ON THE MINIMALITY OF THE POTENTIAL FUNCTION OF A GRADIENT SHRINKING RICCI SOLITON

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1. Gradient Shrinking Ricci Solitons and their $\mathcal{W}$–Entropy

A *gradient shrinking Ricci soliton* is a complete, connected Riemannian manifold (M, g) satisfying the relation

$$\mathrm{Ric} + \nabla^2 f = \frac{g}{2},$$

for some smooth function $f : M \to \mathbb{R}$ which is called *potential function* of the soliton (M, g, f).

It is well known that the quantity $a(g, f) := \mathrm{R} + |\nabla f|^2 - f$ must be constant on M and it is often called the *auxiliary constant*.

We recall the following growth estimates, originally proved by Cao–Zhou and Munteanu [1, 7] and improved by Haslhofer–Müller [5] to the present form.

Proposition 1.1 (Potential and volume growth, Lemma 2.1 and 2.2 in [5]). *Let (M, g, f) be an n–dimensional gradient shrinking Ricci soliton with auxiliary constant $a(g, f)$. Then there exists a point $p \in M$ where f attains its infimum and we have the following estimates for the growth of the potential*

$$\frac{1}{4}\big(d_g(x,p) - 5n\big)_+^2 \leq f(x) - a(g,f) \leq \frac{1}{4}\big(d_g(x,p) + \sqrt{2n}\big)^2.$$

Moreover, we have the volume growth estimate $\mathrm{Vol}(B_r^\infty(p)) \leq V(n)r^n$ for geodesic balls in (M, g) around $p \in M$, where $V(n)$ is a constant depending only on the dimension n of the soliton.

As a consequence of these estimates, $\int_M e^{-f} d\mathrm{Vol}$ is well–defined and the potential function f can always be "normalized" by adding a constant in order that

$$\int_M \frac{e^{-f}}{(4\pi)^{n/2}}\, d\mathrm{Vol} = 1. \tag{1.1}$$

We then call such a potential function f and the resulting soliton (M, g, f) *normalized*.

In all the paper we will always consider complete, connected, normalized, gradient, shrinking Ricci solitons, unless explicitly stated.

Proposition 1.1 implies that every function ϕ satisfying $|\phi(x)| \leq Ce^{\alpha d_g^2(x,p)}$ for some $\alpha < \frac{1}{4}$ and constant C, is integrable with respect to $e^{-f} d\mathrm{Vol}$. In particular, since $0 \leq$

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$\mathrm{R}+|\nabla f|^2 \leq f + a(g,f)$ and $\triangle f = \frac{n}{2} - \mathrm{R}$, this holds for every polynomial in f, $|\nabla f|^2$, R and $\triangle f$. Hence, every gradient shrinking Ricci soliton has a well–defined $\mathcal{W}$*–entropy*

$$\mathcal{W}(g,f) := \int_M \big(\mathrm{R} + |\nabla f|^2 + f - n\big)\frac{e^{-f}}{(4\pi)^{n/2}}\,d\mathrm{Vol}.$$

Let us now collect some properties of shrinking solitons and their $\mathcal{W}$–entropy that we will use in the next sections.

Lemma 1.2. *For every normalized, gradient, shrinking Ricci soliton (M,g) with potential function $f : M \to \mathbb{R}$, the following properties holds:*

(1) *Either the scalar curvature R is positive everywhere or (M,g) is the standard flat $\mathbb{R}^n$ and $f(x) = |x-x_0|^2/4$ for some $x_0 \in \mathbb{R}^n$, called "Gaussian soliton".*
(2) *There holds*

$$\mathcal{W}(g,f) = \int_M \big(\mathrm{R} + 2\triangle f - |\nabla f|^2 + f - n\big)\frac{e^{-f}}{(4\pi)^{n/2}}\,d\mathrm{Vol}.$$

