}(z)$ with $\partial B_{r/2}(z)$ and which is smooth in $B_{r/2}(z) \setminus \{y_0\}$.

More precisely, there is a point $y_0 \in B_{r/4}(z)$ such that

$$\overline{S_u} \cap B_{r/2}(z) = \big\{ y_0 + \rho(\cos\alpha(\rho), \sin\alpha(\rho)) \big\}$$

for some smooth function $\alpha : (0, r/2) \to \mathbb{R}$ which satisfies

$$\rho|\alpha'(\rho)| \le \varepsilon_0 \quad \text{for all } \rho \in (0, r/2), \qquad \lim_{\rho \downarrow 0} \rho|\alpha'(\rho)| = 0. \tag{3.3}$$

In addition, there is a constant C such that, up to a change of sign,

$$u\big(y_0+\rho(\cos\theta,\sin\theta)\big)=\sqrt{\frac{2}{\pi}\rho}\,\sin\Big(\frac{\theta-\alpha(\rho)}{2}\Big)+C+\rho^{1/2}\omega(\rho,\theta)$$

for all $\rho\in(0,r/2)$ and $\theta\in(\alpha(\rho)-\pi,\alpha(\rho)+\pi)$, with $\lim_{\rho\downarrow 0}\sup_\theta|\omega(\rho,\theta)|=0$. Finally,

$$\lim_{\rho\downarrow 0}\frac{1}{\rho}\int_{B_\rho(y_0)}|\nabla u|^2dx=\lim_{\rho\downarrow 0}\frac{1}{\rho}\mathcal{H}^1(\overline{S_u}\cap B_\rho(y_0))=1.$$

As remarked above, the latter theorem does not guarantee that such arc is C^1 *up to the loose end* y_0: in particular it leaves the possibility that the arc spirals infinitely many times around it. Hence, it does not establish the uniqueness of the blowup in the point y_0.

3.2. **Higher integrability of the gradient and the Mumford and Shah conjecture.** Theorem 3.1, or better the characterization of the *singular set* Σ_u in (3.1), can be employed to subdivide Σ_u as follows: $\Sigma_u=\Sigma_u^{(1)}\cup\Sigma_u^{(2)}\cup\Sigma_u^{(3)}$, where

$$\Sigma_u^{(1)}:=\{x\in\Sigma_u:\ \lim_{\rho\downarrow 0}\mathscr{D}_u(x,\rho)=0\},$$

$$\Sigma_u^{(2)}:=\{x\in\Sigma_u:\ \lim_{\rho\downarrow 0}\mathscr{A}_u(x,\rho)=0\},$$

$$\Sigma_u^{(3)}:=\{x\in\Sigma_u:\ \liminf_{\rho\downarrow 0}\mathscr{D}_u(x,\rho)>0,\ \liminf_{\rho\downarrow 0}\mathscr{A}_u(x,\rho)>0\}.$$

According to the Mumford and Shah conjecture 2.20 we should have $\Sigma_u^{(3)}=\emptyset$ if $n=2$. Furthermore, being inspired by the 2d case, we shall refer to $\Sigma_u^{(1)}$ as the set of triple junctions, and to $\Sigma_u^{(2)}$ as the set of crack-tips. Actually, in 2-dimensions we will fully justify the latter terminology in Proposition 4.11 (see also Remark 4.10).

In the general n-dimensional setting De Giorgi conjectured that $\mathcal{H}^{n-2}(\Sigma_u\cap\Omega')<\infty$ for all $\Omega'\subset\subset\Omega$ (cf. [31, conjecture 6]). In particular, the validity of the latter conjecture would imply $\dim_{\mathcal{H}}\Sigma_u\leq n-2$.

A first breakthrough in this direction has been obtained by Ambrosio, Fusco and Hutchinson in [4].

Theorem 3.8 (Ambrosio, Fusco and Hutchinson, [4])**.** *For every $u\in\mathcal{M}(\Omega)$*

$$\dim_{\mathcal{H}}\Sigma_u^{(1)}\leq n-2.$$

Actually, the set $\Sigma_u^{(1)}$ turns out to be countable in 2-dimensions (see Remark 4.10 below).

In the same paper [4], Ambrosio, Fusco and Hutchinson investigated the connection between the higher integrability of ∇u and the Mumford and Shah conjecture.

If conjecture 2.20 does hold, then $\nabla u \in L^p_{loc}$ for all $p < 4$ (cf. with [4, Proposition 6.3] under $C^{1,1}$ regularity assumptions on $\overline{S_u}$, see also Theorem 3.11 below). It was indeed conjectured by De Giorgi in all space dimensions that $\nabla u \in L^p_{loc}$ for all $p < 4$ (cf. with [31, conjecture 1]). So far only a first step into this direction has been established.

Theorem 3.9. *There is $p > 2$ such that $\nabla u \in L^p_{\mathrm{loc}}(\Omega)$ for all $u \in \mathcal{M}(\Omega)$ and for all open sets $\Omega \subseteq \mathbb{R}^n$.*

A proof of Theorem 3.9 in 2-dimension has been given by De Lellis and Focardi in [35], shortly after De Philippis and Figalli established the result without any dimensional limitation in [37]. The exponent p in both papers is not explicitly computed despite some suggestions to do that are also proposed.

The higher integrability can be translated into an estimate for the size of the singular set Σ_u of $\overline{S_u}$ (see [4, Corollary 5.7]) that improves upon the conclusion of Theorem 3.1 (cf. [61], [62], [47] and the survey [63] for related issues for minima of variational integrals and for solutions to nonlinear elliptic systems in divergence form).

Theorem 3.10 (Ambrosio, Fusco and Hutchinson [4])**.** *If $u \in \mathcal{M}(\Omega)$ and $|\nabla u| \in L^p_{loc}(\Omega)$ for some $p > 2$, then*

$$\dim_{\mathcal{H}}\Sigma_u \leq \max\{n-2, n-{}^{p}\!/\!{}_{2}\} \in (0, n-1). \tag{3.4}$$

The estimate $\dim_{\mathcal{H}}\Sigma_u < n-1$ has also been established by David [25] for $n = 2$, and lately by Rigot [71] and by Maddalena and Solimini [55] in general by establishing the porosity of $\Omega \cap \overline{S_u}$. Despite this, it was not related to the higher integrability property of the gradient. In Section 6 below we shall comment more in details on how porosity implies such an estimate and moreover on how porosity can be employed to prove the higher integrability of the gradients of minimizers following De Philippis and Figalli [37] (see Section 7).

For the time being few remarks are in order:

(i) the upper bound $p < 4$ is motivated not only because we need the rhs in the estimate (3.4) to be positive if $n = 2$, but also because explicit examples show that it is the best exponent one can hope for: consider in 2 dimensions a crack-tip minimizer (Bonnet and David [11]), i.e. a function that up to a rigid motion can be written as

$$u(\rho, \theta) = C \pm \sqrt{\frac{2}{\pi}\rho} \cdot \sin(\theta/2)$$

for $\theta \in (-\pi, \pi)$ and $\rho > 0$, and some constant $C \in \mathbb{R}$. Simple calculations imply that crack-tip minimizers satisfy

$$|\nabla u| \in L^p_{loc} \setminus L^4_{loc}(\mathbb{R}^2) \quad \text{for all } p < 4;$$

(ii) If we were able to prove the higher integrability property for every $p < 4$ then we would infer that $\dim_{\mathcal{H}}\Sigma_u \leq n-2$, and actually in 2-dimensions $\dim_{\mathcal{H}}\Sigma_u = 0$. Clearly, this would be a big step towards the solution in positive of the Mumford and Shah conjecture. For further progress in this direction see Theorem 3.11 below.

Given Theorems 3.8 and 3.9 for granted, Theorem 3.10 is a simple consequence of soft measure theoretic arguments. We shall establish Theorem 3.8 in Section 4. Instead, here we prove Theorem 3.10 to underline the role of the higher integrability that, in turn, shall be established in Sections 5 and 7 following two different paths.

Proof of Theorem 3.10. Suppose that $|\nabla u| \in L^p_{loc}(\Omega)$ for some $p > 2$, then for all $s \in (n - p/2, n-1)$ the set

$$\Lambda_s := \left\{ x \in \Omega : \limsup_{\rho} \rho^{-s} \int_{B_\rho(x)} |\nabla u|^p dy = \infty \right\}$$

satisfies $\mathcal{H}^s(\Lambda_s) = 0$ by elementary density estimates for the Radon measure

$$\mu(A) := \int_A |\nabla u|^p dy \quad A \subseteq \Omega \text{ open subset}.$$

Indeed, for all $\delta > 0$, Proposition 2.9 gives that

$$\delta\, \mathcal{H}^s(\Lambda_s) \leq \mu(\Lambda_s) \leq \mu(\Omega) < \infty.$$

Therefore, $\mathcal{H}^s(\Lambda_s) = 0$.

Hence, if we rewrite Σ_u as the disjoint union of $\Sigma_u \cap \Lambda_s$ and of $\Sigma_u \setminus \Lambda_s$, we deduce $\mathcal{H}^s(\Sigma_u \cap \Lambda_s) = 0$ and thus the estimate $\dim_{\mathcal{H}}(\Sigma_u \cap \Lambda_s) \leq s$.

Furthermore, it is easy to prove that $\Sigma_u \setminus \Lambda_s \subseteq \Sigma_u^{(1)}$. If $x \in \Sigma_u \setminus \Lambda_s$ by the higher integrability and Hölder inequality it follows that

$$\mathscr{D}_u(x,\rho) = \rho^{1-n} \int_{B_\rho(x)} |\nabla u|^2 dy \leq \omega_n^{1-\frac{2}{p}} \rho^{\frac{2}{p}\left(s-n+\frac{p}{2}\right)} \left(\rho^{-s} \int_{B_\rho(x)} |\nabla u|^p dy \right)^{\frac{2}{p}} \stackrel{\rho\downarrow 0^+}{\longrightarrow} 0,$$

since $s > n - p/2$. By taking into account Theorem 3.8 we have that $\dim_{\mathcal{H}}(\Sigma_u \setminus \Lambda_s) \leq n-2$.

In conclusion, we infer that for all $s \in (n - p/2, n-1)$

$$\dim_{\mathcal{H}}\Sigma_u = \max\{\dim_{\mathcal{H}}(\Sigma_u \cap \Lambda_s), \dim_{\mathcal{H}}(\Sigma_u \setminus \Lambda_s)\} \leq \max\{n-2, s\},$$

by letting $s \downarrow (n - p/2)$ we are done. □

3.3. **An energetic characterization of the Mumford and Shah conjecture 2.21.** Let us go back to the upper bound $p < 4$ in the higher integrability result. We consider again the crack-tip function in item (i) after Theorem 3.10. A simple calculation shows that its gradient belongs to $L^p_{loc} \setminus L^4_{loc}(\mathbb{R}^2)$ for all $p < 4$. Beyond the scale of L^p spaces something better holds true: $|\nabla u| \in L^{4,\infty}_{loc}(\mathbb{R}^2)$. The latter is a weak-Lebesgue space, i.e. if $U \subseteq \mathbb{R}^2$ is open then $f \in L^{4,\infty}_{loc}(U)$ if and only if for all $U' \subset\subset U$ there exists $K = K(U') > 0$ such that

$$\mathcal{L}^2\big(\{x \in U' : |f(x)| > \lambda\}\big) \le K\lambda^{-4} \quad \text{for all } \lambda > 0.$$

As a side effect of the considerations in [35] one deduces an energetic characterization of the modified Mumford and Shah conjecture 2.21 (see [35, Proposition 5]). Indeed, the validity of the latter is equivalent to a sharp integrability property of the gradient of the minimizers.

Theorem 3.11 (De Lellis and Focardi [35])**.** *If $\Omega \subseteq \mathbb{R}^2$, conjecture 2.21 is true for $u \in \mathcal{M}(\Omega)$ if and only if $\nabla u \in L^{4,\infty}_{loc}(\Omega)$.*

The characterization above would hold for the original Mumford and Shah conjecture 2.20 if C^1 regularity of $\overline{S_u}$ up to crack-tip points had been established.

To prove Theorem 3.11 we need the following preliminary observation.

Lemma 3.12. *Let $f \in L^{4,\infty}_{loc}(\Omega)$, $\Omega \subseteq \mathbb{R}^2$, then for all $\varepsilon > 0$ the set*

$$D_\varepsilon := \left\{ x \in \Omega : \liminf_r \frac{1}{r} \int_{B_r(x)} f^2(y)\, dy \ge \varepsilon \right\} \tag{3.5}$$

is locally finite.

Proof. We shall show in what follows that if $f \in L^{4,\infty}(\Omega)$ then D_ε is finite, an obvious localization argument then proves the general case.

