Nonautonomous Chain Rules in BV with Lipschitz Dependence

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Abstract. The aim of this paper is to state a nonautonomous chain rule in BV with Lipschitz dependence, i.e., a formula for the distributional derivative of the composite function $v(x) = B(x, u(x))$, where $u : \mathbb{R}^N \to \mathbb{R}$ is a scalar function of bounded variation, $B(\cdot, t)$ has bounded variation and $B(x, \cdot)$ is only a Lipschitz continuous function. We present a survey of recent developments on the nonautonomous chain rules in BV. Formulas of this type are an useful tool especially in view to applications to lower semicontinuity for integral functional (see [12, 14, 15, 16]) and to the conservation laws with discontinuous flux (see [8, 10, 11]).

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1. Introduction

In this paper we recall some recent theorems on the nonautonomous chain rule formulas in BV and we prove that, in the scalar case, the nonautonomous chain rule formula for the distributional derivative of the composite function $v(x) = B(x, u(x))$ with $B(\cdot, t)$ and u of bounded variation, proved in [4], holds also by assuming a Lipschitz continuity of $B(x, \cdot)$, instead of a C^1 dependence.

In order to illustrate our formula, we begin with the classical autonomous case $B(x, t) = B(t)$. In the pioneering [25] Vol'pert (see also [26]), in view of applications in the study of quasilinear hyperbolic equations, established a (autonomous) chain rule formula for distributional derivatives of the composite function $v(x) = B(u(x))$, where $u : \Omega \to \mathbb{R}$ has bounded variation in the open subset Ω of $\mathbb{R}^N$ and $B : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. He proved that v has bounded variation and its distributional derivative Dv (which is a Radon measure on Ω) admits an explicit representation in terms of the classical derivative $\partial_t B$ and of the distributional derivative

Du . More precisely, the following equality holds:

$$\begin{aligned} Dv &= (\partial_t B)(u)\nabla u\, \mathcal{L}^N \\ &\quad + (\partial_t B)(\widetilde{u})D^c u + [B(u^+) - B(u^-)]\, \nu_u\, \mathcal{H}^{N-1} \llcorner J_u \,, \end{aligned} \tag{1.1}$$

in the sense of measures, where

$$Du = \nabla u\, \mathcal{L}^N + D^c u + \nu_u\, \mathcal{H}^{N-1} \llcorner J_u$$

is the decomposition of Du in its absolutely continuous part ∇u with respect to the Lebesgue measure $\mathcal{L}^N$, its Cantor part $D^c u$ and its jumping part, which is represented by the restriction of the $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}$ to the jump set J_u . Here, ν_u denotes the measure theoretical unit normal to J_u, $\widetilde{u}$ is the approximate limit and u^+, u^- are the approximate limits from both sides of J_u .

The identity (1.1) holds also in the vectorial case (see Theorem 3.96 in [6]), namely if $u : \mathbb{R}^N \to \mathbb{R}^d$ has bounded variation and $B : \mathbb{R}^d \to \mathbb{R}$ is continuously differentiable.

When B is only a Lipschitz continuous function, this vectorial chain rule is false and a general form of the formula was proved by Ambrosio and Dal Maso in [5] (see also [19]). See Theorem 3.99 in [6] for the scalar case with Lipschitz dependence of B.

In the last years, analogous chain rule formulas are obtained, by admitting an explicit dependence on the space variable x , in view to applications to semicontinuity results for integral convex nonautonomous functionals (see [1, 2, 3, 12, 14, 15, 16]) and to conservation laws with discontinuous flux (see [8, 10, 11]). These formulas describe the distributional derivative of the composite function $v(x) = B(x, u(x))$, where $B(x, \cdot)$ is continuously differentiable and, for every $t \in \mathbb{R}^d$, $B(\cdot, t)$ and u are $W^{1,1}$ and BV functions.

These formulas admit another derivation term due to the presence of the explicit dependence on x and in the case of u and B regular functions, the following pointwise identity holds:

$$\nabla v(x) = (\nabla_x B)(x, u(x)) + (\partial_t B)(x, u(x)) \cdot \nabla u(x)\,, \quad x \in \mathbb{R}^N \,,$$

where all the derivatives are the classical ones.

The first formula of this type is established in [16] for functions $u \in W^{1,1}(\mathbb{R}^N; \mathbb{R}^d)$ by assuming that, for every $t \in \mathbb{R}^d$, $B(\cdot, t)$ is an L^1 function whose distributional divergence belongs to L^1 (in particular it holds if $B(\cdot, t) \in W^{1,1}(\mathbb{R}^N; \mathbb{R}^d)$).

The case of a scalar function $u \in BV(\mathbb{R}^N)$ is studied in the papers [14, 15], where it is considered the particular case

$$B(x, t) = \int_0^t b(x, s) ds \,. \tag{1.2}$$

In [14] the authors have established the validity of the chain rule by requiring a $W^{1,1}$ dependence with respect to the variable x, while in [15] it is assumed only a BV dependence with respect to the variable x (see Theorem 3.1 below) . Moreover in [15] the formula is proved also by assuming that, for every $t \in \mathbb{R}$, $b(\cdot, t)$ is an

L^1 function whose distributional divergence is a Radon measure with bounded total variation and $u \in W^{1,1}(\mathbb{R}^N)$ (the extension of this result to the case $u \in BV(\mathbb{R}^N)$ is studied in [9]). We remark that in these results b is an L^∞ function, hence its integral $B(x,\cdot)$ is a Lipschitz function.

The main difficulty of these results consists in giving sense to the different terms of the formula. Notice that the new term of derivation with respect to x needs a particular attention.

More recently, a very general formula is proven in [4] (see also [8] for $N=1$) for vector functions $u \in BV(\mathbb{R}^N, \mathbb{R}^d)$. In the particular scalar case $d=1$ the setting is the following (see Theorem 4.1 below). The first assumption is a C^1 dependence of $B(x,\cdot)$ with an uniform bound on $(\partial_t B)(x,t)$. Concerning the x-derivative, it is required the existence of a Radon measure σ bounding from above all measures $|D_x B(\cdot,t)|$, uniformly with respect to $t \in \mathbb{R}$ (see assumption (H4) below). With these two bounds it is proved that for any $u \in BV_{\rm loc}$ the composite function $v(x) = B(x,u(x))$ belongs to $BV_{\rm loc}$ and it is shown the existence of a countably $\mathcal{H}^{N-1}$-rectifiable set $\mathcal{N}$, independent of u and containing the jump set of $B(\cdot,t)$ for every $t \in \mathbb{R}$, such that the jump set of v is contained in $\mathcal{N} \cup J_u$. On the other hand, in order to prove the validity of the chain rule formula it is required that $B(x,t)$ satisfies other structural assumptions related to the uniform continuous dependence of the classical derivative $\partial_t B$ and of the diffuse part $\widetilde{D}_x B$ of the measure $D_x B$ with respect to t (see assumptions (H2) and (H3) below).

The aim of this paper is to establish a formula of this type with Lipschitz dependence with respect to the second variable. Since the problem is local, for simplicity we assume $\Omega = \mathbb{R}^N$.

We will obtain the nonautonomous chain rule formula in BV for the distributional derivative of the composite function $v(x) = B(x,u(x))$, where $B(\cdot,t)$ has bounded variation and $B(x,\cdot)$ is Lipschitz continuous and differentiable in $\mathbb{R} \setminus \mathcal{M}_0$ ($\mathcal{M}_0$ independent of x with $\mathcal{L}^1(\mathcal{M}_0) = 0$). In the spirit of [4] we require the existence of a Radon measure σ bounding from above all measures $|D_x B(\cdot,t)|$, uniformly with respect to $t \in \mathbb{R}$. We will prove for any $u \in BV_{\rm loc}$ the composite function $v(x) = B(x,u(x))$ belongs to $BV_{\rm loc}$ and there exists of a countably $\mathcal{H}^{N-1}$-rectifiable set $\mathcal{N}$, independent of t and containing the jump set of $B(\cdot,t)$ for every $t \in \mathbb{R}$, such that the jump set of v is contained in $\mathcal{N} \cup J_u$. Moreover the following chain rule holds (see Theorem 7.2 below for the precise statement) for the distributional derivative of v:

$$\begin{aligned} Dv(x) = {} & D_x B(x,t)|_{t=u(x)} + (\partial_t B)(x,u) \cdot \widetilde{D}u \\ & + [B^*(x,u^+) - B^*(x,u^-)]\nu_u \mathcal{H}^{N-1} \llcorner \mathcal{N} \cup J_u, \end{aligned} \tag{1.3}$$

in the sense of measure, where the measure $D_x B(x,t)$, depending on the parameter t, is computed in $t = u(x)$ in a suitable sense (see Remark 7.4).

The proof uses a regularization argument. We regularize $B(x,t)$ w.r.t. t and we use the chain rule proven in [4]. We need to study for every Borel set $E \subset \mathbb{R}^N$ the map $t \mapsto D_x B(\cdot,t)(E)$ and we will prove that it is Lipschitz continuous, $(\partial_t B)(\cdot,t)$

is a BV function and for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$

$$\partial_t(D_x B(\cdot,t)(E)) = D_x(\partial_t B)(\cdot,t)(E)\,.$$

The last part of the proof consists in analyzing carefully the convergence of all the terms in the formula written for the regularized function, those involving the various parts of the derivative of u and the one containing the derivatives of $B(x,t)$ with respect to x.

Let us conclude this Introduction by presenting the structure of the paper. In Section 2 below we list some definitions and basic fact of the BV functions. In Section 3 and 4 we recall the formulas proven in [15, 4]. Section 5 contains the setting and Section 6 some preliminary results in order to establish the main result (Theorem 7.2 in Section 7). Eventually Section 8 contains its proof.

2. Definitions and preliminaries

In this section we recall some preliminary results and basic definitions (see [6, 17, 18, 27]).