(3) *The $\mathcal{W}$–entropy $\mathcal{W}(g,f)$ is equal to $-a(g,f)$.*
(4) *Two potential functions f^1 and f^2 of the same soliton (M,g) either coincide, or (M,g) is the Riemannian product of the flat $\mathbb{R}^k$, for some $k>1$ with a Riemannian manifold $(\widetilde{M},\widetilde{g})$ which is still a gradient shrinking Ricci soliton with a potential function $\widetilde{f} : M \to \mathbb{R}$ and*

$$f^\ell(x,y) = \widetilde{f}(x) + \tfrac{1}{4}|y-y_\ell|^2_{\mathbb{R}^k},$$

for some points y_1 and $y_2 \in \mathbb{R}^k$.
In particular, if M is compact, there can be only one potential function for the soliton (M,g).
(5) *Any two potential functions f^1 and f^2 of the same soliton (M,g) share the same auxiliary constant, that is $a(g,f^1) = a(g,f^2)$, which implies $\mathcal{W}(g,f^1) = \mathcal{W}(g,f^2)$. Hence, we can speak respectively of* the *auxiliary constant $a(g)$ and* the *$\mathcal{W}$–entropy $\mathcal{W}(g)$ of the soliton (M,g)*
(6) *We have $\mathcal{W}(g) \leq 0$ and $\mathcal{W}(g) = 0$ if and only if the manifold (M,g) is the flat $\mathbb{R}^n$ (Gaussian soliton).*
(7) *If a soliton (M,g,f) is also an Einstein manifold, either it is compact and f is constant, or (M,g,f) is the Gaussian soliton.*

Proof. (1) This is a result of Zhang [16, Theorem 1.3] and Yokota [13, Appendix A.2] (see also Pigola, Rimoldi and Setti [9]).
(2) The necessary partial integration formula

$$\int_M \triangle f e^{-f} d\mathrm{Vol} = \int_M |\nabla f|^2 e^{-f} d\mathrm{Vol}$$

follows from the growth estimates of Proposition 1.1 using a cut–off argument. See Section 2 of Haslhofer–Müller [5] for full detail.

(3) By the auxiliary equation $a(g,f) = \mathrm{R} + |\nabla f|^2 - f$ and the traced soliton equation $\mathrm{R} + \triangle f = \frac{n}{2}$, we have

$$\mathrm{R} + 2\triangle f - |\nabla f|^2 + f - n = -a(g,f)\,,$$

hence, the equality $\mathcal{W}(g,f) = -a(g,f)$ follows from Point 2, by integration.

(4) Since the Hessian of any potential of the soliton is uniquely determined by the soliton equation, the difference function $h := f^1 - f^2$ is either a constant or the vector field ∇h is parallel. In the first case, the constant has to be zero by the normalization condition (1.1). In the second case, by De Rham's splitting theorem, (M,g) isometrically splits off a line (see for instance [3, Theorem 1.16]). Hence, we let $(M,g) = (\widetilde{M},\widetilde{g}) \times (\mathbb{R}^k, \mathrm{can})$, with $1 \leq k \leq n$, such that $\widetilde{M}$ cannot split off a line. Denoting by x the coordinates on $\widetilde{M}$ and by y the coordinates on $\mathbb{R}^k$, the soliton equation implies that both potentials also split as $f^\ell(x,y) = \widetilde{f}^\ell(x) + \frac{1}{4}|y - y_\ell|^2_{\mathbb{R}^k}$ for $\ell = 1,2$, where $y_\ell \in \mathbb{R}^k$. Moreover, $(\widetilde{M},\widetilde{g})$ is still a gradient shrinking Ricci soliton with both functions $\widetilde{f}^\ell : \widetilde{M} \to \mathbb{R}$ as possible potentials, and since $\widetilde{M}$ cannot split off a line, they must coincide. Thus, we have

$$f^\ell(x,y) = \widetilde{f}(x) + \tfrac{1}{4}|y - y_\ell|^2_{\mathbb{R}^k},$$

for some function $\widetilde{f} : \widetilde{M} \to \mathbb{R}$.