Let $\varepsilon > 0$ and consider the set D_ε in (3.5) above. First note that, for any $B_r(x) \subset \Omega$ and any $\lambda > 0$ we have the estimate

$$\begin{aligned}\int_{\{y \in B_r(x) : |f(y)| \ge \lambda\}} f^2(y)\, dy &\le \int_{\{y \in \Omega : |f(y)| \ge \lambda\}} f^2(y)\, dy \\ &= 2\int_\lambda^{+\infty} t\, \mathcal{L}^2\big(\{y \in \Omega : |f(y)| \ge t\}\big) dt \le \int_\lambda^{+\infty} \frac{2K}{t^3} dt = \frac{K}{\lambda^2}.\end{aligned} \tag{3.6}$$

If $x \in D_\varepsilon$ and $r > 0$ satisfy

$$\int_{B_r(x)} f^2(y)\, dy \ge \frac{\varepsilon}{2} r, \tag{3.7}$$

choosing $\lambda = 2(K/r\varepsilon)^{1/2}$ in (3.6) we conclude

$$\int_{\{y\in B_r(x):\,|f(y)|<2(\frac{K}{r\varepsilon})^{1/2}\}} f^2(y)\,dy \geq \frac{\varepsilon}{4}r. \tag{3.8}$$

Furthermore, the trivial estimate

$$\int_{\{y\in B_r(x):\,|f(y)|<\lambda\}} f^2(y)\,dy < \pi\lambda^2 r^2,$$

implies for $\lambda = (\varepsilon/8\pi r)^{1/2}$

$$\int_{\{y\in B_r(x):\,|f(y)|<(\frac{\varepsilon}{8\pi r})^{1/2}\}} f^2(y)\,dy < \frac{\varepsilon}{8}r. \tag{3.9}$$

By collecting (3.8) and (3.9) we infer

$$\int_{\{y\in B_r(x):\,(\frac{\varepsilon}{8\pi r})^{1/2}\leq|f(y)|<2(\frac{K}{r\varepsilon})^{1/2}\}} f^2(y)\,dy \geq \frac{\varepsilon}{8}r,$$

that in turn implies

$$\mathcal{L}^2\big(\{y\in B_r(x):\,|f(y)|\geq(\frac{\varepsilon}{8\pi r})^{1/2}\}\big) \geq \frac{\varepsilon^2 r^2}{32K}. \tag{3.10}$$

Let $\{x_1,\dots,x_N\}\subseteq D_\varepsilon$ and $r>0$ be a radius such that the balls $B_r(x_i)\subseteq\Omega$ are disjoint and (3.7) holds for each x_i. Then, from (3.10) and the fact that $f\in L^{4,\infty}(\Omega)$, we infer

$$N\frac{\varepsilon^2 r^2}{32K} \leq \mathcal{L}^2\big(\{y\in\Omega:\,|f(y)|\geq(\frac{\varepsilon}{8\pi r})^{1/2}\}\big) \leq \frac{K(8\pi r)^2}{\varepsilon^2} \Longrightarrow N \leq \frac{2^{11}K^2\pi^2}{\varepsilon^4},$$

and the conclusion follows at once. □

We are now ready to give the proof of Theorem 3.11. The "if" direction is achieved by first proving that $\overline{S_u}$ has locally finitely many connected components and then invoking the regularity theory developed by Bonnet [10]. In turn, the proof that the connected components are locally finite is a fairly simple application of David's ε-regularity theory (see Theorem 3.4). Vice versa, the "only if" direction is proved by means of Bonnet blow up analysis and standard elliptic regularity theory.

Proof of Theorem 3.11. To prove the direct implication we assume without loss of generality that $\Omega = B_R$ for some $R>1$, being the result local. In addition, we may also suppose that $\overline{S_u}\cap\partial B_1 = \{y_1,\dots,y_M\}$. Theorem 3.4 and Theorem 4.3 in Section 4 below yield that there exists some $\varepsilon_0>0$ such that for all points $x\in B_R\setminus D_{\varepsilon_0}$ the set $\overline{S_u}\cap B_r(x)$ is either empty or diffeomorphic to a minimal cone, for some $r>0$. In particular, in the latter event $B_r(x)\setminus\overline{S_u}$ is not connected.

Supposing that $D_{\varepsilon_0} \cap B_1 = \{x_1, \dots, x_N\}$, and setting

$$\Omega_k := B_{1-1/k} \setminus \bigcup_{i=1}^{N} B_{1/k}(x_i)\,,$$

a covering argument and the last remark give that for every $x \in \Omega_k \cap \overline{S_u}$ there is a continuous arc $\gamma_k : [0,1] \to \overline{S_u}$ with $\gamma_k(0) = x$ and $\gamma_k(1) = y \in \partial\Omega_k$. Then, the sequence $(\widetilde{\gamma}_k)_{k\in\mathbb{N}}$ of reparametrizations of the γ_k's by arc length converges to some arc $\gamma : [0,1] \to \overline{S_u}$ with $\gamma(0) = x$ and $\gamma(1) \in \{x_1, \dots, x_N, y_1, \dots, y_M\}$.

From this, we deduce that $\overline{B_1} \cap \overline{S_u}$ has a finite number of connected components. Bonnet's regularity results [10, Theorems 1.1 and 1.3] then provide the thesis.

To conclude we prove the opposite implication. To this aim we consider $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ and suppose that $\overline{S_u} \cap \Omega''$ is a finite union of C^1 arcs of finite length. Denote by $\{x_1, \dots, x_N\}$ the end points of the arcs in Ω' and let $r > 0$ be such that $B_{4r}(x_i) \subseteq \Omega'$ for all i, and $B_{4r}(x_i) \cap B_{4r}(x_j) = \emptyset$ if $i \neq j$. Theorem 3.3 implies that ∇u has a $C^{0,\alpha}$ extension on both sides of $(\Omega'' \cap \overline{S_u}) \setminus \cup_i \overline{B_r(x_i)}$ for all $\alpha < 1$. In particular, ∇u is bounded on $\overline{\Omega'} \setminus \cup_i B_{2r}(x_i)$.

Next consider the sequence $r_k = r/2^{k-1}$, $k \geq 0$, and fix $i \in \{1, \dots, N\}$. Then, by [26, Proposition 37.8] (or [10, Theorem 2.2]) we can extract a subsequence $k_j \uparrow \infty$ along which the blow up functions $u_j(x) := r_{k_j}^{-1/2}\big(u(x_i + r_{k_j}x) - c_j(x)\big)$ converge to some w in $W^{1,2}_{loc}(B_4 \setminus K)$, for some piecewise constant function $c_j : \Omega \setminus \overline{S_{u_j}} \to \mathbb{R}$, and $(\overline{S_{u_j}})_{j\in\mathbb{N}}$ converges to some set K in the Hausdorff metric.

By Bonnet's blow up theorem [10, Theorem 4.1] only two possibilities occur: either x_i is a triple junction point, i.e., K is a triple junction and w is locally constant on $B_4 \setminus K$, or x_i is a crack-tip, i.e., up to a rotation $K = \{(x,0) : x \leq 0\}$ and $w(\rho,\theta) = C \pm \sqrt{\frac{2}{\pi}\rho} \cdot \sin(\theta/2)$ for $\theta \in (-\pi,\pi)$, $\rho > 0$ and some constant $C \in \mathbb{R}$ (note that in this argument we do not need to know that the blow up limit is unique).

In both cases, we claim that ∇u_j has a $C^{0,\alpha}$ extension on the closure of each connected component of $U_j := (B_3 \setminus \overline{B_1}) \setminus \overline{S_{u_j}}$ with $\sup_j \|\nabla u_j\|_{L^\infty(U_j)} \leq C$. This follows as in [7, Theorem 7.49] (or [26, Proposition 17.15], see also Remark 3.5) locally straightening $\overline{S_{u_j}} \cap (B_4 \setminus \overline{B_{1/2}})$ onto $K \cap (B_4 \setminus \overline{B_{1/2}})$ via a $C^{1,\alpha}$ conformal map, a reflection argument and standard Schauder estimates for the laplacian. Scaling back the previous estimate gives

$$|\nabla u(x)| \leq C\,|x - x_i|^{-1/2} \qquad \text{for } x \in \cup_{j\in\mathbb{N}}(\overline{B_{3r_{k_j}}(x_i)} \setminus B_{r_{k_j}}(x_i)),$$

in turn from this, the maximum principle and Hopf's lemma we infer

$$|\nabla u(x)| \leq C\, r_k^{-1/2} \qquad \text{for } x \in B_{2r}(x_i) \setminus B_{r_k}(x_i).$$

The latter inequality finally implies $\nabla u \in L^{4,\infty}(B_{2r}(x_i))$.

Eventually, we are able to conclude $\nabla u \in L^{4,\infty}(\Omega')$, being on one hand ∇u bounded on $\overline{\Omega'} \setminus \cup_i B_{2r}(x_i)$, and on the other hand belonging to $L^{4,\infty}(\cup_i B_{2r}(x_i))$. $\square$

4. Hausdorff dimension of the set of triple-junctions

In order to prove Theorem 3.8 we need to analyze the asymptotic behavior of MS-minimizer in points of vanishing Dirichlet energy. This issue has been first investigated in [4, Proposition 5.3, Theorem 5.4]. Those results hinge upon the notion of *Almgren's area minimizing sets*, i.e. a $\mathcal{H}^{n-1}$ rectifiable set $S \subset B_1$ such that

$$\mathcal{H}^{n-1}(S) \leq \mathcal{H}^{n-1}(\varphi(S)), \quad \forall \varphi \in \mathrm{Lip}(\mathbb{R}^n, \mathbb{R}^n), \ \{\varphi \neq \mathrm{Id}\} \subset\subset B_1.$$

Following this approach to infer Theorem 3.8 requires a delicate study of the behavior of the composition of SBV functions with Lipschitz deformations that are not necessarily one-to-one, and some specifications on the regularity theory for Almgren's area minimizing sets are needed (cf. [4]). Therefore, following Ambrosio, Fusco and Hutchinson, Theorem 3.10 is a straightforward corollary of a much deeper and technically demanding result (given the higher integrability for granted).

Instead, in Theorem 4.3 below (cf. [35, Proposition 5.1]) we set the analysis into the more natural framework of Caccioppoli partitions.

Definition 4.1. *A Caccioppoli partition of Ω is a countable partition $\mathscr{E} = \{E_i\}_{i=1}^\infty$ of Ω in sets of (positive Lebesgue measure and) finite perimeter with $\sum_{i=1}^\infty \mathrm{Per}(E_i, \Omega) < \infty$.*

For each Caccioppoli partition $\mathscr{E}$ the set of interfaces *is given by $J_{\mathscr{E}} := \bigcup_i \partial^* E_i$.*

The partition $\mathscr{E}$ is said to be minimal *if*

$$\mathcal{H}^{n-1}(J_{\mathscr{E}}) \leq \mathcal{H}^{n-1}(J_{\mathscr{F}})$$

for all Caccioppoli partitions $\mathscr{F}$ for which $\sum_{i=1}^\infty \mathcal{L}^n\left((F_i \triangle E_i) \cap (\Omega \setminus \Omega')\right) = 0$, for some open subset $\Omega' \subset\subset \Omega$.

There is an important correspondence between Caccioppoli partitions and the subspace SBV_0 of "piecewise constant" SBV functions recalled as prototype example in subsection 2.1, in such a way that minimizing the Mumford and Shah energy over SBV_0 corresponds exactly to the minimal area problem for Caccioppoli partitions (see [7, Theorems 4.23, 4.25 and 4.39]).

Existence of minimal Caccioppoli partitions is guaranteed by Ambrosio's SBV closure and compactness Theorem 2.3 without imposing any L^∞ bound simply by composition with $\arctan(t)$, provided the partitions are either equi-finite or ordered, i.e. if $\mathscr{E} = \{E_i\}_{i=1}^\infty$ then $\mathcal{L}^n(E_i) \geq \mathcal{L}^n(E_j)$ for $j \geq i$.

A regularity theory for minimal Caccioppoli partitions has been established by Massari and Tamanini [58]. We limit ourselves here to the ensuing statement.

Theorem 4.2 (Massari and Tamanini [58]). *Let $\mathscr{E}$ be a minimal Caccioppoli partition in Ω,*

$$\omega_{n-1} \leq \liminf_{r\downarrow 0} \frac{\mathcal{H}^{n-1}(J_{\mathscr{E}} \cap B_r(x))}{r^{n-1}} \leq \limsup_{r\downarrow 0} \frac{\mathcal{H}^{n-1}(J_{\mathscr{E}} \cap B_r(x))}{r^{n-1}} \leq n\omega_n.$$

In particular, $J_{\mathscr{E}}$ is essentially closed, i.e. $\mathcal{H}^{n-1}\big((\Omega \cap \overline{J_{\mathscr{E}}}) \setminus J_{\mathscr{E}}\big) = 0$.

Moreover, there exists a relatively closed subset $\Sigma_{\mathscr{E}}$ of $J_{\mathscr{E}}$ such that $J_{\mathscr{E}} \setminus \Sigma_{\mathscr{E}}$ is a $C^{1,1/2}$ hypersurface and $\dim_{\mathcal{H}} \Sigma_{\mathscr{E}} \leq n-2$. If, in addition, $n = 2$, then $\Sigma_{\mathscr{E}}$ is locally finite.

We are now ready to prove a compactness result for sequences of MS-minimizers with vanishing L^1-gradient energy.

Theorem 4.3 (De Lellis and Focardi [35]). *Let $(u_k)_{k\in\mathbb{N}} \subset \mathcal{M}(B_1)$ be such that*

$$\lim_k \|\nabla u_k\|_{L^1(B_1)} = 0. \tag{4.1}$$

Then, (up to the extraction of a subsequence not relabeled for convenience) there exists a minimal Caccioppoli partition $\mathscr{E} = \{E_i\}_{i\in\mathbb{N}}$ such that $(\overline{S_{u_k}})_{k\in\mathbb{N}}$ converges locally in the Hausdorff distance on $\overline{B_1}$ to $\overline{J_{\mathscr{E}}}$ and for all open sets $A \subseteq B_1$

$$\lim_k \mathrm{MS}(u_k, A) = \lim_k \mathcal{H}^{n-1}(S_{u_k} \cap A) = \mathcal{H}^{n-1}(J_{\mathscr{E}} \cap A). \tag{4.2}$$

Though this last statement is, intuitively, quite clear, it is technically demanding, because we do not have any a priori control of the norms $\|u_k\|_{L^1}$, thus preventing the use of Ambrosio's $(G)SBV$ compactness theorem. We can not even expect to gain precompactness via De Giorgi's SBV Poincaré-Wirtinger type inequality, since the latter holds true in a regime of small jumps rather than of small gradients as the current one.