Let E be a measurable subset of $\mathbb{R}^N$. The *density* $D(E;x)$ of E at a point $x \in \mathbb{R}^N$ is defined by

$$D(E;x) = \lim_{\varrho\to 0} \frac{\mathcal{L}^N(E\cap B_\rho(x))}{\omega_N \rho^N},$$

where ω_N is the measure of the unit ball, whenever this limit exists. Hereafter, $B_\rho(x)$ denotes the ball centered at x with radius ρ. The *essential boundary* $\partial^M E$ of E is the Borel set defined as

$$\partial^M E = \mathbb{R}^N \setminus \{x \in \mathbb{R}^N :\, D(E;x) = 0 \ \text{ or } \ D(E;x) = 1\}\,.$$

We say that the set E is of *finite perimeter* in an open set Ω if $\mathcal{H}^{N-1}(\partial^M E\cap\Omega) < \infty$. Notice also that if $\Omega \subset \mathbb{R}^N$ is an open set, the quantity $\mathcal{H}^{N-1}(\partial^M E\cap\Omega)$ agrees with the classical *perimeter of E in* Ω (see [6, Theorem 3.61]).

Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $u : \Omega \to \mathbb{R}$ be a measurable function. The *upper and lower approximate limits* of u at a point $x \in \Omega$ are defined as

$$u^+(x) = \inf\{t \in \mathbb{R} : D(\{u > t\};x) = 0\},$$
$$u^-(x) = \sup\{t \in \mathbb{R} : D(\{u < t\};x) = 0\},$$

respectively. The quantities $u^+(x)$, $u^-(x)$ are well defined (possibly equal to $\pm\infty$) at every $x \in \Omega$, and $u^-(x) \le u^+(x)$. The functions u^+, $u^- : \Omega \to [-\infty,\infty]$ are Borel measurable.

We say that u is *approximately continuous* at a point $x \in \Omega$ if $u^+(x) = u^-(x) \in \mathbb{R}$. In this case, we set $\widetilde{u}(x) = u^+(x) = u^-(x)$ and call $\widetilde{u}(x)$ the *approximate limit* of u at x. The set of all points in Ω where u is approximately continuous is a Borel set which will be denoted by C_u and called the set of *approximate continuity* of u. The set $S_u = \Omega \setminus C_u$ will be referred to as the set of *approximate discontinuity* of u.

Finally, by u^* we denote the *precise representative* of u which is defined by

$$u^*(x) = \frac{u^+(x) + u^-(x)}{2}$$

if $u^+(x), u^-(x) \in \mathbb{R}$, $u^*(x) = 0$ otherwise.

A locally integrable function u is said to be *approximately differentiable* at a point $x \in C_u$ if there exists $\nabla u(x) \in \mathbb{R}^N$ such that

$$\lim_{\rho \to 0} \frac{1}{\rho^{N+1}} \int_{B_\rho(x)} |u(y) - \widetilde{u}(x) - \langle \nabla u(x), y - x \rangle| \, dy = 0 \, .$$

Here, $\langle \cdot, \cdot \rangle$ stands for scalar product in $\mathbb{R}^N$. The vector $\nabla u(x)$ is called the *approximate differential* of u at x. The set of all points in C_u where u is approximately differentiable is denoted by $\mathcal{D}_u$ and is called the set of *approximate differentiability* of u. It can be easily verified that $\mathcal{D}_u$ is a Borel set and that $\nabla u : \mathcal{D}_u \to \mathbb{R}^N$ is a Borel function.

A function $u \in L^1(\Omega)$ is said to be of *bounded variation* if its distributional gradient Du is an $\mathbb{R}^N$-valued Radon measure in Ω and the total variation $|Du|$ of Du is finite in Ω. The space of all functions of bounded variation in Ω is denoted by $BV(\Omega)$, while the notation $BV_{\mathrm{loc}}(\Omega)$ will be reserved for the space of those functions $u \in L^1_{\mathrm{loc}}(\Omega)$ such that $u \in BV(\Omega')$ for every open set $\Omega' \subset\subset \Omega$.

Let $u \in BV(\Omega)$. Then it can be proved that

$$\lim_{\rho \to 0} \fint_{B_\rho(x)} |u(y) - \widetilde{u}(x)| \, dy = 0 \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in C_u$$

and that u is approximately differentiable for $\mathcal{L}^N$-a.e. x. Moreover, the functions u^- and u^+ are finite $\mathcal{H}^{N-1}$-a.e. and for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ there exists a unit vector $\nu_u(x)$ such that

$$\begin{aligned} &\lim_{\rho \to 0} \fint_{B^+_\rho(x;\nu_u(x))} |u(y) - u^+(x)| \, dy = 0, \\ &\lim_{\rho \to 0} \fint_{B^-_\rho(x;\nu_u(x))} |u(y) - u^-(x)| \, dy = 0, \end{aligned} \tag{2.1}$$

where $B^+_\rho(x;\nu_u(x)) = \{y \in B_\rho(x) : \langle y - x, \nu_u(x) \rangle > 0\}$, and $B^-_\varrho(x;\nu_u(x))$ is defined analogously. The set of all points in S_u where the equalities (2.1) are satisfied is called the *jump set* of u and is denoted by J_u.

If u is a BV function, we denote by $D^a u$ the absolutely continuous part of Du with respect to Lebesgue measure. The singular part, denoted by $D^s u$, is split into two more parts, the *jump part* $D^j u$ and the *Cantor part* $D^c u$, defined by

$$D^j u = D^s u \llcorner J_u, \qquad D^c u = D^s u - D^j u \, .$$

Finally, we denote by $\widetilde{D}u$ the *diffuse part* of Du, defined by

$$\widetilde{D}u = D^a u + D^c u \, .$$

We recall the following lemma which contains some useful properties of the characteristic functions of the level sets of a BV function u (see Lemma 2.2 in [15]).

Lemma 2.1. *Let $u : \mathbb{R}^N \to \mathbb{R}$ a measurable function. Then, for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$*

$$u^-(x) > t \implies \chi^*_{\{u>t\}}(x) = 1, \qquad u^+(x) < t \implies \chi^*_{\{u>t\}}(x) = 0\,.$$

Moreover, if $u \in BV(\mathbb{R}^N)$, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ there exists a Borel set $N_t \subset \mathbb{R}^N$, with $\mathcal{H}^{N-1}(N_t) = 0$, such that for any $x \in \mathbb{R}^N \setminus N_t$ the following relations hold:

$$u^-(x) > t \iff \chi^*_{\{u>t\}}(x) = 1, \qquad u^+(x) < t \iff \chi^*_{\{u>t\}}(x) = 0,$$
$$u^-(x) \le t \le u^+(x) \iff \chi^*_{\{u>t\}}(x) = \frac{1}{2}\,.$$

Now, we recall the coarea formula for BV functions.

Theorem 2.2 (Coarea formula). *Let Ω be an open subset of $\mathbb{R}^N$ and let $u \in BV(\Omega)$. Assume that $g : \Omega \to [0, +\infty]$ is a Borel function. Then*

$$\int_\Omega g\, d|Du| = \int_{-\infty}^{+\infty} dt \int_{\partial^M\{u>t\}\cap\Omega} g\, d\mathcal{H}^{N-1}\,. \tag{2.2}$$

An alternative version of formula (2.2) states that

$$\int_\Omega g\, d|Du| = \int_{-\infty}^{+\infty} dt \int_{\{u^-\le t\le u^+\}} g\, d\mathcal{H}^{N-1} \tag{2.3}$$

(see [18, Theorem 4.5.9]).

3. The chain rule proven in [15]

In the paper [15] the authors deal with a general chain rule formula in $BV(\mathbb{R}^N)$ for functions whose dependence in x is BV. More precisely, the following theorem is proved, for particular $B(x,t)$ of the type $\int_0^t b(x,s)ds$.

Theorem 3.1. *Let $b : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a Borel function. Assume that*

(α) the function $b(x,t)$ is locally bounded;
(β) for every $t \in \mathbb{R}$ the function $b(\cdot,t) \in BV(\mathbb{R}^N)$;
(γ) for any compact set $H \subset \mathbb{R}$,

$$\int_H |D_x b(\cdot,t)|(\mathbb{R}^N) dt < +\infty\,,$$

where $D_x b(\cdot,t)$ is the distributional gradient of the map $x \mapsto b(x,t)$.

Then for every $u \in BV(\mathbb{R}^N) \cap L^\infty_{\rm loc}(\mathbb{R}^N)$, the function $v : \mathbb{R}^N \to \mathbb{R}$, defined by

$$v(x) := \int_0^{u(x)} b(x,t)\, dt\,,$$

belongs to $BV_{\rm loc}(\mathbb{R}^N)$ *and for any* $\phi \in C^1_0(\mathbb{R}^N)$ *we have*

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla\phi(x) v(x)\,dx \qquad (3.1)\\
&= -\int_{-\infty}^{+\infty} dt \int_{\mathbb{R}^N} \operatorname{sgn}(t)\chi^*_{\Omega_{u,t}}(x)\phi(x)\,dD_x b(x,t) - \int_{\mathbb{R}^N} \phi(x) b(x,u(x))\nabla u(x)\,dx\\
&\quad - \int_{\mathbb{R}^N} \phi(x)\widetilde{b}(x,\widetilde{u}(x))\,dD^c u - \int_{J_u} \phi(x)\left[\int_{u^-(x)}^{u^+(x)} b^*(x,t)\,dt\right]\nu_u(x)\,d\mathcal{H}^{N-1},
\end{aligned}$$

where J_u *is the jump set of* u, $\Omega_{u,t} = \{x \in \mathbb{R}^N : t$ *belongs to the segment of endpoints* 0 *and* $u(x)\}$ *and* $\chi^*_{\Omega_{u,t}}$ *and* $b^*(\cdot,t)$ *are, respectively, the precise representatives of* $\chi_{\Omega_{u,t}}$ *and* $b(\cdot,t)$.

4. The formula proven in [4]

In this section we recall the following chain rule for scalar BV functions obtained as particular case of the general formula proven in [4] for vector valued functions.

Theorem 4.1 (Theorem 3.2 in [4]). *Let* $B\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *be satisfying:*

(a) $x \mapsto B(x,t)$ *belongs to* $BV_{\rm loc}(\mathbb{R}^N)$ *for all* $t \in \mathbb{R}$;

(b) $t \mapsto B(x,t)$ *is continuously differentiable in* $\mathbb{R}$ *for almost every* $x \in \mathbb{R}^N$;

(H1) *for some constant* M, $|(\partial_t B)(x,t)| \le C$ *for all* $x \in \mathbb{R}^N$ *and* $t \in \mathbb{R}$;

(H2) *for any compact set* $H \subset \mathbb{R}$ *there exists a modulus of continuity* $\widetilde{\omega}_H$ *independent of* x *such that*

$$|(\partial_t B)(x,t) - (\partial_t B)(x,w)| \le \widetilde{\omega}_H(|t-w|)$$

for all $t, w \in H$ *and* $x \in \mathbb{R}^N$;

(H3) *for any compact set* $H \subset \mathbb{R}$ *there exist a positive Radon measure* λ_H *and a modulus of continuity* ω_H *such that*

$$|\widetilde{D}_x B(\cdot,t)(A) - \widetilde{D}_x B(\cdot,w)(A)| \le \omega_H(|t-w|)\lambda_H(A)$$

for all $t, w \in H$ *and* $A \subset \mathbb{R}^N$ *Borel;*

(H4) *the measure*

$$\sigma := \bigvee_{t\in\mathbb{R}} |D_x B(\cdot,t)|$$

is a Radon measure.

