Sharp upper bounds for the density of some invariant measures

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Abstract

We prove sharp upper bounds for invariant measures of Markov processes in $\mathbb{R}^N$ associated with second-order elliptic differential operators with unbounded coefficients.
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1 Introduction

Given a second order elliptic differential operator of the following form

$$A = \sum_{i,j=1}^{N} D_i(a_{ij}D_j) + \sum_{i=1}^{N} F_i D_i,$$

we say that a Borel probability measure μ on $\mathbb{R}^N$ is an *invariant measure* for A if

$$\int_{\mathbb{R}^N} A\phi \, d\mu = 0 \tag{1.1}$$

for every $\phi \in C_c^\infty(\mathbb{R}^N)$ (the local integrability of $A\phi$ with respect to μ will soon be clear). Our assumptions (H0) and (H1) below imply that A, endowed with a suitable domain $D(A)$, generates a semigroup $(T(t))$ in $L^1(\mu)$ and that (1.1) holds for every $\phi \in D(A)$ or, in an equivalent way,

$$\int_{\mathbb{R}^N} T(t)f d\mu = \int_{\mathbb{R}^N} f d\mu \tag{1.2}$$

for every $f \in L^1(\mu)$ and $t \geq 0$. This means that μ is a stationary distribution of the Markov process described by $(A, D(A))$. For this reason, the issues of existence, uniqueness and regularity of invariant measures are investigated both by analytical and probabilistic tools. Here, we are mainly interested in the global regularity of μ and upper bounds for its density with respect to the Lebesgue measure.

Throughout the paper, we assume the following

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(H0) $a_{ij} = a_{ji}, F_i : \mathbb{R}^N \to \mathbb{R}$, $N \geq 2$, with $a_{ij} \in W^{1,q}_{\text{loc}}(\mathbb{R}^N)$, $F_i \in L^q_{\text{loc}}(\mathbb{R}^N)$, for some $q > N$, and

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2,$$

for every $x, \xi \in \mathbb{R}^N$ and some $\lambda > 0$.

Observe that neither a_{ij} nor F are assumed to be bounded. Since for $N = 1$ invariant measures can always be explicitly computed, we assume $N \geq 2$. Under this assumption, it is known that μ is absolutely continuous with respect to the Lebesgue measure and its density ρ belongs to $W^{1,q}_{\text{loc}}(\mathbb{R}^N)$; in particular, ρ is a continuous function. Moreover, it can be proved that ρ is strictly positive everywhere. We refer to [1, Corollaries 2.10, 2.11] for these results. Besides (H0), we assume that

(H1) there exists a function $V \in C^2(\mathbb{R}^N)$ such that

$$\lim_{|x|\to+\infty} V(x) = +\infty, \qquad \lim_{|x|\to+\infty} AV(x) = -\infty, \qquad \lim_{|x|\to+\infty} (\lambda\Delta V + F \cdot \nabla V)(x) = -\infty.$$

A function satisfying the first two conditions in (H1) is called a *Lyapunov function* for the operator A. Thus, requiring (H1) is equivalent to requiring that there is V which is a Lyapunov function both for A and $\lambda\Delta + F \cdot \nabla$. Without loss of generality one can suppose that $V \geq 1$. It is worth observing that, under our regularity assumption (H0), if there exists a Lyapunov function V for A, then there are a unique invariant measure μ for A and a semigroup $(T(t))$ in $L^1(\mu)$ for which (1.2) holds, see [6, Proposition 2.9, Theorem 3.1]. Moreover $AV \in L^1(\mu)$, see [3]. Actually, we need (H1) both for ensuring existence and uniqueness of μ and for performing our method, as we explain later.

The main results of the present paper are the following. We first prove that $\rho \in L^\infty(\mathbb{R}^N)$, basically by adapting the De Giorgi regularity method to our equation (1.1), which, using that $d\mu = \rho\, dx$ and the local regularity of ρ, can be written in a weak form as follows:

$$\int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij} D_i\rho D_j\phi\, dx = \int_{\mathbb{R}^N} F \cdot \nabla\phi\, \rho\, dx, \quad \forall\ \phi \in C^\infty_c(\mathbb{R}^N).$$

Since the coefficients are not bounded, the solution ρ does not belong globally to any Sobolev space, hence the idea is to proceed by approximation. We introduce new diffusion coefficients a^n_{ij} belonging to $C^1_b(\mathbb{R}^N)$ and keep the same drift F, in such a way that the function V given in (H1) is a Lyapunov function for the approximating operators, uniformly in n (see Lemma 2.1). This is possible in view of condition (H1). It follows that each approximating operator admits a unique invariant measure $d\mu_n = \rho_n dx$, whose densities converge pointwise to ρ, as $n \to +\infty$. Then, assuming that $|F| \in L^p(\mu_n)$, for some $p > N$, we prove that the L^∞-norm of ρ_n can be estimated by $\|F\|_{L^p(\mu_n)}$. Finally, we assume that the function V in (H1) is of the form $V = \exp\{|x|^\beta\}$ and that $|F|^p$ can be controlled by V and deduce, in view of a result in [8], that $\|V\|_{L^1(\mu_n)} \leq C$ for every n, and thus conclude that $\rho \in L^\infty(\mathbb{R}^N)$. This is the content of Section 2.

An analogous approach allows, in Section 3, to derive upper pointwise estimates for ρ in terms of $\exp\{-\gamma|x|^\beta\}$. Testing the result obtained on the model operator $A = \Delta - |x|^{\beta-1}\frac{x}{|x|} \cdot \nabla$, we find out that the constants involved are very precise (see Example 3.5). We stress the fact that our results allow for an exponential growth of a_{ij}, $|Da_{ij}|$ and $|F|$ (see Theorem 3.1 for the precise statement).

Concerning the existent literature on the subject, in [8], under weak assumptions on the coefficients of A, comparable to ours, the authors first prove that $\rho \in L^\infty(\mathbb{R}^N)$. Then, by strengthening the hypotheses on a_{ij}, F_i and, in particular, asking that $a_{ij} \in C_b^1(\mathbb{R}^N)$, they study Sobolev regularity of ρ. Finally, under still more restrictive assumptions, they investigate the asymptotic behaviour of ρ, obtaining upper and lower pointwise estimates.

In the paper [2], by refining the techniques of [8] and introducing new tools, Sobolev regularity and upper bounds on the density ρ are proved under slightly more general conditions on the coefficients than in [8]. On the other hand, in [5], lower bounds for densities of stationary distributions are obtained under considerably weaker restrictions on the a_{ij} and F than in [8]. In all these papers, however, the coefficients a_{ij} are bounded and the upper bounds are not as sharp as ours.

