SYMPLECTIC G-CAPACITIES AND INTEGRABLE SYSTEMS

ALESSIO FIGALLI JOSEPH PALMER ÁLVARO PELAYO

ABSTRACT. For any Lie group G, we construct a G-equivariant analogue of symplectic capacities and give examples when $G = \mathbb{T}^k \times \mathbb{R}^{d-k}$, in which case the capacity is an invariant of integrable systems. Then we study the continuity of these capacities, using the natural topologies on the symplectic G-categories on which they are defined.

1. INTRODUCTION

In the 1980s Gromov proved the symplectic non-squeezing theorem [9]. This influential result says that a ball of radius $r > 0$ can be symplectically embedded into a cylinder of radius $R > 0$ only if $r \leqslant R$. This led to the first symplectic capacity, the *Gromov radius*, which is the radius of the largest ball of the same dimension which can be symplectically embedded into a symplectic manifold (M, ω). Symplectic capacities are a class of symplectic invariants introduced by Ekeland and Hofer [6, 11].

In this paper we give a notion of symplectic capacity for symplectic G-manifolds, where G is any Lie group, which we call a *symplectic G-capacity*, and give nontrivial examples. Such a capacity retains the properties of a symplectic capacity (monotonicity, conformality, and an analogue of non-triviality) with respect to symplectic G-embeddings. Symplectic capacities are examples of symplectic G-capacities in the case that G is trivial. In analogy with symplectic capacities, symplectic G-capacities distinguish the symplectic G-type of symplectic G-manifolds. As a first example we construct an equivariant analogue of the Gromov radius where $G = \mathbb{R}^k$ as follows. Let $\mathrm{Symp}^{2n,G}$ denote the category of $2n$-dimensional symplectic G-manifolds. That is, an element of $\mathrm{Symp}^{2n,G}$ is a triple (M, ω, ϕ) where (M, ω) is a symplectic manifold and $\phi\colon G \times M \to M$ is a symplectic G-action. Given integers $0 \leqslant k \leqslant m \leqslant n$ we define the *(m, k)-equivariant Gromov radius*

$$\begin{aligned} (1) \qquad & c_{\mathrm{B}}^{m,k}\colon \mathrm{Symp}^{2n,\mathbb{R}^k} \to [0, \infty] \\ & (M, \omega, \phi) \mapsto \sup\{\, r \geqslant 0 \mid \mathrm{B}^{2m}(r) \xhookrightarrow{\mathbb{R}^k} M \,\}, \end{aligned}$$

where $\xhookrightarrow{\mathbb{R}^k}$ denotes a symplectic $\mathbb{R}^k$-embedding and $\mathrm{B}^{2m}(r) \subset \mathbb{C}^m$ is the standard $2m$-dimensional ball of radius $r > 0$ with $\mathbb{R}^k$-action given by rotation of the first k coordinates. Thanks to the added structure of the $\mathbb{R}^k$-action the proof that $c_{\mathrm{B}}^{m,k}$ is a $\mathbb{R}^k$-capacity for $k \geqslant 1$ uses only elementary techniques.

As an application of symplectic G-capacities to integrable systems we define the *toric packing capacity*

$$\begin{aligned} (2) \qquad & \mathcal{T}\colon \mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{T}^n} \to [0, \infty] \\ & (M, \omega, \phi) \mapsto \left(\frac{\sup\{\, \mathrm{vol}(P) \mid P \text{ is a toric ball packing of } M \,\}}{\mathrm{vol}(\mathrm{B}^{2n})} \right)^{\frac{1}{2n}}, \end{aligned}$$

where $\mathrm{vol}(E)$ denotes the symplectic volume of a subset E of a symplectic manifold, B^{2n} is the standard symplectic unit $2n$-ball, $\mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{T}^n}$ is the category of $2n$-dimensional symplectic toric manifolds, and a toric ball packing P of M is given by a disjoint collection of symplecticly and $\mathbb{T}^n$-equivariantly embedded balls. In analogy we define the *semitoric packing capacity*

$$\mathcal{ST}\colon \mathrm{Symp}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}} \to [0,\infty]$$

on $\mathrm{Symp}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}}$, the category of semitoric manifolds [20], where P in (2) is replaced by a semitoric ball packing of M (Definition 5.2). The following theorem is a combination of Propositions 2.7, 4.2, and 5.5.

Theorem 1.1 (Examples of capacities)**.** *The following hold:*

(i) *The (m,k)-equivariant Gromov radius $c_{\mathrm{B}}^{m,k}\colon \mathrm{Symp}^{2n,\mathbb{R}^k} \to [0,\infty]$ is a symplectic $\mathbb{R}^k$-capacity for $k \geqslant 1$;*

(ii) *The toric packing capacity $\mathcal{T}\colon \mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{T}^n} \to [0,\infty]$ is a symplectic $\mathbb{T}^n$-capacity;*

(iii) *The semitoric packing capacity $\mathcal{ST}\colon \mathrm{Symp}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}} \to [0,\infty]$ is a symplectic $(S^1\times\mathbb{R})$-capacity.*

The continuity of symplectic capacities is discussed in [2, 3, 6, 27]. The semitoric and toric packing capacities are each defined on categories of integrable systems which have a natural topology [15, 18], but we can only discuss the continuity of the (m,k)-equivariant Gromov radius on a subcategory of its domain which has a topology, so we restrict to the case of $(m,k)=(n,n)$. The $\mathbb{T}^n$-action on a symplectic toric manifold may be lifted to an action of $\mathbb{R}^n$. Let $\mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{R}^n}$ be the symplectic category of symplectic toric manifolds each of which is endowed with the $\mathbb{R}^n$-action obtained by lifting the given $\mathbb{T}^n$-action which is a subcategory of $\mathrm{Symp}^{2n,\mathbb{R}^n}$.

Theorem 1.2 (Continuity of capacities)**.** *The following hold:*

(i) *The toric packing capacity $\mathcal{T}\colon \mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{T}^n} \to [0,\infty]$ is everywhere discontinuous and the restriction of $\mathcal{T}$ to the space of symplectic toric $2n$-dimensional manifolds with exactly N fixed points of the $\mathbb{T}^n$-action is continuous for any choice of $N \geqslant 0$;*

(ii) *The semitoric packing capacity $\mathcal{ST}\colon \mathrm{Symp}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}} \to [0,\infty]$ is everywhere discontinuous and the restriction of $\mathcal{ST}$ to the space of semitoric manifolds with exactly N elliptic-elliptic fixed points of the associated $(S^1\times\mathbb{R})$-action is continuous for any choice of $N \geqslant 0$;*

(iii) *The (n,n)-equivariant Gromov radius restricted to the space of symplectic toric manifolds*

$$c_{\mathrm{B}}^{n,n}|_{\mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{R}^n}}\colon \mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{R}^n} \to [0,\infty]$$

is everywhere discontinuous and the restriction of $c_{\mathrm{B}}^{n,n}|_{\mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{R}^n}}$ to the space of symplectic toric $2n$-dimensional manifolds with exactly N fixed points of the $\mathbb{R}^n$-action is continuous for any choice of $N \geqslant 0$.

Theorem 1.2 generalizes [7, Theorem A], which deals with 4-manifolds, and solves [18, Problem 30]. Theorem 1.2 is implied by Theorems 6.3, 7.12, and 7.15.

In Section 2 we give a general notion of symplectic G-capacities and we prove that the (m,k)-equivariant Gromov radius is a capacity. In Section 3 we review facts about Hamiltonian actions and their relation to integrable systems that will be needed in the remainder of the paper. Sections 4 and 5 are devoted to constructing nontrivial symplectic G-capacities when $G=\mathbb{T}^k\times\mathbb{R}^{d-k}$, which include the toric and semitoric packing capacities. In Sections 6 and 7 we discuss the continuity of these symplectic G-capacities.

Acknowledgements. The first author is supported by NSF grants DMS-1262411 and DMS-1361122. The second and third authors are supported by NSF grants DMS-1055897 and DMS-1518420.

2. Symplectic G-capacities

For $n \geqslant 1$ and $r > 0$ let $\mathrm{B}^{2n}(r) \subset \mathbb{C}^n$ be the $2n$-dimensional open symplectic ball of radius r and let

$$\mathrm{Z}^{2n}(r) = \{ (z_i)_{i=1}^n \in \mathbb{C}^n \mid |z_1| < r \}$$

be the $2n$-dimensional open symplectic cylinder of radius r. Both inherit a symplectic structure from their embedding as a subset of $\mathbb{C}^n$ with symplectic form $\omega_0 = \frac{\mathrm{i}}{2}\sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$. We write $\mathrm{B}^{2n} = \mathrm{B}^{2n}(1)$, $\mathrm{Z}^{2n} = \mathrm{Z}^{2n}(1)$, and use $\hookrightarrow$ to denote a symplectic embedding.

2.1. Symplectic capacities.

Let Symp^{2n} be the category of symplectic $2n$-dimensional manifolds with symplectic embeddings as morphisms. A *symplectic category* is a subcategory $\mathcal{C}$ of Symp^{2n} such that $(M, \omega) \in \mathcal{C}$ implies $(M, \lambda\omega) \in \mathcal{C}$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. Let $\mathcal{C} \subset \mathrm{Symp}^{2n}$ be a symplectic category.

The following fundamental notion of symplectic invariant is due to Ekeland and Hofer.

Definition 2.1 ([6, 11])**.** A *generalized symplectic capacity* on $\mathcal{C}$ is a map $c\colon \mathcal{C} \to [0, \infty]$ satisfying:

1. *Monotonicity*: if $(M, \omega), (M', \omega') \in \mathcal{C}$ and there exists a symplectic embedding $M \hookrightarrow M'$ then $c(M, \omega) \leqslant c(M', \omega')$;
2. *Conformality*: if $\lambda \in \mathbb{R} \setminus \{0\}$ and $(M, \omega) \in \mathcal{C}$ then $c(M, \lambda\omega) = |\lambda|\, c(M, \omega)$.

If additionally $\mathrm{B}^{2n}, \mathrm{Z}^{2n} \in \mathcal{C}$ and c satisfies:

3. *Non-triviality*: $0 < c(\mathrm{Z}^{2n}, \omega_0) < \infty$ and $0 < c(\mathrm{B}^{2n}, \omega_0) < \infty$;

then c is a *symplectic capacity.*

2.2. Symplectic G-capacities.

Let G be a Lie group and let $\mathrm{Sympl}(M)$ denote the group of symplectomorphisms of the symplectic manifold (M, ω). A smooth G-action $\phi\colon G \times M \to M$ is *symplectic* if $\phi(g, \cdot) \in \mathrm{Sympl}(M)$ for each $g \in G$. The triple (M, ω, ϕ) is a *symplectic G-manifold.* A *symplectic G-embedding* $\rho\colon (M_1, \omega_1, \phi_1) \hookrightarrow (M_2, \omega_2, \phi_2)$ is a symplectic embedding for which there exists an automorphism $\Lambda\colon G \to G$ of G such that $\rho(\phi_1(g, p)) = \phi_2(\Lambda(g), \rho(p))$ for all $p \in M_1$, $g \in G$, in which case we say that ρ is a symplectic G-embedding with respect to Λ. We write $\overset{G}{\hookrightarrow}$ to denote a symplectic G-embedding. We denote the collection of all $2n$-dimensional symplectic G-manifolds by $\mathrm{Symp}^{2n,G}$. The set $\mathrm{Symp}^{2n,G}$ is a category with morphisms given by symplectic G-embeddings. We call a subcategory $\mathcal{C}_G$ of $\mathrm{Symp}^{2n,G}$ a *symplectic G-category* if $(M, \omega, \phi) \in \mathcal{C}_G$ implies $(M, \lambda\omega, \phi) \in \mathcal{C}_G$ for any $\lambda \in \mathbb{R} \setminus \{0\}$. Let $\mathcal{C}_G \subset \mathrm{Symp}^{2n,G}$ be a symplectic G-category.

Definition 2.2. A *generalized symplectic G-capacity* on $\mathcal{C}_G$ is a map $c\colon \mathcal{C}_G \to [0, \infty]$ satisfying:

1. *Monotonicity*: if $(M, \omega, \phi), (M', \omega', \phi') \in \mathcal{C}_G$ and there exists a symplectic G-embedding $M \overset{G}{\hookrightarrow} M'$ then $c(M, \omega, \phi) \leqslant c(M', \omega', \phi')$;
2. *Conformality*: if $\lambda \in \mathbb{R} \setminus \{0\}$ and $(M, \omega, \phi) \in \mathcal{C}_G$ then $c(M, \lambda\omega, \phi) = |\lambda|\, c(M, \omega, \phi)$.

When the symplectic form and G-action are understood we often write $c(M)$ for $c(M, \omega, \phi)$. Let c be a generalized symplectic G-capacity on a symplectic G-category $\mathcal{C}_G$.

Definition 2.3. For $(N, \omega_N, \phi_N) \in \mathcal{C}_G$ we say that c satisfies *N-non-triviality* or is *non-trivial on* N if $0 < c(N) < \infty$.

Definition 2.4. We say that c is *tamed* by $(N, \omega_N, \phi_N) \in \mathrm{Symp}^{2n,G}$ if there exists some $a \in (0, \infty)$ such that the following two properties hold:
(1) if $M \in \mathcal{C}_G$ and there exists a symplectic G-embedding $M \overset{G}{\hookrightarrow} N$ then $c(M) \leqslant a$;
(2) if $P \in \mathcal{C}_G$ and there exists a symplectic G-embedding $N \overset{G}{\hookrightarrow} P$ then $a \leqslant c(P)$.

The non-triviality condition in Definition 2.1 requires that $\mathrm{B}^{2n}, \mathrm{Z}^{2n} \in \mathcal{C}_G$ and $0 < c(\mathrm{B}^{2n}) \leqslant c(\mathrm{Z}^{2n}) < \infty$, and tameness encodes this second condition without necessarily including the first one. If c is a generalized symplectic G-capacity on $\mathcal{C}_G \subset \mathrm{Symp}^{2n,G}$ we define

$$\begin{aligned}
\mathrm{Symp}_0^{2n,G}(c) &= \{\, N \in \mathrm{Symp}^{2n,G} \mid \inf\{\, c(P) \mid P \in \mathcal{C}_G, N \overset{G}{\hookrightarrow} P \,\} = 0 \,\},\\
\mathrm{Symp}_\infty^{2n,G}(c) &= \{\, N \in \mathrm{Symp}^{2n,G} \mid \sup\{\, c(M) \mid M \in \mathcal{C}_G, M \overset{G}{\hookrightarrow} N \,\} = \infty \,\},\\
\mathrm{Symp}_{\mathrm{tame}}^{2n,G}(c) &= \{\, N \in \mathrm{Symp}^{2n,G} \mid c \text{ is tamed by } N \,\}.
\end{aligned}$$

A generalized symplectic G-capacity gives rise to a decomposition of $\mathrm{Symp}^{2n,G}$.

Proposition 2.5. *Let c be a generalized symplectic G-capacity on a symplectic G-category $\mathcal{C}_G$. Then:*

(a) $\mathrm{Symp}^{2n,G} = \mathrm{Symp}_0^{2n,G}(c) \cup \mathrm{Symp}_\infty^{2n,G}(c) \cup \mathrm{Symp}_{\mathrm{tame}}^{2n,G}(c)$*;*
(b) *the union in part (a) is pairwise disjoint;*
(c) *c is non-trivial on $N \in \mathrm{Symp}^{2n,G}$ if and only if $N \in \mathcal{C}_G \cap \mathrm{Symp}_{\mathrm{tame}}^{2n,G}(c)$.*

Proof. In order to prove item (a) we show that if $N \in \mathrm{Symp}^{2n,G}$ is not in $\mathrm{Symp}_0^{2n,G}(c) \cup \mathrm{Symp}_\infty^{2n,G}(c)$ then it is in $\mathrm{Symp}_{\mathrm{tame}}^{2n,G}(c)$. If $M \overset{G}{\hookrightarrow} N \overset{G}{\hookrightarrow} P$ for some $M, P \in \mathcal{C}_G$ then $M \overset{G}{\hookrightarrow} P$ so $c(M) \leqslant c(P)$. Let $a_1 = \sup\{\, c(M) \mid M \overset{G}{\hookrightarrow} N \,\}$ and $a_2 = \inf\{\, c(P) \mid N \overset{G}{\hookrightarrow} P \,\}$. Since $N \notin \mathrm{Symp}_0^{2n,G}(c) \cup \mathrm{Symp}_\infty^{2n,G}(c)$ we have that $0 < a_1 \leqslant a_2 < \infty$. Pick $a \in [a_1, a_2]$. If $M \in \mathcal{C}_G$ and $M \overset{G}{\hookrightarrow} N$ then $c(M) \leqslant a_1 \leqslant a$ and if $P \in \mathcal{C}_G$ and $N \overset{G}{\hookrightarrow} P$ then $c(P) \geqslant a_2 \geqslant a$ so $N \in \mathrm{Symp}_{\mathrm{tame}}^{2n,G}(c)$. Item (b) follows from a similar argument and (c) is immediate. □

In light of item (c) we view $\mathrm{Symp}_{\mathrm{tame}}^{2n,G}(c)$ as an extension of the set of elements of $\mathrm{Symp}^{2n,G}$ on which c is non-trivial to include those elements outside of the domain of c.

2.3. **Symplectic $(\mathbb{T}^k \times \mathbb{R}^{d-k})$-capacities.** For $1 \leqslant d \leqslant n$ the standard action of $\mathbb{T}^d$ on $\mathbb{C}^n$ is given by

$$\phi_{\mathbb{C}^n}\big((\alpha_i)_{i=1}^d, (z_i)_{i=1}^n\big) = (\alpha_1 z_1, \ldots, \alpha_d z_d, z_{d+1}, \ldots, z_n).$$

This action induces actions of $\mathbb{T}^d = \mathbb{T}^k \times \mathbb{T}^{d-k}$ on B^{2n} and Z^{2n}, which in turn induce the standard actions of $\mathbb{T}^k \times \mathbb{R}^{d-k}$ on B^{2n} and Z^{2n} for $k \leqslant d$. The action of an element of $\mathbb{T}^k \times \mathbb{R}^{d-k}$ is the action of its image under the quotient map $\mathbb{T}^k \times \mathbb{R}^{d-k} \to \mathbb{T}^d$. In the following we endow B^{2n} and Z^{2n} with the standard actions.

Definition 2.6. A generalized symplectic $(\mathbb{T}^k \times \mathbb{R}^{d-k})$-capacity is a *symplectic $(\mathbb{T}^k \times \mathbb{R}^{d-k})$-capacity* if it is tamed by B^{2n} and Z^{2n}.

2.4. **A first example.** The *Gromov radius* $c_{\mathrm{B}} \colon \mathrm{Symp}^{2n} \to (0, \infty]$ is given by

$$c_{\mathrm{B}}(M) := \sup\{\, r > 0 \mid \mathrm{B}^{2n}(r) \hookrightarrow M \,\}.$$

Fix $0 \leqslant k \leqslant m \leqslant n$ and let $c_{\mathrm{B}}^{m,k}$ be as in Equation (1). If $k = 0$ and $m = n$ then $c_{\mathrm{B}} = c_{\mathrm{B}}^{m,k}$.

