OPTIMAL TRANSPORTATION WITH BOUNDARY COSTS AND SUMMABILITY ESTIMATES ON THE TRANSPORT DENSITY

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Abstract. In this paper we analyze a mass transportation problem in a bounded domain with the possibility to transport mass to/from the boundary, paying the transport cost, that is given by the Euclidean distance plus an extra cost depending on the exit/entrance point. This problem appears in import/export model, as well as in some shape optimization problems. We study the L^p summability of the transport density which does not follow from standard theorems, as the target measures are not absolutely continuous but they have some parts which are concentrated on the boundary. We also provide the relevant duality arguments to connect the corresponding Beckmann and Kantorovich problems to a formulation with Kantorovich potentials with Dirichlet boundary conditions.

1. Introduction

In this paper we study a mass transportation problem in a bounded domain where there is the possibility of import/export mass across the boundary paying a tax fee in addition to the transport cost that is assumed to be given by the Euclidean distance. Before entering the details of this variant problem, let us introduce the standard Kantorovich problem.

Let f^+ and f^- be two finite positive measures on a bounded domain $\Omega \subset \mathbb{R}^d$ satisfying the mass balance condition $f^+(\bar{\Omega}) = f^-(\bar{\Omega})$. The classical Kantorovich problem is the following:

Set

$$\Pi(f^+, f^-) := \left\{\gamma \in \mathcal{M}^+(\bar{\Omega} \times \bar{\Omega}) \,:\, (\Pi_x)_{\#}\gamma = f^+,\, (\Pi_y)_{\#}\gamma = f^- \right\},$$

then we minimize the quantity

$$\min\left\{\int_{\bar{\Omega}\times\bar{\Omega}} |x-y| \,\mathrm{d}\gamma \,:\, \gamma \in \Pi(f^+, f^-)\right\} \qquad \text{(KP)}$$

where Π_x and Π_y are the two projections of $\bar{\Omega} \times \bar{\Omega}$ onto $\bar{\Omega}$.

In [12], the authors introduce a variant of (KP). They study a mass transportation problem between two masses f^+ and f^- (which do not have a priori the same total mass) with the possibility of transporting some mass to/from the boundary, paying the transport cost $c(x, y) := |x - y|$ plus an extra cost $g_2(y)$ for each unit of mass that comes out from a point $y \in \partial\Omega$ (the export taxes) or $-g_1(x)$ for each unit of mass that enters at the point $x \in \partial\Omega$ (the import taxes). This means that we can use $\partial\Omega$ as an infinite reserve/repository, we can take as much mass as we wish from the boundary, or send back as much mass as we want, provided that we pay the transportation cost plus the import/export taxes.

In other words, given the set

$$\Pi b(f^+,f^-) := \left\{\gamma\in\mathcal{M}^+(\bar\Omega\times\bar\Omega)\,:\,((\Pi_x)_\#\gamma)_{|\mathring{\Omega}} = f^+,\,((\Pi_y)_\#\gamma)_{|\mathring{\Omega}} = f^-\right\},$$

we minimize the quantity

$$\min\left\{\int_{\bar\Omega\times\bar\Omega}|x-y|\,\mathrm{d}\gamma + \int_{\partial\Omega} g_2\,\mathrm{d}(\Pi_y)_\#\gamma - \int_{\partial\Omega} g_1\,\mathrm{d}(\Pi_x)_\#\gamma\,:\,\gamma\in\Pi b(f^+,f^-)\right\} \qquad \text{(KPb)}.$$

On the other hand, let us consider the following problem

$$\min\left\{|W|(\bar\Omega)\,:\,W\in\mathcal{M}^d(\bar\Omega),\,\nabla\cdot W = f\right\} \qquad \text{(BP)}$$

where $\mathcal{M}^d(\bar\Omega)$ is the space of vector measures and for $W\in\mathcal{M}^d(\bar\Omega)$, $|W|(\bar\Omega)$ denotes the total variation measure (note that $|W|(\bar\Omega)$ is a norm on $\mathcal{M}^d(\bar\Omega)$).

It is well known that if Ω is convex, then (BP) is equivalent to (KP), i.e the values of both problems are equal and we can construct a minimizer for (BP) from a minimizer for (KP) and vice versa.

From the equality $\min\,(\text{KP}) = \min\,(\text{BP})$, it is easy to see that $\min\,(\text{KPb}) = \min\,(\text{BPb})$, where (BPb) is the following problem

$$\min\left\{|W|(\bar\Omega) + \int_{\partial\Omega} g_2\,\mathrm{d}\chi^- - \int_{\partial\Omega} g_1\,\mathrm{d}\chi^+\,:\,W\in\mathcal{M}^d(\bar\Omega),\,\chi\in\mathcal{M}(\partial\Omega),\,\nabla\cdot W = f+\chi\right\}.$$

Before building a minimizer W for (BP), take a minimizer γ for (KP) (which is called *optimal transport plan*) and define the transport density σ associated with γ as follows

$$< \sigma,\varphi > = \int_{\bar\Omega\times\bar\Omega}\mathrm{d}\gamma(x,y)\int_0^1 \varphi(\omega_{x,y}(t))|\dot\omega_{x,y}(t)|\,\mathrm{d}t \quad \text{for all } \varphi\in C(\bar\Omega) \tag{1.1}$$

where $\omega_{x,y}$ is a curve parameterizing the straight line segment connecting x to y.

Then it is easy to check that the vector field W given by $W := -\sigma\nabla u$ is a solution of the above minimization problem (BP), where u is a maximizer (called *Kantorovich potential*) for the following problem

$$\sup\left\{\int_\Omega u\,\mathrm{d}(f^+-f^-)\,:\,u\in\mathrm{Lip}_1\right\} \qquad \text{(DP)}.$$

Actually, it is possible to prove that the maximization problem above is the dual of (KP) and its value equals $\min\,(\text{KP})$.

In addition, (σ,u) solves a particular PDE system, called Monge-Kantorovich system:

$$\begin{cases} -\nabla\cdot(\sigma\nabla u) = f := f^+-f^- & \text{in } \Omega\\ \sigma\nabla u\cdot n = 0 & \text{on } \partial\Omega\\ |\nabla u|\le 1 & \text{in } \Omega,\\ |\nabla u| = 1 & \sigma-\text{a.e.}\end{cases} \tag{1.2}$$

The summability of σ has been the object of intensive research in the last few years, and in particular we have the following:

Proposition 1.1. *Suppose $f^+ \ll \mathcal{L}^d$ or $f^- \ll \mathcal{L}^d$, then the transport density σ is unique (i.e. does not depend on the choice of the optimal transport plan γ) and $\sigma \ll \mathcal{L}^d$. Moreover, if both f^+, $f^- \in L^p(\Omega)$, then σ also belongs to $L^p(\Omega)$.*

These properties are well-known in the literature, and we refer to [5], [6], [7], [10] and [13].

Hence, if $f^+ \ll \mathcal{L}^d$ or $f^- \ll \mathcal{L}^d$, then (BP) is also well-posed in L^1 instead of the space of vector measures. In addition, $W := -\sigma\nabla u$ (which minimizes (BP)) belongs to $L^p(\Omega, \mathbb{R}^d)$ provided that $f \in L^p(\Omega)$.

A variant of this problem, which is already present in [1], [2] and [9], is to complete the Monge-Kantorovich system with a Dirichlet boundary condition. In optimal transport terms, this corresponds to the possibility of transporting some mass to/from the boundary, paying only the transport cost that is given by the Euclidean distance. The easiest version of the system becomes

$$\begin{cases} -\nabla\cdot(\sigma\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u| \le 1 & \text{in } \Omega, \\ |\nabla u| = 1 & \sigma - \text{a.e.} \end{cases} \tag{1.3}$$

Notice that in [8, 3], the same pair (σ, u) (which solves (1.3)) also models (in a statical or dynamical framework) the configuration of stable or growing sandpiles, where u gives the pile shape and σ stands for sliding layer.

But, we can replace also the Dirichlet boundary condition $u = 0$ by $u = g$. In this case, the system becomes

$$\begin{cases} -\nabla\cdot(\sigma\nabla u) = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \\ |\nabla u| \le 1 & \text{in } \Omega, \\ |\nabla u| = 1 & \sigma - \text{a.e.} \end{cases} \tag{1.4}$$

This system describe the growth of a sandpile on a bounded table, with a wall on the boundary of a height g, under the action of a vertical source here modeled by f. Notice that to solve this system, it is clear that g must be 1-Lipschitz.

In [12], the system (1.4) is complemented with the boundary condition $g_1 \le u \le g_2$ instead of $u = g$. The system becomes

$$\begin{cases} -\nabla\cdot(\sigma\nabla u) = f & \text{in } \Omega \\ g_1 \le u \le g_2 & \text{on } \partial\Omega, \\ |\nabla u| \le 1 & \text{in } \Omega, \\ |\nabla u| = 1 & \sigma - \text{a.e.} \end{cases} \tag{1.5}$$

Here, there is no obvious interpretation in terms of sandpiles. However, this corresponds to a mass transportation problem between two masses f^+ and f^- with the possibility of transporting some mass to/from the boundary, paying a transport cost plus two extra costs $-g_1$ and g_2 (the import/export taxes).

The authors of [12] also prove that the dual of (KPb) is the following

$$\sup\left\{\int_\Omega u\,\mathrm{d}(f^+-f^-)\,:\,u\in \mathrm{Lip}_1,\ g_1\le u\le g_2\ \text{ on }\partial\Omega\right\}\qquad \text{(DPb)}.$$

The reader will see later that in this paper we also give an alternative proof for this duality formula that we consider simpler than that in [12].