Finally, let us notice that, in the same vein, analogous estimates for parabolic problems are shown in [4], [9], [10], [11].

Notation We denote by a the $N \times N$ matrix with entries $(a_{ij}(x))$ and we set $a(\xi, \eta) = \sum_{i,j} a_{ij}\xi_i\eta_j$, for any $\xi, \eta \in \mathbb{R}^N$. We denote by $\Lambda(x)$ the maximum eigenvalue of the matrix $(a_{ij}(x))$. For any $u : \mathbb{R}^N \to \mathbb{R}$, u_+ is the positive part of u, i.e. $u_+(x) = \max\{u(x), 0\}$. We denote by $L^p(\mu)$ the space of all functions $u : \mathbb{R}^N \to \mathbb{R}$ which are measurable and p-summable in $\mathbb{R}^N$ with respect to the measure μ. When we write $L^p(\mathbb{R}^N)$ we mean that the underlying measure is the Lebesgue one. If E is measurable, we denote by $|E|$ its Lebesgue measure and by χ_E its characteristic function which takes value 1 in E and 0 in $\mathbb{R}^N \setminus E$.

2 Global boundedness

Let η be a function in $C^1(\mathbb{R})$ satisfying $\eta(t) = 1$ if $|t| \le 1$, $\eta(t) = 0$ if $|t| \ge 2$, $0 \le \eta \le 1$ everywhere and $\eta'(t) \le 0$, if $t \ge 0$. Set

$$\eta_n(x) = \eta\left(\frac{V(x)}{n}\right), \quad x \in \mathbb{R}^N,$$

where V is given in (H1), and define

$$a_{ij}^n = \eta_n a_{ij} + \lambda(1 - \eta_n)\delta_{ij}, \tag{2.1}$$

for every $i, j = 1, \ldots, N$ and $n \in \mathbb{N}$, with the agreement that $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. We introduce the approximating operators

$$A^n = \sum_{i,j=1}^N D_i(a_{ij}^n D_j) + \sum_{i=1}^N F_i D_i \tag{2.2}$$

and we remark that the new coefficients a_{ij}^n verify the ellipticity condition with the same constant λ as in (H0). Let us show that V is a Lyapunov function for every A^n, uniformly with respect to n.

Lemma 2.1 *One has*

$$\lim_{|x|\to+\infty} \sup_n A^n V(x) = -\infty.$$

In particular, V is a Lyapunov function for each A^n.

Proof. By explicit computations we find that

$$\begin{aligned} A^n V &= \eta_n AV + (1-\eta_n)(\lambda \Delta V + F\cdot \nabla V) \\ &\quad + \frac{1}{n}\,\eta'\left(\frac{V(x)}{n}\right)\left(\sum_{i,j=1}^N a_{ij} D_i V D_j V - \lambda |\nabla V|^2\right). \end{aligned}$$

As $\eta'(V(x)/n) \le 0$ and $\sum_{i,j} a_{ij} D_i V D_j V \ge \lambda |\nabla V|^2$, from the previous equation it follows that

$$A^n V \le \eta_n AV + (1-\eta_n)(\lambda \Delta V + F\cdot\nabla V) \tag{2.3}$$

which yields the statement, thanks to (H1). □

In the next corollary, we collect some known facts concerning invariant measures by referring them to the operators A^n.

Corollary 2.2 *Each operator A^n has a unique invariant measure μ_n, which is given by a positive density $\rho_n \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover, $A^n V$ belongs to $L^1(\mu_n)$, and there is $C > 0$ such that $\|A^n V\|_{L^1(\mu_n)} \le C$ for every $n \in \mathbb{N}$.*

Proof. The uniqueness part follows from [6, Proposition 2.9, Theorem 3.1]. For existence and regularity of (μ_n) we refer to [3, Theorem 1.2]. Finally, the estimate

$$\sup_n \|A^n V\|_{L^1(\mu_n)} \le C$$

is equation (1.12) in the proof of [3, Lemma 1.1]. □

The crucial point in the construction above is that the sequence (ρ_n) approximates the function ρ. More precisely, [3, Corollary 1.1] and the uniqueness of μ (which again follows from [6, Proposition 2.9, Theorem 3.1]) yield the following result.

Proposition 2.3 *The densities ρ_n converge to ρ in $L^1(\mathbb{R}^N)$.*

The proposition above clarifies the strategy of our approach to show that ρ is bounded: it suffices to prove that each ρ_n is bounded uniformly with respect to n. By a similar argument, we shall derive pointwise estimates for ρ in the next section.

We start by proving a global regularity property for ρ_n.

Proposition 2.4 *If $|F| \in L^2(\mu_n)$ then $\sqrt{\rho_n} \in W^{1,2}(\mathbb{R}^N)$. Moreover*

$$\int_{\{\rho_n > k\}} \frac{|\nabla \rho_n|^2}{\rho_n} dx \le \frac{1}{\lambda^2} \int_{\{\rho_n > k\}} |F|^2 \rho_n dx \tag{2.4}$$

for every $k \ge 0$, $n \in \mathbb{N}$.

Proof. We basically proceed as in [8, Theorem 3.1]. The local regularity of ρ_n and the invariance of the measure $\rho_n dx$ for the operator A^n lead to

$$\int_{\mathbb{R}^N} a^n(\nabla \rho_n, \nabla \phi) dx = \int_{\mathbb{R}^N} F\cdot \nabla\phi \, \rho_n dx, \tag{2.5}$$

for every $\phi \in C_c^\infty(\mathbb{R}^N)$ and hence, by density, for every $\phi \in W^{1,2}(\mathbb{R}^N)$ with compact support. Note that, as ρ_n is continuous, the function $|F|\rho_n$ belongs to $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. Let us take $\theta \in C_c^\infty(\mathbb{R}^N)$ such that $\theta(x) = 1$ for $|x| \le 1$, $\theta(x) = 0$ if $|x| \ge 2$ and set $\theta_m(x) = \theta(x/m)$. Fix $k > 0$ and consider the functions $\theta_m^2(\log(\rho_n \wedge h) - \log k)_+$, for every $n, m \in \mathbb{N}$ and $h > k$. Note that

$$0 \le (\log(\rho_n \wedge h) - \log k)_+ \le \log \frac{h}{k},$$

and $(\log(\rho_n \wedge h) - \log k)_+ > 0$ if and only if $\rho_n > k$. Moreover

$$\nabla(\log(\rho_n \wedge h) - \log k)_+ = \frac{\nabla \rho_n}{\rho_n} \chi_{\{k < \rho_n < h\}}.$$