Proposition 2.7. *If $k \geqslant 1$, the (m,k)-equivariant Gromov radius $c_{\mathrm{B}}^{m,k} \colon \mathrm{Symp}^{2n,\mathbb{R}^k} \to [0, \infty]$ is a symplectic $\mathbb{R}^k$-capacity.*

Proof. Parts (1) and (2) of Definition 2.2 are immediate. By the standard inclusion map $c_{\mathrm{B}}^{m,k}(\mathrm{B}^{2n}) \geqslant 1$ so we only must show that $c_{\mathrm{B}}^{m,k}(\mathrm{Z}^{2n}) \leqslant 1$. Suppose that for $r > 1$ $\rho \colon \mathrm{B}^{2m}(r) \overset{\mathbb{R}^k}{\hookrightarrow} \mathrm{Z}^{2n}$ is a symplectic $\mathbb{R}^k$-embedding with respect to some $\Lambda \in \mathrm{Aut}(\mathbb{R}^k)$. Let

$$(\eta_1, \ldots, \eta_k) = \Lambda^{-1}(1, 0, \ldots, 0).$$

Since Λ is an automorphism $\eta_{j_0} \neq 0$ for some $j_0 \in \{1, \dots, k\}$. Pick

$$w = (0, \dots, 0, w_{j_0}, 0, \dots, 0) \in \mathrm{B}^{2m}(r)$$

with entries all zero except in the j_0^{th} position and such that $|w_{j_0}| > 1$. Let $u = (u_1, \dots, u_n) = \rho(w)$ and note $|u_1| < 1$. Let $\iota \colon \mathbb{R} \hookrightarrow \mathbb{R}^k$ be given by $\iota(x) = (x, 0, \dots, 0)$. Let $\phi_{\mathrm{B}} \colon \mathbb{R}^k \times \mathrm{B}^{2m}(r) \to \mathrm{B}^{2m}(r)$ and $\phi_{\mathrm{Z}} \colon \mathbb{R}^k \times \mathrm{Z}^{2n} \to \mathrm{Z}^{2n}$ be the standard actions of $\mathbb{R}^k$. Then for $x \in \mathbb{R}$

$$\rho\big(\phi_{\mathrm{B}}(\Lambda^{-1} \circ \iota(x), w)\big) = \phi_{\mathrm{Z}}(\iota(x), \rho(w)) = \phi_{\mathrm{Z}}(\iota(x), u).$$

Thus

$$\rho\big(\{\, (0, \dots, \mathrm{e}^{2\mathrm{i}x\eta_{j_0}} w_{j_0}, 0, \dots, 0) \mid x \in \mathbb{R} \,\}\big) = \{\, (\mathrm{e}^{2\mathrm{i}x} u_1, u_2, \dots, u_n) \mid x \in \mathbb{R} \,\} \tag{3}$$

and since ρ is injective and $\eta_{j_0} \neq 0$ this means that $u_1 \neq 0$. Let

$$S_{\mathrm{B}} = \{\, (0, \dots, 0, \alpha, 0, \dots, 0) \in \mathrm{B}^{2m}(r) \mid |\alpha| < |w_{j_0}| \,\}$$

where α is in the j_0^{th} position and

$$S_{\mathrm{Z}} = \{\, (\beta, u_2, \dots, u_n) \in \mathrm{Z}^{2n} \mid |\beta| < |u_1| \,\}.$$

Equation (3) implies that $\rho(\partial S_{\mathrm{B}}) = \partial S_{\mathrm{Z}}$ and since ρ is an embedding this means $\partial(\rho(S_{\mathrm{B}})) = \partial S_{\mathrm{Z}}$. Since $\rho(S_{\mathrm{B}})$ and S_{Z} have the same boundary, ω_{Z} is closed, and Z^{2n} has trivial second homotopy group,

$$\int_{\rho(S_{\mathrm{B}})} \omega_{\mathrm{Z}} = \int_{S_{\mathrm{Z}}} \omega_{\mathrm{Z}}.$$

Finally, integrating over z we have

$$\frac{\mathrm{i}}{2} \int_{|z|<|w_j|} \mathrm{d}z \wedge \mathrm{d}\bar{z} = \int_{S_{\mathrm{B}}} w_{\mathrm{B}} = \int_{S_{\mathrm{B}}} \rho^* \omega_{\mathrm{Z}} = \int_{\rho(S_{\mathrm{B}})} \omega_{\mathrm{Z}} = \int_{S_{\mathrm{Z}}} \omega_{\mathrm{Z}} = \frac{\mathrm{i}}{2} \int_{|z|<|u_1|} \mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

This implies that $1 < |w_j| = |u_1| < 1$, which is a contradiction.

□

It follows from the proof that $c_{\mathrm{B}}^{m,k}(\mathrm{B}^{2n}) = c_{\mathrm{B}}^{m,k}(\mathrm{Z}^{2n}) = 1$.

Proposition 2.8. *Let $M = (S^2)^n$ with symplectic form $\omega_M = \frac{1}{2} \sum_{i=1}^n \mathrm{d}h_i \wedge \mathrm{d}\theta_i$ where $h_i \in [-1, 1]$, $\theta_i \in [0, 2\pi)$, $i = 1, \dots, n$, are the standard height and angle coordinates. Let $\mathbb{R}^k$, $1 \leqslant k \leqslant n$, act on M by rotating the first k components. Then*

$$c_{\mathrm{B}}^{m,k}(M) = \sqrt{2}$$

for all $m, k \in \mathbb{Z}$ with $1 \leqslant k \leqslant m \leqslant n$.

Proof. The map $\rho \colon \mathrm{B}^{2n}(\sqrt{2}) \xhookrightarrow{\mathbb{R}^n} M$ given by

$$\rho(r_1 e^{\mathrm{i}\theta_1}, \dots, r_n e^{\mathrm{i}\theta_n}) = (\theta_1, r_1^2 - 1, \dots, \theta_n, r_n^2 - 1)$$

is a symplectic $\mathbb{R}^n$-embedding, so $c_{\mathrm{B}}^{n,n}(M) \geqslant \sqrt{2}$.

Fix $k, m, n \in \mathbb{Z}$ satisfying $0 < k \leqslant m \leqslant n$ and let $\rho : \mathrm{B}^{2m}(r) \xhookrightarrow{\mathbb{R}^k} M$ be a symplectic $\mathbb{R}^k$-embedding for some $r > 0$. Let

$$B_j = \{\, (h_i, \theta_i)_{i=1}^n \in M \mid h_i \in \{\pm 1\} \text{ if } i \leqslant k \text{ and } i \neq j \,\}$$

for $j = 1, \dots, k$. For $R \in (0, r)$ let

$$A_R = \{\, (z, 0, \dots, 0) \in \mathrm{B}^{2m}(r) \mid |z| < R \,\}.$$

Every point in A_R, except at the identity, has the same $(k-1)$-dimensional stabilizer in $\mathbb{R}^k$ so there exists $j_0 \leqslant k$ such that $\rho(A_R) \subset B_{j_0}$ for all $R \in (0, r)$. Write $\rho = (H_i, \Theta_i)_{i=1}^n$ and consider coordinates (r, θ) on A_R given by $(r\mathrm{e}^{\mathrm{i}\theta}, 0, \dots, 0) \to (r, \theta)$. For $i \neq j_0$ this means that H_i is constant if $i \leqslant k$ and the $\mathbb{R}^n$-equivariance of ρ implies that H_i and Θ_i are independent of θ if $i > k$. Thus if $i \in \{1, \ldots, n\}$ and $i \neq j_0$ then

$$\int_{\rho(A_R)} \mathrm{d}h_i \wedge \mathrm{d}\theta_i = \int_{A_R} \mathrm{d}H_i \wedge \mathrm{d}\Theta_i = 0$$

for $R \in (0, r)$. Therefore,

$$\pi R^2 = \int_{A_R} \omega_{\mathrm{B}} = \int_{\rho(A_R)} \omega_M = \frac{1}{2} \int_{\rho(A_R)} \mathrm{d}h_{j_0} \wedge \mathrm{d}\theta_{j_0} + \frac{1}{2} \sum_{i \neq j_0} \left(\int_{\rho(A_R)} \mathrm{d}h_i \wedge \mathrm{d}\theta_i \right) \leqslant \frac{1}{2} \int_{S^2} \mathrm{d}h \wedge \mathrm{d}\theta = 2\pi$$

for any $R \in (0, r)$. This implies that $r \leqslant \sqrt{2}$ so

$$\sqrt{2} \leqslant c_{\mathrm{B}}^{n,n}(M) \leqslant c_{\mathrm{B}}^{m,k}(M) \leqslant \sqrt{2}$$

for any $k, m, n \in \mathbb{Z}$ satisfying $0 < k \leqslant m \leqslant n$. $\square$

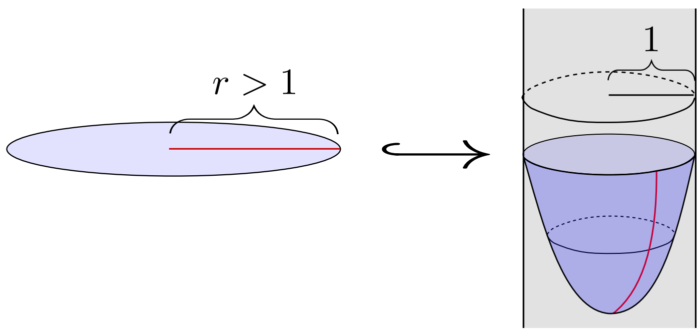

Figure 1. A symplectic $\mathbb{R}$-embedding.

Example 2.9. For $k, n \in \mathbb{Z}_{>0}$ with $k < n$ let $M = \mathrm{Z}^{2n}$ with the standard symplectic form. There are two natural ways in which $\mathbb{R}^k$ can act symplectically on M given by

$$\phi_1((t_i)_{i=1}^k, (z_i)_{i=1}^n) = (\mathrm{e}^{2\mathrm{i}t_1} z_1, \mathrm{e}^{2\mathrm{i}t_2} z_2, \ldots, \mathrm{e}^{2\mathrm{i}t_k} z_k, z_{k+1}, \ldots, z_n)$$

and

$$\phi_2((t)_{i=1}^k, (z_i)_{i=1}^n) = (z_1, \mathrm{e}^{2\mathrm{i}t_1} z_2, \ldots, \mathrm{e}^{2\mathrm{i}t_k} z_{k+1}, z_{k+2}, \ldots, z_n)$$

where $\phi_i \colon \mathbb{R}^k \times M \to M$ for $i = 1, 2$. Let $\rho : M \to M$ be given by

$$\rho((z_i)_{i=1}^n) = \left(\frac{z_{k+1}}{1 + |z_{k+1}|}, \frac{z_1}{1 - |z_1|}, z_2, \ldots, z_k, z_{k+2}, \ldots, z_n \right)$$

similar to the map shown in Figure 1. The map ρ is well-defined because $|z_1| < 1$ and it is an $\mathbb{R}^k$-equivariant diffeomorphism because

$$\begin{aligned} \rho\big(\phi_1((t_i)_{i=1}^k, (z_i)_{i=1}^n)\big) &= \left(\frac{z_{k+1}}{1 + |z_{k+1}|}, \mathrm{e}^{2\mathrm{i}t_1} \frac{z_1}{1 - |z_1|}, \mathrm{e}^{2\mathrm{i}t_2} z_2, \ldots, \mathrm{e}^{2\mathrm{i}t_k} z_k, z_{k+2}, \ldots, z_n \right) \\ &= \phi_2\left((t)_{i=1}^k, \rho((z_i)_{i=1}^n) \right) \end{aligned}$$

for all $t_1, \ldots, t_k \in \mathbb{R}$. Thus the symplectic $\mathbb{R}^k$-manifolds (M, ω, ϕ_1) and (M, ω, ϕ_2) are symplectomorphic via the identity map and $\mathbb{R}^k$-equivariantly diffeomorphic via ρ but they are not $\mathbb{R}^k$-equivariantly symplectomorphic because $c_{\mathrm{B}}^{1,1}(M, \omega, \phi_1) = 1$ and $c_{\mathrm{B}}^{1,1}(M, \omega, \phi_2) = \infty$. $\oslash$

3. Hamiltonian ($\mathbb{T}^k \times \mathbb{R}^{n-k}$)-actions

In this section we review the facts we need for the remainder of the paper about Hamiltonian ($\mathbb{T}^k \times \mathbb{R}^{n-k}$)-actions and their relation to toric and semitoric systems. Let (M,ω) be a symplectic manifold and G a Lie group with Lie algebra $\mathrm{Lie}(G)$ and dual Lie algebra $\mathrm{Lie}(G)^*$. A symplectic G-action is *Hamiltonian* if there exists a map $\mu\colon M \to \mathrm{Lie}(G)^*$, known as the *momentum map*, such that

$$-\mathrm{d}\langle \mu, \mathcal{X}\rangle = \omega(\mathcal{X}_M, \cdot)$$

for all $\mathcal{X} \in \mathrm{Lie}(G)$ where $\mathcal{X}_M$ denotes the vector field on M generated by $\mathcal{X}$ via the action of G. A *Hamiltonian G-manifold* is a quadruple (M,ω,ϕ,μ) where (M,ω,ϕ) is a symplectic G-manifold for which the action of G is Hamiltonian with momentum map μ. Let $\mathrm{Ham}^{2n,G}$ denote the category of $2n$-dimensional Hamiltonian G-manifolds with morphisms given by symplectic G-embeddings which intertwine the momentum maps. Given $f\colon M \to \mathbb{R}$ the associated Hamiltonian vector field is the vector field $\mathcal{X}_f$ on M satisfying $\omega(\mathcal{X}_f,\cdot) = -\mathrm{d}f$.

Definition 3.1. An *integrable system* is a triple (M,ω,F) where (M,ω) is a $2n$-dimensional symplectic manifold and $F = (f_1,\ldots,f_n)\colon M \to \mathbb{R}^n$ is a smooth map such that $f_1,\ldots,f_n$ pairwise Poisson commute, i.e. $\omega(\mathcal{X}_{f_i},\mathcal{X}_{f_j}) = 0$ for all $i,j = 1,\ldots,n$, and the Hamiltonian vector fields $(\mathcal{X}_{f_1})_p,\ldots,(\mathcal{X}_{f_n})_p$ are linearly independent for almost all $p \in M$.

Let $\mathcal{I}^{2n}$ denote the set of all $2n$-dimensional integrable systems for which the Hamiltonian vector fields of the components of the momentum map are complete and define an equivalence relation $\sim_{\mathcal{I}}$ on this space by declaring (M,ω,F) and (M',ω',F') to be equivalent if there exists a symplectomorphism $\phi\colon M \to M'$ such that $F - \phi^*F'\colon M \to \mathbb{R}^n$ is constant. In this paper we always assume the Hamiltonian vector fields of the components of the momentum map are complete, which is automatic if M is compact or if F is proper.

3.1. Hamiltonian $\mathbb{R}^n$-actions and integrable systems.

Let $(M,\omega,F = (f_1,\ldots,f_n))$ be an integrable system such that each $\mathcal{X}_{f_i}$ is complete and for $i = 1,\ldots,n$ let $\psi_i^t\colon M \to M$ denote the flow along $\mathcal{X}_{f_i}$. The *Hamiltonian flow action* $\phi_F\colon \mathbb{R}^n \times M \to M$, given by $\phi_F((t_1,\ldots,t_n),p) = \psi_1^{t_1}\circ\ldots\circ\psi_n^{t_n}(p)$, defines a Hamiltonian $\mathbb{R}^n$-action on M. The action of G on M is *almost everywhere locally free* if the stabilizer of p is discrete for almost all $p \in M$. Let $\mathcal{F}\mathrm{Symp}^{2n,\mathbb{R}^n}$ be the space of $\mathbb{R}^n$-manifolds on which the action of $\mathbb{R}^n$ is Hamiltonian and almost everywhere locally free and let $\sim_{\mathbb{R}^n}$ denote equivalence by $\mathbb{R}^n$-equivariant symplectomorphisms.

Lemma 3.2. *Let $\mathcal{X}_1,\ldots,\mathcal{X}_n$ be vector fields with commuting flows on an m-manifold M, with $n \leqslant m$. Let $\mathbb{R}^n$ act on M by $\phi((t_1,\ldots,t_n),p) = \psi_1^{t_1}\circ\ldots\circ\psi_n^{t_n}(p)$ where ψ_i^t is the flow of $\mathcal{X}_i$. Then, for $p \in M$, the vectors $(\mathcal{X}_1)_p,\ldots,(\mathcal{X}_n)_p \in T_pM$ are linearly independent if and only if the stabilizer of p under the action ϕ is discrete.*

Proof. If $(\mathcal{X}_1)_p,\ldots,(\mathcal{X}_n)_p$ are linearly independent then, since they have commuting flows, there is a chart (U,g), with $U \subset M$ and $g: U \to \mathbb{R}^m$, such that $g^{-1}\colon g(U) \to U$ satisfies

$$g^{-1}(t_1,\ldots,t_n,0,\ldots,0) = \phi((t_1,\ldots,t_n),p)$$

for any $(t_1,\ldots,t_n,0,\ldots,0) \in g(U)$. Thus $g(U)$ is an open neighborhood of the identity in $\mathbb{R}^n$ and there exists no non-zero point in $g(U)$ which fixes p, so the stabilizer of p under the action of $\mathbb{R}^n$ is discrete. On the other hand, if $(\mathcal{X}_1)_p,\ldots,(\mathcal{X}_n)_p$ are linearly dependent, there exist $t_1,\ldots,t_n \in \mathbb{R}$ not all zero such that $\sum_{i=1}^n t_i(\mathcal{X}_i)_p = 0$. Thus $(\alpha t_1,\ldots,\alpha t_n) \in \mathbb{R}^n$ fixes p for all $\alpha \in \mathbb{R}$ and so the stabilizer of p is not discrete. $\square$

Proposition 3.3. *Let ψ be the map which takes an integrable system on M to M equipped with its Hamiltonian flow action. Then*

$$\psi\colon \mathcal{I}^{2n}/\!\sim_{\mathcal{I}} \to \mathcal{F}\mathrm{Symp}^{2n,\mathbb{R}^n}/\!\sim_{\mathbb{R}^n}$$

is a bijection.

Proof. By Lemma 3.2 we know that the Hamiltonian flow action must be almost everywhere locally free because the Hamiltonian vector fields of an integrable system are by definition independent almost everywhere. Next suppose that $\mathbb{R}^n$ acts Hamiltonianly on M in such a way that the action is almost everywhere locally free. Since the action is Hamiltonian there exists a momentum map $\mu\colon M \to \mathrm{Lie}(\mathbb{R}^n)^*$. Define $F = (f_1, \ldots, f_n)\colon M \to \mathbb{R}^n$ by $F = A \circ \mu$ where $A\colon \mathrm{Lie}(\mathbb{R}^n)^* \to \mathbb{R}^n$ is the standard identification which is induced by the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$. These functions Poisson commute because action by the components of $\mathbb{R}^n$ commute and are linearly independent at almost all points because the group action is almost everywhere locally free (Lemma 3.2). Thus, (M, ω, F) is an integrable Hamiltonian system as in Definition 3.1. Let $\{v_1, \ldots, v_n\}$ be the standard basis of $\mathrm{Lie}(\mathbb{R}^n) \cong \mathbb{R}^n$ induced by the standard basis of $\mathbb{R}^n$. Let v_M denote the vector field on M generated by $v \in \mathrm{Lie}(\mathbb{R}^n)$ via the action of G. Then $\langle \mu, v_i \rangle = f_i\colon M \to \mathbb{R}$ so $\mathrm{d}f_i = \omega((v_i)_M, \cdot)$ which means that the Hamiltonian vector field associated to f_i is $(v_i)_M$. Thus the Hamiltonian flow action related to F is the original action of $\mathbb{R}^n$. $\square$

Here we fix the identification between $\mathrm{Lie}(\mathbb{T}^n)^*$ and $\mathbb{R}^n$ that we will use for the remainder of the paper. We specify our convention by choosing an epimorphism from $\mathbb{R}$ to $\mathbb{T}^1$, which we take to be $x \mapsto e^{2\sqrt{-1}x}$.