(5) Integrating the two functions e^{-f^ℓ} of the previous point, by means of Fubini's theorem and the normalization condition (1.1), we get that

$$a(g,f^\ell) = \mathrm{R} + |\nabla f^\ell|^2 - f^\ell = \mathrm{R} + |\nabla \widetilde{f}|^2 - \widetilde{f}$$

which is independent of $\ell = 1,2$.

(6) This point is a result of Yokota (Carrillo–Ni [2] got similar results under more restrictive curvature hypotheses). Our version is equivalent to his statement [13, Corollary 1.1] and [14, Theorem 2].

(7) By point (1) either the scalar curvature is positive, or (M,g,f) is the Gaussian soliton. In the first case (M,g) must be Einstein with a positive constant, hence, it is compact (by Myers's diameter estimate) with constant scalar curvature if $n \geq 3$. If $n = 2$ it is known that the only compact solitons are $\mathbb{S}^2$ and its quotient $\mathbb{RP}^2$ with a constant potential function. If $n \geq 3$ it follows that Δf is constant on M, which is compact, hence, the potential function f (unique by compactness, see point (4)) is constant.

□

2. The Critical Points of the Functional $\mathcal{W}$

We consider the Perelman's functional $\mathcal{W}$ on a normalized, gradient, shrinking Ricci solitons $(M,g,\overline{f})$, freezing the metric g and varying only the function f . We want to discuss the existence of a minimizer or, more generally, of a critical point $f \in C^\infty(M)$, under the constraint $\int_M \frac{e^{-f}}{(4\pi)^{n/2}}\, d\mathrm{Vol} = 1$.

Substituting $u = \frac{e^{-f/2}}{(4\pi)^{n/4}}$ the functional becomes

$$\widetilde{\mathcal{W}}(u) = \int_M \big(\mathrm{R}u^2 + 4|\nabla u|^2 - u^2\log u^2 - u^2\frac{n}{2}\log 4\pi - nu^2\big)\,d\mathrm{Vol}\,,$$

under the constraint $\int_M u^2\,d\mathrm{Vol} = 1$.
Moreover, in studying its properties, we can actually consider the functional $\widetilde{F}$ defined as

$$\widetilde{F}(u) = \int_M \Big(\mathrm{R}u^2 + 4|\nabla u|^2 - u^2\log u^2\Big)\,d\mathrm{Vol}\,,$$

which differs by $\widetilde{\mathcal{W}}(u)$ only for a constant term, by the constraint $\int_M \frac{e^{-f}}{(4\pi)^{n/2}}\,d\mathrm{Vol} = 1$ and we define the infimum

$$\sigma = \inf_{u\in C^\infty(M),\ \int_M u^2\,d\mathrm{Vol}=1} \widetilde{F}(u).$$

Notice that, even in the flat $\mathbb{R}^n$, the functional $\widetilde{F}(u)$ could be unbounded above on $H^1(M,g)$, indeed, consider the functions, for $t>1$, which all satisfy $\int_M u_t^2\,d\mathrm{Vol} = 1$,

$$u_t = \frac{e^{-\frac{|x|^2}{8t}}}{(4\pi t)^{n/4}}\,.$$

We see that

$$|\nabla u_t|^2 = \frac{|x|^2}{16t^2}u_t^2 \qquad\text{and}\qquad -u_t^2\log u_t^2 = \bigg(\frac{|x|^2}{4t} + \frac{n}{2}\log 4\pi t\bigg)u_t^2\,,$$

hence,

$$\widetilde{F}(u_t) = \int_M \bigg(\frac{|x|^2}{2t^2} + \frac{n}{2}\log 4\pi t\bigg)u_t^2\,d\mathrm{Vol} = \int_M \frac{|x|^2}{2t^2}u_t^2\,d\mathrm{Vol} + \frac{n}{2}\log 4\pi t$$

By direct computation, we can see that

$$\int_M |\nabla u_t|^2\,d\mathrm{Vol} = \int_M \frac{|x|^2}{16t^2}u_t^2\,d\mathrm{Vol} = C/t\,,$$

hence, the family of functions u_t, for $t\geq 1$, is uniformly bounded in $H^1(M,g)$ but $\lim_{t\to+\infty}\widetilde{F}(u_t)\to+\infty$.