In addition, if we are able to prove that the vector measure $W := -\sigma\nabla u$ minimizes (BPb), where u is a maximizer for (DPb) and σ is the transport density associated with an optimal transport plan γ for (KPb), then (BPb) is well-posed in $L^1(\Omega,\mathbb{R}^d)$ instead of the space of vector measures as soon as one has $\sigma \ll \mathcal{L}^d$. Here, we need the convexity of Ω to define σ (see (1.1)), but we will show that under some assumptions on g_1 and g_2, we can also use (1.1) to define σ, even if Ω is not convex.

The main object of the present paper is the problem (KPb). First, we give an alternative proof for its dual formulation, which is already proved in [12] and second, we are interested to study the L^p summability of the transport density σ, which does not follow from Proposition 1.1, since in this case the target measures are not in L^p as they have some parts which are concentrated on $\partial\Omega$. Note that in [9], the authors prove that if $g_1 = g_2 = 0$, then the transport density σ belongs to L^p provided that $f \in L^p$. Here, our goal is to prove the same L^p result of [9] but for more general costs g_1 and g_2. First of all, we note that to get a L^p summability on σ, it is natural to suppose that g_1 (resp. g_2) is strictly better than 1-Lip. Indeed, we can find $f \in L^p(\Omega)$ and $\chi \in \mathcal{M}(\partial\Omega)$ such that the transport density σ between $f^+ + \chi^+$ and $f^- + \chi^-$ is not in $L^p(\Omega)$ and in this case, if u is the Kantorovich potential (which is 1-Lip), then (σ, u) solves (1.5) with $g_1 = g_2 = u$.

For this aim, we want to decompose an optimal transport plan γ for (KPb) as a sum of three transport plans γ_{ii}, γ_{ib} and γ_{bi}, where each of these plans solves a particular transport problem. Next, we will study the L^p summability of the transport densities σ_{ii}, σ_{ib} and σ_{bi} associated with these transport plans γ_{ii}, γ_{ib} and γ_{bi}, respectively. In this way, we get the summability of the transport density σ associated with the optimal transport plan γ.

This paper is organized as follows. In Section 2, we prove that it is enough to study the summability of σ_{ib}, to get that of σ and we study duality for (KPb). In Section 3, we study the L^p summability of the transport density σ_{ib}: firstly, we prove it under an assumption on the geometric form of Ω and secondly, we generalize the result to every domain having a uniform exterior ball. In Section 4, we prove direclty the L^p summability of the transport density σ_{ib}, only for the case $g_2 = 0$, by using a geometric lemma. In Section 5, we give the proofs of the key Propositions already used in Section 3, which are very technical and we found better to postpone their presentation.

2. Monge-Kantorovich problems with boundary costs: existence, characterization and duality

In this section, we analyze the problem (KPb). Besides duality questions, we will also decompose it into subproblems. One of this subproblems involves a transport plan γ_{ib} (with its transport density σ_{ib}), where i and b stand for interior and boundary (conversely, we also have a transport plan γ_{bi} with σ_{bi}). We will show that some questions, including summability of σ, reduce to the study of the summability of σ_{ib} and σ_{bi}.

First of all, we suppose that g_1 and g_2 are in $C(\partial\Omega)$ and they satisfy the following inequality

$$g_1(x) - g_2(y) \leq |x-y| \text{ for all } x,\, y \in \partial\Omega. \tag{2.1}$$

Under this assumption, we have the following:

Proposition 2.1. *(KPb) reaches a minimum.*

Proof. Set

$$K(\gamma) := \int_{\bar{\Omega}\times\bar{\Omega}} |x-y|\,\mathrm{d}\gamma \ + \int_{\partial\Omega} g_2\,\mathrm{d}(\Pi_y)_\#\gamma - \int_{\partial\Omega} g_1\,\mathrm{d}(\Pi_x)_\#\gamma, \ \ \forall\, \gamma \in \mathcal{M}(\bar{\Omega}\times\bar{\Omega}).$$

Then, K is continuous with respect to the weak convergence of measures in $\Pi b(f^+, f^-)$. Indeed, if $(\gamma_n)_n$ is a sequence in $\Pi b(f^+, f^-)$ such that $\gamma_n \rightharpoonup \gamma$, then, for every n, there exists $\chi_n^\pm \in \mathcal{M}^+(\partial\Omega)$ such that

$$(\Pi_x)_\#\gamma_n = f^+ + \chi_n^+\,, \ \ (\Pi_y)_\#\gamma_n = f^- + \chi_n^-$$

and

$$\chi_n^\pm \rightharpoonup \chi^\pm,$$

where $(\Pi_x)_\#\gamma = f^+ + \chi^+$ and $(\Pi_y)_\#\gamma = f^- + \chi^-$. As g_1 (resp. g_2) is continuous, then

$$K(\gamma_n) \to K(\gamma).$$

On the other hand, we observe that if $\gamma \in \Pi b(f^+, f^-)$ and $\tilde{\gamma} := \gamma_{|(\partial\Omega\times\partial\Omega)^c}$, then $\tilde{\gamma}$ also belongs to $\Pi b(f^+, f^-)$. In addition, we have

$$\int_{\bar{\Omega}\times\bar{\Omega}} |x-y|\,\mathrm{d}\gamma \ + \int_{\partial\Omega} g_2\,\mathrm{d}(\Pi_y)_\#\gamma \ - \int_{\partial\Omega} g_1\,\mathrm{d}(\Pi_x)_\#\gamma$$

$$= \int_{\partial\Omega\times\partial\Omega} (|x-y| + g_2(y) - g_1(x))\,\mathrm{d}\gamma + \int_{(\partial\Omega\times\partial\Omega)^c} |x-y|\,\mathrm{d}\gamma + \int_{\Omega^\circ\times\partial\Omega} g_2(y)\,\mathrm{d}\gamma - \int_{\partial\Omega\times\Omega^\circ} g_1(x)\,\mathrm{d}\gamma.$$

As

$$|x-y| + g_2(y) - g_1(x) \geq 0,$$

we get

$$\int_{\bar{\Omega}\times\bar{\Omega}} |x-y|\,\mathrm{d}\gamma \ + \int_{\partial\Omega} g_2\,\mathrm{d}(\Pi_y)_\#\gamma - \int_{\partial\Omega} g_1\,\mathrm{d}(\Pi_x)_\#\gamma$$
$$\geq \int_{\bar{\Omega}\times\bar{\Omega}} |x-y|\,\mathrm{d}\tilde{\gamma} \ + \int_{\partial\Omega} g_2\,\mathrm{d}(\Pi_y)_\#\tilde{\gamma} \ - \int_{\partial\Omega} g_1\,\mathrm{d}(\Pi_x)_\#\tilde{\gamma}.$$

Now, let $(\gamma_n)_n \subset \Pi b(f^+, f^-)$ be a minimizing sequence. Then, we can suppose that

$$\gamma_n(\partial\Omega\times\partial\Omega) = 0.$$

In this case, we get

$$\begin{aligned}\gamma_n(\bar\Omega\times\bar\Omega) &\leq \gamma_n(\Omega^0\times\bar\Omega)+\gamma_n(\bar\Omega\times\Omega^0)\\ &= f^+(\bar\Omega)+f^-(\bar\Omega).\end{aligned}$$

Hence, there exist a subsequence $(\gamma_{n_k})_{n_k}$ and a plan $\gamma\in\Pi b(f^+,f^-)$ such that $\gamma_{n_k}\rightharpoonup\gamma$. But, the continuity of K implies that this γ is a minimizer for (KPb). $\square$

Fix a minimizer γ for (KPb) and denote by χ^+ and χ^- the two positive measures concentrated on the boundary of Ω such that $(\Pi_x)_\#\gamma = f^+ + \chi^+$ and $(\Pi_y)_\#\gamma = f^- + \chi^-$. Then, we may see that γ is also a minimizer for the following problem

$$\min\left\{\int_{\bar\Omega\times\bar\Omega}|x-y|\,\mathrm{d}\gamma\,:\,\gamma\in\Pi(\mu^+,\mu^-)\right\}$$

where $\mu^\pm := f^\pm+\chi^\pm$.