3.2. **Hamiltonian $\mathbb{T}^k$-actions.** Atiyah [1] and Guillemin-Sternberg [10] proved that if (M, ω, ϕ, μ) is a compact connected Hamiltonian $\mathbb{T}^k$-manifold, then $\mu(M) \subset \mathrm{Lie}(\mathbb{T}^k)^*$ is the convex hull of the image of the fixed points of the $\mathbb{T}^k$-action. The case in which $k = n$ and the torus action is effective enjoys very special properties, and in such a case (M, ω, ϕ, μ) is called a *symplectic toric manifold*, or a *toric integrable system*. An *isomorphism* of such manifolds is a symplectomorphism which intertwines their respective momentum maps. We denote by $\mathrm{Ham}_{\mathrm{T}}^{2n,\mathbb{T}^n}$ the category of $2n$-dimensional symplectic toric manifolds with morphisms as symplectic $\mathbb{T}^n$-embeddings and we denote equivalence by toric isomorphism by $\approx_{\mathrm{T}}$. In general being an invariant is weaker than being monotonic, but in the case of toric manifolds these are equivalent because symplectic $\mathbb{T}^n$-embeddings between toric manifolds are automatically $\mathbb{T}^n$-equivariant symplectomorphisms. Delzant proved [5] that in this case $\mu(M)$ is a *Delzant polytope*, i.e. simple, rational, and smooth, and that

$$\begin{aligned}\Psi\colon \mathrm{Ham}_{\mathrm{T}}^{2n,\mathbb{T}^n}/{\approx_{\mathrm{T}}} &\to \mathcal{P}_{\mathrm{T}}\\ [(M, \omega, \phi, \mu)] &\mapsto \mu(M)\end{aligned}$$

is a bijection, where $\mathcal{P}_{\mathrm{T}}$ denotes the set of n-dimensional Delzant polytopes. Let $\mathrm{Ham}^{2n,\mathbb{T}^n} \to \mathrm{Symp}^{2n,\mathbb{T}^n}$ be given by $(M, \omega, \phi, \mu) \mapsto (M, \omega, \phi)$ and let $\mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{T}^n}$ denote the image of $\mathrm{Ham}_{\mathrm{T}}^{2n,\mathbb{T}^n}$ under this map. Also let $\sim_{\mathrm{T}}$ denote equivalence on $\mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{T}^n}$ by $\mathbb{T}^n$-equivariant symplectomorphisms.

3.3. **Hamiltonian $(S^1 \times \mathbb{R})$-actions.** We say that an integrable system $(M, \omega, F = (J, H)\colon M \to \mathbb{R}^2)$ is a *semitoric integrable system* or *semitoric manifold* if (M, ω) is a 4-dimensional connected symplectic manifold, J is a proper momentum map for an effective Hamiltonian S^1-action on M, and F has only non-degenerate singularities which have no real-hyperbolic blocks (see [21, Section 4.2.1]). A semitoric integrable system is *simple* if there is at most one singular point of focus-focus type in $J^{-1}(x)$ for each $x \in \mathbb{R}$. Let $(M_i, \omega_i, F_i = (J_i, H_i))$ be a semitoric manifold for $i = 1, 2$. A *semitoric isomorphism* between them is a symplectomorphism $\rho\colon M_1 \to M_2$ such that $\rho^*(J_2, H_2) = (J_1, f(J_1, H_1))$ where $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a smooth function for which $\frac{\partial f}{\partial H_1}$ is everywhere nonzero. Let $\mathrm{Ham}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}}$ denote the category of simple semitoric systems and let $\approx_{\mathrm{ST}}$ denote equivalence by semitoric isomorphism. Let $\mathrm{Symp}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}}$ denote the image of $\mathrm{Ham}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}}$ under the map $\mathrm{Ham}^{4,S^1\times\mathbb{R}} \to \mathrm{Symp}^{4,S^1\times\mathbb{R}}$ given by $(M,\omega,\phi,\mu) \mapsto (M,\omega,\phi)$ and let $\sim_{\mathrm{ST}}$ denote the equivalence on $\mathrm{Symp}^{4,S^1\times\mathbb{R}}_{\mathrm{ST}}$ inherited from $\sim_{\mathrm{ST}}$ on $\mathrm{Ham}^{4,S^1\times\mathbb{R}}_{\mathrm{ST}}$.

The number of focus-focus singular points of an integrable system must be finite [23], and we denote it by m_f.

3.3.1. *Invariant of focus-focus singularities.* It is proven in [22] that the structure in the neighborhood of a fiber over a focus-focus point is determined by a Taylor series. Let $\mathbb{R}[[X,Y]]$ denote the space of real formal Taylor series in two variables X and Y and let $\mathbb{R}[[X,Y]]_0 \subset \mathbb{R}[[X,Y]]$ denote the subspace of series $\sum_{i,j>0}\sigma_{i,j}X^iY^j$ which have $\sigma_{0,0}=0$ and $\sigma_{0,1}\in[0,2\pi)$. The Taylor series invariant consists of m_f elements of $\mathbb{R}[[X,Y]]_0$, one for each focus-focus singular point.

3.3.2. *Affine and twisting-index invariants.* Denote the set of rational polygons in $\mathbb{R}^2$ by $\mathrm{Polyg}(\mathbb{R}^2)$. For $\lambda\in\mathbb{R}$ let ℓ_λ denote the set of $(x,y)\in\mathbb{R}^2$ such that $x=\lambda$, Let $\mathrm{Vert}(\mathbb{R}^2)$ denote the collection of all ℓ_λ as λ varies in $\mathbb{R}$. Let $\pi_i\colon\mathbb{R}^2\to\mathbb{R}$ denote the projection onto the i^{th} coordinate for $i=1,2$. Notice that elements of $\mathrm{Polyg}(\mathbb{R}^2)$ can be non-compact. A *labeled weighted polygon of complexity* $m_f\in\mathbb{Z}_{\geqslant 0}$ is an element

$$\Delta_w = \left(\Delta,(\ell_{\lambda_j},\epsilon_j,k_j)_{j=1}^{m_f}\right) \in \mathrm{Polyg}(\mathbb{R}^2)\times\left(\mathrm{Vert}(\mathbb{R}^2)\times\{-1,+1\}\times\mathbb{Z}\right)^{m_f}$$

with $\min_{s\in\Delta}\pi_1(s)<\lambda_1<\ldots<\lambda_{m_f}<\max_{s\in\Delta}\pi_1(s)$. We denote the space of labeled weighed polygons by $\mathcal{LW}\mathrm{Polyg}(\mathbb{R}^2)$. Let

$$T=\begin{pmatrix}1&0\\1&1\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z}) \tag{4}$$

and for $v_1,\ldots,v_n\in\mathbb{Z}^n$ let $\det(v_1,\ldots,v_n)$ denote the determinant of the matrix with columns given by $v_1,\ldots,v_n$.

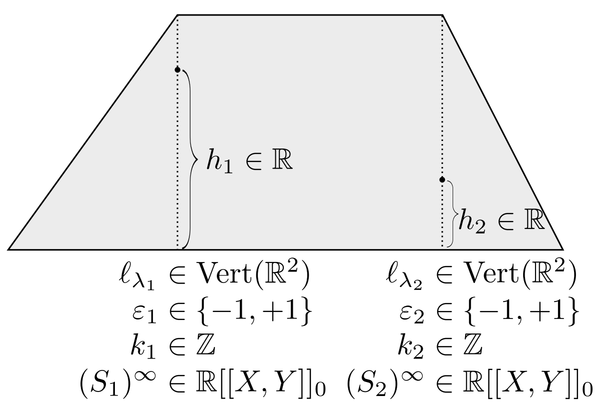

FIGURE 2. The complete invariant of a semitoric system is a collection of these objects.

The *top boundary of* $\Delta\in\mathrm{Polyg}(\mathbb{R}^2)$ is the set $\partial^{\mathrm{top}}\Delta$ of $(x_0,y_0)\in\Delta$ such that y_0 is the maximal $y\in\mathbb{R}$ such that $(x_0,y)\in\Delta$. A point $p\in\partial\Delta$ is a *vertex* of Δ if the edges meeting at p are not co-linear. Let p be a vertex of Δ and let $u,v\in\mathbb{Z}^2$ be primitive vectors directing the edges adjacent to p ordered so that $\det(u,v)>0$. Then we say that:
- p *satisfies the Delzant condition* if $\det(u,v)=1$;
- p *satisfies the hidden condition* if $\det(u,Tv)=1$;
- p *satisfies the fake condition* if $\det(u,Tv)=0$.

We say that Δ has *everywhere finite height* if $\Delta\cap\ell_\lambda$ is either compact or empty for all $\lambda\in\mathbb{R}$. A *primitive semitoric polygon of complexity* $m_f\in\mathbb{Z}_{\geqslant 0}$ [12] is a labeled weighted polygon

$\big(\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^{m_f}\big) \in \mathcal{LW}\mathrm{Polyg}(\mathbb{R}^2)$ such that:
(1) Δ has everywhere finite height;
(2) $\epsilon_j = +1$ for all $j = 1, \ldots, m_f$;
(3) any point in $\partial^{\mathrm{top}}\Delta \cap \ell_{\lambda_j}$ for $j = 1, \ldots, m_f$ satisfies either the hidden or fake condition (and is referred to as either a *hidden corner* or a *fake corner*, respectively);
(4) all other corners satisfy the Delzant condition, and are known as *Delzant corners*. The set of primitive semitoric polygons is denoted by $\mathrm{Polyg}_{\mathrm{ST}}(\mathbb{R}^2)_0$.

For $m_f \in \mathbb{Z}_{\geqslant 0}$ let $G_{m_f} = \{-1, +1\}^{m_f}$ and $\mathcal{G} = \{\, T^k \mid k \in \mathbb{Z}\,\}$ where T is as in Equation (4). For $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$ let $t^k_{\ell_\lambda} : \mathbb{R}^2 \to \mathbb{R}^2$ denote the map which acts as the identity on the left of the line ℓ_λ and acts as T^k relative to an origin placed arbitrarily on the line ℓ_λ to the right of ℓ_λ. Now for $\vec{u} = (u_1, \ldots, u_{m_f}) \in \{-1, 0, 1\}^{m_f}$ and $\vec{\lambda} = (\lambda_1, \ldots, \lambda_{m_f}) \in \mathbb{R}^{m_f}$ define $t^{\vec{u}}_{\vec{\lambda}} \colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$t^{\vec{u}}_{\vec{\lambda}} = t^{u_1}_{\ell_{\lambda_1}} \circ \ldots \circ t^{u_{m_f}}_{\ell_{\lambda_{m_f}}}.$$

We define the action of an element of $G_{m_f} \times \mathcal{G}$ on a labeled weighted polygon by

$$\big((\epsilon'_j)_{j=1}^{m_f}, T^k\big) \cdot \big(\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^{m_f}\big) = \big(t^{\vec{u}}_{\vec{\lambda}} \circ T^k(\Delta), (\ell_{\lambda_j}, \epsilon'_j \epsilon_j, k + k_j)_{j=1}^{m_f}\big)$$

where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_{m_f})$ and $\vec{u} = \left(\frac{\epsilon_j - \epsilon_j \epsilon'_j}{2}\right)_{j=1}^{m_f}$. This action may not preserve the convexity of Δ but it is shown in [20, Lemma 4.2] that the orbit of a primitive semitoric polygon consists only of elements of $\mathcal{LW}\mathrm{Polyg}(\mathbb{R}^2)$.

Definition 3.4 ([20])**.** A *semitoric polygon* is the orbit under $G_{m_f} \times \mathcal{G}$ of a primitive semitoric polygon.

The collection of semitoric polygons is denoted by $\mathrm{Polyg}_{\mathrm{ST}}(\mathbb{R}^2) = (G_{m_f} \times \mathcal{G}) \cdot \mathrm{Polyg}_{\mathrm{ST}}(\mathbb{R}^2)_0$. The orbit of $\Delta_w = \big(\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^{m_f}\big) \in \mathrm{Polyg}_{\mathrm{ST}}(\mathbb{R}^2)_0$ is given by

$$[\Delta_w] = \{\, \big(t^{\vec{u}}_{\vec{\lambda}} \circ T^k(\Delta), (\ell_{\lambda_j}, 1 - 2u_j, k + k_j)_{j=1}^{m_f}\big) \mid \vec{u} \in \{0,1\}^{m_f}, k \in \mathbb{Z}\,\}.$$

The corners of any element of $[\Delta]$ are identified as hidden, fake, or Delzant similar to the case of the primitive semitoric polygon.

3.3.3. *Volume invariant.* For each $j = 1, \ldots, m_f$ we let h_j denote the height of the image of the j^{th} focus-focus point from the bottom of the semitoric polygon. Formally, this amounts to $h_1, \ldots, h_{m_f} \in \mathbb{R}$ satisfying $0 < h_j < \mathrm{length}(\pi_2(\Delta \cap \ell_{\lambda_j}))$ for each $j = 1, \ldots, m_f$.

3.3.4. *Classification.* Semitoric systems are classified by the invariants we have just reviewed. That is, the complete invariant of a semitoric system is an integer m_f, m_f Taylor series, a collection of m_f real numbers, and a labeled weighed semitoric polygon. A single element of this orbit is shown in Figure 2. The complete invariant is an infinite family of such labeled weighted polygons, formed by a countably infinite number of subfamilies of size 2^{m_f} each parameterized by $\vec{\epsilon} \in \{-1, +1\}^{m_f}$ (Figure 3).

Definition 3.5 ([20])**.** A *semitoric list of ingredients* is given by:

(1) *the number of focus-focus singularities invariant:* $m_f \in \mathbb{Z}_{\geqslant 0}$;
(2) *the Taylor series invariant:* a collection of m_f elements of $\mathbb{R}[[X, Y]]_0$;
(3) *the affine and twisting index invariants:* a semitoric polygon of complexity m_f, the $(G_{m_f} \times \mathcal{G})$-orbit of some $\Delta_w = (\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^{m_f}) \in \mathrm{Polyg}_{\mathrm{ST}}(\mathbb{R}^2)_0$;
(4) *the volume invariant:* a collection of real numbers $h_1, \ldots, h_{m_f} \in \mathbb{R}$ such that $0 < h_j < \mathrm{length}(\pi_2(\Delta \cap \ell_{\lambda_j}))$ for each $j = 1, \ldots, m_f$.

Let $\mathbb{I}$ denote the collection of all semitoric lists of ingredients. In [20] the authors prove that semitoric manifolds modulo isomorphisms are classified by semitoric lists of ingredients, that is,

$$\Phi\colon \mathrm{Ham}^{4,S^1\times\mathbb{R}}_{\mathrm{ST}}/\!\approx_{\mathrm{ST}} \to \mathbb{I} \tag{5}$$
$$(M,\omega,(J,H)) \mapsto \big(m_f, ((S_j)^\infty)_{j=1}^{m_f}, [\Delta_w], (h_j)_{j=1}^\infty\big)$$

is a bijection, where Φ is the assignment of the five invariants to the system $(M,\omega,(J,H))$ described in detail in [20].

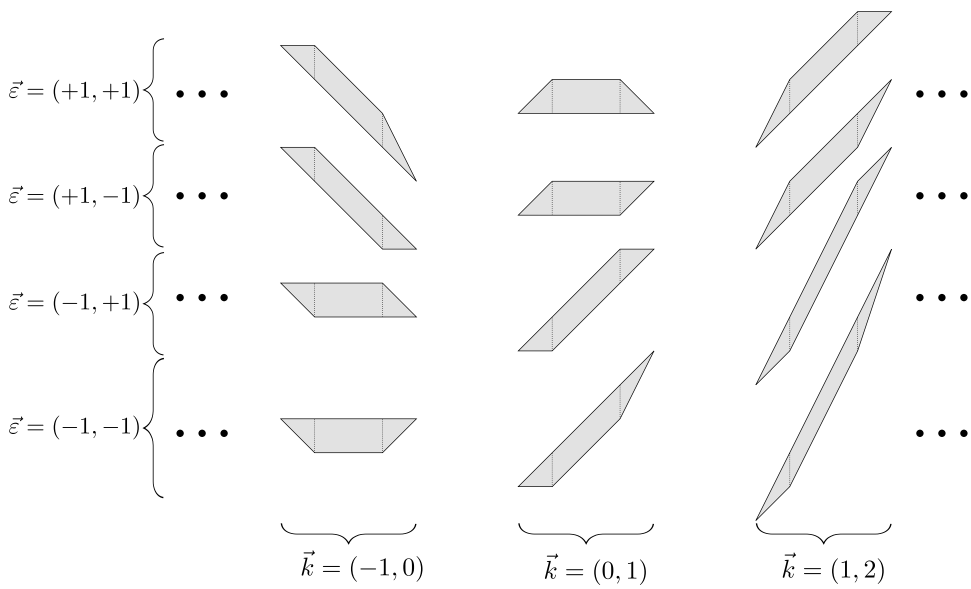

Figure 3. Complete invariant of a semitoric system.

4. Symplectic $\mathbb{T}^n$-capacities

In this section we construct a symplectic $\mathbb{T}^n$-capacity on the space of symplectic toric manifolds. Recall $\mathrm{Ham}^{2n,\mathbb{T}^n}_{\mathrm{T}}/\!\approx_{\mathrm{T}}$ is the moduli space of $2n$-dimensional symplectic toric manifolds up to $\mathbb{T}^n$- -equivariant symplectomorphisms which preserve the moment map. In [7, 16, 17, 19] the authors study the *toric optimal density function* $\Omega\colon \mathrm{Ham}^{2n,\mathbb{T}^n}_{\mathrm{T}}/\!\approx_{\mathrm{T}} \to (0,1]$, which assigns to each symplectic toric manifold the fraction of that manifold which can be filled by equivariantly embedded disjoint open balls. This function is not a capacity because it is not monotonic or conformal. Next we study a modified version of this function which is a capacity.

For $M \in \mathrm{Symp}^{2n,\mathbb{T}^n}$ by a $\mathbb{T}^n$-*equivariantly embedded ball* we mean the image $\phi(\mathrm{B}^{2n}(r))$ of a symplectic $\mathbb{T}^n$-embedding $\phi\colon \mathrm{B}^{2n}(r) \xhookrightarrow{\mathbb{T}^n} M$ for some $r>0$. A *toric ball packing* of M [16] is a disjoint union $P = \bigsqcup_{\alpha\in\mathcal{A}} B_\alpha$ where $B_\alpha \subset M$ is a symplecticly and $\mathbb{T}^n$-equivariantly embedded ball in M for each $\alpha \in \mathcal{A}$, where $\mathcal{A}$ is some index set. That is, for each $\alpha\in\mathcal{A}$ there exists some $r_\alpha > 0$ and some symplectic $\mathbb{T}^n$-embedding $\phi_\alpha\colon \mathrm{B}^{2n}(r_\alpha) \xhookrightarrow{\mathbb{T}^n} M$ such that

$$\phi_\alpha(\mathrm{B}^{2n}(r_\alpha)) = B_\alpha.$$

An example is shown in Figure 4. Recall the toric packing capacity $\mathcal{T}: \mathrm{Symp}^{2n,\mathbb{T}^n}_{\mathrm{T}} \to [0,\infty]$ defined in Equation (2). In the following for $M\in \mathrm{Symp}^{2n,\mathbb{T}^n}$ let $c_{\mathrm{B}}^{n,n}(M)$ be defined by first lifting the action of $\mathbb{T}^n$ on M to an action of $\mathbb{R}^n$ and applying the usual $c_{\mathrm{B}}^{n,n}$ to the resulting symplectic $\mathbb{R}^n$-manifold.