Plugging $\phi = \theta_m^2(\log(\rho_n \wedge h) - \log k)_+$ into (2.5) and arguing as in [8, Theorem 3.1], we can let $m \to +\infty$ and $h \to +\infty$ getting

$$\int_{\{\rho_n > k\}} \frac{a^n(\nabla \rho_n, \nabla \rho_n)}{\rho_n}\, dx \le \frac{1}{\lambda} \int_{\{\rho_n > k\}} |F|^2 \rho_n\, dx.$$

We observe that $(1+|x|^2)^{-1} a_{ij}^n$, $(1+|x|)^{-1} D_j a_{ij}^n$ belong to $L^1(\mu_n)$, as required by [8, Theorem 3.1], since the $a_{ij}^n(x)$ are constant for x large and μ_n is a probability measure. Using the ellipticity condition for the matrix a^n, we have (2.4). Of course if $k \to 0$, (2.4) yields $\sqrt{\rho_n} \in W^{1,2}(\mathbb{R}^N)$. □

Remark 2.5 We point out that a slightly different form of estimate (2.4) can be proved, i.e.,

$$\int_{\{\rho_n > k\}} \frac{a^n(\nabla \rho_n, \nabla \rho_n)}{\rho_n}\, dx \le \int_{\{\rho_n > k\}} |(a^n)^{-\frac{1}{2}} F|^2 \rho_n dx. \tag{2.6}$$

There are only minor changes to do in the proof of Proposition 2.4 in order to get (2.6). We discuss them in the case $k = 0$ by using a formal argument. Taking $\phi = \log \rho_n$ in (2.5), we have

$$\int_{\mathbb{R}^N} \frac{a^n(\nabla \rho_n, \nabla \rho_n)}{\rho_n} dx = \int_{\mathbb{R}^N} F \cdot \frac{\nabla \rho_n}{\rho_n} \rho_n dx.$$

By Hölder inequality we find

$$\int_{\mathbb{R}^N} \frac{a^n(\nabla \rho_n, \nabla \rho_n)}{\rho_n}\, dx \le \left(\int_{\mathbb{R}^N} |(a^n)^{-\frac{1}{2}} F|^2 \rho_n\, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \frac{a^n(\nabla \rho_n, \nabla \rho_n)}{\rho_n}\, dx \right)^{\frac{1}{2}},$$

which implies (2.6), with $k = 0$.

Proposition 2.6 *If $|F| \in L^p(\mu_n)$, with $p > N$ then $\rho_n \in L^\infty(\mathbb{R}^N)$. In particular, there exists a constant $C > 0$, depending only on N, p such that*

$$\|\rho_n\|_\infty \le C \lambda^{-N} \|F\|_{L^p(\mu_n)}^N.$$

Proof. From Proposition 2.4 it follows that $\sqrt{\rho_n} \in W^{1,2}(\mathbb{R}^N)$. Hence, for every $k > 0$, the function $(\sqrt{\rho_n} - \sqrt{k})_+$ still belongs to $W^{1,2}(\mathbb{R}^N)$. If $N > 2$, by the Sobolev embedding theorem the inequality

$$\left(\int_{\{\rho_n > k\}} (\sqrt{\rho_n} - \sqrt{k})_+^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} \le C_S \int_{\{\rho_n > k\}} \frac{|\nabla \rho_n|^2}{\rho_n}\, dx \tag{2.7}$$

holds for a suitable $C_S > 0$. Hence, by Hölder inequality and (2.7) we get

$$\begin{aligned}\int_{\{\rho_n>k\}}(\sqrt{\rho_n}-\sqrt{k}\,)_+^2dx &\leq \left(\int_{\{\rho_n>k\}}(\sqrt{\rho_n}-\sqrt{k}\,)_+^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}}|\{\rho_n>k\}|^{\frac{2}{N}}\\ &\leq C_S\left(\int_{\{\rho_n>k\}}\frac{|\nabla\rho_n|^2}{\rho_n}\,dx\right)|\{\rho_n>k\}|^{\frac{2}{N}}. \qquad (2.8)\end{aligned}$$

If $N = 2$, we apply the inequality

$$\|v\|_{L^{N/(N-1)}(\mathbb{R}^N)} \leq C_S'\|\nabla v\|_{L^1(\mathbb{R}^N)}, \qquad (2.9)$$

which holds true for every $v \in W^{1,1}(\mathbb{R}^N)$, to the function $(\sqrt{\rho_n}-\sqrt{k})_+$ and Hölder inequality getting

$$\begin{aligned}\int_{\{\rho_n>k\}}(\sqrt{\rho_n}-\sqrt{k}\,)_+^2dx &\leq \frac{{C_S'}^2}{4}\left(\int_{\{\rho_n>k\}}\frac{|\nabla\rho_n|}{\sqrt{\rho_n}}\,dx\right)^2\\ &\leq C_S\left(\int_{\{\rho_n>k\}}\frac{|\nabla\rho_n|^2}{\rho_n}\,dx\right)|\{\rho_n>k\}|,\end{aligned}$$

which is the analogue of (2.8) in the case $N = 2$. At this point, by (2.4) we obtain

$$\begin{aligned}\int_{\{\rho_n>k\}}(\sqrt{\rho_n}-\sqrt{k}\,)_+^2dx &\leq \frac{C_S}{\lambda^2}\,|\{\rho_n>k\}|^{\frac{2}{N}}\int_{\{\rho_n>k\}}|F|^2\rho_n dx\\ &\leq \frac{C_S}{\lambda^2}\,|\{\rho_n>k\}|^{\frac{2}{N}}\left(\int_{\mathbb{R}^N}|F|^p\rho_n dx\right)^{\frac{2}{p}}\left(\int_{\{\rho_n>k\}}\rho_n dx\right)^{1-\frac{2}{p}},\end{aligned}$$

where we have used again Hölder inequality with respect to the measure $\rho_n dx$. Fix now $h > k$. Since $\frac{\sqrt{t}}{\sqrt{t}-\sqrt{k}} \leq \frac{\sqrt{h}}{\sqrt{h}-\sqrt{k}}$ if $t > h$ we have

$$\begin{aligned}\int_{\{\rho_n>h\}}\rho_n\,dx &\leq \frac{h}{(\sqrt{h}-\sqrt{k})^2}\int_{\{\rho_n>h\}}(\sqrt{\rho_n}-\sqrt{k})_+^2\,dx\\ &\leq \frac{h}{(\sqrt{h}-\sqrt{k})^2}\int_{\{\rho_n>k\}}(\sqrt{\rho_n}-\sqrt{k})_+^2\,dx\\ &\leq \frac{C_S}{\lambda^2}\frac{h}{(\sqrt{h}-\sqrt{k})^2}\,|\{\rho_n>k\}|^{\frac{2}{N}}\left(\int_{\mathbb{R}^N}|F|^p\rho_n\,dx\right)^{\frac{2}{p}}\left(\int_{\{\rho_n>k\}}\rho_n\,dx\right)^{1-\frac{2}{p}}.\end{aligned}$$