Set

$$\gamma_{ii}:=\gamma_{|\Omega^\circ\times\Omega^\circ},\ \gamma_{ib}:=\gamma_{|\Omega^\circ\times\partial\Omega},\ \gamma_{bi}:=\gamma_{|\partial\Omega\times\Omega^\circ},\ \gamma_{bb}:=\gamma_{|\partial\Omega\times\partial\Omega}=0$$

and

$$\nu^+:=(\Pi_x)_\#\gamma_{ib},\ \nu^-:=(\Pi_y)_\#\gamma_{bi}.$$

Consider the three following problems:

$$\text{(P1)}\qquad \min\left\{\int_{\bar\Omega\times\bar\Omega}|x-y|\,\mathrm{d}\gamma\,:\,\gamma\in\Pi(f^+-\nu^+,f^--\nu^-)\right\}$$

$$\text{(P2)}\qquad \min\left\{\int_{\bar\Omega\times\bar\Omega}|x-y|\,\mathrm{d}\gamma+\int_{\partial\Omega}g_2\,\mathrm{d}\chi^-\,:\,\gamma\in\Pi(\nu^+,\chi^-),\ \mathrm{spt}(\chi^-)\subset\partial\Omega\right\}$$

$$\text{(P3)}\qquad \min\left\{\int_{\bar\Omega\times\bar\Omega}|x-y|\,\mathrm{d}\gamma-\int_{\partial\Omega}g_1\,\mathrm{d}\chi^+\,:\,\gamma\in\Pi(\chi^+,\nu^-),\ \mathrm{spt}(\chi^+)\subset\partial\Omega\right\}.$$

It is not difficult to prove that γ_{ii}, γ_{ib} and γ_{bi} solve (P1), (P2) and (P3), respectively. In addition, we can see that γ_{ib} is of the form $(Id,T_{ib})_\#\nu^+$ and that it solves the following problem

$$\min\left\{\int_{\bar\Omega\times\bar\Omega}|x-y|\,\mathrm{d}\gamma\,:\,\gamma\in\Pi(\nu^+,(T_{ib})_\#\nu^+)\right\},$$

where $T_{ib}(x):=\operatorname{argmin}\{|x-y|+g_2(y),\,y\in\partial\Omega\}$ for all $x\in\bar\Omega$. Similarly, γ_{bi} is of the form $(T_{bi},Id)_\#\nu^-$ and it also solves

$$\min\left\{\int_{\bar\Omega\times\bar\Omega}|x-y|\,\mathrm{d}\gamma\,:\,\gamma\in\Pi((T_{bi})_\#\nu^-,\nu^-)\right\},$$

where $T_{bi}(y):=\operatorname{argmin}\{|x-y|-g_1(x),\,x\in\partial\Omega\}$ for all $y\in\bar\Omega$.

Let σ (resp. σ_{ii}, σ_{ib} and σ_{bi}) be the transport density associated with the optimal transport plan γ (resp. γ_{ii}, γ_{ib} and γ_{bi}), therefore $\sigma=\sigma_{ii}+\sigma_{ib}+\sigma_{bi}$. By Proposition 1.1, if Ω is convex and $f\in L^p(\Omega)$, then σ_{ii} also belongs to $L^p(\Omega)$. Hence, it is enough to study the summability of σ_{ib} (the case of σ_{bi} will be analogous), to get that of σ.

On the other hand, the proof of the duality formula of (KPb), in [12], is based on the Fenchel-Rocafellar duality Theorem and it is decomposed into two steps: firstly, the authors suppose that the inequality in (2.1) is strict and secondly, they use an approximation argument to cover the other case. Here, we want to give an alternative proof for this duality formula, based on a simple convex analysis trick already developed in [1].

Proposition 2.2. *Let g_1 and g_2 be in $C(\partial\Omega)$. Then under the assumption (2.1), we have the following equality*

$$\min\left\{\int_{\bar\Omega\times\bar\Omega}|x-y|\,\mathrm{d}\gamma+\int_{\partial\Omega}g_2\,\mathrm{d}(\Pi_y)_{\#}\gamma-\int_{\partial\Omega}g_1\,\mathrm{d}(\Pi_x)_{\#}\gamma\;:\;\gamma\in\Pi b(f^+,f^-)\right\}\quad\textit{(KPb)}$$
$$=\sup\left\{\int_\Omega\varphi\,\mathrm{d}(f^+-f^-)\;:\;\varphi\in\mathrm{Lip}_1,\;g_1\le\varphi\le g_2\;\textit{on}\;\partial\Omega\right\}\quad\textit{(DPb).}$$

Notice that if (2.1) is not satisfied, then both sides of this equality are $-\infty$.

Proof. For every $(p,q)\in C(\partial\Omega)\times C(\partial\Omega)$, set

$$H(p,q):=-\sup\left\{\int_\Omega\varphi\,\mathrm{d}(f^+-f^-)\;:\;\varphi\in\mathrm{Lip}_1,\;g_1+p\le\varphi\le g_2-q\text{ on }\partial\Omega\right\}.$$

It is easy to see that $H(p,q)\in\mathbb{R}\cup\{+\infty\}$. In addition, we claim that H is convex and l.s.c.

For convexity: take $t\in(0,1)$ and (p_0,q_0), $(p_1,q_1)\in C(\partial\Omega)\times C(\partial\Omega)$ and let φ_0, φ_1 be their optimal potentials. Set

$$p_t:=(1-t)p_0+tp_1,\;q_t:=(1-t)q_0+tq_1$$

and

$$\varphi_t:=(1-t)\varphi_0+t\varphi_1.$$

As

$$g_1+p_0\le\varphi_0\le g_2-q_0\quad\text{and}\quad g_1+p_1\le\varphi_1\le g_2-q_1\text{ on }\partial\Omega,$$

then

$$g_1+p_t\le\varphi_t\le g_2-q_t\text{ on }\partial\Omega.$$

In addition, φ_t is 1-Lip. Consequently, φ_t is admissible in the max defining $-H(p_t,q_t)$ and then,

$$H(p_t,q_t)\le-\int_\Omega\varphi_t\,\mathrm{d}(f^+-f^-)=(1-t)H(p_0,q_0)+tH(p_1,q_1).$$

For semi-continuity: take $p_n\to p$ and $q_n\to q$ uniformly on $\partial\Omega$. Let $(p_{n_k},q_{n_k})_{n_k}$ be a subsequence such that $\liminf_n H(p_n,q_n)=\lim_{n_k}H(p_{n_k},q_{n_k})$ (for simplicity of notation, we still denote this subsequence by $(p_n,q_n)_n$) and let $(\varphi_n)_n$ be their corresponding optimal potentials. As φ_n is 1-Lip and $(p_n)_n$, $(q_n)_n$ are equibounded, then, by Ascoli-Arzelà Theorem, there exist a 1-Lip function φ and a subsequence $(\varphi_{n_k})_{n_k}$ such that $\varphi_{n_k}\to\varphi$ uniformly. As

$$g_1+p_{n_k}\le\varphi_{n_k}\le g_2-q_{n_k}\;\text{ on }\;\partial\Omega,$$

then

$$g_1 + p \leq \varphi \leq g_2 - q \text{ on } \partial\Omega.$$

Consequently, φ is admissible in the max defining $-H(p,q)$ and one has

$$H(p,q) \leq -\int_\Omega \varphi \,\mathrm{d}(f^+ - f^-) = \lim_{n_k} H(p_{n_k}, q_{n_k}) = \liminf_n H(p_n, q_n).$$

Hence, we get that $H^{\star\star} = H$ and in particular, $H^{\star\star}(0,0) = H(0,0)$. But by the definition of H, we have

$$H(0,0) = -\sup\left\{\int_\Omega \varphi \,\mathrm{d}(f^+ - f^-) \,:\, \varphi \in \mathrm{Lip}_1,\, g_1 \leq \varphi \leq g_2 \text{ on } \partial\Omega\right\}.$$

On the other hand, let us compute $H^{\star\star}(0,0)$. Let χ^+, χ^- be in $\mathcal{M}(\partial\Omega)$, then we have the following

$$H^\star(\chi^+, \chi^-) = \sup_{p,\, q \,\in\, C(\partial\Omega)} \left\{\int_{\partial\Omega} p \,\mathrm{d}\chi^+ + \int_{\partial\Omega} q \,\mathrm{d}\chi^- - H(p,q)\right\}$$

$$=$$

$$\sup\left\{\int_{\partial\Omega} p\mathrm{d}\chi^+ + \int_{\partial\Omega} q\mathrm{d}\chi^- + \int_\Omega \varphi\mathrm{d}(f^+ - f^-) : p,\, q \in C(\partial\Omega),\, \varphi \in \mathrm{Lip}_1,\, g_1 + p \leq \varphi \leq g_2 - q \text{ on } \partial\Omega\right\}.$$

If $\chi^+ \notin \mathcal{M}^+(\partial\Omega)$, i.e. there exists $p_0 \in C(\partial\Omega)$ such that $p_0 \geq 0$ and $\int_{\partial\Omega} p_0 \,\mathrm{d}\chi^+ < 0$, we may see that

$$H^\star(\chi^+, \chi^-) \geq -n\int_{\partial\Omega} p_0 \,\mathrm{d}\chi^+ + \int_{\partial\Omega} g_2 \,\mathrm{d}\chi^- - \int_{\partial\Omega} g_1 \,\mathrm{d}\chi^+ \underset{n\to+\infty}{\longrightarrow} +\infty$$

and similarly if $\chi^- \notin \mathcal{M}^+(\partial\Omega)$.