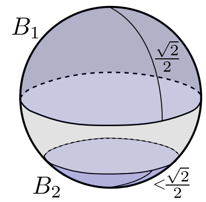

FIGURE 4. Toric ball packing of S^2 by symplectic S^1-disks.

Lemma 4.1. *Let $M \in \mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{T}^n}$, $N \in \mathrm{Symp}^{2n,\mathbb{T}^n}$ be such that the $\mathbb{T}^n$-action on N has $\ell \in \mathbb{Z}_{\geqslant 0}$ fixed points. If there is a symplectic $\mathbb{T}^n$-embedding $M \xhookrightarrow{\mathbb{T}^n} N$ then $\mathcal{T}(M) \leqslant \ell^{1/2n} c_{\mathrm{B}}^{n,n}(N)$.*

Proof. Since the center of $\mathrm{B}^{2n}(r)$, $r > 0$, is a fixed point of the $\mathbb{T}^n$-action we see that the maximal number of such balls that can be simultaneously equivariantly embedded with disjoint images into M is the Euler characteristic $\chi(M)$ of M, which is the number of fixed points of the $\mathbb{T}^n$-action on M. Each of these balls has radius at most $c_{\mathrm{B}}^{n,n}(M)$. For $r > 0$ we have that $\mathrm{vol}(\mathrm{B}^{2n}(r)) = r^{2n}\mathrm{vol}(\mathrm{B}^{2n})$. Therefore

$$(\mathcal{T}(M))^{2n}\mathrm{vol}(\mathrm{B}^{2n}) \leqslant \chi(M)\mathrm{vol}\big(\mathrm{B}^{2n}(c_{\mathrm{B}}^{n,n}(M))\big) = \chi(M)\big(c_{\mathrm{B}}^{n,n}(M)\big)^{2n}\mathrm{vol}(\mathrm{B}^{2n}).$$

Since $\mathbb{T}^n$-embeddings send fixed points to fixed points and $M \xhookrightarrow{\mathbb{T}^n} N$ we know that $\chi(M) \leqslant \ell$. Furthermore, since $M \xhookrightarrow{\mathbb{T}^n} N$ and $c_{\mathrm{B}}^{n,n}$ is a symplectic $\mathbb{T}^n$-capacity by Theorem 1.1 we have that $c_{\mathrm{B}}^{n,n}(M) \leqslant c_{\mathrm{B}}^{n,n}(N)$. Hence $\mathcal{T}(M) \leqslant \ell^{1/2n} c_{\mathrm{B}}^{n,n}(N)$. $\square$

Proposition 4.2. *The toric packing capacity is a symplectic $\mathbb{T}^n$-capacity on $\mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{T}^n}$.*

Proof. Let $M \in \mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{T}^n}$ with $\chi(M) \in \mathbb{Z}_{\geqslant 0}$ fixed points and fix any ordering of these points. Notice that $\mathcal{T}(M)$ is the supremum of

$$\{\, \|\vec{r}\|_{2n} \mid \vec{r} \in \mathbb{R}^{\chi(M)},\ P_M(\vec{r}) \subset M \text{ is a toric packing} \,\}$$

where $\vec{r} = (r_1, \ldots, r_{\chi(M)}) \in \mathbb{R}^{\chi(M)}$,

$$\|\vec{r}\|_{2n} = \left(\sum_{j=1}^{\chi(M)} r_j^{2n} \right)^{1/2n}$$

is the standard ℓ^{2n}-norm, and $P_M(\vec{r}) \subset M$ is the toric ball packing of M in which $\mathrm{B}^{2n}(r_j)$ is embedded at the j^{th} fixed point of M for $j = 1, \ldots, \chi(M)$. Suppose that $\rho\colon \mathrm{B}^{2n}(r) \xhookrightarrow{\mathbb{T}^n} M$ is a symplectic $\mathbb{T}^n$-embedding into (M, ω, ϕ) for some $r > 0$. Then for any $\lambda \in \mathbb{R} \setminus \{0\}$ the map $\rho_\lambda\colon \mathrm{B}^{2n}(|\lambda|\, r) \xhookrightarrow{\mathbb{T}^n} M$ given by

$$\rho_\lambda(z) = \rho(z/|\lambda|)$$

is a symplectic $\mathbb{T}^n$-embedding into $(M, \lambda\omega, \phi)$. Thus if $P_M(\vec{r})$ is a toric packing of (M, ω, ϕ) then $P_M(|\lambda|\, r_1, \ldots, |\lambda|\, r_{\chi(M)})$ is a toric ball packing of $(M, \lambda\omega, \phi)$ for any $\lambda \in \mathbb{R} \setminus \{0\}$. This and the fact that $\|\lambda r\|_{2n} = |\lambda|\, \|r\|_{2n}$ for all $r \in \mathbb{R}^{\chi(M)}$ and $\lambda \in \mathbb{R}$ imply that $\mathcal{T}$ is conformal. Now suppose that $M, M' \in \mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{T}^n}$ and $\rho\colon M \xhookrightarrow{\mathbb{T}^n} M'$. If $P \subset M$ is a toric ball packing of M then $\rho(P) \subset M'$ is a toric ball packing of M' of the same volume so $\mathcal{T}(M) \leqslant \mathcal{T}(M')$ and we see that $\mathcal{T}$ is monotonic. Finally, suppose that there is a symplectic $\mathbb{T}^n$-embedding $M \xhookrightarrow{\mathbb{T}^n} \mathrm{Z}^{2n}$. Then, since Z^{2n} has only one point fixed by the $\mathbb{T}^n$-action and recalling that $c_{\mathrm{B}}^{n,n}(\mathrm{Z}^{2n}) = 1$, it follows from Lemma 4.1 that

$$\mathcal{T}(M) \leqslant (1)^{1/2n} c_{\mathrm{B}}^{n,n}(\mathrm{Z}^{2n}) = 1.$$

Finally, suppose that $\rho\colon \mathrm{B}^{2n} \xhookrightarrow{\mathbb{T}^n} M$ is a symplectic $\mathbb{T}^n$-embedding. Then $P = \rho(\mathrm{B}^{2n}) \subset M$ is a toric ball packing of M and thus

$$\mathcal{T}(M) \geqslant \left(\frac{\mathrm{vol}(P)}{\mathrm{vol}(\mathrm{B}^{2n})}\right)^{1/2n} = 1.$$

Hence $\mathcal{T}$ is tame. $\square$

Example 4.3. Let $M \in \mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{T}^n}$. In [17] it is shown that there exists a $\mathbb{Z}$-valued function $\mathrm{Emb}_M\colon \mathbb{R}_{\geqslant 0} \to [0, n!\chi(M)]$ such that the homotopy type of the space of symplectic $\mathbb{T}^n$-embeddings from $\mathrm{B}^{2n}(r)$ into M is given by the disjoint union of $\mathrm{Emb}_M(r)$ copies of $\mathbb{T}^n$. Thus, for each $r \in \mathbb{R}_{\geqslant 0}$ we may define a symplectic $\mathbb{T}^n$-capacity $\mathcal{E}_r$ on $\mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{T}^n}$ given by

$$\begin{aligned} \mathcal{E}_r\colon \mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{T}^n} &\to [0,\infty] \\ (M,\omega,\phi) &\mapsto (\mathrm{vol}(M))^{\frac{1}{n}} \mathrm{Emb}_M((\mathrm{vol}(M))^{\frac{1}{n}} r). \end{aligned}$$

Since Emb_M is invariant up to $\mathbb{T}^n$-equivariant symplectomorphisms [17] and symplectic embeddings in $\mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{T}^n}$ are automatically symplectomorphisms we see that $\mathcal{E}_r$ is monotonic and it is an exercise to check that it is conformal. It is tame because the space of symplectic $\mathbb{T}^n$-embeddings of B^{2n} into Z^{2n} is homotopic to $n!$ disjoint copies of $\mathbb{T}^n$. $\oslash$

5. Symplectic $(S^1 \times \mathbb{R})$-capacities

In this section we construct a symplectic $(S^1 \times \mathbb{R})$-capacity on the space of semitoric manifolds. Let $(M, \omega, F = (J, H))$ be a simple semitoric manifold with m_f focus-focus singular points and let $\{\lambda_j\}_{j=1}^{m_f} \subset \mathbb{R}$ be the image under J of these points ordered so that $\lambda_1 < \lambda_2 < \ldots < \lambda_{m_f}$. Let (λ_j, y_j) be the image under F of the j^{th} focus-focus singular point and for $\epsilon \in \{\pm 1\}$ let $\ell_{\lambda_j}^{\epsilon}$ be those $(\lambda_j, y) \in \ell_{\lambda_j}$ such that $\epsilon y > \epsilon y_j$. Let $\ell^{\vec{\epsilon}} = \ell_{\lambda_1}^{\epsilon} \cup \ldots \cup \ell_{\lambda_{m_f}}^{\epsilon_{m_f}}$. A homeomorphism

$$f\colon F(M) \to f(F(M)) \subset \mathbb{R}^2$$

is a *straightening map for* M [23] if for some choice of $\vec{\epsilon} \in \{\pm 1\}^{m_f}$ we have the following: $f|_{F(M)\setminus \ell^{\vec{\epsilon}}}$ is a diffeomorphism onto its image; $f|_{F(M)\setminus \ell^{\vec{\epsilon}}}$ is affine with respect to the affine structure $F(M)$ inherits from action-angle coordinates on M and the affine structure $f(F(M))$ inherits as a subset of $\mathbb{R}^2$; f preserves J, i.e. $f(x,y) = (x, f^{(2)}(x,y))$; $f|_{F(M)\setminus \ell^{\vec{\epsilon}}}$ extends to a smooth multi-valued map from $F(M)$ to $\mathbb{R}^2$ such that for any $c = (x_0, y_0) \in \ell^{\vec{\epsilon}}$ we have

$$\lim_{\substack{(x,y)\to c \\ x<x_0}} \mathrm{d}f(x,y) = T \lim_{\substack{(x,y)\to c \\ x>x_0}} \mathrm{d}f(x,y);$$

and the image of f is a rational convex polygon. Recall that T is the matrix given in Equation (4). We say f is associated to $\vec{\epsilon}$.

Let $\mathfrak{T} \subset \mathrm{AGL}_2(\mathbb{Z})$ be the subgroup including powers of T composed with vertical translations. It was proved in [23] that a semitoric system (M, ω, F) has a straightening map $f\colon M \to \mathbb{R}^2$ associated to each $\vec{\epsilon} \in \{\pm 1\}^{m_f}$, unique up to left composition with an element of $\mathfrak{T}$. Define

$$\mathcal{F}_M = \{\, f \circ F \mid f \text{ is a straightening map for } M \,\}. \tag{6}$$

If $V_a\colon \mathbb{R}^2 \to \mathbb{R}^2$ denotes vertical translation by $a \in \mathbb{R}$, then

$$\{\, \widetilde{F}(M) \mid \widetilde{F} \in \mathcal{F}_M \,\} = \{\, V_a(\Delta) \subset \mathbb{R}^2 \mid \Delta \text{ is associated to } M \text{ and } a \in \mathbb{R} \,\}$$

where a polygon is associated to M if it is an element of the affine invariant of M. Up to vertical translations the set $\mathcal{F}_M$ is the orbit of a single non-unique function under the action of $G_{m_f} \times \mathcal{G}$.

If $\widetilde{F} \in \mathcal{F}_M$ then there exists some $\vec{\epsilon} \in \{-1,+1\}^{m_f}$ such that $\widetilde{F}|_{M^{\vec{\epsilon}}}\colon M^{\vec{\epsilon}} \to \mathbb{R}^2$ is a momentum map for a $\mathbb{T}^2$-action $\phi_{\widetilde{F}}\colon \mathbb{T}^2 \times M^{\vec{\epsilon}} \to M^{\vec{\epsilon}}$ where $M^{\vec{\epsilon}} = M \setminus F^{-1}(\ell^{\vec{\epsilon}})$.

Corollary 5.1. *The manifold $M^{\vec{\epsilon}}$ has on it a momentum map for a Hamiltonian $\mathbb{T}^2$-action unique up to $\mathcal{G}$. Thus $M^{\vec{\epsilon}} \in \mathrm{Symp}^{4,\mathbb{T}^2}$ and the given $\mathbb{T}^2$-action is unique up to composing the associated momentum map with an element of $\mathcal{G}$.*

We call such actions of $\mathbb{T}^2$ on $M^{\vec{\epsilon}}$ *induced actions of* $\mathbb{T}^2$. Given any $\rho\colon N \to M$ with $\rho(N) \subset M^{\vec{\epsilon}}$ define $\rho_{\vec{\epsilon}}\colon N \to M^{\vec{\epsilon}}$ by $\rho_{\vec{\epsilon}}(p) = \rho(p)$ for $p \in N$.

Definition 5.2. Let (M,ω,F) be a semitoric manifold and let $(N,\omega_N,\phi) \in \mathrm{Symp}^{4,\mathbb{T}^2}$. A symplectic embedding $\rho\colon N \hookrightarrow M$ is a *semitoric embedding* if there exists $\vec{\epsilon} \in \{\pm 1\}^{m_f}$ and an induced action $\phi_{\vec{\epsilon}} : \mathbb{T}^2 \times M_{\vec{\epsilon}} \to M_{\vec{\epsilon}}$ such that $\rho(N) \subset M^{\vec{\epsilon}}$ and $\rho_{\vec{\epsilon}}\colon (N,\omega_N,\phi) \xhookrightarrow{\mathbb{T}^2} (M^{\vec{\epsilon}},\omega,\phi_{\vec{\epsilon}})$ is a symplectic $\mathbb{T}^2$-embedding.

Let (M,ω,F) be a semitoric manifold. A *semitoric ball packing* of M is a disjoint union $P = \bigsqcup_{\alpha\in\mathcal{A}} B_\alpha$ where $B_\alpha \subset M$ is a semitoricly embedded ball in M. The *semitoric packing capacity* $\mathcal{ST}\colon \mathrm{Symp}^{4,S^1\times\mathbb{R}}_{\mathrm{ST}} \to [0,\infty]$ is given by

$$\mathcal{ST}(M) = \left(\frac{\sup\{\,\mathrm{vol}(P) \mid P \subset M \text{ is a semitoric ball packing of } M\,\}}{\mathrm{vol}(\mathrm{B}^4)}\right)^{\frac{1}{4}}.$$

In order to show that $\mathcal{ST}$ is a $(S^1\times\mathbb{R})$-capacity we need the following lemmas.

Lemma 5.3. *For $i = 1,2$ let (M_i,ω_i) be a symplectic manifold, let $f_i\colon M_i \to \mathbb{R}$ be a function, and let $\mathcal{X}_{f_i}$ denote the Hamiltonian vector field of f_i on M_i. If $\rho\colon M_1 \to M_2$ is a symplectomorphism such that $\rho_*\mathcal{X}_{f_1} = \mathcal{X}_{f_2}$ then $f_1 - \rho^* f_2\colon M_1 \to \mathbb{R}$ is constant.*

Proof. Notice that

$$\begin{aligned}\mathrm{d}(\rho^* f_2) &= \rho^*(\mathrm{d}f_2) = \rho^*(\iota_{\mathcal{X}_{f_2}}\omega_2) = \rho^*(\iota_{\rho_*\mathcal{X}_{f_1}}\omega_2)\\ &= \omega_2(\rho_*\mathcal{X}_{f_1},\rho_*(\cdot)) = (\rho^*\omega_2)(\mathcal{X}_{f_1},\cdot) = \iota_{\mathcal{X}_{f_1}}\omega_1 = \mathrm{d}f_1,\end{aligned}$$

thus f_1 and $\rho^* f_2$ differ by a constant. □

Lemma 5.4. *Let $(M_i,\omega_i,F_i = (J_i,H_i))$ be semitoric manifolds for $i = 1,2$. If $\rho\colon M_1 \xhookrightarrow{S^1\times\mathbb{R}} M_2$ is a symplectic $(S^1\times\mathbb{R})$-embedding with respect to the Hamiltonian flow action on each system, then*

$$\rho^* J_2 = eJ_1 + c_J \quad \text{and} \quad \rho^* H_2 = aJ_1 + bH_1 + c_H$$

for some $e \in \{\pm 1\}$ and $a,b,c_J,c_H \in \mathbb{R}$ such that $b \neq 0$.

Proof. Since ρ is $S^1\times\mathbb{R}$-equivariant there exists $\Lambda \in \mathrm{Aut}(S^1\times\mathbb{R})$ such that $\rho(\phi(g,m_1)) = \phi(\Lambda(g),\rho(m_1))$ for all $g \in S^1\times\mathbb{R}$ and $m_1 \in M_1$. Associate $S^1\times\mathbb{R}$ with $\mathbb{R}/\mathbb{Z}\times\mathbb{R}$ and give it coordinates $(x,y) \in \mathbb{R}^2$. Then $\Lambda \in \mathrm{Aut}(S^1\times\mathbb{R})$ and Λ continuous means that Λ descends from a linear invertible map from $\mathbb{R}^2$ to itself, which we will also denote $\Lambda \in \mathrm{GL}_2(\mathbb{R})$. Write $\Lambda = (\Lambda_{ij})$ for $\Lambda_{ij} \in \mathbb{R}$ and $i,j \in \{1,2\}$. The automorphism Λ sends the identity to itself so $\Lambda\begin{pmatrix} n\\ 0\end{pmatrix} \in \mathbb{Z}\times\{0\}$ for all choices of $n \in \mathbb{Z}$. This implies that $\Lambda_{11} \in \mathbb{Z}$ and $\Lambda_{21} = 0$. Since Λ is invertible and $\Lambda^{-1} \in \mathrm{Aut}(S^1\times\mathbb{R})$ we see that $(\Lambda_{11})^{-1} \in \mathbb{Z}$ and so $\Lambda_{11} = \pm 1$. Since Λ is invertible and upper triangular we know that $\Lambda_{22} \neq 0$.

For a function $f\colon M_i \to \mathbb{R}$ let $\mathcal{X}_f$ denote the associated Hamiltonian vector field on M_i, $i = 1,2$. Also, for $v \in \mathfrak{g} = \mathrm{Lie}(S^1\times\mathbb{R})$, thought of as the tangent space to the identity, let v_{M_i} denote the vector field on M_i generated by v by the group action. Endow $\mathfrak{g}$ with the coordinates (α,β) so that the exponential map will send $(\alpha, \beta) \in \mathfrak{g}$ to $(\alpha, \beta) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$. Now notice that $\mathcal{X}_{J_1} = (1,0)_{M_1}$ and $\mathcal{X}_{H_1} = (0,1)_{M_1}$.