Proof of Theorem 4.3. The sequence $(u_k)_{k\in\mathbb{N}}$ does not satisfy, a priori, any L^p bound, thus in order to gain some insight on the asymptotic behavior of the corresponding jump sets we first construct a new sequence $(w_k)_{k\in\mathbb{N}}$ with null gradients introducing an infinitesimal error on the length of the jump set of w_k with respect to that of u_k. Then, we investigate the limit behavior of the corresponding Caccioppoli partitions.

Step 1. *There exists a sequence $(w_k)_{k\in\mathbb{N}} \subseteq SBV(B_1)$ satisfying*

(i) $\nabla w_k = 0$ $\mathcal{L}^n$ *a.e. on* B_1,
(ii) $\|u_k - w_k\|_{L^\infty(B_1)} \leq 2\|\nabla u_k\|^{1/2}_{L^1(B_1)}$,
(iii) $\mathcal{H}^{n-1}\left(S_{w_k} \setminus (S_{u_k} \cup H_k)\right) = 0$ *for some Borel measurable set* H_k, *with* $\mathcal{H}^{n-1}(H_k) = o(1)$ *as* $k \uparrow \infty$.

Note that in turn item (iii) implies that

$$\mathrm{MS}(w_k) = \mathcal{H}^{n-1}(S_{w_k}) \leq \mathcal{H}^{n-1}(S_{u_k}) + o(1) \leq \mathrm{MS}(u_k) + o(1). \tag{4.3}$$

In Step 2 below we shall eventually show that $|\mathrm{MS}(w_k) - \mathrm{MS}(u_k)| \leq o(1)$.

Recall that the BV Co-Area formula (see [7, Theorem 3.40]) establishes

$$\int_{B_1} |\nabla u_k| dx = |Du_k|(B_1 \setminus S_{u_k}) = \int_{\mathbb{R}} \mathrm{Per}\left(\{u_k \geq t\} \setminus S_{u_k}\right) dt. \tag{4.4}$$

Denote by I_i^k a partition of $\mathbb{R}$ of intervals of equal length $\|\nabla u_k\|^{1/2}_{L^1(B_1)}$. Equation (4.4) and the Mean value Theorem provide the existence of levels $t_i^k \in I_i^k$ satisfying

$$\sum_{i=1}^{\infty} \mathrm{Per}\left(\{u_k \geq t_i^k\} \setminus S_{u_k}\right) \leq \|\nabla u_k\|^{1/2}_{L^1(B_1)}. \tag{4.5}$$

Then define the functions w_k to be equal to t_i^k on $\{u_k \geq t_i^k\} \setminus \{u_k \geq t_{i+1}^k\}$. The choice of the I_i^k's, (4.5) and the very definition yield that w_k belongs to $SBV(B_1)$ and that it satisfies properties (i) and (ii). To conclude, note that $\mathcal{H}^{n-1}\left(S_{w_k} \setminus (\cup_i \partial^* \{u_k \geq t_i^k\} \cup S_{u_k})\right) = 0$ by construction, thus item (iii) follows at once from (4.5).

Step 2. *Compactness for the jump sets.*

Each function w_k determines a Caccioppoli partition $\mathscr{E}_k = \{E_i^k\}_{i\in\mathbb{N}}$ of B_1 (see [18, Lemma 1.11]). In addition, upon reordering the sets E_i^k's, we may assume that $\mathcal{L}^n(E_i^k) \geq \mathcal{L}^n(E_j^k)$ if $i < j$. Then, the compactness theorem for Caccioppoli partitions (see [54, Theorem 4.1, Proposition 3.7] and [7, Theorem 4.19]) provides us with a subsequence (not relabeled) and a Caccioppoli partition $\mathscr{E} := \{E_i\}_{i\in\mathbb{N}}$ such that

$$\lim_j \sum_{i=1}^{\infty} \mathcal{L}^n(E_i^k \triangle E_i) = 0, \quad \text{and} \quad \sum_{i=1}^{\infty} \mathrm{Per}(E_i, A) \leq \liminf_k \sum_{i=1}^{\infty} \mathrm{Per}(E_i^k, A) \tag{4.6}$$

for all open subsets A in B_1. We claim that $\mathscr{E}$ determines a minimal Caccioppoli partition and in proving this we will also establish (4.2).

We start off observing that the first identity (4.6) and the Co-Area formula yield the existence of a set $I \subset (0,1)$ of full measure such that

$$\liminf_k \sum_{i=1}^{\infty} \mathcal{H}^{n-1}\left((E_i^k \triangle E_i) \cap \partial B_\rho\right) = 0 \qquad \forall \rho \in I\,. \tag{4.7}$$

Define the measures μ_k as $\mu_k(A) := \mathrm{MS}(u_k, A) + \mathrm{MS}(w_k, A)$ (A being an arbitrary Borel subset of B_1). Proposition 2.14 and item (iii) in Step 1 ensure that, upon the extraction of a further subsequence, μ_k converges weakly* to a finite measure μ on B_1. W.l.o.g. we may assume that for all $\rho \in I$ we have, in addition, $\mu(\partial B_\rho) = 0$.

Let us now fix a Caccioppoli partition $\mathscr{F} := \{F_i\}_{i\in\mathbb{N}}$ suitable to test the minimality of $\mathscr{E}$, i.e. $\sum_{i=1}^{\infty} \mathcal{L}^n\left((F_i \triangle E_i) \cap (B_1 \setminus \overline{B_t})\right) = 0$ for some $t \in (0,1)$. Moreover, we may also suppose that $\sum_{i=1}^{\infty} \mathcal{H}^{n-1}\left((F_i \triangle E_i) \cap \partial B_\rho\right) = 0$ for all $\rho \in I \cap (t,1)$. Let then ρ and r be radii in $I \cap (t,1)$ with $\rho < r$ and assume, after passing to a subsequence (not relabeled) that the inferior limit in (4.7) is actually a limit for these two radii. We define

$$\omega_k := \begin{cases} w_k & \text{on } B_1 \setminus \overline{B_\rho} \\ t_i^k & \text{on } F_i \cap B_\rho. \end{cases}$$

Note that $\omega_k \in SBV(B_1)$ with $\nabla \omega_k = 0$ $\mathcal{L}^n$ a.e. on B_1, and since $t < \rho \in I$ it follows

$$\mathcal{H}^{n-1}\Big(S_{\omega_k} \setminus \big((J_{\mathscr{F}} \cap B_\rho) \cup (\cup_{i\in\mathbb{N}}(E_i^k \triangle E_i) \cap \partial B_\rho) \cup (S_{w_k} \cap (B_1 \setminus \overline{B_\rho}))\big)\Big) = 0.$$

Consider $\varphi \in \mathrm{Lip} \cap C_c(B_1, [0,1])$ with $\varphi|_{B_r} \equiv 1$, and $|\nabla\varphi| \leq (1-r)^{-1}$ on B_1, and set $v_k := \varphi\, \omega_k + (1-\varphi)\, u_k$. Clearly, v_k is admissible to test the minimality of u_k. Then, simple calculations lead to

$$\begin{aligned}
\mathrm{MS}(u_k) &\leq \mathrm{MS}(v_k)\\
&\leq \mathrm{MS}(\omega_k) + 2\mathrm{MS}(u_k, B_1 \setminus \overline{B_r}) + \frac{2}{(1-r)^2}\|u_k - \omega_k\|^2_{L^2(B_1\setminus\overline{B_r})}\\
&\leq \mathcal{H}^{n-1}\left(J_{\mathscr{F}}\right) + \sum_{i\in\mathbb{N}} \mathcal{H}^{n-1}\left((E_i^k \triangle E_i) \cap \partial B_\rho\right) + \mathcal{H}^{n-1}\left(S_{w_k} \setminus \overline{B_\rho}\right)\\
&+2\mathrm{MS}(u_k, B_1 \setminus \overline{B_r}) + \frac{2}{(1-r)^2}\|u_k - w_k\|^2_{L^2(B_1\setminus\overline{B_r})}\\
&\leq \mathcal{H}^{n-1}\left(J_{\mathscr{F}}\right) + \sum_{i\in\mathbb{N}} \mathcal{H}^{n-1}\left((E_i^k \triangle E_i) \cap \partial B_\rho\right) + 3\mu_k(B_1 \setminus \overline{B_\rho})\\
&+\frac{2}{(1-r)^2}\|u_k - w_k\|^2_{L^\infty(B_1)}.
\end{aligned} \tag{4.8}$$

Note that in the third inequality we have used that ω_k and w_k coincide on $B_1 \setminus \overline{B_\rho}$, and that $\rho < r$. By letting $k \uparrow \infty$ in (4.8), we infer

$$\begin{aligned}
\mathcal{H}^{n-1}(J_{\mathscr{E}}) \leq \liminf_j \mathcal{H}^{n-1}(S_{u_k}) \leq \liminf_j \mathrm{MS}(u_k) \leq \limsup_j \mathrm{MS}(u_k)\\
\leq \limsup_k \mathrm{MS}(v_k) \leq \mathcal{H}^{n-1}\left(J_{\mathscr{F}}\right) + 3\mu(B_1 \setminus \overline{B_\rho}),
\end{aligned}$$

where we have used that r and ρ belong to I, inequality (4.3), the convergence $\mu_k \rightharpoonup^* \mu$, the estimate $\sup_k \mathrm{MS}(u_k, B_1 \setminus B_t) \leq n\omega_n(1 - t^{n-1})$ for all $t \in (0,1)$, that is derived as inequality (2.22) in Proposition 2.14, and the limit (4.7). Finally, by letting $\rho \in I$ tend to 1^- we conclude

$$\mathcal{H}^{n-1}(J_{\mathscr{E}}) \leq \liminf_k \mathcal{H}^{n-1}(S_{u_k}) \leq \liminf_k \mathrm{MS}(u_k) \leq \limsup_k \mathrm{MS}(u_k) \leq \mathcal{H}^{n-1}(J_{\mathscr{F}}), \quad (4.9)$$

which proves the minimality of $\mathscr{E}$. In addition, choosing $\mathscr{E} = \mathscr{F}$, we infer (4.2) for $A = B_1$. Actually, for $\mathscr{E} = \mathscr{F}$ the same same argument employed above gives (4.2) (it suffices to take $v_k = \varphi w_k + (1-\varphi)u_k$).

In particular, $J_{\mathscr{E}}$ is essentially closed (by Theorem 4.2) and it satisfies a density lower bound estimate. Using this and the De Giorgi, Carriero, Leaci density lower bound in formula (2.20) we conclude that $(\overline{S_{u_k}})_{k\in\mathbb{N}}$ converges to $\overline{J_{\mathscr{E}}}$ in the local Hausdorff topology on $\overline{B_1}$. □

Interesting (immediate) consequences of Theorem 4.3 are contained in the ensuing two statements.

Corollary 4.4. *Let $(u_k)_{k\in\mathbb{N}} \subset \mathcal{M}(B_1)$ be as in the statement of Theorem 4.3, then*

$$\lim_k \|\nabla u_k\|_{L^2(B_1)} = 0, \quad \textit{and} \quad \mathcal{H}^{n-1} \llcorner S_{u_k} \stackrel{*}{\rightharpoonup} \mathcal{H}^{n-1} \llcorner J_{\mathscr{E}}.$$

Actually, $\mathcal{H}^{n-1} \llcorner S_{u_k} \rightarrow \mathcal{H}^{n-1} \llcorner J_{\mathscr{E}}$ in the narrow convergence of measures, i.e. in the duality with $C_b(\Omega)$.

Proof. It is an easy consequence of the equalities in (4.2). □

Corollary 4.5. *Let $x \in \Sigma_u^{(1)}$ and $\rho_k \downarrow 0$, then (up to subsequences not relabeled) there exists a minimal Caccioppoli partition $\mathscr{E}$ such that $(S_{u_{x,\rho_k}})_{k\in\mathbb{N}}$, u_{x,ρ_k} defined in (2.8), converges locally in the Hausdorff distance to $\overline{J_{\mathscr{E}}}$ and*

$$\mathcal{H}^{n-1} \llcorner S_{u_{x,\rho_k}} \stackrel{*}{\rightharpoonup} \mathcal{H}^{n-1} \llcorner J_{\mathscr{E}}.$$

Remark 4.6. *Under the assumptions of Corollary 4.5, it is natural to expect the limit partition $\mathscr{E}$ to be* conical, *i.e. $\mathscr{E} = \{E_i\}_{i=1}^{\infty}$ with the E_i's cones with vertices in the origin, as a result of the blow up procedure. In general this latter property can be proven only for suitable sequences $\rho_k \downarrow 0$ by combining a blow up argument and Theorem 4.3 (cf. [4,* Proposition 5.8]*).*

Actually, in 2-dimensions the result is true for every sequence $\rho_k \downarrow 0$ since a structure theorem for minimal Caccioppoli partitions assures that (locally) they are minimal connections (cf. Proposition 5.3). The lack of monotonicity formulas for the Mumford and Shah problem prevents the derivation of such a statement in the general case.