As $|\{\rho_n > k\}| \leq k^{-1}\int_{\{\rho_n>k\}}\rho_n\,dx$, we get

$$\int_{\{\rho_n>h\}}\rho_n\,dx \leq \frac{L}{k^{2/N}}\frac{h}{(\sqrt{h}-\sqrt{k})^2}\left(\int_{\{\rho_n>k\}}\rho_n\,dx\right)^{1-\frac{2}{p}+\frac{2}{N}}, \qquad (2.10)$$

where $L = C_S\lambda^{-2}\|F\|^2_{L^p(\mu_n)}$. Now, take

$$k_m = \overline{k} + \left(1-\frac{1}{2^m}\right)\overline{k}, \qquad m = 0, 1, 2, \ldots$$

with $\overline{k} > 0$ to be determined. Replacing h with k_{m+1} and k with k_m in (2.10), we obtain, after simple computations,

$$y_{m+1} \le \frac{64L}{\overline{k}^{2/N}}\, 2^{2m}\, y_m^{1+\alpha}, \qquad m = 0, 1, 2, \ldots$$

where

$$y_m = \int_{\{\rho_n > k_m\}} \rho_n\, dx, \qquad \alpha = -\frac{2}{p} + \frac{2}{N} > 0.$$

From [7, Lemma 7.1] it follows that if

$$y_0 \le \left(\frac{64L}{\overline{k}^{2/N}}\right)^{-\frac{1}{\alpha}} 4^{-\frac{1}{\alpha^2}},$$

then $\lim_{m\to+\infty} y_m = 0$, which means that $\rho_n \le 2\overline{k}$. It is clear that we can choose $\overline{k}$ large enough in order to fulfil the estimate for y_0. In order to make a quantitative choice of $\overline{k}$, we note that $y_0 \le 1$ because $\rho_n\, dx$ is a probability measure. Hence, we choose $\overline{k}$ such that $1 = \left(\frac{\overline{k}^{2/N}}{64L}\right)^{\frac{1}{\alpha}} 4^{-\frac{1}{\alpha^2}}$, which gives

$$\overline{k} = C\lambda^{-N}\, \|F\|^N_{L^p(\mu_n)},$$

with a suitable constant $C > 0$ depending only on p, N. □

The following lemma follows from a careful inspection of the proof of [8, Proposition 2.4]. We present a complete proof for reader's convenience.

Lemma 2.7 *Assume that there exist $c > 0$, $\beta > 0$ with the properties*

$$\limsup_{|x|\to+\infty} |x|^{1-\beta} F(x)\cdot\frac{x}{|x|} \le -c, \qquad \limsup_{|x|\to+\infty}\left(c\frac{\Lambda(x)}{\lambda} + |x|^{1-\beta} G(x)\cdot\frac{x}{|x|}\right) \le 0, \qquad (2.11)$$

where $G = (G_1, \ldots, G_N)$ and $G_i = F_i + \sum_j D_j a_{ij}$. Then, for $\delta < (\beta\lambda)^{-1}c$ the function $V(x) = \exp\{\delta|x|^\beta\}$ satisfies (H1). Moreover $V \in L^1(\mu_n)$ for all n and the norms $\|V\|_{L^1(\mu_n)}$ are uniformly bounded.

Proof. Assume first that the second limit in (2.11) is less than 0. By a direct computation, we deduce

$$\begin{aligned} AV(x) &= \delta\beta|x|^{\beta-1}e^{\delta|x|^\beta}\left(\frac{\sum_i a_{ii}(x)}{|x|} + \frac{\beta-2}{|x|^3}\sum_{i,j=1}^N a_{ij}(x)x_ix_j \right.\\ &\qquad \left. +\delta\beta|x|^{\beta-3}\sum_{i,j=1}^N a_{ij}(x)x_ix_j + G\cdot\frac{x}{|x|}\right). \end{aligned}$$

Since the quadratic form $|\sum_{i,j} a_{ij}(x)x_ix_j|$ can be estimated by $\Lambda(x)|x|^2$, $AV(x) \to -\infty$ as $|x| \to \infty$ follows by elementary arguments. Analogously, using the first condition in (2.11) we verify the same property for $\lambda\Delta V + F\cdot\nabla V$ and (H1) holds. Next, observe that if $\beta \ge 1$ then $|AV|$ is bigger than V, while if $0 < \beta < 1$ then apply the previous argument to $V_1 = \exp\{(\delta+\varepsilon)|x|^\beta\}$, with $\delta + \varepsilon < (\beta\lambda)^{-1}c$, and check that V_1 is a Lyapunov function such that $|V| \le |AV_1|$ for large $|x|$. The same is true with A replaced by $\lambda\Delta + F\cdot\nabla$. The thesis then follows from (2.3) and Corollary 2.2. Finally, if the second limit in (2.11) is 0, we can fix $\varepsilon > 0$ such that $c - \varepsilon > \delta\beta\lambda$ and apply the same argument with $c - \varepsilon$ in place of c. □

Theorem 2.8 *Assume that there exist $c > 0$, $\beta > 0$ such that (2.11) holds; moreover, suppose that for $|x| \geq 1$,*

$$|F(x)|^p \leq k \exp\{\delta |x|^\beta\},$$

with $k > 0$, $\delta < (\beta\lambda)^{-1}c$ and $p > N$. Then $\rho \in L^\infty(\mathbb{R}^N)$.

PROOF. From Lemma 2.7 we know that the function $V(x) = \exp\{\delta|x|^\beta\}$ (for $|x| \geq 1$) satisfies (H1) and the norms $\|V\|_{L^1(\mu_n)}$ are bounded in n. Then $\|F\|^p_{L^p(\mu_n)} \leq k\|V\|_{L^1(\mu_n)} \leq C$, for every n and so, by Proposition 2.6, the sequence (ρ_n) is uniformly bounded. Finally, Proposition 2.3 yields the statement. □

3 Pointwise estimates

In the present section we provide pointwise upper estimates for the density ρ. The main interest consists in the unboundedness of the diffusion coefficients and in the sharpness of the constants β, γ below. Let us state the main result.