Then for every $u \in BV_{\rm loc}(\mathbb{R}^N) \cap L^\infty_{\rm loc}(\mathbb{R}^N)$ *the composite function* $v(x) := B(x,u(x))$ *belongs to* $BV_{\rm loc}(\mathbb{R}^n)$ *and the following chain rule holds:*

(diffuse) $|Dv| \ll \sigma + |Du|$ *and, for any Radon measure* μ *such that* $\sigma + |Du| \ll \mu$, *it holds*

$$\frac{d\widetilde{D}v}{d\mu} = \frac{d\widetilde{D}_x B(\cdot,\widetilde{u}(x))}{d\mu} + (\partial_t \widetilde{B})(x,\widetilde{u}(x))\frac{d\widetilde{D}u}{d\mu} \qquad \mu\text{-a.e. in } \mathbb{R}^N.$$

(jump) *$J_v \subset \mathcal{N} \cup J_u$ and, denoting by $u^{\pm}(x)$ and $B^{\pm}(x,t)$ the one-sided traces of u and $B(\cdot,t)$ induced by a suitable orientation of $\mathcal{N} \cup J_u$, it holds*

$$D^j v = \big(B^+(x, u^+(x)) - B^-(x, u^-(x)\big)\nu_{\mathcal{N}\cup J_u}\mathcal{H}^{N-1} \llcorner (\mathcal{N} \cup J_u)$$

in the sense of measures.

Moreover for a.e. x the map $y \mapsto B(y, u(x))$ is approximately differentiable at x and

$$\nabla v(x) = \nabla_x B(x, u(x)) + (\partial_t B)(x, u(x))\nabla u(x) \qquad \mathcal{L}^N\text{-a.e. in } \mathbb{R}^N . \tag{4.1}$$

Here the expression

$$\frac{d\widetilde{D}_x B(\cdot, \tilde{u}(x))}{d\mu}$$

means the pointwise density of the measure $\widetilde{D}_x B(\cdot, t)$ with respect to μ, computed choosing $t = \tilde{u}(x)$ (notice that the composition is Borel measurable thanks to the Scorza-Dragoni Theorem and Lemma 3.9 in [4]).

Remark 4.2. Let us note that, since $\sigma \ll \mu$, it holds

$$\frac{d\widetilde{D}v}{d\mu} = \frac{d\widetilde{D}_x B(\cdot, \widetilde{u}(x))}{d\sigma}\frac{d\sigma}{d\mu} + (\partial_t \widetilde{B})(x, \widetilde{u}(x))\frac{d\widetilde{D}u}{d\mu} \qquad \mu\text{-a.e. in } \mathbb{R}^n.$$

Then we have

$$\begin{aligned}\widetilde{D}v &= \frac{d\widetilde{D}_x B(\cdot, \widetilde{u}(x))}{d\sigma}\,\sigma + (\partial_t \widetilde{B})(x, \widetilde{u}(x))\,\widetilde{D}u \\ &= \nabla_x B(x, u(x))\,\mathcal{L}^N + (\partial_t B)(x, u(x))\nabla u(x)\,\mathcal{L}^N \\ &\quad + \frac{dD^c_x B(\cdot, \widetilde{u}(x))}{d\sigma}\,\sigma + (\partial_t \widetilde{B})(x, \widetilde{u}(x))\,D^c u\,,\end{aligned}$$

in the sense of measures.

5. Setting

We will obtain a stronger result than the previous one in the special case of scalar functions. We will weak assumption (b), by requiring only the Lipschitz continuity of $B(x,\cdot)$ instead of the C^1 dependence and we will drop the continuity assumptions (H2).

Let $B : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a locally bounded Borel function such that $B(x, 0) = 0$. We consider the following assumptions:

(I) For all $x \in \mathbb{R}^N$ the function $B(x,\cdot)$ is Lipschitz continuous, there exists a Lebesgue negligible set $\mathcal{M}_0 \subset \mathbb{R}$ such that for all $x \in \mathbb{R}^N$ the function $B(x,\cdot)$ is differentiable in $\mathbb{R} \setminus \mathcal{M}_0$ and there exists a constant $C > 0$ such that

$$|(\partial_t B)(x,t)| \le C, \qquad \forall x \in \mathbb{R}^N,\ t \in \mathbb{R} \setminus \mathcal{M}_0 .$$

(II) For every $t \in \mathbb{R}$ the function $B(\cdot, t) \in BV_{\rm loc}(\mathbb{R}^N)$.

(III) The measure

$$\sigma := \bigvee_{t\in\mathbb{R}\setminus\mathcal{M}_0} |D_x B(\cdot,t)|$$

is a Radon measure, where $\bigvee$ denotes the least upper bound in the space of nonnegative Borel measures.

(IV) For any compact set $H \subset \mathbb{R}$ there exist a positive Radon measure λ_H such that

$$|\widetilde{D}_x B(\cdot,t)(A) - \widetilde{D}_x B(\cdot,w)(A)| \leq |t-w|\lambda_H(A)$$

for all $t, w \in H$ and $A \subset \mathbb{R}^N$ Borel.

Remark 5.1. As in Remark 3.5 in [4], since we will consider $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, condition (III) can be replaced by the following local version:

(III)$_{\mathrm{loc}}$ for every compact set $H \subset \mathbb{R}$ the measure

$$\sigma_H := \bigvee_{t\in H} |D_x B(\cdot,t)|$$

is a Radon measure.

By (III) we have that $\sigma \ll \mathcal{H}^{N-1}$. Let use define

$$\mathcal{N} = \Big\{x \in \mathbb{R}^N : \liminf_{r\to 0} \frac{\sigma(B_r(x))}{r^{N-1}} > 0\Big\}.$$

In [4] it is proved that $\mathcal{N}$ is a $\mathcal{H}^{N-1}$-rectifiable set. This set is independent of t and contains the jump set of the BV function $B(\cdot,t)$ for every $t \in \mathbb{R}$. We omit the dependence of $\mathcal{N}$ and σ on H in the local version. In the following, $\nu_{\mathcal{N}}$ will always denote an oriented normal vector field on $\mathcal{N}$.

We will consider the following assumption:

(V) For every compact set $K \subseteq \mathbb{R}^N$ we have $\mathcal{H}^{N-1}(\mathcal{N} \cap K) < +\infty$.

Remark 5.2. By using (I), we have (see Prop. 4.2 in [4]) that there exists a set $\mathcal{N}_1 \subset \mathbb{R}^N$ with $\mathcal{H}^{N-1}(\mathcal{N}_1) = 0$ such that for all $x \in \mathbb{R}^N \setminus \mathcal{N}_1$ and $t \in \mathbb{R}$ the following limits exist

$$B^{\pm}(x,t) = \lim_{r\to 0} \fint_{B_r^{\pm}(x)} B(y,t)dy$$

and the map $t \mapsto B^{\pm}(x,t)$ is Lipschitz continuous. In particular

$$B^+(x,t) = B^-(x,t) \qquad \mathcal{H}^{N-1}\text{-a.e. } x \in \mathbb{R}^N \setminus \mathcal{N} \text{ and for every } t \in \mathbb{R}.$$

In the same way, see [4, Section 3], for $\mathcal{H}^{N-1}$ almost every point of $\mathbb{R}^N \setminus \mathcal{N}$ and every $t \in \mathbb{R}$ there exists the limit

$$\widetilde{B}(x,t) = \lim_{r\to 0} \fint_{B_r(x)} B(y,t)\,dy,$$

and the map $t \mapsto \widetilde{B}(x,t)$ is Lipschitz continuous. Without loss of generality we shall always assume that $B(x,t) = \widetilde{B}(x,t)$ for $\mathcal{H}^{N-1}$ almost every point of $\mathbb{R}^N \setminus \mathcal{N}$ and every $t \in \mathbb{R}$. For every $t \in \mathbb{R}$ let us denote by

$$B^*(\cdot,t) := \frac{B^+(\cdot,t) + B^-(\cdot,t)}{2}$$

the precise representative of the BV function $x \mapsto B(x,t)$.

Remark 5.3. Under the previous assumptions we have

$$\forall t \in \mathbb{R} \setminus \mathcal{M}_0 \quad (\partial_t B)(\cdot,t) \in BV_{\mathrm{loc}}(\mathbb{R}^N).$$

In fact, let $h \in \mathbb{R}$, for any compact set $H \subset \mathbb{R}$ and $t, t+h \in H$, the functions

$$B_h(x,t) = \frac{B(x,t+h) - B(x,t)}{h}$$

satisfy

$$|B_h(x,t)| \le C, \qquad |D_x B_h(\cdot,t)| \le \lambda_H + C\,\mathcal{H}^{N-1} \llcorner \mathcal{N}.$$

Remark 5.4. For every $t \in \mathbb{R}$ let

$$\psi(\cdot,t) := \frac{d(\widetilde{D}_x B)}{d\sigma}(\cdot,t)$$

be the Radon-Nikodým derivative of the diffuse part $\widetilde{D}_x B(\cdot,t)$ of the measure $(D_x B)(\cdot,t)$ with respect to the measure σ, which is defined σ-a.e.. By Lemma 3.9 in [4] and by (IV) there exists a Borel subset $\mathcal{N}_0$ of $\mathbb{R}^N$ with $\sigma(\mathcal{N}_0) = 0$ such that the following limit

$$\lim_{r \downarrow 0} \frac{\widetilde{D}_x B(\cdot,t)(B_r(x))}{\sigma(B_r(x))} = \frac{d\widetilde{D}_x B(\cdot,t)}{d\sigma}(x)$$

exists for every $x \in \mathbb{R}^N \setminus \mathcal{N}_0$ and for every $t \in \mathbb{R}$ and this equality holds. Moreover, for every $x \notin \mathcal{N}_0$ and every $t, t' \in \mathbb{R}$ we have

$$\begin{aligned} |\psi(x,t) - \psi(x,t')| &= \left| \frac{d[(\widetilde{D}_x B)(\cdot,t) - (\widetilde{D}_x B)(\cdot,t')]}{d\sigma} \right| \\ &= \frac{d|(\widetilde{D}_x B)(\cdot,t) - (\widetilde{D}_x B)(\cdot,t')|}{d\sigma} \le C|t - t'|. \end{aligned} \tag{5.1}$$