A more precise result in the 2d case will be established in Proposition 4.11. Note that no uniqueness is ensured for the limits except for 2d in view of David's ε-regularity result (see Theorem 3.4), and in 3d in view of the analogous result established by Lemenant in [51].

By means of Theorems 4.2, 4.3 and standard blow up arguments we are able to establish Theorem 3.8. Let us first recall a technical lemma.

Lemma 4.7 (Section 3.6 [74]). *Let $s \geq 0$, then*

(i) $\mathcal{H}^s(\Sigma) = 0 \iff \mathcal{H}^{s,\infty}(\Sigma) = 0$, *for all sets* $\Sigma \subseteq \mathbb{R}^n$;

(ii) *if* Σ_j *and* Σ *are compact sets such that* $\sup_{\Sigma_j} \mathrm{dist}\big(\cdot, \Sigma\big) \downarrow 0$ *as* $j \uparrow \infty$, *then*

$$\mathcal{H}^{s,\infty}(\Sigma) \geq \limsup_j \mathcal{H}^{s,\infty}(\Sigma_j);$$

(iii) *if* $\mathcal{H}^s(\Sigma) > 0$, *then for* $\mathcal{H}^s$*-a.e.* $x \in \Sigma$

$$\limsup_{\rho \downarrow 0^+} \frac{\mathcal{H}^{s,\infty}(\Sigma \cap B_\rho(x))}{\omega_s \rho^s} \geq 2^{-s}.$$

The strategy of proof below is essentially that by Ambrosio, Fusco and Hutchinson as reworked by De Lellis, Focardi and Ruffini in light of [36, Theorem 4.2].

Proof of Theorem 3.8. We argue by contradiction: suppose that there exists $s > n - 2$ such that $\mathcal{H}^s\big(\Sigma_u^{(1)}\big) > 0$. From this we infer that $\mathcal{H}^{s,\infty}\big(\Sigma_u^{(1)}\big) > 0$, and moreover that for $\mathcal{H}^s$-a.e. $x \in \Sigma_u^{(1)}$ it holds

$$\limsup_{\rho \downarrow 0^+} \frac{\mathcal{H}^{s,\infty}(\Sigma_u^{(1)} \cap B_\rho(x))}{\rho^s} \geq \frac{\omega_s}{2^s} \tag{4.10}$$

(see for instance [7, Theorem 2.56 and formula (2.43)]). Without loss of generality, suppose that (4.10) holds at $x = 0$, and consider a sequence $\rho_k \downarrow 0$ for which

$$\mathcal{H}^{s,\infty}(\Sigma_u^{(1)} \cap B_{\rho_k}) \geq \frac{\omega_s}{2^{s+1}} \rho_k^s \qquad \text{for all } k \in \mathbb{N}. \tag{4.11}$$

Theorem 4.3 provides a subsequence, not relabeled for convenience, and a minimal Caccioppoli partition $\mathscr{E}$ such that

$$\lim_k \mathcal{H}^{n-1}(S_{u_k} \cap A) = \mathcal{H}^{n-1}(J_{\mathscr{E}} \cap A) \qquad \text{for all open sets } A \subseteq B_1. \tag{4.12}$$

and that

$$\rho_k^{-1}\overline{S_u} \to \overline{J_{\mathscr{E}}} \;\; \text{locally Hausdorff.} \tag{4.13}$$

In turn, from the latter we claim that if $\mathcal{F}$ is any open cover of $\Sigma_{\mathscr{E}} \cap \overline{B}_1$, then for some $h_0 \in \mathbb{N}$

$$\rho_k^{-1}\Sigma_u^{(1)} \cap \overline{B}_1 \subseteq \cup_{F \in \mathcal{F}} F \qquad \text{for all } k \geq k_0. \tag{4.14}$$

Indeed, if this is not the case we can find a sequence $x_{k_j} \in \rho_{k_j}^{-1}\Sigma_u^{(1)} \cap \overline{B}_1$ converging to some point $x_0 \notin \Sigma_{\mathscr{E}}$. If $T_{x_0}^{\mathscr{E}}$ is the tangent plane to $J_{\mathscr{E}}$ at x_0 (which exists by the property of $\Sigma_{\mathscr{E}}$ in Theorem 4.2), then for some ρ_0 we have

$$\rho^{-1-n}\int_{B_\rho(x_0)\cap J_{\mathscr{E}}} \mathrm{dist}^2(y, T_{x_0}^{\mathscr{E}})d\mathcal{H}^{n-1} < \varepsilon_0, \qquad \text{for all } \rho \in (0, \rho_0).$$

In turn, from the latter inequality and the convergence in (4.12), it follows that, for $\rho \in (0, \rho_0 \wedge 1)$,

$$\limsup_{j\uparrow\infty} \rho^{-1-n}\int_{B_\rho(x_{k_j})\cap \rho_{k_j}^{-1}S_u} \mathrm{dist}^2(y, T_{x_0}^{\mathscr{E}})\, d\mathcal{H}^{n-1} < \varepsilon_0.$$

Therefore, as $x_{k_j} \in \rho_{k_j}^{-1}\Sigma_u^{(1)}$, we get for j large enough

$$\limsup_{\rho\downarrow 0} \big(\mathscr{D}(x_{k_j}, \rho) + \mathscr{A}(x_{k_j}, \rho)\big) < \varepsilon_0,$$

a contradiction in view of the characterization of the singular set in (3.1).

To conclude, we note that by (4.14) we get

$$\mathcal{H}^{s,\infty}(\Sigma_{\mathscr{E}} \cap \overline{B_1}) \geq \limsup_{k\uparrow\infty} \mathcal{H}^{s,\infty}(\rho_k^{-1}\Sigma_u^{(1)} \cap \overline{B_1});$$

given this, (4.11) and (4.13) yield that

$$\mathcal{H}^{s}(\Sigma_{\mathscr{E}} \cap \overline{B_1}) \geq \mathcal{H}^{s,\infty}(\Sigma_{\mathscr{E}} \cap \overline{B_1}) \geq \limsup_{k\uparrow\infty} \mathcal{H}^{s,\infty}(\rho_k^{-1}\Sigma_u^{(1)} \cap \overline{B_1}) \geq \frac{\omega_s}{2^{s+1}},$$

thus contradicting Theorem 4.2. □

Corollary 4.8. *If $\Omega \subseteq \mathbb{R}^2$ and $u \in \mathcal{M}(\Omega)$, then $\Sigma_u^{(1)}$ is at most countable.*

Proof. This claim follows straightforwardly from the compactness result Theorem 4.3, David's ε-regularity Theorem 3.4, and a direct application of Moore's triod theorem showing that in the plane every system of disjoint triods, i.e. unions of three Jordan arcs that have all one endpoint in common and otherwise disjoint, is at most countable (see [64, Theorem 1] and [70, Proposition 2.18]). □

Remark 4.9. *Analogously, in 3-dimensions the set of points with blow up a $\mathbb{T}$ cone, i.e. a cone with vertex the origin constructed upon the 1-skeleton of a regular tetrahedron, is at most countable. The latter claim follows thanks to Theorem 4.3, the 3d extension of David's ε-regularity result by Lemenant in* [51, Theorem 8]*, and a suitable extension of Moore's theorem on triods established by Young in* [76].

Let us point out that we employ topological arguments to compensate for the lack of monotonicity formulas. The latter would allow one to exploit Almgren's stratification type *results and get, actually, a more precise picture of the set* $\Sigma_u^{(1)}$ *(cf. with* [75, Theorem 3.2] *and* [42]*).*

Remark 4.10. *In 2d Theorem 4.2 provides the local finiteness of the singular set for minimal Caccioppoli partitions, the blow up limits of* $\rho^{-1}(\overline{S_u} - x)$ *in points* $x \in \Sigma_u^{(1)}$*. This conclusion is far from being established for the set* $\Sigma_u^{(1)}$ *itself. With the results at hand one can prove that every convergent sequence* $(x_j)_{j\in\mathbb{N}} \subset \Sigma_u^{(1)}$ *has a limit* $x_0 \notin \big(\Sigma_u^{(1)} \cup \Sigma_u^{(2)}\big)$*. To show this, first note that* $x_0 \notin \Sigma_u^{(1)}$ *thanks to item (iii) in Proposition 5.3 or Theorem 3.4; moreover* $x_0 \notin \Sigma_u^{(2)}$ *thanks to item (ii) in Proposition 4.11 below and Theorem 3.7. Similarly, any converging sequence* $(x_j)_{j\in\mathbb{N}} \subset \Sigma_u^{(2)}$ *has a limit* $x_0 \notin \big(\Sigma_u^{(1)} \cup \Sigma_u^{(2)}\big)$*. Therefore, in both instances, it might happen that the limit point* x_0 *belong to*

$$\Sigma_u^{(3)} = \{x \in \Sigma_u : \liminf_{\rho\downarrow 0} \mathscr{D}_u(x,\rho) > 0, \liminf_{\rho\downarrow 0} \mathscr{A}_u(x,\rho) > 0\}.$$

We conclude the section by justifying the denomination used for the sets $\Sigma_u^{(1)}$ and $\Sigma_u^{(2)}$ in 2-dimensions.

Proposition 4.11. *Let* $\Omega \subseteq \mathbb{R}^2$ *and* $u \in \mathcal{M}(\Omega)$*, then*

(i) $x \in \Sigma_u^{(1)}$ *if and only if every blow up of* u *in* x *is a triple junction function;*
(ii) $x \in \Sigma_u^{(2)}$ *if and only if every blow up of* u *in* x *is a crack-tip function.*

Proof. We start off recalling that $(u, \Omega\cap\overline{S_u})$ is an *essential pair*, i.e. $\mathcal{H}^1(\overline{S_u}\cap B_r(x)) > 0$ for all $x \in \overline{S_u}$ and $B_r(x) \subseteq \Omega$ (see Theorem 2.6). Then the existence and several properties of blow up limits are guaranteed by [26, Propositions 37.8, 40.9, Corollary 38.48]. More precisely, with fixed a point $x \in \Omega$ and a sequence $\rho_k \downarrow 0$, up to subsequences not relabeled, we may assume that the sets $K_k := \rho_k^{-1}(\overline{S_u} - x)$ locally Hausdorff converges in $\mathbb{R}^2$ to some closed set K as $k \uparrow \infty$. Then there is a subsequence (not relabeled for convenience) and continuous piecewise constant functions c_k on $\mathbb{R}^2 \setminus K_k$, such that the pairs (u_{x,ρ_k}, K_k) with $u_{x,\rho_k}(y) := \rho_k^{-1/2}\big(u(x+\rho_k y) - c_k(y)\big)$ satisfy:

(a) $(u_{x,\rho_k})_k$ converges to some w in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2 \setminus K)$[6],
(b) (w, K) is a global Bonnet minimizer according to Definition 2.22, and an essential pair (see [26, Remark 54.8]),
(c) for $\mathcal{L}^1$ a.e. $r > 0$

$$\lim_k \int_{B_r\setminus K_k} |\nabla u_{x,\rho_k}|^2 dy = \int_{B_r\setminus K} |\nabla w|^2 dy, \qquad \lim_k \mathcal{H}^1\big(B_r \setminus K_k\big) = \mathcal{H}^1\big(B_r \setminus K\big).$$

[6] As $(K_k)_k$ locally Hausdorff converges to K, every $O \subset\subset \mathbb{R}^2 \setminus K$ is contained for k sufficiently big in $\rho_k^{-1}(\Omega - x) \setminus K$, so that the convergence of (u_{x,ρ_k}) in $W^{1,2}(O)$ is well defined.

To prove the direct implication in case (i) note that by Corollary 4.5, the jump set of any blow up limit in a point $x \in \Sigma_u^{(1)}$ is a minimal Caccioppoli partition $\mathscr{E}$, and that 0 is a singular point for it (cf. the argument leading to (4.14) in the proof of Theorem 3.8). Finally, Proposition 5.3 below ensures then that $J_{\mathscr{E}}$ is (locally) a triple junction around 0.