Theorem 3.1 *Assume that there exist $c > 0$, $\beta > 0$ satisfying (2.11) and that there are $0 < \gamma < (\beta\lambda)^{-1}c$, $\alpha > 0$, $p > N$ such that $\alpha p + \gamma < (\beta\lambda)^{-1}c$ and*

$$|F(x)| \leq C_1 \exp\{\alpha|x|^\beta\},$$

$$|x|^{\beta-1}\left[\sum_{i=1}^N \left(\sum_{j=1}^N D_j a_{ij}(x)\right)^2\right]^{\frac{1}{2}} + |x|^{2(\beta-1)} \sum_{i,j=1}^N a_{ij}^2(x) \leq C_2 \exp\{2\alpha|x|^\beta\}, \tag{3.1}$$

for $|x|$ large and for some $C_1, C_2 > 0$, possibly depending on γ, β. Then, there exists a positive constant C such that

$$\rho(x) \leq C \exp\{-\gamma|x|^\beta\}, \qquad \text{for all } x \in \mathbb{R}^N. \tag{3.2}$$

The proof of the theorem above follows as a particular case of a more general situation that we are going to describe. Assume that

(H2) there exists a function $\omega \in C^2(\mathbb{R}^N)$ satisfying

(i) $c_0 \leq \omega \leq cV$,

(ii) $|\nabla V| \leq c\,\omega^{-\frac{1}{p}} V^{1+\frac{1}{p}}$,

(iii) $\omega|F|^p \leq cV$,

(iv) $|aD^2\omega| \leq c\,\omega^{1-\frac{2}{p}} V^{\frac{2}{p}}$,

(v) $|a\nabla\omega| \leq c\,\omega^{1-\frac{1}{p}} V^{\frac{1}{p}}$,

(vi) $\left|\sum_{i,j=1}^N D_j a_{ij} D_i \omega\right| \leq c\,\omega^{1-\frac{2}{p}} V^{\frac{2}{p}}$,

where $c_0, c > 0$ are constants, $p > N$ and V is the function given in (H1).

Remark 3.2 Theorem 3.1 will be proved by taking $\omega(x) = \exp\{\gamma|x|^\beta\}$. We prefer to list the relevant properties in a general form because other choices could be useful. Notice also that if (3.1) holds for every $\alpha > 0$, then clearly (3.2) holds for every $\gamma < (\beta\lambda)^{-1}c$. We point out that particular cases where the hypotheses of Theorem 3.1 are easily checked are presented in Examples 3.5, 3.6.

We start by showing that conditions (iv)-(vi) are preserved by the approximating coefficients a^n_{ij}.

Lemma 3.3 *The coefficients a^n_{ij} defined in (2.1) fulfil hypotheses (H2)(iv)-(vi) uniformly in n.*

Proof. We first observe that, since a is invertible, we have

$$|D^2\omega| = |a^{-1}aD^2\omega| \le \lambda^{-1}|aD^2\omega|, \qquad |\nabla\omega| \le \lambda^{-1}|a\nabla\omega|.$$

Therefore, (H2)(iv) implies that $|a^nD^2\omega| \le \eta_n|aD^2\omega| + \lambda(1-\eta_n)|D^2\omega| \le c\,\omega^{1-\frac{2}{p}}V^{\frac{2}{p}}$. Using a similar argument, from (H2)(v) it follows that $|a^n\nabla\omega| \le c\,\omega^{1-\frac{1}{p}}V^{\frac{1}{p}}$. In order to check (H2)(vi) for the a^n_{ij}, we first compute

$$\sum_{i,j=1}^N D_ja^n_{ij}D_i\omega = a(\nabla\omega, \nabla\eta_n) + \eta_n\sum_{i,j=1}^N D_ja_{ij}D_i\omega - \lambda\nabla\eta_n\cdot\nabla\omega.$$

From now on, we denote by c a constant which may change from line to line but remains independent of n. Since $\nabla\eta_n(x) = \frac{1}{n}\eta'\left(\frac{V(x)}{n}\right)\nabla V(x)$ and $\eta'(t) \neq 0$ only if $1 \le |t| \le 2$ we can estimate

$$|a(\nabla\omega, \nabla\eta_n)| \le c\,\omega^{1-\frac{1}{p}}V^{\frac{1}{p}}\|\eta'\|_\infty\frac{|\nabla V|}{n} \le c\|\eta'\|_\infty\frac{|\nabla V|}{V}\omega^{1-\frac{1}{p}}V^{\frac{1}{p}} \le c\,\omega^{1-\frac{2}{p}}V^{\frac{2}{p}},$$

taking (H2)(ii) into account. In a similar way, one can see that $|\nabla\eta_n\cdot\nabla\omega| \le c\,\omega^{1-\frac{2}{p}}V^{\frac{2}{p}}$. Hence

$$\left|\sum_{i,j=1}^N D_ja^n_{ij}D_i\omega\right| \le c\,\omega^{1-\frac{2}{p}}V^{\frac{2}{p}},$$

and the proof is complete. □

We now consider equation (2.5) for the operator A^n defined in (2.2). By a density argument, it is readily seen that we can take $\psi \in C^2_c(\mathbb{R}^N)$ as a test function. Let $\psi \in C^2_c(\mathbb{R}^N)$. Plugging $\phi = \psi\omega$ in (2.5), ω being given by (H2), and integrating by parts we obtain

$$\int_{\mathbb{R}^N}\sum_{i,j=1}^N a^n_{ij}D_iu_nD_j\psi = \int_{\mathbb{R}^N}f_n\psi + \int_{\mathbb{R}^N}h_n\cdot\nabla\psi.$$

where

$$u_n = \rho_n\omega \tag{3.3}$$

and

$$\begin{aligned} f_n &= \rho_n\Big(F\cdot\nabla\omega + \sum_{i,j=1}^N D_ja^n_{ij}D_i\omega + \sum_{i,j=1}^N a^n_{ij}D_{ij}\omega\Big), \\ h_n &= \rho_n\big(\omega F + 2a^n\nabla\omega\big). \end{aligned} \tag{3.4}$$

The next lemma provides an explicit estimate for the L^∞-norm of u_n.