In order to prove this fact, it is sufficient to fix a dense countable subset T of $\mathbb{R}$ and for every $t, t' \in T$ to consider the sets $D_{t,t'} \subseteq \mathbb{R}^N$, with $\sigma(D_{t,t'}) = 0$, such that for every $x \notin D_{t,t'}$ condition (5.1) holds. Hence, if we define $\mathcal{N}_0 = \bigcup_{t,t' \in T} D_{t,t'}$, we have $\sigma(\mathcal{N}_0) = 0$ and by (IV) condition (5.1) holds for every $x \notin \mathcal{N}_0$. By (5.1) for every $x \notin \mathcal{N}_0$ the function $t \mapsto \psi(x,t)$ is Lipschitz continuous. For every $t \in \mathbb{R}$ the following decomposition formula holds

$$(D_x B)(\cdot,t) = \psi(x,t)\,\sigma + \left[B^+(x,t) - B^-(x,t)\right] \nu_{\mathcal{N}}\, \mathcal{H}^{N-1} \llcorner \mathcal{N},$$

in the sense of measures.

Moreover, we have that the limit

$$\lim_{r \downarrow 0} \frac{D_x^c B(\cdot,t)(B_r(x))}{\sigma(B_r(x))} = \frac{dD_x^c B(\cdot,t)}{d\sigma}(x)$$

exists for every $x \in \mathbb{R}^N \setminus \mathcal{N}_0$ and for every $t \in \mathbb{R}$ and this equality holds, where $\frac{dD_x^c B(\cdot,t)}{d\sigma}(x)$ is Radon-Nikodým derivative at x of the Cantor part of the measure $D_x B(\cdot,t)$ w.r.t. σ.

On the other hand, by Proposition 4.4 of [4] there exists a Borel set $\mathcal{N}_2 \subset \mathbb{R}^N$ such that $\mathcal{L}^N(\mathcal{N}_2) = 0$ and the approximate gradient $\nabla_x B(x,t)$ of the function $y \mapsto B(y,t)$ at x exists for every $x \in \mathbb{R}^N \setminus \mathcal{N}_2$ and for every $t \in \mathbb{R}$ and

$$\frac{dD_x B(\cdot,t)}{d\mathcal{L}^N}(x) = \nabla_x B(x,t).$$

Moreover, we have

$$\frac{dD_x^j B(\cdot,t)}{d\mathcal{H}^{N-1}}(x) = [B^+(x,t) - B^-(x,t)]\nu_{\mathcal{N}}(x) \tag{5.2}$$

for every $x \in \mathcal{N} \setminus \mathcal{N}_1$ and for every $t \in \mathbb{R}$.

6. Preliminaries

Let us define the function $b(x,t) := (\partial_t B)(x,t)$ for every $x \in \mathbb{R}^N$ and $t \in \mathbb{R} \setminus \mathcal{M}_0$.

Proposition 6.1. *Let $B(x,t)$ satisfying (I) – (IV).*

(j) *For every Borel bounded set $E \subset \mathbb{R}^N$, the map $t \mapsto D_x B(\cdot,t)(E)$ is Lipschitz continuous, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$*

$$\partial_t(D_x B(\cdot,t)(E)) = D_x b(\cdot,t)(E)$$

and $D_x b(\cdot,t) \ll \sigma$.

(k) *For every Borel bounded function u with compact support, the map*

$$t \mapsto \int_{\mathbb{R}^N} u(x)\, dD_x B(\cdot,t)$$

is Lipschitz continuous with derivative

$$\int_{\mathbb{R}^N} u(x)\, dD_x b(\cdot,t)\,,$$

for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$.

(l) *For $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ and $\mathcal{H}^{N-1}$-a.e. $x \in \mathcal{N}$ there exists the derivative of the function $t \mapsto B^+(x,t) - B^-(x,t)$ and*

$$\partial_t \left(B^+(x,t) - B^-(x,t)\right) = b^+(x,t) - b^-(x,t) \quad \textit{for } \mathcal{H}^{N-1}\textit{-a.e. } x \in \mathcal{N}\,.$$

(m) *For $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ and for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N \setminus \mathcal{N}$ there exist the derivative of the function $t \mapsto \nabla_x B(x,t)$ and the approximate gradient $\nabla_x(\partial_t B)(x,t)$ of the function $y \mapsto (\partial_t B)(y,t)$ at x, i.e.,*

$$\frac{dD_x(\partial_t B)(\cdot,t)}{d\mathcal{L}^N}(x) = \nabla_x(\partial_t B)(x,t),$$

and for every Borel set $E \subseteq \mathbb{R}^N \setminus \mathcal{N}$ with $\mathcal{L}^N(E) > 0$ we have

$$\partial_t \int_E \nabla_x B(x,t)\, dx = \int_E \nabla_x(\partial_t B)(x,t)\, dx. \tag{6.1}$$

(n) *For $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ and for σ-a.e. $x \in \mathbb{R}^N \setminus \mathcal{N}$ there exist the derivative of the function $t \mapsto \frac{dD^c_x B}{d\sigma}(x,t)$ and the Radon-Nikodým derivative $\frac{dD^c_x(\partial_t B)(\cdot,t)}{d\sigma}$ at x of the Cantor part of the measure $D_x(\partial_t B)(\cdot,t)$ w.r.t. σ, i.e., the following limit*
$$\lim_{r\downarrow 0}\frac{D^c_x(\partial_t B)(\cdot,t)(B_r(x))}{\sigma(B_r(x))} = \frac{dD^c_x(\partial_t B)(\cdot,t)}{d\sigma}(x)$$
exists for σ-a.e. $x \in \mathbb{R}^N$ and for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ and for every Borel set $E \subseteq \mathbb{R}^N \setminus \mathcal{N}$ with $\mathcal{L}^N(E) = 0$ we have
$$\partial_t \int_E \frac{dD^c_x B}{d\sigma}(x,t)\, d\sigma = \int_E \frac{dD^c_x(\partial_t B)}{d\sigma}(x,t)\, d\sigma.$$

Proof. We will we prove (j). Firstly, we prove that if Ω is an open bounded subset of $\mathbb{R}^N$, then for a.e. $t \in \mathbb{R}$ the function $t \mapsto D_x B(\cdot,t)(\Omega)$ is Lipschitz continuous and
$$\partial_t(D_x B(\cdot,t))(\Omega) = D_x b(\cdot,t)(\Omega)\,, \tag{6.2}$$
for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$. Let ψ_h be an increasing sequence of functions in $C^1_0(\Omega)$ converging to the characteristic function of Ω. Then for every $\phi \in C^1_0(\mathbb{R})$ we have
$$\begin{aligned}\int_{\mathbb{R}} \phi'(t) D_x B(\cdot,t)(\Omega)\, dt &= \lim_{h\to+\infty}\int_{\mathbb{R}} dt \int_\Omega \phi'(t)\psi_h(x)\, d\,D_x B(\cdot,t)\\ &= -\lim_{h\to+\infty}\int_{\mathbb{R}} dt \int_\Omega \phi'(t)\nabla\psi_h(x)\, B(x,t)\, dx\\ &= \lim_{h\to+\infty}\int_\Omega \nabla\psi_h(x)\int_{\mathbb{R}} \phi(t)\, b(x,t)\, dt\, dx\\ &= \lim_{h\to+\infty}\int_{\mathbb{R}} \phi(t)\int_\Omega \nabla\psi_h(x)\, b(x,t)\, dx\, dt\\ &= -\lim_{h\to+\infty}\int_{\mathbb{R}} \phi(t)\int_\Omega \psi_h(x)\, d\,D_x b(\cdot,t)\, dt\\ &= -\int_{\mathbb{R}} \phi(t)\, D_x b(\cdot,t)(\Omega)\, dt\,.\end{aligned}$$

In order to prove (j), let us fix a ball B and consider the family $\mathcal{F}$ of Borel sets $E \subseteq B$ such that the function $t \mapsto D_x B(\cdot,t)(E)$ is Lipschitz continuous and
$$\partial_t(D_x B(\cdot,t))(E) = D_x b(\cdot,t)(E)\,,$$
for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$. Let E_h be an increasing sequence in $\mathcal{F}$ such that $E_h \uparrow E$. Then for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ and for every $h \in \mathbb{N}$ we have
$$\partial_t(D_x B(\cdot,t))(E_h) = D_x b(\cdot,t)(E_h)\,.$$
Moreover setting for every $t \notin \mathcal{M}_0$
$$f_h(t) := D_x B(\cdot,t)(E_h)\,, \quad f(t) := D_x B(\cdot,t)(E)$$
we have that f_h converges to f in $L^1_{\rm loc}(\mathbb{R})$. Therefore, since for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$
$$\partial_t(D_x B(\cdot,t))(E_h) \to \partial_t(D_x B(\cdot,t))(E),$$
we obtain
$$\partial_t f(t) = \partial_t(D_x B(\cdot,t))(E) = D_x b(\cdot,t)(E)$$

and E belongs to $\mathcal{F}$. Then by using Proposition 1.8 (coincidence criterion) and Remark 1.9 in [6], we can conclude that $\mathcal{F}$ coincides with the σ-algebra of Borel sets contained in B. Since

$$\partial_t \left(\int_E \frac{D_x B(\cdot,t)}{d\sigma}(x)\, d\sigma \right) = D_x b(\cdot,t)(E)\,,$$

we have $D_x b(\cdot,t) \ll \sigma$. This prove (j).

In order to prove (k) we remark that if $u = \chi_E$ and E is a Borel set, then by (j) for every $\phi \in C^1_c(\mathbb{R})$ we have

$$\begin{aligned}\int_{\mathbb{R}} \phi'(t) \int_{\mathbb{R}^N} \chi_E(x)\, dD_x B(\cdot,t)\, dt &= \int_{\mathbb{R}} \phi'(t) D_x B(\cdot,t)(E)\, dt \\ = \int_{\mathbb{R}} \phi(t)\, D_x b(\cdot,t)(E)\, dt &= \int_{\mathbb{R}} \phi(t) \int_{\mathbb{R}^N} \chi_E(x) dD_x b(\cdot,t)\, dt\,.\end{aligned}$$

The conclusion then follows since $u(x) = \sum_{i=1}^{+\infty} \frac{1}{i}\chi_{E_i}(x)$, where E_i is a sequence of bounded Borel sets.