For $m_i \in M_i$, $i = 1, 2$, such that $\rho(m_1) = m_2$ we have

$$\rho_* \mathcal{X}_{J_1}(m_2) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Big(\rho\big(\phi((t,0), m_1)\big)\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Big(\phi(\Lambda[(t,0)], m_2)\Big) = \big(\mathrm{T}\Lambda(1,0)\big)_{M_2}(m_2)$$

Notice that $\mathrm{T}_{(1,0)} = (\Lambda_{11}, 0) \in \mathfrak{g}$. Then $\rho_* \mathcal{X}_{J_1} = \big(\mathrm{T}\Lambda(1,0)\big)_{M_2} = \Lambda_{11}(1,0)_{M_2} = \Lambda_{11}\mathcal{X}_{J_2}$. Similarly we see that $\rho_* \mathcal{X}_{H_1} = \Lambda_{12}\mathcal{X}_{J_2} + \Lambda_{22}\mathcal{X}_{H_2}$. By Lemma 5.3 this implies that

$$\rho^* J_2 = \frac{1}{\Lambda_{11}} J_1 + c_J \qquad \text{and} \qquad \rho^* H_2 = \frac{-\Lambda_{12}}{\Lambda_{11}\Lambda_{22}} J_1 + \frac{1}{\Lambda_{22}} H_1 + c_H$$

for some $c_J, c_H \in \mathbb{R}$. Recalling that $\Lambda_{11} \in \{\pm 1\}$ and $\Lambda_{11}, \Lambda_{22} \neq 0$ take $e = (\Lambda_{11})^{-1}$, $a = \frac{-\Lambda_{12}}{\Lambda_{11}\Lambda_{22}}$, and $b = (\Lambda_{22})^{-1}$ to complete the proof. ☐

Proposition 5.5. *The semitoric packing capacity, $\mathcal{ST}$, is a symplectic $(S^1 \times \mathbb{R})$-capacity on $\mathrm{Symp}_{\mathrm{ST}}^{4, S^1 \times \mathbb{R}}$.*

Proof. The proof that $\mathcal{ST}$ is conformal and non-trivial is analogous to the proof of Proposition 4.2, so we must only show that $\mathcal{ST}$ is monotonic. Let (M_i, ω_i, F_i) be semitoric for $i = 1, 2$ and suppose $\phi \colon M_1 \xhookrightarrow{S^1 \times \mathbb{R}} M_2$ is a symplectic $(S^1 \times \mathbb{R})$-embedding. Recall that action-angle coordinates are local Darboux charts in which the flow of the Hamiltonian vector fields are linear. Since ϕ is symplectic, $(S^1 \times \mathbb{R})$-equivariant, and $\phi^*(F_2) = A \circ F_1$ where $A \colon \mathbb{R}^2 \to \mathbb{R}^2$ is affine (Lemma 5.4), this means that ϕ sends action-angle coordinates to action-angle coordinates. Since semitoric embeddings are those which respect the action-angle coordinates, given any semitoric embedding $\rho \colon \mathrm{B}^{2n}(r) \hookrightarrow M_1$ the map $\phi \circ \rho \colon \mathrm{B}^{2n}(r) \hookrightarrow M_2$ is a semitoric embedding. It follows that $\mathcal{ST}(M_1) \leqslant \mathcal{ST}(M_2)$. ☐

6. Continuity of symplectic $\mathbb{T}^n$-capacities

In this section we study the continuity of the symplectic $\mathbb{T}^n$-capacity constructed in Section 4. We will outline the procedure used in [18] to construct a natural metric on the moduli space of toric manifolds. Since $\Psi \colon \mathrm{Ham}_{\mathrm{T}}^{2n, \mathbb{T}^n}/{\approx_{\mathrm{T}}} \to \mathcal{P}_{\mathrm{T}}$ is a bijection we can define a metric space structure on $\mathrm{Ham}_{\mathrm{T}}^{2n, \mathbb{T}^n}/{\approx_{\mathrm{T}}}$ by defining a metric on $\mathcal{P}_{\mathrm{T}}$ and pulling it back via Ψ. A natural metric on $\mathcal{P}_{\mathrm{T}}$ is given by the volume of the symmetric difference. For $A, B \subset \mathbb{R}^n$ let $A * B = (A \setminus B) \cup (B \setminus A)$ denote the symmetric difference and let λ denote the Lebesgue measure on $\mathbb{R}^n$. For $\Delta_1, \Delta_2 \in \mathcal{P}_{\mathrm{T}}$ define $d_{\mathcal{P}}(\Delta_1, \Delta_2) = \lambda(\Delta_1 * \Delta_2)$. Now let $d_{\mathrm{T}} = \Psi^* d_{\mathcal{P}}$. In [18] the authors show that $(\mathrm{Ham}_{\mathrm{T}}^{2n, \mathbb{T}^n}/{\approx_{\mathrm{T}}}, d_{\mathrm{T}})$ is a non-locally compact non-complete metric space.

The map

$$\mathrm{Ham}_{\mathrm{T}}^{2n, \mathbb{T}^n}/{\approx_{\mathrm{T}}} \to \mathrm{Symp}_{\mathrm{T}}^{2n, \mathbb{T}^n}/{\sim_{\mathrm{T}}}$$

given by $[(M, \omega, \phi, \mu)] \mapsto [(M, \omega, \phi)]$ is a quotient map and thus we can endow $\mathrm{Symp}_{\mathrm{T}}^{2n, \mathbb{T}^n}/{\sim_{\mathrm{T}}}$ with the quotient topology. Since $\mathrm{Symp}_{\mathrm{T}}^{2n, \mathbb{T}^n}/{\sim_{\mathrm{T}}}$ is a quotient of $\mathrm{Symp}_{\mathrm{T}}^{2n, \mathbb{T}^n}$ we can pull the topology up from $\mathrm{Symp}_{\mathrm{T}}^{2n, \mathbb{T}^n}/{\sim_{\mathrm{T}}}$ to $\mathrm{Symp}_{\mathrm{T}}^{2n, \mathbb{T}^n}$ by declaring that a set in $\mathrm{Symp}_{\mathrm{T}}^{2n, \mathbb{T}^n}$ is open if and only if it is the preimage of an open set from $\mathrm{Symp}_{\mathrm{T}}^{2n, \mathbb{T}^n}/{\sim_{\mathrm{T}}}$ under the natural projection. Two points in $\mathrm{Symp}_{\mathrm{T}}^{2n, \mathbb{T}^n}$ are not separable if and only if they are $\mathbb{T}^n$-equivariantly symplectomorphic. Thus a map $c \colon \mathrm{Symp}_{\mathrm{T}}^{2n, \mathbb{T}^n} \to [0, \infty]$ which descends to a well-defined map ϕ on $\mathrm{Symp}_{\mathrm{T}}^{2n, \mathbb{T}^n}/{\sim_{\mathrm{T}}}$ is continuous if and only if the map

$$\hat{c} \colon \mathrm{Ham}_{\mathrm{T}}^{2n, \mathbb{T}^n}/{\approx_{\mathrm{T}}} \to [0, \infty]$$

is continuous, where $\hat{c}$ is defined by the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Ham}^{2n,\mathbb{T}^n}_{\mathrm{T}} \arrow[r] \arrow[d] & \mathrm{Symp}^{2n,\mathbb{T}^n}_{\mathrm{T}} \arrow[r, "c"] \arrow[d] & {[0,\infty]} \\
\mathrm{Ham}^{2n,\mathbb{T}^n}_{\mathrm{T}}/{\approx_{\mathrm{T}}} \arrow[r] \arrow[rru, bend right, "\hat{c}"'] & \mathrm{Symp}^{2n,\mathbb{T}^n}_{\mathrm{T}}/{\sim_{\mathrm{T}}} \arrow[ru, "\phi"] &
\end{tikzcd}$$

Next we define an operation on Delzant polytopes. Let $n \in \mathbb{Z}_{>0}$. For $x_0 \in \mathbb{R}^n$, $w_1, \ldots, w_n \in \mathbb{Z}^n$, and $\varepsilon > 0$ define

$$\mathcal{H}^{\varepsilon}_{x_0}(w_1, \ldots, w_n) = \{\, x_0 + \textstyle\sum_j t_j w_j \mid t_1, \ldots, t_n \in \mathbb{R}_{\geqslant 0}, \sum_j t_j \geqslant \varepsilon \,\}. \tag{7}$$

Suppose that $\Delta \in \mathcal{P}_{\mathrm{T}}$ and $x_0 \in \mathbb{R}^n$ is a vertex of Δ. Let $u_i \in \mathbb{Z}^n$, $i = 1, \ldots, n$, denote the primitive vectors along which the edges adjacent to x_0 are aligned. The *ε-corner chop of Δ at x_0* is the polygon $\Delta^{\varepsilon}_{x_0} \in \mathcal{P}_{\mathrm{T}}$ given by $\Delta^{\varepsilon}_{x_0} = \Delta \cap \mathcal{H}^{\varepsilon}_{x_0}(u_1, \ldots, u_n)$ where ε is sufficiently small so that $\Delta^{\varepsilon}_{x_0}$ has exactly one more face than Δ does as is shown in Figure 5. One can check that if $\Delta \in \mathcal{P}_{\mathrm{T}}$

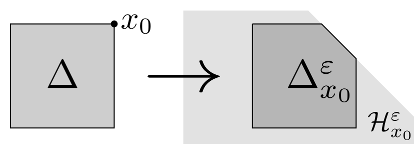

FIGURE 5. An ε-corner chop at a vertex x_0 of Δ for some $\varepsilon > 0$.

then $\Delta^{\varepsilon}_{x_0} \in \mathcal{P}_{\mathrm{T}}$. Notice that $\lim_{\varepsilon \to 0} d_{\mathcal{P}}(\Delta, \Delta^{\varepsilon}_{x_0}) = 0$. This means that given any element of $\mathcal{P}_{\mathrm{T}}$ with N vertices, corner chopping can be used to produce other polygons which are close in $d_{\mathcal{P}}$ and all polygons produced in this way will have more than N vertices. Let $\mathcal{P}^N_{\mathrm{T}}$ denote the set of Delzant polygons in $\mathbb{R}^n$ with exactly N vertices. We will later need the following.

Proposition 6.1 ([7])**.** *Let $N \in \mathbb{Z}_{>0}$ and $\Delta \in \mathcal{P}^N_{\mathrm{T}}$. Any sufficiently small neighborhood of Δ is a subset of $\cup_{(N' \geqslant N)} \mathcal{P}^{N'}_{\mathrm{T}}$.*

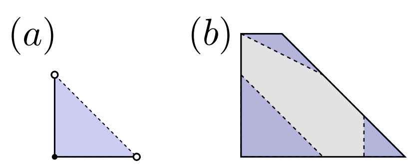

FIGURE 6. (a) An image of $\Delta(1) \subset \mathbb{R}^2$. (b) An image of an admissible, but not maximal, packing.

We study ball packing problems about symplectic toric manifolds by instead studying packings of the associated Delzant polygon. Let $\Delta \in \mathcal{P}_{\mathrm{T}}$ be a Delzant polytope. Let $\mathrm{AGL}_n(\mathbb{Z}) = \mathrm{GL}_n(\mathbb{Z}) \ltimes \mathbb{R}^n$ denote the group of affine transformations in $\mathbb{R}^n$ with linear part in $\mathrm{GL}_n(\mathbb{Z})$. For $r > 0$ let $\Delta(r) = \mathrm{Conv}\{\, re_1, \ldots, re_n, 0 \,\} \setminus \mathrm{Conv}\{\, re_1, \ldots, re_n \,\}$ where $\mathrm{Conv}(E)$ denotes the convex hull of the set $E \subset \mathbb{R}^n$ and $\{\, e_1, \ldots, e_n \,\}$ denote the standard basis vectors in $\mathbb{R}^n$. Following [16], a subset Σ of Δ is an *admissible simplex of radius $r > 0$ with center at a vertex x_0 of Δ* if there exists some $A \in \mathrm{AGL}_n(\mathbb{Z})$ such that:
(1) $A(\Delta(r^{1/2})) = \Sigma$;

(2) $A(0) = x_0$;
(3) A takes the edges of $\Delta(r^{1/2})$ meeting at the origin to the edges of Δ meeting at x_0.

An *admissible packing of* Δ is a disjoint union $R = \bigsqcup_{\alpha\in\mathcal{A}} \Sigma_\alpha \subset \Delta$ where each Σ_α is an admissible simplex for Δ. This is illustrated in Figure 6. The half-plane $\mathcal{H}^\varepsilon_{x_0}$ given in Equation (7) is designed so that that an ε-corner chop on a Delzant polytope corresponds to the removal of an admissible simplex of radius ε.

The function $\Omega\colon \mathrm{Symp}^{2n,\mathbb{T}^n}_{\mathrm{T}}/{\sim_{\mathrm{T}}} \to (0,1]$ given by

$$\Omega(M) = \frac{\sup\{\,\mathrm{vol}(P) \mid P \text{ is a toric ball packing of } M\,\}}{\mathrm{vol}(M)},$$

known as the *optimal toric density function*, has been studied in [7, 16, 19]. In particular, in [7] the first and third authors of the present article studied the regions of continuity of Ω and proved the $n = 2$ case of Theorem 1.2 part (i). They stated the theorem in terms of Ω, while we state it in terms of $\mathcal{T}$.

Let $\mathrm{vol}\colon \mathrm{Symp}^{2n,\mathbb{T}^n}_{\mathrm{T}} \to \mathbb{R}$ denote the total symplectic volume of a symplectic toric manifold and let $\mathrm{vol}_{\mathcal{P}}\colon \mathcal{P}_{\mathrm{T}} \to \mathbb{R}$ denote Euclidean volume function of a polytope in $\mathbb{R}^n$. Let $(\mathrm{B}^{2n}(r), \omega_{\mathrm{B}}, \phi_{\mathrm{B}}, \mu_{\mathrm{B}}) \in \mathrm{Ham}^{2n,\mathbb{T}^n}_{\mathrm{T}}$ denote the standard ball of radius $r > 0$ in $\mathbb{C}^n$ with the standard action of $\mathbb{T}^n$ and suppose that $(M, \omega, \phi, \mu) \in \mathrm{Ham}^{2n,\mathbb{T}^n}_{\mathrm{T}}$. Let $\Delta_{\mathrm{B}} = \mu_{\mathrm{B}}(\mathrm{B}^{2n}(r))$ and $\Delta = \mu(M)$. Then, as shown in [16], $\mathrm{vol}(M) = n!\pi^n \mathrm{vol}_{\mathcal{P}}(\Delta)$ and if $f\colon \mathrm{B}^{2n}(r) \xhookrightarrow{\mathbb{T}^n} M$ is a symplectic $\mathbb{T}^n$-embedding then

$$\mathrm{vol}(\mathrm{B}^{2n}(r)) = \mathrm{vol}(f(\mathrm{B}^{2n}(r))) = n!\pi^n \mathrm{vol}_{\mathcal{P}}(\mu \circ f(\mathrm{B}^{2n}(r))) = n!\pi^n \mathrm{vol}_{\mathcal{P}}(\Delta_{\mathrm{B}}).$$

Theorem 6.2 ([16])**.** *Let $(M, \omega, \phi, \mu) \in \mathrm{Ham}^{2n,\mathbb{T}^n}_{\mathrm{T}}$ and let $\Delta = \mu(M)$. Suppose $\phi\colon \mathrm{B}^{2n}(r) \hookrightarrow M$ is a symplectic $\mathbb{T}^n$-embedding for some $r > 0$. Then $\mu(\phi(\mathrm{B}^{2n}(r))) \subset \Delta$ is an admissible simplex of radius r^2. Conversely, if $\Sigma \subset \Delta$ is an admissible simplex of radius r^2 then there exists a symplectic $\mathbb{T}^n$-embedding $\phi\colon \mathrm{B}^{2n}(r) \hookrightarrow M$ such that $\mu(\phi(\mathrm{B}^{2n}(r))) = \Sigma$.*

Moreover, if P is a toric ball packing of M, then $\mu(P) \subset \Delta$ is an admissible packing of Δ. Conversely, if R is an admissible packing of Δ then there exists a toric ball packing P of M such that $\mu(P) = R$.

Since there is a toric ball packing P of M related to an admissible packing R of Δ by $\mu(P) = R$, it follows that $\mathrm{vol}(P) = n!\pi^n \mathrm{vol}_{\mathcal{P}}(R)$. To study packing of the manifold we will study packing of the polygon. Thus, we define $\pi_{\mathrm{T}}\colon \mathcal{P}_{\mathrm{T}} \to (0, \infty)$ by

$$\pi_{\mathrm{T}}(\Delta) = \sup\{\,\mathrm{vol}_{\mathcal{P}}(R) \mid R \text{ is an admissible packing of } \Delta\,\}.$$

Suppose that $\Delta \in \mathcal{P}^N_{\mathrm{T}}$ with vertices $v_1, \ldots, v_N \in \mathbb{R}^n$ and let $\pi^i_{\mathrm{T}}(\Delta)$ be the supremum of $\mathrm{vol}_{\mathcal{P}}(R)$ over all admissible packings $\mathcal{R}$ of Δ in which $v_i \notin \mathcal{R}$.

The following result generalizes [7, Theorem 7.1] to the case $n \geqslant 3$.

Theorem 6.3. *Fix $n \in \mathbb{Z}_{>0}$. For $N \in \mathbb{Z}_{\geqslant 1}$ and let $\mathcal{P}^N_{\mathrm{T}}$ denote the set of Delzant polygons in $\mathbb{R}^n$ with exactly N vertices. Then:*

(1) *π_{T} is discontinuous at each point in $\mathcal{P}_{\mathrm{T}}$;*
(2) *the restriction $\pi_{\mathrm{T}}|_{\mathcal{P}^N_{\mathrm{T}}}$ is continuous for each $N \geqslant 1$;*
(3) *if $\Delta \in \mathcal{P}^N_{\mathrm{T}}$ then $\mathcal{P}^N_{\mathrm{T}}$ is the largest neighborhood of Δ in $\mathcal{P}_{\mathrm{T}}$ in which π_{T} is continuous if and only if $\pi^i_{\mathrm{T}}(\Delta) < \pi_{\mathrm{T}}(\Delta)$ for all $1 \leqslant i \leqslant N$.*

Proof. First we show (1). Let $\Delta \in \mathcal{P}^N_{\mathrm{T}}$ and for any small enough $\varepsilon > 0$ perform an ε-corner chop (as in Section 6) at each corner to produce $\Delta_\varepsilon \in \mathcal{P}^{2N}_{\mathrm{T}}$. Any admissible packing of Δ_ε can have at most $2N$ simplices and each simplex must have one side with length at most ε while the other sides are universally bounded by the maximal side length of Δ. The size of such simplices decreases to zero as ε does, so $\lim_{\varepsilon\to 0}\pi_{\mathrm{T}}(\Delta_\varepsilon)=0$. Hence

$$\lim_{\varepsilon\to 0} d_{\mathcal{P}}(\Delta,\Delta_\varepsilon)=0$$

but

$$\lim_{\varepsilon\to 0}|\pi_{\mathrm{T}}(\Delta)-\pi_{\mathrm{T}}(\Delta_\varepsilon)|=\pi_{\mathrm{T}}(\Delta)>0,$$

so π_{T} is discontinuous at Δ.