Lemma 3.4 *Assume (H2) with $V \in L^1(\mu_n)$. Then, the function u_n, given by (3.3), is bounded and there is $C > 0$, independent of n, such that $\|u_n\|_\infty \le C\|V\|_{L^1(\mu_n)}$.*

PROOF. *First Step.* We assume in addition that ω is bounded. Since $|F|^p \le cV$, Proposition 2.6 applies, and $\rho_n \in L^\infty(\mathbb{R}^N)$. Then, u_n is bounded too, but we want to provide an estimate of the L^∞-norm independent of $\|\omega\|_\infty$ and with the explicit dependence on n. Let us prove that $f_n \in L^{p/2}(\mathbb{R}^N)$ and $h_n \in L^p(\mathbb{R}^N)$, where f_n and h_n are defined in (3.4). To this aim, recalling that ω verifies (H2), by Lemma 3.3 we get

$$\Big|\sum_{i,j=1}^N a_{ij}^n D_{ij}\omega\Big|^{\frac{p}{2}} \rho_n^{\frac{p}{2}} \le c|a^n D^2\omega|^{\frac{p}{2}}\rho_n^{\frac{p}{2}} \le c\|u_n\|_\infty^{\frac{p}{2}-1}\rho_n V,$$

$$\Big|\sum_{i,j=1}^N D_j a_{ij}^n D_i\omega\Big|^{\frac{p}{2}} \rho_n^{\frac{p}{2}} \le c\|u_n\|_\infty^{\frac{p}{2}-1}\rho_n V.$$

and

$$|F\cdot\nabla\omega|^{\frac{p}{2}}\rho_n^{\frac{p}{2}} \le cV^{\frac{1}{2}}\omega^{-\frac{1}{2}}\omega^{\frac{p-1}{2}}V^{\frac{1}{2}}\rho_n^{\frac{p}{2}} \le c\|u_n\|_\infty^{\frac{p}{2}-1}\rho_n V.$$

By the previous estimates we obtain

$$\|f_n\|_{L^{p/2}(\mathbb{R}^N)} \le c\|u_n\|_\infty^{1-\frac{2}{p}}\|\rho_n V\|_{L^1(\mathbb{R}^N)}^{\frac{2}{p}}. \tag{3.5}$$

We proceed in a similar way for h_n. So

$$|\omega F|^p\rho_n^p \le c\,\omega^{p-1}V\rho_n^p \le c\,\|u_n\|_\infty^{p-1}\rho_n V$$

and

$$|a^n\nabla\omega|^p\rho_n^p \le c\,\omega^{p-1}V\rho_n^p \le c\,\|u_n\|_\infty^{p-1}\rho_n V,$$

which yield

$$\|h_n\|_{L^p(\mathbb{R}^N)} \le c\,\|u_n\|_\infty^{1-\frac{1}{p}}\|\rho_n V\|_{L^1(\mathbb{R}^N)}^{\frac{1}{p}}. \tag{3.6}$$

In order to apply Theorem 4.1, we need to show that $u_n \in W^{1,2}(\mathbb{R}^N)$. Clearly

$$\|\rho_n\omega\|_{L^2(\mathbb{R}^N)} \le \|\rho_n\omega\|_\infty^{\frac{1}{2}}\|\rho_n\omega\|_{L^1(\mathbb{R}^N)}^{\frac{1}{2}} \le c\|u_n\|_\infty^{\frac{1}{2}}\|\rho_n V\|_{L^1(\mathbb{R}^N)}^{\frac{1}{2}}. \tag{3.7}$$

Furthermore, since $\nabla(\rho_n\omega) = \omega\nabla\rho_n + \rho_n\nabla\omega$, we have

$$|\nabla\omega|^2\rho_n^2 \le c\omega^{\frac{2(p-1)}{p}}V^{\frac{2}{p}}\rho_n^2 \le c\|\omega\|_\infty^{\frac{2(p-1)}{p}}\|\rho_n\|_\infty\rho_n V \in L^1(\mathbb{R}^N),$$

and

$$|\omega\nabla\rho_n|^2 \le \|\rho_n\|_\infty\|\omega\|_\infty^2\frac{|\nabla\rho_n|^2}{\rho_n} \in L^1(\mathbb{R}^N),$$

by Proposition 2.4. Then, from Theorem 4.1 and estimates (3.5), (3.6) and (3.7) it follows that (now the constant C may change from line to line, but is independent of n)

$$\|u_n\|_\infty \le C\Big(\|u_n\|_\infty^{\frac{1}{2}}\|\rho_n V\|_{L^1(\mathbb{R}^N)}^{\frac{1}{2}} + \|u_n\|_\infty^{1-\frac{2}{p}}\|\rho_n V\|_{L^1(\mathbb{R}^N)}^{\frac{2}{p}} + \|u_n\|_\infty^{1-\frac{1}{p}}\|\rho_n V\|_{L^1(\mathbb{R}^N)}^{\frac{1}{p}}\Big),$$

and therefore

$$\|u_n\|_\infty^p \le C\Big(\|u_n\|_\infty^{\frac{p}{2}}\|\rho_n V\|_{L^1(\mathbb{R}^N)}^{\frac{p}{2}} + \|u_n\|_\infty^{p-2}\|\rho_n V\|_{L^1(\mathbb{R}^N)}^2 + \|u_n\|_\infty^{p-1}\|\rho_n V\|_{L^1(\mathbb{R}^N)}\Big).$$

Setting $\|u_n\|_\infty = \alpha\,\|\rho_n V\|_{L^1(\mathbb{R}^N)}$, with $\alpha > 0$, we obtain

$$\alpha^p \le C\big(\alpha^{\frac{p}{2}} + \alpha^{p-2} + \alpha^{p-1}\big).$$

Then, α is bounded from above, which means that there exists a possibly different constant C, such that $\alpha \le C$. Hence

$$\|u_n\|_\infty = \alpha\,\|\rho_n V\|_{L^1(\mathbb{R}^N)} \le C\,\|V\|_{L^1(\mu_n)}.$$

Second Step. If ω is not bounded, then we consider $\omega_\varepsilon = \omega/(1+\varepsilon\omega)$. A straightforward computation shows that ω_ε satisfies hypothesis (H2) with a constant c independent of ε. Therefore, from the first step we obtain

$$\|\rho_n\omega_\varepsilon\|_\infty \le C\,\|V\|_{L^1(\mu_n)}$$

for a possibly different constant C independent of ε. Letting $\varepsilon \to 0$, the statement follows. □

Proof of Theorem 3.1. Set $\delta = \alpha p + \gamma$ and introduce ω, V smooth positive functions such that, for $|x|$ large,

$$\omega = \exp\{\gamma|x|^\beta\}, \qquad V = \exp\{\delta|x|^\beta\}. \tag{3.8}$$

As $\delta < (\beta\lambda)^{-1}c$ and (2.11) holds, V fulfils assumption (H1). Hence, by Lemma 2.7, V is integrable with respect to μ_n uniformly in n: $\|V\|_{L^1(\mu_n)} \le C$. Moreover, Theorem 2.8 applies and we have that $\|\rho_n\|_\infty \le C$ for every $n \in \mathbb{N}$.