Now in order to prove (l), for every Borel set $E \subseteq \mathcal{N}$ and for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ we have

$$D_x B(\cdot,t)(E) = \int_{E\cap\mathcal{N}} \big(B^+(x,t) - B^-(x,t)\big)\, \nu_{\mathcal{N}}\, d\mathcal{H}^{N-1}\,.$$

Then by (j) we have

$$\partial_t (D_x B(\cdot,t))(E) = D_x b(\cdot,t)(E)\,,$$

and

$$\begin{aligned}&\partial_t \int_{E\cap\mathcal{N}} \big(B^+(x,t) - B^-(x,t)\big)\, \nu_{\mathcal{N}}\, d\mathcal{H}^{N-1} \\ &= \int_{E\cap\mathcal{N}} \big(b^+(x,t) - b^-(x,t)\big)\, \nu_{\mathcal{N}}\, d\mathcal{H}^{N-1}\,.\end{aligned}$$

This implies that for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ and for $\mathcal{H}^{N-1}$-a.e. $x \in \mathcal{N}$ there exists the derivative of the function $t \mapsto B^+(x,t) - B^-(x,t)$ and

$$\partial_t \big(B^+(x,t) - B^-(x,t)\big) = b^+(x,t) - b^-(x,t)\,.$$

Now in order to prove (m), for every Borel set $E \subseteq \mathbb{R}^N \setminus \mathcal{N}$ and for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ we have

$$\widetilde{D}_x B(\cdot,t)(E) = D_x B(\cdot,t)(E) \qquad \widetilde{D}_x b(\cdot,t)(E) = D_x b(\cdot,t)(E).$$

Then by (k) we have

$$\partial_t(\widetilde{D}_x B(\cdot,t)(E)) = \widetilde{D}_x b(\cdot,t)(E)\,,$$

For every Borel set $E \subseteq \mathbb{R}^N \setminus \mathcal{N}$ with $\mathcal{L}^N(E) > 0$ we have

$$\partial_t(D^a_x B(\cdot,t))(E) = D^a_x b(\cdot,t)(E)\,.$$

Hence

$$\partial_t \int_E \nabla_x B(x,t)\, dx = D^a_x b(\cdot,t)(E)\,.$$

This implies that there exists the approximate gradient $\nabla_x(\partial_t B)(x,t)$ of the function $y \mapsto (\partial_t B)(y,t)$ at x for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, i.e.,

$$D^a_x b(\cdot,t)(E) = \int_E \frac{dD_x(\partial_t B)(\cdot,t)}{d\mathcal{L}^N}(x)\, dx = \int_E \nabla_x(\partial_t B)(x,t)\, dx,$$

and we can conclude that for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$

$$\partial_t \int_E \nabla_x B(x,t)\, dx = \int_E \nabla_x(\partial_t B)(x,t)\, dx.$$

Now, we will prove (n). For every Borel set $E \subseteq \mathbb{R}^N \setminus \mathcal{N}$ with $\mathcal{L}^N(E) = 0$ we have

$$\partial_t(D^c_x B(\cdot,t)(E)) = D^c_x b(\cdot,t)(E)\,.$$

Hence

$$\partial_t \left(\int_E \frac{dD^c_x B(\cdot,t)}{d\sigma}(x)\, d\sigma \right) = D^c_x b(\cdot,t)(E) \tag{6.3}$$

and $D^c_x b(\cdot,t) \ll \sigma$. This implies that there exists the Radon-Nikodým derivative of the Cantor part of the measure $D_x(\partial_t B)(\cdot,t)$ w.r.t. σ, i.e., the following limit

$$\lim_{r\downarrow 0} \frac{D^c_x(\partial_t B)(\cdot,t)(B_r(x))}{\sigma(B_r(x))} = \frac{dD^c_x(\partial_t B)(\cdot,t)}{d\sigma}(x)$$

exists for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ and for σ-a.e. $x \in \mathbb{R}^N$ and

$$D^c_x b(\cdot,t)(E) = \int_E \frac{dD_x(\partial_t B)(\cdot,t)}{d\sigma}(x)\, d\sigma = \int_E \frac{dD^c_x(\partial_t B)}{d\sigma}(x,t)\, d\sigma\,.$$

We can conclude that for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$

$$\partial_t \int_E \frac{dD^c_x B}{d\sigma}(x,t)\, d\sigma = \int_E \frac{dD^c_x(\partial_t B)}{d\sigma}(x,t)\, d\sigma. \qquad \square$$

Corollary 6.2. *For $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ we have*

$$\begin{aligned} D_x B(x,t) &= \left[\int_0^t \frac{dD_x(\partial_t B)}{d\sigma}(x,w)\, dw \right] d\sigma \\ &= \left[\int_0^t \nabla_x(\partial_t B)(x,w)\, dt \right] d\mathcal{L}^N + \left[\int_0^t \frac{dD^c_x(\partial_t B)}{d\sigma}(x,w)\, dw \right] d\sigma \\ &\quad + \left[\int_0^t [(\partial_t B)^+(x,w) - (\partial_t B)^-(x,w)]\, dw \right] \nu_{\mathcal{N}}\, d\mathcal{H}^{N-1}\,. \end{aligned}$$

7. The new formula

In this section we present the main result of this paper. We need to introduce some further conditions.

We recall that by (l) of Proposition 6.1 we have that there exist a set $\overline{\mathcal{N}} \subset \mathbb{R}^N$ with $\mathcal{H}^{N-1}(\overline{\mathcal{N}}) = 0$ and a set $\mathcal{M}_1 \subset \mathbb{R}$ with $\mathcal{L}^1(\mathcal{M}_1) = 0$ such that for every $x \in \mathcal{N} \setminus \overline{\mathcal{N}}$ and for every $t \in \mathbb{R} \setminus (\mathcal{M}_0 \cup \mathcal{M}_1)$ there exists the derivative of the function $t \mapsto B^+(x,t) - B^-(x,t)$. On the other hand, by Remark 5.2, for every $x \in \mathcal{N} \setminus \mathcal{N}_1$, since the functions $B^\pm(x,\cdot)$ are Lipschitz continuous, there exists a set

$\mathcal{M}_1(x) \subset \mathbb{R}$ such that for every $t \in \mathbb{R} \setminus (\mathcal{M}_0 \cup \mathcal{M}_1(x))$ there exists the derivatives of the functions $t \mapsto B^{\pm}(x,t)$.

However, in the following we need to consider the following stronger condition

(VI) for $\mathcal{H}^{N-1}$-a.e. $x \in \mathcal{N}$ and every $t \in \mathbb{R}\setminus(\mathcal{M}_0 \cup \mathcal{M}_1)$ there exist the derivatives of the functions $t \mapsto B^{\pm}(x,t)$.

Remark 7.1. Similarly we need also that for $\mathcal{H}^{N-1}$-a.e. $x \in \mathbb{R}^N \setminus \mathcal{N}$ there exists a set $\mathcal{M}_2 \subset \mathbb{R}$ (independent of x) such that for all $t \in \mathbb{R} \setminus (\mathcal{M}_0 \cup \mathcal{M}_2)$ there exists the derivative of the function $t \mapsto \widetilde{B}(x,t)$. This condition is satisfied since we assume that for $\mathcal{H}^{N-1}$-a.e. $x \in \mathbb{R}^N \setminus \mathcal{N}$ and for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ we have $B(x,t) = \widetilde{B}(x,t)$.

Theorem 7.2. *Let $B : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a function such that $B(x,0) = 0$ and* (I) – (VI) *hold. Then for every $u \in BV_{\mathrm{loc}}(\mathbb{R}^N) \cap L^{\infty}_{\mathrm{loc}}(\mathbb{R}^N)$ the composite function $v(x) := B(x, u(x))$, $x \in \mathbb{R}^N$, belongs to $BV_{\mathrm{loc}}(\mathbb{R}^N)$ and for any $\phi \in C^1_0(\mathbb{R}^N)$ we have*

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla\phi(x) v(x)\, dx \\
&= -\int_{\mathbb{R}^N} \phi(x)(\nabla_x B)(x,u(x))\, dx - \int_{\mathbb{R}^N} \phi(x)(\partial_t B)(x,u(x)) \cdot \nabla u(x)\, dx \\
&\quad - \int_{\mathbb{R}^N} \phi(x) \frac{dD^c_x B}{d\sigma}(x,\widetilde{u}(x))\, d\sigma - \int_{\mathbb{R}^N} \phi(x)(\partial_t \widetilde{B})(x,\widetilde{u}(x)) \cdot dD^c u(x) \\
&\quad - \int_{\mathcal{N}\cup J_u} \phi(x)[B^+(x,u^+(x)) - B^-(x,u^-(x))]\, \nu_{\mathcal{N}\cup J_u}\, d\mathcal{H}^{N-1},
\end{aligned} \tag{7.1}$$

where it is understood that for $\mathcal{H}^{N-1}$-a.e. $x \in \mathcal{N} \cap J_u$ the normal $\nu_{\mathcal{N}\cup J_u}$ is chosen equal to $\nu_{\mathcal{N}}$.

Remark 7.3. The formula make sense provided the terms $(\partial_t B)(x,u(x)) \cdot \nabla u(x)$ and $(\partial_t \widetilde{B})(x,\widetilde{u}(x)) \cdot dD^c u(x)$ are interpreted to be zero whenever $\nabla u = 0$ and $D^c u = 0$ respectively, irrespective of whether $(\partial_t \widetilde{B})(x,u(x))$ is defined, i.e., $(\partial_t \widetilde{B})(x,\widetilde{u}(x)) \cdot d\widetilde{D}u$ is interpreted to vanish on sets where $\widetilde{D}u$ vanishes (see Prop. 3.92 in [6]).