For the direct implication in item (ii), as by the very definition of $\Sigma_u^{(2)}$ for $\mathcal{L}^1$ a.e. $r > 0$

$$\lim_k \mathscr{A}_u(x, r\,\rho_k) = \lim_k \mathscr{A}_{u_{x,\rho_k}}(0, r) = \mathscr{A}_w(0, r) = 0,$$

$\overline{S_w}$ is actually contained in a 1-dimensional vector space. In this case a result by Léger [50] ensures that $\overline{S_w}$ is either empty or a line or a half line (cf. [26, Theorem 64.1]). Therefore, the energy upper bound in (2.22) and item (iii) above yield for $\mathcal{L}^1$ a.e. $r > 0$

$$\lim_k \mathscr{D}_u(x, r\,\rho_k) = \lim_k \mathscr{D}_{u_{x,\rho_k}}(0, r) = \mathscr{D}_w(0, r) \in [\varepsilon_0, 2\pi\, r]. \tag{4.15}$$

The possibility that $\overline{S_w} = \emptyset$ is ruled out as follows: in such a case $|\nabla w|^2$ would be subharmonic on $\mathbb{R}^2$, being w harmonic there, and thus we would deduce that

$$\sup_{B_{r/2}} |\nabla w|^2 \leq \frac{4}{\pi r^2} \int_{B_r} |\nabla w|^2 dx \overset{(2.22)}{\leq} \frac{8}{r}.$$

By letting $r \uparrow \infty$ we would conclude w to be constant, in contrast to (4.15). Analogously, if $\overline{S_w}$ would be a line, w would be harmonic in $\mathbb{R}^2 \setminus \overline{S_w}$. Considering the restriction of w to one of the two half-spaces forming $\mathbb{R}^2 \setminus \overline{S_w}$ and performing an even reflection, since $\frac{\partial w}{\partial \nu} = 0$ on $\overline{S_w}$ we would get an harmonic function $\widetilde{w}$ on $\mathbb{R}^2$ satisfying for all $r > 0$

$$\int_{B_r} |\nabla \widetilde{w}|^2 dx \leq 4\pi r.$$

Arguing as before $\widetilde{w}$ would be constant. Hence, w would be locally constant on $\mathbb{R}^2 \setminus \overline{S_w}$, leading again to a contradiction to (4.15). Therefore, $\overline{S_w}$ is a half-line, that up to a rotation can be written as $\overline{S_w} = \{(x, 0) : x \leq 0\}$. Then the map $\widetilde{w} : \{z \in \mathbb{C} : \mathrm{Re} z \geq 0\} \to \mathbb{R}$ defined by $\widetilde{w}(z) := w(z^2)$ is harmonic, $\frac{\partial \widetilde{w}}{\partial \nu} = 0$ on $\{\mathrm{Re} z = 0\}$ and it satisfies

$$\int_0^r \int_{-\pi/2}^{\pi/2} \Big(\rho \Big|\frac{\partial \widetilde{w}}{\partial r}\Big|^2 + \frac{1}{\rho}\Big|\frac{\partial \widetilde{w}}{\partial \theta}\Big|^2\Big) d\rho\, d\theta = \int_{B_{r^2}} |\nabla w|^2 \leq 2\pi r^2. \tag{4.16}$$

In view of (4.16), the even extension of $\widetilde{w}$ on B_1, that we still denote by $\widetilde{w}$, has Fourier decomposition

$$\widetilde{w}(r, \theta) = \alpha_0 + \beta_1\, r\, \sin\theta.$$

Changing back coordinates and taking into account the minimality of w we get

$$w(r\cos\theta, r\sin\theta) = \alpha_0 \pm \sqrt{\frac{2}{\pi} r}\, \sin\frac{\theta}{2},$$

the conclusion follows at once.

The reverse implications in both cases are easily concluded. Indeed, in case (i) if $(\rho_k)_k$ satisfies

$$\lim_k \mathscr{D}_u(x,\rho_k) = \limsup_{\rho\downarrow 0} \mathscr{D}_u(x,\rho),$$

then the blow up limit w of $(u_{x,\rho_k})_k$ is a triple junction function by assumption. By taking into account item (c) above we infer for $\mathcal{L}^1$ a.e. $r>0$

$$\lim_k \mathscr{D}_u(x, r\rho_k) = \lim_k \mathscr{D}_{u_{x,\rho_k}}(0,r) = \mathscr{D}_w(0,r) = 0,$$

thus implying that $x \in \Sigma_u^{(1)}$.

Similarly, one can prove the reverse implication in case (ii). □

Actually, in the first instance of Proposition 4.11 uniqueness of the blow up limit is ensured by Theorem 3.4. As already remarked, case (ii) is still open.

5. Higher integrability of the gradient in dimension 2

The higher integrability of the gradient has been first established by De Lellis and Focardi [35] in dimension 2. Following a classical path, the key ingredient to establish Theorem 3.9 is a reverse Hölder inequality for the gradient, which we state independently (see [35, Theorem 1.3]).

Theorem 5.1 (De Lellis and Focardi [35]). *For all $q \in (1,2)$ there exist $\rho \in (0,1)$ and $C>0$ such that*

$$\|\nabla u\|_{L^2(B_\rho)} \le C \|\nabla u\|_{L^q(B_1)} \qquad \text{for any } u \in \mathcal{M}(B_1). \tag{5.1}$$

Using the obvious scaling invariance of (2.3), Theorem 5.1 yields a corresponding reverse Hölder inequality for balls of arbitrary radius. Theorem 3.9 is then a consequence of a by now classical result.

Theorem 5.2 (Giaquinta and Modica [49]). *Let $v \in L^q_{loc}(\Omega)$, $q>1$, be nonnegative such that for some constants $\beta>0$, $\lambda \ge 1$ and $R_0>0$*

$$\left(\fint_{B_r(z)} v^q\,dy\right)^{1/q} \le \beta \fint_{B_{\lambda r}(z)} v\,dy$$

for all $z\in\Omega$, $r \in \big(0, R_0 \wedge \mathrm{dist}(z,\partial\Omega)\big)$.

Then $v \in L^p_{loc}(\Omega)$ for some $p>q$ and there is $C = C(\beta,n,q,p,\lambda)>0$ such that

$$\left(\fint_{B_r(z)} v^p\,dy\right)^{1/p} \le C\left(\fint_{B_{2r}(z)} v^q\,dy\right)^{1/q}.$$

The exponent p could be explicitly estimated in terms of q, C and ρ. However, since our argument for Theorem 5.1 is indirect, we do not have any explicit estimate for C (ρ can instead be computed). Hence, combining Theorem 3.9 with [4] we can only conclude that the dimension of the singular set of $\overline{S_u}$ is strictly smaller than 1. Guy David pointed out that the corresponding dimension estimate could be made explicit. In fact, he suggested to the Authors of [35] that also the constant C in Theorem 5.1 might be estimated: a viable strategy would combine the core argument of this paper with some ideas from [26] (the proof of Theorem 5.1 given here makes already a fundamental use of the paper [26], but depends only on the ε-regularity theorems for "triple junctions" and "segments" stated in Section 3). However, the resulting estimate would give an extremely small number, whereas the proof would very likely become much more complicated.

In spite of the dimensional restriction, the indirect proof has as interesting side results Theorem 4.3 and its related consequences highlighted in Section 4. No dimensional limitation is present in Theorem 4.3, instead dimension 2 enters dramatically in the proof of Theorem 5.1 as the structure of minimal Caccioppoli partitions in $\mathbb{R}^2$ can be described precisely via minimal connections. Recall that a *minimal connection* of $\{q_1,\dots,q_N\}\subset\mathbb{R}^2$ is any minimizer of the Steiner problem

$$\min\left\{\mathcal{H}^1(\Gamma):\,\Gamma\text{ closed and connected, and }q_1,\dots,q_N\in\Gamma\right\}.$$

Proposition 5.3 (Proposition 11, Lemma 12 [35]). *Let $\mathscr{E}$ be a minimal Caccioppoli partition in $\Omega\subset\mathbb{R}^2$, then*

(i) $\mathcal{H}^0(\overline{J_{\mathscr{E}}}\cap\partial B_\rho(x))<+\infty$ *if* $B_\rho(x)\subset\subset\Omega$*;*

(ii) $\mathcal{H}^0(K\cap\partial B_\rho(x))\geq 2$ *for each connected component K of $\overline{J_{\mathscr{E}}}\cap\overline{B}_\rho(x)$, and it is a minimal connection of $K\cap\partial B_\rho(x)$;*

(iii) *if $\Omega=B_1$, then there exists $\rho_0\in(0,1)$ such that for all $t\in(0,\rho_0)$*

$$\mathcal{H}^0(\overline{J_{\mathscr{E}}}\cap\partial B_t)\leq 3,\quad\textit{and}\quad\mathcal{H}^1(\overline{J_{\mathscr{E}}}\cap B_t)\leq 3t.$$

We are now ready to sketch the proof of Theorem 5.1 in 2-dimensions following De Lellis and Focardi [35].

Proof of Theorem 5.1. We fix an exponent $q\in(1,2)$ and a suitable radius ρ (whose choice will be specified later) for which (5.1) is false, that is

$$\|\nabla u_k\|_{L^2(B_\rho)}\geq k\|\nabla u_k\|_{L^q(B_1)}\quad\text{for a sequence }(u_k)_{k\in\mathbb{N}}\in\mathcal{M}(B_1).\tag{5.2}$$

Since the Mumford and Shah energy of any $u\in\mathcal{M}(B_1)$ can be easily bounded a priori by 2π (cf. Proposition 2.14), we have $\|\nabla u_k\|_{L^q(B_1)}\to 0$. Theorem 4.3 and Proposition 5.3 then show that:

(i) The L^2 energy of the gradients of u_k converge to 0;

(ii) $\overline{S_{u_k}}$ converge in the local Hausdorff metric to the (closure of) set of interfaces of a minimal Caccioppoli partition $\overline{J_{\mathscr{E}}}$;

(iii) $\mathcal{H}^0(\overline{J_{\mathscr{E}}} \cap \partial B_t) \leq 3$ for $t \in (0, \rho_0)$.

An elementary argument shows the existence of $t \geq \rho_0/4$ such that

(a) either $\overline{J_{\mathscr{E}}} \cap B_t = \emptyset$;

(b) or $\overline{J_{\mathscr{E}}} \cap B_t$ is a segment and $\partial B_t \setminus \overline{J_{\mathscr{E}}}$ is the union of two arcs each with length $< \frac{4\pi}{3} t$;

(c) or $\overline{J_{\mathscr{E}}} \cap B_t$ is a triple junction and $\partial B_t \setminus \overline{J_{\mathscr{E}}}$ the union of three arcs each with length $< (2\pi - \frac{1}{8})t$.

In any case we set $\rho := \rho_0/9$ (cf. with (5.2)). By Theorem 3.4 (we keep the notation introduced there), we may find a constant $\beta \in (0, 1/3)$ such that for all k sufficiently big one of the following alternatives happens

(a$_1$) $\overline{S_{u_k}} \cap B_t = \emptyset$;

(b$_1$) For each $s \in ((1-\beta)t, t)$, $\partial B_s \setminus \overline{S_{u_k}}$ is the union of two arcs γ_1^k and γ_2^k each with length $< (2\pi - \frac{1}{9})s$, whereas $\overline{S_{u_k}} \cap B_s$ is connected and divides B_s in two components B_1^k, B_2^k with $\partial B_i^k = \gamma_i^k \cup (\overline{S_{u_k}} \cap \overline{B_s})$;

(c$_1$) For each $s \in ((1-\beta)t, t)$, $\partial B_s \setminus \overline{S_{u_k}}$ is the union of three arcs γ_1^k, γ_2^k and γ_3^k each with length $< (2\pi - \frac{1}{9})s$, whereas $\overline{S_{u_k}} \cap B_s$ is connected and divides B_s in three connected components B_1^k, B_2^k and B_3^k with $\partial B_i^k \subset \gamma_i^k \cup (\overline{S_{u_k}} \cap \overline{B_s})$.

Choose then $r \in (2/3 t, t)$ and a subsequence (not relabeled) such that

$$g_k := u_k|_{\partial B_r} \in W^{1,q}(\gamma, \mathcal{H}^1) \qquad \text{for any connected component of } \partial B_r \setminus S_{u_k}$$

and

$$\int_{\partial B_r \setminus S_{u_k}} |g_k'|^q d\mathcal{H}^1 \leq \frac{3}{t} \int_{B_t} |\nabla u_k|^q \, dx \leq \frac{12}{\rho_0} \int_{B_1} |\nabla u_k|^q \, dx.$$

Let us first deal with the (easier) case (a). By compactness, as $\overline{J_{\mathscr{E}}} \cap B_t = \emptyset$ then

$$\overline{S_{u_k}} \cap B_t = \emptyset \quad \text{for } k \gg 1.$$

Hence, being $u_k \in \mathcal{M}(B_1)$ we get that u_k is the harmonic extension of its trace in B_t. In conclusion, as $\rho = \rho_0/9 < 2/3\, t < r$ we have

$$\int_{B_{\rho_0/9}} |\nabla u_k|^2 dx \leq \int_{B_r} |\nabla u_k|^2 dx \leq C \min_\lambda \|g_k - \lambda\|^2_{H^{1/2}(\partial B_r)}$$
$$\overset{W^{1,q} \hookrightarrow H^{1/2}}{\leq} C \left(\int_{\partial B_r} |g_k'|^q \, d\mathcal{H}^1 \right)^{2/q} \leq C \left(\frac{12}{\rho_0} \int_{B_1} |\nabla u_k|^q \, dx \right)^{2/q},$$

for some $C > 0$ (independent of k), contradicting (5.2).

In case (b) or (c) hold the construction is similar. Denote by K_k the minimal connection relative to $\overline{S_{u_k}} \cap \partial B_r$. Then K_k splits $\overline{B_r}$ into two (case (b$_1$)) or three (case (c$_1$)) regions denoted by B^i_r. Let γ^i be the arc of ∂B_r contained in the boundary of B^i_r. Having all the arcs length uniformly bounded from below, it is easy to check that for all i we can find a function $w^i_k \in W^{1,2}(B_r)$ with boundary trace g_k and satisfying for some absolute constant $C > 0$

$$\int_{B_r} |\nabla w^i_k|^2 \, dx \le C \left(\int_{\gamma^i} |g'_k|^q \, d\mathcal{H}^1 \right)^{2/q} \tag{5.3}$$

(cf. [35, Lemma 7]). Denote by w_k the function equal to w^i_k on B^i_k, then $w_k \in SBV(B_r)$ and $S_{w_k} \subseteq K_k$. The minimality of u_k implies then that

$$\int_{B_{\rho_0/9}} |\nabla u_k|^2 \le \int_{B_r} |\nabla u_k|^2 \le \int_{B_r} |\nabla w_k|^2 + \mathcal{H}^1(K_k) - \mathcal{H}^1(S_{u_k} \cap B_r)$$
$$\le \int_{B_r} |\nabla w_k|^2 \overset{(5.3)}{\le} C \left(\int_{\partial B_r \setminus S_{u_k}} |g'_k|^q \, d\mathcal{H}^1 \right)^{2/q} \le C \left(\frac{12}{\rho_0} \int_{B_1} |\nabla u_k|^q \, dx \right)^{2/q},$$

contradicting (5.2). □

6. Higher integrability of the gradient in any dimension: Porosity of the Jump set

A central role in establishing the higher integrability of the gradient in any dimension shall be played by the following improvement of Theorem 3.1.