Furthermore, as a consequence of the assumptions on the coefficients of the operator A given in the statement, the functions ω and V defined by (3.8) verify (H2). Therefore Lemma 3.4 applies and shows that the functions $u_n = \rho_n\omega$ are uniformly bounded. By Proposition 2.3 the proof is complete. □

In order to test the sharpness of Theorem 3.1, we consider some special cases.

Example 3.5 Let us consider the following class of operators

$$A = \Delta + F\cdot\nabla,$$

with $\limsup_{|x|\to+\infty} |x|^{1-\beta}F(x)\cdot\dfrac{x}{|x|} = -c$, for some $\beta, c > 0$. Fix $0 < \gamma < \beta^{-1}c$ and choose $\alpha > 0$ and $p > N$ such that $\alpha p + \gamma < \beta^{-1}c$ and $|F(x)| \le C_1\exp\{\alpha|x|^\beta\}$, for some $C_1 > 0$. Then Theorem 3.1 yields a constant $C > 0$ such that

$$\rho(x) \le C\exp\left\{-\gamma|x|^\beta\right\}. \tag{3.9}$$

If

$$F(x) = -|x|^{r-1}\frac{x}{|x|}, \tag{3.10}$$

with $r > 0$, then we choose $\beta = r$, $c = 1$, and (3.9) becomes

$$\rho(x) \le C\exp\left\{-\gamma|x|^r\right\}, \tag{3.11}$$

with $0 < \gamma < 1/r$. On the other hand, since (3.10) holds, one can compute explicitly the density ρ which is given by $\rho(x) = c \exp\left\{-\frac{|x|^r}{r}\right\}$.

Thus, Theorem 3.1 provides the correct constant $1/r$ up to an arbitrary small $\varepsilon > 0$. However, this $\varepsilon > 0$ is needed, in general. For, consider the differential operator A_1 defined by

$$A_1 = \Delta - \left(|x|^{r-1}\frac{x}{|x|} - |x|^{k-1}\frac{x}{|x|}\right)\cdot\nabla,$$

where $k < r$. Then we still have $\beta = r$ and $c = 1$, so that we get estimate (3.11) again. But now $\rho(x) = c\exp\left\{-\frac{|x|^r}{r} + \frac{|x|^k}{k}\right\}$ and $\gamma < 1/r$ is necessary.

Example 3.6 Consider the following operator with unbounded diffusion coefficients

$$A = (1 + |x|^s)\Delta - |x|^{r-1}\frac{x}{|x|}\cdot\nabla,$$

with $r, s \geq 0$ and such that $r > s$. Fix $0 < \gamma < (r-s)^{-1}$. Then there exists $c \in (0,1)$ such that $\gamma < c(r-s)^{-1}$. With such a choice of c and $\beta = r - s$, it is readily seen that assumption (2.11) is satisfied. Then, by Theorem 3.1 we get that there is a suitable constant $C > 0$ such that

$$\rho(x) \leq C\exp\{-\gamma|x|^{r-s}\}.$$

4 Appendix: An auxiliary result

In this section we show an auxiliary estimate that is classical in spirit, but we could not find in the literature in the exact form we need. For this reason, we present a complete proof, even though we do not claim that it is original.

Let us consider the following differential operator in divergence form

$$A_0 = \sum_{i,j=1}^N D_i(a_{ij}D_j),$$

whose coefficients $a_{ij} = a_{ji}$ belong to $L^\infty(\mathbb{R}^N)$ and satisfy the uniform ellipticity condition

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2,$$

for every $x, \xi \in \mathbb{R}^N$ and some $\lambda > 0$. Take $f \in L^{r/2}(\mathbb{R}^N)$ and $h_i \in L^r(\mathbb{R}^N)$, $i = 1, \ldots, N$, with $r > N$. Set $h = (h_1, \ldots, h_N)$.

Theorem 4.1 *Let $u \in W^{1,2}(\mathbb{R}^N)$ be a weak solution of the equation $A_0u = \mathrm{div}h - f$, i.e.*

$$\int_{\mathbb{R}^N}\sum_{i,j=1}^N a_{ij}D_iuD_j\phi = \int_{\mathbb{R}^N} f\phi + \int_{\mathbb{R}^N} h\cdot\nabla\phi, \tag{4.1}$$

for every $\phi \in C_c^\infty(\mathbb{R}^N)$. Then u is bounded and there exists a constant $C > 0$, depending only on λ, N, r, such that

$$\|u\|_\infty \leq C(\|u\|_{L^2(\mathbb{R}^N)} + \|f\|_{L^{r/2}(\mathbb{R}^N)} + \|h\|_{L^r(\mathbb{R}^N)}).$$

Proof. Let us first assume that $\|u\|_{L^2(\mathbb{R}^N)}, \|f\|_{L^{r/2}(\mathbb{R}^N)}, \|h\|_{L^r(\mathbb{R}^N)} \leq 1$. By a standard density argument, it is readily seen that equation (4.1) is also satisfied by any $\phi \in W^{1,2}(\mathbb{R}^N)$ with compact support.

Fix $k > 0$ and consider the function $(u-k)_+$ which is in $W^{1,2}(\mathbb{R}^N)$. Plugging $\phi = \theta_n^2 (u-k)_+$ in (4.1), where θ_n is a standard sequence of cutoff functions, we get

$$\int_{\mathbb{R}^N} a(\nabla(u-k)_+, \nabla(u-k)_+)\theta_n^2 dx + \int_{\mathbb{R}^N} 2\theta_n a(\nabla(u-k)_+, \nabla\theta_n)(u-k)_+ dx$$
$$= \int_{\mathbb{R}^N} f\theta_n^2 (u-k)_+ dx + \int_{\mathbb{R}^N} \theta_n^2 h \cdot \nabla(u-k)_+ dx + \int_{\mathbb{R}^N} 2\theta_n (u-k)_+ h \cdot \nabla\theta_n dx$$

Now, by the Cauchy-Schwartz inequality we have

$$\left| \int_{\mathbb{R}^N} 2\theta_n a(\nabla(u-k)_+, \nabla\theta_n)(u-k)_+ dx \right| \leq \frac{1}{2} \int_{\mathbb{R}^N} a(\nabla(u-k)_+, \nabla(u-k)_+)\theta_n^2 dx + \frac{C\|a\|_\infty}{n^2} \int_{\mathbb{R}^N} (u-k)_+^2.$$