We first discuss when the functional $\widetilde{F}$ is bounded below on $H^1(M,g)$. Notice that if (M,g) is different by the flat $\mathbb{R}^n$, the function R is everywhere positive and actually bounded above by

$$\mathrm{R} \leq \overline{f} + a(g) \leq 2a(g) + \frac{1}{4}\big(d_g(x,p) + \sqrt{2n}\big)^2$$

for some point $p\in M$, by the results of the previous section. Hence, we only need to uniformly bound the integrand $u^2\log u^2$ (notice that the function $h(t) = t^2\log t^2$ is C^1, defining $h(0)=0$) and bounded below by $1/e$) , hence, we will need that Sobolev inequalities hold. This is assured by the following result of Varopoulos in [12] (see also [4]), assuming that the Ricci tensor is uniformly bounded below and the soliton is non–collapsed.

Proposition 2.1 (Theorem 3.2 in [6]). *Let (M, g) be a smooth, complete, n–dimensional Riemannian manifold with Ricci tensor bounded below and*

$$\inf_{x\in M} \mathrm{Vol}(B_1(x)) > 0\,,$$

where $B_1(p)$ is the unit geodesic ball in (M, g) of center $x \in M$.
Then, the Sobolev embeddings $W^{1,q}(M, g) \hookrightarrow L^p(M, g)$ holds for every $q \in [1, n)$, where $1/p = 1/q - 1/n$.

As we do not know whether every normalized, gradient, shrinking Ricci soliton has a bound from below on the Ricci tensor and/or it must be non–collapsed. What we know is that, by Perelman's work [8], all the gradient, shrinking Ricci solitons coming from a blow–up of a compact Ricci flow satisfy these conditions.

From now on we will assume, unless differently specified, that all the solitons we are going to consider are non–collapsed and with Ricci tensor bounded below. As a consequence, Sobolev embeddings hold, in particular, there exists a constant C_M such that

$$\int_M u^{2^*}\, d\mathrm{Vol} \leq \Big(C_M \int_M (|\nabla u|^2 + u^2)\, d\mathrm{Vol}\Big)^{\frac{n}{n-2}}\,,$$

where $2^* = \frac{2n}{n-2}$, for every $u \in H^1(M, g)$ (when $n = 2$ we can take 2^* to be whatever value in $(2, +\infty)$, by Theorem 3.7 in [6]).

Proposition 2.2. *On $H^1(M, g)$ the functional $\widetilde{F}$ (hence also W and $\widetilde{W}$) is uniformly bounded below, that is, $\sigma > -\infty$.*

Proof. Clearly, since we know that $\mathrm{R} \geq 0$ it is sufficient to show that the integral $\int_M u^2 \log u^2\, d\mathrm{Vol}$ is uniformly bounded above.

For any $u \in H^1(M, g)$, by applying Jensen inequality with respect to the probability measure $u^2\, d\mathrm{Vol}$, one has

$$\begin{aligned}\int_M u^2 \log u^2\, d\mathrm{Vol} &= \frac{n-2}{2}\int_M u^2 \log u^{\frac{4}{n-2}}\, d\mathrm{Vol}\\ &\leq \frac{n-2}{2}\log\Big(\int_M u^2 u^{\frac{4}{n-2}}\, d\mathrm{Vol}\Big)\\ &= \frac{n-2}{2}\log\Big(\int_M u^{\frac{2n}{n-2}}\, d\mathrm{Vol}\Big)\\ &= \frac{n-2}{2}\log\Big(\int_M u^{2^*}\, d\mathrm{Vol}\Big)\,.\end{aligned}$$