Theorem 6.1 (Rigot [71], Maddalena and Solimini [55]). *There are dimensional constants $\varepsilon(n)$, $C_0(n) > 0$ such that for every $\varepsilon \in (0, \varepsilon(n))$ there exists $\alpha_\varepsilon \in (0, 1/2)$ such that if $u \in \mathcal{M}(B_2)$ and $B_\rho(x) \subset \Omega$, with $x \in \overline{S_u}$ and $\rho \in (0, 1)$, then there exists a ball $B_r(y) \subset B_\rho(x)$ with radius $r \in (\alpha_\varepsilon \rho, \rho)$ such that*

(i) *$\mathscr{D}_u(y, r) + \mathscr{A}_u(y, r) < \varepsilon_0$, $\varepsilon_0 > 0$ the constant in Theorem 3.1;*
(ii) *$\overline{S_u} \cap B_r(y)$ is a $C^{1,\gamma}$ graph, for all $\gamma \in (0, 1)$, containing y;*
(iii)
$$r \|\nabla u\|^2_{L^\infty(B_r(y))} \le C_0 \, \varepsilon. \tag{6.1}$$

We can restate the result above by saying that Σ_u is $(\alpha_\varepsilon, 1)$-*porous* in $\overline{S_u}$ according to the following definition.

Definition 6.2. *Given a metric space (X, d_X), a subset K is (α, δ)-*porous *in X, with $\alpha \in (0, 1/2)$ and $\delta > 0$, if for every $x \in X$ and $\rho \in (0, \delta)$ we can find $y \in B_\rho(x)$ and*

$r \in (\alpha\rho, \rho)$ *such that*

$$B_r(y) \subset B_\rho(x) \setminus K.$$

Clearly, in our case $X = \overline{S_u}$, $K = \Sigma_u$ and d_X is the metric induced by the Euclidean one. The Hausdorff dimension estimate in the papers by David [25], Rigot [71] and Maddalena and Solimini [55] follows from the porosity property in Theorem 6.1 and Theorem 6.4 below.

To this aim we recall that, given an Alfhors regular metric space (X, d_X) of dimension ℓ, i.e. (X, d_X) is complete and there is $\Lambda > 0$ such that

$$\Lambda^{-1} r^\ell \leq \mathcal{H}^\ell(B_r(z)) \leq \Lambda r^\ell \quad \text{for all } z \in X \text{ and } r > 0,$$

the *lower/upper Minkowski dimension* of K is defined as

$$\underline{\dim}_{\mathcal{M}} K := \inf\{s \in (0, \ell] : \mathcal{M}^s_*(K) = 0\}, \quad \overline{\dim}_{\mathcal{M}} K := \inf\{s \in (0, \ell] : \mathcal{M}^{*\,s}(K) = 0\},$$

where the *lower/upper Minkowski content* is given by

$$\mathcal{M}^s_*(K) := \liminf_{r \downarrow 0} \frac{\mathcal{H}^\ell\big((K)_r\big)}{r^{\ell - s}}, \qquad \mathcal{M}_{*\,s}(K) := \limsup_{r \downarrow 0} \frac{\mathcal{H}^\ell\big((K)_r\big)}{r^{\ell - s}}.$$

and

$$(K)_r := \{x \in X : d_X(x, K) < r\}. \tag{6.2}$$

Let us first relate the 2-dimensions introduced above.

Lemma 6.3. *For all sets $K \subset X$*

$$\dim_{\mathcal{H}} K \leq \dim_{\mathcal{M}} K.$$

Proof. We may assume $\dim_{\mathcal{M}} K < \ell$ since otherwise the inequality is trivial.

Let $N(K, r)$ be the minimal number of balls of radius r covering K, and $P(K, r)$ be the maximal number of disjoint balls with centers belonging to K, then

$$N(K, 2r) \leq P(K, r)^7.$$

Indeed, set $N := N(K, 2r)$, $P := P(K, r)$ and let $\mathscr{B} = \{B_r(x_i)\}_{i=1}^P$ be the corresponding maximal family of disjoint balls with $x_i \in K$. If there exists $x \in K \setminus \cup_{i=1}^P B_{2r}(x_i)$, then $\{B_r(x)\} \cup \mathscr{B}$ is a disjoint family of balls with centers on K, a contradiction.

Therefore, for all $s \in (\dim_{\mathcal{M}} K, \ell)$

$$N\omega_\ell r^\ell \leq P\omega_\ell r^\ell \leq \mathcal{H}^\ell\big((K)_r\big) \Longrightarrow N\omega_\ell r^s \leq \frac{\mathcal{H}^\ell\big((K)_r\big)}{r^{\ell - s}},$$

from which one easily conclude that $2^{-s} \frac{\omega_\ell}{\omega_s} \mathcal{H}^s(K) \leq \mathcal{M}^s_*(K) = 0.$ ∎

[7] One can also prove that $P(K, r) \leq N(K, r/2)$

We are now ready to state an estimate on the Hausdorff dimension of porous sets.

Theorem 6.4 (David and Semmes [28]). *If (X, d_X) is Alfhors regular of dimension ℓ, then every (α, δ)-porous subset K of X of diameter d satisfies*

$$\mathcal{H}^\ell\big((K)_r\big) \leq C\, r^{\ell-\eta} \quad \forall r \in (0, d),$$

for some constant $C = C(\ell, \delta, d) > 0$ and $\eta = \eta(\alpha, d, \Lambda) \in [0, \ell)$. Hence,

$$\dim_{\mathcal{H}} K \leq \dim_{\mathcal{M}} K \leq \eta.$$

The higher integrability property of the gradient for MS-minimizers will be (essentially) a consequence of the result above. Actually, we cannot take advantage directly of Theorem 6.4 since we are not able to relate $(\Sigma_u)_r$ and $\{|\nabla u|^2 \geq r^{-1}\}$. Therefore, we shall prove a suitable version of Theorem 6.4 and establish its links with the higher integrability property in the Section 7 following the approach by De Philippis and Figalli [37].

In passing, we mention that recently porosity has been employed in several instances to estimate the Hausdorff dimension of singular sets of solutions to variational problems (cf. [45], [48], [38]).

In the rest of the present section we shall comment on porosity in the more standard Euclidean setting, i.e. $X = \mathbb{R}^n$, and prove analogous results to those of interest for us. This is done to get more acquainted with porosity and it is intended as a warm up to the proof of Theorem 3.9 by De Philippis and Figalli.

Let then $K \subseteq \mathbb{R}^n$ be a (α, δ)-porous set. Few remarks are in order:

(i) By Lebesgue's differentiation theorem clearly $\mathcal{L}^n(K) = 0$;

(ii) K is nowhere dense, i.e. $\mathrm{int}\overline{K} = \emptyset$, since the latter is equivalent to: for every $x \in X$ and $\rho > 0$ there exists $y \in X$ and $r > 0$ such that $B_r(y) \subset B_\rho(x) \setminus K$.

(iii) Zajíček [77] actually proved that there are non-porous sets which are nowhere dense and with zero Lebesgue measure.

An elementary covering argument actually provides an estimate on the Hausdorff dimension of K and therefore improves item (i) above.

Proposition 6.5 (Salli [72]). *Suppose that K is a bounded (α, δ)-porous set in $\mathbb{R}^n$ with* $\mathrm{diam}K \leq d$*, then*

$$\mathcal{L}^n\big((K)_r\big) \leq C\, r^{n-\gamma} \qquad \forall r \in (0, d). \tag{6.3}$$

for some constants $C = C(n, \delta, d) > 0$ and $\gamma = \gamma(\alpha, n) < n$. In particular,

$$\dim_{\mathcal{H}} K \leq \dim_{\mathcal{M}} K \leq \gamma. \tag{6.4}$$

Proof. The building step of the argument goes as follow: consider a cube Q of $\mathrm{diam}Q < \delta$ and center x_Q, then by taking into account the (α,δ)-porosity of K, we find a point y_Q such that $B_{\alpha\,\mathrm{diam}Q/2\sqrt{n}}(y_Q) \subset B_{\mathrm{diam}Q/2\sqrt{n}}(x_Q) \setminus K$. If $\{Q_i\}_i$ is a covering of Q of k^n sub cubes with $\mathrm{diam}Q_i = \mathrm{diam}Q/k$, $k \in \mathbb{N}$, then at least one of those cubes does not intersect K if k is sufficiently big. Indeed, it suffices to impose

$$\alpha\,\mathrm{diam}Q > 2\frac{\mathrm{diam}Q}{k} \Longleftrightarrow \alpha\,k > 2.$$

Hence, we may choose $k = k(\alpha)$ for which the previous condition is satisfied.

Therefore, given a covering of K of $m = m(\delta, d)$ cubes with diameter $\delta/2$, we can construct another covering made of $m(k^n - 1) = mk^\gamma$ cubes of diameter $\delta/2k$, where $\gamma = \gamma(\alpha, n) \in (0, n)$ is such that $k^\gamma = k^n - 1$.

Clearly, we can iterate this procedure in each of the new cubes, so that for all $N \in \mathbb{N}$ we may find a covering of K made of $mk^{N\gamma}$ cubes $\{Q^k_i(x^k_i)\}_{i=1}^{mk^{N\gamma}}$ of diameter $\delta/2k^N$. In particular, each ball $B_{\delta/4k^N}(x^k_i)$ contains Q^k_i, and their union covers K. Thus,

$$(K)_{\delta/4k^N} \subset \cup_{i=1}^{mk^{N\gamma}} B_{\delta/2k^N}(x^k_i).$$

Hence

$$\mathcal{L}^n\left((K)_{\delta/4k^N}\right) \le \omega_n m k^{N\gamma}\left(\frac{\delta}{2k^N}\right)^n = o(1) \quad N \uparrow \infty.$$

Estimate (6.3) follows at once by a simple dyadic argument on the radii, i.e. given $r > 0$ choosing k such that $r \in [\frac{\delta}{2k^{N+1}}, \frac{\delta}{2k^N})$.

Instead, estimate (6.4) is an easy consequence of Lemma 6.3. □

Remark 6.6. *Theorem 6.4 can be proved exactly as Proposition 6.5 once the existence of a family of dyadic cubes in (X, d_X) has been established (cf. [28, Lemma 5.8]). By this, we mean a collection $\{\triangle_j\}_{\mathbb{Z}\ni j<j_0}$ of families of measurable subsets of X, $j_0 = \infty$ if* $\mathrm{diam}X = \infty$ *and otherwise $j_0 \in \mathbb{Z}$ such that $2^{j_0} \le \mathrm{diam}X < 2^{j_0+1}$, having the following properties:*

(i) *each $\triangle_j$ is a partition of X, i.e. $X = \cup_{Q\in\triangle_j} Q$ for any j as above;*
(ii) *$Q \cap Q' = \emptyset$ whenever Q, $Q' \in \triangle_j$ and $Q \neq Q'$;*
(iii) *if $Q \in \triangle_j$ and $Q' \in \triangle_k$ for $k \ge j$, then either $Q \subseteq Q'$ or $Q \cap Q' = \emptyset$;*
(iv) *$\lambda^{-1}\,2^j \le \mathrm{diam}Q \le \lambda\,2^j$ and $\lambda^{-1}2^{j\ell} \le \mathcal{H}^\ell(Q) \le \lambda 2^{j\ell}$ for all j and all $Q \in \triangle_j$;*
(v) *for all j and all $Q \in \triangle_j$, and $\tau > 0$*

$$\mathcal{H}^\ell\left(\{x \in Q : \mathrm{dist}(x, X\setminus Q) \le \tau 2^j\}\right) + \mathcal{H}^\ell\left(\{x \in X\setminus Q : \mathrm{dist}(x, Q) \le \tau 2^j\}\right) \le \lambda\tau^{1/\lambda}\mathcal{H}^\ell(Q).$$

with $\lambda = \lambda(\ell, \Lambda)$ (for the existence of such families see [24] and [73]).

With fixed a given porosity $\alpha \in (0, 1/2)$, we are then interested in analyzing the worst case, i.e.

$$D(\alpha, n) := \sup\{\dim_{\mathcal{H}} K : K \subset \mathbb{R}^n \text{ is } (\alpha, \delta)\text{-porous for some } \delta > 0\}.$$

We can easily deduce the estimate

$$n - 1 \leq D(\alpha, n) \leq \gamma(\alpha, n) < n,$$

as $(n-1)$-dimensional vector spaces are (α, δ)-porous for all $\delta > 0$ and $\alpha \in (0, 1/2)$. Furthermore, Mattila (see, for instance, [59, Theorem 11.14]) has shown that

$$\lim_{\alpha \uparrow 1/2} D(\alpha, n) = n - 1. \tag{6.5}$$

Remark 6.7. *Salli [72] has actually improved upon the previous result by showing that*

$$n - 1 + \frac{B(n)}{|\ln(1 - 2\alpha)|} \leq D(\alpha, n) \leq n - 1 + \frac{A(n)}{|\ln(1 - 2\alpha)|} \qquad \text{for all } \alpha \in (0, 1/2),$$

for some strictly positive dimensional constants A and B.