Set $A_k = \{u > k\}$. Applying Hölder inequality and the Sobolev embedding theorem to the function $(u-k)_+$, when $N > 2$, we get a constant $C_S > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^N} |f|\theta_n^2 (u-k)_+ \, dx &\leq \|f\|_{L^{r/2}(\mathbb{R}^N)} \|(u-k)_+\|_{L^{2^*}(\mathbb{R}^N)} |A_k|^{1-\frac{2}{r}-\frac{1}{2^*}} \\ &\leq C_S \|\nabla(u-k)_+\|_{L^2(\mathbb{R}^N)} |A_k|^{\frac{1}{2}-\frac{2}{r}+\frac{1}{N}}. \end{aligned} \tag{4.2}$$

If $N = 2$, it suffices to apply estimate (2.9) and Hölder inequality to get

$$\begin{aligned} \int_{\mathbb{R}^2} |f|\theta_n^2 (u-k)_+ \, dx &\leq \|f\|_{L^{r/2}(\mathbb{R}^2)} \|(u-k)_+\|_{L^2(\mathbb{R}^2)} |A_k|^{1-\frac{2}{r}-\frac{1}{2}} \\ &\leq C_S \|\nabla(u-k)_+\|_{L^2(\mathbb{R}^2)} |A_k|^{1-\frac{2}{r}}, \end{aligned}$$

which is (4.2) for $N = 2$.

By Fatou lemma, the previous estimates imply that $\int_{\mathbb{R}^N} |f|(u-k)_+ \, dx$ is finite. The same conclusion holds for $\int_{\mathbb{R}^N} |h||\nabla(u-k)_+| \, dx$, as we can estimate

$$\begin{aligned} \int_{\mathbb{R}^N} \theta_n^2 |h||\nabla(u-k)_+| \, dx &\leq \|h\|_{L^r(\mathbb{R}^N)} \|\nabla(u-k)_+\|_{L^2(\mathbb{R}^N)} |A_k|^{\frac{1}{2}-\frac{1}{r}} \\ &\leq \|\nabla(u-k)_+\|_{L^2(\mathbb{R}^N)} |A_k|^{\frac{1}{2}-\frac{1}{r}}. \end{aligned}$$

Finally

$$\left| \int_{\mathbb{R}^N} 2\theta_n (u-k)_+ h \cdot \nabla\theta_n dx \right| \leq \frac{C}{n} \|h\|_{L^r(\mathbb{R}^N)} \|(u-k)_+\|_{L^2(\mathbb{R}^N)} |A_k|^{\frac{1}{2}-\frac{1}{r}}.$$

Collecting all the estimates so far and letting $n \to +\infty$ we get

$$\begin{aligned} \int_{\mathbb{R}^N} a(\nabla(u-k)_+, \nabla(u-k)_+) dx &\leq 2\int_{\mathbb{R}^N} |f|(u-k)_+ dx + 2\int_{\mathbb{R}^N} |h||\nabla(u-k)_+| dx \\ &\leq \|\nabla(u-k)_+\|_{L^2(\mathbb{R}^N)} \big(2C_S |A_k|^{\frac{1}{2}-\frac{2}{r}+\frac{1}{N}} + 2|A_k|^{\frac{1}{2}-\frac{1}{r}}\big). \end{aligned}$$

Using the ellipticity condition we derive

$$\lambda\|\nabla(u-k)_+\|_{L^2(\mathbb{R}^N)} \leq 2C_S|A_k|^{\frac{1}{2}-\frac{2}{r}+\frac{1}{N}} + 2|A_k|^{\frac{1}{2}-\frac{1}{r}}.$$

If $k \geq 1$ then $|A_k| \leq 1$ and, consequently, $|A_k|^{\frac{1}{2}-\frac{2}{r}+\frac{1}{N}} \leq |A_k|^{\frac{1}{2}-\frac{1}{r}}$, since $\frac{1}{2}-\frac{2}{r}+\frac{1}{N} > \frac{1}{2}-\frac{1}{r}$, by the assumption on r. It follows that

$$\lambda\|\nabla(u-k)_+\|_{L^2(\mathbb{R}^N)} \leq \big(2C_S+2\big)|A_k|^{\frac{1}{2}-\frac{1}{r}}.$$

Now, it turns out that for every $h > k$ we can write

$$\begin{aligned}(h-k)^2|A_h| &\leq \int_{A_h}(u-k)_+^2\,dx \leq \int_{A_k}(u-k)_+^2\,dx \leq \|(u-k)_+\|^2_{L^{2^*}(\mathbb{R}^N)}|A_k|^{1-\frac{2}{2^*}}\\ &\leq C_S\|\nabla(u-k)_+\|^2_{L^2(\mathbb{R}^N)}|A_k|^{\frac{2}{N}} \leq L\,|A_k|^{1-\frac{2}{r}+\frac{2}{N}}, \qquad (4.3)\end{aligned}$$

where $L = C_S\lambda^{-2}\big(2C_S+2\big)^2$. At this point, we take

$$k_n = \overline{k} + \Big(1-\frac{1}{2^n}\Big)\overline{k}, \qquad n = 0,1,2,\ldots$$

with $\overline{k} \geq 1$ to be determined. Replacing h with k_{n+1} and k with k_n in (4.3), we obtain, for every $n \in \mathbb{N}$,

$$y_{n+1} \leq \frac{4L}{\overline{k}^2}\,2^{2n}\,y_n^{1+\alpha},$$

where

$$y_n = |A_{k_n}|, \qquad \alpha = -\frac{2}{r}+\frac{2}{N} > 0.$$

From [7, Lemma 7.1] it follows that if

$$y_0 \leq \left(\frac{4L}{\overline{k}^2}\right)^{-\frac{1}{\alpha}} 4^{-\frac{1}{\alpha^2}},$$

then $\lim_{n\to+\infty} y_n = 0$, which means that $u \leq 2\overline{k}$, a.e. in $\mathbb{R}^N$. As $y_0 \leq 1$, it suffices to choose $\overline{k} = \max\{1, 2^{1+\frac{1}{\alpha}}\sqrt{L}\,\}$. By linearity, interchanging u by $-u$ we get

$$\|u\|_\infty \leq 2\max\{1, 2^{1+\frac{1}{\alpha}}\sqrt{L}\,\} =: C.$$

Finally, in the general case, we consider $\tilde{f} = f/M$ and $\tilde{h} = h/M$, where $M = \|u\|_{L^2(\mathbb{R}^N)} + \|f\|_{L^{r/2}(\mathbb{R}^N)} + \|h\|_{L^r(\mathbb{R}^N)}$. Then, by linearity, the solution corresponding to the new data is $\tilde{u} = u/M$. From the first part of the proof it follows that $\|\tilde{u}\|_\infty \leq C$, i.e., $\|u\|_\infty \leq CM$, which is the thesis. □

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