Suppose $\chi^\pm \in \mathcal{M}^+(\partial\Omega)$. As $g_1 + p \leq \varphi \leq g_2 - q$ on $\partial\Omega$, we should choose the largest possible p and q, i.e $p(x) = \varphi(x) - g_1(x)$ and $q(y) = g_2(y) - \varphi(y)$ for all x, $y \in \partial\Omega$. Hence, we have

$$H^\star(\chi^+, \chi^-) = \sup\left\{\int_{\bar{\Omega}} \varphi \,\mathrm{d}(f + \chi) \,:\, \varphi \in \mathrm{Lip}_1\right\} + \int_{\partial\Omega} g_2 \,\mathrm{d}\chi^- - \int_{\partial\Omega} g_1 \,\mathrm{d}\chi^+,$$

where $f := f^+ - f^-$ and $\chi := \chi^+ - \chi^-$. By [15, Theorem 1.14], we get

$$H^\star(\chi^+, \chi^-) = \min\left\{\int_{\bar{\Omega}\times\bar{\Omega}} |x-y| \,\mathrm{d}\gamma \,:\, \gamma \in \Pi(f^+ + \chi^+, f^- + \chi^-)\right\} + \int_{\partial\Omega} g_2 \,\mathrm{d}\chi^- - \int_{\partial\Omega} g_1 \,\mathrm{d}\chi^+$$

$$= \min\left\{\int_{\bar{\Omega}\times\bar{\Omega}} |x-y| \,\mathrm{d}\gamma + \int_{\partial\Omega} g_2 \,\mathrm{d}(\Pi_y)_\#\gamma - \int_{\partial\Omega} g_1 \,\mathrm{d}(\Pi_x)_\#\gamma \,:\, \gamma \in \Pi(f^+ + \chi^+, f^- + \chi^-)\right\}.$$

Finally, we have

$$H^{\star\star}(0,0) = \sup\left\{-H^\star(\chi^+, \chi^-) \,:\, \chi^+,\, \chi^- \in \mathcal{M}^+(\partial\Omega)\right\}$$

$$= -\min\left\{\int_{\bar{\Omega}\times\bar{\Omega}} |x-y| \,\mathrm{d}\gamma + \int_{\partial\Omega} g_2 \,\mathrm{d}(\Pi_y)_\#\gamma - \int_{\partial\Omega} g_1 \,\mathrm{d}(\Pi_x)_\#\gamma \,:\, \gamma \in \Pi b(f^+, f^-)\right\}. \quad \square$$

Let u be a maximizer for (DPb). Then, we have the following:

Proposition 2.3. *The potential u is also a Kantorovich potential for the following problem*

$$\sup\left\{\int_{\bar{\Omega}}\varphi\,\mathrm{d}(\mu^+-\mu^-)\,:\,\varphi\in\mathrm{Lip}_1\right\}$$

where $\mu^\pm := f^\pm+\chi^\pm$.

Proof. Let v be a Kantorovich potential for this dual problem. Then, we have

$$\int_\Omega u\,\mathrm{d}(f^+-f^-)+\int_{\partial\Omega}g_1\,\mathrm{d}\chi^+-\int_{\partial\Omega}g_2\,\mathrm{d}\chi^-\leq\int_{\bar{\Omega}}u\,\mathrm{d}(\mu^+-\mu^-)\leq\int_{\bar{\Omega}}v\,\mathrm{d}(\mu^+-\mu^-).$$

By Proposition 2.2 and the fact that (see [15, Theorem 1.14])

$$\sup\left\{\int_{\bar{\Omega}}\varphi\,\mathrm{d}(\mu^+-\mu^-)\,:\,\varphi\in\mathrm{Lip}_1\right\}=\min\left\{\int_{\bar{\Omega}\times\bar{\Omega}}|x-y|\,\mathrm{d}\gamma\,:\,\gamma\in\Pi(\mu^+,\mu^-)\right\},$$

we infer that these inequalities are in fact equalities and u is a Kantorovich potential for this dual problem. □

Suppose that Ω is convex and set $W := -\sigma\nabla u$, where we recall that σ is the transport density associated with the optimal transport plan γ. Then, from Proposition 2.3 and the fact that $\min$ (BPb) $=\min$ (KPb), we can conclude that W and χ solve together (BPb). Moreover, the same result will be true, even if Ω is not convex, by using the following:

Proposition 2.4. *Suppose that*

$$|g_1(x)-g_2(y)|\leq|x-y|\ \textit{ for all }(x,y)\in\partial\Omega\times\partial\Omega,$$

i.e. $g_1=g_2:=g$ and g is 1-Lip. Then there exists a minimizer $\gamma^\star$ for (KPb) such that for all $(x,y)\in\mathrm{spt}(\gamma^\star)$, we have $[x,y]\subset\bar{\Omega}$. In addition, if g is λ-Lip with $\lambda<1$, then for any minimizer γ of (KPb) and for all $(x,y)\in\mathrm{spt}(\gamma)$, $[x,y]\subset\bar{\Omega}$.

Proof. Let γ be a minimizer for (KPb) and set

$$E:=\{(x,y)\in\bar{\Omega}\times\bar{\Omega},\ [x,y]\subset\bar{\Omega}\},$$
$$h_1:\bar{\Omega}\times\bar{\Omega}\mapsto\bar{\Omega}\times\partial\Omega$$
$$(x,y)\mapsto(x,y')$$

where y' is the first point of intersection between the segment $[x,y]$ and the boundary if $(x,y)\notin E$ and $y'=y$ else. Also set

$$h_2:\bar{\Omega}\times\bar{\Omega}\mapsto\partial\Omega\times\bar{\Omega}$$
$$(x,y)\mapsto(x',y)$$

where x' is the last point of intersection between the segment $[x,y]$ and the boundary if $(x,y)\notin E$ and $x'=x$ else.

Now, set

$$\gamma^\star := \gamma_{|E} + (h_1)_\#(\gamma_{|E^c}) + (h_2)_\#(\gamma_{|E^c}).$$

It is clear that $\gamma^\star \in \Pi b(f^+, f^-)$. In addition, we have

$$\int_{\bar\Omega\times\bar\Omega} |x-y|\,\mathrm{d}\gamma^\star + \int_{\partial\Omega} g(y)\,\mathrm{d}(\Pi_y)_\#\gamma^\star - \int_{\partial\Omega} g(x)\,\mathrm{d}(\Pi_x)_\#\gamma^\star$$

$$= \int_E |x-y|\,\mathrm{d}\gamma + \int_{E^c} (|x-y'|+|x'-y|+g(y')-g(x'))\,\mathrm{d}\gamma + \int_{\partial\Omega} g(y)\,\mathrm{d}(\Pi_y)_\#\gamma - \int_{\partial\Omega} g(x)\,\mathrm{d}(\Pi_x)_\#\gamma.$$

Yet,

$$|x-y'| + |x'-y| + g(y') - g(x') \le |x-y'| + |x'-y| + |x'-y'| = |x-y|.$$

Consequently, $\gamma^\star$ is a minimizer for (KPb) and for all $(x,y) \in \mathrm{spt}(\gamma^\star)$, we have $[x,y] \subset \bar\Omega$. The second statement follows directly from the last inequality, which becomes strict. $\square$

3. L^p summability on the transport density

In this section, we will study the L^p summability of the transport density σ_{ib}, under the assumption that Ω satisfies a uniform exterior ball and by supposing that g_2 is λ-Lipschitz with $\lambda < 1$ and semi-concave. First, we will suppose that Ω has a very particular shape, i.e. its boundary is composed of parts of a sphere of radius r (such domains are called *round polyhedra*), and then, by an approximation argument, we are able to generalize the result to any domain having a uniform exterior ball.

To do that, let us consider the following transport problem

$$\text{(P2)} \quad \min\left\{\int_{\bar\Omega\times\bar\Omega} |x-y|\,\mathrm{d}\gamma + \int_{\partial\Omega} g\,\mathrm{d}\chi \,:\, \gamma \in \Pi(f,\chi),\ \mathrm{spt}(\chi) \subset \partial\Omega\right\}.$$

Suppose that g is λ-Lipschitz with $\lambda < 1$ and set

$$T(x) := \mathrm{argmin}\,\{|x-y| + g(y),\ y \in \partial\Omega\} \text{ for all } x \in \bar\Omega.$$

Then, we have the following:

Proposition 3.1. *$T(x)$ is a singleton Lebesgue-almost everywhere.*

Proof. Set

$$h(x) := \min\{|x-y| + g(y),\ y \in \partial\Omega\} \text{ for all } x \in \bar\Omega.$$

It is clear that h is 1-Lip, therefore it is differentiable Lebesgue-almost everywhere. Let x_0 be in $\mathring{\Omega}$ and suppose that there exist y_0 and $y_1 \in \partial\Omega$ such that

$$h(x_0) = |x_0 - y_0| + g(y_0) = |x_0 - y_1| + g(y_1).$$

As

$$h(x) - |x - y_0| \le g(y_0) \text{ for all } x \in \bar\Omega,$$

then the function: $x \mapsto h(x) - |x - y_0|$ reaches a maximum at x_0. Hence, if it is differentiable at $x_0 \in \mathring{\Omega}$, then $\nabla h(x_0) = \frac{x_0-y_0}{|x_0-y_0|}$. In the same way, we get $\nabla h(x_0) = \frac{x_0-y_1}{|x_0-y_1|}$.

Consequently, we have $\frac{x_0-y_0}{|x_0-y_0|} = \frac{x_0-y_1}{|x_0-y_1|}$, which is a contradiction as y_1 is in the half line with vertex x_0 and passing through y_0, indeed in this case, one has

$$\begin{aligned} |\,y_0 - y_1| &= ||x_0 - y_0| - |x_0 - y_1|| \\ &= |\,g(y_0) - g(y_1)| \\ &\leq \lambda |\,y_0 - y_1|. \end{aligned}$$

Note that if Ω is convex, we can prove the same result without using the fact that g is λ-Lip with $\lambda < 1$, indeed a half line with vertex in the interior of Ω cannot intersect $\partial\Omega$ at two different points. □

Proposition 3.2. *If $x \in \Omega$ and $y \in T(x)$, then $(x, y) \cap \partial\Omega = \emptyset$.*

Proof. Suppose that this is not the case, i.e there exist $x \in \Omega$, $y \in T(x)$ and some point $z \in (x, y) \cap \partial\Omega$. By definition of T, we have

$$|x - y| + g(y) \leq |x - z| + g(z).$$

Then

$$|z - y| = |x - y| - |x - z| \leq g(z) - g(y) \leq \lambda |z - y|,$$

which is a contradiction. □

Let S be the set of all the points $x \in \bar{\Omega}$ where $T(x)$ is a singleton. Then, we have the following:

Proposition 3.3. *If $x \in S$ and $y \in [x, T(x)]$, then $y \in S$ and $T(y) = T(x)$.*

Proof. For every $z \neq T(x) \in \partial\Omega$, we have

$$\begin{aligned} |y - T(x)| + g(T(x)) &= |x - T(x)| - |x - y| + g(T(x)) \\ &< |x - z| + g(z) - |x - y| \\ &\leq |y - z| + g(z). \quad \square \end{aligned}$$

We observe that if $x \in S$, then the image of y and x through T is the same. This is a well-known principle in optimal transport with distance cost, as y is on the same transport ray as x (i.e. $T(y) = T(x)$).