Now we prepare to show part (2). For any $v_1,\dots,v_n\in\mathbb{Z}^n$ let $[v_1,\dots,v_n]$ denote the $n\times n$ integer matrix with i^{th} column given by v_i for $i=1,\dots,n$. Let $\eta\colon \mathrm{SL}_n(\mathbb{Z})\to\mathrm{GL}_n(\mathbb{R})$ given by

$$\eta([v_1,\dots,v_n])=\left[\frac{v_1}{|v_1|},\dots,\frac{v_n}{|v_n|}\right]$$

take a nonsingular integer matrix to its *column normalization*. Notice for any $A=[v_1,\dots,v_n]\in\mathrm{SL}_n(\mathbb{Z})$ that

$$\det(A)=|v_1|\cdots|v_n|\cdot\det(\eta(A)).$$

Suppose $\Delta\in\mathcal{P}_{\mathrm{T}}$ is n-dimensional. In a neighborhood around each vertex the polytope is described by a collection of vectors $v_1,\dots,v_n\in\mathbb{Z}^n$ with $\det(v_1,\dots,v_n)=1$ along which the edges adjacent to this vertex are directed. So, associated to any vertex of a Delzant polytope, there is a matrix $A\in\mathrm{SL}_n(\mathbb{Z})$ given by $A=[v_1,\dots,v_n]$ which is unique up to even permutations of its columns and thus, though A is not unique, the values determined by $\det(A)$ and $\det(\eta(A))$ associated to a vertex are well-defined. Fix $\Delta\in\mathcal{P}_{\mathrm{T}}^N$ and $\{\,\Delta_j\,\}_{j=1}^\infty\subset\mathcal{P}_{\mathrm{T}}^N$ such that

$$\lim_{j\to\infty} d_{\mathcal{P}}(\Delta,\Delta_j)=0. \tag{8}$$

For j large enough for each vertex V of Δ there must be a corresponding vertex V_j of Δ_j so that $V_j\to V$ as $j\to\infty$. Let $A\in\mathrm{SL}_n(\mathbb{Z})$ be a matrix corresponding to V and let $A_j\in\mathrm{SL}_n(\mathbb{Z})$ be a matrix corresponding to V_j for $j\in\mathbb{Z}$ large enough. In particular, convergence in $d_{\mathcal{P}}$, which is convergence in $\mathrm{L}^1(\mathbb{R}^n)$, implies that locally these vertices must converge, so Equation (8) implies that

$$\lim_{j\to\infty}|\det(\eta(A))-\det(\eta(A_j))|=0.$$

Now we are ready to prove (2) by showing that the collection of possible vertices of Delzant polytopes is discrete. Fix $\Delta\in\mathcal{P}_{\mathrm{T}}^N$ with a vertex V at the origin and let $\varepsilon>0$. Choose $\delta>0$ small enough so that if $\Delta'\in\mathcal{P}_{\mathrm{T}}^N$ with a vertex V' at the origin then $d_{\mathcal{P}}(\Delta,\Delta')<\delta$ implies that

$$\big|\det(\eta(A))-\det(\eta(A'))\big|<\varepsilon, \tag{9}$$

where $A\in\mathrm{SL}_n(\mathbb{Z})$ is a matrix associated to V and $A'\in\mathrm{SL}_n(\mathbb{Z})$ is a matrix associated to V'. Suppose that $\varepsilon<\det(\eta(A))$. Now let $A'=[w_1,\dots,w_n]$ for $w_i\in\mathbb{Z}^n$, $i=1,\dots,n$. These are all nonzero integer vectors so $|w_i|\geqslant 1$ for $i=1,\dots,n$. For each i we have

$$1=\det(A')=|w_1|\,|w_2|\dots|w_n|\det(\eta(A'))\geqslant|w_i|\det(\eta(A'))$$

and so by Equation (9)

$$|w_i|\leqslant\frac{1}{\det(\eta(A'))}\leqslant\frac{1}{\det(\eta(A))-\varepsilon}.$$

Thus each $w_i\in\mathbb{Z}^n$ has length at most $(\det(\eta(A))-\varepsilon)^{-1}$, a value which does not depend on Δ', and so to be within δ of Δ the vectors directing the edges coming out from the vertex V' of Δ' must be chosen from only finitely many options. This means the set of possible local neighborhoods of vertices is discrete. Thus, for small enough $\delta>0$ we conclude that $d_{\mathcal{P}}(\Delta,\Delta')<\delta$ implies that there exist open sets $U,U'\subset\mathbb{R}^n$ around the vertices V and V' such that

$$\Delta\cap U=F_c(\Delta'\cap U')$$

where $F_c\colon \mathbb{R}\to\mathbb{R}$ is a translation by some fixed $c\in\mathbb{R}^n$. Now, let $\Delta\in\mathcal{P}_{\mathrm{T}}^N$ be any Delzant polytope in $\mathbb{R}^n$ with N vertices. In a sufficiently small $d_{\mathcal{P}}$-neighborhood of Δ all polytopes must have the same angles at the finitely many vertices by the argument above. Thus they are all related to Δ by translating its faces in a parallel way (which includes as a special case rescaling the polytope), which continuously changes π_{T}. This proves (2) because π_{T} is continuous on such families.

Finally we show (3). Let $\Delta\in\mathcal{P}_{\mathrm{T}}^N$ and assume that $\pi_{\mathrm{T}}(\Delta)=\pi_{\mathrm{T}}^i(\Delta)$ for some $i\in\{1,\ldots,N\}$. Then there is an optimal packing of Δ which avoids the i^{th} vertex. For $\varepsilon>0$ let $\Delta_\varepsilon\in\mathcal{P}_{\mathrm{T}}^{N+1}$ be the ε-corner chop of Δ at the i^{th} vertex. Since the optimal packing of Δ avoids the i^{th} vertex, we see that $\lim_{\varepsilon\to0} d_{\mathcal{P}}(\Delta,\Delta_\varepsilon)=0$ and $\lim_{\varepsilon\to0}\pi_{\mathrm{T}}(\Delta)=\pi_{\mathrm{T}}(\Delta_\varepsilon)$ so there is a set larger than $\mathcal{P}_{\mathrm{T}}^N$ on which π_{T} is continuous around Δ.

Conversely assume that $\Delta\in\mathcal{P}_{\mathrm{T}}^N$ satisfies $\pi_{\mathrm{T}}^i(\Delta)<\pi_{\mathrm{T}}(\Delta)$ for all $i=1,\ldots,n$. By Proposition 6.1 we know that any small enough neighborhood of Δ only includes polytopes with N vertices and polytopes with more than N vertices, which are produced from corner chops of Δ. We must now only show that π_{T} cannot be continuous on any neighborhood of Δ which includes any such polygons. For $\varepsilon>0$ let $\Delta_\varepsilon\in\mathcal{P}_{\mathrm{T}}^{N+1}$ be the ε-corner chop of Δ at the i^{th} vertex. Then $\lim_{\varepsilon\to0}\pi_{\mathrm{T}}(\Delta_\varepsilon)=\pi_{\mathrm{T}}^i(\Delta)<\pi_{\mathrm{T}}$ so for small enough corner chops $\pi_{\mathrm{T}}(\Delta_\varepsilon)$ is bounded away from $\pi_{\mathrm{T}}(\Delta)$. Thus any set on which π_{T} is continuous around Δ cannot include any corner chops of Δ. From this we conclude that any such set cannot include polytopes with greater than N vertices. The result follows since is continuous on all of $\mathcal{P}_{\mathrm{T}}^N$. $\square$

Theorem 1.2 part (i) follows from Theorem 6.2 and Theorem 6.3. In addition, these Theorems also imply the following result. Let $N\geqslant 1$ and let $\mathrm{Symp}_{\mathrm{T},N}^{2n,\mathbb{T}^n}$ denote the set of symplectic toric manifolds with exactly N points fixed by the $\mathbb{T}^n$-action. For $(M,\omega,\phi)\in\mathrm{Symp}_{\mathrm{T},N}^{2n,\mathbb{T}^n}$ with fixed points $p_1,\ldots,p_N\in M$ let

$$\mathcal{T}^i(M)=\left(\frac{\sup\{\,\mathrm{vol}(P)\mid P\text{ is a toric ball packing of }M\text{ such that }p_i\notin P\,\}}{\mathrm{vol}(\mathrm{B}^{2n})}\right)^{\frac{1}{2n}}.$$

Proposition 6.4. *The space $\mathrm{Symp}_{\mathrm{T},N}^{2n,\mathbb{T}^n}$ is the largest neighborhood of M in $\mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{T}^n}$ in which $\mathcal{T}$ is continuous if and only if $\mathcal{T}^i(M)<\mathcal{T}(M)$ for every $1\leqslant i\leqslant N$.*

Theorem 1.2 part (i) and Proposition 6.4 are illustrated in Figure 7. If $n=2$ Proposition 6.4 was proved in [7].

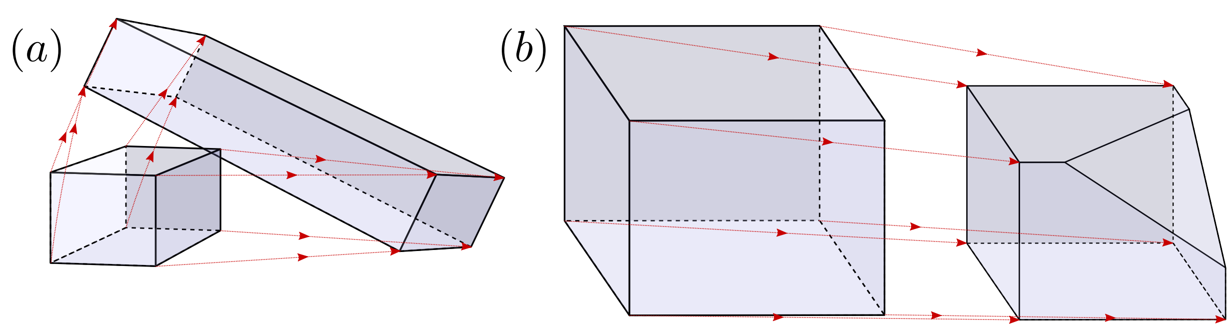

FIGURE 7. Continuous families of Delzant polygons on which (a) $\mathcal{T}$ is continuous and (b) $\mathcal{T}$ is not continuous.

7. Continuity of symplectic $(S^1 \times \mathbb{R})$-capacities

In this section we study the continuity of the symplectic $(S^1 \times \mathbb{R})$-capacity constructed in Section 5. In [15] the second author defines a metric space structure on the moduli space of simple semitoric systems and in this section we will review this structure. We are only interested in the topology of $\mathrm{Symp}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}}/{\sim_{\mathrm{ST}}}$ so, as is suggested in [15, Remark 1.31(3)], we will use a simplified version of the metric. It is shown that while the simplified version produces a different metric space structure on $\mathrm{Symp}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}}/{\sim_{\mathrm{ST}}}$ it induces the same topology on $\mathrm{Symp}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}}/{\sim_{\mathrm{ST}}}$ as the full metric [15, Section 2.6].

Let us recall how the metric is constructed, since it is essential for the proofs of the upcoming results. One has a metric for every invariant (Definition 3.5) and then [15] constructs a "joint" metric from these. The first metric is the one on the Taylor series invariants, which is given as follows. A sequence $\{b_n\}_{n=0}^{\infty}$ with $b_n \in (0,\infty)$ is said to be *linear summable* if $\sum_{n=0}^{\infty} n b_n < \infty$. Let $\{b_n\}$ be any such sequence and define $d_{\mathbb{R}[[X,Y]]_0}^{\{b_n\}_{n=0}^{\infty}}\big((S^1)^\infty,(S^2)^\infty\big)$ to be

$$\sum_{i,j\geqslant 0,(i,j)\neq(0,1)} \min\big(\left|\sigma_{i,j}^1-\sigma_{i,j}^2\right|, b_{i+j}\big) + \min\big(\left|\sigma_{0,1}^1-\sigma_{0,1}^2\right|, 2\pi - \left|\sigma_{0,1}^1-\sigma_{0,1}^2\right|, b_1\big)$$

where $(S^\ell)^\infty = \sum_{i,j\geqslant 0} \sigma_{i,j}^\ell X^i Y^j \in \mathbb{R}[[X,Y]]_0$ for $\ell = 1,2$.

We denote the Lebesgue measure by λ and use $*$ to denote the symmetric difference. A measure ν on $\mathbb{R}^2$ is *admissible* if it is in the same measure class as λ (i.e. $\nu \ll \lambda$ and $\lambda \ll \nu$) and there exists some $g\colon \mathbb{R}\to\mathbb{R}$ such that the Radon-Nikodym derivative of ν with respect to λ satisfies $\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}(x,y) = g(x)$ for all $x,y\in\mathbb{R}$, where g is bounded and bounded away from zero.

Fix an admissible measure ν. For $m_f \in \mathbb{Z}_{\geqslant 0}$ and $\vec{k}\in\mathbb{Z}^{m_f}$ let $\mathrm{Polyg}_{\mathrm{ST}}^{m_f,\vec{k}}(\mathbb{R}^2)_0$ denote the set of primitive semitoric polygons with complexity m_f and twisting index $\vec{k}$ and let $\mathrm{Polyg}_{\mathrm{ST}}^{m_f,\vec{k}}(\mathbb{R}^2)$ denote the set of semitoric polygons which are the orbit of a primitive semitoric polygon in $\mathrm{Polyg}_{\mathrm{ST}}^{m_f,\vec{k}}(\mathbb{R}^2)_0$. We may define $d_{\mathcal{P}}^{\nu}\colon \mathrm{Polyg}_{\mathrm{ST}}^{m_f,\vec{k}}(\mathbb{R}^2)\times\mathrm{Polyg}_{\mathrm{ST}}^{m_f,\vec{k}}(\mathbb{R}^2)\to[0,\infty)$ by showing how it acts on orbits $[\Delta_w^i]$ elements $\Delta_w^i = \big(\Delta^i,(\ell_{\lambda_j^i},+1,k_j)_{j=1}^{m_f}\big)\in\mathrm{Polyg}_{\mathrm{ST}}^{m_f,\vec{k}}(\mathbb{R}^2)_0$. If $m_f>0$,

$$d_{\mathcal{P}}^{\nu}\big([\Delta_w^1],[\Delta_w^2]\big) = \sum_{\vec{u}\in\{0,1\}^{m_f}} \nu\big(t_{\vec{\lambda}^1}^{\vec{u}}(\Delta^1) * t_{\vec{\lambda}^2}^{\vec{u}}(\Delta^2)\big)$$

and, if $m_f = 0$,

$$d_{\mathcal{P}}^{\nu}\big([\Delta_w^1],[\Delta_w^2]\big) = \nu\big(\Delta^1 * \Delta^2\big).$$

For $I^i = \big(m_f,((S_j^i)^\infty)_{j=1}^{m_f},[\Delta_w^i],(h_j^i)_{j=1}^{m_f}\big)\in\mathbb{I}$, $i=1,2$ define $d_{m_f,\vec{k}}^{\nu,\{b_n\}_{n=0}^{\infty}}(I^1,I^2)$ to be

$$d_{\mathcal{P}}^{\nu}([\Delta_w^1],[\Delta_w^2]) + \sum_{j=1}^{m_f}\left(d_{\mathbb{R}[[X,Y]]_0}^{\{b_n\}_{n=0}^{\infty}}\big((S_j^1)^\infty,(S_j^2)^\infty\big) + \left|h_j^1-h_j^2\right|\right)$$

if $I^1,I^2\in\mathbb{I}_{m_f,\vec{k}}$ for some $m_f\in\mathbb{Z}_{\geqslant 0},\vec{k}\in\mathbb{Z}^{m_f}$ and otherwise define $d_{m_f,\vec{k}}^{\nu,\{b_n\}_{n=0}^{\infty}}(I^1,I^2)=\infty$ (so systems in different $\mathbb{I}_{m_f,\vec{k}}$ will be in different components of the resulting topological space). The *metric* $\mathcal{D}_{\mathrm{ST}}$ *on* $\mathrm{Symp}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}}/{\sim_{\mathrm{ST}}}$ is the pullback of this one by Φ. It was shown in [15, Theorem A] that the topology induced on $(\mathrm{Symp}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}}/{\sim_{\mathrm{ST}}},\mathcal{D}_{\mathrm{ST}})$ by the metric does not depend on the choice of ν or $\{b_n\}_{n=0}^{m_f}$.

Since $\mathrm{Ham}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}}/{\approx_{\mathrm{ST}}}$ is a quotient of $\mathrm{Ham}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}}$ we can pull the topology up from $\mathrm{Ham}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}}/{\approx_{\mathrm{ST}}}$ to $\mathrm{Ham}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}}$ by declaring that a set in $\mathrm{Ham}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}}$ is open if and only if it is the preimage of an open set from $\mathrm{Ham}^{4,S^1\times\mathbb{R}}_{\mathrm{ST}}/{\approx_{\mathrm{ST}}}$ under the natural projection. We endow $\mathrm{Symp}^{4,S^1\times\mathbb{R}}_{\mathrm{ST}}$ with the quotient topology relative to the map $\mathrm{Ham}^{4,S^1\times\mathbb{R}}_{\mathrm{ST}} \to \mathrm{Symp}^{4,S^1\times\mathbb{R}}_{\mathrm{ST}}$ which forgets the momentum map. Thus a map $c\colon \mathrm{Symp}^{4,S^1\times\mathbb{R}}_{\mathrm{ST}} \to [0,\infty]$ which descends to a well-defined map ϕ on $\mathrm{Symp}^{4,S^1\times\mathbb{R}}_{\mathrm{ST}}/{\sim_{\mathrm{ST}}}$ is continuous if and only if the map $\hat{c}\colon \mathrm{Ham}^{4,S^1\times\mathbb{R}}_{\mathrm{ST}}/{\approx_{\mathrm{ST}}} \to [0,\infty]$ is continuous where $\hat{c}$ is defined by the commutative diagram:

$$\begin{tikzcd}
\mathrm{Ham}^{4,S^1\times\mathbb{R}}_{\mathrm{ST}} \arrow[r] \arrow[d] & \mathrm{Symp}^{4,S^1\times\mathbb{R}}_{\mathrm{ST}} \arrow[r, "c"] \arrow[d] & {[0,\infty]} \\
\mathrm{Ham}^{4,S^1\times\mathbb{R}}_{\mathrm{ST}}/{\approx_{\mathrm{ST}}} \arrow[r] \arrow[urr, bend right, "\hat{c}"'] & \mathrm{Symp}^{4,S^1\times\mathbb{R}}_{\mathrm{ST}}/{\sim_{\mathrm{ST}}} \arrow[ur, "\phi"] &
\end{tikzcd}$$

Let $\Delta_w = (\Delta, (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f})$ be a primitive semitoric polygon, and let $v \in \Delta$ be a vertex.

Definition 7.1. An *admissible semitoric simplex of radius $r > 0$ with center at v* is a subset Σ of Δ such that there exist some $A \in \mathrm{AGL}_2(\mathbb{Z})$ and $\vec{u} \in \{0,1\}^{m_f}$ satisfying:
- $A(\Delta(r^{1/2})) = t^{\vec{u}}_{\vec{\lambda}}(\Sigma)$;
- $A(0) = t^{\vec{u}}_{\vec{\lambda}}(v)$;
- A takes the edges of $\Delta(r^{1/2})$ meeting at the origin to the edges of $t^{\vec{u}}_{\vec{\lambda}}(\Delta)$ meeting at $t^{\vec{u}}_{\vec{\lambda}}(v)$;
- $\Sigma \subset \Delta^{\vec{u}}$ where

$$\Delta^{\vec{u}} = \Delta \setminus \left\{ (x,y) \in \Delta \;\middle|\; \begin{array}{l} x = \lambda_j \text{ and } (-2\vec{u}+1)y \geqslant \min_{(\lambda_j, y_0)} y_0 + h_j \\ \text{for some } j \in \{1, \dots, m_f\} \end{array} \right\}.$$

An *admissible semitoric packing of Δ_w* is a disjoint union $R = \bigsqcup_{\alpha\in\mathcal{A}} \Sigma_\alpha$ where each Σ_α is an admissible simplex of some radius, where the radii of the simplices are allowed to be different.

Such a simplex cannot exist at a fake corner.

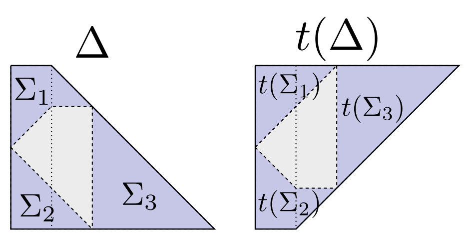

FIGURE 8. An admissible semitoric packing. Here t denotes $t^{\vec{u}}_{\vec{\lambda}}$.