On the other hand,

$$\log\Big(\int_M u^{2^*}\,d\mathrm{Vol}\Big)\leq \log\Big[\Big(C_M\int_M(|\nabla u|^2+u^2)\,d\mathrm{Vol}\Big)^{\frac{n}{n-2}}\Big]$$
$$=\frac{n}{n-2}\log\Big(C_M\int_M(|\nabla u|^2+u^2)\,d\mathrm{Vol}\Big)\,,$$

where C_M is the Sobolev constant of (M,g).
Putting together these two inequalities we get

$$-\int_M u^2\log u^2\,d\mathrm{Vol}\geq -\log\Big(C_M\int_M(|\nabla u|^2+u^2)\,d\mathrm{Vol}\Big)\geq -4\int_M(|\nabla u|^2+u^2)\,d\mathrm{Vol}-C'_M\,,$$

for some positive constant C'_M depending only on (M,g). Hence,

$$\begin{aligned}\widetilde{\mathcal{F}}(u)&=\int_M\Big(\mathrm{R}u^2+4|\nabla u|^2-u^2\log u^2\Big)\,d\mathrm{Vol}\\ &\geq 4\int_M|\nabla u|^2\,d\mathrm{Vol}-4\int_M(|\nabla u|^2+u^2)\,d\mathrm{Vol}-C'_M\\ &=-4\int_M u^2\,d\mathrm{Vol}-C'_M\\ &=-4-C'_M\,,\end{aligned}$$

where in the last passage we used that $\int_M u^2\,d\mathrm{Vol}=1$. This shows that $\sigma>-\infty$. □

2.1. **Critical Points and Minima.** By means of compact perturbations, every critical point of the functionals $\widetilde{F}$ or $\widetilde{W}$ satisfies the Euler equation

$$-4\Delta u+\mathrm{R}u-2u\log u=\lambda u$$

for some constant λ coming from the constraint. Indeed, as usual, multiplying by u and integrating, we get

$$\widetilde{\mathcal{W}}(u)=\int_M\big(\mathrm{R}u^2+4|\nabla u|^2-u^2\log u^2-u^2\frac{n}{2}\log 4\pi-nu^2\big)\,d\mathrm{Vol}=\lambda-\frac{n}{2}\log 4\pi-n\,,$$

that is,

$$\lambda=\widetilde{\mathcal{W}}(u)+\frac{n}{2}\log 4\pi+n\,.$$

Rereading all this in terms of the function $f=-\log\big[(4\pi)^{n/2}u^2\big]$ we get, equivalently,

$$2\Delta f-|\nabla f|^2+\mathrm{R}+f=\lambda-\frac{n}{2}\log 4\pi$$

for every constrained critical point $f:M\to\mathbb{R}$ of the functional $\mathcal{W}$ such that $u=\frac{e^{-f/2}}{(4\pi)^{n/4}}\in H^1(M,g)$ and $\int_M\frac{e^{-f}}{(4\pi)^{n/2}}\,d\mathrm{Vol}=1$. Moreover,

$$\lambda=\mathcal{W}(f)+\frac{n}{2}\log 4\pi+n\,,$$

hence, setting $\mu=\lambda-\frac{n}{2}\log 4\pi$, we conclude as follows.

Proposition 2.3. *A constrained critical point of the functional $\mathcal{W}$ on the set of functions $f : M \to \mathbb{R}$ such that $u = \frac{e^{-f/2}}{(4\pi)^{n/4}} \in H^1(M,g)$ and $\int_M \frac{e^{-f}}{(4\pi)^{n/2}}\, d\mathrm{Vol} = 1$ satisfies*

$$2\Delta f - |\nabla f|^2 + \mathrm{R} + f = \mu$$

where the constant μ is given by

$$\mu = \mathcal{W}(f) + n\,.$$

The potential function $\overline{f}$, satisfies $\Delta\overline{f} + \mathrm{R} = n/2$ and $\int_M e^{-\overline{f}}\, d\mathrm{Vol} = 1$, moreover, $\mathrm{R} + |\nabla\overline{f}|^2 - \overline{f} = a(g,\overline{f}) = a(g)$, hence,

$$2\Delta\overline{f} - |\nabla\overline{f}|^2 + \mathrm{R} + \overline{f} = n - a(g) = n + \mathcal{W}(g) = n + \mathcal{W}(\overline{f})$$

which implies that $\overline{f}$ is a critical point of the functional $\mathcal{W}$.