Proposition 3.4. *The multi-valued map T has a Borel selector function.*

Proof. To prove that T has a Borel selector function, it is enough to show that the graph of T is closed (see for instance Chapter 3 in [4]). Take a sequence (x_n, y_n) in the graph of T such that $(x_n, y_n) \to (x, y)$. As $y_n \in T(x_n)$, then we have

$$|x_n - y_n| + g(y_n) \leq |x_n - z| + g(z) \ \text{ for all } \ z \in \partial\Omega.$$

Passing to the limit, we get

$$|x - y| + g(y) \leq |x - z| + g(z) \ \text{ for all } z \in \partial\Omega$$

and then, $y \in T(x)$. □

For simplicity of notation, we still denote this selector by T.

We recall that the plan $\gamma_T := (Id, T)_{\#}f$ is the unique minimizer for (P2). In addition, γ_T solves the following problem

$$\min\left\{\int_{\bar\Omega\times\bar\Omega}|x-y|\,\mathrm{d}\gamma\,:\,\gamma\in\Pi(f,(T)_{\#}f)\right\}.$$

For simplicity of notation, we will denote this minimizer by γ instead of γ_T.

Let σ be the transport density associated with the transport of f into $(T)_{\#}f$. By the definition of σ (see (1.1)), we have that for all $\phi \in C(\bar\Omega)$

$$\begin{aligned}<\sigma,\phi> &= \int_{\bar\Omega\times\bar\Omega}\int_0^1|x-y|\phi((1-t)x+ty)\,\mathrm{d}t\,\mathrm{d}\gamma(x,y)\\ &= \int_{\bar\Omega}\int_0^1|x-T(x)|\phi((1-t)x+tT(x))f(x)\,\mathrm{d}t\,\mathrm{d}x.\end{aligned}$$

Then

$$\sigma=\int_0^1\mu_t\,\mathrm{d}t,$$

where

$$<\mu_t,\phi>:=\int_{\bar\Omega}|x-T(x)|\phi(T_t(x))f(x)\,\mathrm{d}x\ \text{ for all }\ \varphi\in C(\bar\Omega)$$

and

$$T_t(x):=(1-t)x+tT(x)\ \text{ for all }\ x\in\bar\Omega.$$

Notice that in the definition of μ_t, differently from what done in [13], we need to keep the factor $|x-T(x)|$, which will be essential in the estimates. In addition, we have that $\mu_t \ll \mathcal{L}^d$ as soon as one has $f \ll \mathcal{L}^d$ (see [13]).

Now, we will introduce the two following propositions, whose proofs, for simplicity of exposition, are postponed to Section 5.

Proposition 3.5. *Suppose that Ω is a round polyhedron and g is in $C^2(\partial\Omega)$ with $|\nabla g| < 1$. Then, the closure N of $\Omega\backslash S$ is negligible. Moreover, T is a C^1 function on $\Omega\backslash N$.*

Now, we want to give an explicit formula of μ_t in terms of f and T. Let ϕ be in $C(\bar\Omega)$, then we have

$$\int_{\bar\Omega}\phi(y)\,\mathrm{d}\mu_t(y)=\int_\Omega\phi(T_t(x))|x-T(x)|f(x)\,\mathrm{d}x.$$

Take a change of variable $y := T_t(x)$. By Propositions 3.3 & 3.5, we infer that

$$x=\frac{y-tT(y)}{1-t}\ \text{ and }\ |x-T(x)|=\frac{|y-T(y)|}{1-t}.$$

Consequently, we have

$$\int_\Omega\phi(y)\mu_t(y)\,\mathrm{d}y=\int_{\Omega_t}\phi(y)\frac{|y-T(y)|}{1-t}f\left(\frac{y-tT(y)}{1-t}\right)|J(y)|\,\mathrm{d}y,$$

where $\Omega_t := T_t(\Omega)$ and $J(y) := (\det(DT_t(x)))^{-1}$ for all $y = T_t(x) \in \Omega_t$. Finally, this implies that

$$\mu_t(y) = \frac{|y - T(y)|}{1-t} f\left(\frac{y - tT(y)}{1-t}\right) |J(y)| 1_{\Omega_t}(y) \text{ for a.e. } y \in \Omega.$$

Notice that for all $y \in \Omega_t$, we have $|y - T(y)| \le (1-t)l(y)$ where $l(y)$ is the length of the transport ray containing y, i.e.

$$l(y) := \sup\left\{|x - T(x)| : T(x) = T(y),\, x \in \bar{\Omega} \cap \{T(y) + s(y - T(y)),\, s \ge 1\}\right\}.$$

Proposition 3.6. *Suppose that Ω is a round polyhedron and g is in $C^2(\partial\Omega)$ with $|\nabla g| \le \lambda < 1$. Then, there exists a constant $C := C(d, diam(\Omega), \lambda, r, M) > 0$, where $D^2 g \le MI$, such that for a.e. $x \in \Omega$, we have the following estimate*

$$|\det(DT_t(x))| \ge C(1-t).$$

We are now ready to prove the L^p summability of the transport density σ. Then, under the assumption that Ω is a round polyhedron, we have the following result.

Proposition 3.7. *Suppose that Ω is a round polyhedron and $g \in C^2(\partial\Omega)$ with $|\nabla g| \le \lambda < 1$. Then, the transport density σ belongs to $L^\infty(\Omega)$ provided that $f \in L^\infty(\Omega)$.*

Proof. By Proposition 3.6, we have

$$\begin{aligned}
\| \sigma \|_{L^\infty(\Omega)} &:= \sup_{y \in \Omega} \left(\int_0^1 \mu_t(y)\, \mathrm{d}t\right) \\
&= \sup_{y \in \Omega} \left(\int_0^{1 - \frac{|y-T(y)|}{l(y)}} \frac{|y - T(y)| f(\frac{y - tT(y)}{1-t})}{(1-t)|\det(DT_t(x))|} 1_{\Omega_t}(y)\, \mathrm{d}t\right) \\
&\le C^{-1} \| f \|_{L^\infty(\Omega)} \left(\int_0^{1 - \frac{|y-T(y)|}{l(y)}} \frac{|y - T(y)|}{(1-t)^2}\, \mathrm{d}t\right).
\end{aligned}$$

Yet, it is easy to see that

$$\sup_{y \in \Omega} \left(\int_0^{1 - \frac{|y-T(y)|}{l(y)}} \frac{|y - T(y)|}{(1-t)^2} \mathrm{d}t\right) \le \mathrm{diam}(\Omega).$$

Then,

$$\| \sigma \|_{L^\infty(\Omega)} \le C \| f \|_{L^\infty(\Omega)},$$

for some constant C depending only on d, $\mathrm{diam}(\Omega)$, λ, r and M, where M is any constant such that $D^2 g \le MI$. □

Proposition 3.8. *Let Ω be a round polyhedron, $g \in C^2(\partial\Omega)$ with $|\nabla g| < 1$ and suppose $f \in L^p(\Omega)$ for some $p \in [1, +\infty]$. Then, the transport density σ also belongs to $L^p(\Omega)$.*

Proof. We observe that as the transport is between f and $(T)_\# f$, then the transport density σ linearly depends on f: in this case, L^p estimates could be obtained via interpolation as soon as one has L^1 and L^∞ estimates (see for instance [11]). In order to get an L^1 estimate, it is enough to note that the implication $f \in L^1(\Omega) \Rightarrow \sigma \in L^1(\Omega)$ is well-known (see, for instance, [6]), and that we have

$$||\sigma||_{L^1(\Omega)} \le \mathrm{diam}(\Omega)\, ||f||_{L^1(\Omega)}.$$

In addition, the L^∞ estimates follow from Proposition 3.7. □

Remark 3.1. *The same proof as in Proposition 3.7 could also be adapted to proving Proposition 3.8, but a suitable use of a Hölder inequality would be required.*

We will now generalize, via a limit procedure, the result of Proposition 3.8 to arbitrary domain having a uniform exterior ball. But before that, we will give a definition of such a domain.

Definition 3.1. *We say that* Ω *has a uniform exterior ball of radius* $r > 0$ *if for all* $y \in \partial\Omega$*, there exists some* $x \in \mathbb{R}^d$ *such that* $B(x, r) \cap \Omega = \emptyset$ *and* $|x - y| = r$*.*

We suppose that Ω is such a domain, then we have the following :

Proposition 3.9. *The transport density* σ *between* f *and* $(T)_{\#}f$ *belongs to* $L^p(\Omega)$ *provided that* $f \in L^p(\Omega)$ *and* g *is* λ*-Lip with* $\lambda < 1$ *and semi-concave.*

Proof. This proposition can be proven using the same ideas as in [9, Proposition 3.4]. To do that, take a sequence of domains $(\Omega_k)_k$ such that : the boundary of each Ω_k is a union of parts of sphere of radius r, $\partial\Omega_k \to \partial\Omega$ in the Hausdorff sense and $\Omega \subset \Omega_k \subset \tilde{\Omega}$ for some large compact set $\tilde{\Omega}$.