Lemma 7.2 ([17]). *Let F^B be a momentum map for the usual $\mathbb{T}^n$-action on $\mathrm{B}^{2n}(r)$, $r > 0$, and let (M, ω, ϕ, F) be a Hamiltonian $\mathbb{T}^n$-manifold of dimension $2n$. If $\rho\colon \mathrm{B}^{2n}(r) \hookrightarrow M$ is a symplectic $\mathbb{T}^n$-embedding with respect to some $\Lambda \in \mathrm{Aut}(\mathbb{T}^n)$ then there exists some $x \in \mathbb{R}^n$ such that the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{B}^{2n}(r) \arrow[r, "\rho"] \arrow[d, "F^B"'] & M \arrow[d, "F"] \\
\mathbb{R}^2 \arrow[r, "(\Lambda^t)^{-1}+x"] & \mathbb{R}^2
\end{tikzcd}$$

where $(\Lambda^t)^{-1} + x$ is the affine map with linear part $(\Lambda^t)^{-1}$ which takes 0 to x.

In [13] a *proper Hamiltonian $\mathbb{T}^n$-manifold* is a quadruple $(Q, \omega^Q, F^Q, \Gamma)$ where (Q, ω^Q) is a connected $2n$-dimensional symplectic manifold with momentum map F^Q for an action of $\mathbb{T}^n$ and $\Gamma \subset \mathrm{Lie}(\mathbb{T}^n)^*$ is an open convex subset with $F^Q(Q) \subset \Gamma$ and such that F^Q is proper as a map to Γ. A proper Hamiltonian $\mathbb{T}^n$-manifold is *centered about* $p \in \Gamma$ if p is an element of each component of $F^Q(Q^K)$ for each subgroup $K \subset \mathbb{T}^n$, where Q^K is the set of all points in Q which are fixed by the action of all elements of K.

Lemma 7.3 ([13])**.** *Let $(Q, \omega^Q, F^Q, \Gamma)$ be a proper Hamiltonian $\mathbb{T}^n$-manifold of dimension $2n$. If $(Q, \omega^Q, F^Q, \Gamma)$ is centered about $p \in \Gamma$ and $(F^Q)^{-1}(\{p\}) = \{q\}$, then Q is equivariantly symplectomorphic to $\{\, z \in \mathbb{C}^n \mid p + \sum_{j=1}^n |z_j|^2\, \eta_j^q \in \Gamma \,\}$, where $\eta_1^q, \ldots, \eta_m^q \in \mathrm{Lie}(\mathbb{T}^n)^*$ are the weights of the isotropy representation of $\mathbb{T}^n$ on T_qQ.*

We use Lemma 7.2 and Lemma 7.3 to prove the following.

Proposition 7.4. *Let $(M, \omega, F = (J, H))$ be a semitoric manifold such that*

$$\Phi\big((M, \omega, F)\big) = \big(m_f, ((S_j)^\infty)_{j=1}^{m_f}, [\Delta_w], (h_j)_{j=1}^\infty\big)$$

where $\Delta_w = (\Delta, (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f})$ is primitive with associated momentum map $\widetilde{F} \in \mathcal{F}_M$ such that $\widetilde{F}(M) = \Delta$. Then:

1. *Suppose $\rho\colon \mathrm{B}^4(r) \hookrightarrow M$ is a semitoric embedding for some $r > 0$. Then $\widetilde{F}(\rho(\mathrm{B}^4(r))) \subset \Delta$ is an admissible semitoric simplex with radius r^2. Conversely, if $\Sigma \subset \Delta$ is an admissible semitoric simplex with radius r^2 then there exists a semitoric embedding $\rho\colon \mathrm{B}^4(r) \hookrightarrow M$ such that $\widetilde{F}(\rho(\mathrm{B}^4(r))) = \Sigma$.*
2. *Let P be a semitoric ball packing of M. Then $\widetilde{F}(P) \subset \Delta$ is an admissible packing of Δ_w. Conversely, if R is an admissible packing of Δ_w then there exists a semitoric ball packing P of M such that $\widetilde{F}(P) = R$.*

Proof. Part (2) follows from Part (1) since the semitoric simplices associated to disjoint semitoricly embedded balls are disjoint. This follows from the fact that $\widetilde{F}^{-1}(p)$ is a 2-dimensional submanifold of M for any regular point $p \in \Delta$ and the embedded balls are 2-dimensional.

Suppose that $B \subset M$ is a semitoricly embedded ball of radius $r > 0$. Then for some $\vec{\epsilon} \in \{-1, +1\}^{m_f}$ the map $\rho_{\vec{\epsilon}}\colon \mathrm{B}^4(r) \hookrightarrow M^{\vec{\epsilon}}$ is a $\mathbb{T}^2$-embedding with respect to some $\Lambda \in \mathrm{Aut}(\mathbb{T}^2)$. Recall $M^{\vec{\epsilon}}$ is a Hamiltonian $\mathbb{T}^2$-manifold and denote a momentum map for this action by $F^{\vec{\epsilon}}$. Let $p = F^{\vec{\epsilon}}(\rho(0))$ and let $\Delta^{\vec{\epsilon}} = F^{\vec{\epsilon}}(M^{\vec{\epsilon}})$. Hence by Lemma 7.2 the diagram

$$\begin{tikzcd}
\mathrm{B}^4(r) \arrow[r, "\rho"] \arrow[d, "F_{\mathrm{B}}"'] & M^{\vec{\epsilon}} \arrow[d, "F^{\vec{\epsilon}}"] \\
\Delta_{\mathrm{B}} \arrow[r, "(\Lambda^t)^{-1}+x"] & \Delta^{\vec{\epsilon}}
\end{tikzcd}$$

commutes for some $x \in \mathrm{Lie}(\mathbb{T}^2)^*$. Since Λ is an automorphism so is $(\Lambda^t)^{-1}$, hence it sends the weights of the isotropy representation of $\mathbb{T}^2$ on $T_0(\mathrm{B}^4(r))$ to the weights of the isotropy representation on T_pM. Since $(\Lambda^t)^{-1}$ is linear and Δ_{B} is the convex hull of the isotropy weights of the representation on $T_0(\mathrm{B}^4(r))$ and the origin, we find that

$$\Sigma^{\vec{\epsilon}} := [(\Lambda^t)^{-1} + x](\Delta_{\mathrm{B}})$$

is the convex hull of p, $p + r^2\alpha_1$, and $p + r^2\alpha_2$, minus the convex hull of $p + r^2\alpha_1$ and $p + r^2\alpha_2$, where α_1 and α_2 are the weights of the isotropy representation of $\mathbb{T}^2$ on T_pM. For $\vec{u} = \frac{1}{2}(1 - \vec{\epsilon})$ recall that $t^{\vec{u}}_{\vec{\lambda}}(\Delta) = \Delta^{\vec{\epsilon}}$ and let $\Sigma = \left(t^{\vec{u}}_{\vec{\lambda}}\right)^{-1}(\Sigma^{\vec{\epsilon}})$. Notice that $\Sigma = \widetilde{F}(\rho(\mathrm{B}^4(r))) \subset \Delta$ and is an admissible semitoric simplex.

To prove the converse let $\Sigma \subset \Delta$ be an admissible semitoric simplex. This means that there exists some $\vec{\epsilon} \in \{-1,+1\}^{m_f}$ such that

$$\Sigma' := t^{\vec{u}}_{\vec{\lambda}}(\Sigma)$$

satisfies the requirements of Definition 7.1, where $\vec{u} = \frac{1}{2}(1-\vec{\epsilon})$. Let $\Delta' = t^{\vec{u}}_{\vec{\lambda}}(\Delta)$. Let p be the unique vertex of Σ'. Thus, Σ' is the convex hull of p, $p + r^2\alpha_1$, and $p + r^2\alpha_2$, minus the convex hull of $p+r^2\alpha_1$ and $p+r^2\alpha_2$, for some $\alpha_i \in \mathbb{R}^2$, $i = 1, 2$. Let $\Gamma \subset \mathbb{R}^2$ be the unique open half plane satisfying $\Gamma \cup \Delta' = \Sigma'$. Let $N = \widetilde{F}^{-1}(\Sigma)$ and let $\omega^N = \omega|_N$. We can see that $N \subset M$ is open and by the proof of the Atiyah-Guillemin-Sternberg Convexity Theorem [1, 10] we know that N is connected. The map $\widetilde{F}$ is proper because its first component, J, is proper and thus $\widetilde{F}^N := t^{\vec{u}}_{\vec{\lambda}}\left(\widetilde{F}|_N\right) : N \to \Sigma'$ is proper. Therefore $\widetilde{F}^N : N \to \Gamma$ is proper because $(\widetilde{F}^N)^{-1}(\Gamma \setminus \Sigma') = \varnothing$, and hence $(N, \omega^N, \widetilde{F}^N, \Gamma)$ is a proper Hamiltonian $\mathbb{T}^2$-manifold. Since $(N, \omega^N, \widetilde{F}^N, \Gamma)$ is centered about $p \in \mathbb{R}^2$ by Lemma 7.3 we conclude that N is equivariantly symplectomorphic to

$$\{\, z \in \mathbb{C}^2 \mid p + |z_1|^2\,\alpha_1 + |z_2|^2\,\alpha_2 \in \Gamma \,\} = \mathrm{B}^4(r).$$

It follows that there exists a symplectic $\mathbb{T}^2$-embedding $\rho\colon \mathrm{B}^4(r) \hookrightarrow M^{\vec{\epsilon}}$ with image N so $\widetilde{F}(\rho(\mathrm{B}^4(r))) = \widetilde{F}(N) = \Sigma$. $\square$

Define the *optimal semitoric polygon packing function* $\pi_{\mathrm{ST}}\colon \mathrm{Polyg}_{\mathrm{ST}}(\mathbb{R}^2) \to [0,\infty]$ by

$$\pi_{\mathrm{ST}}([\Delta_w]) = \sup\{\, \mathrm{vol}_{\mathcal{P}}(P) \mid P \text{ is an admissible semitoric packing of } \Delta_w \,\}.$$

It is well-defined because any two primitive semitoric polygons in the same orbit are related to one another by a transformation in $G_{m_f} \times \mathcal{G}$ which sends semitoric packings to semitoric packings and preserves volume.

Definition 7.5. We call $\alpha \in (0, \pi)$ a *smooth angle* if it can be obtained as an angle in a Delzant polygon.

Equivalently, $\alpha \in (0, \pi)$ is smooth if and only if it is the angle at the origin of $A_\alpha(\Delta(1))$ for some $A_\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

Lemma 7.6. *The set of smooth angles is discrete in $(0, \pi) \subset \mathbb{R}$.*

Proof. Fix a smooth angle $\alpha \in (0, \pi)$ and fix some $\varepsilon > 0$ small enough so that $(\alpha-\varepsilon, \alpha+\varepsilon) \subset (0, \pi)$. Let

$$B_\varepsilon(\alpha) = \{\, \beta \in (0, \pi) \mid \beta \text{ is a smooth angle and } |\alpha - \beta| < \varepsilon \,\}$$

and let $\delta_\varepsilon > 0$ be such that if $\beta \in B_\varepsilon(\alpha)$ then $|\sin(\alpha) - \sin(\beta)| < \delta_\varepsilon$. Now fix any $\beta \in B_\varepsilon(\alpha)$. This means there exists some $A_\beta \in \mathrm{SL}_2(\mathbb{Z})$ such that β is the angle at the origin of $\Delta = A_\beta(\Delta(1))$. Let $\ell_1, \ell_2 \in \mathbb{R}$ denote the lengths of two edges of the simplex Δ which are adjacent to the vertex at the origin. These each represent the magnitude of a vector in $\mathbb{Z}^n$ so $\ell_i \geqslant 1$ for $i = 1, 2$. By the choice of δ_ε we have that $\sin(\beta) > \sin(\alpha) - \delta_\varepsilon$. Since Δ has area $1/2$ we know that $\frac{\ell_1\ell_2\sin(\beta)}{2} = \frac{1}{2}$ and so for $i = 1, 2$ we conclude that $1 = \ell_1\ell_2\sin(\beta) \geqslant \ell_i\sin(\beta)$ which implies that

$$\ell_i \leqslant \frac{1}{\sin(\beta)} < \frac{1}{\sin(\alpha) - \delta_\varepsilon}.$$

Therefore associated to each $\beta \in B_\varepsilon(\alpha)$ there is a pair of vectors in $\mathbb{Z}^2$ each with length less than $(\sin(\alpha) - \delta_\varepsilon)^{-1}$, a value which does not depend on β. There are only finitely many such vectors. $\square$

The proof of Lemma 7.6 is taken from the proof of [7, Theorem 7.1] and is a two-dimensional version of the strategy used in Theorem 6.3. Let $\alpha \in (0,\pi)$ be called a *hidden smooth angle* if it can be obtained as a hidden corner in a primitive semitoric polygon.

Corollary 7.7. *The set of hidden smooth angles is discrete in $(0,\pi) \subset \mathbb{R}$.*

It is important to notice that a sequence of smooth angles can approach π. This must be the case, for example, if a semitoric polygon has infinitely many vertices.

Definition 7.8. We say that a vertex v of $(\Delta, (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f}))$ is *non-fake* if it is either Delzant or hidden in one, and hence all, elements of the affine invariant. For $N \geqslant 1$ let $\mathrm{Polyg}^N_{\mathrm{ST}}(\mathbb{R}^2)_0$ denote the set of primitive polygons with exactly N non-fake vertices and let $\mathrm{Polyg}^N_{\mathrm{ST}}(\mathbb{R}^2)$ denote the set of $(G_{m_f} \times \mathcal{G})$-orbits of elements of $\mathrm{Polyg}^N_{\mathrm{ST}}(\mathbb{R}^2)_0$. Let $\mathbb{I}^N$ be the set of all semitoric ingredients for which the affine invariant is an element of $\mathrm{Polyg}^N_{\mathrm{ST}}(\mathbb{R}^2)$ and let

$$\mathrm{Symp}^{4,S^1\times\mathbb{R}}_{\mathrm{ST},N} = \Phi^{-1}(\mathbb{I}^N)$$

where Φ is as in Equation (5).

Recall $\mathcal{H}^{\varepsilon}_p(v)$ defined in Equation (7). The following are two operations which can be performed on $[\Delta_w]$ to produce a new element of $\mathrm{Polyg}_{\mathrm{ST}}(\mathbb{R}^2)_0$.

Definition 7.9. Let $\Delta_w = (\Delta, (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f})$. Let $p \in \Delta$ be a vertex and let $v_1, v_2 \in \mathbb{Z}^2$ be the primitive inwards pointing normal vectors to the two edges which meet at p ordered so that $\det(v_1, v_2) > 0$.

If p is a Delzant vertex of Δ_w then the *ε-corner chop of Δ_w at p* is the primitive semitoric polygon

$$\Delta_w^{p,\varepsilon} = \left(\Delta \cap \mathcal{H}^{\varepsilon}_p(v_1+v_2), (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f}\right).$$

Similarly, given $[\Delta_w]$ we say that $[\Delta_w^{p,\varepsilon}]$ is the *ε-corner chop of $[\Delta_w]$ at p.*

Suppose p is a hidden corner of Δ_w and thus there exists $j \in \{1, \ldots, m_f\}$ such that $p \in \ell_{\lambda_j}$. The *ε-hidden corner chop of Δ_w at p* is the primitive semitoric polygon

$$\Delta_w^{p,\varepsilon} = \left(\Delta \cap t^{-1}_{\ell_{\lambda_j}}\left(\mathcal{H}^{\varepsilon}_p(v_1+v_2)\right), (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f}\right).$$

We say that $[\Delta_w^{p,\varepsilon}]$ is the *ε-hidden corner chop of $[\Delta_w]$ at p.*

The hidden corner chop of a hidden corner amounts to acting on the polygon with $t^1_{\ell_{\lambda_j}}$ to transform the hidden corner into a Delzant corner, performing the usual corner chop on this Delzant corner, and then transforming the polygon back with $t^{-1}_{\ell_{\lambda_j}}$. This is shown in Figure 9.

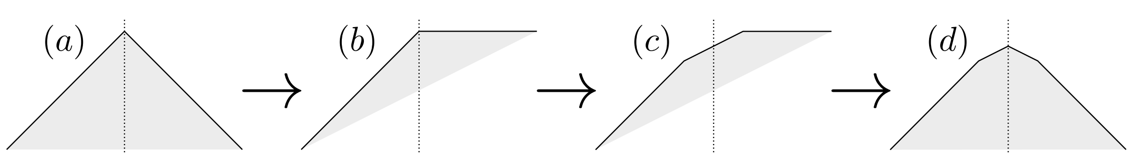

FIGURE 9. In (a) a hidden corner is shown. In (b) we unfold it by reversing the sign of the associated ϵ_i resulting in a Delzant corner. In (c) we perform corner chop on this corner and in (d) the ϵ_i returns to its original sign.

Lemma 7.10. *Fix $N \in \mathbb{Z}_{\geqslant 0}$. Each $[\Delta_w] \in \mathrm{Polyg}^N_{\mathrm{ST}}(\mathbb{R}^2)$ has an open neighborhood in $\mathrm{Polyg}^N_{\mathrm{ST}}(\mathbb{R}^2)$ which consists exclusively of transformations of $[\Delta_w]$ in which its sides are moved in a parallel way. Moreover, any sufficiently small neighborhood of $[\Delta_w]$ in $\mathrm{Polyg}_{\mathrm{ST}}(\mathbb{R}^2)$ is contained in $\cup_{(N' \geqslant N)} \mathrm{Polyg}^{N'}_{\mathrm{ST}}(\mathbb{R}^2)$.*

Proof. The angles of non-fake corners are discrete by Lemma 7.6 and Corollary 7.7. This means that there exists a neighborhood of $[\Delta_w]$ in which all elements which have N non-fake vertices must have all of the same angles as $[\Delta_w]$. This is the open neighborhood described in the Lemma. Any semitoric polygon with fewer non-fake vertices than $[\Delta_w]$ is bounded away from $[\Delta_w]$ because the only ways to change the number of non-fake vertices are a corner chop or introducing a smooth angle into an edge of infinite length but by Lemma 7.6 smooth angles are discrete. $\square$

Lemma 7.11. *The map $\pi_{\mathrm{ST}} \colon \mathrm{Polyg}_{\mathrm{ST}}(\mathbb{R}^2) \to [0, \infty]$ is discontinuous at every point.*

Proof. Primitive semitoric polygons must have at least one non-fake vertex. Let

$$[\Delta_w] = [(\Delta, (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f})]$$

be a semitoric polygon. First assume that $[\Delta_w] \in \mathrm{Polyg}^N_{\mathrm{ST}}(\mathbb{R}^2)$ for some $N \geqslant 1$ and that $\pi_{\mathrm{ST}}([\Delta_w]) < \infty$. Then for $\varepsilon > 0$ small enough define $[\Delta^\varepsilon_w]$ to be the semitoric polygon produced by performing an ε-corner chop at each non-fake vertex of $[\Delta_w]$. We have that

$$\lim_{\varepsilon \to 0} d^{\mathcal{P}}_{\mathrm{ST}}([\Delta], [\Delta^\varepsilon_w]) = 0. \tag{10}$$

A packing of $[\Delta^\varepsilon_w]$ has at most $2N$ disjoint admissible simplices. Since their side lengths are determined by the lengths of the adjacent edges, one of which is length ε, we have that $\lim_{\varepsilon \to 0} \pi_{\mathrm{ST}}([\Delta^\varepsilon_w]) = 0$. Since every semitoric polygon has positive optimal packing we have

$$\lim_{\varepsilon \to 0} |\pi_{\mathrm{ST}}([\Delta_w]) - \pi_{\mathrm{ST}}(\Delta^\varepsilon_w)| = \pi_{\mathrm{ST}}([\Delta_w]) > 0$$

and thus, in light of Equation (10), π_{ST} is discontinuous at $[\Delta_w]$.