In the following, we want to discuss whether other critical point or minimizers of $\mathcal{W}$ actually exist and their relation with the potential function of the soliton (M,g).

2.2. The Compact Case.

Proposition 2.4. *If the soliton (M,g) is compact, the infimum σ is achieved by a minimizer $u \in C^\infty(M)$, moreover, $u > 0$ everywhere on M.*

Proof. The same argument showing that $\sigma > -\infty$, gives that any minimizing sequence $u_i \in C^\infty(M)$, with $\|u\|_{L^2} = 1$, is uniformly bounded in the space $H^1(M,g)$. Hence, we can extract a subsequence (not relabeled) weakly converging in $H^1(M,g)$ and strongly converging in $L^{2+\varepsilon}(M)$, for some $\varepsilon > 0$, to some function u (by compactness of (M,g)) (the Sobolev compact embeddings hold on a compact Riemannian manifold, see [6]). Clearly, by the the $L^{2+\varepsilon}$–convergence and the compactness of (M,g), we have $\int_M u^2\, d\mathrm{Vol} = 1$ and we can also assume $u \geq 0$, by the definition of $\widetilde{\mathcal{F}}$.
It is easy to see that the functional $\widetilde{\mathcal{F}}$ is lower semicontinuous with respect to the weak convergence in $H^1(M,g)$, as the term $u^2 \log u^2$ is *subcritical* (and the function $h(t) = t^2 \log t^2$ is continuous) hence its integral is continuous.
Then, a limit function $u : M \to \mathbb{R}$ is a nonnegative, constrained minimizer of $\widetilde{\mathcal{F}}$ in $H^1(M,g)$.
The Euler–Lagrange equation for u read

$$-4\Delta u + \mathrm{R}u - (u \log u^2 + u) = Cu\,,$$

for some constant C. It can be rewritten as

$$\Delta u = \mathrm{R}u/4 + Cu - u \log u\,, \tag{2.1}$$

to be intended in $H^1(M,g)$.
As u is in $H^1(M,g)$ and the term $u^2 \log u$ is subcritical, a bootstrap argument together with standard elliptic estimates gives that $u \in C^{1,\alpha}$.

Rothaus proved in [11] that a solution of equation (2.1) is positive or identically zero. this second possibility is obviously excluded by the constraint $\int_M u^2\, d\mathrm{Vol} = 1$.
Finally, as the function $h(t) = t^2 \log t^2$ is smooth in $\mathbb{R} \setminus \{0\}$, again by a bootstrap argument, we can conclude that the function u is actually smooth. ☐

Assume that $f : M \to \mathbb{R}$ is a critical point, that is, f satisfies $2\Delta f - |\nabla f|^2 + \mathrm{R} + f =$ constant, then we have

$$
\begin{aligned}
&g^{kj}\nabla_k \left[2(\mathrm{R}_{ij} + \nabla^2_{ij} f - g_{ij}/2)e^{-f}\right]\\
&\quad= (\nabla_i \mathrm{R} + 2\Delta\nabla_i f)e^{-f} - 2[(\mathrm{R}_{ij} + \nabla^2_{ij} f - g_{ij}/2)g^{jk}\nabla_k f]e^{-f}\\
&\quad= (\nabla_i \mathrm{R} + 2\nabla_i\Delta f + 2\mathrm{R}_{is}\nabla^s f)e^{-f} - 2[(\mathrm{R}_{ij} + \nabla^2_{ij} f - g_{ij}/2)g^{jk}\nabla_k f]e^{-f}\\
&\quad= (\nabla_i \mathrm{R} + 2\nabla_i\Delta f - 2g^{jk}\nabla^2_{ij} f\nabla_k f + \nabla_i f)e^{-f}\\
&\quad= \nabla_i(\mathrm{R} + 2\Delta f - |\nabla f|^2 + f)e^{-f}\\
&\quad= 0\,.
\end{aligned}
$$