First of all, we extend f by 0 outside Ω and we suppose that $g \in C^2(\bar{\Omega})$. Let γ_k be an optimal transport plan between f and $(T^k)_{\#}f$, i.e γ_k solves

$$\min\left\{\int_{\tilde{\Omega}\times\tilde{\Omega}} |x - y| \,\mathrm{d}\gamma \;:\; \gamma \in \Pi(f, (T^k)_{\#}f)\right\},$$

where $T^k(x) := \operatorname{argmin}\{|x - y| + g(y),\, y \in \partial\Omega_k\}$.

Let σ_k be the transport density associated with the optimal transport plan γ_k. From Proposition 3.8, we have

$$\sigma_k \in L^p(\Omega_k)$$

and

$$\| \sigma_k \|_{L^p(\Omega_k)} \leq C \| f \|_{L^p(\Omega)},$$

for some constant $C := C(d, \operatorname{diam}(\Omega), \lambda, r, M)$, where M is a constant such that $D^2 g \leq MI$. Then, up to a subsequence, we can assume that $\sigma_k \rightharpoonup \sigma$ weakly in $L^p(\tilde{\Omega})$. Moreover, we have the following estimate

$$\| \sigma \|_{L^p(\Omega)} \leq \liminf_k \| \sigma_k \|_{L^p(\Omega_k)} \leq C \| f \|_{L^p(\Omega)} .$$

Hence, it is sufficient to show that this σ is in fact the transport density associated with the transport of f into $(T)_{\#}f$.

Firstly, we observe that for a given x, $(T_k(x))_k$ converges, up to a subsequence, to a point $y \in \partial\Omega$ such that $y \in \operatorname{argmin}\{|x - z| + g(z),\, z \in \partial\Omega\}$. Since this point is unique for a.e. x, we get (with no need of passing to a subsequence) :

$$T^k(x) \to T(x)$$

and

$$(T^k)_{\#}f \rightharpoonup (T)_{\#}f \text{ in the sense of measures.}$$

By [15, Theorem 5.20], we get that

$$\gamma_k \rightharpoonup \gamma,$$

where γ solves

$$\min\left\{\int_{\bar\Omega\times\bar\Omega}|x-y|\,\mathrm{d}\gamma\ :\ \gamma\in\Pi(f,(T)_{\#}f)\right\}.$$

Let σ_γ be the unique transport density between f and $(T)_{\#}f$. As $\gamma_k\rightharpoonup\gamma$, we find that $\sigma_k\rightharpoonup\sigma_\gamma$ (see (1.1)). Consequently, $\sigma_\gamma=\sigma\in L^p(\Omega)$ and we have the following estimate

$$\|\,\sigma_\gamma\,\|_{L^p(\Omega)}\le C\,\|\,f\,\|_{L^p(\Omega)},$$

for some constant $C:=C(d,\mathrm{diam}(\Omega),\lambda,r,M)$, where M is a constant such that $D^2g\le MI$.

The approximation of a semi-concave function g with smoother functions is also standard. Then, it is not difficult to check again that our result is still true for a semi-concave function g. □

4. The case $g=0$

In the particular case $g=0$, we are able to prove Proposition 3.6 via a geometric argument which will not be available for the general case.

Lemma 4.1. *Let $P_{\partial\Omega}$ be the projection on the boundary of Ω, i.e*

$$P_{\partial\Omega}(x):=\mathit{argmin}\{|x-y|,\,y\in\partial\Omega\}\ \text{for all}\ x.$$

Then, $P_{\partial\Omega}$ is the gradient of a convex function. In addition, if Ω has a uniform exterior ball of radius $r>0$, then for a.e $x\in\Omega$, the positive symmetric matrix $DP_{\partial\Omega}(x)$ has $d-1$ eigenvalues larger than $\frac{r}{r+d(x,\partial\Omega)}$.

Proof. Set

$$u(x):=\sup\left\{x\cdot y-\frac{1}{2}|y|^2,\,y\in\partial\Omega\right\}.$$

As we can rewrite $u(x)$ as follows

$$u(x)=\sup\left\{-\frac{1}{2}|x-y|^2+\frac{1}{2}|x|^2,\,y\in\partial\Omega\right\},$$

then the supremum is attained at $P_{\partial\Omega}(x)$ and $\nabla u(x)=P_{\partial\Omega}(x)$ for a.e x. On the other hand, take $x_0\in\mathring{\Omega}$ and let y_0 be the center of a ball $B(y_0,r)$ such that $B(y_0,r)\cap\Omega=\emptyset$ and $|y_0-P_{\partial\Omega}(x_0)|=r$. Then x_0, $P_{\partial\Omega}(x_0)$ and y_0 are aligned. Indeed, if not, we get $|x_0-y_0|<|x_0-P_{\partial\Omega}(x_0)|+r$, but $|x_0-y_0|=|x_0-z|+|z-y_0|$ for some $z\in[x_0,y_0]\cap\partial\Omega$, which is a contradiction as $|x_0-z|\ge|x_0-P_{\partial\Omega}(x_0)|$ and $|z-y_0|\ge r$.

Moreover, we have

$$\begin{aligned}
u(x) &= \sup\left\{\frac{1}{2}|x|^2-\frac{1}{2}|x-y|^2,\,y\in\partial\Omega\right\}\\
&\ge \frac{1}{2}|x|^2-\frac{1}{2}|x-y^\star|^2,\ \text{for some}\ y^\star\in[x,y_0]\cap\partial\Omega\\
&\ge \frac{1}{2}|x|^2-\frac{1}{2}(|x-y_0|-r)^2\ :=v(x).
\end{aligned}$$

As $u(x_0)=v(x_0)$, then the function: $x\mapsto u(x)-v(x)$ has a minimum at x_0. Hence, we get that $D^2u(x_0)\ge D^2v(x_0)$ and the eigenvalues of $D^2u(x_0)$ are bounded from below by those of

$D^2v(x_0)$. Yet, it is easy to show that

$$D^2v(x_0) = \frac{r}{r+d(x_0,\partial\Omega)}\left(I - e(x_0)\otimes e(x_0)\right),$$

where $e(x_0) := \frac{x_0-y_0}{|x_0-y_0|}$.

Then, we conclude by observing that the eigenvalues of this matrix are 0 and $\frac{r}{r+d(x_0,\partial\Omega)}$ (with multiplicity $d-1$). □

Set

$$P_t(x) := (1-t)x + tP_{\partial\Omega}(x).$$

By Lemma 4.1, we have

$$\det(DP_t(x)) \geq (1-t)\left(1-t+t\frac{r}{r+d(x,\partial\Omega)}\right)^{d-1}.$$

Set $y := P_t(x)$. As $d(y,\partial\Omega) = (1-t)d(x,\partial\Omega)$, then the Jacobian at y satisfies the following estimate

$$\begin{aligned} J(y) := \frac{1}{\det(DP_t(x))} &\leq \frac{1}{1-t}\left(\frac{r+d(x,\partial\Omega)}{r+(1-t)d(x,\partial\Omega)}\right)^{d-1} \\ &= \frac{1}{(1-t)^d}\left(\frac{(1-t)r+d(y,\partial\Omega)}{r+d(y,\partial\Omega)}\right)^{d-1}. \end{aligned}$$

Suppose $f \in L^\infty(\Omega)$ and let σ be the transport density between f and $(P_{\partial\Omega})_\# f$, then we have the following pointwise inequality

$$\sigma(y) \leq \| f \|_{L^\infty(\Omega)} \int_0^{1-\frac{d(y,\partial\Omega)}{l(y)}} \frac{d(y,\partial\Omega)}{(1-t)^{d+1}}\left(\frac{(1-t)r+d(y,\partial\Omega)}{r+d(y,\partial\Omega)}\right)^{d-1} \mathrm{d}t.$$

Hence,

$$\sigma(y) \leq C \| f \|_{L^\infty(\Omega)} \int_0^{1-\frac{d(y,\partial\Omega)}{l(y)}} \frac{d(y,\partial\Omega)}{(1-t)^{d+1}} \frac{(1-t)^{d-1}\, r^{d-1} + (d(y,\partial\Omega))^{d-1}}{(r+d(y,\partial\Omega))^{d-1}} \mathrm{d}t$$

$$= \frac{C\, d(y,\partial\Omega)\, \| f \|_{L^\infty(\Omega)}}{(r+d(y,\partial\Omega))^{d-1}}\left(\int_0^{1-\frac{d(y,\partial\Omega)}{l(y)}} \frac{1}{(1-t)^2}\,\mathrm{d}t + (d(y,\partial\Omega))^{d-1}\int_0^{1-\frac{d(y,\partial\Omega)}{l(y)}} \frac{1}{(1-t)^{d+1}}\,\mathrm{d}t\right).$$

But, it is easy to see that

$$\int_0^{1-\frac{d(y,\partial\Omega)}{l(y)}} \frac{1}{(1-t)^2}\,\mathrm{d}t + (d(y,\partial\Omega))^{d-1}\int_0^{1-\frac{d(y,\partial\Omega)}{l(y)}} \frac{1}{(1-t)^{d+1}}\,\mathrm{d}t \leq \frac{C}{d(y,\partial\Omega)}.$$

Consequently,

$$\sigma(y) \leq \frac{C \| f \|_{L^\infty(\Omega)}}{(r+d(y,\partial\Omega))^{d-1}}.$$

This provides a very useful and pointwise estimate on σ. It shows that σ is bounded as soon as $r > 0$, or we are far from the boundary $\partial\Omega$. As a particular case, we get the results of Section 3 in the case $g = 0$ whenever $r > 0$.