Finally, we note that the analogous property in (6.5) in the case of interest for us, if true, would then let us conclude another characterization of the conjectured estimate on the Hausdorff dimension of Σ_u: *If Σ_u is (α, δ)-porous in $\overline{S_u}$ for all $\alpha \in (0, 1/2)$ and some $\delta = \delta(\alpha) > 0$, then $\dim_{\mathcal{H}} \Sigma_u \leq n - 2$.*

7. Higher integrability of the gradient in any dimension: the proof

In this section we shall prove the higher integrability property of the gradient following De Philippis and Figalli [37]. We shall first establish in Proposition 7.3 below a particular case of Theorem 6.4 that is sufficient for our purposes.

To this aim we recall that the conclusions of Theorem 2.7 and Proposition 2.14 show the Alfhors regularity of $\Omega \cap \overline{S_u}$: for some constants $C_0 = C_0(n) > 0$, $\rho_0 = \rho_0(n) > 0$

$$C_0^{-1} r^{n-1} \leq \mathcal{H}^{n-1}(\overline{S_u} \cap B_r(z)) \leq C_0 r^{n-1} \tag{7.1}$$

for all $z \in \overline{S_u}$, and all $r \in (0, \rho_0 \wedge \mathrm{dist}(z, \partial\Omega))$, $u \in \mathcal{M}(B_2)$.

To prove Proposition 7.3 we need two technical lemmas. The first one is obtained via De Giorgi's slicing/averaging principle.

Lemma 7.1. *There are dimensional constants M_1, C_1 such that if $M \geq M_1$ for every $u \in \mathcal{M}(B_2)$ we can find three decreasing sequences of radii such that*

(i) $1 \geq R_h \geq S_h \geq T_h \geq R_{h+1}$;

(ii) $8M^{-(h+1)} \leq R_h - R_{h+1} \leq M^{-(h+1)/2}$, *and* $S_h - T_h = T_h - R_{h+1} = 4M^{-(h+1)}$;

(iii) $\mathcal{H}^{n-1}\big(\overline{S_u}\cap(\overline{B}_{S_h}\setminus\overline{B}_{R_{h+1}})\big)\leq C_1M^{-(h+1)/2}$;

(iv) $R_\infty=S_\infty=T_\infty\geq 1/2$.

Proof. Let $R_1=1$, given R_h we construct S_h, T_h and R_{h+1} as follows.

Set $N_h:=\lfloor M^{(h+1)/2}/8\rfloor\in\mathbb{N}$ and fix $M_1\in\mathbb{N}$ such that $N_h\geq\lfloor M^{(h+1)/2}/16\rfloor$ for $M\geq M_1$. Here, $\lfloor\alpha\rfloor$ denotes the integer part of $\alpha\in\mathbb{R}$.

The annulus $B_{R_h}\setminus\overline{B}_{R_h-8M^{-(h+1)/2}}$ contains the N_h disjoint sub annuli $\overline{B}_{R_h-8(i-1)M^{-(h+1)}}\setminus\overline{B}_{R_h-8iM^{-(h+1)}}$, $i\in\{1,\dots,N_h\}$, of equal width $8\,M^{-(h+1)}$. By averaging we can find an index $i_h\in\{1,\dots,N_h\}$ such that

$$\begin{aligned}&\mathcal{H}^{n-1}\big(K\cap(\overline{B}_{R_h-8(i_h-1)M^{-(h+1)/2}}\setminus\overline{B}_{R_h-8i_h\,M^{-(h+1)/2}})\big)\\&\qquad\leq\frac{1}{N_h}\,\mathcal{H}^{n-1}\big(K\cap(\overline{B}_{R_h}\setminus\overline{B}_{R_h-8M^{-(h+1)/2}})\big)\overset{\text{d.u.b. in (7.1)}}{\leq}C_0\,\frac{R_h^{n-1}}{N_h}\leq C_1\,M^{-(h+1)/2},\end{aligned}$$

so that (iii) is established. Finally, set

$$S_h:=R_h-8(i_h-1)M^{-(h+1)},\ R_{h+1}:=R_h-8i_h\,M^{-(h+1)},\ T_h:=\frac{1}{2}(S_h+R_{h+1}),$$

then items (i) and (ii) follow by the very definition, and item (iv) from (ii) if M_1 is sufficiently big. □

The second lemma has a geometric flavor.

Lemma 7.2. *Let $f:\mathbb{R}^{n-1}\to\mathbb{R}$ be Lipschitz with*

$$f(0)=0,\quad\|\nabla f\|_{L^\infty}\leq\eta.\tag{7.2}$$

If $G:=\mathrm{graph}(f)\cap B_2$ and $\eta\in(0,1/15]$, then for all $\delta\in(0,1/2)$ and $x\in(\overline{B}_{1+\delta}\setminus B_1)\cap G$

$$\mathrm{dist}(x,(\overline{B}_{1+2\delta}\setminus B_{1+\delta})\cap G)\leq\frac{3}{2}\delta.$$

Proof. Clearly by (7.2) we get

$$\|f\|_{W^{1,\infty}(B_2)}\leq 3\eta.$$

Let $x=(y,f(y))\in(\overline{B}_{1+\delta}\setminus B_1)\cap G$ and $\hat{x}:=(\lambda\,y,f(\lambda\,y))$, with λ to be chosen suitably. Note that as $|x|\geq 1$ we have

$$\begin{aligned}&|f(\lambda\,y)-\lambda\,f(y)|\leq|f(\lambda\,y)-f(y)|+|\lambda-1|\,|f(y)|\\&\qquad\qquad\leq|\lambda-1|\,\big(\|\nabla f\|_{L^\infty}\,|y|+\|f\|_{L^\infty}\big)\leq 3\eta|\lambda-1|\,|x|.\end{aligned}$$

Hence,

$$|\hat{x}-x|\leq|\hat{x}-\lambda\,x|+|\lambda-1||x|\leq\big(3\eta+1\big)|\lambda-1|\,|x|.$$

It is easy to check that the choice $\lambda=1+\frac{5}{4}\delta|x|^{-1}$ gives the conclusion. □

We are now ready to prove the version of Theorem 6.4 of interest for our purposes.

Proposition 7.3 (De Philippis and Figalli [37]). *Let C_0, C_1, M_1 be the constants in (7.1) and Lemma 7.1, respectively.*

There exist dimensional constants C_2, $M_2 > 0$ and $\alpha \in (0, 1/4)$, $\beta \in (0, 1/4)$, with $M_2 \geq M_1$, such that for every $M \geq M_2$, $u \in \mathcal{M}(B_2)$, we can find families $\mathcal{F}_j$ of disjoint balls

$$\mathcal{F}_j = \left\{B_{\alpha M^{-j}}(y_i) : y_i \in \overline{S_u},\, 1 \leq j \leq N_j\right\}$$

such that for all $h \in \mathbb{N}$

(i) *B, $B' \in \cup_{j=1}^h \mathcal{F}_j$ are distinct balls, then $(B)_{4M^{-(h+1)}} \cap (B')_{4M^{-(h+1)}} = \emptyset$;*

(ii) *if $B_{\alpha M^{-j}}(y_i) \subset \mathcal{F}_j$, then $\overline{S_u} \cap B_{2\alpha M^{-j}}(y_i)$ is a $C^{1,\gamma}$ graph, $\gamma \in (0,1)$ any, containing y_i,*

$$\mathscr{D}_u\big(y_i, 2\alpha M^{-j}\big) + \mathscr{A}_u\big(y_i, 2\alpha M^{-j}\big) < \varepsilon_0;$$
$$\|\nabla u\|_{L^\infty\left(B_{2\alpha M^{-j}}(y_i)\right)} < M^{j+1}; \tag{7.3}$$

(iii) *let $\{R_h\}$, $\{S_h\}$, $\{T_h\}$ be the sequences of radii in Lemma 7.1, and let*

$$K_h := (\overline{S_u} \cap \overline{B}_{S_h}) \setminus \left(\cup_{j=1}^h \cup_{\mathcal{F}_j} B\right),$$

(note that by construction $K_{h+1} \subset K_h \setminus \cup_{\mathcal{F}_{h+1}} B$), and

$$\widetilde{K_h} := (\overline{S_u} \cap \overline{B}_{T_h}) \setminus \left(\cup_{j=1}^h \cup_{\mathcal{F}_j} (B)_{2M^{-(h+1)}}\right) \subset K_h.$$

Then, there exists a finite set of points $\mathcal{C}_h := \{x_i\}_{i \in I_h} \subset \widetilde{K_h}$ such that

$$|x_j - x_k| \geq 3M^{-(h+1)} \quad \forall j, k \in I_h,\, j \neq k; \tag{7.4}$$

$$(K_h \cap \overline{B}_{R_{h+1}})_{M^{-(h+1)}} \subset \cup_{i \in I_h} B_{8M^{-(h+1)}}(x_i); \tag{7.5}$$

$$\mathcal{H}^{n-1}(K_h) \leq C_1\, h\, M^{-2h\beta}; \tag{7.6}$$

$$\mathcal{L}^n\left((K_h \cap \overline{B}_{R_{h+1}})_{M^{-(h+1)}}\right) \leq C_2\, h\, M^{-h(1+2\beta)-1}. \tag{7.7}$$

(iv) *$\Sigma_u \cap B_{1/2} \subset K_h$ for all $h \in \mathbb{N}$ and*

$$\mathcal{L}^n\big((\Sigma_u \cap B_{1/2})_r\big) \leq C_2\, r^{1+\beta} \qquad \forall r \in (0, 1/2]. \tag{7.8}$$

In particular, $\dim_{\mathcal{M}}(\Sigma_u \cap B_{1/2}) \leq n - 1 - \beta$.

Proof. For notational convenience we set $K = \overline{S_u}$. In what follows we shall repeatedly use Theorem 6.1 with $\varepsilon \in (0,1)$ fixed and sufficiently small.

We split the proof in several steps.

Step 1. *Inductive definition of the families $\mathcal{F}_j$.*

For $h = 1$ we define

$$\mathcal{F}_1 := \emptyset,\ K_1 = K \cap \overline{B}_{S_1},\ \widetilde{K_1} = K \cap \overline{B}_{T_1},$$

and choose $\mathcal{C}_1$ to be a maximal family of points at distance $3M^{-2}$ from each other. Of course, properties (i) and (ii) and (7.4) are satisfied. To check the others, one can argue as in the verification below.

Suppose that we have built the families $\{\mathcal{F}_j\}_{j=1}^h$ as in the statement, to construct $\mathcal{F}_{h+1}$ we argue as follows. Let $\mathcal{C}_h = \{x_i\}_{i\in I_h} \subset \widetilde{K_h}$ be a family of points satisfying (7.4), i.e. $|x_i - x_k| \geq 3M^{-(h+1)}$ for all $j,\, k \in I_h$ with $j \neq k$, and consider

$$\mathcal{G}_{h+1} := \{B_{M^{-(h+1)}}(x_i)\}_{i\in I_h}.$$

By the porosity assumption on K for every ball $B_{M^{-(h+1)}}(x_i) \in \mathcal{G}_{h+1}$ we can find a sub-ball $B_{2\alpha\, M^{-(h+1)}}(y_i) \subset B_{M^{-(h+1)}}(x_i) \setminus K$, $\alpha \in (0, 1/4)$ for which the theses of Theorem 6.1 are satisfied. Then, define

$$\mathcal{F}_{h+1} := \{B_{\alpha\, M^{-(h+1)}}(y_i)\}_{i\in I_h}.$$

By condition (7.4), the balls $B_{\frac{3}{2}M^{-(h+1)}}(x_i)$ are disjoint and do not intersect

$$\cup_{j=1}^h \cup_{\mathcal{F}_j} (B)_{\frac{1}{2}M^{-(h+1)}}$$

by the very definition of $\widetilde{K_h}$. Thus, item (i) and (ii) are satisfied.

Hence, we can define K_{h+1}, $\widetilde{K}_{h+1}$ and $\mathcal{C}_{h+1}$ as in the statement.

Step 2. *Proof of* (7.5).

Let $x \in (K_h \cap \overline{B}_{R_{h+1}})_{M^{-(h+2)}}$ and let z be a point of minimal distance from $K_h \cap \overline{B}_{R_{h+1}}$. In case $z \in \widetilde{K}_{h+1}$, by maximality there is $x_i \in \mathcal{C}_{h+1}$ such that $|z - x_i| \leq 3M^{-(h+2)}$ and thus we conclude $x \in B_{5M^{-(h+2)}}(x_i)$. Instead, if $z \in (K_h \cap \overline{B}_{R_{h+1}}) \setminus \widetilde{K}_{h+1}$, the definitions of K_{h+1} and $\widetilde{K}_{h+1}$ yield the existence of a ball $\widetilde{B} \in \cup_{j=1}^{h+1}\mathcal{F}_j$ for which $z \in (K \cap (\widetilde{B})_{2M^{-(h+2)}}) \setminus \widetilde{B}$. In view of property (ii), a rescaled version of Lemma 7.2 gives a point y satisfying

$$y \in \big(K \cap (\widetilde{B})_{4M^{-(h+2)}}\big) \setminus (\widetilde{B})_{2M^{-(h+2)}}, \quad \text{and} \quad |z - y| \leq 3M^{-(h+2)}.$$

Therefore, as $z \in \overline{B}_{R_{h+2}}$ and $T_{R_{h+1}} = R_{h+2} + 4M^{-(h+2)}$ we get by property (i) and the definition of $\widetilde{K}_{h+1}$

$$y \in \big(K \cap (\widetilde{B})_{4M^{-(h+2)}} \cap \widetilde{B})_{4M^{-(h+2)}}\big) \setminus (\widetilde{B})_{2M^{-(h+2)}}.$$

Finally, by maximality we may find $x_i \in \mathcal{C}_{h+1}$ such that $|y - x_i| \leq 3M^{-(h+2)}$. In conclusion, we have

$$|x - x_i| \leq |x - z| + |z - y| + |y - x_i| \leq 7M^{-(h+2)},$$

so that (7.5) follows at once.