Remark 7.4. By using (5.2) it is easy to check that

$$\begin{aligned}
&[B^+(x,u^+(x)) - B^-(x,u^-(x))]\nu_{\mathcal{N}\cup J_u} \\
&= \Big[B^*(x,u^+(x)) - B^*(x,u^-(x)) \\
&\quad + \frac{B^+(x,u^+(x)) - B^-(x,u^+(x))}{2} + \frac{B^+(x,u^-(x)) - B^-(x,u^-(x))}{2}\Big]\nu_{\mathcal{N}\cup J_u} \\
&= \Big[B^*(x,u^+(x)) - B^*(x,u^-(x))\Big]\nu_{\mathcal{N}\cup J_u} \\
&\quad + \frac{1}{2}\Big[\frac{dD^j_x B(\cdot,t)}{d\mathcal{H}^{N-1}}(x)_{t=u^+(x)} + \frac{dD^j_x B(\cdot,t)}{d\mathcal{H}^{N-1}}(x)_{t=u^-(x)}\Big].
\end{aligned}$$

Then the representation formula in Theorem 7.2 can be written as the following equality in the sense of measures:

$$\begin{aligned} Dv(x) = {} & (\nabla_x B)(x,u(x))\mathcal{L}^N + (\partial_t B)(x,u(x)) \cdot \nabla u(x)\mathcal{L}^N \\ & + \frac{dD_x^c B}{d\sigma}(x,\widetilde{u}(x))\,\sigma + (\partial_t \widetilde{B})(x,\widetilde{u}(x)) \ \cdot dD^c u \\ & + \left[B^*(x,u^+(x)) - B^*(x,u^-(x))\right] \nu_{\mathcal{N}\cup J_u}\, \mathcal{H}^{N-1} \llcorner J_u \\ & + \frac{1}{2}\Big[\frac{dD_x^j B(\cdot,t)}{d\mathcal{H}^{N-1}}(x)\big|_{t=u^+(x)} + \frac{dD_x^j B(\cdot,t)}{d\mathcal{H}^{N-1}}(x)\big|_{t=u^-(x)}\Big]\mathcal{H}^{N-1} \llcorner \mathcal{N}\cup J_u \,. \end{aligned}$$

Hence

$$\begin{aligned} Dv(x) = {} & D_x B(x,t)|_{t=u(x)} + (\partial_t \widetilde{B}(x,u) \cdot \widetilde{D}u \\ & + [B^*(x,u^+) - B^*(x,u^-)]\nu_{\mathcal{N}\cup J_u}\mathcal{H}^{N-1}\llcorner \mathcal{N}\cup J_u, \end{aligned} \tag{7.2}$$

where

$$\begin{aligned} D_x B(\cdot,t)|_{t=u(x)} := {} & \frac{1}{2}\left[\left[\frac{dD_x B(\cdot,t)}{d\sigma}\right]_{t=u^+(x)} + \left[\frac{dD_x B(\cdot,t)}{d\sigma}\right]_{t=u^-(x)}\right]\sigma \\ = {} & (\nabla_x B)(x,u(x))\mathcal{L}^N + \frac{dD_x^c B}{d\sigma}(x,\widetilde{u}(x))\,\sigma \\ & + \frac{1}{2}\Big[\frac{dD_x^j B(\cdot,t)}{d\mathcal{H}^{N-1}}(x)\big|_{t=u^+(x)} \\ & + \frac{dD_x^j B(\cdot,t)}{d\mathcal{H}^{N-1}}(x)\big|_{t=u^-(x)}\Big]\mathcal{H}^{N-1}\llcorner \mathcal{N}\cup J_u \end{aligned} \tag{7.3}$$

and with the compact notation

$$D_x B(\cdot,t)|_{t=u(x)} = \frac{1}{2}\left[D_x B(\cdot,u^+(x)) + D_x B(\cdot,u^-(x))\right] .$$

Remark 7.5. A preliminary problem related to the chain rule formula is to find sufficient conditions assuring that the composite function belongs to the space BV of functions of bounded variation. For $N=1$ it is well known that the autonomous superposition operator $u \mapsto Tu$, defined by $(Tu)(x) = B(u(x))$, maps the space BV of functions of one variable of bounded variation in the sense of Jordan into itself if and only if the function B which generates the operator is locally Lipschitz (see [22, 23, 24]).

In the nonautonomous case, one of the well-known results ensuring that superposition operator $u \mapsto Tu$, defined by $(Tu)(x) = B(x,u(x))$, maps the space BV into itself is the theorem by A.G. Ljamin (see [20]). According to that theorem it suffices to consider the class of functions which are uniformly Lipschitz w.r.t. the second variable and of uniformly bounded variation w.r.t. the first variable. Unfortunately, Ljamin's result is false. Recently, Maćkowiak in [21] gives an example contradicting sufficiency of those conditions. However, a very interesting sufficient condition is given in [7] and we extend this result to the multidimensional case: in fact, as in Proposition 3.7 of [4], by using assumptions (I) – (III), we have that for every

$u \in BV_{\rm loc}(\mathbb{R}^N) \cap L^\infty_{\rm loc}(\mathbb{R}^N)$ the composite function $v(x) := B\,(x, u(x))$, $x \in \mathbb{R}^N$, belongs to $BV_{\rm loc}(\mathbb{R}^N)$.

8. Proof of Theorem 7.2

Proof. We denote by $\varrho_\delta(t)$ a standard mollifier and we define

$$B_\delta(x,t) := \int_{\mathbb{R}} \varrho_\delta(t-s)\, B(x,s)\, ds$$

for all $x \in \mathbb{R}^N$, $t \in \mathbb{R}$. Hence for all $x \in \mathbb{R}^N$, $t \in \mathbb{R} \setminus \mathcal{M}_0$ the function $B_\delta(x,\cdot)$ is continuously differentiable in $\mathbb{R} \setminus \mathcal{M}_0$ and

$$\partial_t B_\delta(x,t) = (\partial_t B)_\delta(x,t) := \int_{\mathbb{R}} \varrho_\delta(t-s)\, \partial_t\, B(x,s)\, ds\,. \tag{8.1}$$

Moreover $|(\partial_t B)_\delta(x,t)| \le C$ for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R} \setminus \mathcal{M}_0$. Hence $B_\delta(x,t)$ satisfies hypotheses (b) and (H1) of Theorem 4.1. Let us define the convolution of the Radon-Nikodým derivative $\frac{dD_x B}{d\sigma}(x,t)$ w.r.t. the variable $t \in \mathbb{R}$

$$\left(\frac{dD_x B}{d\sigma}\right)_\delta (x,t) := \int_{\mathbb{R}} \varrho_\delta(t-s)\, \frac{dD_x B}{d\sigma}(x,s)\, ds\,.$$

We claim that for every $t \in \mathbb{R} \setminus \mathcal{M}_0$ the function $B_\delta(\cdot,t) \in BV_{\rm loc}(\mathbb{R}^N)$ (hence $B_\delta(x,t)$ satisfies hypothesis (a) of Theorem 4.1), $D_x B_\delta(\cdot,t) \ll \sigma$ and for σ-a.e. $x \in \mathbb{R}^N$ and for every $t \in \mathbb{R} \setminus \mathcal{M}_0$

$$\frac{dD_x B_\delta}{d\sigma}(x,t) = \left(\frac{dD_x B}{d\sigma}\right)_\delta (x,t). \tag{8.2}$$

In fact, for every test function $\phi \in C^1_0(\mathbb{R}^N)$

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla\phi(x) B_\delta(x,t) dx = \int_{\mathbb{R}^N} \nabla\phi(x) \left(\int_{\mathbb{R}} \varrho_\delta(t-s)\, B(x,s)\, ds \right) dx \\
&= \int_{\mathbb{R}} \varrho_\delta(t-s) \left(\int_{\mathbb{R}^N} \nabla\phi(x)\, B(x,s)\, dx \right) ds \\
&= -\int_{\mathbb{R}} \varrho_\delta(t-s) \left(\int_{\mathbb{R}^N} \phi(x)\, dD_x B(x,s) \right) ds \\
&= -\int_{\mathbb{R}} \varrho_\delta(t-s) \left(\int_{\mathbb{R}^N} \phi(x)\, \frac{dD_x B(x,s)}{d\sigma} d\sigma \right) ds \\
&= -\int_{\mathbb{R}^N} \phi(x) \left(\int_{\mathbb{R}} \varrho_\delta(t-s) \frac{dD_x B(x,s)}{d\sigma}\, ds \right) d\sigma \\
&= -\int_{\mathbb{R}^N} \phi(x) \left(\frac{dD_x B}{d\sigma} \right)_\delta (x,t)\, d\sigma\,.
\end{aligned}$$

Then for every $t \in \mathbb{R} \setminus \mathcal{M}_0$ the following equality holds

$$D_x B_\delta(\cdot,t) = \left(\frac{dD_x B}{d\sigma}\right)_\delta (\cdot,t)\ \sigma$$

in the sense of measures. Finally

$$|D_x B_\delta(\cdot,t)| \le \sigma$$

and for every $\delta > 0$

$$\sigma_\delta := \bigvee_{t\in\mathbb{R}} |D_x B_\delta(\cdot,t)| \le \sigma,$$

then it is a Radon measure. Hence $B_\delta(x,t)$ satisfies hypothesis (H4) of Theorem 4.1. For any compact set $H \subset \mathbb{R}$ for all $t, w \in H \setminus \mathcal{M}_0$ and $x \in \mathbb{R}^N$

$$\begin{aligned}
&|(\partial_t B_\delta)(x,t) - (\partial_t B_\delta)(x,w)| = |(\partial_t B)_\delta(x,t) - (\partial_t B)_\delta(x,w)| \\
&\le \int_{\mathbb{R}} |\partial_t\, \varrho_\delta(s)||B(x,t-s) - B(x,w-s)|\, ds \\
&\le |t-w| \int_{\mathbb{R}} \partial_t\, \varrho_\delta(s)\, ds = C_\delta |t-w|.
\end{aligned}$$

Hence $B_\delta(x,t)$ satisfies hypothesis (H2) of Theorem 4.1.