By interpolation (see [11]), we get also that σ belongs to $L^p(\Omega)$ provided that $f \in L^p(\Omega)$.

5. Technical proofs

In this section, we want to give the proofs of Propositions 3.5 & 3.6. First of all, suppose that Ω is a round polyhedron and set

$$\Omega_i := \{x = (x_1, x_2, ..., x_d) \in \bar{\Omega} : T(x) \in F_i\},$$

where T is the Borel selector function, which was mentioned earlier in Proposition 3.4, $F_i \subset \partial B(b_i, r)$ is the ith part in the boundary of Ω, contained in a sphere centered at b_i and with radius $r > 0$.

Then, we have the following:

Proposition 5.1. *For all $x \in \mathring{\Omega}$, there does not exist $i \neq j$ such that $T(x) \in F_i \cap F_j$.*

Proof. Suppose that this is not the case at some point $x \in \mathring{\Omega}$, i.e there exist two different faces F_i and F_j such that $T(x) \in F_i \cap F_j$. By Proposition 3.2, the segment $[x, T(x)]$ cannot intersect the boundary of Ω at another point $z \neq T(x)$. Then taking into account the geometric form of Ω (see the proof of the proposition 3.9), we can assume that there exist $\dot{\gamma_1}(0)$ and $\dot{\gamma_2}(0)$ two tangent vectors in $T(x)$ on F_i and F_j respectively in such a way that the angle between them is less than 180° (γ_1 and γ_2 are two curves plotted on F_i and F_j respectively) and

$$x - T(x) = \alpha\dot{\gamma_1}(0) + \beta\dot{\gamma_2}(0)$$

for some two positive constants α and β.

Set

$$f_1(t) := |x - \gamma_1(t)| + g(\gamma_1(t))$$

and

$$f_2(t) := |x - \gamma_2(t)| + g(\gamma_2(t)).$$

By optimality of $T(x) = \gamma_1(0) = \gamma_2(0)$, we can deduce that $\dot{f_1}(0) \geq 0$ and $\dot{f_2}(0) \geq 0$. Hence, we have

$$-\frac{x - T(x)}{|x - T(x)|} \cdot \dot{\gamma_1}(0) + \nabla g(T(x)) \cdot \dot{\gamma_1}(0) \geq 0$$

and

$$-\frac{x - T(x)}{|x - T(x)|} \cdot \dot{\gamma_2}(0) + \nabla g(T(x)) \cdot \dot{\gamma_2}(0) \geq 0.$$

We multiply the first inequality by α, the second one by β and we take the sum, we get

$$-|x - T(x)| + \nabla g(T(x)) \cdot (x - T(x)) \geq 0$$

and

$$1 \leq |\nabla g(T(x))| \leq \lambda,$$

which is a contradiciton. $\square$

Proposition 5.2. *For every $x \in \Omega_i \cap S$, there exists a neighborhood of x contained in Ω_i.*

Proof. Suppose that this is not the case at some point x. Then, there exists a sequence $(x_n)_n$ such that $x_n \to x$ and $T(x_n) \in F_j$ for some $j \neq i$. Yet, up to a subsequence, we can assume that $T(x_n) \to y \in F_j$. By definition of T, we have

$$|x_n - T(x_n)| + g(T(x_n)) \leq |x_n - z| + g(z) \text{ for all } z \in \partial\Omega.$$

Passing to the limit, we get

$$|x - y| + g(y) \leq |x - z| + g(z) \text{ for all } z \in \partial\Omega,$$

which is in contradiction with Proposition 5.1. □

Consider Ω_1 (eventually it will be the same for the other Ω_i) and recall that

$$\Omega_1 := \{x = (x_1, ..., x_d) \in \bar{\Omega} : T(x) \in F_1\}.$$

Suppose that Proposition 3.5 is true and fix $x \in \Omega_1 \backslash N$. After a translation and rotation of axis, we can suppose that the tangent space at $T(x)$ on F_1 is contained in the plane $x_d = 0$ and that there exists $\varphi : U \mapsto \mathbb{R}$, where $U \subset \mathbb{R}^{d-1}$, a parameterization of F_1, i.e. for any $z := (z_1, ..., z_d) \in F_1$, we have $\bar{z} := (z_1, ..., z_{d-1}) \in U$ and $z_d = \varphi(\bar{z})$ (notice that an explicit formula of φ is not needed for the sequel).

For simplicity of notation, we denote $\alpha(x) := |x - T(x)|$.

Set

$$c(z) := \sqrt{|\bar{x} - z|^2 + (x_d - \varphi(z))^2} + g(z, \varphi(z)) \text{ for all } z \in U.$$

For any $i \in \{1, ..., d-1\}$,

$$\frac{\partial c}{\partial z_i}(z) = \frac{(z_i - x_i) - (x_d - \varphi(z))\frac{\partial \varphi}{\partial z_i}(z)}{\sqrt{|\bar{x} - z|^2 + (x_d - \varphi(z))^2}} + \frac{\partial g}{\partial z_i}(z, \varphi(z)) + \frac{\partial g}{\partial z_d}(z, \varphi(z))\frac{\partial \varphi}{\partial z_i}(z).$$

Set $T(x) := (\bar{T}(x), \varphi(\bar{T}(x)))$, where $\bar{T}(x) := (T_1(x), ..., T_{d-1}(x))$. Then, we have

$$\bar{T}(x) = \operatorname{argmin}\{c(z),\ z \in U\}.$$

By Proposition 5.1, $\bar{T}(x) \in \mathring{U}$. Hence, we get

$$\frac{\partial c}{\partial z_i}(\bar{T}(x)) = 0 \text{ for all } i \in \{1, ..., d-1\}$$

or equivalently,

$$\frac{T_i(x) - x_i}{\alpha(x)} + \frac{\partial g}{\partial z_i}(T(x)) - \frac{(x_d - \varphi(\bar{T}(x)))}{\alpha(x)}\frac{\partial \varphi}{\partial z_i}(\bar{T}(x)) + \frac{\partial g}{\partial z_d}(T(x))\frac{\partial \varphi}{\partial z_i}(\bar{T}(x)) = 0 \tag{5.1}$$

for all $i \in \{1, ..., d-1\}$.

By Propositions 3.5 & 5.2, the equality in (5.1) holds in a neighborhood of x. Then, differentiating (5.1) with respect to x_j and taking into account the fact that in this new system of coordinates we have

$$\frac{\partial \varphi}{\partial z_i}(\bar{T}(x)) = 0 \ \text{ for all } \ i \in \{1, ..., d-1\},$$

we get

$$\begin{aligned} &\frac{\partial T_i}{\partial x_j}(x) - \frac{(x_i - T_i(x))}{(\alpha(x))^2}\sum_{k=1}^{d-1}(x_k - T_k(x))\frac{\partial T_k}{\partial x_j} + \alpha(x)\sum_{k=1}^{d-1}\frac{\partial^2 g}{\partial z_i \partial z_k}(T(x))\frac{\partial T_k}{\partial x_j}(x) \\ &- \ (x_d - \varphi(\bar{T}(x)))\sum_{k=1}^{d-1}\frac{\partial^2 \varphi}{\partial z_i \partial z_k}(\bar{T}(x))\frac{\partial T_k}{\partial x_j}(x) + \alpha(x)\frac{\partial g}{\partial z_d}(T(x))\sum_{k=1}^{d-1}\frac{\partial^2 \varphi}{\partial z_i \partial z_k}(\bar{T}(x))\frac{\partial T_k}{\partial x_j}(x) \\ &= \delta_{ij} \ - \ \frac{(x_i - T_i(x))(x_j - T_j(x))}{(\alpha(x))^2} \end{aligned} \tag{5.2}$$

for all $i,\ j \in \{1, ..., d-1\}$.

On the other hand, we have

$$DT_t(x) = (1-t)I + tDT(x) = \begin{pmatrix} 1 - t + t\frac{\partial T_1}{\partial x_1} & t\frac{\partial T_1}{\partial x_2} & ... & t\frac{\partial T_1}{\partial x_d} \\ t\frac{\partial T_2}{\partial x_1} & 1-t+t\frac{\partial T_2}{\partial x_2} & ... & t\frac{\partial T_2}{\partial x_d} \\ ... & & & \\ 0 & ... & 0 & 1-t \end{pmatrix}.$$

Then,

$$|DT_t(x)| = (1-t)|\det(A)|,$$

where $A := \left((1-t)\delta_{ij} + t\frac{\partial T_i}{\partial x_j}(x)\right)_{i,j=1,....,d-1}$.