Step 3. *the K_h's satisfy a suitable d.l.b. as that of K in (7.1).*

We claim that for every $h \in \mathbb{N}$

$$K_h \cap B_{M^{-(h+1)}}(x_i) = K \cap B_{M^{-(h+1)}}(x_i) \qquad \text{for all } x_i \in \mathcal{C}_h. \tag{7.9}$$

In particular, from the latter we infer the conclusion of this step.

The equality above is proven by contradiction: assume we can find $x_i \in \mathcal{C}_h$ and

$$x \in (K \setminus K_h) \cap B_{M^{-(h+1)}}(x_i).$$

As $x_i \in \widetilde{K}_h$ then $x_i \in B_{T_h}$, in turn implying $x \in B_{S_h}$ since $S_h - T_h = 4M^{-(h+1)}$. Therefore $x \in \big(K \setminus K_h\big) \cap \overline{B}_{S_h}$, and by definition of K_h we can find a ball $B \in \mathcal{F}_j$, $j \le h$, such that $x \in B$. We conclude that

$$\mathrm{dist}(x_i, B) \le |x - x_i| \le M^{-(h+1)},$$

contradicting that $x_i \in \widetilde{K}_h$.

Step 4. *Proof of* (7.6)*.*

We get first a lower bound for $\#(I_h)$: use (7.5) and the d.u.b in (7.1) to get

$$\mathcal{H}^{n-1}\big(K_h \cap \overline{B}_{R_{h+1}}\big) = \mathcal{H}^{n-1}\big(K_h \cap \overline{B}_{R_{h+1}} \cap \cup_{i\in I_h} B_{8M^{-(h+1)}}(x_i)\big) \le C_0\, \#(I_h)\big(8M^{-(h+1)}\big)^{n-1}$$

that is

$$\#(I_h) M^{-(h+1)(n-1)} \ge 8^{1-n} C_0^{-1} \mathcal{H}^{n-1}\big(K_h \cap \overline{B}_{R_{h+1}}\big). \tag{7.10}$$

Thus, we estimate as follows

$$\begin{aligned}
\mathcal{H}^{n-1}(K_{h+1}) &\le \mathcal{H}^{n-1}\left(K_h \setminus \cup_{\mathcal{F}_{h+1}} B\right) \overset{\text{disjoint balls}}{=} \mathcal{H}^{n-1}(K_h) - \sum_{\mathcal{F}_{h+1}} \mathcal{H}^{n-1}(K_h \cap B) \\
&\overset{\text{d.l.b. in (7.1), (7.9)}}{\le} \mathcal{H}^{n-1}(K_h) - \frac{\alpha^{n-1}}{C_0}\#(I_h) M^{-(h+1)(n-1)} \overset{(7.10)}{\le} \mathcal{H}^{n-1}(K_h) - \frac{8^{1-n}\alpha^{n-1}}{C_0^2}\mathcal{H}^{n-1}\big(K_h \cap \overline{B}_{R_{h+1}}\big) \\
&= (1-\eta)\mathcal{H}^{n-1}(K_h) + \eta\left(\mathcal{H}^{n-1}(K_h) - \mathcal{H}^{n-1}\big(K_h \cap \overline{B}_{R_{h+1}}\big)\right) \\
&\overset{\text{def. of } K_h}{\le} (1-\eta)\mathcal{H}^{n-1}(K_h) + \eta\, \mathcal{H}^{n-1}\big(K \cap (\overline{B}_{S_h} \setminus \overline{B}_{R_{h+1}})\big) \overset{\text{(iii) Lemma 7.1}}{\le} (1-\eta)\mathcal{H}^{n-1}(K_h) + C_1 M^{-\frac{h+1}{2}},
\end{aligned} \tag{7.11}$$

where we have set $\eta := {}^{8^{1-n}\alpha^{n-1}}/{C_0^2}$.

By iteration of (7.11), we find by Young inequality

$$\mathcal{H}^{n-1}(K_h) \le C_1 \sum_{i=0}^{h} (1-\eta)^{h-i} M^{-i/2} \le C_1\, h\, \max\{(1-\eta)^h, M^{-h/2}\}.$$

Choose $\beta \in (0, 1/4)$ such that $(1-\eta) \le M^{-2\beta}$, the previous estimate then yields (7.6),

$$\mathcal{H}^{n-1}(K_h) \le C_1\, h\, \max\{M^{-2h\beta}, M^{-h/2}\} = C_1\, h\, M^{-2h\beta}.$$

Step 5. *Proof of* (7.7).

Then, we exploit (7.5) to get

$$\begin{aligned}
\mathcal{L}^n\left((K_{h+1}\cap\overline{B}_{R_{h+2}})_{M^{-(h+2)}}\right) &\le \mathcal{L}^n\left(\cup_{i\in I_{h+1}}B_{8M^{-(h+2)}}(x_i)\right)\le \#(I_{h+1})\big(8M^{-(h+2)}\big)^n\\
&\overset{\text{d.l.b. in (7.1), (7.9)}}{\le} \frac{8^n}{C_0}M^{-(h+2)}\sum_{i\in I_{h+1}}\mathcal{H}^{n-1}\big(K_{h+1}\cap B_{M^{-(h+2)}}(x_i)\big)\\
&\overset{\text{disjoint balls}}{\le} \frac{8^n}{C_0}M^{-(h+2)}\mathcal{H}^{n-1}\big(K_{h+1}\big)\overset{(7.6)}{\le}\frac{8^n\,C_1}{C_0}\,(h+1)\,M^{-2(h+1)\beta-(h+2)}. \qquad (7.12)
\end{aligned}$$

Step 6. *Proof of* (7.8)

By construction we have that $\Sigma_u\cap B_{1/2}\subseteq K_h$. Therefore, (7.7) gives as $R_h\ge R_\infty\ge 1/2$

$$\mathcal{L}^n\big((\Sigma_u\cap B_{1/2})_{M^{-(h+1)}}\big)\le\mathcal{L}^n\left((K_h\cap\overline{B}_{R_{h+1}})_{M^{-(h+1)}}\right)\le C_2\,h\,M^{-h(1+2\beta)-1}.$$

Hence, if $r\in(M^{-(h+2)},M^{-(h+1)}]$ we get

$$\mathcal{L}^n\big((\Sigma_u\cap B_{1/2})_r\big)\le C_2\,h\,M^{-h(1+2\beta)-1}\le C_2\,M^{-h(1+\beta)-1}\le C_2\,r^{1+\beta}.$$

□

Remark 7.4. *Apart from Step 2, all the arguments in the other steps of Proposition 7.3 employ only the Alfhors regularity of $\Omega\cap\overline{S_u}$ and its consequence Lemma 7.1.*

In addition, note that one can easily infer the (more) intrinsic estimates

$$\mathcal{H}^{n-1}\left(\overline{S_u}\cap(K_h\cap\overline{B}_{R_{h+1}})_{M^{-(h+1)}}\right)\le C_2\,h\,M^{-2h\beta},$$

and

$$\mathcal{H}^{n-1}\big(\overline{S_u}\cap(\Sigma_u\cap B_{1/2})_r\big)\le C_2\,r^\beta \qquad \forall r\in(0,1/2].$$

Indeed, by arguing as in (7.12) *we get*

$$\begin{aligned}
\mathcal{H}^{n-1}\left(\overline{S_u}\cap(K_{h+1}\cap\overline{B}_{R_{h+2}})_{M^{-(h+2)}}\right)&\le\mathcal{H}^{n-1}\left(\overline{S_u}\cap\cup_{i\in I_{h+1}}B_{8M^{-(h+2)}}(x_i)\right)\\
\le\#(I_{h+1})\big(8M^{-(h+2)}\big)^{n-1}&\overset{\textit{d.l.b. in (7.1), (7.9)}}{\le}\frac{8^{n-1}}{C_0}\sum_{i\in I_{h+1}}\mathcal{H}^{n-1}\big(K_{h+1}\cap B_{M^{-(h+2)}}(x_i)\big)\\
&\overset{\textit{disjoint balls}}{\le}\frac{8^{n-1}}{C_0}\mathcal{H}^{n-1}\big(K_{h+1}\big)\overset{(7.6)}{\le}\frac{8^{n-1}\,C_1}{C_0}\,(h+1)\,M^{-2(h+1)\beta}.
\end{aligned}$$

As outlined in Section 6 the former result leads to the higher integrability of the gradient for MS-minimizers in any dimension.

Theorem 7.5 (De Philippis and Figalli [37]). *There is $p>2$ such that $\nabla u\in L^p_{\rm loc}(\Omega)$ for all $u\in\mathcal{M}(\Omega)$ and for all open sets $\Omega\subseteq\mathbb{R}^n$.*

Proof. Clearly, it is sufficient for our purposes to prove a localized estimate. Hence, for the sake of simplicity we suppose that $\Omega = B_2$.

We keep the notation of Proposition 7.3 and furthermore denote for all $h \in \mathbb{N}$

$$A_h := \big\{x \in B_2 \setminus K : |\nabla u(x)|^2 > M^{h+1}\big\}. \tag{7.13}$$

We claim that

$$A_{h+2} \cap B_{R_{h+2}} \subset (K_h \cap B_{R_{h+1}})_{M^{-(h+1)}}. \tag{7.14}$$

Given this for granted we conclude as follows: we use (7.7) to deduce that

$$\mathcal{L}^n(A_{h+2} \cap B_{R_{h+2}}) \le \mathcal{L}^n\big((K_h \cap B_{R_{h+1}})_{M^{-(h+1)}}\big) \le C_2\, h\, M^{-h(1+2\beta)-1}. \tag{7.15}$$

Therefore, recalling that ${}^1\!/\!{}_2 \le R_\infty \le R_h$, in view of (7.15) and Cavalieri's formula for $q > 1$ we get that

$$\int_{B_{1/2}} |\nabla u|^{2q} dx = q \int_0^\infty t^{q-1} \mathcal{L}^n\big(\big\{x \in B_{1/2} \setminus K : |\nabla u(x)|^2 > t\big\}\big) dt$$

$$\le q \sum_{h \ge 3} \int_{M^h}^{M^{h+1}} t^{q-1} \mathcal{L}^n\big(\big\{x \in B_{1/2} \setminus K : |\nabla u(x)|^2 > t\big\}\big) dt + M^{3q} \mathcal{L}^n(B_{1/2})$$

$$\le \sum_{h \ge 0} M^{(h+4)q} \mathcal{L}^n\big(A_{h+2} \cap B_{1/2}\big) + M^{3q} \mathcal{L}^n(B_{1/2}) \le C_2 \sum_{h \ge 0} h\, M^{(h+4)q - h(1+2\beta)-1} + M^{3q} \mathcal{L}^n(B_{1/2}).$$

The conclusion follows at once by taking $q \in (1, 1+2\beta)$ and $p = 2q$.

Let us now prove formula (7.14) in two steps.

Step 1. *For all $M > n$ and $R \in (0,1]$ we have that*

$$A_h \cap B_{R-2M^{-h}} \subset \big(K \cap B_R\big)_{M^{-h}} \qquad \text{for all } h \in \mathbb{N}. \tag{7.16}$$

Indeed, for $x \in A_h \cap B_{R-2M^{-h}}$ let $z \in K$ be such that $\mathrm{dist}(x,K) = |x-z|$. If $|x-z| > M^{-h}$ then $B_{M^{-h}}(x) \cap K = \emptyset$ so that u is harmonic on $B_{M^{-h}}(x)$. Therefore, by subharmonicity of $|\nabla u|^2$ on the same set and the d.u.b. in (7.1) we infer that

$$M^{h+1} \overset{x \in A_h}{\le} |\nabla u(x)|^2 \le \fint_{B_{M^{-h}}(x)} |\nabla u|^2 \le n\, M^h,$$

that is clearly impossible for $M > n$.

Finally, as $x \in B_{R-2M^{-h}}$ and $|x-z| \le M^{-h}$ we conclude that $z \in B_R$.

Step 2. *Proof of* (7.14).

Since $R_{h+1} - R_{h+2} \ge 8M^{-(h+2)}$ (cf. (i) Lemma 7.1), we apply Step 1 to A_{h+2} and $R = R_{h+1}$ and then (7.16) implies that

$$A_{h+2} \cap B_{R_{h+2}} \subset \big(K \cap B_{R_{h+1}}\big)_{M^{-(h+1)}}.$$

Let $x \in A_{h+2} \cap B_{R_{h+2}}$, $z \in K \cap B_{R_{h+1}}$ be a point of minimal distance, and suppose that $z \in K \setminus K_h$.

Since $R_{h+1} \leq S_h$, by the very definition of K_h we find a ball $B \in \cup_{j=1}^{h} \mathcal{F}_j$ such that $z \in B$. Since $B = B_t(y)$ for some y and with the radius $t \geq \alpha\, M^{-h}$, then $x \in B_{2t}(y)$ as $|x - z| \leq M^{-(h+1)}$ for M sufficiently large. Thus, estimate $|\nabla u(x)|^2 < M^{h+1}$ follows from (7.3) in item (ii) of Proposition 7.3. This is in contradiction with $x \in A_{h+2}$. $\square$

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