Hence,

$$\mathrm{div}[(\mathrm{Ric} + \nabla^2 f - g/2)e^{-f}] = 0$$

and

$$
\begin{aligned}
&\mathrm{div}[(\nabla_k f - \nabla_k \overline{f})\, g^{kj}(\mathrm{R}_{ij} + \nabla^2_{ij} f - g_{ij}/2)e^{-f}]\\
&\qquad\qquad= (\nabla^2_{lk} f - \nabla^2_{lk}\overline{f})g^{kj}g^{li}(\mathrm{R}_{ij} + \nabla^2_{ij} f - g_{ij}/2)e^{-f}\\
&\qquad\qquad= (\nabla^2_{lk} f + \mathrm{R}_{lk} - g_{lk}/2)g^{kj}g^{li}(\mathrm{R}_{ij} + \nabla^2_{ij} f - g_{ij}/n)e^{-f}\\
&\qquad\qquad= |\mathrm{R}_{ij} + \nabla^2_{ij} f - g_{ij}/n|^2 e^{-f}\,,
\end{aligned}
$$

where, passing from the second to the third line, we substituted $\nabla^2_{lk}\overline{f}$ with $g_{lk}/2 - \mathrm{R}_{lk}$, by the soliton relation.
Hence, we conclude that, setting $\mathrm{T} = (\nabla_k f - \nabla_k\overline{f})g^{kj}(\mathrm{R}_{ij} + \nabla^2_{ij} f - g_{ij}/2)e^{-f}$, we have

$$0 \le Q = |\mathrm{Ric} + \nabla^2 f - g/2|^2 e^{-f} = \mathrm{div}\,\mathrm{T}\,.$$

Being M compact, integrating Q on M, we immediately get that $Q = 0$, since it is nonnegative.
This says that the function f is a potential for the gradient Ricci soliton (M, g), then, by point (4) of Lemma 1.2 it must coincide with $\overline{f}$.

Proposition 2.5. *If the soliton (M, g) is compact its potential function is the unique constrained critical point of the functional $\mathcal{W}$, it is smooth and minimizes $\mathcal{W}$.*

2.3. **The Noncompact Case.** The noncompact case is more delicate, in particular it is quite more difficult to obtain the existence of a minimizer of the functional $\widetilde{F}$. In [15] Zhang showed that there exists complete, noncollapsed *manifolds* with bounded Riemann tensor such that the functional $\widetilde{F}$ does not have an extremal.

Anyway, Carrillo and Ni [2] where able to show that on every gradient shrinking soliton, the potential function is a constrained minimizer of the functional $\mathcal{W}$. Zhang [15] and, recently, with weaker hypotheses on the geometry of the manifold, Rimoldi and Veronelli [10] showed the existence of extremals for $\mathcal{W}$ on a generic manifold, under a condition at infinity. In particular, in this latter paper, the authors use such conclusion to show that a general shrinking soliton (not a priori *gradient*) with bounded Ricci tensor and injectivity radius uniformly bounded below, actually admits a gradient soliton structure (under the above mentioned condition at infinity).

As a consequence of the work of Carrillo and Ni [2], arguing as Rimoldi and Veronelli [10], if we consider, as in the compact case, the tensor $\mathrm{T} = (\nabla_k f - \nabla_k \overline{f}) g^{kj}(\mathrm{R}_{ij} + \nabla^2_{ij} f - g_{ij}/2)e^{-f}$,

$$0 \leq Q = |\mathrm{Ric} + \nabla^2 f - g/2|^2 e^{-f} = \mathrm{div}\, \mathrm{T}\,,$$

under the hypotheses of bound on the Ricci tensor and on the injectivity radius of the manifold, the function Q can be integrated on M and we can conclude that $Q = 0$.

Proposition 2.6. *If a gradient shrinking Ricci soliton (M, g) has uniformly bounded Ricci tensor and injectivity radius (this latter from below), the unique constrained critical point of the functional $\mathcal{W}$ is the potential function of the soliton and minimizes $\mathcal{W}$.*

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