Set

$$F := \left(\delta_{ij} - \frac{(x_i - T_i(x))(x_j - T_j(x))}{(\alpha(x))^2}\right)_{ij}$$

and

$$N := \left(\alpha(x)\frac{\partial^2 g}{\partial z_i \partial z_j}(T(x)) - (x_d - \varphi(\bar{T}(x)))\frac{\partial^2 \varphi}{\partial z_i \partial z_j}(\bar{T}(x)) + \alpha(x)\frac{\partial g}{\partial z_d}(T(x))\frac{\partial^2 \varphi}{\partial z_i \partial z_j}(\bar{T}(x))\right)_{ij}.$$

Suppose that $F+N$ is invertible for a.e $x \in \Omega$ (see Proposition 5.3 below). From (5.2), we observe that

$$\left(\frac{\partial T_i}{\partial x_j}(x)\right)_{i,j=1,...,d-1} = (F+N)^{-1}F.$$

Hence,

$$A = (1-t)I + t(F+N)^{-1}F = (F+N)^{-1}(F+(1-t)N)$$

and

$$\det(A) = \frac{\det(F+(1-t)N)}{\det(F+N)}.$$

We note that the matrix $F+N = \alpha(x)D^2c(\bar{T}(x))$ and so, by optimality of $\bar{T}(x)$, it is non-negative. On the other hand, as $D^2\varphi(\bar{T}(x)) = \frac{-1}{r}I$ and $D^2 g \leq MI$, then

$$F+N \leq C(d, \mathrm{diam}(\Omega), \lambda, r, M)I$$

and so,

$$\det(F+N) \leq C(d, \mathrm{diam}(\Omega), \lambda, r, M). \tag{5.3}$$

From (5.1), we have

$$\frac{x_i - T_i(x)}{|x-T(x)|} = \frac{\partial g}{\partial z_i}(T(x)) \text{ for any } i \in \{1, ..., d-1\}.$$

Then,

$$|\bar{x} - \bar{T}(x)| \leq \lambda |x - T(x)|.$$

Yet, this implies that

$$\begin{aligned} < F\xi, \xi > &= |\xi|^2 - \left(\frac{\bar{x} - \bar{T}(x)}{|x-T(x)|} \cdot \xi\right)^2 \\ &\geq |\xi|^2 - \frac{|\bar{x} - \bar{T}(x)|^2}{|x-T(x)|^2}|\xi|^2 \\ &\geq (1-\lambda^2)|\xi|^2. \end{aligned}$$

Hence, $F \geq (1-\lambda^2)I$. Now, we are ready to give a lower bound for $\det(A)$. First, if $t \geq \frac{1}{2}$, we have

$$F + (1-t)N = tF + (1-t)(F+N) \geq \frac{1}{2}F \geq \frac{(1-\lambda^2)}{2}I$$

and so,

$$\det(F+(1-t)N) \geq \left(\frac{1-\lambda^2}{2}\right)^{d-1}. \tag{5.4}$$

If $t < \frac{1}{2}$, then one has

$$F + (1-t)N \geq (1-t)(F+N) \geq \frac{1}{2}(F+N),$$

which implies that

$$\det(F+(1-t)N) \geq \frac{1}{2^{d-1}} \det(F+N). \tag{5.5}$$

Combining (5.3), (5.4) & (5.5), we infer that there is a constant $C>0$ depending only on d, $\operatorname{diam}(\Omega)$, λ, r and M, for some constant M with $D^2 g \leq MI$, such that

$$\det(A) \geq C$$

or equivalently,

$$|\det(DT_t(x))| \geq C(1-t).$$

Now, we introduce the proof of the Proposition 3.5.

Proof. Fix $a \in \Omega_1 \cap S$ and, without loss of generality, suppose that the tangent space at $T(a)$ on F_1 is contained in the plane $x_d = 0$. Let φ be a parameterization of F_1 and for any $i \in \{1, ..., d-1\}$, set

$$h_i(x,y) := \frac{y_i - x_i}{\sqrt{|\bar{x}-y|^2 + (x_d - \varphi(y))^2}} + \frac{\partial g}{\partial z_i}(y,\varphi(y)) - \frac{x_d - \varphi(y)}{\sqrt{|\bar{x}-y|^2 + (x_d-\varphi(y))^2}} \frac{\partial \varphi}{\partial z_i}(y)$$

$$+ \frac{\partial g}{\partial z_d}(y,\varphi(y)) \frac{\partial \varphi}{\partial z_i}(y)$$

for all $(x,y) \in \Omega_1 \times U$.

Set $h := (h_i)_i$. By Proposition 5.3, we can assume that the matrix $(\frac{\partial h_i}{\partial y_j}(a, \bar{T}(a)))_{1 \leq i,j \leq d-1}$ is invertible. As $h(a, \bar{T}(a)) = 0$, then by the Implicit Function Theorem, there exist an open neighborhood K of $(a, \bar{T}(a))$ in $\Omega_1 \times U$, a neighborhood V of a in Ω_1 and a function $q : V \to \mathbb{R}^{d-1}$ of class C^1 such that for all $(x,y) \in K$, we have

$$h(x,y) = 0 \Leftrightarrow y = q(x).$$

By Proposition 5.2 and the fact that T is continuous at a, we infer that there exists a small open neighborhood $v(a) \subset \Omega_1$ of a such that $(x, \bar{T}(x)) \in K$ for every $x \in v(a)$. But $h(x, \bar{T}(x)) = 0$ for every $x \in v(a)$, then $\bar{T}(x) = q(x)$ and T is a C^1 function on $v(a)$. Moreover, we can assume also that $v(a) \subset S$. Indeed, if this is not the case, then there exists a sequence $(a_n)_n$ such that $a_n \to a$ and for all n, $a_n \notin S$ (i.e. for all n, there exist z_n, $w_n \in \operatorname{argmin}\{|a_n - y| + g(y),\, y \in \partial\Omega\}$ such that $z_n \neq w_n$). As $a \in S$, then $(z_n)_n$ and $(w_n)_n$ converge to $T(a)$. But $h(a_n, \bar{z}_n) = h(a_n, \bar{w}_n) = 0$, then $\bar{z}_n = \bar{w}_n = q(a_n)$, which is a contradiction.

Consequently, the closure N of $\Omega \backslash S$ is negligible. In addition, T is a C^1 function on $\Omega \backslash N$. □

It remains to prove the following:

Proposition 5.3. *The matrix $F+N$ is invertible for a.e. $x \in \Omega$.*

Proof. It is enough to prove that the determinant of $F+N$ only vanishes at a countable number of points on each transport ray, since it is well-known that a set that meets each transport ray at a countable number of points is negligible and this is due to the fact that the direction of the transport rays is countably Lipschitz (for more details about the proof of this property, we can see for instance [14, Chapter 3]). To do that, fix $x_0 \in \Omega$ and set $x := T_t(x_0)$, where $t \in (0,1]$. Then, we have

$$F(x) = F(x_0) \text{ and } N(x) = (1-t)N(x_0).$$

Hence,

$$F(x) + N(x) = F(x_0) + (1-t)N(x_0) > 0.$$

Consequently, $F+N$ is invertible at a point x as soon as x is not a lower boundary point of some transport ray. □

References

[1] G. Bouchitté and G. Buttazzo, Characterization of optimal shapes and masses through Monge-Kantorovich equation *J. Eur. Math. Soc.* 3 (2), 139–168, 2001.

[2] G. Buttazzo, É. Oudet and E. Stepanov. Optimal transportation problems with free Dirichlet regions. In *Variational methods for discontinuous structures*, 41–65, vol 51 of *PNLDE*, Birkhäuser, Basel, 2002.

[3] P. Cannarsa and P. Cardaliaguet, Representation of equilibrium solutions to the table problem for growing sandpiles, *J. Eur. Math. Soc.* 6, 1-30.

[4] C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions.* Springer-Verlag. Berlin. Heidelberg. New York 1977.

[5] L. De Pascale, L. C. Evans and A. Pratelli, integral estimates for transport densities, *Bull. of the London Math. Soc.*. 36, n. 3,pp. 383-395, 2004.

[6] L. De Pascale and A. Pratelli, Regularity properties for Monge Transport Density and for Solutions of some Shape Optimization Problem, *Calc. Var. Par. Diff. Eq.* 14, n. 3, pp.249-274, 2002.

[7] L. De Pascale and A. Pratelli, Sharp summability for Monge Transport density via Interpolation, *ESAIM Control Optim. Calc. Var.* 10, n. 4, pp. 549-552, 2004.

[8] S. Dumont and N. Igbida, On a dual formulation for the growing sandpile problem *Euro. J. Appl. Math.,* 2009, 20 (02), pp.169-185.

[9] S. Dweik and F. Santambrogio, Summability estimates on transport densities with Dirichlet regions on the boundary via symmetrization techniques, *preprint arXiv:1606.00705, 2016.*

[10] M. Feldman and R. McCann, Uniqueness and transport density in Monge's mass transportation problem, *Calc. Var. Par. Diff. Eq.* 15, n. 1, pp. 81-113, 2002.

[11] J.Marcinkiewicz, Sur l'interpolation d'operations, *C. R. Acad. Sci.* Paris, vol. 208 (1939) pp. 1272-1273.

[12] J.M.Mazon, J.Rossi and J.Toledo. An optimal transportation problem with a cost given by the euclidean distance plus import/export taxes on the boundary. *Rev. Mat. Iberoam.* 30 (2014), no. 1, 1-33.

[13] F. Santambrogio, Absolute continuity and summability of transport densities: simpler proofs and new estimates, *Calc. Var. Par. Diff. Eq.* (2009) 36: 343-354.

[14] F. Santambrogio *Optimal Transport for Applied Mathematicians* in *Progress in Nonlinear Differential Equations and Their Applications* 87, Birkhäuser Basel (2015).

[15] C. Villani. *Topics in Optimal Transportation.* Graduate Studies in Mathematics. Vol. 58, 2003.

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