On the other hand, we will prove that for any compact set $H \subset \mathbb{R}$ by using (IV) we have

$$|\widetilde{D}_x B_\delta(\cdot,t)(A) - \widetilde{D}_x B_\delta(\cdot,w)(A)| \le |t-w|\sigma_H(A)$$

for all $t, w \in H \setminus \mathcal{M}_0$ and $A \subset \mathbb{R}^N \setminus \mathcal{N}$ Borel. In fact, by (8.2)

$$\begin{aligned}
&|\widetilde{D}_x B_\delta(\cdot,t)(A) - \widetilde{D}_x B_\delta(\cdot,w)(A)| \\
&\le \int_A \left| \frac{d\widetilde{D}_x B_\delta}{d\sigma}(x,t) - \frac{d\widetilde{D}_x B_\delta}{d\sigma}(x,w) \right| d\sigma \\
&= \int_A \left| \left(\frac{d\widetilde{D}_x B}{d\sigma} \right)_\delta (x,t) - \left(\frac{d\widetilde{D}_x B}{d\sigma} \right)_\delta (x,w) \right| d\sigma \\
&\le \int_A \int_{\mathbb{R}} \rho_\delta(s) \left| \frac{d\widetilde{D}_x B}{d\sigma}(x,s-t) - \frac{d\widetilde{D}_x B}{d\sigma}(x,s-w) \right| ds\, d\sigma \\
&\le \int_{\mathbb{R}} \rho_\delta(s) \int_A |t-w|\, d\sigma\, ds \le |t-w|\sigma(A).
\end{aligned}$$

Hence $B_\delta(x,t)$ satisfies hypothesis (H3) of Theorem 4.1. We prove that by Remark 5.2 for every $x \in \mathbb{R}^N \setminus \mathcal{N}_1$ and for every $t \in \mathbb{R} \setminus \mathcal{M}_0$ we have

$$(B^\pm)_\delta(x,t) = (B_\delta)^\pm(x,t). \tag{8.3}$$

In fact, for every $t \in \mathbb{R} \setminus \mathcal{M}_0$ we have

$$\begin{aligned}
&\lim_{r\downarrow 0} \fint_{B_r^\pm(x)} |B_\delta(y,t) - (B^\pm)_\delta(x,t)| dy \\
&= \lim_{r\downarrow 0} \fint_{B_r^\pm(x)} \left| \int_{\mathbb{R}} \varrho_\delta(t-s)\,[B(y,s) - B^\pm(x,s)]\, ds \right| dy \\
&\le \lim_{r\downarrow 0} \int_{\mathbb{R}} \varrho_\delta(t-s) \left[\fint_{B_r^\pm(x)} \left| B(y,s) - B^\pm(x,s) \right|\, dy \right] ds = 0\,.
\end{aligned}$$

Therefore $(B_\delta)^{\pm}(x,t)$ tends to $B^{\pm}(x,t)$ for every $x \in \mathbb{R}^N \setminus \mathcal{N}_1$ and for every $t \in \mathbb{R} \setminus \mathcal{M}_0$. Similarly, we have

$$\widetilde{B}_\delta(x,t) = \widetilde{B_\delta}(x,t). \tag{8.4}$$

Given a function $u \in BV_{\rm loc}(\mathbb{R}^N) \cap L^\infty_{\rm loc}(\mathbb{R}^N)$ for every $x \in \mathbb{R}^N$ we define

$$v_\delta(x) := B_\delta(x, u(x)).$$

Since the function $t \to B_\delta(x,t)$ satisfies all the hypotheses of Theorem 4.1 by using the chain rule formula we have that $v_\delta \in BV_{\rm loc}(\mathbb{R}^N)$ and

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla\phi(x) v_\delta(x)\, dx \\
&= -\int_{\mathbb{R}^N} \phi(x)(\nabla_x B_\delta)(x,u(x))\, dx - \int_{\mathbb{R}^N} \phi(x)(\partial_t B_\delta)(x,u(x)) \cdot \nabla u(x)\, dx \\
&\quad - \int_{\mathbb{R}^N} \phi(x) \frac{dD^c_x B_\delta}{d\sigma}(x,\widetilde{u}(x))\, d\sigma - \int_{\mathbb{R}^N} \phi(x)(\partial_t \widetilde{B_\delta})(x,\widetilde{u}(x)) \ \cdot dD^c u \\
&\quad - \int_{\mathcal{N}\cup J_u} \phi(x)[(B_\delta)^+(x,u^+(x)) - (B_\delta)^-(x,u^-(x))]\, \nu_{\mathcal{N}\cup J_u}(x)\, d\mathcal{H}^{N-1}.
\end{aligned}$$

Step 1. Let us now prove that

$$\lim_{\delta\to 0^+} \int_{\mathbb{R}^N} \nabla\phi(x) B_\delta(x,u(x))\, dx = \int_{\mathbb{R}^N} \nabla\phi(x)\, B(x,u(x))\, dx\,. \tag{8.5}$$

It is enough to observe that for every $x \in \mathbb{R}^N$ the functions $B_\delta(x,\cdot)$ converge in $L^1(\mathbb{R})$ to $B(x,\cdot)$ and to use the Lebesgue's dominated convergence theorem.

Step 2. We shall prove the convergence of the diffuse parts, i.e.,

$$\lim_{\delta\to 0^+} \int_{\mathbb{R}^N} \phi(x)(\partial_t \widetilde{B_\delta})(x,\widetilde{u}(x))\, d\widetilde{D}u = \int_{\mathbb{R}^N} \phi(x)(\partial_t \widetilde{B})(x,\widetilde{u}(x))\, d\widetilde{D}u. \tag{8.6}$$

By recalling that, by (8.4) and by repeating the argument in (8.1), for every $x \in \mathbb{R}^N \setminus \mathcal{N}$ and for every $t \in \mathbb{R} \setminus \mathcal{M}_0$ we have

$$(\partial_t \widetilde{B_\delta})(x,t) = (\partial_t \widetilde{B}_\delta)(x,t) = (\partial_t \widetilde{B})_\delta(x,t)$$

and by using the coarea formula (2.3), we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} \phi(x)(\partial_t \widetilde{B_\delta})(x,\widetilde{u}(x))\, d\widetilde{D}u \qquad (8.7)\\
&= \int_{C_u} \phi(x)(\partial_t \widetilde{B})_\delta(x,\widetilde{u}(x)) \frac{d\widetilde{D}u}{d|Du|}(x)\, d|Du| \\
&= \int_{-\infty}^{+\infty} dt \int_{\{u^- \le t \le u^+\}} \phi(x)(\partial_t \widetilde{B})_\delta(x,\widetilde{u}(x)) \chi_{C_u}(x) \frac{d\widetilde{D}u}{d|Du|}(x)\, d\mathcal{H}^{N-1} \\
&= \int_{-\infty}^{+\infty} dt \int_{\{\widetilde{u}=t\}\cap C_u} \phi(x)(\partial_t \widetilde{B})_\delta(x,t) \frac{d\widetilde{D}u}{d|Du|}(x)\, d\mathcal{H}^{N-1}\,.
\end{aligned}$$

For $\mathcal{H}^{N-1}$-a.e. $x \in \mathbb{R}^N \setminus \mathcal{N}$, by Remark 7.1 the map $\widetilde{B}(x,\cdot)$ is differentiable for every $t \in \mathbb{R} \setminus (\mathcal{M}_0 \cup \mathcal{M}_2)$. Hence

$$(\partial_t \widetilde{B})_\delta(x,t) \to (\partial_t \widetilde{B})(x,t)$$

as $\delta \to 0$. Therefore, for every $t \in \mathbb{R} \setminus (\mathcal{M}_0 \cup \mathcal{M}_2)$, we have

$$\begin{aligned}
&\lim_{\delta\to 0} \int_{\{\widetilde{u}=t\}\cap C_u} \phi(x)(\partial_t \widetilde{B})_\delta(x,t) \frac{d\widetilde{D}u}{d|Du|}\, d\mathcal{H}^{N-1} \\
&= \int_{\{\widetilde{u}=t\}\cap C_u} \phi(x)(\partial_t \widetilde{B})(x,t) \frac{d\widetilde{D}u}{d|Du|}\, d\mathcal{H}^{N-1}\,.
\end{aligned}$$

From this equation, using the boundedness of $\partial_t B$ and the fact that, by the coarea formula (2.3),

$$\int_{-\infty}^{+\infty} \mathcal{H}^{N-1}\left(\{\widetilde{u}=t\}\cap C_u\right) dt = |Du|(C_u) < \infty\,,$$

we can pass to the limit in (8.7) and by the Lebesgue's dominated convergence theorem we get

$$\begin{aligned}
&\lim_{\delta\to 0} \int_{\mathbb{R}^N} \phi(x)(\partial_t \widetilde{B_\delta})(x,\widetilde{u}(x))\, d\widetilde{D}u \\
&= \int_{-\infty}^{+\infty} dt \int_{\{\widetilde{u}=t\}\cap C_u} \phi(x)(\partial_t \widetilde{B})(x,t) \frac{d\widetilde{D}u}{d|Du|}(x)\, d\mathcal{H}^{N-1}\,.
\end{aligned}$$

From this equation, using the coarea formula (2.3) again, we immediately get (8.6).

Step 3. We shall prove the convergence of the jump parts, i.e.,

$$\begin{aligned}
&\lim_{\delta\to 0^+} \int_{\mathcal{N}\cup J_u} \phi(x)[(B^+)_\delta(x,u^+(x)) - (B^-)_\delta(x,u^-(x))]\, \nu_{\mathcal{N}\cup J_u}(x)\, d\mathcal{H}^{N-1} \\
&= \int_{\mathcal{N}\cup J_u} \phi(x)[B^+(x,u^+(x)) - B^-(x,u^-(x))]\, \nu_{\mathcal{N}\cup J_u}(x)\, d\mathcal{H}^{N-1}\,.
\end{aligned} \tag{8.8}$$

Firstly, we shall prove that for $\mathcal{H}^{N-1}$-a.e. $x \in \mathcal{N}\cup J_u$

$$\lim_{\delta\to 0}[(B_\delta)^+(x,u^+(x)) - (B_\delta)^-(x,u^-(x))] = B^+(x,u^+(x)) - B^-(x,u^-(x)).$$