Suppose $[\Delta_w] \in \mathrm{Polyg}^N_{\mathrm{ST}}(\mathbb{R}^2)$ for some $N \geqslant 1$ and $\pi_{\mathrm{ST}}([\Delta_w]) = \infty$. Since $[\Delta_w]$ has only finitely many non-fake vertices, any admissible packing has only finitely many admissible simplices. Hence there is a vertex at which an arbitrarily large simplex fits. The only possible case is that $N = 1$ and the polygon is of complexity zero. Taking a corner chop of any size at the single non-fake vertex produces a polygon on which π_{ST} evaluates to a finite number, so π_{ST} is discontinuous at $[\Delta_w]$.

Now suppose that $\pi_{\mathrm{ST}}([\Delta_w]) < \infty$ and $[\Delta_w] \in \mathrm{Polyg}_{\mathrm{ST}}(\mathbb{R}^2) \setminus \bigcup_{N \geqslant 1} \mathrm{Polyg}^N_{\mathrm{ST}}(\mathbb{R}^2)$. For $i \in \mathbb{Z}_{\geqslant 1}$ let $I_i \subset \mathbb{R}$ be given by $I_i = [-n, n] \setminus (-(n-1), n-1)$ and let $N_i \in \mathbb{Z}_{\geqslant 0}$ denote the number of non-fake vertices of $[\Delta_w]$ with x-coordinate in I_i. This number is finite by the definition of a convex polygon and it is invariant under the action of $G_{m_f} \times \mathcal{G}$. For $\varepsilon > 0$ small enough let $[\Delta^\varepsilon_w]$ be a semitoric polygon which has a small corner chop at each non-fake vertex such that, at each vertex in I_i for $i \in \mathbb{Z}_{\geqslant 1}$, the largest possible admissible simplex that can fit into that vertex has volume at most $\varepsilon/(N_i S^{i+1})$. Then an admissible packing R of $[\Delta^\varepsilon_w]$ satisfies

$$\mathrm{vol}_{\mathcal{P}}(R) \leqslant \sum_{i=1}^{\infty} \frac{\varepsilon}{N_i 2^{i+1}} 2N_i = \varepsilon.$$

Therefore

$$\lim_{\varepsilon \to 0} d^{\mathcal{P}}_{\mathrm{ST}}([\Delta_w], [\Delta^\varepsilon_w]) = 0$$

while

$$\lim_{\varepsilon \to 0} |\pi_{\mathrm{ST}}([\Delta_w]) - \pi_{\mathrm{ST}}([\Delta^\varepsilon_w])| = \pi_{\mathrm{ST}}([\Delta_w]) > 0$$

and thus π_{ST} is not continuous at $[\Delta_w]$. $\square$

For $[\Delta_w] = [(\Delta, (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f})] \in \mathrm{Polyg}_{\mathrm{ST}}^N(\mathbb{R}^2)$ with non-fake vertices $v_1, \ldots, v_N$, let $\pi_{\mathrm{ST}}^{\mathcal{P},i}(\Delta)$ be the total volume of the optimal packing excluding all packings which have a simplex centered at v_i.

Theorem 7.12. *Let $\pi_{\mathrm{ST}} \colon \mathrm{Polyg}_{\mathrm{ST}}(\mathbb{R}^2) \to [0, \infty]$ be the optimal semitoric polygon packing function. Then:*

(1) *π_{ST} is discontinuous at each point in $\mathrm{Polyg}_{\mathrm{ST}}(\mathbb{R}^2)$;*
(2) *the restriction $\pi_{\mathrm{ST}}|_{\mathrm{Polyg}_{\mathrm{ST}}^N(\mathbb{R}^2)}$ is continuous for each $N \in \mathbb{Z}_{\geqslant 1}$;*
(3) *if $[\Delta_w] \in \mathrm{Polyg}_{\mathrm{ST}}^N(\mathbb{R}^2)$ then $\mathrm{Polyg}_{\mathrm{ST}}^N(\mathbb{R}^2)$ is the largest neighborhood of Δ_w in $\mathrm{Polyg}_{\mathrm{ST}}^N(\mathbb{R}^2)$ in which π_{ST} is continuous if and only if $\pi_{\mathrm{ST}}^i([\Delta_w]) < \pi_{\mathrm{ST}}([\Delta_w])$ for all $1 \leqslant i \leqslant N$.*

Proof. Part (1) is the content of Lemma 7.11.

By Lemma 7.10, given any $[\Delta_w] \in \mathrm{Polyg}_{\mathrm{ST}}^N(\mathbb{R}^2)$, there exists a neighborhood of $[\Delta_w]$ in $\mathrm{Polyg}_{\mathrm{ST}}^N(\mathbb{R}^2)$ containing exclusively orbits of polygons formed by translating the sides of Δ_w in a parallel way. Hence part (2) follows from this because π_{ST} is continuous on such transformations.

For Part (3) suppose first that $\pi_{\mathrm{ST}}([\Delta_w]) = \pi_{\mathrm{ST}}^i([\Delta_w])$ for some $i \in \{1, \ldots, N\}$. This means that there exists some optimal packing avoiding the i^{th} non-fake vertex. For $\varepsilon > 0$ let $[\Delta_w^\varepsilon]$ be the result of an ε-corner chop at the i^{th} vertex and notice that $\lim_{\varepsilon \to 0} d_{\mathrm{ST}}^{\mathcal{P}}([\Delta_w], [\Delta_w^\varepsilon]) = 0$ and $\lim_{\varepsilon \to 0} \pi_{\mathrm{ST}}([\Delta_w^\varepsilon]) = \pi_{\mathrm{ST}}([\Delta_w])$. Thus there exists some set larger than $\mathrm{Polyg}_{\mathrm{ST}}^N(\mathbb{R}^2)$ on which π_{ST} is continuous, as shown in Figure 10.

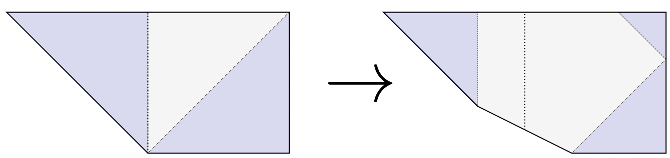

FIGURE 10. Corner chop of a corner not used in the optimal packing.

Finally, to show the converse assume that $[\Delta_w]$ satisfies $\pi_{\mathrm{ST}}^{\mathcal{P},i}([\Delta_w]) < \pi_{\mathrm{ST}}^{\mathcal{P}}([\Delta_w])$ for all $1 \leqslant i \leqslant N$. By Lemma 7.10 there is an open set around $[\Delta_w]$ in which the only elements not in $\mathrm{Polyg}_{\mathrm{ST}}^N(\mathbb{R}^2)_0$ are obtained from $[\Delta_w]$ by iterations of corner chops, parallel translations of the edges, and introducing a smooth angle into an edge of infinite length. For $\varepsilon > 0$ let $[\Delta_w^\varepsilon]$ be any ε-corner chop at the i^{th} non-fake vertex of $[\Delta_w]$. Then

$$\lim_{\varepsilon \to 0} \pi_{\mathrm{ST}}([\Delta_w^\varepsilon]) = \pi_{\mathrm{ST}}^i([\Delta_w]) < \pi_{\mathrm{ST}}([\Delta_w])$$

and the result follows. □

Notice that the quotient map $\mathrm{Symp}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}} \to \mathrm{Polyg}_{\mathrm{ST}}(\mathbb{R}^2)$ is continuous and the metric on $\mathrm{Symp}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}}$ is the sum of the metric on $\mathrm{Polyg}_{\mathrm{ST}}(\mathbb{R}^2)$ and the metric on the remaining components. Thus, Theorem 1.2 part (ii) follows from Theorem 7.12. For $(M, \omega, F) \in \mathrm{Symp}_{\mathrm{ST},N}^{4,S^1\times\mathbb{R}}$ with fixed points $p_1, \ldots, p_N \in M$ let

$$\mathcal{ST}^i(M) = \left(\frac{\sup\{\, \mathrm{vol}(P) \mid P \subset M \text{ is a semitoric ball packing of } M \text{ and } p_i \notin P \,\}}{\mathrm{vol}(\mathrm{B}^4)} \right)^{\frac{1}{4}}.$$

Proposition 7.13. *Let $N \geqslant 1$. If $(M, \omega, F) \in \mathrm{Symp}_{\mathrm{ST},N}^{4,S^1\times\mathbb{R}}$ then $\mathrm{Symp}_{\mathrm{ST},N}^{4,S^1\times\mathbb{R}}$ is the largest neighborhood of M in $\mathrm{Symp}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}}$ in which $\mathcal{ST}$ is continuous if and only if $\mathcal{ST}^i(M) < \mathcal{ST}(M)$ for all $1 \leqslant i \leqslant N$.*

Theorem 1.2 part (ii) and Proposition 7.13 are illustrated in Figure 11.

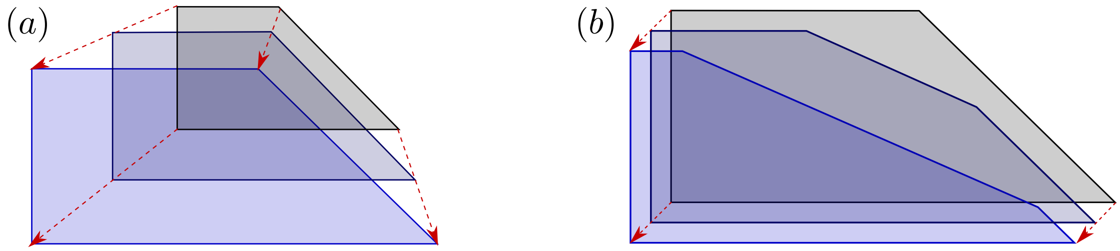

FIGURE 11. Continuous families of primitive semitoric polygons on which (a) $\mathcal{ST}$ is continuous and (b) $\mathcal{ST}$ is not continuous.

Definition 7.14. The *semitoric radius capacity* is the symplectic $(S^1\times\mathbb{R})$-capacity $\mathcal{ST}_{\mathrm{rad}}\colon \mathrm{Symp}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}} \to [0,\infty]$ given by

$$\mathcal{ST}_{\mathrm{rad}}(M) = \sup\{\, r>0 \mid \text{there exists a semitoric embedding } \mathrm{B}^4(r) \hookrightarrow M \,\}.$$

It can be shown that $\mathcal{ST}_{\mathrm{rad}}$ is a $(S^1\times\mathbb{R})$-capacity in the same way that it was shown that $\mathcal{ST}$ is a $(S^1\times\mathbb{R})$-capacity. Recall that $\mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{R}^n}$ is the symplectic $\mathbb{R}^n$-category which is the collection of toric manifolds with their $\mathbb{T}^n$-action lifted to an $\mathbb{R}^n$-action. Let $\mathrm{Symp}_{\mathrm{T},N}^{2n,\mathbb{R}^n}$ denote those systems with exactly N points fixed by the $\mathbb{R}^n$-action. By repeating the proofs of the continuity results Theorem 1.2 part (i), Proposition 6.4, Theorem 1.2 part (ii), and Proposition 7.13 we immediately have the following result, that yields Theorem 1.2 part (iii).

Theorem 7.15. *The maps $c_{\mathrm{B}}^{n,n}|_{\mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{R}^n}}$ and $\mathcal{ST}_{\mathrm{rad}}$ are discontinuous everywhere on their domains and the restrictions $c_{\mathrm{B}}^{n,n}|_{\mathrm{Symp}_{\mathrm{T},N}^{2n,\mathbb{R}^n}}$ and $\mathcal{ST}_{\mathrm{rad}}|_{\mathrm{Symp}_{\mathrm{ST},N}^{4,S^1\times\mathbb{R}}}$ are both continuous. For $(M,\omega,F)\in \mathrm{Symp}_{\mathrm{T},N}^{2n,\mathbb{R}^n}$ the set $\mathrm{Symp}_{\mathrm{T},N}^{2n,\mathbb{R}^n}$ is not the largest neighborhood of M in $\mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{R}^n}$ in which $c_{\mathrm{B}}^{n,n}|_{\mathrm{Symp}_{\mathrm{T}}^{2n,\mathbb{R}^n}}$ is continuous and for $(M,\omega,F)\in \mathrm{Symp}_{\mathrm{ST},N}^{4,S^1\times\mathbb{R}}$ the set $\mathrm{Symp}_{\mathrm{ST},N}^{4,S^1\times\mathbb{R}}$ is the largest neighborhood of M in $\mathrm{Symp}_{\mathrm{ST}}^{4,S^1\times\mathbb{R}}$ in which $\mathcal{ST}_{\mathrm{rad}}$ is continuous if and only if $N=1$.*

Remark 7.16. There are many examples of classical symplectic capacities (see for instance [3]), and it would be of interest to adapt these capacities to the equivariant category. It would also be useful to construct symplectic G-capacities for more general integrable systems. In particular, integrable systems where a complete list of invariants is not known (that is, the vast majority).

In [8] the authors give a lower bound on the number of fixed points of a circle action on a compact almost complex manifold M with nonempty fixed point set, under the condition that the Chern number $c_1c_{n-1}[M]$ vanishes. These results apply to a class of manifolds which do not support any Hamiltonian circle action with isolated fixed points, and which includes all symplectic Calabi-Yau manifolds [26] (see [8, Proposition 2.15]). The class of symplectic Calabi-Yau manifolds is thus of particular interest because they do not admit integrable systems of toric or semitoric type. Also, there is work extending the classification in [20] and related results to higher dimensions [24], so one could extend the semitoric packing capacity to higher dimensional semitoric systems, for which there is currently no classification.

Another interesting direction would be to generalize the work in [14] to our setting. There, the author constructs infinite dimensional symplectic capacities for a general class of Hamiltonian PDEs. In case the PDEs preserves some G-action, one may expect to construct also G-capacities in such infinite dimensional setting, and this may give new interesting result on the long time behavior of solutions.

Symplectic capacities are also of interest from a physical view point, for instance in [4] the authors describe interrelations between symplectic capacities and the uncertainty principle. It would be interesting to explore similar connections to symplectic G-capacities. $\oslash$

Remark 7.17. In this paper G can be a compact Lie group (like in the case of symplectic toric manifolds) or a non-compact Lie group (like in the case of semitoric systems). In general there are obstructions to the existence of effective G-actions on compact and non-compact manifolds, even in the case that the G-action is only required to be smooth. For instance, in [25, Corollary in page 242] it is proved that if N is an n-dimensional manifold on which a compact connected Lie group G acts effectively and there are $\sigma_1, \ldots, \sigma_n \in \mathrm{H}^1(M, \mathbb{Q})$ such that $\sigma_1 \cup \ldots \cup \sigma_n \neq 0$ then G is a torus and the G-action is locally free. In [25] Yau also proves several other results giving restrictions on G, M, and the fixed point set M^G. If the G-action is moreover assumed to be symplectic or Kähler, there are even more non-trivial constraints. Therefore the class of symplectic manifolds for which one can define a notion of symplectic G-capacity with G non-trivial is in general much more restrictive than the class of all symplectic manifolds. $\oslash$

References

1. M. Atiyah, *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. **14** (1982), 1–15.
2. S. M. Bates, *Some simple continuity properties of symplectic capacities*, The Floer memorial volume, Progr. Math., vol. 133, Birkhäuser, Basel, 1995, pp. 185–193.
3. K. Cieliebak, H. Hofer, J. Latschev, and F. Schlenk, *Quantitative symplectic geometry*, Dynamics, ergodic theory, and geometry, Math. Sci. Res. Inst. Publ., vol. 54, Cambridge Univ. Press, Cambridge, 2007, pp. 1–44.
4. M. de Gosson and F. Luef, *Symplectic capacities and the geometry of uncertainty: the irruption of symplectic topology in classical and quantum mechanics*, Phys. Rep. **484** (2009), no. 5, 131–179.
5. T. Delzant, *Hamiltoniens périodiques et image convex de l'application moment*, Bull. Soc. Math. France **116** (1988), 315–339.
6. I. Ekeland and H. Hofer, *Symplectic topology and Hamiltonian dynamics*, Math. Z. **200** (1989), no. 3, 355–378.
7. A. Figalli and Á. Pelayo, *On the density function on moduli spaces of toric 4-manifolds*, Adv. Geom. **16** (2016), no. 3, 291–300.
8. L. Godinho, Á. Pelayo, and S. Sabatini, *Fermat and the number of fixed points of periodic flows*, Commun. Number Theory Phys. **9** (2015), no. 4, 643–687. MR 3506787
9. M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), no. 2, 307–347.
10. V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), 491–513.
11. H. Hofer, *Symplectic capacities*, Geometry of low-dimensional manifolds, 2 (Durham, 1989), London Math. Soc. Lecture Note Ser., vol. 151, Cambridge Univ. Press, Cambridge, 1990, pp. 15–34.
12. D. M. Kane, J. Palmer, and Á. Pelayo, *Classifying toric and semitoric fans by lifting equations from* $\mathrm{SL}_2(\mathbb{Z})$, arXiv:1502.07698.
13. Y. Karshon and S. Tolman, *The Gromov width of complex Grassmannians*, Alg. and Geo. Topology **5** (2005), 911–922.
14. S. Kuksin, *Infinite-dimensional symplectic capacities and a squeezing theorem for Hamiltonian PDEs*, Comm. Math. Phys. **167** (1995), no. 3, 531–552.
15. J. Palmer, *Moduli spaces of semitoric systems*, J. Geom. Phys., (to appear) arXiv:1502.07296.
16. Á Pelayo, *Toric symplectic ball packing*, Topology and its Appl. **157** (2006), 3633–3644.
17. Á. Pelayo, *Topology of spaces of equivariant symplectic embeddings*, Proc. Amer. Math. Soc. **135** (2007), no. 1, 277–288.
18. Á. Pelayo, A.R. Pires, T. Ratiu, and S. Sabatini, *Moduli spaces of toric manifolds*, Geometriae Dedicata **169** (2014), 323–341.
19. Á Pelayo and B. Schmidt, *Maximal ball packings of symplectic-toric manifolds*, Intern. Math. Res. Not. (2008), 24p, ID rnm139.
20. Á. Pelayo and S. Vũ Ngọc, *Constructing integrable systems of semitoric type*, Acta Math. **206** (2011), 93–125.
21. ———, *Symplectic theory of completely integrable Hamiltonian systems*, Bull. Amer. Math. Soc. **48** (2011), 409–455.
22. S. Vũ Ngọc, *On semi-global invariants of focus-focus singularities*, Topology **42** (2003), no. 2, 365–380.

23. ———, *Moment polytopes for symplectic manifolds with monodromy*, Adv. Math. **208** (2007), no. 2, 909–934.
24. C. Wacheux, *Asymptotics of action variables near semi-toric singularities*, J. Geom. Phys. **98** (2015), 28–39.
25. S. T. Yau, *Remarks on the group of isometries of a Riemannian manifold*, Topology **16** (1977), no. 3, 239–247.
26. ———, *A survey of Calabi-Yau manifolds*, Geometry and analysis. No. 2, Adv. Lect. Math. (ALM), vol. 18, Int. Press, Somerville, MA, 2011, pp. 521–563.
27. K. Zehmisch and F. Ziltener, *Discontinuous symplectic capacities*, J. Fixed Point Theory Appl. **14** (2013), no. 1, 299–307.

Alessio Figalli
University of Texas at Austin
Mathematics Dept. RLM 8.100
2515 Speedway Stop C1200
Austin TX, 78712-1082, USA.
E-mail: figalli@math.utexas.edu

Joseph Palmer
Álvaro Pelayo
University of California, San Diego
Department of Mathematics
9500 Gilman Drive #0112
La Jolla, CA 92093-0112, USA.
E-mail: j5palmer@ucsd.edu
E-mail: alpelayo@ucsd.edu