We recall that by Proposition 6.1 (l) the one-sided limits $(\partial_t B)^+(x,t)$ and $(\partial_t B)^-(x,t)$ defined by

$$\lim_{r\downarrow 0} \fint_{B_r^\pm(x)} |(\partial_t B)(y,t) - (\partial_t B)^\pm(x,t)| dy = 0$$

exist for $\mathcal{H}^{N-1}$-a.e. $x \in \mathbb{R}^N$ and for every $t \in \mathbb{R}\setminus\mathcal{M}_0$. Moreover we have

$$\partial_t((B_\delta)^\pm)(x,t) = \partial_t((B^\pm)_\delta)(x,t) = (\partial_t(B^\pm))_\delta(x,t),$$

where the first equality is due to (8.3) and the last one can be obtained as in (8.1). Therefore, for every $\delta > 0$ we have

$$\begin{aligned}&(B_\delta)^+(x,u^+(x)) - (B_\delta)^-(x,u^-(x))\\&= \int_0^{u^+(x)} \partial_t((B_\delta)^+)(x,t)\,dt - \int_0^{u^-(x)} \partial_t((B_\delta)^-)(x,t)\,dt\\&= \int_0^{u^+(x)} (\partial_t(B^+))_\delta(x,t)\,dt - \int_0^{u^-(x)} (\partial_t(B^-))_\delta(x,t)\,dt.\end{aligned}$$

By condition (VI) for $\mathcal{H}^{N-1}$-a.e. $x \in \mathbb{R}^N$, the maps $B^\pm(x,\cdot)$ are differentiable for every $t \in \mathbb{R} \setminus (\mathcal{M}_0 \cup \mathcal{M}_1)$. Then

$$\begin{aligned}&\lim_{\delta\to 0}[(B_\delta)^+(x,u^+(x)) - (B_\delta)^-(x,u^-(x))]\\&= \int_0^{u^+(x)} \partial_t(B^+)(x,t)\,dt - \int_0^{u^-(x)} \partial_t(B^-)(x,t)\,dt\\&= B^+(x,u^+(x)) - B^-(x,u^-(x))\,.\end{aligned}\tag{8.9}$$

We estimate

$$\begin{aligned}&\Big|\int_{\mathcal{N}\cup J_u} \phi(x)\big[(B_\delta)^+(x,u^+) - (B_\delta)^-(x,u^-)\\&\qquad -[B^+(x,u^+) - B^-(x,u^-)]\big]\nu_{\mathcal{N}\cup J_u}\,d\mathcal{H}^{N-1}\Big|\\&\le \|\phi\|_\infty \int_{\mathcal{N}\cup J_u\cap\{u^+-u^-<1/h\}} \Big|(B_\delta)^+(x,u^+) - (B_\delta)^-(x,u^-)\\&\qquad -[B^+(x,u^+) - B^-(x,u^-)]\Big|\,d\mathcal{H}^{N-1}\\&\quad +\|\phi\|_\infty \int_{\mathcal{N}\cup J_u\cap\{u^+-u^-\ge 1/h\}} \Big|(B_\delta)^+(x,u^+) - (B_\delta)^-(x,u^-)\\&\qquad -[B^+(x,u^+) - B^-(x,u^-)]\Big|\,d\mathcal{H}^{N-1}\,.\end{aligned}\tag{8.10}$$

Notice that for all $\delta > 0$ and $h \in \mathbb{N}$

$$\begin{aligned}&\int_{\mathcal{N}\cup J_u\cap\{u^+-u^-<1/h\}} \Big|(B_\delta)^+(x,u^+) - (B_\delta)^-(x,u^-)\\&\qquad -[B^+(x,u^+) - B^-(x,u^-)]\Big|\,d\mathcal{H}^{N-1}\\&\le 2C \int_{\mathcal{N}\cup J_u\cap\{u^+-u^-<1/h\}} |u^+(x) - u^-(x)|\,d\mathcal{H}^{N-1}\,.\end{aligned}$$

On the other hand, the integral

$$\begin{aligned}&\int_{\mathcal{N}\cup J_u\cap\{u^+-u^-\ge 1/h\}} \Big|(B_\delta)^+(x,u^+) - (B_\delta)^-(x,u^-)\\&\qquad -[B^+(x,u^+) - B^-(x,u^-)]\Big|\,d\mathcal{H}^{N-1}\end{aligned}$$

is infinitesimal, as $\delta \to 0$, by (8.9), since $B^{\pm}$ and $B^{\pm}_{\delta}$ are bounded and since for any h by (V) we have that $\mathcal{N} \cup J_u \cap \{x \in \mathbb{R}^N : u^+(x) - u^-(x) \geq 1/h\}$ is a set of finite $\mathcal{H}^{N-1}$ measure. Therefore, from (8.10), letting first δ tend to zero and then h tend to ∞, we immediately obtain (8.8).

We recall that by Corollary 6.2 for σ-a.e. $x \in \mathbb{R}^N$ and for every $w \in \mathbb{R} \setminus \mathcal{M}_0$

$$\frac{dD_xB}{d\sigma}(x,w) = \int_0^w \frac{dD_x(\partial_t B)}{d\sigma}(x,t)\,dt.$$

Similarly we have

$$\frac{dD_x(B_\delta)}{d\sigma}(x,w) = \int_0^w \frac{dD_x(\partial_t (B_\delta))}{d\sigma}(x,t)\,dt.$$

Moreover, as in (8.2) for every $t \in \mathbb{R} \setminus \mathcal{M}_0$ the following equality holds

$$D_x(\partial_t B)_\delta(\cdot,t) = \left(\frac{dD_x(\partial_t B)}{d\sigma}\right)_\delta (\cdot,t)\ \sigma$$

in the sense of measures. Then for every $w \in \mathbb{R} \setminus \mathcal{M}_0$ we have

$$\begin{aligned}\frac{dD_xB_\delta}{d\sigma}(x,w) &= \int_0^w \frac{dD_x(\partial_t B_\delta)}{d\sigma}(x,t)\,dt \\ &= \int_0^w \frac{dD_x(\partial_t B)_\delta}{d\sigma}(x,t)\,dt = \int_0^w \left(\frac{dD_x(\partial_t B)}{d\sigma}\right)_\delta (x,t)\,dt\end{aligned}$$

In particular, for every $w \in \mathbb{R} \setminus \mathcal{M}_0$ we have

$$\nabla_x B_\delta(x,w) = \int_0^w \left(\nabla_x(\partial_t B)\right)_\delta (x,t)\,dt \quad \text{for } \mathcal{L}^N - \text{a.e. } x \in \mathbb{R}^N \tag{8.11}$$

and

$$\frac{dD^c_xB_\delta}{d\sigma}(x,w) = \int_0^w \left(\frac{dD^c_x(\partial_t B)}{d\sigma}\right)_\delta (x,t)\,dt \quad \text{for } \sigma - \text{a.e. } x \in \mathbb{R}^N. \tag{8.12}$$

STEP 4. We shall prove that

$$\lim_{\delta\to 0^+} \int_{\mathbb{R}^N} \phi(x)(\nabla_x B_\delta)(x,u(x))\,dx = \int_{\mathbb{R}^N} \phi(x)(\nabla_x B)(x,u(x))\,dx. \tag{8.13}$$

By Proposition 6.1 (m) there exists a Borel set $\hat{\mathcal{N}_2} \subset \mathbb{R}^N$ such that $\mathcal{L}^N(\hat{\mathcal{N}_2}) = 0$ and the approximate gradient $\nabla_x(\partial_t B)(x,t)$ of the function $y \mapsto (\partial_t B)(y,t)$ at x exists for every $x \in \mathbb{R}^N \setminus \hat{\mathcal{N}_2}$ and for every $t \in \mathbb{R} \setminus \mathcal{M}_0$ and

$$\frac{dD_x(\partial_t B)(\cdot,t)}{d\mathcal{L}^N}(x) = \nabla_x(\partial_t B)(x,t)$$

for every $x \in \mathbb{R}^N \setminus (\mathcal{N} \cup \hat{\mathcal{N}_2})$ and for every $t \in \mathbb{R} \setminus \mathcal{M}_0$. By (8.11) we get

$$\nabla_x B_\delta(x,w) = \int_0^w \nabla_x((\partial_t B)_\delta)(x,t)\,dt = \int_0^w (\nabla_x(\partial_t B))_\delta(x,t)\,dt$$

for every $x \in \mathbb{R}^N \setminus (\mathcal{N} \cup \hat{\mathcal{N}_2})$ and for every $w \in \mathbb{R} \setminus \mathcal{M}_0$. This term tends to

$$\int_0^w \nabla_x(\partial_t B)(x,t)\,dt = \nabla_x B(x,w),$$

where the last equality is due to (6.1). The conclusion follows by Lebesgue Dominated Convergence Theorem.

STEP 5. We shall prove that

$$\lim_{\delta\to 0^+}\int_{\mathbb{R}^N}\phi(x)\frac{dD^c_x B_\delta}{d\sigma}(x,\widetilde{u}(x))\,d\sigma=\int_{\mathbb{R}^N}\phi(x)\frac{dD^c_x B}{d\sigma}(x,\widetilde{u}(x))\,d\sigma\,. \tag{8.14}$$

By Proposition 6.1 (n), there exists a Borel set $\hat{\mathcal{N}_0}\subseteq\mathbb{R}^N$ such that $\sigma(\hat{\mathcal{N}_0})=0$ such that the following limit

$$\lim_{r\downarrow 0}\frac{D^c_x(\partial_t B)(\cdot,t)(B_r(x))}{\sigma(B_r(x))}=\frac{dD^c_x(\partial_t B)(\cdot,t)}{d\sigma}(x)$$

exists for every $x\in\mathbb{R}^N\setminus\hat{\mathcal{N}_0}$ and for every $t\in\mathbb{R}\setminus\mathcal{M}_0$ and this equality holds, where $\frac{dD^c_x(\partial_t B)(\cdot,t)}{d\sigma}(x)$ is Radon-Nikodým derivative at x of the Cantor part of the measure $D_x(\partial_t B)(\cdot,t)$ w.r.t. σ. By (8.12) we get

$$\frac{dD^c_x B_\delta}{d\sigma}(x,w)=\int_0^w\frac{dD^c_x(\partial_t B)_\delta}{d\sigma}(x,t)\,dt=\int_0^w\left(\frac{dD^c_x(\partial_t B)}{d\sigma}\right)_\delta(x,t)\,dt. \tag{8.15}$$

for every $x\in\mathbb{R}^N\setminus(\mathcal{N}\cup\hat{\mathcal{N}_0})$ and for every $w\in\mathbb{R}\setminus\mathcal{M}_0$. By (6.3), this term tends to

$$\int_0^w\frac{dD^c_x(\partial_t B)}{d\sigma}(x,t)\,dt=\frac{dD^c_x B}{d\sigma}(x,w).$$

Then by Dominated Lebesgue Convergence Theorem, condition (8.14) holds.

Therefore, the assertion follows at once from (8.5), (8.6), (8.8), (8.13), (8.14) and from equation (8.5). □

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